Università degli Studi di Padova

FACOLTÀ DI SCIENZE MATEMATICHE, FISICHE E NATURALI
Corso di Laurea Specialistica in Fisica

Black Holes and Taub-NUT spacetimes in N=2 Supergravity

Tesi di Laurea

Relatore
Prof. PAOLO PASTI

Laureando
SIMONE SORGATO

Correlatore
Prof. TOMÁS ORTÍN

Contents

Notation and conventions

We use Planck units, $c = \hbar = G = k_B = 1$, except for some parts in the Introduction and in Chapter 2 (every change of units is specified in the text). We always consider theories in four dimensions. Our conventions about the indices of curved and flat space, together with the related metrics which raise and lower them, are

indices	space	metric	signature
$\mu, \nu \ldots$	curved 4-d	$g_{\mu\nu}$	$(+,-,-,-)$
$\underline{m}, \underline{n}, \ldots$	curved 3-d	$\gamma_{\underline{mn}}$	$(+,+,+)$
$a, b, \ldots$	flat 4-d	η_{ab}	$(+,-,-,-)$
$m, n, \ldots$	flat 3-d	δ_{mn}	$(+,+,+)$

The determinant of the metric $g_{\mu\nu}$ is denoted with g and is negative, that of $\gamma_{\underline{mn}}$ with γ and is positive. The Vielbein basis that we use is reported in Appendix A. The 4-dimensional spacetime covariant derivative, given in terms of the Levi-Civitá connection, is denoted with ∇_μ, the 3-dimensional one with $\nabla_{\underline{m}}$. We will use very often the following relations

$$\begin{aligned}
\nabla_{\mu_1} T_{4d}^{\mu_1 \ldots \mu_k} &= \frac{1}{\sqrt{|g|}}\, \partial_{\mu_1} \left(\sqrt{|g|}\, T_{4d}^{\mu_1 \ldots \mu_k} \right) \\
\nabla_{\underline{m}_1} T_{3d}^{\underline{m}_1 \ldots \underline{m}_k} &= \frac{1}{\sqrt{|\gamma|}}\, \partial_{\underline{m}_1} \left(\sqrt{|\gamma|}\, T_{3d}^{\underline{m}_1 \ldots \underline{m}_k} \right)
\end{aligned}$$

where $T_{4d}^{\mu_1 \ldots \mu_k}$ and $T_{3d}^{\underline{m}_1 \ldots \underline{m}_k}$ are respectively 4-dimensional and 3-dimensional completely antisymmetric tensors of rank k. Other covariant derivatives, when needed, are specified in the text.

The antisymmetric Levi-Civitá symbol, in tangent space, is defined as

$$\epsilon^{0123} = +1 \qquad \epsilon_{0123} = -1$$

and in curved space (underlining the indices to stress that they are curved) it is

$$\epsilon^{\underline{0123}} = +1 \qquad \epsilon_{\underline{0123}} = g = -|g|.$$

Given a completely antisymmetric tensor $F_{(k)}$ of rank k, its *Hodge dual* is defined as the rank $4-k$ completely antisymmetric tensor ${}^\star F_{(4-k)}$, with components

$$
{}^\star F_{(4-k)}^{\mu_1\cdots\mu_{(4-k)}} = \frac{1}{k!\sqrt{|g|}}\epsilon^{\mu_1\cdots\mu_{(4-k)}\mu_{(4-k+1)}\cdots\mu_4}F_{(k)\,\mu_{(4-k+1)}\cdots\mu_4}.
$$

If F is of rank 2, in four dimensions we have

$$
{}^{\star\star}F = -F.
$$

Real scalar fields are denoted with ϕ^i, $i = 1, \ldots, n$ (see Chapter 3); complex scalar fields are indicated with z^i and their complex conjugates are $\overline{z}^{i^*}$; in general, a bar over a quantity denotes its complex conjugate. Vector fields (when there are more than one) bear a capital Greek index, $\Lambda, \Sigma = 1, \ldots, n_V$, with $n_V = n + 1$. Spinors will appear very rarely in Chapter 3: they are always 4-component Dirac spinors, with spinor indices $\alpha, \beta, \ldots$.

Chapter 1

Introduction

A black hole is a closed region of spacetime containing a singularity (a point, or more generally a surface, wherein physical laws cease to be valid) from which light, and therefore particles, cannot escape to infinity; it is enclosed by a semi-permeable surface, named *event horizon*, that can be crossed coming from infinity but not in the opposite sense.
Although originally (the Schwarzschild's solution was found already in 1916) black holes were considered only a troublesome curiosity in General Relativity, in the years between 1960 and 1975 they started to be object of a great deal of theoretical and observational effort, owing to the realization that they could actually have a physical significance as trustful descriptions of certain extreme states of matter, likely to exist in astrophysical contexts (as consequences of the gravitational collapse of massive objects). In that period, people like Hawking, Israel, Carter, Wheeler, between the others, contributed to the development of a general theory of black holes, new such solutions (for example Kerr's) were found and various results (no-hair theorem, black hole thermodynamics, Hawking radiation,. . .) established. One of the more outstanding achievements was undoubtedly the Bekenstein-Hawking formula ([1], [2]), relating the area A of the event horizon to the entropy S of the black hole:

$$S = k_B \frac{Ac^3}{4\hbar G} \tag{1.1}$$

where k_B is the Boltzmann constant. The particularity of the above equation is that it relates a geometric quantity to a thermodynamic one; as we will see, the (1.1) is a consequence of a series of correspondences between the laws of black hole physics and those of thermodynamics.

Subsequent to these developments was the discovery of supersymmetric theories, in which symmetry under the Poincaré group is extended by adding to the Poincaré algebra a set of anticommuting spinor generators, which transform fermions into bosons and vice versa, these transformations being global (independent on spacetime coordinates); and then of their local versions, with the supersymmetry transformations depending on the points of spacetime, i.e. Supergravity. This latter is a generalisation of General Relativity, in which gravity is supersymmetrically coupled to other fields, and then comprises all the classical solutions of Einstein's relativity, so that black holes fit naturally into the supergravity background; clearly their structure gets more complicated, since requirement of supersymmetry translates in the presence of additional bosonic and fermionic fields (here we will consider only bosonic configurations). Further on, Supergravity theories are low energy approximations of the more fundamental Superstring theory, that could turn out to be the final theory of quantum gravity. The fact that $\hbar$ is involved in (1.1) reveals the quantum-mechanical origin of the entropy, and it would be auspicable an explanation of (1.1) based on statistical methods, i.e. determining and counting the black hole microstates; this is a field in which Superstrings have given some promising results (progress in this field started to be achieved in the middle of the '90es, with the work of Vafa and Strominger, [8]). In this thesis, however, we will not be concerned at all with the quantum mechanical side of the problem. Instead, for all the black holes we will consider the radius of the horizon is much larger that the string scale, so that they can be well described by the supergravity approximation; moreover, we will limit ourselves to deal with *large* black holes, for which (1.1) is valid. When this is not the case and the area of the horizon vanishes, one speaks of *small* black holes; for them, calculation of entropy need quantum corrections.

A particular attention has been devoted through the years to the so-called *extremal* black holes, which we present in Chapter 2 in the context of electrically charged black holes (Reissner-Nordström's). It will be shown that the event horizon exists only when the bound[1] $M \geq |Q|$ is satisfied (*extremality bound*; M and Q are the mass and the electric charge; other parameters characterizing the solution can appear in more general cases, for example angular momentum or other conserved charges, see Chapter 5); if this does not happen, there is not an horizon covering the singularity and a *naked singularity* appears. Extremal black holes are those for which the above bound is saturated, $M = |Q|$; they possess some particular features which make them easier to deal with in the Supergravity and String theory framework (so far statistical computation of black hole entropy has been successfully done only for extremal solutions). They are the only stable black

[1]In chapter 2 this expression will be slightly different due to the normalization of the action used there.

hole configurations, since loss of mass via Hawking radiation would bring them beyond the extremal limit; consequently they cannot radiate and we will see that their temperature is zero.
The feature of extremal black holes that we are most interested in in this work is the *attractor mechanism*: in certain extremal black hole configurations of gravity coupled to matter (in particular to scalar and vector fields) in supersymmetric theories, scalars (or *moduli*) exhibit an attractive behaviour near the horizon, in the sense that they run to fixed points while approaching the black hole. In most of the cases, the values that scalars assume on the horizon are independent from their initial conditions at infinity and the attractive points can be found as functions of the conserved electric and magnetic charges of the theory. This however is not true in general, since examples have been found in which an (we could say incomplete) attractor mechanism is present but some of the fixed points do not lose the dependence on their own boundary conditions. It seems that the complete attractor mechanism is proper of extremal black holes that are also supersymmetric (these are named *BPS-extremal black holes*), i.e. that preserve a certain fraction of the original supersymmetry. In any case, the existence of this attractor behaviour has strong consequences on the formula (1.1): as we will report in Chapter 4, the entropy of extremal black holes turns out to depend only on electric and magnetic charges.

The purpose of this thesis is to analyse how the approach to the attractor mechanism for static black holes coupled to N=2, d=4 vector supermultiplets depicted in the article *Black holes and critical points in moduli space* by Ferrara, Gibbons and Kallosh ([15]) can be generalized to stationary metrics. The importance of the study performed in this paper is that the arising of an attractor mechanism for extremal static black holes is deduced without use of supersymmetry, as it was done in the pioneeristic papers [14] and [23], where attractors were first discovered. The crucial point in this approach is that the equations governing the system can be reformulated in terms of two quantities which depend only on scalar fields: a symmetric matrix $\mathcal{G}_{ij}$, which is the matrix controlling the scalar sector of the action (i.e. the metric of the space spanned by the scalar fields), and the so-called *black hole potential* V_{BH}: we will see in Chapter 4 how it can be defined for static black holes.
It would be interesting to extend this treatment to rotating, charged black holes (Kerr-Newman's solutions, see Chapter 5), but the complexity of the problem suggests to start with a simpler case. We then will consider a general stationary metric of the form

$$ds^2 = e^{2U}(dt+\omega)^2 - e^{-2U}\gamma_{\underline{mn}}dx^{\underline{m}}dx^{\underline{n}} \tag{1.2}$$

also called *conforma-stationary metric*, and study the equations of motion of scalar and vector fields (belonging to vector supermultiplets of N=2, d=4 Supergravity) coupled to gravity in the background of metric (1.2), trying to determine a suitable black hole potential. To work with a specific spacetime, we will introduce in Chapter 5 a particular stationary solution of General Relativity, the Taub-NUT solution, showing that it can be put in the form (1.2). This solution is in reality a rather strange object, with a certain number of particular features, which nevertheless plays an important role in many fields of theoretical physics. For our purposes, it will be useful to see how non-diagonal terms in the metric enters in the equations of motion and how they could affect the emerging of a possible attractor mechanism. For what concerns the applicability of the calculations that we will perform to the Kerr-Newman case, a good starting point would be to put the Kerr-Newman metric in the conformastationary form (1.2), and this already seems to be problematic.

The thesis is organized as follows:

Chapter 2 introduces static black holes, starting with the Schwarzschild's solution and then moving to the Reissner-Nordström's charged black hole. Some topics like no-hair theorems and black hole thermodynamics are discussed, and some useful changes of coordinates are performed.

Chapter 3 reviews the basic properties of black holes embedded in Supergravity, especially referring to N=2, d=4 theories, introducing some formalism and quantities which will enter in the description of attractors.

Chapter 4 reports the calculations done in [15], and a review of the attractor mechanism for extremal black holes is given.

Chapter 5 after a brief description of the Kerr-Newman black hole, introduces the Taub-NUT solution. The related Einstein equations are worked out and solved, to prepare the ground for the general case.

Chapter 6 contains the original calculation for the equations of motion of gravity coupled to supergravity vector multiplets in a Taub-NUT spacetime. The possible occurring of an attractor mechanism is discussed.

Chapter 2

Static Black Holes

In this chapter we review some properties of static black holes, starting with the Schwarzschild's solution and then moving to its charged version (Reissner-Nordström's black hole). We also perform some changes of coordinates which put the solutions in a form that will be useful for the calculations done in the next chapters. Only in this chapter, we will use the normalisations and notations of chapter 8.2 of [3]: $c = 1$ and the electric and magnetic charges, which will be defined in the following, have dimensions of mass.

2.1 The Schwarzschild black hole

The concept of Black Hole (BH) arises from the analysis of Schwarzschild's solution of General Relativity. Schwarzschild metric is the only solution of vacuum Einstein equations which describes the spacetime outside a massive static and spherically symmetric body.[1] The metric is expected to reflect the symmetries of the source, so a static (time-independent: a timelike Killing vector does exist and it is orthogonal to some family of hypersurfaces, so that there are not mixed terms like $dt\,dr, \ldots$ in the metric) and spherically symmetric (i.e. $SO(3)$-invariant[2]) solution is sought. Einstein equations in absence of matter are:

$$R_{\mu\nu} - \frac{1}{2}R = 0 \tag{2.1}$$

[1]The uniqueness of the solution is a consequence of Birkhoff's theorem; the statement remains true also relaxing the staticity condition, since the region outside the source does not depend on its internal dynamical properties.

[2]This means that the metric should admit three Killing vectors whose commutators give the algebra of $SO(3)$.

and a way to solve them is to make an ansatz about the form of the solution, which, using *Schwarzschild's coordinates*, in this case is

$$ds^2 = f(r)\,dt^2 - g(r)dr^2 - h(r)^2\left(d\theta^2 + \sin^2\theta\,d\varphi^2\right) \tag{2.2}$$

with $f(r)$, $g(r)$ and $h(r)$ undetermined functions of the radial coordinate r. Inserting the ansatz (2.2) in the equation of motion and imposing asymptotic flatness one obtains $g(r) = f(r)^{-1}$ and $h(r) = r$; the form of $f(r)$ can be completely specified requiring that, at large distance (large r), in the weak field regime, the motion of a massive test particle be that of a body in a Newtonian gravitational field produced by a spherically symmetric source located at $r = 0$. Schwarzschild's solution is then

$$ds^2 = \left(1 - \frac{2MG}{r}\right)dt^2 - \left(1 - \frac{2MG}{r}\right)^{-1}dr^2 - r^2\left(d\theta^2 + \sin^2\theta\,d\varphi^2\right) \tag{2.3}$$

and M can be identified with the mass of the body generating the gravitational field; it represents also the total mass (energy) of the spacetime: for an asymptotically flat spacetime this quantity is denoted ADM mass M_{ADM}, from Arnowitt, Deser, Misner (see for example [3] or [4]). The metric is valid in the radial interval going from $r = r_e$, with r_e the radius of the source, to infinity.[3]

It is evident from (2.3) that this metric is singular at $r = 0$ and $r = 2MG \equiv r_S$, where r_S is the Schwarzschild radius of the source: in the case of a spherically symmetric body in equilibrium both the points will be inside the matter-filled interior, so the presence of singularities in the metric does not really have physical implications; however, it is also well known that bodies with a sufficiently large mass undergo complete gravitational collapse. In this case, the behaviour of the Schwarzschild solution becomes significative in the $r \le r_e$ region. The first issue is then to identify the true nature of the metric singularities, i.e. to understand if the divergences of the metric for $r = 0$ and $r = r_S$ reflect real singularities of the underlying spacetime geometry or if they are instead caused by the failure of the chosen coordinate system to describe correctly some regions of the spacetime (in this case they would be *coordinate singularities*). In the current case, calculation of curvature scalars (such as the *Kretschmann invariant* $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$) shows that the singularity at $r = 0$ is truly physical, not depending on the choice of coordinates, while the one for $r = r_S$ is simply a divergence related to the coordinate system that we are using. However, an analysis of the motion along radial null

[3] For $r < r_e$ the Einstein equations must be solved considering an energy-momentum tensor which describes the interior of the spherically symmetric body; this leads to the *Schwarzschild's interior solutions.*

curves (describing the motion of massless particles, let's say a photon) reveals interesting characteristics of the zone near r_S: considering radial null curves ($\theta, \varphi =$ constant, $ds^2 = 0$) we get

$$ds^2 = \left(1 - \frac{r_S}{r}\right) dt^2 - \left(1 - \frac{r_S}{r}\right)^{-1} dr^2 = 0 \tag{2.4}$$

and then

$$\frac{dt}{dr} = \pm\left(1 - \frac{r_S}{r}\right)^{-1};$$

this last quantity measures the slope of light cones in a $r-t$ diagram: for large r they clearly assume the form they would have in flat space, but for $r \sim r_S$ it happens that $dt/dr = \pm\infty$ and the light cones close up; it seems that the photon keeps approaching indefinitely the surface at $r = r_S$ without reaching it. In reality, this is the situation as seen by a (sufficiently) distant observer, whose proper time can be identified with the coordinate t. It can be easily calculated ([3]) that, when measured by the proper time a free-falling particle (massive or not), the amount of time needed to go from a point $r > r_S$ to, for example, $r = 0$ is finite and nothing strange happens when crossing r_S. If now we switch to *Eddington-Finkelstein coordinates* (u, r, θ, φ), where t is substituted by

$$U = t + r + r_S \log\left(\frac{r}{r_S} - 1\right)$$

the metric (2.3) becomes

$$ds^2 = \left(1 - \frac{r_S}{r}\right) du^2 - 2du\, dr - r^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right); \tag{2.5}$$

the condition for radial null curves gives, for ingoing motion,

$$\frac{du}{dr} = 0 \tag{2.6}$$

while for outgoing photons

$$\frac{du}{dr} = 2\left(1 - \frac{r_S}{r}\right)^{-1}. \tag{2.7}$$

From (2.6) it can be seen that the coordinate u is constant on radial null geodesic. Particles coming from outside can cross the surface at $r = r_S$; but from (2.7) it can also be deduced that, for $r < r_S$, the light cones tilt over so that motion can occur only towards smaller radii. In the end, once a particle has crossed the surface at r_S (which is light-like), it is obliged to keep travelling inward, until it reaches the (spacelike) singularity at $r = 0$, while nothing can pass r_S going outward. For this reason the $r = r_S$ hypersurface is named *event horizon* and the region of spacetime described by the Schwarzschild metric without matter in the interval $0 < r \leq r_S$ is called a *black hole*.

As already said, the region corresponding to $0 < r \leq r_e$ can be properly described by a Schwarzschild (vacuum) metric in the case of gravitational collapse of the massive source. In particular, when the radius of the collapsing distribution of matter becomes less than r_s, the spacetime for $r < r_s$ will exhibit the behaviour presented above, and so it seems to acquire a physical significance. Nowadays it is well known that black holes can form during the final stages of the gravitational collapse of sufficiently massive stars[4]; there is also evidence of the presence of supermassive black holes in the centre of galaxies, where they could have originated from the collapse of the core of a dense star cluster; another possibility is the formation of primordial black holes in the first stages of the evolution of the universe, due to matter density inhomogeneities.

Isotropic form for the Schwarzschild metric. For future purposes, it is convenient to rewrite the metric (2.3) in a different coordinate system with a new radial coordinate τ, in which the horizon is reached when $\tau \to -\infty$. Let's set $G = 1$ and $M \equiv r_0$; to arrive at the desired form we firstly perform the change

$$r = (\rho + r_0/2)^2/\rho \tag{2.8}$$

which puts (2.3) in the spatially isotropic form

$$ds^2 = \left(1 - \frac{r_0/2}{\rho}\right)^2 \left(1 + \frac{r_0/2}{\rho}\right)^{-2} dt^2 - \left(1 + \frac{r_0/2}{\rho}\right)^4 \left(d\rho^2 + \rho^2 d\Omega^2_{(2)}\right) \tag{2.9}$$

where $d\Omega^2_{(2)} = d\theta^2 + \sin^2\theta\, d\varphi^2$ is the metric on the unitary 2-sphere; the horizon is at $\rho = r_0/2 \equiv \rho_h$. We note that, in the limit $r_0 \to 0$ (the so-called *extremal limit*), the metric reduces to Minkowski's. The parameter that goes to zero in the extremal limit is called the *non-extremality parameter*, and in the Schwarzschild case it is the mass M (the real importance of this limit will be clear when dealing with more general, charged black holes). Now, switching to the radial coordinate τ, defined by

$$\rho = -\frac{r_0}{2\tanh\frac{r_0\tau}{2}}, \tag{2.10}$$

[4]Taking into account the various effects which can affect the collapse of a star, such as rotation, mass loss, possibility to form a neutron star and so on, a rough estimation of the mass range in which a star is likely to collapse to a black hole is $2M_\odot \leq M \leq 100M_\odot$, with $M_\odot$ the solar mass.

the (2.9) takes the form

$$ds^2 = e^{2U}dt^2 - e^{-2U}\gamma_{\underline{mn}}dx^{\underline{m}}dx^{\underline{n}} \tag{2.11}$$

where the spatial three-dimensional metric is

$$\gamma_{\underline{mn}}dx^{\underline{m}}dx^{\underline{n}} = \frac{r_0^4}{\sinh^4 r_0\tau}d\tau^2 + \frac{r_0^2}{\sinh^2 r_0\tau}d\Omega^2_{(2)}. \tag{2.12}$$

Actually, the (2.11) is valid for the exterior region of any static black hole, with $r_0 > 0$, depending on the form of the function $U = U(\tau)$; for the Schwarzschild black hole it specializes to $U(\tau) = r_0\tau$. The range of the coordinate τ is $(-\infty, 0)$, which corresponds to the zone between, respectively, the horizon and spatial infinity. The use of a coordinate which reaches $-\infty$ at the event horizon is common when studying the *attractors mechanism*, as will be shown in Chapter 4.

2.2 The Reissner-Nordström black hole

The second example of black hole is the Reissner-Nordström's, the simplest BH-type solution of General Relativity in presence of matter fields, which in this case are a single abelian, massless vector field A_μ, whose associated field strength is $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$. The action for gravity coupled to electromagnetism is the *Einstein-Maxwell action*

$$S_{EM} = \frac{1}{16\pi G}\int d^4x\sqrt{|g|}\left(R - \frac{1}{4}F^2\right) \tag{2.13}$$

from which the following equations of motion for the metric $g_{\mu\nu}$ and the vector field A_μ descend

$$G_{\mu\nu} = \frac{1}{2}T_{\mu\nu} \qquad \text{Einstein's equation} \tag{2.14}$$

$$\nabla_\mu F^{\mu\nu} = 0 \qquad \text{Maxwell's equation,} \tag{2.15}$$

with

$$T_{\mu\nu} = F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}$$

the energy-momentum tensor of the electromagnetic field. In the action (2.13) a source for the electromagnetic field does not appear, however it is always possible to define the total electric charge of spacetime through (see, for example, [3])

$$q = \frac{1}{16\pi G}\int_{S^2_\infty} {}^\star F \tag{2.16}$$

where a constant-time hypersurface with the topology of a 2-sphere at infinity has been considered.
As in Schwarzschild's case, a static, spherically symmetric solution is looked for: it seems obvious then to reconsider the ansatz (2.2), always using Schwarzschild coordinates; moreover, the following suggestion for the form of the electromagnetic field strength

$$F_{tr} \propto \pm \frac{1}{h(r)} \tag{2.17}$$

follows from the requirement that the solution be a point-like charged, static object. Then, inserting the ansatzs (2.2) and (2.17) in the equations of motion, the *electric Reissner-Nordström solution* is found:[5]

$$ds^2 = \frac{(r-r_+)(r-r_-)}{r^2}dt^2 - \frac{r^2}{(r-r_+)(r-r_-)}dr^2 - r^2 d\Omega^2_{(2)} \tag{2.18}$$

$$F_{tr} = \frac{4Gq}{r^2} \tag{2.19}$$

where

$$r_\pm = GM \pm r_0 \qquad\qquad r_0 = G\sqrt{M^2 - 4q^2}. \tag{2.20}$$

Here M is the total mass of the Reissner-Nordström's spacetime (the ADM mass), containing all the contributions to the energy: that associated with matter and the ones related to the electromagnetic and gravitational fields. Even if the (2.18) is a solution for every value of M and q, we shall restrict ourselves to real and positive values of $r_\pm$, the case which corresponds to $M \geq 2|q|$. The metric is then singular at $r = 0$ and $r = r_+, r_-$; by computing some curvature invariants one realizes that the divergence at $r = 0$ is a curvature singularity, while those for $r = r_+, r_-$ are not. The value r_+ is where the event horizon (with area $A = 4\pi r_+^2$) is located, while the hypersurface corresponding to r_- is a *Cauchy horizon*, i.e. a light-like surface which acts as a boundary for the domain of validity of a Cauchy problem.[6] It can be seen that (2.18), properly extended in the beyond-the-horizon region and with the matter concentrated in the origin describes a black hole, with a timelike singularity in $r = 0$ that can be avoided by an observer which enters the event horizon.

[5] And, again, thanks to Birkhoff's theorem it is possible to prove that this is the unique family of spherically symmetric solutions of the Einstein-Maxwell system. Moreover, it is easy to see that if the electric charge is zero we recover Schwarzschild's metric.

[6] For a rigorous discussion, see [4].

Extremal Reissner-Nordström black holes. In the special case $M = 2|q|$, the two horizons coincide being $r_+ = r_- = GM$; the resulting object is an *extreme* (or *extremal*) Reissner-Nordström black hole, and the (now vanishing) non-extremality parameter is r_0, defined in (2.20). It can be seen that the causal structure of an extremal black hole is profoundly different from that of a non-extremal one [3]. The physical properties of an extremal Reissner-Nordström solution are in fact quite different from those of a standard one, even for infinitesimal deviations from the extremal limit $M = 2|q|$. The metric becomes

$$ds^2 = \left(1 - \frac{r_+}{r}\right)^2 dt^2 - \left(1 - \frac{r_+}{r}\right)^{-2} dr^2 - r^2 d\Omega^2_{(2)} \tag{2.21}$$

and the horizon area is given by $A_{extreme} = 4\pi r_+^2 = 4\pi (GM)^2$. A particular feature which will have a role in the attractor mechanism presented in Chapter 4 is that the proper distance of every point from the horizon, along radial directions and at constant time, is infinite

$$\lim_{r_2 \to r_+} \int_{r_1}^{r_2} dr \left(1 - \frac{r_+}{r}\right)^{-1} = \infty. \tag{2.22}$$

Now we can shift the radial coordinate $\rho = r - GM$ in the metric (2.21) and obtain

$$ds^2 = \left(1 + \frac{GM}{\rho}\right)^{-2} dt^2 - \left(1 + \frac{GM}{\rho}\right)^2 \left(d\rho^2 + \rho^2 d\Omega^2_{(2)}\right) \tag{2.23}$$

the horizon being now placed at $\rho = 0$. Taking the near-horizon limit $\rho \to 0$, the metric looks like

$$ds^2 = \left(\frac{\rho}{GM}\right)^2 dt^2 - \left(\frac{\rho}{GM}\right)^{-2} d\rho^2 - (GM)^2 d\Omega^2_{(2)} \tag{2.24}$$

and this goes under the name of *Robinson-Bertotti* (or also $AdS_2 \times S^2$) metric. The space described by (2.24) is the direct product of a two-dimensional anti-de Sitter space (AdS_2) with radius $R_{AdS} = GM$ ($t - \rho$ part of the metric) and isometry group $SO(1,2)$, and of a 2-sphere S^2 with the same radius (and clearly $SO(3)$-invariant). This direct product is then invariant under $SO(1,2) \times SO(3)$, so the near-horizon spacetime possesses more symmetries if compared with the metric (2.21), which has isometry group $SO(1,1) \times SO(3)$ ($SO(1,1) \sim \mathbb{R}^+ \times \mathbb{Z}_2$ represents shifts in time and time inversions).

Another peculiarity of extremal black holes is the fact that their temperature is zero, as we will see when talking about black hole thermodynamics. The importance of the extremal limit when black holes are considered in extended Supergravity theories will be clarified in the next chapters.

Dyonic Reissner-Nordström black holes. The metric (2.18) can be generalized including a magnetic charge p, which is defined as the integral of the Hodge dual of ${}^{\star}F$

$$p \equiv -\frac{1}{16\pi G}\int_{S^2_\infty} F. \tag{2.25}$$

The magnetic charge can be inserted in the RN metric modifying the definition of r_0 which becomes

$$r_0 = G\sqrt{M^2 - 4\left(p^2 + q^2\right)} \tag{2.26}$$

so that now

$$\frac{(r-r_+)(r-r_-)}{r^2} = 1 - \frac{2GM}{r} + \frac{4G^2(p^2+q^2)}{r^2};$$

the electromagnetic field is given by

$$F_{tr} = \frac{4Gq}{r^2} \qquad\qquad F_{\theta\varphi} = -4Gp\sin\theta \tag{2.27}$$

or, more symmetrically

$$F_{tr} = \frac{4Gq}{r^2} \qquad\qquad ({}^{\star}F)_{tr} = \frac{4Gp}{r^2}. \tag{2.28}$$

This gives a *dyonic* black hole. Apart from the shifting in the position of the horizons $r_\pm$, the structure of this more general black hole is identical to the electrically charged one.

τ-coordinate version of the Reissner-Nordström metric. Also for this black hole it is possible to express the metric in terms of the radial coordinate τ introduced in the Schwarzschild case: setting again $G = 1$, we firstly define the coordinate ρ by

$$r = \frac{\rho^2 + M\rho + \frac{r_0^2}{4}}{\rho}$$

obtaining the spatially isotropic metric

$$ds^2 = \frac{\left(1 - \frac{r_0/2}{\rho}\right)^2\left(1 + \frac{r_0/2}{\rho}\right)^2}{\left(1 + \frac{\rho_+/2}{\rho}\right)^2\left(1 + \frac{\rho_-/2}{\rho}\right)^2}dt^2 - \left(1 + \frac{\rho_+/2}{\rho}\right)^2\left(1 + \frac{\rho_-/2}{\rho}\right)^2\left(d\rho^2 + \rho^2 d\Omega^2_{(2)}\right) \tag{2.29}$$

where

$$\rho_\pm = M \pm 2|q|$$

and the horizons are at $\rho = \pm\frac{r_0}{2}$. Then, performing the reparametrization

$$\rho = -\frac{r_0}{2\tanh\frac{r_0\tau}{2}},$$

we obtain again a metric of the general form (2.11), but this time with

$$e^{-2U} = e^{-2r_0\tau}\left[\frac{r_+}{2r_0} - \frac{r_-}{2r_0}e^{2r_0\tau}\right]^2. \tag{2.30}$$

As expected, if $q = 0$ the (2.30) reduces to $e^{-2U} = e^{-2r_0\tau}$, which gives the Schwarzschild metric.

2.3 Cosmic censorship and no-hair conjectures

If negative values for the mass M of the Schwarzschild black hole are allowed, there is no more an event horizon covering the singularity in $r = 0$. Analogously, if we admitted negative ($M < -2|q|$) and complex ($-2|q| < M < 2|q|$) values for the $r_\pm$ of the Reissner-Nordström's one, we would find in both cases that there are no event horizons surrounding $r = 0$; both the curvature singularities would be *naked* and they could be experienced by all observers. However, there are strong arguments suggesting to exclude the possibility for a naked singularity to exist as the endpoint of the gravitational collapse of a star (or of an object with a physically acceptable energy-momentum tensor): for example, the study of linear perturbations of the Schwarzschild metric shows that the collapse of a system with small deviations from spherical symmetry produces a black hole, not a naked singularity [5]. This idea is condensed in Penrose's *cosmic censorship conjecture* [6], which in its weak form states that the complete gravitational collapse of a body always ends with a black hole, or, in other words, that singularities deriving from the collapse are hidden by an event horizon and cannot be seen by distant observers.

Another hypothesis about the general features of static (but also stationary) BH-type solutions of the Einstein-Maxwell equations is the *no-hair conjecture*: any such black hole is fully described by the three parameters mass M, angular momentum J and electric charge Q (and other locally conserved charges). Analysis of the collapse of a perturbed Schwarzschild metric shows that the final object is a Schwarzschild black hole, whose only parameter is the mass; in the same way, calculations relative to the collapse of a star with small departures from spherical symmetry and small non-zero electric charge Q prove that the final state is a black hole with external fields determined solely by M, Q and J. More generally, there exists a theorem (*uniqueness* *theorem*) whose formulation was suggested by Israel, Penrose and Wheeler, which states that:

- the only black hole with charge and angular momentum both vanishing is Schwarzschild's;
- with mass and electric charge is Reissner-Nordström's;
- the only black hole with mass and angular momentum is Kerr's. Then all the stationary solutions of the Einstein-Maxwell system are comprised in the Kerr-Newman family (see Chapter 5).

Moreover, there are not black holes with non-constant scalar fields.[7] Generally, the presence of non-constant scalar fields and of higher order multipole momenta of the electromagnetic and gravitational field (that is, different from charge, mass and angular momentum) is associated to the absence of an event horizon, so that the corresponding solution is actually not a black hole. It must be emphasized that the above statements work only for Einstein-Maxwell black holes: a generalized version of the no-hair conjecture could suggest that, in the presence of other types of matter (for example, in Einstein-Yang-Mills systems), stationary black holes are described only by a set of global charges, but this does not happen. For a review, see [9].

2.4 Black hole Thermodynamics

In this section we will point out some crucial features of black holes which permit to consider them as (nearly) ordinary thermodynamic systems. First, it was shown by Hawking [10] that, as a consequence of Einstein equations, the area A of an event horizon never decreases with time: this result clearly resembles the second law of thermodynamics, and it could suggest some analogy between the area of a black hole and the entropy, the two being both never decreasing quantities. Secondly, the event horizon of stationary black holes is a *Killing horizon* (at least for the cases of our interest): that is, it is invariant under one isometry of the metric with generating Killing vector k^μ and the modulus of k^μ vanishes on the horizon. On Killing horizons the *surface gravity*

$$\kappa = -\frac{1}{2}\left(\nabla^\mu k^\nu\right)\left(\nabla_\mu k_\nu\right)\big|_{\text{horizon}} \tag{2.31}$$

can be defined and it turns out to be constant over the event horizon of stationary black holes (for example, see [4]). This brings to a second analogy

[7] For more information about the above statements, see references in [3] and [7].

with thermodynamic system, remembering that the zeroth law of thermodynamics states that the temperature is constant at every point of a body in thermodynamic equilibrium. Furthermore, in the specific Schwarzschild's case, it can be shown ([11]) that the following relation between mass (energy), surface gravity and horizon area holds

$$dM = \frac{1}{8\pi G}\kappa\, dA \tag{2.32}$$

and this could be viewed as a black hole version of the first law of thermodynamics

$$dE = T\, dS$$

so that the identifications $A \leftrightarrow S$ and $\kappa \leftrightarrow T$ seem to be plausible. The discovery of Hawking's radiation ([12]) allowed to fix the proportionality in the above relations, since the temperature can be expressed by the surface gravity through[8]

$$T = \frac{1}{k_B}\frac{\hbar\kappa}{2\pi c} \tag{2.33}$$

leading to the *Bekenstein-Hawking entropy-area* formula:

$$S = k_B\,\frac{Ac^3}{4\hbar G} \tag{2.34}$$

or also

$$S = \frac{k_B}{l_P^2}\frac{A}{4} \tag{2.35}$$

where l_P^2 is the square of the Planck length, $l_P^2 = \hbar G/c^3$. The above formulas are referred to the Schwarzschild black hole; they can be readily generalized to the Reissner-Nordström case. In (2.32) it has to be introduced a new term to keep into account the effects of charge variations on the changes in the mass:

$$dM = \frac{1}{8\pi G}\kappa\, dA + \phi^h\, dq \tag{2.36}$$

with ϕ^h the electrostatic potential on the horizon

$$\phi^h = \phi(r_+) = \frac{4G\, q}{r_+},$$

while the relations (2.33) and (2.34) keep the same form: specifying them with the values of κ and A proper of Reissner-Nordström black holes we have

[8] In the next three equations we temporarily restore c and use also the Boltzmann constant k_B, so that the temperature is measured in kelvin and the entropy in joule/kelvin.

($\hbar = c = k_B = 1$)

$$T = \frac{r_0}{2\pi r_+^2} = \frac{1}{2\pi G} \frac{\sqrt{M^2 - 4q^2}}{\left(M + \sqrt{M^2 - 4q^2}\right)^2} \tag{2.37}$$

$$S = \frac{\pi r_+^2}{G} = \pi G \left(M + \sqrt{M^2 - 4q^2}\right)^2. \tag{2.38}$$

From (2.37) it is evident that extremal black holes have $T = 0$. Moreover the following relation can be deduced from (2.37) and (2.38):

$$r_0 = G\, 2ST. \tag{2.39}$$

In the case of a stationary (rotating) black hole, with angular momentum J, the (2.36) can be further generalized in

$$dM = \frac{1}{8\pi G}\kappa\, dA + \phi^h\, dq + \Omega_h\, dJ \tag{2.40}$$

where the constant Ω_h is the angular velocity of the horizon. The new terms in (2.36) and(2.40) have the form of the work term PdV of the first law of thermodynamics. Finally, we mention that there are good arguments and explicit computations suggesting that the third law of thermodynamics has its black hole counterpart: it is impossible to reduce the surface gravity κ to zero through a finite sequence of operations. In the end we can summarize the above considerations in the following *laws of black hole thermodynamics*[9]:

zeroth law the surface gravity is constant over the horizon of stationary black holes;

first law the variation of the mass in terms of changes in the horizon area, charge and angular momentum is

$$dM = \frac{1}{8\pi G}\kappa\, dA + \phi^h\, dq + \Omega_h\, dJ;$$

second law the area of the horizon never decreases with time;

third law it is not possible to achieve $\kappa = 0$ by a finite number of physical processes.

It is worth mentioning that the alternate version of the third law, which in the thermodynamic case says that the entropy S vanishes when the temperature reaches the absolute zero, does not apply to black holes, since A could remain finite when $\kappa \to 0$.

[9]The following statements are taken from [4].

Chapter 3

Black holes and N=2, d=4 Supergravity

Black holes find a natural description in the context of Supergravity theories, since the latter ones are supersymmetric extensions of General Relativity. In some cases, when their gravitational field is particularly strong, they would need a theory of quantum gravity to be thoroughly studied and understood; however, their description within Supergravity is sufficiently correct when the radius of the horizon is much larger than the string scale, i.e. in the limit of large charges, which is the situation to which we confine ourselves.

In the present chapter we present some aspects and results of the theory of black holes in extended Supergravity, considering in particular N=2, d=4 theories. From now on, we work in Planck units (then $c = \hbar = G = k_B = 1$).

3.1 Supersymmetry and extremality

When black holes are embedded in N-extended Supergravities, their structure becomes richer as supersymmetry imposes the presence of additional vector, fermionic and scalar fields in certain proportions.[1] It is remarkable that in $N \geq 2$ theories the (properly generalized) extremality bound[2]

$$M \geq |Q| \tag{3.1}$$

can be directly deduced from the supersymmetry algebra, so that the cosmic censorship conjecture is always satisfied. A generalization of the constraint (3.1) comes from the inclusion of scalars in the theory: referring to the case of

[1] We will always consider black holes as bosonic configurations and set fermions to zero.

[2] Q is the electric charge in general, without taking into account extra numerical coefficients due to conventions, see previous chapter.

four dimensional theories, as we will always do, the part of the supersymmetry algebra regarding the supercharges is

$$\left\{\bar{Q}_{A\alpha}, \bar{Q}_{B\beta}\right\} = -(C\gamma^{\mu})_{\alpha\beta} P_{\mu}\delta_{AB} + i(C\,\mathbb{Z}_{AB})_{\alpha\beta} \tag{3.2}$$

with supersymmetry indices $A, B = 1, \ldots, 2p$ (specifying the different supercharges that can be taken under consideration) and α, β spinor indices; the supersymmetry charges $\bar{Q}_A \equiv Q_A^{\dagger}\gamma_0 = Q_A^T C$ are Majorana spinors and C is the charge conjugation matrix; P_μ is the four-momentum operator and finally $\mathbb{Z}_{AB}$ is an antisymmetric tensor defined as

$$\mathbb{Z}_{AB} = \mathrm{Re}\,(Z_{AB}) + i\gamma^5\,\mathrm{Im}\,(Z_{AB}) \tag{3.3}$$

where the complex antisymmetric matrix $Z_{AB} = -Z_{BA}$ is the central charge operator, with $p = N/2$ (when N is even) or $p = (N-1)/2$ (for N odd) complex eigenvalues, the *central charges*[3] Z_m. For $N = 2$, Z_{AB} can be written as $\epsilon_{AB}Z$, where ϵ_{AB} is the 2×2 antisymmetric matrix, and there is only one central charge. Without going into the detailed proof (which can be found in [13]) we report that for generic N it turns out that the following relation holds:

$$M \geq |Z_m| \quad \forall\, Z_m, \quad m = 1, \ldots, p \tag{3.4}$$

which extends (3.1), replacing the electric charge with the central charges. For $M = |Z_m|$ (considering the maximum among the central charges) the black hole is naturally referred to as extremal.

As already pointed out in Chapter 2, extremal black holes share a number of important features; further new characteristics emerge when supersymmetric coupling to matter (vector) supermultiplets is investigated. One (and maybe the most important) of these is the phenomenon called *attractor mechanism*: when the dynamics of scalar fields is considered in an extremal black hole background, it happens that, approaching the horizon, scalars run to a fixed point ϕ_h, which sometimes does not depend at all on their initial values ϕ_∞ (which we assume to be given at spatial infinity), but only on certain combinations of the electric and magnetic charges. This fact can be seen as a realization of the no-hair theorem: black hole solutions near the horizon are characterized only by discrete parameters associated to conserved charges related to gauge symmetries, while they are not sensitive to the values of scalars at spatial infinity. As we will see, this mechanism has strong

[3]These are additional generators which appear in the supersymmetry algebra of $N \geq 2$ theories; they commute with every other generator of the theory, so they give *central extensions* of the algebra.

implications for the Bekenstein-Hawking entropy formula (2.34): the near horizon geometry of an extremal black hole is that of a Robinson-Bertotti space (Eq. (2.24)) and the horizon mass parameter $M \equiv M_{RB}$ turns out to be equal to (the modulus of) the maximum eigenvalue Z_{max} of the central charge of the theory, for a generic N-extended Supergravity, evaluated at the fixed point:

$$M_{RB} = |Z_{max}(\phi_h, p, q)| \tag{3.5}$$

and so it is also

$$S_{BH} = \frac{A_{RB}(p,q)}{4} = \pi |Z_{max}(\phi_h, p, q)|^2. \tag{3.6}$$

Initially, the appearance of an attractor mechanism was realized in the context of *BPS*[4] *extremal black holes*, that is extremal black holes which are also supersymmetric, in the sense that they preserve a fraction (1/2, 1/4 or 1/8) of the original supersymmetries of the theory: for them, the bound (3.4) is automatically saturated (this means that *supersymmetry implies extremality*, while the converse is not true). Formally, the presence of unbroken supersymmetries means that there exists a projector S acting on the supercharges Q such that

$$(S\,Q)|BPS >= 0 \tag{3.7}$$

where $|BPS>$ is a BPS state; S selects some of the supercharges and the action of the resulting operators on the BPS state is its annihilation. From a practical perspective, if the BPS state is realized as a background described by a certain configuration of fields, equation (3.7) requires that the supersymmetric variations of all the fields are zero. If one considers only bosonic configurations, the bosons already satisfy (3.7), since they supersymmetric variations are proportional to fermions; imposing that fermionic transformations are in turn zero provides a set of first order differential equations for the bosonic fields, called *Killing spinor equations*. By use of this procedure, in [14] the attractor mechanism for N=2 BPS extremal black holes was first recognized; later on it was realized ([15]) that this behaviour is typical of generic extremal black holes, not necessarily BPS (although in the non-BPS black holes it is not true in general that the scalars at the horizon lose completely they dependence on initial conditions). A remarkable characteristic of BPS extremal black holes is that the extremality identity $M = |Q|$ remains valid also when quantum corrections are taken into account and the Supergravity perturbative approximation is no more justified.

[4]Bogomol'nyi-Prasad-Sommerfield.

3.2 N=2, d=4 Supergravity

The action that we will consider in this work is[5]

$$S = \int dx^4 \sqrt{|g|}\left(R + \mathcal{G}_{ij}\partial_\mu\phi^i\partial^\mu\phi^j + 2\,\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}\,F^\Lambda_{\mu\nu}F^{\Sigma\,\mu\nu} - 2\,\mathrm{Re}\,\mathcal{N}_{\Lambda\Sigma}\,F^\Lambda_{\mu\nu}{}^\star F^{\Sigma\,\mu\nu}\right). \tag{3.8}$$

It can describe the bosonic sector of all four-dimensional ungauged supergravities, previous specification of the metric $\mathcal{G}_{ij}$, defined on the scalar (or moduli) space $\mathcal{M}_{scalar}$, and of the symmetric *period matrix* $\mathcal{N}$, which in general depends on the moduli fields, $\mathcal{N} = \mathcal{N}(\phi)$ (ϕ indicates a real scalar, further on we will consider also complex moduli). The $F^\Lambda_{\mu\nu} = 2\partial_{[\mu}A^\Lambda_{\nu]}$ are the field strengths of vector fields A^Λ_μ whose number and meaning we will specify in a moment. The imaginary part of $\mathcal{N}_{\Lambda\Sigma}$ (which we will call $I_{\Lambda\Sigma}$) generalizes the square coupling constant appearing in ordinary gauges theories and has to be negative definite in order to guarantee the positivity of energy; instead, the real part (from now on $R_{\Lambda\Sigma}$) is a generalization of the theta-angle of quantum chromodynamics. In particular we will consider a theory with N=2 supersymmetries (which means 2 supercharges: every set of particles related by supersymmetry will have three levels of spin, for example 1, 1/2, 0) coupled to n vector supermultiplets, so the total particle contents is given by:

- the supergravity multiplet, with the spin-2 metric tensor $g_{\mu\nu}$ (*graviton*), two spin-3/2 *gravitinos* $\psi_{I\,\mu}$ ($I = 1, 2$), and the massless spin-1 *graviphoton* $\tilde{A}_\mu$: this last will be considered on the same level of the vector fields of the vector supermultiplets, so henceforth we will drop the tilde;

- the n vector multiplets, labelled by the indices $i, j, \ldots$; each is formed by a complex scalar z^i with $i = 1, \ldots, n$ (its complex conjugate is written as $\overline{z}^{i^*}$), two spin-1/2 fermions -the *gauginos*- denoted as λ^{Ii} ($I = 1, 2$) and a massless vector field A^i_μ.

As already said, we work with all the fermions set to zero. In the end we have n scalars and $n + 1 \equiv n_V$ vectors. The n_V vectors are then better labelled by the indices $\Lambda, \Sigma, ... = 1, \ldots, n_V$. In $N \geq 2$ theories the structure of the scalar manifold $\mathcal{M}_{scalar}$ is subject to certain conditions arising from the coupling of scalars to vectors through the matrix $\mathcal{N}$, which we will briefly examine in this section, referring to $N = 2$ theories. Our main goal is to introduce the symplectic formalism of [16] and give some expressions for the central and matter charges of the theory, together with some useful relations

[5]Our conventions about the action, the supersymmetry transformation rules, etc. . . are those of [18].

involving them; they will enter in the discussion of the attractor mechanism in Supergravity, presented in the next chapter.

3.2.1 Duality transformations

If we consider n vector multiplets, in the (3.8) there are n scalars and $n_V = n+1$ vectors, so $\Lambda = 1, \dots, n_V$. The (vector) field equations descending from (3.8) are invariant under electric/magnetic duality: that is, defining the dual field strengths

$$G_\Lambda \equiv -\frac{1}{4\sqrt{|g|}}\frac{\delta S}{\delta\, {}^\star F^\Lambda} \qquad {}^\star G_\Lambda \equiv -\frac{1}{4\sqrt{|g|}}\frac{\delta S}{\delta F^\Lambda} \tag{3.9}$$

i.e.

$$G_\Lambda = R_{\Lambda\Sigma}F^\Sigma + I_{\Lambda\Sigma}\, {}^\star F^\Sigma \qquad {}^\star G = R_{\Lambda\Sigma}{}^\star F^\Sigma - I_{\Lambda\Sigma}F^\Sigma \tag{3.10}$$

the Bianchi identities and Maxwell equations take the form

$$\nabla_\mu {}^\star F^{\Lambda\,\mu\nu} = 0 \tag{3.11}$$
$$\nabla_\mu {}^\star G^{\mu\nu}_\Lambda = 0 \tag{3.12}$$

or, in a vectorial notation

$$\nabla_\mu \begin{bmatrix} {}^\star F^{\Lambda\,\mu\nu} \\ {}^\star G^{\mu\nu}_\Lambda \end{bmatrix} \equiv \nabla \mathbf{V} = 0. \tag{3.13}$$

Invariance of the equations under electric/magnetic duality manifest itself in the fact that the (3.11) and (3.12) can be rotated into each other by a linear constant transformation $S \in GL(2n_V, \mathbb{R})$, acting on the $2n_V \times n_V$ vector $\mathbf{V}$:

$$\mathbf{V}' = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \mathbf{V} \tag{3.14}$$

and the transformed vector $\mathbf{V}'$ satisfies the same equations. Now, since F and G are related as in (3.10), consistency requires that the matrix $\mathcal{N}$ does transform as well

$$G'_\Lambda = R'_{\Lambda\Sigma}F'^\Sigma + I'_{\Lambda\Sigma}\, {}^\star F'^\Sigma \tag{3.15}$$

with

$$\mathcal{N}' = (C + D\mathcal{N})\,(A + B\mathcal{N})^{-1}. \tag{3.16}$$

Furthermore, the requirement that the symmetry of $\mathcal{N}$ be preserved by the duality transformation gives the constraints

$$D^T B = B^T D \qquad A^T C = C^T A \qquad D^T A - A^T D = \mathbb{I}_{n_V}$$

that is, $S \in Sp(2n_V, \mathbb{R})$. So S must be a symplectic matrix, satisfying

$$S^T \Omega\, S = \Omega$$

where Ω is the symplectic invariant $2n_V \times 2n_V$ matrix

$$\Omega = \begin{bmatrix} 0 & \mathbb{I}_{n_V} \\ -\mathbb{I}_{n_V} & 0 \end{bmatrix}. \tag{3.17}$$

Duality is then realized by linear symplectic (constant) transformations. If some sources were present, it is evident from (3.11) and (3.12) that we would have to consider electric as well as magnetic charges to preserve the symmetric form of the equations; the charges should obviously transform in a consistent manner, in particular they should form a symplectic vector (we will define it in the end of this chapter).

Now, since vectors are coupled to scalars through the moduli-dependent matrix $\mathcal{N}$, it is clear that, when a duality transformation is performed on the vector field strengths, the scalars should transform in a related fashion, under the action of some diffeomorphism on $\mathcal{M}_{scalar}$. These diffeomorphisms have to be also isometries of the scalar metric $\mathcal{G}_{ij}$, in order to leave the scalar sector invariant and so guarantee the invariance of the field equations under dualities. In the end one has to conclude that there is a homomorphism

$$Iso(\mathcal{M}_{scalar}) \longrightarrow Sp(2n_V, \mathbb{R}).$$

In N=2 theories this considerations lead to the fact that the scalar manifold[6] is a *special Kähler manifold*, an object which we will define in the next section. It seems [19] that for this class of manifolds all isometries are induced by symplectic transformations, although a proof is still to be given. Finally, let us mention that the symmetry under duality is referred to the field equations and Bianchi identities, but in general it cannot be extended to a symmetry of the action; moreover, the action of dualities is easily represented working with the field strengths, while it would be more problematic to define them on the vector potentials, since these transform non-locally.

3.2.2 Special Kähler Manifolds

Kähler manifolds enter in the description of scalar manifolds in supersymmetric theories.[7] Since the scalar which appears in the (n) vector supermultiplets

[6] We are referring to the scalars of the vector multiplet: the moduli of a possible hypermultiplet turn out to be inessential to the description of black holes [13].

[7] For the various concepts of differential geometry which will appear in the following, see for example [17].

is complex, $\mathcal{M}_{scalar}$ is a complex manifold, with complex dimension n; it can be parametrized by the holomorphic complex coordinates z_i, $\overline{z}_{i^*}$, with i, $i^* = 1, \ldots, n$ (representing the scalars).

A Kähler manifold is a complex manifold (with complex dimension n) whose holonomy group is a subgroup of $U(n)$ ([22]). Its metric can be written locally as

$$\mathcal{G}_{ij^*} = \mathcal{G}_{j^*i} = \partial_i \partial_{j^*} \mathcal{K} \tag{3.18}$$

where $\mathcal{K}$ is a real function called the *Kähler potential*; moreover

$$\mathcal{G}_{ij} = \mathcal{G}_{i^*j^*} = 0.$$

and the Levi-Civitá connection on the Kähler manifold is

$$\Gamma^i_{jk} = \mathcal{G}^{ii^*} \partial_j \, \mathcal{G}_{i^*k} \qquad \Gamma^{i^*}_{j^*k^*} = \mathcal{G}^{i^*i} \partial_{j^*} \, \mathcal{G}_{ik^*}. \tag{3.19}$$

The relation (3.18) is preserved under the so-called *Kähler transformations*:

$$\mathcal{K}(z, \overline{z}) \longmapsto \mathcal{K}(z, \overline{z}) + h(z) + \overline{h}(\overline{z}) \tag{3.20}$$

where h is a complex holomorphic function of the scalars.

In $N = 1$ Supergravity the scalar manifold which has to be considered is a *Hodge-Kähler* manifold: a Kähler manifold endowed with a complex line bundle $\mathcal{L} \to \mathcal{M}_{scalar}$ (with an associated $U(1)$-bundle) whose first Chern class is equal to the cohomology class of the Kähler 2-form K:

$$c_1(\mathcal{L}) = [K]$$

where

$$K = i \mathcal{G}_{ij^*} \, dz^i \wedge d\overline{z}^{j^*}.$$

This means that, locally, we can define the ($U(1)$) connection (*Kähler 1-form*) as

$$\mathcal{Q} = -\frac{i}{2} \left(\partial_i \mathcal{K} \, dz^i - \partial_{i^*} \mathcal{K} \, d\overline{z}^{i^*} \right).$$

This connection is needed to define the covariant derivative with respect to the Kähler transformations, (3.20):

$$D\Phi = (d + ip\mathcal{Q}) \, \Phi \tag{3.21}$$

where $\phi = \phi(z, \overline{z})$ is a smooth function transforming under Kähler transformations with weight p, $\phi \mapsto e^{-ph(z)} \phi$ (a *section* of the $U(1)$-bundle). In components (3.21) is

$$D_i \Phi = \left(\partial_i + \frac{1}{2} p \, \partial_i \mathcal{K} \right) \Phi \qquad D_{i^*} \Phi = \left(\partial_{i^*} - \frac{1}{2} p \, \partial_{i^*} \mathcal{K} \right) \Phi.$$

A covariantly holomorphic section is defined by $D_{i^*}\Phi = 0$. Truly holomorphic sections can be obtained from covariantly holomorphic ones with the position

$$\tilde{\Phi} = e^{-p\mathcal{K}/2}\Phi$$

which gives

$$D_i\tilde{\Phi} = (\partial_i + p\,\partial_i\mathcal{K})\,\tilde{\Phi} \qquad D_{i^*}\tilde{\Phi} = \partial_{i^*}\tilde{\Phi}.$$

If the object on which the derivative acts bears also scalar indices i, i^*, the covariant derivative has to include the Levi-Civitá connection (3.19).

Special Kähler manifolds are called for in $N = 2$ Supergravity. They can be defined in several equivalent ways [20] and some of these make use of an auxiliary holomorphic function $F(z)$, called prepotential, from which the Kähler potential can be derived.[8] Here we give a prepotential-free definition, following [13], which puts on evidence the symplectic structure of the theory. A special Kähler manifold is a Hodge-Kähler manifold which is also the base space of a flat symplectic vector bundle with structure group $Sp(2(n+1),\mathbb{R})$; this means that we can introduce the following (scalar-dependent) vectors

$$V = \left(f^\Lambda,\ h_\Sigma\right)$$

which are defined as vectors transforming under (global) $Sp(2(n+1),\mathbb{R})$ transformations (they are the sections of the symplectic vector bundle).[9] In addition, these sections are requested to be covariantly holomorphic

$$0 = D_{i^*}V = \left(\partial_{i^*} - \frac{1}{2}\,\partial_{i^*}\mathcal{K}\right)V \equiv V_{i^*} = \left(f^\Lambda_{i^*},\ h_{\Sigma\,i^*}\right) \tag{3.22}$$

and obey the additional condition

$$i\left\langle \overline{V}, V\right\rangle = i\left(\overline{f}^\Lambda h_\Lambda - \overline{h}_\Lambda f^\Lambda\right) = 1 \tag{3.23}$$

where we made use of the product $\langle V, W\rangle \equiv V^T\Omega\,W$.
We can define the following three-index tensor[10]

$$C_{ijk} \equiv \langle D_iV_j\,, V_k\rangle\,; \tag{3.24}$$

[8] In general the theory can be formulated without a prepotential; however, it has been shown [20] that every formulation without prepotential can be obtained through a symplectic transformation starting from one in which a prepotential can be individuated.

[9] They also tranform under (local) $U(1)$ (with weight $p = 1$), since the Hodge-Kähler structure is mantained.

[10] $D_iV \equiv V_i$: every time a quantity bears a scalar pedix i (or i^*) , it has been acted on by D_i (D_{i^*}).

in [18] it is shown that this implies

$$D_i V_j = i C_{ijk}\, \mathcal{G}^{kk^*}\, \overline{V}_{k^*} \tag{3.25}$$

and that C_{ijk} is completely symmetric in its three indices. It is also seen how the two relations

$$D_i \overline{V}_{j^*} = \mathcal{G}_{ij^*} \overline{V} \tag{3.26}$$

$$D_i \overline{V} = 0 \tag{3.27}$$

follows from the definition of Special Kähler manifold, through the requirements (3.22) and (3.23) (the (3.27) is immediate). Equations (3.25),(3.26) and (3.27) will be useful when discussing the attractor mechanism in the next chapter.

The Kähler potential $\mathcal{K}$ can be expressed in a symplectically invariant form: introducing, as in the Hodge-Kähler example above, the holomorphic sections

$$\begin{aligned} &v(z,\overline{z}) = e^{-\mathcal{K}/2} V = e^{-\mathcal{K}/2} \left(f^\Lambda \,, h_\Sigma \right) \equiv \left(X^\Lambda \,, M_\Sigma \right) \\ &\partial_{i^*}\, v = 0 \end{aligned}$$

it follows that

$$\mathcal{K} = -\ln i \, \langle \overline{v}, v \rangle = -\ln i \left(\overline{X}^\Lambda M_\Lambda - \overline{M}_\Lambda X^\Lambda \right).$$

One can define the complex, symmetric $(n+1) \times (n+1)$ matrix $\mathcal{N}$ by

$$h_\Lambda = \mathcal{N}_{\Lambda\Sigma} f^\Sigma \qquad h_{\Lambda\, i^*} = \overline{\mathcal{N}}_{\Lambda\Sigma} f^\Sigma_{i^*} \tag{3.28}$$

and it turns out that this is the same matrix $\mathcal{N}$ appearing in the vector sector of the action (3.8): it is easy to check that, when the sections V are transformed by a $S \in Sp(2(n+1), \mathbb{R})$ with the same structure as in (3.14), owing to the definitions (3.28) the matrix $\mathcal{N}$ transforms as

$$\mathcal{N}'(f',h') = \left(C + D\mathcal{N}(f,h) \right) \left(A + B\mathcal{N}(f,h) \right)^{-1} \tag{3.29}$$

that is, exactly as in (3.16).

3.2.3 Central and matter charges

The formalism that we are using provides explicitly symplectic expressions for the field strengths of the graviphoton and of the matter vector fields, and these in turn give the central charge and the matter charges of the theory.

First, recalling the vector field strengths F^Λ and their duals G_Λ defined in (3.10), in the presence of electric and magnetic sources[11] it is possible to define the magnetic and electric charges

$$p^\Lambda = \frac{1}{4\pi}\int_{S^2} F^\Lambda \qquad q_\Lambda = \frac{1}{4\pi}\int_{S^2} G_\Lambda \tag{3.30}$$

where S^2 is a 2-sphere. These should be viewed as *bare* charges, while the *physical* ones, those entering in the interacting theory, can be deduced by the supersymmetry transformation laws of the fermions. Here, in fact, appear the interacting field strengths of the vector fields (called matter vector field strengths) and of the graviphoton and they are linear combinations of the F^Λ and G_Λ, the coefficients being functions of the scalars. The physical graviphoton field strength $T_{\mu\nu}$ can be identified in the transformation law of its fermionic superpartners, the gravitinos $\psi_{I\mu}$:

$$\delta_\xi\, \psi_{I\mu} = \mathfrak{D}_\mu \xi_I + T_{\mu\nu}\gamma^\nu \epsilon_{IJ}\,\xi^J + \ldots . \tag{3.31}$$

Here, the ξ is the spinorial parameter of the supersymmetry transformation; ϵ_{IJ} is the 2-dimensional antisymmetric tensor $(I, J = 1, 2)$. The derivative $\mathfrak{D}_\mu$, acting on fermions, is defined as

$$\mathfrak{D}_\mu \xi_I = \left(\mathcal{D}_\mu + \frac{i}{2}\mathcal{Q}_\mu\right)\xi_I;$$

$\mathcal{D}_\mu$ is the spacetime derivative covariant with respect to the spin connection $\omega_\mu^{\ ab}$

$$\mathcal{D}_\mu \xi_I = \left(\partial_\mu - \frac{1}{4}\omega_\mu^{\ ab}\gamma_{ab}\right)\xi_I \tag{3.32}$$

where $\gamma_{ab} = \gamma_{[a}\gamma_{b]}$, while $\mathcal{Q}_\mu$ is

$$\mathcal{Q}_\mu = -\frac{i}{2}\left(\partial_\mu z^i\,\partial_i\mathcal{K} - \partial_\mu \overline{z}^{i^*}\,\partial_{i^*}\mathcal{K}\right). \tag{3.33}$$

The dots represent trilinear fermion terms. It has be shown [21] that $T_{\mu\nu}$ can be constructed by means of the symplectic section $\left(f^\Lambda\,, h_\Sigma\right)$ and vector $\left(F^\Lambda\,, G_\Sigma\right)$:

$$T = h_\Lambda F^\Lambda - f^\Lambda G_\Lambda; \tag{3.34}$$

the central charge Z (field theory representation of the central charge appearing in the supersymmetry algebra (3.2)) is then defined as the integral over S^2 of T ([21]):

$$Z \equiv -\frac{1}{4\pi}\int_{S^2} T = f^\Lambda\left(z,\overline{z}\right) q_\Lambda - h_\Lambda\left(z,\overline{z}\right) p^\Lambda. \tag{3.35}$$

[11]But also when there are not explicit sources in the action, see [3].

In a similar manner, from the gauginos λ^{Ii} transformation law we can determine the matter vector field strengths $T_{i\,\mu\nu}$

$$\delta_\xi \, \lambda^{Ii} = i\gamma^\mu \partial_\mu z^i \xi^I + \mathcal{G}^{ij^*} T_{j^*\,\mu\nu} \gamma^{\mu\nu} \epsilon^{IJ} \xi_J + \dots \tag{3.36}$$

and, again, T_j turns out to be given by a symplectically invariant combination

$$T_{i^*} = \overline{h}_{\Lambda\, i^*} F^\Lambda - \overline{f}^\Lambda_{i^*} G_\Lambda \longrightarrow T_i = h_{\Lambda\, i} F^\Lambda - f^\Lambda_i G_\Lambda . \tag{3.37}$$

The matter charges are then

$$Z_i \equiv -\frac{1}{4\pi} \int_{S^2} T_i = f^\Lambda_i \,(z, \overline{z})\, q_\Lambda - h_{\Lambda\, i}\,(z, \overline{z})\, p^\Lambda \tag{3.38}$$

and since $D_i V = \left(f^\Lambda_i \,, h_{\Lambda\, i} \right)$ we get the further relation

$$Z_i = D_i Z. \tag{3.39}$$

Finally, a sum rule has been derived for the Z and Z_i [16]

$$|Z|^2 + |Z_i|^2 = |Z|^2 + Z_i \, \mathcal{G}^{ij^*} \, \overline{Z}_{j^*} = -\frac{1}{2} Q^T \mathcal{M} \, Q \tag{3.40}$$

where Q is a symplectic vector built with the electric and magnetic charges

$$Q = \begin{bmatrix} p^\Lambda \\ q_\Lambda \end{bmatrix} \tag{3.41}$$

and the matrix $\mathcal{M}$ is given by

$$\mathcal{M} \equiv \begin{bmatrix} (I + RI^{-1}R)_{\Lambda\Sigma} & -(RI^{-1})_\Lambda^{\ \Sigma} \\ \\ -(I^{-1}R)^\Lambda_{\ \Sigma} & (I^{-1})^{\Lambda\Sigma} \end{bmatrix}. \tag{3.42}$$

The quantities defined in (3.35), (3.38) as well as the identity (3.40) will be used in the next chapter, when dealing with the attractor mechanism in Supergravity. We will also see that the matrix (3.42) plays a crucial role in this context; in Chapter 6 we will work out how the same matrix appears in the equations of motion descending from (3.8).

Chapter 4

Attractor mechanism in static black holes

The attractor mechanism was first described by Ferrara, Kallosh and Strominger in [14] in the context of a N=2 supersymmetric extremal, magnetically charged black hole, and then extended by Strominger [23] to the case with both electric and magnetic charges (always BPS). These two derivations rely on supersymmetry, in particular [14] deduces the damped geodesic equation for the scalar fields from the vanishing of the gravitino and gaugino local supersymmetry transformations. Here we consider the derivation proposed by Ferrara, Gibbons and Kallosh in [15], which does not make any use of supersymmetry. Instead, they reduce the dynamics of scalar and vector fields in a static black hole background to a one dimensional geodesic motion of conveniently defined scalar fields; the attractor mechanism is then explained specifying requirements of regularity for the (extremal) configurations of the solution. In the following chapters we will extend this procedure to the more general case of a stationary metric characterized by an additional parameter N (Taub-NUT metric), working out all the calculations, so now we will report only the main passages; when dealing with the stationary case it will be evident how setting the parameter N equal to zero (i.e. returning to the static metric) one recovers the results that we are going to present now.
In the last section of the chapter we will examine some features of the attractor mechanism when it is considered in the context of N=2, d=4 Supergravity.

4.1 Equations of motion from 4 to 1 dimensions

We consider theories with action as in (3.8),

$$S = \int dx^4 \sqrt{|g|} \left(\hat{R} + \mathcal{G}_{ij} \partial_\mu \phi^i \partial^\mu \phi^j + 2 I_{\Lambda\Sigma} \hat{F}^\Lambda_{\mu\nu} \hat{F}^{\Sigma\,\mu\nu} - 2 R_{\Lambda\Sigma} \hat{F}^\Lambda_{\mu\nu} {}^\star\hat{F}^{\Sigma\,\mu\nu} \right) \tag{4.1}$$

where $\mathcal{G} \equiv \mathcal{G}(\phi)$, $R_{\Lambda\Sigma} \equiv \mathrm{Re}\,\mathcal{N}_{\Lambda\Sigma}(\phi)$ and $I_{\Lambda\Sigma} \equiv \mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}(\phi)$; the indices $i, j = 1, \ldots, n$ specify the (real, for the moment) scalars while $\Lambda, \Sigma = 1, \ldots, n_V = n+1$ refer to the vectors. The hatted tensors $\hat{F}, \hat{R}$ are 4-dimensional.[1] We will look for static solutions and use the following ansatz for the metric

$$ds^2 = e^{2U} dt^2 - e^{-2U} \gamma_{\underline{mn}} dx^{\underline{m}} dx^{\underline{n}}, \qquad \underline{m}, \underline{n} = 1, 2, 3. \tag{4.2}$$

The equations of motion following from the action (4.1) are

$$\hat{G}_{\mu\nu} + \mathcal{G}_{ij} \left(\partial_\mu \phi^i \partial_\nu \phi^j - \frac{1}{2} g_{\mu\nu} \partial_\rho \phi^i \partial^\rho \phi^j \right) + 4 I_{\Lambda\Sigma} \left(\hat{F}^{\Lambda\,\rho}_\mu \hat{F}^\Sigma_{\nu\rho} - \frac{1}{4} g_{\mu\nu} \hat{F}^\Lambda_{\rho\sigma} \hat{F}^{\Sigma\,\rho\sigma} \right) = 0 \qquad \text{(Einstein)};$$

$$\nabla_\mu \left(I_{\Lambda\Sigma} \hat{F}^{\Sigma\,\mu\nu} - R_{\Lambda\Sigma} {}^\star\hat{F}^{\Sigma\,\mu\nu} \right) = 0 \qquad \text{(Maxwell)};$$

$$\nabla_\mu \left(\mathcal{G}_{ij} \partial^\mu \phi^j \right) - \frac{1}{2} \partial_i \mathcal{G}_{kl}\, \partial_\mu \phi^k \partial^\mu \phi^l - \left(\partial_i I_{\Lambda\Sigma} \hat{F}^\Lambda_{\mu\nu} \hat{F}^{\Sigma\,\mu\nu} - \partial_i R_{\Lambda\Sigma} \hat{F}^\Lambda_{\mu\nu} {}^\star\hat{F}^{\Sigma\,\mu\nu} \right) = 0 \qquad \text{(scalars)}.$$

As we will prove in Chapter 6, the Maxwell equations can be written

$$\nabla_{\underline{m}} \left\{ e^{-2U} \begin{bmatrix} (I + R I^{-1} R)_{\Lambda\Sigma} & -(R I^{-1})_\Lambda{}^\Sigma \\ -(I^{-1} R)^\Lambda{}_\Sigma & (I^{-1})^{\Lambda\Sigma} \end{bmatrix} \partial^{\underline{m}} \begin{bmatrix} \psi^\Sigma \\ \chi_\Sigma \end{bmatrix} \right\} = 0; \tag{4.3}$$

ψ^Λ and χ_Λ are $2n_V$ scalars representing respectively the electric and magnetic potentials. With the identifications

$$\mathcal{M}_{MN} \equiv \begin{bmatrix} (I + R I^{-1} R)_{\Lambda\Sigma} & -(R I^{-1})_\Lambda{}^\Sigma \\ -(I^{-1} R)^\Lambda{}_\Sigma & (I^{-1})^{\Lambda\Sigma} \end{bmatrix} \qquad \Psi^M \equiv \begin{bmatrix} \psi^\Sigma \\ \chi_\Sigma \end{bmatrix} \tag{4.4}$$

[1] In the calculation it will be useful to distinguish between four- and three-dimensional quantities; the latter ones are without hat.

the (4.3) reads

$$\nabla_{\underline{m}}\left[e^{-2U}\mathcal{M}_{MN}\,\partial^{\underline{m}}\Psi^N\right]=0. \tag{4.5}$$

For what concerns the Einstein equations, considering them in tangent space indices, their 00, $0m$ and mn components turn out to be, in order

$$R+2(\partial U)^2-4\nabla^2U+\mathcal{G}_{ij}\,\partial_{\underline{m}}\phi^i\partial^{\underline{m}}\phi^j-4e^{-2U}\mathcal{M}_{MN}\partial_{\underline{m}}\Psi^M\partial^{\underline{m}}\Psi^N=0 \tag{4.6}$$

$$\partial_{[m}\psi^{\Lambda}\,\partial_{n]}\chi_{\Lambda}=0 \tag{4.7}$$

$$\begin{aligned}G_{mn}+2\left(\partial_mU\partial_nU-\frac{\delta_{mn}}{2}(\partial U)^2\right)+\mathcal{G}_{ij}\left(\partial_m\phi^i\partial_n\phi^j-\frac{\delta_{mn}}{2}\,\partial_{\underline{r}}\phi^i\partial^{\underline{r}}\phi^j\right)&\\+4e^{-2U}\mathcal{M}_{MN}\left(\partial_m\Psi^M\partial_n\Psi^N-\frac{\delta_{mn}}{2}\partial_{\underline{r}}\Psi^M\partial^{\underline{r}}\Psi^N\right)&=0.\end{aligned} \tag{4.8}$$

Finally, the equation for scalars becomes

$$\nabla_{\underline{m}}\left(\mathcal{G}_{ij}\partial^{\underline{m}}\phi^j\right)-\frac{1}{2}\partial_i\mathcal{G}_{kl}\,\partial_{\underline{m}}\phi^k\partial^{\underline{m}}\phi^l-2\,e^{-2U}\partial_i\mathcal{M}_{MN}\,\partial_{\underline{m}}\Psi^M\partial^{\underline{m}}\Psi^N=0. \tag{4.9}$$

All the scalar and vector quantities appearing in the expressions from (4.3) to (4.9) should be understood as 3-dimensional entities (see also Appendix A): thanks to the hypothesis of staticity the 4-dimensional problem can be reduced to a system of equations living in the 3-dimensional space. Now, taking the trace of (4.8) and inserting it in (4.6), rearranging a bit the various coefficients, the system takes the form

$$\nabla_{\underline{m}}\left[4e^{-2U}\mathcal{M}_{MN}\,\partial^{\underline{m}}\Psi^N\right]=0 \tag{4.10}$$

$$\partial_{\underline{m}}\left(2\,\partial^{\underline{m}}U\right)+4e^{-2U}\mathcal{M}_{MN}\partial_{\underline{m}}\Psi^M\partial^{\underline{m}}\Psi^N=0 \tag{4.11}$$

$$\partial_{[m}\psi^{\Lambda}\,\partial_{n]}\chi_{\Lambda}=0 \tag{4.12}$$

$$R_{mn}+2\,\partial_mU\partial_nU+\mathcal{G}_{ij}\partial_m\phi^i\partial_n\phi^j+4e^{-2U}\mathcal{M}_{MN}\partial_m\Psi^M\partial_n\Psi^N=0 \tag{4.13}$$

$$\nabla_{\underline{m}}\left(\mathcal{G}_{ij}\partial^{\underline{m}}\phi^j\right)-\frac{1}{2}\partial_i\mathcal{G}_{kl}\,\partial_{\underline{m}}\phi^k\partial^{\underline{m}}\phi^l-\frac{1}{2}\partial_i\left(4\,e^{-2U}\mathcal{M}_{MN}\right)\,\partial_{\underline{m}}\Psi^M\partial^{\underline{m}}\Psi^N=0. \tag{4.14}$$

We define the block-diagonal matrix

$$\mathcal{G}_{AB}\equiv\begin{bmatrix}\mathcal{G}_{UU}&&\\&\mathcal{G}_{ij}&\\&&\mathcal{G}_{MN}\end{bmatrix}\equiv\begin{bmatrix}2&&\\&\mathcal{G}_{ij}&\\&&4\,e^{-2U}\mathcal{M}_{MN}\end{bmatrix} \tag{4.15}$$

and group the scalar fields which appear in the equations in a single vector

$$\tilde{\phi}^A = \left(U \ \phi^i \ \psi^\Lambda \ \chi_\Lambda\right); \tag{4.16}$$

this permits to write the above system as

$$\partial_{\underline{m}}\left(\mathcal{G}_{AB}\,\partial^{\underline{m}}\tilde{\phi}^B\right) - \frac{1}{2}\partial_A\mathcal{G}_{BC}\,\partial_{\underline{m}}\tilde{\phi}^B\partial^{\underline{m}}\tilde{\phi}^C = 0\,; \tag{4.17}$$

$$R_{mn} + \mathcal{G}_{AB}\,\partial_m\tilde{\phi}^A\partial_n\tilde{\phi}^B = 0\,; \tag{4.18}$$

$$\partial_{[m}\psi^\Lambda\,\partial_{n]}\chi_\Lambda = 0. \tag{4.19}$$

The first two equations can be derived from the effective 3-dimensional action

$$\int d^3x\,\sqrt{|\gamma|}\left(R\left[\gamma\right] + \mathcal{G}_{AB}\,\partial_{\underline{m}}\tilde{\phi}^A\partial^{\underline{m}}\tilde{\phi}^B\right) \tag{4.20}$$

while the third acts as an additional constraint on the potentials.
To proceed we need to specify the 3-dimensional metric: in addition to staticity, we make the hypothesis of spherical symmetry. The metric is assumed to be of the form

$$\gamma_{\underline{mn}}dx^{\underline{m}}dx^{\underline{n}} = \frac{r_0^4}{\sinh^4 r_0\tau}d\tau^2 + \frac{r_0^2}{\sinh^2 r_0\tau}d\Omega^2_{(2)}; \tag{4.21}$$

determinant and Christoffel symbols as well as curvature and Ricci tensor are listed in the Appendix A. If we also ask for the scalar fields to depend only on the radial coordinate τ, the (4.19) is automatically satisfied, while the (4.17) and (4.18) become:

$$\frac{d}{d\tau}\left(\mathcal{G}_{AB}\,\frac{d\tilde{\phi}^B}{d\tau}\right) - \frac{1}{2}\partial_A\mathcal{G}_{BC}\frac{d\tilde{\phi}^B}{d\tau}\frac{d\tilde{\phi}^C}{d\tau} = 0 \tag{4.22}$$

$$\mathcal{G}_{AB}\frac{d\tilde{\phi}^A}{d\tau}\frac{d\tilde{\phi}^B}{d\tau} = 2r_0^2. \tag{4.23}$$

The (4.22) can be rewritten as a pure geodesic equation

$$\frac{d^2}{d\tau^2}\tilde{\phi}^A + \Gamma^A_{BC}\,\frac{d\tilde{\phi}^B}{d\tau}\frac{d\tilde{\phi}^C}{d\tau} = 0 \tag{4.24}$$

once we make the identification

$$\Gamma^C_{AB} = \frac{1}{2}\mathcal{G}^{CD}\left(\partial_A\mathcal{G}_{DB} + \partial_B\mathcal{G}_{DA} - \partial_D\mathcal{G}_{AB}\right)$$

i.e. once we define the Christoffel symbols for what we could call, following [15], enlarged scalar manifold.
Remarkably, the initial problem has been reformulated in terms of the one-dimensional geodesic radial equation (4.22) which can be derived from the one-dimensional geodesic action

$$\int d\tau\, \mathcal{G}_{AB} \frac{d\tilde{\phi}^A}{d\tau} \frac{d\tilde{\phi}^B}{d\tau}. \tag{4.25}$$

Now, since the metric does not depend on ψ^Λ and χ_Λ, and given the form of (4.5), it is possible to define the $2n_V$ constants of motion[2]:

$$q_\Lambda \equiv 4\alpha\, e^{-2U} \left(\mathcal{M}_{\Lambda\Sigma} \frac{d\psi^\Sigma}{d\tau} + \mathcal{M}_\Lambda^{\ \Sigma} \frac{d\chi_\Sigma}{d\tau} \right) \tag{4.26}$$

$$p^\Lambda \equiv -4\alpha\, e^{-2U} \left(\mathcal{M}^\Lambda_{\ \Sigma} \frac{d\psi^\Sigma}{d\tau} + \mathcal{M}^{\Lambda\Sigma} \frac{d\chi_\Sigma}{d\tau} \right); \tag{4.27}$$

through these definitions, the electric and magnetic potentials can be eliminated from the Einstein and scalar equations. In fact, gathering the charges in the (symplectic) vector

$$Q^M \equiv \begin{bmatrix} p^\Lambda \\ q_\Lambda \end{bmatrix} \tag{4.28}$$

it is easy to see that

$$Q^M = \alpha\, \Omega^{MN} \mathcal{G}_{NP} \frac{d\Psi^P}{d\tau} \tag{4.29}$$

where

$$\Omega^{MN} = \begin{bmatrix} 0 & -\mathbb{I}_{n_V} \\ \mathbb{I}_{n_V} & 0 \end{bmatrix} \quad \text{is the inverse of} \quad \Omega_{MN} = \begin{bmatrix} 0 & \mathbb{I}_{n_V} \\ -\mathbb{I}_{n_V} & 0 \end{bmatrix};$$

then we have

$$\frac{d\Psi^M}{d\tau} = \frac{1}{\alpha}\, \mathcal{G}^{MN} \Omega_{NP} Q^P \equiv \frac{1}{\alpha} \mathcal{G}^{MN} Q_N. \tag{4.30}$$

[2]Since the two quantities on the right side of (4.26) and (4.27) are conserved, it seems natural to assume that they are proportional to the electric and magnetic charges, through the normalisation constant α. Moreover, the way they transform under symplectic transformations is the same of the charges.

Writing the (4.22) and the (4.23) with the explicit fields and making use of the (4.30), we obtain (prime means differentiation with respect to τ)

$$2(U')^2 + \mathcal{G}_{ij}\,\frac{d\phi^i}{d\tau}\frac{d\phi^j}{d\tau} + \frac{1}{\alpha^2}\,\mathcal{G}^{MN}Q_M Q_N = 2r_0^2 \tag{4.31}$$

$$2U'' + \frac{1}{\alpha^2}\,\mathcal{G}_{MN}\mathcal{G}^{MP}\mathcal{G}^{NQ}Q_P Q_Q = 0 \tag{4.32}$$

$$\frac{d}{d\tau}\left(\mathcal{G}_{ij}\frac{d\phi^j}{d\tau}\right) - \frac{1}{2}\partial_i\mathcal{G}_{jk}\frac{d\phi^j}{d\tau}\frac{d\phi^k}{d\tau} - \frac{1}{2\alpha^2}\,\partial_i\mathcal{G}_{MN}\mathcal{G}^{MP}\mathcal{G}^{NQ}Q_P Q_Q = 0 \tag{4.33}$$

which, with the definition of the *black hole potential*[3]

$$V_{BH}(\phi,p,q) \equiv \frac{1}{2\alpha^2}\,e^{-2U}\mathcal{G}^{MN}Q_M Q_N, \tag{4.34}$$

become

$$2(U')^2 + \mathcal{G}_{ij}\,\frac{d\phi^i}{d\tau}\frac{d\phi^j}{d\tau} + 2e^{2U}V_{BH} = 2r_0^2 \tag{4.35}$$

$$U'' + e^{2U}V_{BH} = 0 \tag{4.36}$$

$$\frac{d}{d\tau}\left(\mathcal{G}_{ij}\frac{d\phi^j}{d\tau}\right) - \frac{1}{2}\partial_i\mathcal{G}_{jk}\frac{d\phi^j}{d\tau}\frac{d\phi^k}{d\tau} + \partial_i\left[e^{2U}V_{BH}\right] = 0. \tag{4.37}$$

The last two equations can be derived from the effective action

$$S_{\text{eff}}\left[U,\phi^i\right] = \int d\tau\left[(U')^2 + \frac{1}{2}\mathcal{G}_{ij}\frac{d\phi^i}{d\tau}\frac{d\phi^j}{d\tau} - e^{2U}V_{BH}\right]. \tag{4.38}$$

In the end, we have found that the properties of the system are completely determined by the metric on the scalar manifold $\mathcal{G}_{ij}$ and by the scalar-dependent potential V_{BH}. In the above derivation, a crucial role was played by the assumptions of staticity and spherical symmetry. In the stationary case, which will be analysed later, things will be more complicated.
The procedure of transforming a n-dimensional problem (in this case the system of equations, but the same could be done also directly on the action) to a (n-1)-dimensional one (or also less) goes under the name of *dimensional*

[3]This definition of the black hole potential differs by a sign from that of Gibbons, Kallosh and Ferrara in [15], and it is definite-negative.

reduction, and it is a special case of a more general and articulated technique called *Kaluza-Klein compactification*. By the way, the possibility of performing this reduction is of fundamental importance for the proof of no-hair theorems in the case of stationary black holes, see for example [9].

4.2 Attractors in extremal black holes

Now we consider the very special case in which the extremality parameter vanishes, $r_0^2 = 2ST = 0$, so to have an extremal black hole; the metric then becomes

$$ds^2 = e^{2U}dt^2 - e^{-2U}\left(\frac{d\tau^2}{\tau^4} + \frac{1}{\tau^2}d\Omega^2_{(2)}\right) \tag{4.39}$$

In order to have a solution with finite area of the horizon,

$$A = \int_{\tau=\tau_+} \sqrt{g_{\theta\theta}g_{\varphi\varphi}}\, d\theta d\varphi = \int_{\tau=\tau_+} \frac{e^{-2U}}{\tau^2}\sin\theta\, d\theta d\varphi = 4\pi\frac{e^{-2U(\tau_+)}}{\tau_+^2} \tag{4.40}$$

we have to require that, for $\tau \to -\infty$, the metric function behaves as

$$e^{-2U} \longrightarrow \left(\frac{A}{4\pi}\right)\tau^2 \tag{4.41}$$

where A is the area of the horizon. We shall specify also a regularity condition for the scalar fields: in particular, approaching the horizon, it will have to be

$$\mathcal{G}_{ij}\frac{d\phi^i}{d\tau}\frac{d\phi^j}{d\tau}e^{2U}\tau^4 < \infty, \tag{4.42}$$

condition which, on top of (4.41), gives

$$\mathcal{G}_{ij}\frac{d\phi^i}{d\tau}\frac{d\phi^j}{d\tau}\left(\frac{4\pi}{A}\right)\tau^2 \longrightarrow X^2 \quad \text{when } \tau \to -\infty \tag{4.43}$$

with X^2 finite. With these requirements, the constraint (4.31) in the near-horizon limit becomes

$$A + \frac{1}{2}\frac{A^2}{4\pi}X^2 + 4\pi V_{BH}(\phi_h, p, q) = 0 \tag{4.44}$$

where $\phi_h = \phi(-\infty)$; it follows that

$$A \leq -4\pi V_{BH}(\phi_h, p, q). \tag{4.45}$$

In the case of *double-extremal black holes*, black holes with constant scalar fields, the above relation is saturated. We also note that with the hypothesis we made the near-horizon geometry assumes the expression

$$dt^2 = \frac{4\pi}{A}\frac{dt^2}{\tau^2} - \frac{A}{4\pi}\frac{d\tau^2}{\tau^2} - \frac{A}{4\pi}d\Omega^2_{(2)}$$

which is the metric of a Robinson-Bertotti space, introduced in (2.24), as it becomes after the reparametrization $\tau = 1/\rho$; the mass parameter M_{RB} is

$$M^2_{RB} = \frac{A}{4\pi}. \tag{4.46}$$

Following [15] we can say that the extremal black hole solution interpolates between the asymptotic, Minkoskwi's flat vacuum and the near-horizon, Robinson-Bertotti space, i.e. between two different vacua with higher symmetry.
More can be said about the relation (4.45). The new radial coordinate

$$\omega = -\log(-\tau)$$

goes from $-\infty$ at the horizon to $+\infty$ at spatial infinity. The hypothesis (4.42) in the new coordinate reads

$$\mathcal{G}_{ij}\frac{d\phi^i}{d\omega}\frac{d\phi^j}{d\omega}\left(\frac{4\pi}{A}\right) \longrightarrow X^2 \quad \text{when } \omega \to -\infty; \tag{4.47}$$

now, if it were

$$\frac{d\phi^i}{d\omega} = \text{constant} \quad \text{for } \omega \to -\infty$$

the scalars would be linear in ω, but this is incompatible with the requirement of finiteness of the moduli near the horizon. We are forced to set

$$\frac{d\phi^i}{d\omega} = 0 \quad \text{for } \omega \to -\infty$$

so that approaching the horizon the scalars assume a finite, constant value ϕ^i_h. Here emerges the fact that, in the context of extremal black holes, the dynamics of the moduli seems to exhibit an attractive behaviour[4]: the trajectories of scalars start at spatial infinity and end on the horizon, where there is a fixed point ϕ_h, with zero velocity. In some cases that we will consider later on, the value of the moduli at the horizon does depend only

[4] A fixed point ϕ_{fix} is a point in which the phase velocity vanishes; it is named *attractor* if $\lim_{t\to\infty}\phi(t) = \phi_{fix}$.

on the charges of the theory, while it is not sensitive to their asymptotic values: when the scalars reach the horizon, they lose memory about the initial conditions. When this happens, we speak of an *attractor mechanism*.
Thanks to the vanishing of the scalar velocity near the horizon, we can reconsider the relation (4.45) and write also for single extremal black holes

$$A = -4\pi V_{BH}(\phi_h, p, q) \tag{4.48}$$

and remembering (4.46), it is also

$$M_{RB}^2 = -V_{BH}(\phi_h, p, q). \tag{4.49}$$

If now we recall the entropy-area formula presented in (2.34), the (4.48) immediately gives

$$S = -\pi V_{BH}(\phi_h, p, q); \tag{4.50}$$

the entropy of an extremal black hole depends only on the values of the scalars at the horizon and on conserved quantities as the electric and magnetic charges.
The issue of determining the fixed points of the scalars can be faced noticing that the equation of motion for scalars (4.37) can be rewritten in the formalism of equation (4.24) (keeping the radial coordinate τ):

$$\frac{d^2\phi^i}{d\tau^2} + \Gamma^i_{jk}\frac{d\phi^j}{d\tau}\frac{d\phi^k}{d\tau} = -e^{2U}\mathcal{G}^{ij}\partial_j V_{BH}; \tag{4.51}$$

since $0 = d\phi^i/d\omega = \tau d\phi^i/d\tau$, multiplying (4.51) by τ^2 and considering it in the near-horizon limit, we obtain

$$\frac{d^2\phi^i}{d\tau^2} = -\frac{4\pi}{A\tau^2}\left(\mathcal{G}^{ij}\partial_j V_{BH}\right)\Big|_{\phi_h}$$

with solution

$$\phi^i = \frac{4\pi}{A}\left(\mathcal{G}^{ij}\partial_j V_{BH}\right)\Big|_{\phi_h}\log(-\tau) + \alpha\tau + \phi_h.$$

We drop the term linear in τ since it would lead to a singularity at the horizon. Moreover, it is clear that the only way to have a regular scalar field at $\tau \to \infty$ is to impose

$$\left.\frac{dV_{BH}}{d\phi^i}\right|_h = 0. \tag{4.52}$$

So the fixed values of the scalars are defined as those points in the moduli space which extremize the black hole potential. Since the latter depends only on the moduli and the (given) charges, the (4.52) translates into a set of n

equations with n variables, the ϕ_h^i, and it would be possible to determine the fixed, attractive points in terms of the electric and magnetic charges only. We cannot exclude, however, that the potential possesses some flat directions in the moduli space, i.e. directions in which its derivative with respect to some scalars vanishes identically. So, a priori, we should expect that the fixed points may depend also on the asymptotic values of the moduli.

Until now we have analysed the problem with the non-extremality parameter r_0 set to zero. What if we do not make this assumption? It turns out that the attractor mechanism does not work. First of all, requirement of finite horizon area causes the metric function to behave as

$$e^{-2U} \longrightarrow \frac{A}{4\pi}\frac{\sinh^2 r_0\tau}{r_0^2} \qquad \text{when } \tau \to -\infty.$$

With the change of coordinates

$$\rho = 2e^{r_0\tau}$$

in the near-horizon limit it should be

$$e^{-2U} \longrightarrow \frac{A}{4\pi}\frac{1}{(\rho r_0)^2}$$

so that the full metric reads

$$ds^2 = \frac{4\pi}{A}(\rho r_0)^2 dt^2 - \frac{A}{4\pi}\left(d\rho^2 + d\Omega^2_{(2)}\right)$$

and the event horizon is now located at $\rho = 0$. The new radial coordinate ρ measures the physical distance from the horizon in units of $r_0 = \sqrt{A/4\pi}$ and, in contrast to the extremal case (see eq.(2.22)) this quantity turns out to be finite for every point at a finite ρ_0 from the horizon

$$\int_0^{\rho_0} \sqrt{\frac{A}{4\pi}}\, d\rho = \sqrt{\frac{A}{4\pi}}\,\rho_0.$$

Then a first indication that the analogue of the extremal attractor mechanism cannot happen when $r_0 \neq 0$ comes from the fact that scalars approaching the horizon do not have enough distance to travel to lose memory about their initial values, while in the extremal case their radial distance from the horizon always diverges. Furthermore, the requirement of regularity of scalars near the horizon results in the statement that they should admit a Taylor expansion in ρ around $\rho = 0$:

$$\phi^i = \phi_h^i + \left.\frac{\partial\phi^i}{\partial\rho}\right|_{\rho=0} \rho + \frac{1}{2}\frac{4\pi}{A}\left.\left(\mathcal{G}^{ij}\partial_j V_{BH}\right)\right|_{\phi_h}\rho^2 + O(\rho^3)$$

and this puts no constraints, aside from finiteness, on their derivatives at the horizon, so an attractor mechanism is not necessarily involved. Attractors are typically related to extremality.

4.3 Attractors in N=2, d=4 Supergravity

In this section we specialize the previous treatment to the case in which the action (4.1) describes the bosonic sector of a N=2, d=4 Supergravity: the scalar manifold $\mathcal{M}_{scalar}$ is then a special Kähler manifold parametrized by the complex coordinates z^i, $\overline{z}^{i^*}$ and metric given by $\mathcal{G}_{ij^*} = \partial_i \partial_{j^*} \mathcal{K}$, with $\mathcal{K} \equiv \mathcal{K}(z, \overline{z})$ the Kähler potential introduced in (3.18). The action is

$$S = \int dx^4 \sqrt{|g|} \left(\hat{R} + 2\,\mathcal{G}_{ij^*}\, \partial_\mu z^i\, \partial^\mu \overline{z}^{j^*} + 2 I_{\Lambda\Sigma}\, \hat{F}^\Lambda_{\mu\nu} \hat{F}^{\Sigma\,\mu\nu} - 2 R_{\Lambda\Sigma}\, \hat{F}^\Lambda_{\mu\nu}{}^\star \hat{F}^{\Sigma\,\mu\nu} \right) \tag{4.53}$$

and the equations coming from it, (4.35), (4.36) and (4.37), become

$$(U')^2 + \mathcal{G}_{ij^*} \frac{dz^i}{d\tau} \frac{d\overline{z}^{j^*}}{d\tau} + e^{2U} V_{BH} = r_0^2 \tag{4.54}$$

$$U'' + e^{2U} V_{BH} = 0 \tag{4.55}$$

$$\frac{d^2 z^i}{d\tau} + \mathcal{G}^{ij^*} \partial_k \mathcal{G}_{lj^*} \frac{dz^k}{d\tau} \frac{dz^l}{d\tau} + e^{2U} \mathcal{G}^{ij^*} \partial_{j^*} V_{BH} = 0. \tag{4.56}$$

4.3.1 Supersymmetric attractors

The crucial fact which occurs when considering the above problem in the context of Supergravity is that, as pointed out in [15], the black hole potential has exactly the same form[5] of the symplectically invariant quantity introduced in (3.40)

$$-V_{BH} = -\frac{1}{2} Q^T \mathcal{M}\, Q = |Z|^2 + |Z_i|^2 \tag{4.57}$$

so it is directly related to the central and matter charges of the theory.[6] This identification allows to study easily the critical points of the potential using relations of special geometry, and also to gain insight into the difference of behaviour between extremal BPS and non-BPS black holes. In fact, given

[5] To make direct contact with the notation of [15], and also to avoid extra numerical factors, we set $\alpha^2 = 1/4$ in (4.34)

[6] This is true in all supergravity theories, see for example [13].

(4.57), the lagrangian appearing in the effective action (4.38) can be written ([15], [25])

$$\mathcal{L}_{\text{eff}} = \left(\frac{dU}{d\tau} \pm e^U|Z|\right)^2 + \mathcal{G}_{ij^*}\left(\frac{dz^i}{d\tau} \pm e^U\mathcal{G}^{ik^*}D_{k^*}\overline{Z}\right)\left(\frac{d\overline{z}^{j^*}}{d\tau} \pm e^U\mathcal{G}^{j^*l}D_lZ\right) \mp 2\frac{d}{d\tau}\left(e^U|Z|\right) \quad (4.58)$$

and the second order equations of motion can be solved assuming that the following first order equations[7], coming from $\mathcal{L}_{\text{eff}}$, are satisfied:

$$\frac{dU}{d\tau} = e^U|Z| \quad (4.59)$$

$$\frac{dz^i}{d\tau} = e^U\mathcal{G}^{ij^*}D_{j^*}\bar{Z}. \quad (4.60)$$

It can be shown ([26]) that (4.59) and (4.60) are derivable from the N=2 Killing spinor equations, so that the corresponding solutions are actually supersymmetric (BPS). Evaluating them at the horizon, $\tau \to -\infty$, gives (remembering (4.41))

$$\left(\frac{A}{4\pi}\right)^{\frac{1}{2}} = |Z(z_h, \overline{z}_h, p, q)| \quad (4.61)$$

$$D_iZ|_h = 0. \quad (4.62)$$

Using the (4.61), an expression for the entropy of an extremal supersymmetric black hole is readily found

$$S = \pi|Z(z_h, \overline{z}_h, p, q)|^2 \quad (4.63)$$

and we can also justify eq. (3.5): from (4.49), (4.57) and (4.62) it is clear that

$$M_{RB} = |Z_{max}(z_h, \overline{z}_h, p, q)|\,. \quad (4.64)$$

Moreover, equation (4.62) permits to find the fixed points z_h, $\overline{z}_h$ only in terms of the charges p^Λ, q_Λ. In fact it has been shown ([14], [23] and [24]) that, for supersymmetric configurations, the fixed points of scalars do not depend on their values at infinity; we can conclude that there is an attractor mechanism at work (specifically, the fixed points are *supersymmetric attractors*, following [25]).

[7] These are also called *flow equations*.

The (4.62) is also equivalent ([24]) to requiring $\partial_i |Z||_h = 0$: this expresses the so-called *minimal area principle*, stating that the area of the black hole horizon is proportional to an extremum of the modulus of the central charge, extremized with respect to the moduli. We will see that this extremum is a minimum.

We can give an argument for the fact that use of equations (4.59), (4.60) means requirement of supersymmetry. If we evaluate the constraint (4.54) at spatial infinity $\tau \to 0^-$, owing to (4.57) we get (without writing the dependence of the central charge on the electric and magnetic charges)

$$M_{ADM}^2(z_\infty, \overline{z}_\infty) + \mathcal{G}_{ij^*}\, \Sigma^i \overline{\Sigma}^{j^*} - |Z(z_\infty, \overline{z}_\infty)|^2 - |Z_i(z_\infty, \overline{z}_\infty)|^2 = r_0^2 \tag{4.65}$$

where

$$\Sigma^i \equiv \left.\frac{dz^i}{d\tau}\right|_\infty .$$

It is evident that setting $r_0 = 0$ (i.e. imposing extremality) in (4.65) does not imply the saturation of the BPS bound. Instead, this is reached if we use the two equations (4.59), (4.60), evaluated at spatial infinity:

$$M_{ADM} = |Z(z_\infty, \overline{z}_\infty)| \tag{4.66}$$

$$\Sigma^i = \mathcal{G}^{ij^*} D_{j^*} \bar{Z}\big|_\infty ; \tag{4.67}$$

the first relation already provides the BPS condition; inserting the second one in (4.65) when $r_0 = 0$ gives the same. This is an example of the general truth that supersymmetry implies extremality.

Now, we can exploit the identity (4.57) in order to study the critical points of the black hole potential. Through the following relations, which are immediate consequences of the (3.25), (3.26) and (3.27) introduced in the previous chapter and of the definition of Z, (3.35),

$$D_i D_j Z = i C_{ijk}\, \mathcal{G}^{kk^*}\, D_{k^*} \bar{Z} \tag{4.68}$$

$$D_i D_{j^*} \bar{Z} = \mathcal{G}_{ij^*} \bar{Z} \tag{4.69}$$

$$D_{i^*} Z = 0 \tag{4.70}$$

it is immediate to calculate

$$-\,\partial_i V_{BH} = 2\bar{Z} D_i Z + i C_{ijk}\, \mathcal{G}^{jj^*}\, \mathcal{G}^{kk^*} D_{j^*} \bar{Z} D_{k^*} \bar{Z}. \tag{4.71}$$

This means that the critical points of the central charge are also critical points for the potential:

$$D_i Z = D_{i^*} \bar{Z} = 0 \qquad \Longrightarrow \qquad \partial_i V_{BH} = \partial_{i^*} V_{BH} = 0. \tag{4.72}$$

The equation (4.62) tells that a critical point of the central charge occurs at the horizon. The nature of this extremum was analysed in [15]: in particular, being $D_i Z = D_{i^*} \bar{Z} = 0$ at the horizon, it turns out that

$$(\partial_i \partial_j |Z|)_h = 0$$

while the mixed second derivatives are

$$(\partial_i \partial_{j^*} |Z|)_h = \frac{1}{2} \mathcal{G}_{ij^*} |Z|_h \tag{4.73}$$

so, if the scalar metric is positive at this critical point, the central charge, and then the BPS mass, is correspondingly minimized. For what concerns the extremization of the potential at the horizon, equations (4.52), (4.57) and (4.62) say that

$$(\partial_i V_{BH})_h = (\partial_{i^*} V_{BH})_h = 0 \qquad - (V_{BH})_h = |Z|_h^2 \tag{4.74}$$

and, as calculated in [15],

$$(\partial_i \partial_{j^*} V_{BH})_h = -2 (\mathcal{G}_{ij^*} V_{BH})_h \tag{4.75}$$

so again, if the scalar metric does not change sign at this critical point, the black hole potential reaches a minimum (remembering that in our notation $V_{BH} < 0$).

4.3.2 Non-supersymmetric attractors

The above discussion showed that extremal supersymmetric configurations exhibit an attractor mechanism in their scalar sector; however, we already anticipated that attractors as we have defined them are not automatically introduced when we restrict to extremal solutions. The point is that the (4.58) is not the unique way one can rewrite the effective lagrangian; the (4.38) can be rewritten as in (4.58) but substituting $|Z|$ with another function W (called "superpotential", [39]) depending on scalar fields and electric/magnetic charges. The choice $W = |Z|$ is the only case with the superpotential related to the eigenvalue of the central charge, and the corresponding black hole solutions are supersymmetric. In general, we can consider different superpotentials W, which will not give supersymmetric configurations. Equations (4.63) for the entropy and (4.66) for the ADM mass are still valid but with Z replaced by W; the complete attractor mechanism does not always take place, since the fixed points of scalars could depend on their values at spatial infinity, owing to the presence of flat directions in the black hole potential (an example of this is provided in [25]). In this case the critical points of the potential are named *non-supersymmetric attractors*. For the time being, it seems that the complete attractor mechanism takes place in general only in BPS configurations; however, it has been shown that in all extremal cases the entropy depends only on the charges [27].

Chapter 5

Stationary BH-type solutions of General Relativity

In this chapter we move from static to stationary solutions of General Relativity: first we briefly review the Kerr-Newman black hole, then we introduce the spacetime which will be central in the following treatment, the Taub-NUT solution. The motivation for studying the latter is that it would be interesting to extend the calculation developed in the previous chapter (the rewriting of the equations of motion of gravity coupled to vector supermultiplets in a geodesic-like form and the determination of a black hole potential) to stationary black holes (i.e. Kerr-Newman's), in order to analyse, for example, if an attractor mechanism is at work. Since doing this in the Kerr-Newman case seems pretty difficult, we will try to do the same in the simpler situation of a Taub-NUT metric, which is maybe the most manageable stationary, black hole-type metric (in the sense that an event horizon is present); facing the issue of coupling scalars and vectors to this metric can give some insight into how to handle the same problem in the Kerr-Newman's case.
We will also see that some particular features of this solution don't allow us to classify it as real black hole, although it shares some properties with the Reissner-Nordström solution.

5.1 Stationary Black Holes: the Kerr-Newman solution

Schwarzschild's (Reissner-Nordström's) metric is the only static, spherically symmetric black hole-type solution of the Einstein (Einstein-Maxwell) system. If the staticity condition is relaxed and one contents himself only with stationarity, spherical symmetry will not hold any more and the spacetime will be axisymmetric.[1] Stationary, axisymmetric metrics are characterized by two Killing vectors ∂_t (generating time translations) and ∂_φ (generating rotations around the z-axis) which are not mutually orthogonal, so that the $g_{t\varphi}$ component of the metric is different from zero. The only black hole-type stationary solution of the Einstein-Maxwell system is the *Kerr-Newman* metric [28]:

$$\begin{aligned} ds^2 = &\left(1 - \frac{2Mr - 4q^2}{\Sigma}\right) dt^2 + 2\,\frac{a(2Mr - 4q^2)\sin^2\theta}{\Sigma} dt\, d\varphi \\ &- \frac{\Sigma}{\Delta} dr^2 - \Sigma d\theta^2 - \frac{\mathcal{A}}{\Sigma}\sin^2\theta\, d\varphi^2 \end{aligned} \tag{5.1}$$

$$\Sigma = r^2 + a^2\cos^2\theta \qquad \Delta = r^2 - 2Mr + 4q^2 + a^2 \tag{5.2}$$

$$\mathcal{A} = \Sigma\left(r^2 + a^2\right) + \left(2Mr - 4q^2\right) a^2 \sin^2\theta \tag{5.3}$$

$$A_\mu = \frac{4qr}{\Sigma}\left(\delta_{\mu t} - \delta_{\mu\varphi}\, a \sin^2\theta\right) \tag{5.4}$$

where $a = J/M$ is the angular momentum per unit mass; the coordinates (t, r, θ, φ) are called *Boyer-Lindquist coordinates*. The metric (5.1) has three parameters, M, a and the total electric charge of spacetime q: in the cases $a = q = 0$ and $a = 0$ it reduces respectively to the Schwarzschild and Reissner-Nordström solutions, while when $q = 0$ we obtain its vacuum version, the Kerr metric. If finally $M = q = 0$, $a \neq 0$ the spacetime is Minkowski's, in spheroidal coordinates.

The expression (5.1) is evidently singular when $\Sigma = 0$ and $\Delta = 0$. Analysis of curvature invariants shows that, when $M = 0$, the divergence given by $\Sigma = 0$ (i.e. $r = 0$, $\theta = \pi/2$: a ring[2]) is the true physical singularity; instead, only when $M^2 > 4q^2 + a^2$ the equation $\Delta = 0$ has two distinct solutions, $r_\pm = M \pm \sqrt{M^2 - 4q^2 - a^2}$: the surface at $r = r_+$ is the event horizon, while the one at r_- is a sort of “inner” horizon; both the horizons cover the ring singularity. In the extremal case, $M^2 = 4q^2 + a^2$ (we can then identify the parameter of extremality, $r_0^2 = M^2 - 4q^2 - a^2$), the two horizons coincide (as in the Reissner-Nordström black hole); the remaining option is the absence of event horizons so that the ring singularity is naked, and the resulting object is not a black-hole (this cased is ruled out by the cosmic censorship hypothesis).

[1] This is a consequence of a set of theorems by Hawking, Israel and Carter. References and reviews can be found in [7].

[2] As it is clear taking the $M \to 0$ limit, Boyer-Lindquist coordinates resemble spheroidal coordinates.

5.2 The Taub-NUT solution

The Kerr-Newman family of metrics exhausts the possible black hole-type stationary solutions of General Relativity coupled to electromagnetism; now we examine a further stationary solution which, strictly speaking, is not a black hole (in compliance with the uniqueness theorems). The Taub-NUT metric was discovered in 1951 by A. H. Taub [29] as an exact solution to the vacuum Einstein equations, and then extensions of it were found in 1963 [30]. The charged version was given by Brill [31]. The main feature of this metric is the presence of a new parameter, the NUT charge N which can be interpreted as a dual ("magnetic") mass. The structure of the singularities related to the new charge is identical to that of the Dirac monopole vector field, and this results in a relation between the time coordinate and the NUT charge which resembles the Dirac quantization condition.

5.2.1 The vacuum Taub-NUT metric

The expression of the Taub-NUT metric in Schwarzschild-like coordinates is

$$ds^2 = f(r)(dt + 2N\cos\theta\, d\varphi)^2 - f^{-1}(r)dr^2 - (r^2 + N^2)d\Omega^2_{(2)} \tag{5.5}$$

$$f(r) = \frac{(r - r_+)(r - r_-)}{r^2 + N^2} \tag{5.6}$$

$$r_\pm = M \pm r_0 \qquad r_0^2 = M^2 + N^2. \tag{5.7}$$

It reduces to the Schwarzschild solution for $N = 0$. M can be identified with the mass of the solution. The new parameter N, called *NUT charge*, can be interpreted noticing that the only off-diagonal term in the metric is, for large r, $g_{t\varphi} \sim 2N\cos\theta$: the same form of the vector potential of a magnetic monopole with charge proportional to N. Following this analogy, the NUT charge could be considered a magnetic mass, dual to the mass M: the Taub-NUT field is composed by a diagonal part which can be put in relation with the Newtonian potential, given, for large r, by

$$\phi = \frac{g_{tt} - 1}{2} \sim -\frac{M}{r}$$

(this also confirms that M really represents the mass); and then there is a off-diagonal, "gravitomagnetic" component, with strength controlled by N. This is the reason for calling the solution a *gravitational dyon* (see also [32]). For $N \neq 0$ the spacetime is not asymptotically flat: one evident first reason is given by the presence of the off-diagonal term in the metric, which does not vanish at infinity. Moreover, as pointed out by Misner in [33], although all curvature invariants vanish for $r \to \infty$ (being $R_{\mu\nu\rho\sigma} = O(1/r^3)$), it is not possible to find coordinate systems for which $g_{\mu\nu} - \eta_{\mu\nu} \to 0$ when $r \to \infty$.

Symmetries and periodic time coordinate. For $N \neq 0$ the above metric is free of curvature singularities[3]; in particular it is regular in $r = 0$, so, although (5.5) somewhat generalizes Schwarzschild's metric, there is not a singularity that could possibly be hidden by some event horizon: we are not dealing with a black hole. But, as thoroughly discussed by Misner in [33], the metric presents "wire" singularities (*Misner strings*) for $\theta = 0, \pi$, which are not the standard singularities of spherical coordinates; instead, they are related to the off-diagonal term containing the NUT charge.[4] This turns out clearly considering rectangular coordinates xyz and analysing the corresponding expression for the one-form $dt + 2N \cos\theta \, d\varphi$; given the absence of physical singularities, Misner strings have to be a topological defect. The situation is similar to the case of a magnetic monopole placed in the origin: a vector potential describing appropriately the magnetic field can be determined locally, but it will be singular along some string-like region (usually these strings- *Dirac strings* -are chosen to stay on the negative or positive part of the z-axis). In the approach proposed by Misner[5], the two singularities do not have a physical significance, since they can be removed introducing two distinct coordinate patches:

- one covers smoothly the $0 \leq \theta < \pi$ region: here, shifting the time to

  $$t_N = t + 2N\varphi,$$

  the metric becomes

  $$ds_N^2 = f(r)\left[dt_N - 2N(1-\cos\theta)d\varphi\right]^2 - f^{-1}(r)dr^2 - (r^2+N^2)d\Omega_{(2)}^2$$

  and is regular everywhere except for the south pole, $\theta = \pi$;

- the other parametrizes the region $0 < \theta \leq \pi$, where the time coordinate is changed to

  $$t_S = t - 2N\varphi$$

[3] A calculation of the curvature can be found in [33].

[4] In the first treatment by Newman, Tamburino and Unti ([30]) the singularity at $\theta = 0$ was removed with a redefinition of the $dtd\varphi$ term.

[5] An alternative approach, suggested by Bonnor in [34], considers the $\theta = \pi$ singularity (after having removed the $\theta = 0$ one through the redefinition mentioned above) as physical: the source of the TN field is composed by a spherically symmetric mass plus a semi-infinite source of angular momentum along the symmetry axis. This interpretation, as opposed to Misner's, preserves the axial symmetry of the system.

so that the metric is

$$ds^2_S = f(r)\,[dt_S + 2N(1+\cos\theta)d\varphi]^2 - f^{-1}(r)dr^2 - (r^2+N^2)d\Omega^2_{(2)}$$

and is regular at the south pole but singular at $\theta = 0$.

By unifying these two patches one can obtain a metric free of singularities. In the region $0 < \theta < \pi$, where the two parametrizations overlap, $t_N = t_S + 4N\varphi$ holds; being φ an angular coordinate with period 2π, for consistency also time has to be periodic, with period $8N\pi$. This resembles the way the Dirac quantization condition can be obtained by trying to eliminate the wire singularities of the magnetic monopole. Furthermore, the group of motion that the metric admits is determined by four Killing vectors[6]: three of these (the spatial ones) have the commutation rules of the So(3) Lie algebra. It can be shown that, when the time coordinate has period $8N\pi$, these So(3)-generating vectors can be integrated to give an unitary representation of the global SO(3) and consequently the spacetime is spherically symmetric.[7] Finally, it turns out ([33]) that constant-r hypersurfaces have the topology of a three-sphere, S^3, with $(t/2N,\, \theta,\, \varphi)$ Euler angles.

Horizons and structure of the spacetime. The function $f(r)$ vanishes for $r = r_\pm$, where the metric has two coordinate singularities. The two hypersurfaces at $r = r_\pm$, which are Killing horizons, are two horizons analogous to the ones located at $r = r_\pm$ in the Reissner-Nordström black hole, and it is possible to find extensions through them similar to the Eddington-Finkelstein ones ([38]). In the following, we will refer to the outer surface as to the event horizon.
The two surfaces also act as Cauchy horizons for the region $r_- < r < r_+$, where t is spacelike and r timelike: here the Taub-NUT metric describes a closed, anisotropic, singularity-free spacetime ("Taub universe": the original Taub solution described this region). Following [31], this can be interpreted as a universe "held together by its content of gravitational radiation, which is present in its lowest possible mode". Instead, for $r < r_-$ and $r > r_+$ (the regions described by the extension provided by Newman, Tamburino and Unti), the role of t and r is exchanged as t is timelike and r spacelike: here it is evident that the Taub-NUT metric is somewhat an extension of Schwarzschild's. Owing to the periodicity of the time coordinate the metric here possesses closed timelike curves (CRCs), and this is another obstacle to

[6]See [30], [33]. One of these Killing vectors is related to time translations.

[7]See Hurst, [35] for the general case; Dowker in [36] faces the problem in the gravitational Taub-NUT case.

the interpretation of the Taub-NUT solution as a black hole. The presence of CRCs is not a Taub-NUT's exclusive feature but is typical of gravitational solutions with some sort of dual mass [37]. CRCs could be avoided by dropping the periodicity condition on the time coordinate, but at the price of keeping the wire singularities. This pathologies cause no possibility for the Taub-NUT metric to have some part in the description of macroscopical objects; instead, it deserves some importance in the context of quantum gravity theories.

5.2.2 The charged Taub-NUT solution

The electrically charged version of the TN solution was found by Brill in 1963 [31]:

$$ds^2 = f(r)(dt + 2N\cos\theta\, d\varphi)^2 - f^{-1}(r)dr^2 - (r^2+N^2)d\Omega^2_{(2)} \tag{5.8}$$

$$f(r) = \frac{(r-r_+)(r-r_-)}{r^2+N^2} \tag{5.9}$$

$$F_{tr} = \frac{4q\,(r^2-N^2)}{(r^2+N^2)^2} \qquad\qquad ({}^\star F)_{tr} = \frac{8qNr}{(r^2+N^2)^2} \tag{5.10}$$

$$r_\pm = M \pm r_0 \qquad r_0^2 = M^2+N^2-4q^2 \tag{5.11}$$

where q is the electric charge. When $N=0$ we recover the Reissner-Nordström spacetime. The solution can be further generalized with the inclusion of a magnetic charge p: the form of the metric is the same as in (5.8), but r_0 (which now can be identified with the non-extremality parameter) has to be modified:

$$r_0^2 = M^2+N^2-4(p^2+q^2).$$

and the electromagnetic field strength becomes correspondingly

$$F_{tr} = \frac{4q\,(r^2-N^2)-8pNr}{(r^2+N^2)^2} \qquad ({}^\star F)_{tr} = \frac{8qNr+4p(r^2-N^2)}{(r^2+N^2)^2}. \tag{5.12}$$

The spacetime described by the metric (5.8) has a structure similar to the vacuum Taub-NUT space: again, for $r_- < r < r_+$, t is spacelike and r timelike, and the metric describes a closed cosmological model whose electromagnetic field contents are given by (5.12). For $r < r_-$ and $r > r_+$, the situation is analogous to the vacuum case. If the non-extremality parameter r_0 is zero, the two horizons coincide.

Conforma-stationary metrics. Also in this case it is possible and convenient to write the metric in an isotropic form, with a new radial coordinate τ

which covers the region from spatial infinity (where it will be $\tau = 0$) to the horizon ($\tau = -\infty$). Firstly, we rewrite the (5.8) in a slightly more general way

$$ds^2 = f(r)(dt + \omega)^2 - f^{-1}dr^2 - (r^2 + N^2)d\Omega^2_{(2)} \tag{5.13}$$

keeping the identifications (5.9) and (5.10). The ω should be intended as a 1-form in 3-dimensional space; in the Taub-NUT case its only non-vanishing component is $\omega_\varphi = 2N\cos\theta$. Now, performing the change of coordinate

$$r = \frac{\rho^2 + M\rho + \frac{r_0^2}{4}}{\rho}$$

the following spatially isotropic form is reached

$$\begin{aligned} ds^2 = & \frac{\left(1 - \frac{r_0/2}{\rho}\right)^2\left(1 + \frac{r_0/2}{\rho}\right)^2}{\left(1 + \frac{\rho_+/2}{\rho}\right)^2\left(1 + \frac{\rho_-/2}{\rho}\right)^2 + \frac{N^2}{\rho^2}}(dt + \omega)^2 \\ & - \left[\left(1 + \frac{\rho_+/2}{\rho}\right)^2\left(1 + \frac{\rho_-/2}{\rho}\right)^2 + \frac{N^2}{\rho^2}\right]\left(d\rho^2 + \rho^2 d\Omega^2_{(2)}\right) \end{aligned}$$

where

$$\rho_\pm = M \pm \sqrt{4q^2 - N^2}$$

and the horizons are at $\rho = \pm\frac{r_0}{2}$. Next, with the reparametrization

$$\rho = -\frac{r_0}{2\tanh\frac{r_0\tau}{2}},$$

we reach the expression

$$ds^2 = e^{2U}(dt + \omega)^2 - e^{-2U}\gamma_{\underline{mn}}dx^{\underline{m}}dx^{\underline{n}} \tag{5.14}$$

where the three dimensional metric $\gamma_{\underline{mn}}$ is that of (2.12). The metric function is

$$e^{-2U} = e^{-2r_0\tau}\left[\frac{r_+^2 + N^2}{4r_0^2} + \frac{r_-^2 + N^2}{4r_0^2}e^{4r_0\tau} - \frac{2q^2}{r_0^2}e^{2r_0\tau}\right]. \tag{5.15}$$

It is easy to see that the (5.15) reduces to the Reissner-Nordström metric function (2.30) when the NUT charge is zero. If instead it is the electric charge which vanishes, we have the function for the vacuum Taub-NUT metric:

$$e^{-2U} = \frac{r_+}{2r_0}e^{-2r_0\tau} - \frac{r_-}{2r_0}e^{2r_0\tau}. \tag{5.16}$$

Metrics with the form (5.14) are known as *conforma-stationary metrics*. Notice that neither (5.15) nor (5.16) depend on the angular coordinates, but only on τ. In the next section and in Chapter 6 we will take advantage of this, making the assumption of spherical symmetry for the scalar U (and obviously also for the other fields that will be coupled to gravity).

5.3 Einstein equations for the vacuum Taub-NUT metric

In order to see in which way the new off-diagonal components of the metric enter in the calculations we are going to perform, let us see firstly how the equations of motion look like in the case of a pure gravitational system. So we consider the general form of conformastationary metrics

$$ds^2 = e^{2U}(dt+\omega)^2 - e^{-2U}\gamma_{\underline{mn}}dx^{\underline{m}}dx^{\underline{n}} \tag{5.17}$$

and in this case we will have $\omega = \omega_\varphi d\varphi$ and $\omega_\varphi = 2N\cos\theta$. The spatial 3-dimensional metric is

$$\gamma_{\underline{mn}}dx^{\underline{m}}dx^{\underline{n}} = \frac{r_0^4}{\sinh^4 r_0\tau}d\tau^2 + \frac{r_0^2}{\sinh^2 r_0\tau}d\Omega^2_{(2)}. \tag{5.18}$$

We want to solve the Einstein equation for the system whose action is[8]

$$S = \int dx^4\sqrt{|g|}\,\hat{R}, \tag{5.19}$$

from which the vacuum Einstein equation follows

$$\hat{G}_{\mu\nu} = \hat{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\hat{R} = 0, \tag{5.20}$$

with the ansatz (5.17) for the metric, in order to determine the form of the function e^{2U}. As said before, we shall assume that the function U depends only on the radial coordinate τ. It is convenient to work in the tangent space, using the choice for the Vielbein basis reported in Appendix A, where also the curvature and the Ricci tensor and scalar can be found. We need to calculate the various component of the Ricci tensor: the first thing to be determined is the tangent space expression of the antisymmetric tensor $W_{\underline{mn}} \equiv 2\partial_{[\underline{m}}\omega_{\underline{n}]}$. The 3-dimensional part of the Vielbein basis can be easily specified using the fact that the spatial metric (5.18) is diagonal; thus a convenient choice for the spatial *dreibein* is:

$$v^1{}_\tau = \sqrt{\gamma_{\tau\tau}} = \frac{r_0^2}{\sinh^2 r_0\tau} \tag{5.21}$$

$$v^2{}_\theta = \sqrt{\gamma_{\theta\theta}} = \frac{r_0}{\sinh r_0\tau} \tag{5.22}$$

$$v^3{}_\varphi = \sqrt{\gamma_{\varphi\varphi}} = \frac{r_0\sin\theta}{\sinh r_0\tau}. \tag{5.23}$$

[8] We keep working with the normalization and notations of (4.1), distinguishing between four- and three-dimensional tensors.

Since the only non-vanishing component of the 1-form ω is the φ's one, the matrix W (in curved indices) has only two non-zero components:

$$W_{\theta\varphi} = -W_{\varphi\theta} = \partial_\theta \omega_\varphi = -2N \sin\theta \tag{5.24}$$

and the same antisymmetric structure will be preserved switching to the tangent space basis:

$$W_{23} = {v_2}^\theta {v_3}^\varphi \, W_{\theta\varphi} = -\frac{2N \sinh^2 r_0 \tau}{r_0^2}$$

with $W_{23} = -W_{32}$. The Ricci tensor in flat indices is then

$$\hat{R}_{00} = -e^{2U} \left(\nabla^2 U + 2\, e^{4U} N^2 \gamma^{\tau\tau} \right) \tag{5.25}$$

$$\hat{R}_{0m} = 0 \tag{5.26}$$

$$\hat{R}_{mn} = e^{2U} \left(R_{mn} + 2\, \partial_m U \partial_n U - \delta_{mn} \nabla^2 U - \frac{e^{4U}}{2} (W_{m2} W_{n2} + W_{m3} W_{n3}) \right) \tag{5.27}$$

and the Ricci scalar

$$\hat{R} = -e^{2U} \left(R - 2\, e^{4U} N^2 \gamma^{\tau\tau} - 2\nabla^2 U + 2 (\partial U)^2 \right) .$$

Now we can specify the various components of the Einstein equation, which in flat indices are

$$\hat{G}_{00} = \hat{R}_{00} - \frac{1}{2} \eta_{00} \hat{R} = 0 \Rightarrow R + 2\, (\partial U)^2 - 4 \nabla^2 U - 6 e^{4U} N^2 \gamma^{\tau\tau} = 0 \tag{5.28}$$

$$\hat{G}_{0m} = \hat{R}_{0m} - \frac{\eta_{0m}}{2} \hat{R} = 0 \tag{5.29}$$

$$\begin{aligned} \hat{G}_{mn} = 0 \Rightarrow G_{mn} &+ 2\, \partial_m U \partial_n U - \delta_{mn} (\partial U)^2 \\ &- \frac{e^{4U}}{2} \left(W_{mp} W_{np} - 2 \delta_{mn} N^2 \gamma^{\tau\tau} \right) = 0 \end{aligned} \tag{5.30}$$

where in the last equation $G_{mn} = R_{mn} - \frac{1}{2} \delta_{mn} R$, with all 3-d tensors. The two equations that we have obtained

$$R + 2\, (\partial U)^2 - 4 \nabla^2 U - 6 e^{4U} N^2 \gamma^{\tau\tau} = 0 \tag{5.31a}$$

$$G_{mn} + 2\, \partial_m U \partial_n U - \delta_{mn} (\partial U)^2 - \frac{e^{4U}}{2} \left(W_{mp} W_{np} - 2 \delta_{mn} N^2 \gamma^{\tau\tau} \right) = 0 \tag{5.31b}$$

can be further manipulated taking the trace of (5.31b), which is

$$R + 2\, (\partial U)^2 + 2\, e^{4U} N^2 \gamma^{\tau\tau} = 0$$

and putting it in (5.31a), obtaining

$$\nabla^2 U + 2\, e^{4U} N^2 \gamma^{\tau\tau} = 0. \tag{5.32}$$

Now we'll go back to curved (3-dimensional) space. To proceed, we need the explicit form of the 3-dimensional Ricci tensor and scalar; the Christoffel symbols of the metric (5.18) are listed in Appendix A and, knowing them, it is only a matter of time to compute the non-zero components of the Riemann tensor

$$\begin{aligned} R_{\tau\theta\tau}{}^{\theta} &= R_{\tau\varphi\tau}{}^{\varphi} = -r_0^2 \\ R_{\theta\varphi\theta}{}^{\varphi} &= \sinh^2 r_0\tau \end{aligned}$$

and then find that the only non-vanishing term in the Ricci tensor is

$$R_{\tau\tau} = -2r_0^2$$

so that for the Ricci scalar we have

$$R = R_{\underline{mn}}\gamma^{\underline{mn}} = R_{\tau\tau}\gamma^{\tau\tau} = -2\frac{\sinh^4 r_0\tau}{r_0^2}.$$

Changing from flat to curved indices, the function $\nabla^2 U$ becomes $\frac{\sin\theta}{\sqrt{|\gamma|}}\, U''(\tau) = \gamma^{\tau\tau} U''(\tau)$ and the (5.32) reaches its final form

$$U'' + 2e^{4U} N^2 = 0. \tag{5.33}$$

For what concerns the (5.31b), all its three non-zero components give equations proportional to

$$(U')^2 + e^{4U}\, N^2 = r_0^2. \tag{5.34}$$

The (5.33) and (5.34) form the system of differential equations that we need to solve in order to determine $U(\tau)$ (or, better, the second could be seen as a constraint on the solutions to the first one). Actually, the first equation can be obtained from the second upon differentiation with respect to τ, so in fact it is sufficient to solve the first one; the integration constant which will arise in the solution can be determined imposing asymptotic flatness. Solving (5.34) we have:

$$\begin{aligned} (U')^2 + e^{4U}\, N^2 = r_0^2 \implies U' &= \sqrt{r_0^2 - e^{4U} N^2} \\ \implies \int \frac{U'\, d\tau}{\sqrt{r_0^2 - e^{4U} N^2}} &= \int d\tau = \tau + k. \end{aligned}$$

The left-side integral in the last line can be solved with the substitution $t^2 = r_0^2 - e^{4U}N^2$; the result is

$$-\frac{1}{4r_0}\ln\left|\frac{r_0+\sqrt{r_0^2-e^{4U}N^2}}{r_0-\sqrt{r_0^2-e^{4U}N^2}}\right| = \tau + k \tag{5.35}$$

and then we get the expression for the metric function, apart from the presence of the yet unknown integration constant k:

$$e^{-2U} = \pm\frac{N}{r_0}\cosh 2r_0(\tau+k). \tag{5.36}$$

We choose the sign $+$, considering the modulus of N. To determine the final form of the solution we need k; the only way to do this seems to be the imposition of asymptotic flatness[9], in the limit $\tau \to 0$:

$$\frac{|N|}{r_0}\cosh 2r_0(\tau+k) \xrightarrow{\tau\to 0} 1 - 2M\tau \tag{5.37}$$

where M is the mass. For $\tau \to 0$ we have[10]

$$\frac{|N|}{r_0}\cosh 2r_0(\tau+k) \xrightarrow{\tau\to 0} \frac{|N|}{r_0}(\cosh 2r_0k + 2r_0\tau\sinh 2r_0k) \tag{5.38}$$

and taking into account (5.37) we get

$$\frac{r_0}{|N|} = \cosh 2r_0k \qquad |N|\sinh 2r_0k = -M. \tag{5.39}$$

From these relations, we have

$$N^2(\cosh^2 2r_0\tau - 1) = M^2 \implies r_0^2 = M^2 + N^2$$

which fixes the relation between the mass, the NUT charge N and the non-extremality parameter r_0. Returning to the (5.36), using the two (5.39), the final form of the metric function is reached:

$$\begin{aligned}
e^{-2U} &= \frac{|N|}{r_0}\cosh 2r_0(\tau+k) = \frac{|N|}{r_0}\left(\cosh 2r_0\tau\frac{r_0}{|N|} - \sinh 2r_0\tau\frac{M}{|N|}\right)\\
&= \frac{|N|}{r_0}\left(\frac{e^{2r_0\tau}+e^{-2r_0\tau}}{2}\frac{r_0}{|N|} - \frac{e^{2r_0\tau}-e^{-2r_0\tau}}{2}\frac{M}{|N|}\right)\\
&= \left(\frac{r_0+M}{2r_0}e^{-2r_0\tau} + \frac{r_0-M}{2r_0}e^{2r_0\tau}\right)
\end{aligned}$$

[9] More correctly, we impose that, at spatial infinity, the function e^{-2U} behaves in the same way of $f(r)$ which appears in (5.5); we have seen that Taub-NUT spacetimes are not asymptotically flat.

[10] $\cosh 2r_0(\tau+k) = \cosh 2r_0\tau\cosh 2r_0k + \sinh 2r_0\tau\sinh 2r_0k$.

and through the identifications (5.7) we have finally

$$e^{-2U} = \left(\frac{r_+}{2r_0} e^{-2r_0\tau} - \frac{r_-}{2r_0} e^{2r_0\tau} \right) \tag{5.40}$$

which in fact is the metric function for the Taub-NUT vacuum metric.
It seems that, with respect to the static case, the presence of the additional parameter N modifies the equations of motion only with the term proportional to $e^{4U}N^2$. In the next chapter, where we will consider the equations of motion of bosonic matter in a Taub-NUT background, the above calculation will be useful to keep things under control, at least for what concerns the purely gravitational sector.

Chapter 6

N=2 vector multiplets in the Taub-NUT spacetime

In this chapter, we will face the problem of writing the equations of motion for a system with action (3.8) in the background of a Taub-NUT spacetime. In particular we will try to put the equations into a one-dimensional form, as it was done for the static case in [15] and as we reported in Chapter 4; this will be possible due to independence on time of the system and to the assumption of spherical symmetry for the various fields. Having done this, the possible appearance of an attractor mechanism can be discussed. We will find some complications due to the form assumed by the Maxwell equations, and getting rid of vectors will not be as easy as in the static case; however, at least formally the thing can be done, even though the resulting black hole potential will have a more complicated form.
The calculations we will perform here are a generalisation of those of Chapter 4. As anticipated there, we have chosen to work out in detail the more general case with NUT charge: setting this charge to zero provides exactly the equations of motion that we presented in Chapter 4 without proof. As a test of consistence we can also use the calculation done in the previous chapter, where the equations of the pure gravitational Taub-NUT system were solved.

6.1 Equations of motion

We are always interested in theories whose action is of the form (3.8)

$$S = \int dx^4 \sqrt{|g|} \left(\hat{R} + \mathcal{G}_{ij}\partial_\mu \phi^i \partial^\mu \phi^j + 2I_{\Lambda\Sigma}\, \hat{F}^{\Lambda}_{\mu\nu} \hat{F}^{\Sigma\,\mu\nu} - 2R_{\Lambda\Sigma}\, \hat{F}^{\Lambda}_{\mu\nu}{}^{\star}\hat{F}^{\Sigma\,\mu\nu} \right) \tag{6.1}$$

where $\mathcal{G} \equiv \mathcal{G}(\phi)$, $R_{\Lambda\Sigma} \equiv \mathrm{Re}\,\mathcal{N}_{\Lambda\Sigma}(\phi)$ and $I_{\Lambda\Sigma} \equiv \mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}(\phi)$; the vector field strengths are given in terms of the vector potentials $\hat{F}^{\Lambda}_{\mu\nu} = 2\partial_{[\mu}\hat{A}^{\Lambda}_{\nu]}$; the conventions about three- and four-dimensional quantities are those explained in Chapter 4. The spacetime is given by a conforma-stationary metric (5.14)

$$ds^2 = e^{2U}(dt+\omega)^2 - e^{-2U}\gamma_{\underline{mn}}dx^{\underline{m}}dx^{\underline{n}} \tag{6.2}$$

and for the spatial 3-dimensional metric we keep using the expression (2.12):

$$\gamma_{\underline{mn}}dx^{\underline{m}}dx^{\underline{n}} = \frac{r_0^4}{\sinh^4 r_0\tau}d\tau^2 + \frac{r_0^2}{\sinh^2 r_0\tau}d\Omega^2_{(2)}. \tag{6.3}$$

Now, in the following calculation it is convenient to consider the components $\omega_{\underline{m}}$ of the 3-dimensional 1-form ω appearing in the metric as the components of an independent vector field (*Taub-NUT field*); we will specify them later.

We recall that the equations of motion descending from (6.1) are

$$\hat{G}_{\mu\nu} + \mathcal{G}_{ij}\left(\partial_\mu\phi^i\partial_\nu\phi^j - \frac{1}{2}g_{\mu\nu}\partial_\rho\phi^i\partial^\rho\phi^j\right) + 4I_{\Lambda\Sigma}\left(\hat{F}^{\Lambda}_{\mu}{}^{\rho}\hat{F}^{\Sigma}_{\nu\rho} - \frac{1}{4}g_{\mu\nu}\hat{F}^{\Lambda}_{\rho\sigma}\hat{F}^{\Sigma\,\rho\sigma}\right) = 0 \qquad \text{(Einstein)};$$

$$\nabla_\mu\left(I_{\Lambda\Sigma}\,\hat{F}^{\Sigma\,\mu\nu} - R_{\Lambda\Sigma}\,{}^\star\hat{F}^{\Sigma\,\mu\nu}\right) = 0 \qquad \text{(Maxwell)};$$

$$\nabla_\mu\left(\mathcal{G}_{ij}\partial^\mu\phi^j\right) - \frac{1}{2}\partial_i\mathcal{G}_{kl}\,\partial_\mu\phi^k\partial^\mu\phi^l - \left(\partial_i I_{\Lambda\Sigma}\,\hat{F}^{\Lambda}_{\mu\nu}\hat{F}^{\Sigma\,\mu\nu} - \partial_i R_{\Lambda\Sigma}\,\hat{F}^{\Lambda}_{\mu\nu}{}^\star\hat{F}^{\Sigma\,\mu\nu}\right) = 0 \qquad \text{(scalars)}.$$

Since there is not dependence on time, we can rewrite the above equations in the 3-dimensional space, performing a from-four-to-three dimensional reduction[1]: the 4×4 metric is decomposed into a 3×3 metric (which is $-e^{-2U}\gamma_{\underline{mn}}$), a three dimensional vector ($\omega_{\underline{m}}$) and a scalar (e^U); each 4-dimensional vector field $\hat{A}^{\Lambda}_{\mu}$ instead gives origin to a 3-dimensional vector field, let us name it $V^{\Lambda}_{\underline{m}}$, and a scalar, which will be its time component $\hat{A}^{\Lambda}_t$ (with 3-dimensional vectors and scalars we mean objects that transform in the corresponding ways under $SO(3)$). We are going to obtain this dimensional reduction of the equations of motion using the formalism of Scherz and Schwarz (as described in [3]), with time playing the role of the compactified

[1]This amounts to consider only the Fourier zero mode of the 4-dimensional metric and fields in the so-called *Kaluza-Klein compactification* of the theory, in which the compactified dimension would be the time. See [3].

(although the term is imprecise) coordinate.
The form (6.2) already makes clear the above decomposition of the metric; the Scherk and Schwarz procedure makes use of the Vielbein formalism and we will use the Vielbein basis reported in Appendix A (already used in Chapter 5). Now, it is known that when a coordinate is compactified, some of the general coordinate transformations of the higher-dimensional spacetime appears as internal gauge transformations in the lower-dimensional one. To determine the 3-dimensional vectors coming from the decomposition of the 4-dimensional vector fields, we can exploit the Vielbeins: we define the following object, which is a scalar under diffeomorphisms (for clarity we temporarily drop $\mathcal{N}$-matrix indices)

$$e_m^{\,\underline{n}}\, V_{\underline{n}} \equiv e_m^{\,\mu}\, \hat{A}_\mu = -e^U v_m^{\,\underline{n}}\, \omega_{\underline{n}} \hat{A}_t + e^U v_m^{\,\underline{n}}\, \hat{A}_{\underline{n}} \tag{6.4}$$

and this gives ($\hat{A}_t \equiv \psi$)

$$V_{\underline{m}} = \hat{A}_{\underline{m}} - \psi \omega_{\underline{m}}; \tag{6.5}$$

the quantity $V_{\underline{m}}$ is a 3-dimensional vector, invariant under the internal symmetry coming from the dimensional reduction. This and the scalar $\hat{A}_t$ are the fields which origin from the decomposition of the vectors.
Now, taking the 3-dimensional part of the $\hat{A}_\mu$'s field strength and inserting (6.5) we get

$$\begin{aligned}\hat{F}_{\underline{mn}} \equiv F_{\underline{mn}} = 2\partial_{[\underline{m}} V_{\underline{n}]} + 2\psi\, \partial_{[\underline{m}} \omega_{\underline{n}]} + 2\partial_{[\underline{m}} \psi\, \omega_{\underline{n}]} \\ \equiv G_{\underline{mn}} + \psi\, W_{\underline{mn}} + 2\partial_{[\underline{m}} \psi\, \omega_{\underline{n}]}.\end{aligned} \tag{6.6}$$

As will emerge from explicit calculation, the 3-dimensional field strength will be related to the quantity

$$H_{mn} \equiv e_m^\mu e_n^\nu \hat{F}_{\mu\nu} \tag{6.7}$$

which, using (6.6), turns out to be

$$H_{mn} = e_m^{\underline{m}} e_n^{\underline{n}} \left(G_{\underline{mn}} + \psi\, W_{\underline{mn}} \right) \equiv e_m^{\underline{m}} e_n^{\underline{n}}\, H_{\underline{mn}}. \tag{6.8}$$

The field strength $H_{\underline{mn}}$ is invariant under both ordinary local gauge and internal gauge transformations. Finally, $V_{\underline{m}}$ and then $H_{\underline{mn}}$ carry a period matrix index (as $\hat{A}_\mu$ does).

The next step is to re-express all the equations of motion in terms of the new 3-dimensional fields $\hat{A}_{\underline{m}}$, $\omega_{\underline{m}}$ and ψ, using the relations (6.5) and (6.6). Moreover, since we want to obtain a final expression for the equations in terms of only scalar quantities, it will be necessary to introduce appropriate new scalar fields. Let us consider separately the various equations.

Maxwell's equation. Let us start with the time component of Maxwell's equation

$$\nabla_{\underline{m}}\left(I_{\Lambda\Sigma}\,\hat{F}^{\Sigma\,\underline{mt}} - R_{\Lambda\Sigma}\,{}^{\star}\hat{F}^{\Sigma\,\underline{mt}}\right) = 0 \tag{6.9}$$

where time derivatives can be ignored because of independence on time. The $\hat{F}^{\underline{mt}}$, expressed in terms of the 3-dimensional field strengths, is (we temporarily drop the multiplet index Λ)

$$\hat{F}^{\underline{mt}} = -\gamma^{\underline{mn}}\left[F_{\underline{n}t} - e^{4U}\omega^2\,F_{\underline{n}t} + e^{4U}\omega^{\underline{r}}F_{\underline{nr}} + e^{4U}\omega_{\underline{n}}\,\omega^{\underline{r}}F_{\underline{r}t}\right] \tag{6.10}$$

and, using (6.6), it becomes simply

$$\hat{F}^{\underline{mt}} = -\partial^{\underline{m}}\psi + e^{4U}\omega_{\underline{n}}\left(G^{\underline{nm}} + \psi\,W^{\underline{nm}}\right) = -\partial^{\underline{m}}\psi + e^{4U}\omega_{\underline{n}}H^{\underline{nm}}. \tag{6.11}$$

On top of this, the (6.9) in 3-dimensional form is[2]

$$\frac{e^{2U}}{\sqrt{|\gamma|}}\partial_{\underline{m}}\left\{e^{-2U}\sqrt{|\gamma|}\,I_{\Lambda\Sigma}\left[-\partial^{\underline{m}}\psi^{\Sigma} + e^{4U}\omega_{\underline{n}}H^{\Sigma\,\underline{nm}}\right]\right.$$
$$\left. + R_{\Lambda\Sigma}\frac{\epsilon^{\underline{mnr}}}{2}\left[H^{\Sigma}_{\underline{nr}} + 2\partial_{\underline{n}}\psi^{\Sigma}\omega_{\underline{r}}\right]\right\} = 0. \tag{6.12}$$

For what concerns the spatial components

$$\nabla_{\underline{m}}\left(I_{\Lambda\Sigma}\,\hat{F}^{\Sigma\,\underline{mn}} - R_{\Lambda\Sigma}\,{}^{\star}\hat{F}^{\Sigma\,\underline{mn}}\right) = 0, \tag{6.13}$$

we use the following

$$\hat{F}^{\underline{mn}} = e^{4U}\left(\omega^{\underline{m}}\,\partial^{\underline{n}}\psi - \omega^{\underline{n}}\,\partial^{\underline{m}}\psi\right) + e^{4U}\,F^{\underline{mn}}; \tag{6.14}$$

inserting (6.6) in (6.14), it turns out that it is simply $\hat{F}^{\underline{mn}} = e^{4U}H^{\underline{mn}}$; thus the spatial component of Maxwell equation is

$$\frac{e^{2U}}{\sqrt{|\gamma|}}\partial_{\underline{m}}\left(e^{2U}\sqrt{|\gamma|}\,I_{\Lambda\Sigma}\,H^{\Sigma\,\underline{mn}} + R_{\Lambda\Sigma}\,\epsilon^{\underline{mnr}}\,\partial_{\underline{r}}\psi^{\Sigma}\right) = 0. \tag{6.15}$$

Now, we would like to express these equations (and also the following ones) in terms of some scalars; two of these have already been determined, U, which could be called the metric scalar, and ψ^{Λ}, the electric potential (actually, there are n_V of this). The next one, let us name it the magnetic potential χ_{Λ}, can be introduced dualizing the quantity between parenthesis in (6.15):

$$\epsilon^{\underline{mnr}}\,\partial_{\underline{r}}\chi_{\Lambda} \equiv e^{2U}\sqrt{|\gamma|}\,I_{\Lambda\Sigma}\,H^{\Sigma\,\underline{mn}} + R_{\Lambda\Sigma}\,\epsilon^{\underline{mnr}}\,\partial_{\underline{r}}\psi^{\Sigma} \tag{6.16}$$

[2] Using $\hat{F}^{\Sigma\,\underline{mt}} = \frac{\epsilon^{\underline{mtnr}}}{2\sqrt{|g|}}\hat{F}^{\Sigma}_{\underline{nr}} = -\frac{\epsilon^{\underline{mnr}}}{2e^{-2U}\sqrt{|\gamma|}}F^{\Sigma}_{\underline{nr}}$.

which in turn gives

$$H^{\Lambda}_{\underline{mn}} = \frac{e^{-2U}}{\sqrt{|\gamma|}}\epsilon_{\underline{mnr}}\left[\left(I^{-1}\right)^{\Lambda\Sigma}\partial^{\underline{r}}\chi_{\Sigma} - \left(I^{-1}R\right)^{\Lambda}_{\ \Sigma}\partial^{\underline{r}}\psi^{\Sigma}\right]. \tag{6.17}$$

With the definition (6.16) the (6.15) is already satisfied. We need an alternative equation: for example the Bianchi identity for $H_{\underline{mn}}$, which, using (6.6) and (6.8), gives

$$\epsilon^{\underline{mnr}}\,\partial_{\underline{r}}H^{\Lambda}_{\underline{mn}} = \epsilon^{\underline{mnr}}\,\partial_{\underline{r}}\psi^{\Lambda}\,W_{\underline{mn}}. \tag{6.18}$$

Inserting in this last equation the expression (6.17), we get a first equation for the electric and magnetic potentials[3]:

$$\nabla_{\underline{m}}\left\{e^{-2U}\left[\left(I^{-1}R\right)^{\Lambda}_{\ \Sigma}\partial^{\underline{m}}\psi^{\Sigma} - \left(I^{-1}\right)^{\Lambda\Sigma}\partial^{\underline{m}}\chi_{\Sigma}\right]\right\} = -\frac{\epsilon^{\underline{mnr}}}{2\sqrt{|\gamma|}}W_{\underline{mn}}\,\partial_{\underline{r}}\psi^{\Lambda}. \tag{6.19}$$

Moreover, plugging (6.17) in the time component of Maxwell equation as written in (6.12), we have a second equation:

$$\nabla_{\underline{m}}\left\{e^{-2U}\left[\left(I + RI^{-1}R\right)_{\Lambda\Sigma}\partial^{\underline{m}}\psi^{\Sigma} - \left(RI^{-1}\right)_{\Lambda}^{\ \Sigma}\partial^{\underline{m}}\chi_{\Sigma}\right]\right\} = -\frac{\epsilon^{\underline{mnr}}}{2\sqrt{|\gamma|}}W_{\underline{mn}}\,\partial_{\underline{r}}\chi_{\Lambda}. \tag{6.20}$$

The all thing can be conveniently written in matricial form

$$\boxed{\nabla_{\underline{m}}\left\{e^{-2U}\begin{bmatrix}(I + RI^{-1}R)_{\Lambda\Sigma} & -(RI^{-1})_{\Lambda}^{\ \Sigma} \\ -(I^{-1}R)^{\Lambda}_{\ \Sigma} & (I^{-1})^{\Lambda\Sigma}\end{bmatrix}\partial^{\underline{m}}\begin{bmatrix}\psi^{\Sigma} \\ \chi_{\Sigma}\end{bmatrix}\right\} = \frac{W_{\underline{mn}}\,\epsilon^{\underline{mnr}}}{2\sqrt{|\gamma|}}\partial_{\underline{r}}\begin{bmatrix}-\chi_{\Lambda} \\ \psi^{\Lambda}\end{bmatrix}}$$

and this explains how the matrix $\mathcal{M}$, first introduced in (3.42), is built from the equations of motion.

Einstein Equations. They are given by:

$$\hat{G}_{\mu\nu} + \mathcal{G}_{ij}\left(\partial_{\mu}\phi^{i}\partial_{\nu}\phi^{j} - \frac{1}{2}g_{\mu\nu}\partial_{\rho}\phi^{i}\partial^{\rho}\phi^{j}\right) + 4I_{\Lambda\Sigma}\left(\hat{F}^{\Lambda\,\rho}_{\mu}\hat{F}^{\Sigma}_{\nu\rho} - \frac{1}{4}g_{\mu\nu}\hat{F}^{\Lambda}_{\rho\sigma}\hat{F}^{\Sigma\,\rho\sigma}\right) = 0. \tag{6.21}$$

[3]In three dimensions, with metric $\gamma_{\underline{mn}}$, one has $\epsilon^{\underline{mnl}}\epsilon_{\underline{mnr}} = 2\,\delta^{l}_{\underline{r}}\,\gamma$.

We will consider them in flat indices. Starting with the 00-component, we have:

- for the pure gravitational section:

$$\hat{G}_{00} = \hat{R}_{00} - \frac{1}{2}\eta_{00}\hat{R} = \frac{e^{2U}}{2}\left(R + 2(\partial U)^2 - 4\nabla^2 U - \frac{3}{4}e^{4U}W^2\right);$$

- for the scalar part

$$\partial_0\phi^i\partial_0\phi^j - \frac{1}{2}\eta_{00}\partial_\rho\phi^i\partial^\rho\phi^j = \frac{e^{2U}}{2}\partial_{\underline{m}}\phi^i\partial^{\underline{m}}\phi^j$$

since

$$\partial_0\phi = e^{\mu}_{\ 0}\,\partial_\mu\phi = e^{t}_{\ 0}\,\partial_t\phi + e^{\underline{m}}_{\ 0}\,\partial_{\underline{m}}\phi = 0 \tag{6.22}$$

and

$$\partial_\rho\phi^i\partial^\rho\phi^j = g^{\underline{mn}}\partial_{\underline{m}}\phi^i\partial_{\underline{n}}\phi^j = -e^{2U}\partial_{\underline{m}}\phi^i\partial^{\underline{m}}\phi^j;$$

- dealing with the vector part, we need to evaluate $\eta^{ab}\hat{F}^{\Lambda}_{a0}\hat{F}^{\Sigma}_{b0}$ and $\hat{F}^{\Lambda}_{\mu\nu}\hat{F}^{\Sigma\,\mu\nu}$. For the first one, we have

$$\hat{F}_{0a} = e^{\mu}_{\ 0}\hat{F}_{\mu a} = e^{t}_{\ 0}\hat{F}_{ta} + e^{\underline{m}}_{\ 0}\hat{F}_{\underline{m}a} = e^{-U}e^{\underline{n}}_{\ a}\hat{F}_{t\underline{n}}$$

and then

$$\eta^{ab}\hat{F}^{\Lambda}_{a0}\hat{F}^{\Sigma}_{b0} = e^{-2U}\eta^{ab}\,e^{\underline{m}}_{\ a}\,e^{\underline{n}}_{\ b}\,\hat{F}^{\Lambda}_{\underline{m}t}\,\hat{F}^{\Sigma}_{\underline{n}t} = e^{-2U}g^{\underline{mn}}\,\hat{F}^{\Lambda}_{\underline{m}t}\,\hat{F}^{\Sigma}_{\underline{n}t} = -\partial_{\underline{m}}\psi^{\Lambda}\,\partial^{\underline{m}}\psi^{\Sigma}.$$

Calculating the scalar $\hat{F}^{\Lambda}_{\mu\nu}\hat{F}^{\Sigma\,\mu\nu}$, we have firstly

$$\hat{F}^{\Lambda}_{\mu\nu}\hat{F}^{\Sigma\,\mu\nu} = 2\hat{F}^{\Lambda}_{\underline{m}t}\hat{F}^{\Sigma\,\underline{m}t} + \hat{F}^{\Lambda}_{\underline{mn}}\hat{F}^{\Sigma\,\underline{mn}};$$

using (6.5), (6.11) and (6.14), we arrive to

$$2\hat{F}^{\Lambda}_{\underline{m}t}\hat{F}^{\Sigma\,\underline{m}t} = -2\partial_{\underline{m}}\psi^{\Lambda}\,\partial^{\underline{m}}\psi^{\Sigma} + 2e^{4U}\partial_{\underline{m}}\psi^{\Lambda}\,\omega_{\underline{n}}H^{\Sigma\,\underline{nm}}$$

$$\hat{F}^{\Lambda}_{\underline{mn}}\hat{F}^{\Sigma\,\underline{mn}} = e^{4U}H^{\Lambda}_{\underline{mn}}H^{\Sigma\,\underline{mn}} - 2e^{4U}\partial_{\underline{m}}\psi^{\Lambda}\,\omega_{\underline{n}}H^{\Sigma\,\underline{nm}}$$

and then

$$\hat{F}^{\Lambda}_{\mu\nu}\hat{F}^{\Sigma\,\mu\nu} = -2\partial_{\underline{m}}\psi^{\Lambda}\,\partial^{\underline{m}}\psi^{\Sigma} + e^{4U}H^{\Lambda}_{\underline{mn}}H^{\Sigma\,\underline{mn}}. \tag{6.23}$$

Putting all together, we have an intermediate form for the 00-component:

$$\frac{e^{2U}}{2}\left(R+2(\partial U)^2-4\nabla^2 U-\frac{3}{4}e^{4U}W^2\right)+\frac{e^{2U}}{2}\mathcal{G}_{ij}\partial_{\underline{m}}\phi^i\partial^{\underline{m}}\phi^j \\ -I_{\Lambda\Sigma}\left(2\partial_{\underline{m}}\psi^\Lambda\,\partial^{\underline{m}}\psi^\Sigma+e^{4U}H^\Lambda_{\underline{mn}}H^{\Sigma\,\underline{mn}}\right)=0. \quad (6.24)$$

Now we use the expression (6.17) for $H^\Lambda_{\underline{mn}}$ in the last term of (6.24): this gives

$$I_{\Lambda\Sigma}\left(2\partial_{\underline{m}}\psi^\Lambda\,\partial^{\underline{m}}\psi^\Sigma+e^{4U}H^\Lambda_{\underline{mn}}H^{\Sigma\,\underline{mn}}\right)=$$

$$2\partial_{\underline{m}}\psi^\Lambda\,\partial^{\underline{m}}\psi^\Sigma\big(I+RI^{-1}R\big)_{\Lambda\Sigma}-2\partial_{\underline{m}}\psi^\Lambda\partial^{\underline{m}}\chi_\Sigma\big(RI^{-1}\big)_\Lambda^{\ \Sigma}$$

$$-2\partial_{\underline{m}}\chi_\Lambda\partial^{\underline{m}}\psi^\Sigma\big(I^{-1}R\big)^\Lambda_{\ \Sigma}+2\partial_{\underline{m}}\chi_\Lambda\partial^{\underline{m}}\chi_\Sigma\big(I^{-1}\big)^{\Lambda\Sigma}=$$

$$\partial_{\underline{m}}\begin{bmatrix}\psi^\Lambda & \chi_\Lambda\end{bmatrix}\begin{bmatrix}(I+RI^{-1}R)_{\Lambda\Sigma} & -(RI^{-1})_\Lambda^{\ \Sigma}\\ -(I^{-1}R)^\Lambda_{\ \Sigma} & (I^{-1})^{\Lambda\Sigma}\end{bmatrix}\partial^{\underline{m}}\begin{bmatrix}\psi^\Sigma\\ \chi_\Sigma\end{bmatrix}\equiv\mathcal{M}_{MN}\partial_{\underline{m}}\Psi^M\partial^{\underline{m}}\Psi^N.$$

Thus, in the end, the 00-component of the Einstein equation is

$$\boxed{R+2(\partial U)^2-4\nabla^2U+\mathcal{G}_{ij}\,\partial_{\underline{m}}\phi^i\partial^{\underline{m}}\phi^j= \frac{3}{4}e^{4U}W^2+4e^{-2U}\mathcal{M}_{MN}\partial_{\underline{m}}\Psi^M\partial^{\underline{m}}\Psi^N}$$

Next, we have the 0m-component:

- first of all, the Einstein tensor reduces to

$$\hat{G}_{0m}=\hat{R}_{0m}=\frac{1}{2}\nabla_n\left(e^{4U}W_{nm}\right);$$

- the scalar part does not contribute (see (6.22));
- the vector part contributes with the term (apart from a factor of $4I_{\Lambda\Sigma}$)

$$\eta^{ab}\hat{F}^\Lambda_{a0}\hat{F}^\Sigma_{bm}=e^{2U}\left(\gamma^{\underline{mn}}\hat{F}^\Lambda_{\underline{mt}}\hat{F}^\Sigma_{\underline{nt}}\,\omega_m-\omega^r v_r^{\underline{m}}\,v_m^{\underline{n}}\,\hat{F}^\Lambda_{\underline{mt}}\hat{F}^\Sigma_{\underline{nt}}+\gamma^{\underline{mr}}\,v_m^{\underline{n}}\,\hat{F}^\Lambda_{\underline{mt}}\hat{F}^\Sigma_{\underline{nr}}\right);$$

  expressing it in three dimensions (through (6.5)) we find

$$\eta^{ab}\hat{F}^\Lambda_{a0}\hat{F}^\Sigma_{bm}=e^{2U}\partial_n\psi^\Lambda\,H^\Sigma_{mn} \quad (6.25)$$

and the equation assumes the form:

$$\nabla_n \left(e^{4U} W_{nm}\right) - 8 I_{\Lambda\Sigma}\, e^{2U} \partial_n \psi^\Lambda \, H^\Sigma_{mn} = 0. \tag{6.26}$$

Now, in order to have it expressed in term of the electric and magnetic potentials, we use (6.17): it follows

$$I_{\Lambda\Sigma}\, e^{2U} \partial_n \psi^\Lambda \, H^\Sigma_{mn} = \epsilon_{mnr} \left(\partial_n \psi^\Lambda \partial_r \chi_\Lambda - R_{\Lambda\Sigma}\, \partial_n \psi^\Lambda \partial_r \psi_\Sigma\right)$$

but the term with $R_{\Lambda\Sigma}$ is zero due to the symmetry of the period matrix combined with the antisymmetry of the Levi-Civitá tensor. The 0m-component is then

$$\boxed{\nabla_n \left(e^{4U} W_{nm}\right) - 8\epsilon_{mnr} \partial_n \psi^\Lambda \partial_r \chi_\Lambda = 0}$$

Finally, we have the mn-component

$$\hat{G}_{mn} + \mathcal{G}_{ij} \left(\partial_m \phi^i \partial_n \phi^j + \frac{1}{2} \delta_{mn} \partial_\rho \phi^i \partial^\rho \phi^j\right) + 4 I_{\Lambda\Sigma} \left(\hat{F}^{\Lambda\, a}_m \hat{F}^\Sigma_{na} + \frac{1}{4} \delta_{mn} \hat{F}^\Lambda_{\rho\sigma} \hat{F}^{\Sigma\, \rho\sigma}\right) = 0. \tag{6.27}$$

- The mn components of the Einstein tensor are

$$\begin{aligned}\hat{G}_{mn} = \hat{R}_{mn} + \frac{\delta_{mn}}{2} \hat{R} &= e^{2U} \left(R_{mn} + 2\, \partial_m U \partial_n U - \delta_{mn} \nabla^2 U - \frac{1}{2}\, e^{4U} W_{mp} W_{np}\right) \\ &\quad - \frac{e^{2U}}{2} \left(R - \frac{1}{4}\, e^{4U} W^2 - 2\nabla^2 U + 2(\partial U)^2\right) \\ = e^{2U} \left[G_{mn} + 2\left(\partial_m U \partial_n U - \frac{\delta_{mn}}{2} (\partial U)^2\right)\right] &- \frac{e^{6U}}{2} \left(W_{mp} W_{np} - \frac{\delta_{mn}}{4} W^2\right)\end{aligned} \tag{6.28}$$

where in the last line the 3-dimensional Einstein tensor is $G_{mn} = R_{mn} - \frac{1}{2} \delta_{mn} R$.

- The scalar part is

$$\mathcal{G}_{ij} \left(\partial_m \phi^i \partial_n \phi^j + \frac{1}{2} \delta_{mn} \partial_\rho \phi^i \partial^\rho \phi^j\right) = e^{2U} \mathcal{G}_{ij} \left(\partial_m \phi^i \partial_n \phi^j - \frac{\delta_{mn}}{2}\, \partial_{\underline{r}} \phi^i \partial^{\underline{r}} \phi^j\right). \tag{6.29}$$

- The vector sector, apart from the term with the scalar $\hat{F}^{\Lambda}_{\mu\nu}\hat{F}^{\Sigma\,\mu\nu}$ which has already been calculated, presents the quantity $\eta^{ab}\hat{F}^{\Lambda}_{am}\hat{F}^{\Sigma}_{bm}$; using

$$\hat{F}^{\Lambda}_{ma} = e^{U}\omega_m\, e_a^{\underline{m}}\, \hat{F}^{\Lambda}_{\underline{m}t} + e^{U} v_m^{\underline{n}}\, e_a^{t}\, \hat{F}^{\Lambda}_{\underline{n}t} + e^{U} v_m^{\underline{m}}\, e_a^{\underline{n}}\, \hat{F}^{\Lambda}_{\underline{mn}}$$

an easy but tedious calculation yields

$$\eta^{ab}\hat{F}^{\Lambda}_{am}\hat{F}^{\Sigma}_{bn} = \partial_m\psi^{\Lambda}\partial_n\psi^{\Sigma} - e^{4U} H^{\Lambda}_{mr}H^{\Sigma}_{nr}$$

and then the vectorial part in 3-dimensional fields is

$$4I_{\Lambda\Sigma}\Big[\left(\partial_m\psi^{\Lambda}\partial_n\psi^{\Sigma} - e^{4U}H^{\Lambda}_{mr}H^{\Sigma}_{nr}\right) - \frac{\delta_{mn}}{4}\left(2\,\partial_{\underline{m}}\psi^{\Lambda}\partial^{\underline{m}}\psi^{\Sigma} - e^{4U}\,H^{\Lambda}_{\underline{mn}}H^{\Sigma\,\underline{mn}}\right)\Big]. \quad (6.30)$$

Now we need to rewrite $H^{\Lambda}_{mr}H^{\Sigma}_{nr}$ using (6.17); this brings to

$$\begin{aligned} H^{\Lambda}_{mr}H^{\Sigma}_{nr} &= e^{-4U}\epsilon_{mrp}\epsilon_{nrq}\left[\left(I^{-1}\right)^{\Lambda\Gamma}\partial_p\chi_{\Gamma} - \left(I^{-1}R\right)^{\Lambda}_{\ \Gamma}\partial_p\psi^{\Gamma}\right] \\ &\quad\times\left[\left(I^{-1}\right)^{\Sigma\Pi}\partial_q\chi_{\Pi} - \left(I^{-1}R\right)^{\Sigma}_{\ \Pi}\partial_q\psi^{\Pi}\right] \\ &\equiv e^{-4U}\left(\delta_{mn}\delta_{pq} - \delta_{mq}\delta_{pn}\right)\Big[\ldots\Big]^{\Lambda}_{p}\Big[\ldots\Big]^{\Sigma}_{q} \end{aligned}$$

and with the same formalism we have also

$$H^{\Lambda}_{\underline{mn}}H^{\Sigma\,\underline{mn}} = 2\,e^{-4U}\Big[\ldots\Big]^{\Lambda}_{r}\Big[\ldots\Big]^{\Sigma}_{r}$$

so that (6.30) becomes

$$4I_{\Lambda\Sigma}\Bigg[\bigg(\partial_m\psi^{\Lambda}\partial_n\psi^{\Sigma} + \Big[\ldots\Big]^{\Lambda}_{m}\Big[\ldots\Big]^{\Sigma}_{n} - \delta_{mn}\Big[\ldots\Big]^{\Lambda}_{r}\Big[\ldots\Big]^{\Sigma}_{r}\bigg) - \frac{\delta_{mn}}{2}\bigg(\partial_{\underline{m}}\psi^{\Lambda}\partial^{\underline{m}}\psi^{\Sigma} - \Big[\ldots\Big]^{\Lambda}_{r}\Big[\ldots\Big]^{\Sigma}_{r}\bigg)\Bigg]. \quad (6.31)$$

Then, a calculation nearly identical to the one in the 00-component gives for the vector part

$$4\left(\mathcal{M}_{MN}\partial_m\Psi^M\partial_n\Psi^N - \frac{\delta_{mn}}{2}\mathcal{M}_{MN}\partial_{\underline{m}}\Psi^M\partial^{\underline{m}}\Psi^N\right)$$

and we have the final form for the mn-component

$$\boxed{\begin{aligned} G_{mn} + 2\left(\partial_m U \partial_n U - \frac{\delta_{mn}}{2}(\partial U)^2\right) - \frac{e^{4U}}{2}\left(W_{mp}W_{np} - \frac{\delta_{mn}}{4}W^2\right) \\ + \mathcal{G}_{ij}\left(\partial_m \phi^i \partial_n \phi^j - \frac{\delta_{mn}}{2}\,\partial_{\underline{r}}\phi^i \partial^{\underline{r}}\phi^j\right) \\ + 4e^{-2U}\mathcal{M}_{MN}\left(\partial_m \Psi^M \partial_n \Psi^N - \frac{\delta_{mn}}{2}\partial_{\underline{r}}\Psi^M \partial^{\underline{r}}\Psi^N\right) = 0 \end{aligned}}$$

Scalar equation. We are left with the equation

$$\nabla_\mu\left(\mathcal{G}_{ij}\partial^\mu\phi^j\right) - \frac{1}{2}\partial_i\mathcal{G}_{kl}\,\partial_\mu\phi^k\partial^\mu\phi^l - \left(\partial_i I_{\Lambda\Sigma}\,\hat{F}^\Lambda_{\mu\nu}\hat{F}^{\Sigma\,\mu\nu} - \partial_i R_{\Lambda\Sigma}\,\hat{F}^\Lambda_{\mu\nu}{}^\star\hat{F}^{\Sigma\,\mu\nu}\right) = 0.$$

Switching to 3-dimensional indices, we have first of all

$$\nabla_\mu\left(\mathcal{G}_{ij}\partial^\mu\phi^j\right) = -e^{2U}\nabla_{\underline{m}}\left(\mathcal{G}_{ij}\partial^{\underline{m}}\phi^j\right).$$

Next, we must take care of the term with the Hodge dual of the field strength:

$$\partial_i R_{\Lambda\Sigma}\,\hat{F}^\Lambda_{\mu\nu}{}^\star\hat{F}^{\Sigma\,\mu\nu} = \partial_i R_{\Lambda\Sigma}\left(2\hat{F}^\Lambda_{\underline{m}t}{}^\star\hat{F}^{\Sigma\,\underline{m}t} + \hat{F}^\Lambda_{\underline{mn}}{}^\star\hat{F}^{\Sigma\,\underline{mn}}\right) = -\frac{2}{\sqrt{|g|}}\epsilon^{\underline{mnr}}\hat{F}^\Lambda_{\underline{m}t}\hat{F}^\Sigma_{\underline{nr}}$$

thanks to the symmetry of $R_{\Lambda\Sigma}$. Inserting (6.6) and using again (6.17) we get

$$\begin{aligned} \partial_i R_{\Lambda\Sigma}\,\hat{F}^\Lambda_{\mu\nu}{}^\star\hat{F}^{\Sigma\,\mu\nu} = 4\,\partial_i R_{\Lambda\Sigma}\,\partial_{\underline{m}}\psi^\Lambda\left[\left(I^{-1}R\right)^\Sigma{}_\Pi\,\partial^{\underline{m}}\psi^\Pi - \left(I^{-1}\right)^{\Sigma\Pi}\partial^{\underline{m}}\chi_\Pi\right] \\ - 4\,\partial_i R_{\Lambda\Sigma}\,\frac{\epsilon^{\underline{mnr}}}{\sqrt{|g|}}\,\partial_{\underline{m}}\psi^\Lambda\,\partial_{\underline{n}}\psi^\Sigma\omega_{\underline{r}}. \quad (6.32) \end{aligned}$$

The last term is zero by the combination of symmetry in the period matrix indices and antisymmetry in the spacetime ones. Now, using (6.32), (6.23) and, as usual, (6.17) we find

$$\partial_i R_{\Lambda\Sigma}\,\hat{F}^\Lambda_{\mu\nu}{}^\star\hat{F}^{\Sigma\,\mu\nu} - \partial_i I_{\Lambda\Sigma}\,\hat{F}^\Lambda_{\mu\nu}\hat{F}^{\Sigma\,\mu\nu} =$$

$$2\,\partial_{\underline{m}}\psi^\Lambda\partial^{\underline{m}}\psi^\Sigma\left[\partial_i I_{\Lambda\Sigma} - \left(RI^{-1}\partial_i I\,I^{-1}R\right)_{\Lambda\Sigma} + \left(\partial_i R\,I^{-1}R\right)_{\Lambda\Sigma} + \left(RI^{-1}\partial_i R\right)_{\Lambda\Sigma}\right]$$

$$+2\,\partial_{\underline{m}}\psi^\Lambda\partial^{\underline{m}}\chi_\Sigma\left[2\left(RI^{-1}\partial_i I\,I^{-1}\right)_\Lambda{}^\Sigma - 2\left(\partial_i R\,I^{-1}\right)_\Lambda{}^\Sigma\right]$$

$$-2\,\partial_{\underline{m}}\chi_\Lambda\partial^{\underline{m}}\chi_\Sigma\left(I^{-1}\partial_i I\,I^{-1}\right)^{\Lambda\Sigma}$$

and this is nothing but the derivative of the matrix $\mathcal{M}$ with respect to the scalars, multiplied by the spatial derivatives of the electric and magnetic potentials:

$$\partial_i R_{\Lambda\Sigma}\,\hat{F}^{\Lambda}_{\mu\nu}\,{}^{\star}\hat{F}^{\Sigma\,\mu\nu} - \partial_i I_{\Lambda\Sigma}\,\hat{F}^{\Lambda}_{\mu\nu}\hat{F}^{\Sigma\,\mu\nu} = 2\,\partial_{\underline{m}}\Psi^M\,\partial_i\mathcal{M}_{MN}\,\partial^{\underline{m}}\Psi^N. \tag{6.33}$$

The equation of motion for scalars is then

$$\boxed{\nabla_{\underline{m}}\left(\mathcal{G}_{ij}\partial^{\underline{m}}\phi^j\right) - \frac{1}{2}\partial_i\mathcal{G}_{kl}\,\partial_{\underline{m}}\phi^k\partial^{\underline{m}}\phi^l - 2\,e^{-2U}\partial_i\mathcal{M}_{MN}\,\partial_{\underline{m}}\Psi^M\partial^{\underline{m}}\Psi^N = 0}$$

6.2 Taub-NUT and black hole potentials

6.2.1 The Taub-NUT potential

Let us recall all the equations that we have obtained so far:

$$\nabla_{\underline{m}}\left\{e^{-2U}\begin{bmatrix}(I+RI^{-1}R)_{\Lambda\Sigma} & -(RI^{-1})_{\Lambda}{}^{\Sigma}\\ -(I^{-1}R)^{\Lambda}{}_{\Sigma} & (I^{-1})^{\Lambda\Sigma}\end{bmatrix}\partial^{\underline{m}}\begin{bmatrix}\psi^{\Sigma}\\ \chi_{\Sigma}\end{bmatrix}\right\} = \frac{\epsilon^{\underline{mnr}}}{2\sqrt{|\gamma|}}W_{\underline{mn}}\,\partial_{\underline{r}}\begin{bmatrix}-\chi_{\Lambda}\\ \psi^{\Lambda}\end{bmatrix} \tag{6.34}$$

$$R + 2(\partial U)^2 - 4\nabla^2 U + \mathcal{G}_{ij}\,\partial_{\underline{m}}\phi^i\partial^{\underline{m}}\phi^j = \frac{3}{4}e^{4U}W^2 + 4e^{-2U}\mathcal{M}_{MN}\partial_{\underline{m}}\Psi^M\partial^{\underline{m}}\Psi^N \tag{6.35}$$

$$\nabla_n\left(e^{4U}W_{nm}\right) - 8\epsilon_{mnr}\partial_n\psi^{\Lambda}\partial_r\chi_{\Lambda} = 0 \tag{6.36}$$

$$\begin{aligned}G_{mn} + 2\left(\partial_m U\partial_n U - \frac{\delta_{mn}}{2}(\partial U)^2\right) - \frac{e^{4U}}{2}\left(W_{mp}W_{np} - \frac{\delta_{mn}}{4}W^2\right)\\ + \mathcal{G}_{ij}\left(\partial_m\phi^i\partial_n\phi^j - \frac{\delta_{mn}}{2}\,\partial_{\underline{r}}\phi^i\partial^{\underline{r}}\phi^j\right)\\ + 4e^{-2U}\mathcal{M}_{MN}\left(\partial_m\Psi^M\partial_n\Psi^N - \frac{\delta_{mn}}{2}\partial_{\underline{r}}\Psi^M\partial^{\underline{r}}\Psi^N\right) = 0\end{aligned} \tag{6.37}$$

$$\nabla_{\underline{m}}\left(\mathcal{G}_{ij}\partial^{\underline{m}}\phi^j\right)-\frac{1}{2}\partial_i\mathcal{G}_{kl}\,\partial_{\underline{m}}\phi^k\partial^{\underline{m}}\phi^l-2\,e^{-2U}\partial_i\mathcal{M}_{MN}\,\partial_{\underline{m}}\Psi^M\partial^{\underline{m}}\Psi^N=0 \quad (6.38)$$

It is evident that if the Taub-NUT field strength $W_{\underline{mn}}$ vanishes (which is equivalent to set to zero the NUT charge N) we obtain the equations (4.5), (4.6), (4.7), (4.8) and (4.9). The fields which appear are all scalars, except for the Taub-NUT field strength $W_{\underline{mn}}$. We will replace it with a scalar field α, which could be named Taub-NUT potential. Concentrating on the (6.36), we set to zero the second term[4], so that, switching to curved indices, the equation

$$\nabla_{\underline{m}}\left(e^{4U}W^{\underline{mn}}\right)=0 \quad (6.39)$$

can be solved with the introduction of the potential α, given by:

$$\epsilon^{\underline{mnr}}\,\partial_{\underline{r}}\alpha=\sqrt{|\gamma|}\,e^{4U}W^{\underline{mn}} \quad (6.40)$$

and clearly

$$W^{\underline{mn}}=\frac{e^{-4U}}{\sqrt{|\gamma|}}\epsilon^{\underline{mnr}}\,\partial_{\underline{r}}\alpha. \quad (6.41)$$

The equation of motion for α can be obtained from the Bianchi identity for the tensor W; it results

$$\nabla_{\underline{m}}\left(e^{-4U}\partial^{\underline{m}}\alpha\right)=0. \quad (6.42)$$

Through repeated use of (6.41) and subtracting from (6.35) the trace of (6.37), the equations (6.34)-(6.38) reach the form

$$\nabla_{\underline{m}}\left\{e^{-2U}\begin{bmatrix}(I+RI^{-1}R)_{\Lambda\Sigma} & -(RI^{-1})_{\Lambda}^{\ \Sigma}\\ -(I^{-1}R)^{\Lambda}_{\ \Sigma} & (I^{-1})^{\Lambda\Sigma}\end{bmatrix}\partial^{\underline{m}}\begin{bmatrix}\psi^{\Sigma}\\ \chi_{\Sigma}\end{bmatrix}\right\}=e^{-4U}\,\partial_{\underline{m}}\alpha\,\partial^{\underline{m}}\begin{bmatrix}-\chi_{\Lambda}\\ \psi^{\Lambda}\end{bmatrix} \quad (6.43)$$

$$2\nabla^2U+e^{-4U}\left(\partial\alpha\right)^2+4e^{-2U}\mathcal{M}_{MN}\partial_{\underline{m}}\Psi^M\partial^{\underline{m}}\Psi^N=0 \quad (6.44)$$

$$\nabla_{\underline{m}}\left(e^{-4U}\partial^{\underline{m}}\alpha\right)=0 \quad (6.45)$$

$$R_{mn}+2\partial_mU\partial_nU+\frac{e^{-4U}}{2}\partial_m\alpha\,\partial_n\alpha + \mathcal{G}_{ij}\partial_m\phi^i\partial_n\phi^j+4e^{-2U}\mathcal{M}_{MN}\,\partial_m\Psi^M\partial_n\Psi^N=0 \quad (6.46)$$

[4]This follows from comparison with the static case, but it is also automatic if (as we will do) one assumes spherical symmetry of the configuration.

$$\nabla_{\underline{m}}\left(\mathcal{G}_{ij}\partial^{\underline{m}}\phi^j\right)-\frac{1}{2}\partial_i\mathcal{G}_{kl}\,\partial_{\underline{m}}\phi^k\partial^{\underline{m}}\phi^l-2\,e^{-2U}\partial_i\mathcal{M}_{MN}\,\partial_{\underline{m}}\Psi^M\partial^{\underline{m}}\Psi^N=0. \quad (6.47)$$

Now, we can first integrate the (6.45): assuming spherical symmetry of α and U, with explicit use of the three-dimensional metric γ the (6.45) becomes

$$\frac{d}{d\tau}\left(e^{-4U}\frac{d\alpha}{d\tau}\right)=0$$

and then

$$e^{-4U}\frac{d\alpha}{d\tau}=C\longrightarrow\alpha=C\int e^{4U}d\tau+K.$$

We take $C=-2N$.[5] The second constant K does not appear in the other equations since α is always differentiated with respect to τ. Substituting $\frac{d\alpha}{d\tau}=-2Ne^{4U}$ everywhere, assuming spherical symmetry for all the remaining fields, the system becomes (prime means differentiation with respect to τ):

$$\begin{aligned}\frac{d}{d\tau}\left\{e^{-2U}\begin{bmatrix}(I+RI^{-1}R)_{\Lambda\Sigma} & -(RI^{-1})_{\Lambda}{}^{\Sigma}\\ -(I^{-1}R)^{\Lambda}{}_{\Sigma} & (I^{-1})^{\Lambda\Sigma}\end{bmatrix}\frac{d}{d\tau}\begin{bmatrix}\psi^{\Sigma}\\ \chi_{\Sigma}\end{bmatrix}\right\}&=-2N\,\frac{d}{d\tau}\begin{bmatrix}-\chi_{\Lambda}\\ \psi^{\Lambda}\end{bmatrix}\\ &=2N\begin{bmatrix}0 & \mathbb{I}_{n_V}\\ -\mathbb{I}_{n_V} & 0\end{bmatrix}\frac{d}{d\tau}\begin{bmatrix}\psi^{\Lambda}\\ \chi_{\Lambda}\end{bmatrix}\end{aligned}\quad (6.48)$$

$$U''+2e^{-2U}\mathcal{M}_{MN}\frac{d\Psi^M}{d\tau}\frac{d\Psi^N}{d\tau}+2\,e^{4U}N^2=0 \quad (6.49)$$

$$\alpha=-2N\int e^{4U}d\tau+G \quad (6.50)$$

$$(U')^2+\frac{1}{2}\mathcal{G}_{ij}\,\frac{d\phi^i}{d\tau}\frac{d\phi^j}{d\tau}+2e^{-2U}\mathcal{M}_{MN}\frac{d\Psi^M}{d\tau}\frac{d\Psi^N}{d\tau}+e^{4U}N^2=r_0^2 \quad (6.51)$$

$$\frac{d}{d\tau}\left(\mathcal{G}_{ij}\frac{d\phi^j}{d\tau}\right)-\frac{1}{2}\partial_i\mathcal{G}_{kl}\frac{d\phi^k}{d\tau}\frac{d\phi^l}{d\tau}-2\,\partial_i\left(e^{-2U}\mathcal{M}_{MN}\right)\frac{d\Psi^M}{d\tau}\frac{d\Psi^N}{d\tau}=0. \quad (6.52)$$

[5]The motivation for this is the following: in reality we already know the form of the field strength $W_{\underline{mn}}$ in the Taub-NUT case, see (5.24) in the previous chapter. It is then straightforward to check that, from (6.40), turns out that $\frac{d\alpha}{d\tau}=-2e^{4U}N$; this also shows that the assumption $\alpha=\alpha(\tau)$ is correct, since the θ- and φ-derivatives of α vanish. In the end, the introduction of the potential α, in this case, is somehow pleonastic; however it shows how to proceed in the situation where the off-diagonal component(s) of the metric depends on some undetermined fields.

6.2.2 Determination of the black hole potential

It is now crucial to find a way to eliminate the vector fields from the above equations and possibly to define a reasonable generalization of the black hole potential found in Chapter 4. Given the way the vectors enter in (6.48) this seems quite difficult: however, there are at least two ways to proceed, although the result won't be as satisfactory as before. The first one, a direct generalization of the procedure followed in the static case, is the following: noticing that the (6.48) can be written in a more compact way in the notations of Chapter 4,

$$\frac{d}{d\tau}\left\{e^{-2U}\mathcal{M}_{MN}\frac{d}{d\tau}\Psi^N\right\} = 2N\,\Omega_{MN}\frac{d}{d\tau}\Psi^N \tag{6.53}$$

we can always define the constants of motion[6]

$$2\,Q_M \equiv 2\,\Omega_{MN}Q^N \equiv \kappa\left(e^{-2U}\mathcal{M}_{MN}\frac{d}{d\tau}\Psi^N - 2N\,\Omega_{MN}\Psi^N\right) \tag{6.54}$$

where we introduced the symplectic vector

$$Q^M = \begin{bmatrix} p^\Lambda \\ q_\Lambda \end{bmatrix} \tag{6.55}$$

constituted by the conserved electric/magnetic charges; what we have done is to observe that the constant quantities in the (last) right side of (6.54) should be proportional, through κ, to the electric and magnetic charges, for the same reasons explained in Chapter 4. Here, however, the situation is more complicated since we are forced to keep explicitly the vectors Φ^M (in the alternative approach we will eliminate them but other problems will emerge). The derivatives of the electromagnetic potentials are then

$$\frac{d}{d\tau}\Psi^M = 2\,e^{2U}\left(\mathcal{M}^{-1}\right)^{MN}\Omega_{NP}\left(N\Psi^P + \frac{Q^P}{\kappa}\right); \tag{6.56}$$

expressions will look more readable if we consider the normalized charge vector $\hat{Q}^M \equiv Q^M/\kappa$. The (6.56) permits to rewrite the matricial products of equations (6.49), (6.51) and (6.52) as (at the same time we define the black hole potential)

$$2e^{-2U}\mathcal{M}_{MN}\frac{d\Psi^M}{d\tau}\frac{d\Psi^N}{d\tau} = 8\,e^{2U}\left(N\Psi+\hat{Q}\right)_M\left(\mathcal{M}^{-1}\right)^{MN}\left(N\Psi+\hat{Q}\right)_N$$
$$\equiv e^{2U}V_{BH}(\phi,\Psi,\hat{Q},N) \tag{6.57}$$

[6]The factor 2 is for pure convenience, while κ is a proportionality constant.

$$-2\,\partial_i\left(e^{-2U}\mathcal{M}_{MN}\right)\frac{d\Psi^M}{d\tau}\frac{d\Psi^N}{d\tau} = 8\,e^{2U}\left(N\Psi+\hat{Q}\right)_M\partial_i\left(\mathcal{M}^{-1}\right)^{MN}\left(N\Psi+\hat{Q}\right)_N$$
$$\equiv e^{2U}\partial_i V_{BH}(\phi,\Psi,\hat{Q},N) \quad (6.58)$$

so that, finally, the evolution equation for the metric function U, (6.49), the constraint (6.51) (analogous to the constraint (4.35)) and the dynamical equation for the scalars are

$$U'' + e^{2U}V_{BH} + 2\,e^{4U}N^2 = 0 \qquad (6.59)$$

$$(U')^2 + \frac{1}{2}\mathcal{G}_{ij}\frac{d\phi^i}{d\tau}\frac{d\phi^j}{d\tau} + e^{2U}V_{BH} + e^{4U}N^2 = r_0^2 \qquad (6.60)$$

$$\frac{d}{d\tau}\left(\mathcal{G}_{ij}\frac{d\phi^j}{d\tau}\right) - \frac{1}{2}\partial_i\mathcal{G}_{kl}\frac{d\phi^k}{d\tau}\frac{d\phi^l}{d\tau} + e^{2U}\partial_i V_{BH} = 0. \qquad (6.61)$$

They have to be completed with the equation (6.56) for the potentials. Again, the four-dimensional equations of motion have been reduced to a one-dimensional set of radial equations, which, apart from the presence of the terms proportional to N^2, are identical to the equations obtained in the static case, (4.35), (4.36) and (4.37). Then it is reasonable to expect that we can repeat in the current case the analysis performed in Chapter 4.2, at least as long as the explicit form of the black hole potential is not involved (and with the possible differences arising from the new terms in N^2). But before moving further, we shortly describe the alternative method that can be employed to define the black hole potential. If we rescale the potentials

$$\Phi^M \equiv \left(\Psi + \frac{\hat{Q}}{N}\right)^M$$

equation (6.56) can be rewritten

$$\frac{d}{d\tau}\Phi^M = 2N\,e^{2U}\left(\mathcal{M}^{-1}\right)^{MN}\Omega_{NP}\Phi^P \equiv \mathcal{A}^M_N\,\Phi^N \qquad (6.62)$$

where the matrix $\mathcal{A}$ depends on τ. The equation (6.62) is identical, for example, to the differential equation which describes the evolution in time of a non-conservative system in quantum mechanics. It is well known that it can be formally integrated by an iterative procedure; once one has defined what, in this case, could be called a τ-ordered product

$$\mathcal{T}\left[f(\tau_1)f(\tau_2)\right] := \begin{cases} f(\tau_1)f(\tau_2) & \text{if } \tau_1 > \tau_2, \\ f(\tau_2)f(\tau_1) & \text{if } \tau_2 > \tau_1 \end{cases}$$

it turns out that the exact solution for (6.62) is (dropping the matricial indices)

$$\Phi(\tau) = \mathcal{T}\left[\exp\left(\int_{\tau_0}^{\tau} \mathcal{A}(\sigma)\, d\sigma\right)\right]\, \Phi(\tau_0) \tag{6.63}$$

where we could take $\tau_0 = 0$. The rescaled potentials evaluated at spatial infinity, $\Phi(\tau_0)$, can be expressed in terms of the constant electric and magnetic charges and then one could get rid of the vector fields in equations (6.49), (6.51) and (6.52) using

$$\frac{d}{d\tau}\Phi^M = \mathcal{A}_N^M\, \mathcal{T}\left[\exp\left(\int_{\tau_0}^{\tau} \mathcal{A}(\sigma)\, d\sigma\right)\right]_P^N \Phi^P(\tau_0) \equiv \mathcal{A}_N^M\, \mathcal{T}_P^N\, \Phi^P(\tau_0).$$

The black hole potential could then be defined as (the radial derivatives of Φ and Ψ are identical since they differ by a constant shift)

$$V_{BH}(\phi, \hat{Q}, N) \equiv 8N^2\, \Phi^M(\tau_0)\big[\mathcal{T}\Omega\, \mathcal{M}^{-1}\Omega\mathcal{T}\big]_{MN}\Phi^N(\tau_0). \tag{6.64}$$

The equations of motion then take the same form obtained with the definition (6.57). This second definition has the particularity that the vector fields are no more involved, since only their constant asymptotic values appear, and this would be the best situation; on the other hand, the solution of (6.62) given through the path-ordered exponential is formally correct but difficult to deal with from a practical perspective. If we do not need the explicit form of the black potential, we could equally use both the definitions; we choose the first one, remembering that the scalar potentials of the vectors appear explicitly and their behaviour should be kept under control.

6.3 Extremal limit and attractors

Now we want to repeat the considerations about finite horizon area and regularity of scalars that in the static case lead to a probable attractor mechanism. Considering extremal configurations, with $r_0 = 0$, the metric becomes

$$ds^2 = e^{2U}(dt+\omega)^2 - e^{-2U}\left(\frac{d\tau^2}{\tau^2} + \frac{1}{\tau^2}d\Omega^2_{(2)}\right) \tag{6.65}$$

and there is a unique horizon, since $r_+ = r_- \equiv r_h$. Its area is given by (referring to Schwarzschild's coordinates)

$$A = \int_{r=r_h} \sqrt{g_{\theta\theta}g_{\varphi\varphi}}\, d\theta d\varphi \tag{6.66}$$

so, in the parametrization we are using,

$$A = \int_{\tau=\tau_h} \left(\frac{e^{-4U} \sin^2\theta}{\tau^4} - \frac{4N^2 \cos^2\theta}{\tau^2} \right)^{\frac{1}{2}} d\theta d\varphi.$$

Since $\tau_h = -\infty$, the second term between parenthesis vanishes. The area will then be regular if we require (as in the static case)

$$\frac{e^{-2U}}{\tau^2} \longrightarrow \frac{A}{4\pi} \quad \text{for} \quad \tau \longrightarrow -\infty. \tag{6.67}$$

We also ask for the scalar fields to be regular at the horizon, imposing

$$\mathcal{G}_{ij} \frac{d\phi^i}{d\tau} \frac{d\phi^j}{d\tau} e^{2U} \tau^4 < \infty, \tag{6.68}$$

condition that, together with (6.67), ensures that

$$\mathcal{G}_{ij} \frac{d\phi^i}{d\tau} \frac{d\phi^j}{d\tau} \left(\frac{4\pi}{A} \right) \tau^2 \longrightarrow X^2 \quad \text{when } \tau \to -\infty \tag{6.69}$$

with X^2 finite. Now, it is immediate to see that the above requirements translate in the following behaviour of the constraint (6.60) in the vicinity of the horizon[7]: for $\tau \to -\infty$

$$\frac{4\pi}{A} + \frac{1}{2} X^2 + \left(\frac{4\pi}{A} \right)^2 V_{BH}(\phi_h, \Psi_h) + \left(\frac{4\pi}{A} \right)^3 \frac{N^2}{\tau^2} = 0 \tag{6.70}$$

and the last term goes to zero, so in the end (we do not indicate that dependence of V_{BH} on the NUT and electric/magnetic charges)

$$\frac{4\pi}{A} + \frac{1}{2} X^2 + \left(\frac{4\pi}{A} \right)^2 V_{BH}(\phi_h, \Psi_h) = 0 \tag{6.71}$$

holds, near the horizon, also for the case with NUT charge. This readily gives

$$A \leq -4\pi V_{BH}(\phi_h, \Psi_h)\,; \tag{6.72}$$

it is now clear that, since (6.71) is the same relation found in the static case (equation (4.44)) and also the condition (6.68) does not change, one can simply repeat the deduction done in Chapter 4, finding that ($\omega = -\log(-\tau)$)

$$\frac{d\phi^i}{d\omega} = 0 \quad \text{when } \omega \to -\infty \text{ (horizon)}.$$

[7] It should also be assumed that the electromagnetic potentials which the black hole potential depends on are regular at the horizon; using the alternative definition of V_{BH}, the one with the path ordered exponentials, this would not be necessary.

Also in this case the scalars run into a fixed point, so an attractor mechanism is likely to exist. This in turn gives

$$A = -4\pi V_{BH}(\phi_h, \Psi_h, \hat{Q}). \tag{6.73}$$

Finally, the structure of the equation of motion for scalars, eq. (6.61) does not change with respect to the static case. Then it follows that the black hole potential reaches a critical point at the horizon

$$\left.\frac{\partial V_{BH}}{\partial \phi^i}\right|_{(\phi_h, \Psi_h)} = 0$$

exactly as before. It seems that, apart from the complications in the definition of the black hole potential, when scalar and vector fields are considered in a Taub-NUT background it is possible to obtain the same results of the static case for what concerns the behaviour of scalars and the existence of critical points of V_{BH} at the horizon. Also the area-horizon relation is still valid. There are some indications that an attractor mechanism may be at work; on the other hand there is the novelty that now the critical values of the scalars could also be related to the horizon values of the electric and magnetic potentials. This seems unavoidable if we use the definition (6.57), which we report:

$$V_{BH}(\phi, \Psi, \hat{Q}, N) \equiv 8 \left(N\Psi + \hat{Q}\right)_M \left(\mathcal{M}^{-1}\right)^{MN} \left(N\Psi + \hat{Q}\right)_N, \tag{6.74}$$

while it would not happen with the alternative definition (6.64).

The next step would be to specify the problem to the case in which the action is that of N=2 Supergravity: we should then consider complex scalars $z^i, \overline{z}^{i^*}$ as coordinates on a Special Kähler manifold and equations (6.59), (6.60) and (6.61) would become

$$U'' + e^{2U} V_{BH} + 2\, e^{4U} N^2 = 0 \tag{6.75}$$

$$(U')^2 + \mathcal{G}_{ij^*} \frac{dz^i}{d\tau} \frac{d\overline{z}^{j^*}}{d\tau} + e^{2U} V_{BH} + e^{4U} N^2 = r_0^2 \tag{6.76}$$

$$\frac{d^2 z^i}{d\tau} + \mathcal{G}^{ij^*} \partial_k \mathcal{G}_{lj^*} \frac{dz^k}{d\tau} \frac{dz^l}{d\tau} + e^{2U} \mathcal{G}^{ij^*} \partial_{j^*} V_{BH} = 0. \tag{6.77}$$

Next, it would be useful to have an identity like the (4.57) between the black hole potential and a combination of the central and matter charges, but it is evident that, given the form of V_{BH} in (6.74), only the term without electromagnetic potentials can be proportional to $|Z|^2 + |Z_i|^2$, and it seems difficult to find another suitable invariant of special geometry, given the presence of the potentials which are not, in principle, objects of special geometry.

6.4 Conclusion

The above computation showed a way to treat the equations of motion of vectors and scalars coupled to gravity in a stationary Taub-NUT background. Some comments are in order. The new technical problem, with respect to the static case, was the presence of the off-diagonal term in the metric. Some initial attempts showed that, using explicitly the $g_{t\varphi}$ component, the pretty symmetric form of the equations was systematically broken and it was complicated to check if the static case was obtained for $N = 0$, or also to see if the matrix $\mathcal{M}$ was again involved (as it was expected). Things started to work well when we considered the possibility of performing the dimensional reduction explained in 6.1, using at the same time the general form (6.2) for the metric, with $\omega_{\underline{m}}$ an independent field. Once the right 3-dimensional fields (6.5) and (6.8) coming from the reduction were determined, the calculations were similar to the static case ones, only with the complication given by the non-diagonal metric. The nice fact is that the new equations look as natural generalizations of the static ones, and, once one introduces the NUT charge N, this gives rise to a new, additive term; the contact with the static case is then immediate.
The arguments which in Chapter 4 and in [15] led to the (loosely speaking) attractor mechanism can be reformulated and bring to the same conclusions. It seems then that attractors are a constant feature of extremal configurations in which scalars are present; it could then be investigated if the difference between non-supersymmetric and supersymmetric, complete attractors still holds.
On the other hand, some negative points also emerged: it has been shown that the definition of the black hole potential V_{BH} is more problematic and that in the end there is not a satisfactory relation between V_{BH} and an invariant of special geometry, since when $N \neq 0$ vectors cannot be eliminated.

An important observation is that, for the reasons explained in Chapter 5, we have assumed again spherical symmetry of the fields. This assumption greatly simplifies things, but cannot be valid if we tried to solve the same problem in the Kerr-Newman background (which would be the next step and also a more significative problem from the physical point of view). In this last case, indeed, some attempts suggest that also the Kerr-Newman metric can be put in the general conforma-stationary form (and this is an essential point if we want to apply the above machinery), but with the metric scalar U now depending on τ and θ. It would not be possible then to rely on spherical symmetry and calculations would be more complicated, in particular the arguments used to deduced the attractor mechanism and the existence of critical points of the black hole potential may not be sufficient.

Appendix A

Conforma-stationary metrics

A conforma-stationary metric has the general form

$$ds^2 = e^{2U}(dt + \omega)^2 - e^{-2U}\gamma_{\underline{mn}} dx^{\underline{m}} dx^{\underline{n}}, \qquad \underline{m}, \underline{n} = 1, 2, 3 \tag{A.1}$$

with all components independent of the time coordinate t. $\omega = \omega_{\underline{m}} dx^{\underline{m}}$ is a 3-dimensional one-form; $\gamma_{\underline{mn}}$ is the 3-dimensional metric (which in the cases of our interest is always diagonal). The spatial part of the 4-dimensional metric is

$$g_{\underline{mn}} = e^{2U}\omega_{\underline{m}}\,\omega_{\underline{n}} - e^{-2U}\gamma_{\underline{mn}}$$

and

$$g^{\underline{mn}} = -e^{2U}\gamma^{\underline{mn}}.$$

In matricial form and Schwarzschild-like coordinates, they are

$$g_{\mu\nu} =$$

$$\begin{bmatrix} e^{2U} & e^{2U}\omega_\tau & e^{2U}\omega_\theta & e^{2U}\omega_\varphi \\ e^{2U}\omega_\tau & -e^{-2U}\gamma_{\tau\tau} + e^{2U}\omega_\tau\omega_\tau & e^{2U}\omega_\tau\omega_\theta & e^{2U}\omega_\tau\omega_\varphi \\ e^{2U}\omega_\theta & e^{2U}\omega_\theta\omega_\tau & -e^{-2U}\gamma_{\theta\theta} + e^{2U}\omega_\theta\omega_\theta & e^{2U}\omega_\theta\omega_\varphi \\ e^{2U}\omega_\varphi & e^{2U}\omega_\varphi\omega_\tau & e^{2U}\omega_\varphi\omega_\theta & -e^{-2U}\gamma_{\varphi\varphi} + e^{2U}\omega_\varphi\omega_\varphi \end{bmatrix}$$

$$g^{\mu\nu} = \begin{bmatrix} e^{-2U} - e^{2U}\omega^2 & e^{2U}\omega^{\tau} & e^{2U}\omega^{\theta} & e^{2U}\omega^{\varphi} \\ e^{2U}\omega^{\tau} & -e^{2U}\gamma^{\tau\tau} & 0 & 0 \\ e^{2U}\omega^{\theta} & 0 & -e^{2U}\gamma^{\theta\theta} & 0 \\ e^{2U}\omega^{\varphi} & 0 & 0 & -e^{2U}\gamma^{\varphi\varphi} \end{bmatrix}$$

and the structure of the metric relevant for the dimensional reduction is evident (see Chapter 6). When working in tangent space, we use the following Vielbein basis

$$\left(e^{a}{}_{\mu}\right) = \begin{pmatrix} e^{U} & e^{U}\omega_{\underline{m}} \\ 0 & e^{-U}v_{\underline{m}}{}^{n} \end{pmatrix} \qquad \left(e^{\mu}{}_{a}\right) = \begin{pmatrix} e^{-U} & -e^{U}\omega_{m} \\ 0 & e^{U}v_{m}{}^{\underline{n}} \end{pmatrix}$$

with

$$\gamma_{\underline{mn}} = v_{\underline{m}}{}^{p} v_{\underline{n}}{}^{q}\delta_{pq} \qquad \omega_m = v_m{}^{\underline{n}}\,\omega_{\underline{n}}.$$

With the definitions

$$\partial_m \equiv v_m{}^{\underline{n}}\partial_{\underline{n}}, \qquad W_{\underline{mn}} \equiv 2\partial_{[\underline{m}}\omega_{\underline{n}]}, \qquad W_{mn} = v_m{}^{\underline{p}}v_n{}^{\underline{q}}\,W_{\underline{pq}}$$

the components of the spin connection are

$$\begin{aligned} \omega_{00m} &= -e^{U}\partial_m U, & \omega_{0mn} &= \frac{1}{2}e^{3U}W_{mn}, \\ \omega_{m0n} &= \omega_{0mn}, & \omega_{mnp} &= -e^{U}\varpi_{mnp} - 2e^{U}\delta_{m[n}\partial_{p]}U \end{aligned}$$

where ϖ_m^{np} is the 3-dimensional spin connection. The components of the Riemann tensor are (the hatted, left-hand side ones are components of 4-dimensional tensors; all quantities in the right-hand side are 3-dimensional objects; 3-dimensional covariant derivatives and curvature are built using the 3-dimensional spin connection)

$$\begin{aligned} \hat{R}_{0m0n} &= \frac{1}{2}\nabla_m\partial_n e^{2U} + e^{2U}\partial_m U\partial_n U - e^{2U}\delta_{mn}(\partial U)^2 + \frac{1}{4}\nabla_m e^{6U}W_{mp}W_{np}, \\ \hat{R}_{0mnp} &= -\frac{1}{2}\nabla_m\left(e^{4U}W_{np}\right) + \frac{1}{2}W_{m[n}\partial_{p]}e^{4U} - \frac{1}{4}\delta_{m[n}W_{p]l}\partial_q e^{4U}, \\ \hat{R}_{mnpq} &= -e^{2U}R_{mnpq} + \frac{1}{2}e^{6U}\left(W_{mn}W_{pq} - W_{p[m}W_{n]q}\right) - \\ &\quad - 2\,e^{2U}\delta_{mn,pq}(\partial U)^2 + 4\,e^{U}\delta_{[m}{}^{[p}\nabla_{n]}\partial^{q]}e^{U} \end{aligned}$$

where $2\delta_{mn,pq} \equiv \delta_{mp}\delta_{nq} - \delta_{mq}\delta_{np}$. It is then straightforward to calculate the components of the Ricci tensor:

$$\hat{R}_{00} = -e^{2U}\nabla^2 U - \frac{1}{4}\,e^{6U}W^2 \tag{A.2}$$

$$\hat{R}_{0m} = \frac{1}{2}\nabla_n\left(e^{4U}W_{nm}\right) \tag{A.3}$$

$$\hat{R}_{mn} = e^{2U}\left(R_{mn} + 2\,\partial_m U\partial_n U - \delta_{mn}\nabla^2 U - \frac{1}{2}\,e^{4U}W_{mp}W^{np}\right) \tag{A.4}$$

and the Ricci scalar

$$\hat{R} = -e^{2U}\left(R - \frac{1}{4}\,e^{4U}W^2 - 2\nabla^2 U + 2(\partial U)^2\right). \tag{A.5}$$

All the above expressions are used also in the static case, when the 1-form ω is zero, simply setting $\omega = W = 0$ everywhere.

The 3-dimensional metric that we use throughout the thesis is

$$\gamma_{\underline{mn}}dx^{\underline{m}}dx^{\underline{n}} = \frac{r_0^4}{\sinh^4 r_0\tau}d\tau^2 + \frac{r_0^2}{\sinh^2 r_0\tau}d\Omega^2_{(2)};$$

with $d\Omega^2_{(2)} = d\theta^2 + \sin^2\theta\, d\varphi^2$. Its non-vanishing Christoffel symbols are

$$\Gamma^{\tau}_{\tau\tau} = -2r_0\frac{\cosh r_0\tau}{\sinh r_0\tau} \qquad \Gamma^{\tau}_{\theta\theta} = \frac{\sinh r_0\tau\ \cosh r_0\tau}{r_0} \qquad \Gamma^{\tau}_{\varphi\varphi} = \sin^2\theta\,\Gamma^{\tau}_{\theta\theta}$$
$$\Gamma^{\theta}_{\varphi\varphi} = -\sin\theta\cos\theta \qquad \Gamma^{\theta}_{\tau\theta} = -r_0\,\frac{\cosh r_0\tau}{\sinh r_0\tau}$$
$$\Gamma^{\varphi}_{\tau\varphi} = \Gamma^{\theta}_{\tau\theta} = -r_0\,\frac{\cosh r_0\tau}{\sinh r^0\tau} \qquad \Gamma^{\varphi}_{\theta\varphi} = \frac{\cos\theta}{\sin\theta}.$$

The non-zero components of the Riemann tensor are then

$$R_{\tau\theta\tau}{}^{\theta} = R_{\tau\varphi\tau}{}^{\varphi} = -r_0^2$$
$$R_{\theta\varphi\theta}{}^{\varphi} = \sinh^2 r_0\tau$$

and the Ricci tensor has only one non-vanishing component

$$R_{\tau\tau} = -2r_0^2$$

so that the Ricci scalar is

$$R = R_{\underline{mn}}\gamma^{\underline{mn}} = R_{\tau\tau}\gamma^{\tau\tau} = -2\,\frac{\sinh^4 r_0\tau}{r_0^2}.$$

Acknowledgements

I am extremely grateful to professor Tomás Ortín for accepting to be my advisor and for following me very patiently during (and also after) my staying at the Instituto de Física Teórica at the Universidad Autonoma de Madrid. I also have to thank my officemates, Alberto Palomo Lozano and Carlos Shahin Shahbazi Alonso, for their help and for the crazy and relaxed mood of the office B08 at the IFT.
Thanks to prof. Gianguido Dall'Agata for his interest and support, and to professors Paolo Pasti and Kurt Lechner, especially for the clarifying discussions.
Grazie infinite ai miei genitori, che sempre mi hanno supportato nelle mie (spesso non condivise) scelte, e a mio fratello Nicolas.
Finally, I thank everyone of my friends, and all the people I met during my Erasmus staying in Madrid; well, actually not all of them, but the idea should be clear.

Bibliography

[1] S. W. Hawking, *Gravitational radiation from colliding black holes*, Phys. Rev. Lett. **26** (1971) 1344.

[2] J. D. Bekenstein, *Black holes and entropy*, Phys. Rev. D **7** (1973) 2333.

[3] T. Ortín (2004), *Gravity and Strings*, Cambridge University Press.

[4] R. M. Wald (1984), *General Relativity*, The University of Chicago Press.

[5] R. H. Price, Phys. Rev. **D5** (1972) 2419-2438.
R. H. Price, Phys. Rev. **D5** (1972) 2439-2454.

[6] R. Penrose, Riv. Nuovo Cim. **1** (1969) 252-276.

[7] C. W. Misner, K. S. Thorne, J. A. Wheeler (1973) *Gravitation*, W. H. Freeman and Company, San Francisco.

[8] A. Strominger and C. Vafa, *Microscopic Origin of the Bekenstein- Hawking Entropy*, Phys. Lett. B **379** (1996) 99, `arXiv:hep-th/9601029`.

[9] Heusler, M., *Stationary Black Holes: Uniqueness and Beyond*, Living Rev. Relativity, 1, (1998), 6. `http://www.livingreviews.org/Articles/Volume1/1998-6heusler/`.

[10] S. W. Hawking, Phys. Rev. **D13** (1976) 191-197.

[11] J. Bekenstein, Ph.D. Thesis, Princeton University (unpublished).

[12] S. W. Hawking, Nature **248** (1974) 30.
S. W. Hawking, Commun. Math. Phys. **43** (1975) 199-220.

[13] L. Andrianopoli, R. D'Auria, S. Ferrara and M. Trigiante, *Extremal Black Holes in Supergravity*, `arXiv:hep-th/0611345`.

[14] S. Ferrara, R. Kallosh, A. Strominger, *N=2 extremal black holes*, Phys. Rev. D **52** (1995) 5412, `arXiv:hep-th/9508072`.

[15] S. Ferrara, G. W. Gibbons, R. Kallosh, *Black holes and critical points in moduli space*, Nucl. Phys. B **500** (1997) 75, `arXiv:hep-th/9702103`.

[16] A. Ceresole, R. D'Auria, S. Ferrara, *The Symplectic Structure of N=2 Supergravity and its Central Extension*, Proceedings of the Trieste workshop on Mirror Symmetry and S-Duality, Trieste 1995, Eds. K.S. Narain and E. Gava; `arXiv:hep-th/9509160`.

[17] M. Nakahara, *Geometry, Topology and Physics*, Institute of Physics Publishing, The Institute of Physics, London.

[18] P. Meessen, T. Ortín, *The supersymmetric configurations of N=2, d=4 supergravity coupled to vector supermultiplets*, Nucl. Phys. **B749** (2006) 291-324, `arXiv:hep-th/0603099`.

[19] A. Van Proeyen, $N = 2$ *supergravity in* $d = 4, 5, 6$ *and its matter couplings*, lectures given at the Institute Henri Poincaré, Paris, November 2000, `http://itf.fys.kuleuven.be/~toine/LectParis.pdf`

[20] B. Craps, f. Roose, W. Troost, A. Van Proeyen, *What is special Kähler Geometry?*, `arXiv:hep-th/9703082`.

[21] L. Andrianopoli, R. D'Auria, S. Ferrara, *Central Extension of Extended Supergravity in Diverse Dimensions*, Int. J. Mod. Phys. A **12** 3759 (1997), `arXiv:hep-th/9608015`.

[22] L. Andrianopoli, M. Bertolini, A. Ceresole, R. D'Auria, S. Ferrara, P. Fré and T. Magri, *N = 2 supergravity and N = 2 super Yang-Mills theory on general scalar manifolds: Symplectic covariance, gaugings and the momentum map*, J. Geom. Phys. **23** (1997) 111, `arXiv:hep-th/9605032`.

[23] A. Strominger, *Macroscopic entropy of N=2 extremal black holes*, Phys. Lett. **B383**, 39, 1996, `hep-th/9602111`.

[24] S. Ferrara and R. Kallosh, *Supersymmetry and attractors*, Phys. Rev. **D54**, 1514 (1996), `arXiv:hep-th/9602136`.

[25] P. Galli, T. Ortín, J. Perz, C. S. Shahbazi, *Non-extremal black holes of* $N = 2, d = 4$ *supergravity*, `arXiv:hep-th/1105.3311`.

[26] J. Bellorín, T. Ortín, P. Meessen, *Supersymmetry, attractors and cosmic censorship*, Nucl. Phys. **B762** (2007) 229-255, `arXiv:hep-th/0606201`.

[27] A. Sen, JHEP **0603** 008 (2006), `arXiv:hep-th/0508042`.

[28] R. P. Kerr, Phys. Rev. Lett. **11** (1963) 237.
E. T. Newman, E. Couch, K. Chinnapared, A. Exton, A. Prakash and R. Torrence, J. Math. Phys.**6** (1965) 918.

[29] A. H. Taub, Ann. Math. **53**, 472 (1951).

[30] E. Newman, L. Tambourino, and T. Unti, *Empty-Space Generalization of the Schwarzschild Metric*, J. Math. Phys. **4** (1963) 915.

[31] D. T. Brill, *Electromagnetic Fields in a Homogeneous, Nonisotropic Universe*, Phys. Rev. **133** (1964) 3B B845-B848.

[32] M. Demianski and E. T. Newman, *A Combined Kerr-NUT Solution of the Einstein Field Equations*, Bull. Acad. Pol. Sci. Ser. Sci. Math. Astron. Phys. **14** (1966) 653.

[33] C. Misner, *The Flatter Regions of Newman, Unti and Tamburino's Generalized Schwarzschild Spacetime*, J. Math. Phys. **4**, 924 (1963).

[34] W. B. Bonnor, (1969) Proc. Camb. Phil. Soc. **66**, 145.

[35] C. A. Hurst, Ann. Phys. **50** (1968) 51.

[36] J. S. Dowker, *The NUT Solution as a Gravitational Dyon*, GRG vol.5 No.5 (1974) 603-613.

[37] E. Radu, *Gravitating nonabelian solution with NUT charge*, `arXiv: hep/th-0211120`.

[38] S. W. Hawking, G. F. R. Ellis, *The large scale structure of space-time*, Cambridge University Press.

[39] A. Ceresole and G. Dall'Agata, *Flow Equations for non-BPS Extremal Black Holes*, JHEP **0703** (2007) 110, `arXiv:hep-th/0702088`.