Supergravity, black holes and holography

Memoria de Tesis Doctoral realizada por

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¿Trampas para ratones cómo van a cazar lobos?
SDJ Crew

Los guerreros vencedores ganan primero, y luego van a la guerra;
mientras que los perdedores van a la guerra primero, y luego pretenden ganar.
孫子兵法　孫子

Esta tesis es el resultado de mis años como estudiante de doctorado en el Instituto de Física Teórica UAM/CSIC bajo la supervision del Profesor Tomás Ortín Miguel. Llegando a su fin como está esta etapa, y puesto que el *agradecimiento es la memoria del corazón*[1], es momento de mirar atrás y agradecer a todas aquellas personas que de una forma u otra han posibilitado que haya llegado hasta *aquí*.

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[1] Lao-Tsé.
[2] Seguramente no están todos los que son, pero seguro que sí son todos los que están.

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Madrid, 26 de febrero de 2015

Publications

The contents of this thesis are based on the following publications:

1. Pablo Bueno and Robert C. Myers,
"Corner contributions to holographic entanglement entropy", in preparation.

2. Pablo Bueno, Robert C. Myers and William Witczak-Krempa,
"Universality of corner entanglement in gapless quantum matter", in preparation.

3. Pablo Bueno, Tomas Ortín and C. S. Shahbazi,
"Non-extremal branes", Phys. Lett. **B743** (2015), 301-305. [arXiv:1412.5547 [hep-th]].

4. Pablo Bueno, Patrick Meessen, Tomas Ortín and Pedro F. Ramírez, "$\mathcal{N}=2$ Einstein-Yang-Mills' static two-center solutions", JHEP **1412** (2014) 093. [arXiv:1410.4160 [hep-th]].

5. Pablo Bueno and Pedro F. Ramírez, "Higher-curvature corrections to holographic entanglement entropy in geometries with hyperscaling violation", JHEP **1412** (2014) 078. [arXiv:1408.6380 [hep-th]].

6. Pablo Bueno and C. S. Shahbazi, "The violation of the No-Hair Conjecture in four-dimensional ungauged Supergravity", Class. Quantum Grav. **31** (2014) 145005. [arXiv:1310.6379 [hep-th]].

7. Pablo Bueno, Pietro Galli, Patrick Meessen and Tomas Ortín "Black holes and equivariant charge vectors in N=2,d=4 supergravity", JHEP **1309** (2013) 010. [arXiv:1305.5488 [hep-th]].

8. Pablo Bueno and C. S. Shahbazi, "Non-perturbative black holes in Type-IIA String Theory vs. the No-Hair conjecture", Class. Quantum Grav. **31** (2013) 015023. [arXiv:1304.8079 [hep-th]].

9. Pablo Bueno, Wissam Chemissany and C. S. Shahbazi, "On hvLif-like solutions in gauged Supergravity," Eur. Phys. J. C (2014) **74:2684**. [arXiv:1212.4826 [hep-th]].

10. Pablo Bueno, Rhys Davies and C. S. Shahbazi, "Quantum black holes in Type-IIA String Theory", JHEP **1301** (2013) 089. [arXiv:1210.2817 [hep-th]].

11. Pablo Bueno, Wissam Chemissany, Patrick Meessen, Tomas Ortín and C. S. Shahbazi, "Lifshitz-like solutions with hyperscaling violation in ungauged supergravity", JHEP **1301** (2013) 189. [arXiv:1209.4047 [hep-th]].

Contents

1
Introduction

In this section we introduce several concepts related to the three topics which give name to this thesis, namely: supergravity, black holes and holography. We have tried to include here the material which has played a more prominent rôle in the research carried out throughout these years, and that is essential for a good understanding of the results obtained in the papers which in this thesis have been compiled.

1.1 Supergravity

The first topic is supergravity (SUGRA), the framework in which essentially all my research has been developed. After a short reflection on string theory, and a small general introduction to supersymmetry and supergravity, we study in some detail certain aspects of $\mathcal{N} = 2$, $d = 4$ SUGRA theories, which are of most relevance for the results obtained in the next chapters.

1.1.1 Two words on string theory

For some time now, string theory (ST) [33, 214, 215, 353, 365, 366] has established itself as the most prominent (if not the only) candidate for consistently describing the quantum nature of gravity, along with all the rest of known (and surely some unknown) interactions in a unified framework. Some of the highlights of ST, leaving mathematics aside, can be enumerated as follows

- Provides a(n arguably) quantum mechanically consistent UV completion for gravity.
- Not only unifies gauge interactions and gravity, but it makes them inseparable.
- Produces numerous candidates for grand unification gauge groups. It is even possible to get just the standard model at low energies from it [249].
- Incorporates (world-sheet and spacetime) supersymmetry, which in many cases provides natural candidates for dark matter as well as helping to make sense of other issues such as the hierarchy problem, the discrepancy in the anomalous magnetic moment of the muon, etc.
- It has no free parameters. Besides, as oppossed to QFTs, where you have the freedom to choose a particular gauge group, the fields you include in the theory, the representations under which they transform, etc., in some sense, there is a unique

string theory (there are five consistent superstring theories which are supposed to correspond to different limits of a single (*M*-)theory, and which are related to each other through different kinds of *dualities*).

- Satisfactorily accounts for the microscopic entropy of certain extremal and near-extremal black holes [403], producing results which match the macroscopic result obtained through the Bekenstein-Hawking formula.
- Provides a majestic realization of the holographic principle as well as a window into the strongly coupled regime of certain QFTs by means of the AdS/CFT correspondence [311].
- Predicts a wrong number of spacetime dimensions (10 or 11), which suggests that some of them might be compact and small (or something alternative, like in the *brane-worlds* avatar [376]).
- It seems to possess a huge amount of 4D vacua, the so-called *String landscape* (see, *e.g.*, [368]). This has raised some doubts on the capability of the theory to address certain fundamental issues in theoretical physics, such as the cosmological constant problem. Some people, however, consider that an anthropic view of the problem (according to which we just happen to live in a particular locus of the landscape which is suitable enough to support life so as to have us here wondering about what the reason for this *highly unlikely* suitability might be) can be acceptable, and even the best we can get.

From a down-top perspective, ST is very attractive. Indeed, physics beyond the standard model is crying out for solutions to several problems which could be nicely addressed by supersymmetry (some of them, such as the hierarchy problem, less nicely the higher the bounds for the masses of the SUSY partners are set at accelerators). Now, if SUSY is a fundamental symmetry of nature, some supergravity theory will be responsible for describing the massless modes of the corresponding theory (many of which will not be really massless because of broken symmetries at certain scales such as the vacuum expectation value of the Higgs, $\Lambda_{\rm GUT}$, etc.). Actually, the consistent quantization of a spin 3/2 massless field (a.k.a. *Rarita-Schwinger field*) imposes the theory under consideration to be symmetric under local supersymmetry transformations (*i.e.*, to be a SUGRA theory), just like similar arguments for spin 1 and 2 fields lead to gauge and diffeomorphism invariances. As it turns out, the low energy limits of the different String Theories (which are all related through *dualities* of different kinds) correspond to different SUGRA theories, and so do their corresponding compactified versions to 4 dimensions. Hence, it is extremely tempting to think that some of these effective theories might be adequate for describing nature[1].

In spite of these achievements and hints, our knowledge of ST is still rather limited, and a significantly deeper understanding of the theory will still probably require (in case such a goal is pursued) several generations of theorists. Such understanding could eventually lead us to know whether ST is the right framework for describing the physics of our

[1]It is convenient to note that, as we will briefly comment, SUGRA theories contain solitonic SUSY solutions, which due to their supersymmetric properties are protected from receiving stringy corrections when we move from the effective SUGRA action, and which correspond to the long range fields produced by bound states of non-perturbative ST objects (like *D-branes* [367]). This means that SUGRA theories also contain non-perturbative information about the corresponding ST.

universe or not. In any event, it is fair to say that even if that a goal is never fully realized, ST has already provided us with numerous fascinating insights concerning the physics of gravity, quantum mechanics and spacetime (as well as mathematics). An example of this is, as we have already mentioned, the celebrated AdS/CFT correspondece, which not only unveils a deep connection between certain gravity (and string) theories and some quantum field theories, but also provides a realization of the holographic principle.

1.1.2 SUSY and SUGRA

Supersymmetry is one of the big ideas of the second half of twentieth-century theoretical physics. After the observation by Coleman and Mandula [142] that under very generic assumptions, any Lie group of symmetries of QFT S-matrices containing the Poincaré group P and some additional group of internal symmetries G must be a direct product of both in order to produce non-trivial physics, Golfand and Likhtman showed [212] that this no-go theorem could be circumvented by generalizing the concept of Lie algebra to that of a $\mathbb{Z}_2$-graded Lie algebra, which included new anticommuting (fermionic) generators. Haag, Lopuszanski and Sohnius [223] proved that the most general non-trivial extension of the Poincaré group in 4-dimensions including this kind of generators could be obtained by considering $4\mathcal{N}$ (for some integer $\mathcal{N}$) of these: Q^I_α, $\hat{Q}^I_{\dot\alpha}$, $I = 1, ..., \mathcal{N}$, $\alpha, \dot\alpha = 1, 2$, transforming as left (right) Weyl spinors, and satisfying the following (anti)commutation relations

$$[Q^I_\alpha, J_{\mu\nu}] = (\sigma_{\mu\nu})_\alpha^{\beta} Q^I_\beta \,, \tag{1.1}$$

$$[Q^I_\alpha, P_\mu] = [\bar{Q}^I_{\dot\alpha}, P_\mu] = 0 \tag{1.2}$$

$$\{Q^I_\alpha, \bar{Q}_{\dot\beta J}\} = 2(\sigma^\mu)_{\alpha\dot\beta} P_\mu \delta^I_J \,, \tag{1.3}$$

$$\{Q^I_\alpha, Q^J_\beta\} = \epsilon_{\alpha\beta} Z^{IJ} \,, \tag{1.4}$$

where $Z^{IJ} = -Z^{JI}$ are the so-called *central charges*, commuting with all the generators of the superalgebra, $J_{\mu\nu}$ are the generators of boosts and rotations of the Poincaré group, and P_μ those of translations (among them, they satisfy the usual commutation relations of the Poincaré algebra). The group H of automorphisms of the superalgebra is the so-called *R-symmetry group*. When $Z^{IJ} = 0$, H=U($\mathcal{N}$), and when $Z^{IJ} \neq 0$, H is a subgroup of U($\mathcal{N}$).

Studying the massless linear *irreps* of Poincaré's superalgebra [375], one finds that each multiplet is composed of $2^{\mathcal{N}}$ states of helicities $\lambda_0 + k/2$ ($k = 0, ..., \mathcal{N}$), ($\binom{\mathcal{N}}{k}$ for each k) and with the same number of fermionic and bosonic states in each case. Let us consider for example the case $\lambda_0 = -2$, $\mathcal{N} = 8$, which in fact corresponds to the field content of $\mathcal{N} = 8$, $d = 4$ SUGRA: we have $\binom{8}{0} = 1$ states of helicities ± 2, $\binom{8}{1} = \binom{8}{7} = 8$ states of helicities $\pm 3/2$, $\binom{8}{2} = \binom{8}{6} = 28$ states of helicities ± 1, $\binom{8}{3} = \binom{8}{5} = 56$ states of helicities $\pm 1/2$ and $\binom{8}{4} = 70$ states of helicities 0. So the number of bosonic states $2 \times 1 + 2 \times 28 + 70 = 128$ equals the number of fermionic states $2 \times 8 + 2 \times 56 = 128$.

Assuming $|\lambda \leq 2|$[2], the maximum number of supersymmetries in 4-dimensional theories is given by $\mathcal{N} = 8$. In addition, renormalizability becomes problematic for $\mathcal{N} \geq 4$ [375], although some recent intriguing results [58, 59, 265] indicating the possible quantum

[2]Particles of helicities larger than 2 are usually problematic in standard QFTs. Higher-spin theories [416] are a different kettle of fish.

finiteness of $\mathcal{N} = 8$, $d = 4$ SUGRA might jeopardize the widespread belief that no field theory of quantum gravity in 4D can be quantum mechanically finite.

SUSY transformations in field space are given by $\delta_\epsilon \sim \epsilon_L^\alpha \mathcal{Q}_\alpha^L$, being ϵ_L the fermionic parameter of the transformations[3], and they generically act on bosonic (B) and fermionic (F) fields as

$$\delta_\epsilon B \sim \bar{\epsilon} F\,, \tag{1.5}$$
$$\delta_\epsilon F \sim B\epsilon\,. \tag{1.6}$$

Now, a field theory is said to be supersymmetric if it is invariant under the action of the corresponding SUSY transformations. If we allow the transformation parameters to depend on the spacetime coordinates $\epsilon_L = \epsilon_L(x)$, all the fields will necessarily be coupled to the gravitational field, and the theories invariant under the action of such local transformations

$$\delta_\epsilon B \sim \bar{\epsilon} F\,, \tag{1.7}$$
$$\delta_\epsilon F \sim B\epsilon + \partial\epsilon\,, \tag{1.8}$$

will be called *supergravity theories*. The composition of two local SUSY transformations produces an infinitesimal general coordinate transformation, which immediatly makes us think about General Relativity (GR). The jump is, however, not completely straightforward since SUGRA theories contain fermions, whereas the original framework of GR does not allow for such a possibility. This is because only the group of diffeomorphisms of a given spacetime $\mathcal{M}$, Diff($\mathcal{M}$), acts naturally on the fields of the theory, and this does not have finite-dimensional spinorial representations. Actually, SUGRA theories can be seen as particular cases of the Cartan-Sciama-Kibble theory (see [353,380] for reviews), which is precisely a generalization of GR suitable for the coupling of fermions to gravity. In classical field theories, bosonic fields B transform under tensorial representations of the Lorentz group $\mathrm{SO}(1,3)$, so in GR they correspond to spacetime tensors. However, fermionic fields F transform under spinorial representations, which correspond, *e.g.*, to the fundamental of the universal cover of $\mathrm{SO}(1,3)$, $\mathrm{Spin}(1,3)$. These cannot be identified with any section of the tangent or cotangent budles of $\mathcal{M}$ (as opposed to the bosonic fields), so some further structure must be added in order to incorporare them when gravity is present (a more detailed discussion of this issues can be found in [386]). Such a structure is known as *spin bundle*, of which fermions would be sections. In case $\mathcal{M}$ admits this structure, one can apply the *first-order formalism* of Cartan-Sciama-Kibble. In this, we use the Vierbein $\mathbf{e}$ instead of the spacetime metric $\mathbf{g}$ as the dynamical field associated to gravity (being both related by $\mathbf{g} = \eta(\mathbf{e},\mathbf{e})$ where η is the flat frame metric). Using a Vierbein we can make the theory manifestly invariant under local $\mathrm{Spin}(1,3)$ transformations, and allows for the definition of a *spin connection* ω, which enables one to define covariant derivatives $\mathcal{D} \sim \partial + \omega$ with respect to the action of that group. This is the needed machinery for including fermions into the theory. The kinetic terms of the spinors are constructed now using the covariant derivative associated to the spin connection, which is considered an auxiliary field (in the sense that its equation of motion will act as a constraint for the other fields).

[3]Remarkably, the commutator of two of these is related to the spacetime derivative (this reflects nothing but the fact that the commutator of two $\mathcal{Q}$ is proportional to the translations generator in (1.3)) as $[\delta_1, \delta_2] \propto \gamma^\mu \partial_\mu$.

Let us consider, for example, pure $\mathcal{N}=1$, $d=4$ SUGRA, whose field content is given by the Vierbein, and the gravitino, which is a vector of Majorana or Weyl spinors, so we have (we choose to show explicitly spacetime and frame indices, and omit the spinorial ones): $\left\{e^a_\mu, \psi_\mu\right\}$. The action reads [353]

$$I[e^a_\mu, \omega_\mu{}^{ab}, \psi_\mu] = \frac{1}{16\pi G}\int d^4x\, e\left[R(e,\omega) + 2e^{-1}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_5\gamma_\nu\mathcal{D}_\rho\psi_\sigma\right]\,, \tag{1.9}$$

where $e \equiv \det(\mathbf{e})$, the Lorentz-covariant derivative of the Rarita-Schwinger field reads

$$\mathcal{D}_\rho\psi_\sigma \equiv \partial_\rho\psi_\sigma - \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\psi_\sigma\,, \tag{1.10}$$

and

$$R(e,\omega) \equiv e^\mu_a e^\nu_b R_{\mu\nu}{}^{ab}(\omega)\,, \tag{1.11}$$

where $R_{\mu\nu}{}^{ab}(\omega)$ is the curvature of the spin connection ω^{ab}_μ,

$$R_{\mu\nu a}{}^{b}(\omega) \equiv 2\partial_{[\mu}\omega_{\nu]a}{}^{b} - 2\omega_{[\mu|a}{}^{c}\omega_{|\nu]c}{}^{b}\,. \tag{1.12}$$

The above action, which only contains first derivatives to the power of 1 (hence the name *first-order formalism*), is invariant under general coordinate transformations, local Lorentz transformations, as well as local $\mathcal{N}=1$ SUSY transformations[4]

$$\delta_\epsilon e^a_\mu = -i\bar{\epsilon}\gamma^a\psi_\mu\,, \quad \delta_\epsilon\psi_\mu = \mathcal{D}_\mu\epsilon\,. \tag{1.13}$$

As a general fact, the invariance of SUSY and SUGRA theories under the corresponding supersymmetric transformations constrains the field content as well as the geometric structure of the different multiplets. We will review how this works for a particular theory, namely: $\mathcal{N}=2$, $d=4$ ungauged SUGRA, which is of particular interest for the results of this thesis. Before doing so, let us introduce the general structure of the theory.

1.1.3 $\mathcal{N}=2$, $d=4$ SUGRA

A $\mathcal{N}=2$, $d=4$ SUGRA is a field theory invariant under the action of two independent local supersymmetry transformations generated by two spinors (Weyl or Majorana). According to this definition, there are SUGRA theories which include terms of arbitrarily high order in derivatives in their action. However, we will restrict ourselves to second-order Lagrangians. Let us also assume for the moment that none of the global symmetries of the theory has been gauged.

The field theory content of any ungauged *classical* SUGRA[5] is then the following

- A gravity multiplet: $(\mathbf{e}, \psi_I, A^0)$, where $\mathbf{e}$ is the Vierbein, ψ_I an SU(2) (which is the R-symmetry group of the theory) doublet of gravitini and A^0 is the graviphoton 1-form.

[4] All this is a bit more subtle, because ω is an independent field here. It is often convenient to work in the so-called *1,5 formalism*, according to which we mantain the first-order form of the action, but impose the relations satisfied by ω (which come from its equation of motion) on its variations. We will not mess up with this here.

[5] Classical in the sense that it does not include higher-order terms in derivatives, which arise from stringy and quantum corrections, so that the bosonic sector is a particular case of GR

- n_v *vector multiplets*: (A^i, λ^{iI}, z^i), where A^i, $i = 1, ..., n_v$ are 1-forms, λ^i_I are spinors and the Z^i are complex scalars. These parametrize a n_v-dimensional *Special Kähler manifold* [145] (see below).
- n_h *hypermultiplets*: $(\chi_\alpha, \chi^\alpha, q^u)$ where the χ_α, $\alpha = 1, ..., 2n_h$ are spinors, and the q^u, $u = 1, ..., 4n_h$ real scalars parametrizing a $4n_h$-dimensional *quaternionic Kähler manifold* [63].

Every supersymmetric Lagrangian, and the $\mathcal{N} = 2$ $d = 4$ SUGRA in particular, is invariant under a $\mathbb{Z}_2$ symmetry which makes $B \to B$, $F \to -F$. As a consequence, truncating all fermions is always consistent[6]. Since we will be interested in purely bosonic configurations, let us truncate all fermions from now on. The action corresponding to the bosonic sector of ungauged $\mathcal{N} = 2$ $d = 4$ SUGRA is given by

$$I = \frac{1}{16\pi G}\int d^4x\sqrt{|g|}\,\Big\{R + h_{uv}(q)\partial_\mu q^u\partial^\mu q^v + \mathcal{G}_{i\bar{j}}(Z,\bar{Z})\partial_\mu Z^i\partial^\mu \bar{Z}^{\bar{j}} \\ +2\Im_{\Lambda\Sigma}(Z,\bar{Z})F^{\Lambda}{}_{\mu\nu}F^{\Sigma\,\mu\nu} - 2\Re_{\Lambda\Sigma}(Z,\bar{Z})F^{\Lambda}{}_{\mu\nu}\star F^{\Sigma\,\mu\nu}\Big\}\,, \tag{1.14}$$

where we used the symplectic index $\Lambda = (0, i)$ for all vector fields A^Λ, $\Lambda = 0, ..., n_v$. The first term is the usual Ricci scalar[7]. The second is the kinetic term of the hyperscalars, which parametrize a quaternionic manifold of metric $h_{uv}(q)$. It is easy to show that these fields can be fixed to a constant value $q_u = q_u^0$ in a consistent manner, and this we assume henceforth: their equations of motion do not involve more fields than themselves, and in their SUSY transformation all terms contain some dependence on them. The third term is a *non-linear sigma model* corresponding to the kinetic term of the vector multiplets' complex scalars, which parametrize a (complex) n_v-dimensional Special Kähler manifold with metric $\mathcal{G}_{i\bar{j}}(Z,\bar{Z})$. The two remaining terms correspond to the kinetic and *CP-violating like* terms corresponding to the n_v+1 1-forms. $\Im_{\Lambda\Sigma} \equiv \Im m\mathcal{N}_{\Lambda\Sigma}$ (which is negative-definite) and $\Re_{\Lambda\Sigma} \equiv \Re e\mathcal{N}_{\Lambda\Sigma}$ are the imaginary and real parts of some symplectic matrix depending on the scalars Z^i, $\mathcal{N}_{\Lambda\Sigma}(Z,\bar{Z})$.

We will study different models of ungauged $\mathcal{N} = 2$ $d = 4$ SUGRA in the forthcoming chapters. Our interest on them will be in all cases related to the search for new solutions. In chapters 2, 3 and 4, we will focus on black holes, whereas in the first part of 7 we will look for a different kind of solutions, namely, *hvLf-like* (see below).

Let us now explain the rôle played by symplectic and Special Kähler geometries on the structure of $\mathcal{N} = 2$ $d = 4$ SUGRA theories.

Electric-magnetic duality and symplectic covariance

Consider the action (1.14) without hypermultiplets. If we define a tensor dual to F^Λ, as[8]

$$\tilde{F}_{\Lambda\mu\nu} \equiv -\frac{1}{4\sqrt{|g|}}\frac{\delta S}{\delta \star F^{\Lambda\mu\nu}} = \Re e\mathcal{N}_{\Lambda\Sigma}F^{\Sigma}_{\mu\nu} + \Im m\mathcal{N}^*_{\Lambda\Sigma}F^{\Sigma}_{\mu\nu}\,, \tag{1.15}$$

the equations of motion for the n_v+1 1-forms can be written as $\mathcal{E}^\mu_\Lambda \equiv \nabla_\nu \star \tilde{F}^{\nu\mu}_\Lambda = 0$, which clearly resembles the Bianchi identity for the F^Λ: $\mathcal{B}^{\Lambda\mu} \equiv \nabla_\nu \star F^{\Lambda\nu\mu} = 0$. If we define the

[6] This means that any solution of the truncated theory is a solution of the complete theory.

[7] Once we truncate the fermions, the first-order formalism looses a great part of it usefulness. From now on, we come back to the usual second-order formulation.

[8] The results of this subsections are based on [129, 186, 206].

doublet

$$\mathcal{E}^M_\mu \equiv \begin{pmatrix} \mathcal{B}^\Lambda_\mu \\ \mathcal{E}_{\Lambda\mu}, \end{pmatrix}, \tag{1.16}$$

Maxwell equations and Bianchi identities can be simply written as $\mathcal{E}^M = 0$, so they admit as a symmetry, an arbitrary $\mathrm{GL}(2n_v + 2, \mathbb{R})$ rotation on the M index: $\mathcal{E}^M_\mu = 0 \to m^M{}_N \mathcal{E}^N_\mu = 0$, $m^M{}_N \in \mathrm{GL}(2n_v + 2, \mathbb{R})$. These transformations act analogously on the 2-forms F^Λ and $\tilde{F}_\Lambda$ as

$$F^M_\mu \equiv \begin{pmatrix} F^\Lambda \\ \tilde{F}_\Lambda \end{pmatrix}, F'^M = m^M{}_N F^N. \tag{1.17}$$

However, F^Λ and $\tilde{F}_\Lambda$ are not independent, as it can be seen from (1.15). Imposing (1.15) to be valid after the rotation clearly imposes that the matrix $\mathcal{N}_{\Lambda\Sigma}(Z, \bar{Z})$ must be transformed under $\mathrm{GL}(2n_v + 2, \mathbb{R})$ as well. Then, one needs to impose an action of this group on the scalars in a way such that $\mathcal{N}'_{\Lambda\Sigma}(Z, \bar{Z})$ has the desired structure. In order to do this, we consider a diffeomorphism $\xi \in \mathrm{Diff}(\mathcal{M}_{\mathrm{escalar}})$ on the scalar manifold $\mathcal{M}_{\mathrm{escalar}}$, as well as the existence of a group homomorphism as

$$\mathfrak{i} : \mathrm{Diff}(\mathcal{M}_{\mathrm{scalar}}) \to \mathrm{GL}(2n_v + 2, \mathbb{R}), \tag{1.18}$$

such that, for every $\xi \in \mathrm{Diff}(\mathcal{M}_{\mathrm{scalar}})$ it assigns a transformation $\mathfrak{i}(\xi) \in \mathrm{GL}(2n_v + 2)$. This construction allows us to define the simultaneous action of ξ on all the fields of the theory $\{Z, F^M, \mathcal{N}_{\Sigma\Lambda}(Z)\} \xrightarrow{\xi} \{\xi(Z), (\mathfrak{i}(\xi))^M{}_N F^N, \mathcal{N}'_{\Sigma\Lambda}(\xi(Z))\}$. The consistency condition on $\mathcal{N}'_{\Lambda\Sigma}(Z, \bar{Z})$, which can be written as

$$\tilde{F}'_{\Lambda\mu\nu} \equiv -\frac{1}{4\sqrt{|g|}} \frac{\delta S'}{\delta \star F'^{\Lambda\mu\nu}}, \tag{1.19}$$

translates into the fact that the transformations $m^M{}_N$ must belong to the subgroup $\mathrm{Sp}(2n_v + 2, \mathbb{R})$, so the homomorphism $\mathfrak{i}$ reduces to $\mathfrak{i} : \mathrm{Diff}(\mathcal{M}_{\mathrm{escalar}}) \to \mathrm{Sp}(2n_v + 2, \mathbb{R})$, and that the period matrix $\mathcal{N}_{\Lambda\Sigma}(Z, \bar{Z})$ transforms as

$$\mathcal{N}' = (A\mathcal{N} + B)(C\mathcal{N} + D)^{-1}, \tag{1.20}$$

where A, B, C y D are $(n_v + 1) \times (n_v + 1)$ matrices such that

$$m \equiv \begin{pmatrix} D & C \\ B & A \end{pmatrix} \in \mathrm{Sp}(2n_v + 2, \mathbb{R}). \tag{1.21}$$

Thus, whenever the period matrix satisfies (1.20), we can define these *symplectic electric-magnetic duality transformations* acting as symmetries of the Maxwell and Bianchi equations. On the other hand, it is convenient to stress that not all these transformations are symmetries of the action (1.14) (not even of the sector corresponding to the 1-forms). However, we can make of these transformations symmetries of all the equations of motion of (1.14) by restricting the diffeomorphisms $\xi \in \mathrm{Diff}(\mathcal{M}_{\mathrm{scalar}})$ associated to the duality transformations to be isometries of the scalar manifold metric $\mathcal{G}_{i\bar{j}}$. This means that the homomorphism reduces even more to

$$\mathfrak{i} : \mathrm{Isometries}(\mathcal{M}_{\mathrm{scalar}}, \mathcal{G}_{ij}) \to \mathrm{Sp}(2n_v + 2, \mathbb{R}). \tag{1.22}$$

Summing up, the symplectic duality transformations which are global symmetries of the equations of motion correspond to isometries of the scalar manifold which act on the scalars as diffeomorphisms, and on the 1-forms through (1.22), provided the period matrix satisfies (1.20).

Finally, it actually happens that these transformations correspond to symmetries of the action (1.14) in some cases. The condition for this to be so reads: $B = C = 0$.

A crash course in Special Kähler geometry

As we have mentioned already, the geometry underlying the vector multiplets of any $\mathcal{N} = 2$, $d = 4$ SUGRA is called *Special Kähler* [155, 402]. Let us now review what it is about[9].

Let $(\mathcal{M}, J, \mathcal{G})$ be a d-dimensional Hermitean manifold with complex structure J and Hermitean metric $\mathcal{G}$. By definition, they satisfy

$$\mathcal{J} \equiv \frac{1}{2} J_{mn} dX^m \wedge dX^n = J_{i\bar{j}} dZ^i \wedge d\bar{Z}^{\bar{j}} = 2i\mathcal{G}_{i\bar{j}} dZ^i \wedge d\bar{Z}^{\bar{j}} \,, \tag{1.23}$$

where we have introduced the *Kähler 2-form*, $\mathcal{J}$, and we have used real, X^m, $m = 1, ..., 2d$, and (anti-) holomorphic complex coordinates ($\bar{Z}^i$) Z^i, $i = 1, ..., d$. $(\mathcal{M}, J, \mathcal{G})$ is a *Kähler manifold* iff its *Kähler form* is closed

$$d\mathcal{J} = 0 \,. \tag{1.24}$$

Therefore, a Kähler manifold is a Riemannian ($\mathcal{G}$), complex (J) and symplectic ($\mathcal{J}$) manifold which nicely incorporates in a compatible way the three basic structures of differential geometry ($\mathcal{J}(.,.) = \mathcal{G}\,(J.,.)$).

The defininig property of a Kähler manifold immediatly leads to the conditions $\partial_{[k}\mathcal{G}_{i]\bar{j}} = \partial_{[\bar{k}|}\mathcal{G}_{i|\bar{j}]} = 0$, which in turn imply the vanishing of the torsion and the identification of the Hermitean and Levi-Civita connections. These equations are solved in a local patch $U_{(x)}$ by

$$\mathcal{G}_{i\bar{j}} = \partial_i \partial_{\bar{j}} \mathcal{K}_{(x)} \,, \tag{1.25}$$

or $\mathcal{J} = 2i\partial\bar{\partial}\mathcal{K}_{(x)}$, for some function $\mathcal{K}_{(x)}(Z, \bar{Z})$, called *Kähler potential*. This is not uniquely defined. Indeed, if we construct a new Kähler potential as

$$\mathcal{K}'_{(x)}(Z, \bar{Z}) = \mathcal{K}_{(x)}(Z, \bar{Z}) + \lambda(Z) + \bar{\lambda}(\bar{Z}) \,, \tag{1.26}$$

it is trivial to see that the metric constructed from it equals the one constructed from the original one. Similarly, in compact manifolds, Kähler potentials defined on two different patches $U_{(x)}$, $U_{(y)}$ are related through

$$\mathcal{K}_{(x)} = \mathcal{K}_{(y)} + \lambda_{xy}(Z) + \bar{\lambda}_{xy}(\bar{Z}) \,, \tag{1.27}$$

in the overlap $U_{(x)} \cap U_{(y)}$.

In a Kähler manifold it is possible to define objects transforming under the Kähler transformations (1.26). Such an object $\Psi(Z, \bar{Z})$ is said to have Kähler weight $(q, \bar{q})$ if, under (1.26), it transforms as

$$\Psi' = e^{-(q\lambda + \bar{q}\bar{\lambda})/2}\Psi \,. \tag{1.28}$$

[9] We follow the conventions of [320, 353].

One can define a Kähler-covariant derivative acting on these objects as

$$\mathcal{D}_i \equiv \nabla_i + iq\mathcal{Q}_i\,, \qquad \mathcal{D}_{\bar{i}} \equiv \nabla_{\bar{i}} - i\bar{q}\mathcal{Q}_{\bar{i}}\,, \tag{1.29}$$

where ∇ is the Levi-Civita (Hermitean) connection and $\mathcal{Q}$ is the so-called *Kähler 1-form*, which is defined as

$$\mathcal{Q}_{(x)} \equiv \frac{1}{2i}\left(\partial - \bar{\partial}\right)\mathcal{K}_{(x)}\,, \tag{1.30}$$

so in each patch $\mathcal{J} = 2d\mathcal{Q}_{(x)}$. Under a Kähler transformation (1.26), this object transforms as

$$\mathcal{Q}'_i = \mathcal{Q}_i + \frac{1}{2i}\partial_i\lambda\,. \tag{1.31}$$

A situation of particular interest for our purposes is that of fields with $\bar{q} = -q$, whose Kähler transformations are Z-dependent U(1) transformations:

$$\Psi' = e^{-iq\Im\mathfrak{m}\lambda(z)}\Psi\,. \tag{1.32}$$

In particular, for $q = 1$, the structure that supports these fields is that of a U(1) bundle associated to an holomorphic line bundle (*i.e.*, a complex line bundle whose projection is holomorphic) $\mathcal{L} \to \mathcal{M}$ over the Kähler manifold, being the consistency condition of this construction that the first Chern class of the line bundle (which can be obtained from the Ricci 2-form $\mathfrak{R}$ of the fiber's Hermitean metric) equals the Kähler 2-form $\mathcal{J}$. Manifolds with this additional structure are called *Kähler-Hodge manifolds*, and they play an important rôle in SUGRA theories. In particular, the manifolds parametrized by the complex scalars of the chiral multiplets of $\mathcal{N} = 1$, $d = 4$ SUGRA must be of this kind. The same occurs for the manifolds parametrized by the complex scalars of the vector multiplets of $\mathcal{N} = 2$, $d = 4$ SUGRA (because of which we are talking about this), although these must satisfy further constraints defining what is called Special Kähler geometry, which we are finally able to review now.

Let us consider a Kähler-Hodge manifold of complex dimension n_v (this is a suitable name as n_v will indeed correspond to the number of vector multiplets of a $\mathcal{N} = 2$, $d = 4$ SUGRA) and a flat $2(n_v + 1)$-dimensional vector bundle $E \to \mathcal{M}$ with structure group $\mathrm{Sp}(2(n_v + 1); \mathbb{R})$. The product bundle $E \otimes \mathcal{L} \to \mathcal{M}$ will be a *special Kähler manifold* if there is a section $\mathcal{V}$ of it, called *covariantly-holomorphic canonical symplectic section*, which satisfies the following properties

$$\begin{aligned} -\langle \mathcal{V}|\bar{\mathcal{V}}\rangle &= i\,, & (1.33)\\ \mathcal{D}_{\bar{i}}\mathcal{V} &= 0\,, & (1.34)\\ \langle \mathcal{U}_i|\mathcal{V}\rangle &= 0\,, & (1.35) \end{aligned}$$

where

$$\mathcal{U}_i \equiv \mathcal{D}_i\mathcal{V} = \left(\partial_i + \frac{1}{2}\partial_i\mathcal{K}\right)\mathcal{V}\,,\quad \mathcal{D}_{\bar{i}}\mathcal{V} = \left(\partial_{\bar{i}} - \frac{1}{2}\partial_{\bar{i}}\mathcal{K}\right)\mathcal{V}\,, \tag{1.36}$$

and the symplectic product $\langle .|.\rangle$ is defined as

$$\langle \mathcal{A}|\mathcal{B}\rangle \equiv \mathcal{A}_M\mathcal{B}^M = -\mathcal{A}^N\Omega_{NM}\mathcal{B}^M = \mathcal{B}^\Lambda\mathcal{A}_\Lambda - \mathcal{B}_\Lambda\mathcal{A}^\Lambda, \tag{1.37}$$

where we have used symplectic indices $M, N = 1, ..., 2(n_v+1)$ or, equivalently, pairs of one upper and one lower index $\Lambda, \Sigma = 1, ..., n_v+1$ and $\Omega_{MN} = \mathrm{antidiag}[\mathbb{I}, -\mathbb{I}]$ is the symplectic metric.

The canonical section $\mathcal{V}$, which is usually written in components as $\mathcal{V} = (\mathcal{L}^\Lambda, \mathcal{M}_\Sigma)^T$, completely defines a $\mathcal{N}=2$, $d=4$ SUGRA model in the absence of hypermultiplets, as we will see. An important object we can define is the period matrix

$$\mathcal{M}_\Lambda = \mathcal{N}_{\Lambda\Sigma}\mathcal{L}^\Sigma\,, \quad h_{\Lambda i} = \bar{\mathcal{N}}_{\Lambda\Sigma} f_i^\Sigma\,, \tag{1.38}$$

where $h_{\Lambda i}$, f_i^Λ are the components of $\mathcal{U}_i$: $\mathcal{U}_i = (f_i^\Lambda, h_{\Sigma i})^T$. It can be seen that the defining properties of the Special Kähler manifold (1.33), (1.34), (1.35) imply $\mathcal{N}_{\Lambda\Sigma}$ to be symmetric and its imaginary part $\Im\mathfrak{m}\mathcal{N}_{\Lambda\Sigma}$ to be negative-definite (which is exactly what we want for the matrix appearing in (1.14)!).

There are many other useful objects one can construct in a Special Kähler manifold. In particular, it is convenient to consider another holomorphic section, Ω, which is defined as

$$\Omega \equiv e^{-\mathcal{K}/2}\mathcal{V} \equiv \left(\mathcal{X}^\Lambda, \mathcal{F}_\Sigma\right)^T\,. \tag{1.39}$$

Since $e^{-\mathcal{K}/2}$ and $\mathcal{V}$ have Kähler weights $(1,1)$ and $(1,-1)$ respectively, this is a weight $(2,0)$ section. The following expression for the Kähler potential can be now obtained

$$e^{-\mathcal{K}} = -2\Im\mathfrak{m}\mathcal{N}_{\Lambda\Sigma}\mathcal{X}^\Lambda\bar{\mathcal{X}}^\Sigma\,. \tag{1.40}$$

Now, assuming that the lower components of Ω, $\mathcal{F}_\Lambda$, depend on the complex coordinates Z^i only through the upper components $\mathcal{X}$, it can be seen [353] that (1.40) implies the following relation,

$$\mathcal{F}_\Lambda = \frac{\partial\mathcal{F}}{\partial\mathcal{X}^\Lambda}\,, \tag{1.41}$$

where

$$\mathcal{F}(\mathcal{X}) \equiv \frac{1}{2}\mathcal{X}^\Sigma\mathcal{F}_\Sigma(\mathcal{X})\,, \tag{1.42}$$

is the so-called *prepotential*, which is a homogeneous function of second degree in the $\mathcal{X}^\Lambda$s, as is clear from (1.41) and (1.42). It might be that the prepotential does not exist for a given holomorphic section Ω. However, it can be shown that there always exists a symplectic transformation of Ω such that the prepotential exists. From the SUGRA perspective, this transformation will correspond to a change of coordinates in the scalar manifold so, for practical purposes, the existence of a prepotential can in general be assumed. From the prepotential, one can reconstruct the metric of the Special Kähler manifold and the period matrix as

$$\mathcal{G}_{i\bar{j}} = -\partial_i\partial_{\bar{j}}\ln\left[i\left[\bar{\mathcal{X}}^\Lambda\partial_\Lambda\mathcal{F} - \mathcal{X}^\Lambda\partial_{\Lambda}\bar{\mathcal{F}}\right]\right]\,, \tag{1.43}$$

$$\mathcal{N}_{\Lambda\Sigma} = \partial_\Lambda\partial_\Sigma\bar{\mathcal{F}} + 2i\frac{\Im m(\partial_\Lambda\partial_{\Lambda'}\bar{\mathcal{F}})\mathcal{X}^{\Lambda'}\Im m(\partial_\Sigma\partial_{\Sigma'}\bar{\mathcal{F}})\mathcal{X}^{\Sigma'}}{\mathcal{X}^\Omega\Im m(\partial_\Omega\partial_{\Omega'}\bar{\mathcal{F}})\mathcal{X}^{\Omega'}}\,. \tag{1.44}$$

At this point, it is not difficult to guess what the above objects correspond to in the bosonic action (1.14) of $\mathcal{N}=2$, $d=4$ SUGRA. Indeed, SUSY imposes $\mathcal{G}_{i\bar{j}}(Z,\bar{Z})$ and $\mathcal{N}_{\Lambda\Sigma}(Z,\bar{Z})$ to be identified with the Hermitean metric and the period matrix of a Special Kähler manifold parametrized by the complex scalars of the vector multiplets, and satisfying all the relations presented in this subsection. In particular, the period matrix satisfies (1.20), so the equations of motion of $\mathcal{N}=2$, $d=4$ ungauged SUGRA can be invariant under the symplectic duality transformations explained in the previous subsection.

Therefore, after truncating hyperscalars and fermions, the theory gets completely determined by the election of some holomorphic symplectic section for the bundle $\mathcal{SM}$

with structure group $\mathrm{Sp}(2n_v+2,\mathbb{R})$ defined on the scalar manifold or, equivalently in the case it exists, by the election of an holomorphic and homogeneous function of second degree (the prepotential) $\mathcal{F}$, defined in (1.42), and from which the scalar metric and the period matrix can be constructed using (1.43), and (1.44). Examples of prepotentials we will study in the following chapters are

- The $\overline{\mathbb{CP}}^n$ model (see chapters 2 and 5), which contains n scalar fields given by $Z^i \equiv \mathcal{X}^i/\mathcal{X}^0$, which parametrize the symmetric space $\mathrm{U}(1,n)/(\mathrm{U}(1)\times\mathrm{U}(n))$

$$\mathcal{F} = -\tfrac{i}{4}\eta_{\Lambda\Sigma}\mathcal{X}^\Lambda\mathcal{X}^\Sigma\,, \qquad (\eta_{\Lambda\Sigma}) = \mathrm{diag}(+-\cdots-)\,. \tag{1.45}$$

- The $ST[2,n]$ model (see chapter 5), which is an example of a cubic model

$$\mathcal{F} \;=\; -\frac{1}{3!}\,d_{ijk}\,\frac{\mathcal{X}^i\mathcal{X}^j\mathcal{X}^k}{\mathcal{X}^0}\,, \tag{1.46}$$

  where d is completely symmetric in its indices, characterized by $d_{1\alpha\beta}=\eta_{\alpha\beta}$ where $(\eta_{\alpha\beta}) = \mathrm{diag}(+-\cdots-)$ and where the indices α,β take n values between 2 and $n+1$. In this model, the scalar $Z^1=\mathcal{X}^1/\mathcal{X}^0$ plays a special role and parametrizes a $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ coset space, whereas the other n scalars, $Z^\alpha = \mathcal{X}^\alpha/\mathcal{X}^0$ ($\alpha = 2,\cdots,n$), parametrize a $\mathrm{SO}(2,n)/(\mathrm{SO}(2)\times\mathrm{SO}(n))$ coset space.

- Type-IIA string theory compactified to 4D on a Calabi-Yau manifold up to second-order in derivatives (see chapters 3 and 4) [109–111]

$$\mathcal{F} = -\frac{1}{3!}\kappa^0_{ijk}z^iz^jz^k + \frac{ic}{2} + \frac{i}{(2\pi)^3}\sum_{d_i} n_{d_i} Li_3\left(e^{2\pi i d_i z^i}\right)\,, \tag{1.47}$$

  where z^i, $i=1,\ldots,n_v=h^{1,1}$, are the scalars in the vector multiplets, $c=\frac{\chi\zeta(3)}{(2\pi)^3}$ with χ the Euler characteristic of the C.Y. three-fold, (given by $\chi = 2(h^{1,1}-h^{2,1})$), κ^0_{ijk} are the *classical intersection numbers*, $d_i\in\mathbb{Z}^+$ is a n_v-dimensional summation index and

$$Li_3(x) \equiv \sum_{j=1}^{\infty}\frac{x^j}{j^3} \tag{1.48}$$

  is the third polylogarithmic function.

$\mathcal{N}=2$, $d=4$ SEYM

In general (ungauged) $\mathcal{N}=2$, $d=4$ SUGRA theories, the global symmetry group G can be written as [353]

$$\mathrm{G} = \mathrm{G_V}\times\mathrm{G_{hyper}}\times\mathrm{SU(2)_R}\times\mathrm{U(1)_R}\,, \tag{1.49}$$

where $\mathrm{G_V}$ and $\mathrm{G_{hyper}}$ stand for the isometry groups of the Special and quaternionic Kähler manifolds respectively. $\mathcal{N}=2$, $d=4$ *gauged SUGRA theories* are those in which some subgroup of G has been promoted to a local symmetry group through the corresponding gauging procedure while preserving the supersymmetric structure of the theory.

When a (necessarily non-Abelian) subgroup of $\mathrm{G_V}$ is gauged the scalar potential is positive semidefinite, which tipically allows for asymptotically de-Sitter and asymptotically flat solutions. This is in contradistinction to theories in which a subgroup of $\mathrm{SU(2)_R}$ (or the complete $\mathrm{SU}(2)_{\mathrm{R}}$) is gauged via Fayet-Iliopoulos (FI) terms[10] in whose case the scalar potential becomes negative definite, the solutions thus being typically asymptotically anti-de Sitter. This is the kind of theories we will deal with in the second part of chapter 7. In particular, a short review of $\mathcal{N}=2$, $d=4$ SUGRA theories with FI gaugings can be found in section 7.8.

$\mathcal{N}=2$, $d=4$ *SEYM theories* can be seen as the simplest $\mathcal{N}=2$ supersymmetrization of the Einstein-Yang-Mills (EYM) or Einstein-Yang-Mills-Higgs (EYMH) theories. They are nothing but theories of $\mathcal{N}=2$, $d=4$ SUGRA coupled to n_v vector multiplets in which a non-Abelian[11] subgroup of the isometry group of the (Special Kähler) scalar manifold has been gauged using some of the vector fields of the theory as gauge fields[12].

The bosonic sector of the theory, in the absence of hypers, has the following action

$$\begin{aligned} S[g_{\mu\nu}, A^{\Lambda}{}_{\mu}, Z^i] \;\; = \;\; & \int d^4x\sqrt{|g|}\,\big[R + 2\mathcal{G}_{ij^*}\mathfrak{D}_{\mu}Z^i\mathfrak{D}^{\mu}Z^{*\,j^*} + 2\Im\mathfrak{m}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,\mu\nu}F^{\Sigma}{}_{\mu\nu} \\ & -2\Re\mathfrak{e}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,\mu\nu}\star F^{\Sigma}{}_{\mu\nu} - V(Z,Z^*)\big]\,. \end{aligned} \tag{1.50}$$

where $\mathfrak{D}_{\mu}Z^i$ is the gauge-covariant derivative

$$\mathfrak{D}_{\mu}Z^i = \partial_{\mu}Z^i + gA^{\Lambda}{}_{\mu}k_{\Lambda}{}^{i}\,, \tag{1.51}$$

$F^{\Lambda}{}_{\mu\nu}$ is the vector field strength

$$F^{\Lambda}{}_{\mu\nu} = 2\partial_{[\mu}A^{\Lambda}{}_{\nu]} - gf_{\Sigma\Gamma}{}^{\Lambda}A^{\Sigma}{}_{\mu}A^{\Gamma}{}_{\nu}\,, \tag{1.52}$$

and $V(Z,Z^*)$ is the scalar potential

$$V(Z,Z^*) = -\tfrac{1}{4}g^2\Im\mathfrak{m}\mathcal{N}^{\Lambda\Sigma}\mathcal{P}_{\Lambda}\mathcal{P}_{\Sigma}\,, \tag{1.53}$$

which is positive-semidefinite, since the imaginary part of the period matrix is negative-definite. In the above equations, $k_{\Lambda}{}^{i}(Z)$ are the holomorphic Killing vectors of the isometries that have been gauged[13] and $\mathcal{P}_{\Lambda}(Z,Z^*)$ the corresponding momentum maps, which are related to the Killing vectors and to the Kähler potential $\mathcal{K}$ by

$$i\mathcal{P}_{\Lambda} \;\; = \;\; k_{\Lambda}{}^{i}\partial_i\mathcal{K} - \lambda_{\Lambda}\,, \tag{1.54}$$

$$k_{\Lambda\,i^*} \;\; = \;\; i\partial_{i^*}\mathcal{P}_{\Lambda}\,, \tag{1.55}$$

for some holomorphic functions $\lambda_{\Lambda}(Z)$. Furthermore, the holomorphic Killing vectors and the generators T_{Λ} of the gauge group satisfy the Lie algebras

$$[k_{\Lambda}, k_{\Sigma}] = -f_{\Lambda\Sigma}{}^{\Gamma}k_{\Gamma}\,, \qquad [T_{\Lambda}, T_{\Sigma}] = +f_{\Lambda\Sigma}{}^{\Gamma}T_{\Gamma}\,. \tag{1.56}$$

[10]The overall $\mathrm{U}(1)_R$ group cannot be gauged in this way. The Abelian gaugings discussed in the literature deal with a subgroup $\mathrm{U}(1) \in \mathrm{SU}(2)_{\mathrm{R}}$.

[11] The theory becomes identical to the ungauged one when the gauge group is Abelian.

[12] A global symmetry group can be gauged if it acts on the vector fields in the adjoint representation. Furthermore, it is required to be a symmetry of the prepotential; see e.g., [245] for more details.

[13] The employed notation associates a Killing vector to each value of the index Λ in order to avoid the introduction of yet another class of indices and the embedding tensor (see e.g., the reviews [413]); it is understood that not all the k_{Λ} need to be non-vanishing.

For the gauge group SU(2), which is the only one we are going to consider, we use lowercase indices[14] $a, b, c = 1, 2, 3$ and the structure constants are $f_{ab}{}^{c} = -\varepsilon_{abc}$, so

$$[k_a, k_b] = +\varepsilon_{abc} k_c\,, \qquad [T_a, T_b] = -\varepsilon_{abc} T_c\,. \tag{1.57}$$

We said before that these theories could be seen as the simplest SUSY version of Einstein-Yang-Mills (EYM) or Einstein-Yang-Mills-Higgs (EYMH) theories. The main differences of these w.r.t. the SUGRA theories we are dealing with are the complexification of the Higgs field and the presence of a non-trivial period matrix. A further difference is the possibility of having more general scalar manifolds, which is reflected in the expressions of the gauge-covariant derivatives of the scalar fields.

In chapter 2, we will consider several $\mathcal{N} = 2$, $d = 4$ SEYM models, for which we will be able to construct the first examples of regular multi-center black-hole-monopole solutions of any model of gravity coupled to non-Abelian Yang-Mills(-Higgs) fields as well as new *global-monopole* solutions.

More aspects of $\mathcal{N} = 2$, $d = 4$ SUGRA theories will be highlighted in some of the following chapters as they are needed for the results presented there. For the moment, let us close here our introduction to (certain aspects of) supergravity and start dealing with our second topic of interest, to wit: black holes.

1.2 Black holes

In this section we review some basics about black holes, their thermodynamic properties and their rôle as supersymmetric objects in SUGRA theories and string theory. We also introduce the H-FGK formalism, which facilitates the characterization and construction of static and spherically symmetric black holes of $\mathcal{N} = 2$, $d = 4$ SUGRA.

1.2.1 Asymptotically flat black holes in General Relativity

In the context of Newtonian mechanics, it was a British geologist, John Michell, who first conceived the idea of a massive object so dense that its escape velocity would exceed c, so light would not be able to scape from it. Such an observation was made by Michell in a letter sent to Cavendish in 1783 and although at this point the interaction between light and matter was very far from understood, so the comment can be regarded as not much more than a curiosity, the intuition brought up by Michell is certainly remarkable[15]. The issue of these *black* objects was broadly ignored until Einstein's formulation of GR [167]. A few months after his seminal papers appeared, Schwarzschild found the first exact solution to Einstein's vacuum equations, which described the gravitational field outside an spherical distribution of mass, and which would turn out to hide many surprises. In fact, many of the properties of the Schwarzschild solution were understood many years after, and those involving the regime in which $\hbar$ cannot be neglected are still far from being understood. Schwarzschild's solution inaugurated the era of a new class of physical objects: the *black holes*.

[14] These will be a certain subset of those represented by $\Lambda, \Sigma, \ldots$.

[15] Obviously, this *Newtonian version* of black holes hardly shares any of the properties of GR black holes.

The precise definition of black hole in GR is rather technical and requires the introduction of several concepts which we are about to review[16]. In particular, we will start focusing in the definition of *asymptotically flat* black holes, which have been of prominent importance throughout my Ph.D. studies[17]. So let us get started with the definitions, no need for anaesthesia.

Two metrics $\mathbf{g}$ and $\tilde{\mathbf{g}}$ defined on some spacetime $\mathcal{M}$ are conformally related if there exists an scalar function Ω on $\mathcal{M}$ such that $\tilde{\mathbf{g}} = \Omega^2 \mathbf{g}$. A *conformal compactification* of $(\mathcal{M}, \mathbf{g})$ consists of an election of a metric $\tilde{\mathbf{g}}$ such that $(\mathcal{M}, \tilde{\mathbf{g}})$ can be *isometrically embedded* (*i.e.*, $\tilde{\mathbf{g}}$ is given by the pullback of $\mathbf{g}'$) in a compact domain U' of a new spacetime $(\mathcal{M}', \mathbf{g}')$.

A spacetime $\mathcal{M}$ is said to be *asymptotically simple* iff it admits a conformal compactification $\mathcal{M}'$ and all null geodesics in $\mathcal{M}$ have future and past endpoints in $\partial\mathcal{M}'$. Asymptotically simple spaces include Minkowski and the asymptotically flat spaces containing bounded objects such as stars which have not collapsed. They do not include, however, black holes, as in these there are null geodesics which do not have endpoints in $\partial\mathcal{M}'$ (esentially because of the presence of singularities). We need a more refined definition for them. A spacetime $\mathcal{M}$ is said to be *weakly asymptotically simple* if there exists an asymptotically simple spacetime $\mathcal{M}'$ and a neighbourhood U' of $\partial\mathcal{M}'$ in $\mathcal{M}'$ such that $\mathcal{M}' \cap U'$ is isometric to a subset of $\mathcal{M}$.

Now, $\mathcal{M}$ is said to be *asymptotically flat* if it is weakly asymptotically simple and its metric in the neighborhood of the boundary of the conformal compactification $\partial\mathcal{M}'$ satisfies Einstein's vacuum equations. Conceptually, a spacetime of this kind corresponds to the general relativistic version (in the absence of cosmological constant) of an isolated system. In a spacetime of this kind, there exists a region far away from any energy density in which curvature becomes arbitrarily small, and Minkowski geometry is recovered asymptotically.

Given an asymptotically flat spacetime $\mathcal{M}$, the *black hole* region $\mathcal{B} \subset \mathcal{M}$ is defined as

$$\mathcal{B} \equiv \mathcal{M} - I^-(\mathcal{I}^+)\,, \tag{1.58}$$

where $\mathcal{I}^+$ stands for the future null infinity (the set of points asymptotically approached by null geodesics which can escape to spatial infinity) and I^- is the chronological past.

The *event horizon* $\mathcal{H}$ of a black hole is defined as the boundary of $\mathcal{B}$. Hence, $\mathcal{H}$ is the boundary of the past of $\mathcal{I}^+$.

A black hole therefore consists of a set of points in $\mathcal{M}$ from which null geodesics cannot escape to infinity. Thus, an observer in $\mathcal{B}$ cannot have any causal influence on anything happening outside the horizon: no information sent from the interior of $\mathcal{H}$ can escape from it. Similarly, all information sent into a black hole from the exterior is inevitably lost forever[18]. It is convenient to stress that $\mathcal{H}$ has no local significance (in fact, the curvature can be arbitrarily small on $\mathcal{H}$): the whole history of spacetime's future must be known in order to determine the location of an even horizon.

If an asymptotically flat spacetime contains a black hole, this is said to be *stationary* if it admits a timelike Killing vector field. In such a spacetime, the components of the metric can be chosen locally so that they are all independent of the time coordinate. A

[16] Some of the contents in this section are based on [235, 419, 421]

[17] However, many of the general properties of black holes we will present afterwards will apply as well for other kinds of beasts, such as asymptotically AdS black holes.

[18] All these statements apply only in the framework of GR, and are modified when $\hbar$ is at work.

static black hole is stationary, with the additional property that there exists a family of space-like hypersurfaces Σ orthogonal to the Killing field.

A null surface $\mathcal{K}$ whose generators coincide with the orbits of a uniparametric group of isometries is called a *Killing horizon*. A result due to Hawking [235] establishes that for electrovacuum solutions of 4D Einstein's equations, the event horizon of any stationary black hole must be a Killing horizon. Now let $\mathcal{K}$ be a Killing horizon (not necessarily an event horizon) with orthogonal Killing field ξ^μ. Given that $\nabla^\mu \xi^2$ is also orthogonal to $\mathcal{K}$, both vectors must be proportional in every point of the horizon. Hence, there must exist a function κ, on $\mathcal{K}$, known as *surface gravity* of $\mathcal{K}$, defined through $\nabla^\mu \xi^2 = -2\kappa\xi^\mu$. The surface gravity can be thought of as the force required at infinity to hold a unit mass particle at rest near the horizon.

A final definition useful for our purposes is the following: a 4-dimensional spacetime is *spherically symmetric* if its isometry group has an SO(3) subgroup.

1.2.2 Black hole thermodynamics

All we have said so far about black holes holds in the world of General Relativity. It is also in this framework where the results we are going to review now, known as the *laws of black hole mechanics*, are found. Remarkably, these laws strikingly resemble the laws of thermodynamics. When the machinery of quantum fields in curved spaces is introduced to analyze this issue, one finds that this apparently accidental analogy is not just that. Black holes turn out to be thermodynamic objects whose macroscopic thermodynamic information is encoded in their geometric stucture in a very nice way. Let us go by parts.

Bardeen, Carter and Hawking [27] showed that in case Einstein's equation is satisfied for some stress-energy tensor satisfying the *dominant energy condition* (which requires that all physical observers measure a speed of energy flow less than or equal to the speed of light), then κ must be constant on any Killing horizon. This result is usually known as *the zeroth law of black hole mechanics*, in an analogy with thermodynamics' zeroth law, which establishes that the temperature along a system in thermal equilibrium is constant.

The *first law of black hole mechanics* [27] is an identity which relates the variations of mass M (in analogy with energy E in the thermodynamic case), area of the horizon A (in analogy with the entropy S as we will see in a moment), angular momentum J, electric charge Q, and other magnitudes when a black hole is perturbed. At first order, this variations turn out to satisfy

$$\delta M = \frac{1}{8\pi}\kappa\delta A + \Omega\delta J + \Phi\delta Q + \dots , \tag{1.59}$$

where Ω is the angular velocity, and Φ the electrostatic potential.

If Einstein's equation is satisfied for a matter content satisfying the null energy condition (which states that for every future-pointing null vector field V^μ, the stress tensor satisfies: $T_{\mu\nu}V^\mu V^\nu \geq 0$) and the black hole is *strongly future asymptotically predictable*, *i.e.*, if there exists a globally hyperbolic region (such that it contains a *Cauchy surface*, *i.e.*, one which is crossed by every inextensible null and time-like curve once, and only once) containing $I^-(\mathcal{I}^+) \cup \mathcal{H}$ (where $\mathcal{H}$ is the horizon of a black hole in that spacetime), it can be shown that the expansion θ (which measures how much geodesics infinitesimally close to each other in a congruence expand in average) satisfies $\theta \geq 0$ in all points of $\mathcal{H}$. As a consequence, the area of the event horizon A corresponding to a black hole contained in a spacetime of this kind can never decrease with time, as discovered by Hawking [233].

$$\delta A \geq 0\,. \tag{1.60}$$

This is the *second law of black hole mechanics*, in analogy with the second law of thermodynamics, which establishes that the entropy of an isolated system cannot decrease.

The mathematical analogy between these and the laws of thermodynamics is broken with the Planck-Nernst form of the third law of thermodynamcis, which establishes that $S \to 0$ as $T \to 0$. Indeed, there are *extremal* black holes (*i.e.*, those with $\kappa = 0$) with finite area. In any case, the analogy seems to hold in the formulation according to which it is not possible for a thermodynamic system (black hole) to reach $T \to 0$ ($\kappa \to 0$) by means of a finite number of physical processes.

In spite of the suggesting analogy we have established between the laws satisfied by magnitudes characterizing black holes, and thermodynamics, the strict interpretation of M, A or κ as thermodynamic quantities does not make sense in GR. Although identifying the mass of the black hole M with the energy looks fine, the temperature of a black hole in GR is strictly zero, since it does not radiate anything. This of course difficults the identification of T and κ. As a consequence, identifying the conjugate variable of T, S with the area of the black hole looks also dubious.

Interestingly enough, when quantum effects are taken into account, the situation turns out to change dramatically. Indeed, as discovered by Hawking in 1974 [234], black holes emit radiation as perfect black bodies at a temperature

$$T = \frac{\kappa}{2\pi}\,. \tag{1.61}$$

Thus, surface gravity is after all related to the physical temperature of the black hole. This is also supporting evindence for the proportionality between the area of the black hole and its physical entropy. Using the first law of black hole mechanics and (1.61) one can fix the proportionality constant to be $1/4$, so $S_{\rm bh} = \frac{A}{4G}$.

Actually, some time before Hawking discovered the relationship between surface gravity and temperature, Bekenstein [39, 40] had already proposed a generalization of the second law of thermodynamics for systems including a black hole as a subsystem. The motivation for this arised, in addition to the formal analogies between black holes mechanics and thermodynamics, from the fact that the absortion of matter by a black hole would produce a decrease in the total entropy of the Universe. In order to overcome this problem, Bekenstein had proposed that black holes actually have an entropy proportional to its area, so the second law of thermodynamics would read now $\delta S' \geq 0$, where $S' \equiv S + S_{\rm bh}$, with $S_{\rm bh}$ the entropy of the black hole, and S that of the rest of the Universe.

In summary, the laws of black hole mechanics can be regarded as particular cases of the laws of thermodynamics applied to systems which contain black holes. Now, just like the laws of statistical physics underlie the laws of thermodynamics, it is reasonable to expect that black hole mechanics is determined by the dynamics of certain microscopic degrees of freedom. In the context of string theory, the counting of microstates can be achieved for certain families of extremal and near-extremal black holes [34, 54, 80, 81, 239, 312, 403] as observed by Strominger and Vafa for the first time in [403] (see more at the end of section (3.4)).

1.2.3 Black holes and supersymmetry

In a SUGRA theory[19], a configuration of fields is said to be supersymmetric or *BPS* (from Bogomol'ny-Prasad-Sommefield) if it preserves some supersymmetry, *i.e.*, if (see the previous section for notation) [354]

$$\delta_\epsilon B \sim \bar{\epsilon} F = 0\,, \tag{1.62}$$

$$\delta_\epsilon F \sim \partial\epsilon + B\epsilon = 0\,, \tag{1.63}$$

for at least one spinor ϵ. For purely bosonic configurations, the first equation is trivially satisfied, while the second is known as *Killing spinor equation* (KSE) and ϵ is a Killing spinor if there exists a solution for the corresponding equation. If a given configuration is invariant under the maximum number of independent Killing spinors, it is said to be *maximally supersymmetric*.

It is possible to couple a Rarita-Schwinger spin 3/2 field Ψ_μ to gravity [183, 353]. The resulting theory is pure $\mathcal{N} = 1$, $d = 4$ SUGRA. If we now truncate the gravitino, the bosonic solutions of the original theory (vacuum GR) will still be solutions of the SUGRA theory. In particular, Schwarzschild's metric

$$\mathbf{g} = \left(1 - \frac{2M}{r}\right) dt \otimes dt - \left(1 - \frac{2M}{r}\right)^{-1} dr \otimes dr - r^2 \mathbf{h}_{S^2}\,, \tag{1.64}$$

$$\mathbf{h}_{S^2} = d\theta \otimes d\theta + \sin^2\theta d\phi \otimes d\phi\,, \tag{1.65}$$

which is the unique static and spherically symmetric solution of the vacuum Einstein equation (Birkhoff's theorem) will also be a solution of $\mathcal{N} = 1$, $d = 4$ SUGRA. However, it is not supersymmetric, because the corresponding KSE

$$\delta_\epsilon \Psi_\mu|_{\text{Schw.}} = 0 \tag{1.66}$$

has no solutions. On the other hand, Minkowski spacetime is maximally supersymmetric, as it preserves *four supersymmetries*, corresponding to the components of a 4D Majorana spinor. Obviously, in the asymptotic region $r \to \infty$, the supersymmetries are recovered for Schwarzschild's solution, a characteristic which will be common to all asymptotically flat solutions.

Let us consider now the *Reissner-Nordström* (RN) solution, which is the only asymptotically flat, static and spherically symmetric solution of the Einstein-Maxwell system

$$\mathbf{g} = \left(1 - \frac{2M}{r} + \frac{q^2}{r^2}\right) dt \otimes dt - \left(1 - \frac{2M}{r} + \frac{q^2}{r^2}\right)^{-1} dr \otimes dr - r^2 \mathbf{h}_{S^2}\,, \tag{1.67}$$

where q is the electric charge of the solution. In this case, the solution has two horizons at $r_\pm = M \pm \sqrt{M^2 - q^2}$: the inner horizon r_-, which is a *Cauchy horizon*, and the event horizon r_+.

According to the *cosmic censorship hypothesis*, it is not possible that a *naked singularity* (*i.e.*, one which is not hidden behing an event horizon) is produced dynamically by means of any physical process with a physically reasonable matter content (such that

[19] A nice introduction to the relation between black holes and supersymmetry in SUGRA theories can be found in [47].

it satisfies certain energy conditions). In the case of the RN black hole, the condition $M^2 \geq q^2$ must be fulfilled in order to avoid the absence of event horizon. This relation is remarkably similar to the *BPS bound* relative to the stability of solitons in gauge theories [373]. When the bound is saturated, $M^2 = q^2$, both horizons coincide, and the black hole becomes extremal, because

$$T = \frac{\kappa}{2\pi} = \frac{r_+ - r_-}{4\pi r_+^2} = \frac{\sqrt{M^2 - q^2}}{2\pi^2 r_+^2} = 0\,. \tag{1.68}$$

From the point of view of SUSY, what happens is the following. The Einstein-Maxwell theory can be consistently embedded in pure $\mathcal{N} = 2$, $d = 4$ SUGRA by introducing two gravitini Ψ_μ^L, $L = 1, 2$ (and nothing else). For generic values of M and q, RN's black hole does not preserve any of the 8 supercharges. However, the saturation of the bound $M^2 = q^2$ makes the KSE

$$\delta_\epsilon \Psi_\mu|_{\text{RN extr.}} = 0 \tag{1.69}$$

have four independent solutions, making a $\frac{1}{2}$-*BPS* solution of it. In addition, the solution can be interpreted as a soliton, as it interpolates between two maximally supersymmetric vacua of the theory, namely: the near horizon solution $AdS_2 \times S^2$, and the asymptotic solution, Minkowski.

An interesting property of an extremal solution is that its entropy depends only on the quantized electric and magnetic charges of the solution, which is a key feature that allows for a comparison between the macroscopic (Bekenstein-Hawking) and the microscopic (string theory) entropy. In addition, in supersymmetric black holes all the information relative to the asymptotic values of the fields is lost in the horizon. In that case, if we understand the trip between spatial infinity and the horizon as a flow, the scalars lose all memory about the *initial configuration* and are *attracted* by certain configurations known as *attractors*. In non-supersymmetric extremal cases, the system also contains fixed points at the horizon, but these depend in general on the asymptotic values of the scalars. All these properties are a direct consequence of the so-called *attractor mechanism* [46, 48, 127, 173–177, 210, 384, 414] for extremal black holes, which establishes that the scalars of any extremal black-hole solution of a broad class of theories flow from arbitrary values at spatial infinity to others completely fixed in the horizon, and which in the supersymmetric cases are determined uniquely by the quantized charges of the black hole. This mechanism is rather general, and turns out to work for any theory whose action can be described by (1.14), particularly in SUGRA theories.

It can be seen that the most general form of a static and spherically symmetric black hole for an action of the form (1.14) is given by [173]

$$\begin{aligned} \mathbf{g} &= e^{2U(\tau)} dt \otimes dt - e^{-2U(\tau)} \gamma_{\underline{mn}} dx^{\underline{m}} \otimes dx^{\underline{n}}\,, \\ \gamma_{\underline{mn}} dx^{\underline{m}} \otimes dx^{\underline{n}} &= \frac{r_0^2}{\sinh^2 r_0 \tau} \left[\frac{r_0^2}{\sinh^2 r_0 \tau} d\tau \otimes d\tau + h_{S^2} \right]\,, \end{aligned} \tag{1.70}$$

where τ is a (inverse) radial coordinate and r_0 is the *non-extremality parameter*, which vanishes for extremal configurations. In such a case, the outer horizon is covered by $\tau \in (-\infty, 0)$, with the horizon at $\tau \to -\infty$ and spatial infinity at $\tau \to 0^-$. The inner Cauchy horizon is covered by (τ_S, ∞), with the horizon at $\tau \to \infty$, and the singularity at some positive and finite τ_S [190]. Under the assumption that the spacetime is static and spherically symmetric, all the fields of a theory given by an action of the form (1.14) depend only on τ. Maxwell equations can be explicitly integrated, so the vector fields can be obtained as functions of the radial coordinate and the electric q_Λ and magnetic charges of the solution p^Λ. Solving the equations of motion of the theory turns out to be equivalent to solving the one-dimensional equations of motion for $U(\tau)$ and $Z^i(\tau)$ corresponding to the so-called *FGK* effective action [173]

$$I_{\rm FGK}[U, z^i] = \int d\tau \left\{ (\dot{U})^2 + \mathcal{G}_{i\bar{j}}\dot{Z}^i\dot{\bar{Z}}^{\bar{j}} - e^{2U}V_{\rm bh}(Z,\bar{Z},\mathcal{Q}) \right\} , \tag{1.71}$$

together with the *Hamiltonian constraint*

$$(\dot{U})^2 + \mathcal{G}_{i\bar{j}}\dot{Z}^i\dot{\bar{Z}}^{\bar{j}} + e^{2U}V_{\rm bh}(Z,\bar{Z},\mathcal{Q}) = r_0^2 , \tag{1.72}$$

where $V_{\rm bh}(Z,\bar{Z},\mathcal{Q})$ is the so-called *black hole potential*, which is defined as

$$V_{\rm bh}(Z,\bar{Z},\mathcal{Q}) \;\; \equiv \;\; \frac{1}{2}\mathcal{M}_{MN}(\mathcal{N})\mathcal{Q}^M\mathcal{Q}^N , \tag{1.73}$$

$\mathcal{Q}^M$ being the symplectic $(2n_V+2)-$dimensional vector of charges

$$(\mathcal{Q}^M) = \begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix} , \tag{1.74}$$

and $\mathcal{M}_{MN}(\mathcal{N})$ is a symplectic and symmetric matrix defined in terms of $I \equiv \Im\mathfrak{m}(\mathcal{N})$ and $R \equiv \Re\mathfrak{e}(\mathcal{N})$ as

$$(\mathcal{M}_{MN}(\mathcal{N})) \equiv \begin{pmatrix} I + RI^{-1}R & -RI^{-1} \\ & \\ -I^{-1}R & I^{-1} \end{pmatrix} . \tag{1.75}$$

Thus, this dimensional reduction allows to simplify the problem of finding black-hole solution of this kind to a mechanical problem for the $(2n_V+1)$ variables corresponding to the complex scalars Z^i and the metric factor U.

Let us now see how the attractor mechanism works in this formalism. In the extremal limit, $r_0 \to 0$, (1.70) is given by

$$\mathbf{g} = e^{2U(\tau)}dt\otimes dt - e^{-2U(\tau)}[\delta_{ab}dx^a\otimes dx^b] , \tag{1.76}$$

where x^a $(a = 1,2,3)$ are Euclidean coordinates. It can be seen that in this situation,

$$\lim_{\tau\to-\infty} e^{-2U} = \frac{A}{4\pi}\lim_{\tau\to-\infty}\tau^2 , \quad \lim_{\tau\to-\infty}\tau\frac{d\phi^i}{d\tau} = 0 , i = 1, ..., n_v , \tag{1.77}$$

where A is the area of the horizon. Using this equation and assuming that the scalars do not diverge at the horizon, it is easy to show from the 1-dimensional equation of motion corresponding to these fields that [386]

$$\lim_{\tau\to-\infty}\phi^i = \phi^i_h , \quad \mathcal{G}^{ij}(\phi_h)\partial_j V_{\rm bh}(\phi_h) = 0 , \tag{1.78}$$

thus, assuming the metric of the scalar manifold not to be degenerate, it follows that

$$\partial_j V_{\rm bh}(\phi_h) = 0 , \tag{1.79}$$

i.e., the possible values of the scalars at the horizon correspond to critical points of the black-hole potential. If $V_{\rm bh}$ has no flat directions (something untrue in general), then (1.79) is a compatible system of n_v independent equations, the value of all the scalars gets fixed in the horizon in terms of the black hole charges. Similarly, in the extremal case (supersymmetric or not), it can be seen [386] that the entropy of the solution is given by

$$S = \pi V_{\rm bh}(\phi_h(\mathcal{Q})) = 0\,. \tag{1.80}$$

And not only that: regardless of whether the scalars at the horizon depend on their asymptotic values (if $V_{\rm bh}$ has flat directions) or not, the entropy of any extremal solution depends only on the quantized charges of the solution, without any dependence on these asymptotic moduli. An additional property of extremal black holes entropy is the fact that it turns out to equal the square-root of the products of the entropies of the inner and outer horizons, so this product remains moduli-independent for non-extremal solutions (see *e.g.*, [211] for details). Examples of all these properties can be found in chapters 2, 3 and 5.

We mentioned before that Schwarzschild's is the only static and spherically-symmetric solution of Einstein's equation in the vacuum. Similar uniqueness theorems exist for less symmetric spacetimes, such as Kerr-Newmann [318] as the only stationary, axisymmetric and electrovacuum solution, and possible extensions to systems with more fields have been studied intensively [38, 41, 143]. In the context of SUGRA theories, there is a common belief that given a black-hole solution with some degree of symmetry and all the charges and fields active, that must be the unique solution of the kind. In chapter 4 we show how this is not the case in general for $\mathcal{N}=2$, $d=4$ SUGRA theories, by constructing a static and spherically symmetric black-hole solution (to a rather exotic, but otherwise well- and uniquely-defined model) whose physical fields depend on a function which is bivalued in the real numbers, something that allows one to choose one of the two branches (which are not symmetric in any sense) to construct two different solutions which share the same physical charges as well as asymptotic values of the scalar fields (as we will see, our construction seems to entail certain stability issues though).

1.2.4 The H-FGK formalism

In Refs. [324,331] it was shown that the problem of finding static, single-center, spherically-symmetric black-hole solutions of any ungauged $\mathcal{N}=2$, $d=4$ SUGRA theory coupled to n_v vector multiplets can be reduced to that of finding solutions to the effective action for the $2(n_v+1)$ real variables $H^M(\tau)$

$$-\,I_{\text{H-FGK}}[H] = \int d\tau \left\{\tfrac{1}{2}g_{MN}\dot{H}^M\dot{H}^N - V\right\}\,, \tag{1.81}$$

subject to the *Hamiltonian constraint*

$$\tfrac{1}{2}g_{MN}\dot{H}^M\dot{H}^N + V + r_0^2 = 0\,, \tag{1.82}$$

where r_0 is again the non-extremality parameter. The H-FGK action plus Hamiltonian constraint are equivalent to the FGK formulation (see the previous section), although now we have an extra variable which introduces a gauge freedom in the system [189]. The equations of motion that follow from the above action read [189, 331]

$$g_{MN}\ddot{H}^N + (\partial_N g_{PM} - \tfrac{1}{2}\partial_M g_{NP})\dot{H}^N\dot{H}^P + \partial_M V = 0\,. \tag{1.83}$$

The metric $g_{MN}(H)$ and the potential $V(H)$ of the H-FGK effective action are given in terms of the so-called *Hesse potential* $\mathsf{W}(H)$ by

$$g_{MN}(H) \equiv \partial_M\partial_N \log \mathsf{W} - 2\frac{H_M H_N}{\mathsf{W}^2}\,, \tag{1.84}$$

$$V(H) \equiv \left\{-\tfrac{1}{4}\partial_M\partial_N \log \mathsf{W} + \frac{H_M H_N}{\mathsf{W}^2}\right\}\mathcal{Q}^M\mathcal{Q}^N\,, \tag{1.85}$$

where $\mathcal{Q}^M = (p^\Lambda, q_\Lambda)^T$ is the symplectic vector of magnetic and electric charges. The Hesse potential contains all the information needed to characterize the $\mathcal{N}=2, d=4$ SUGRA theory under consideration, and defines it (at least in this context) just like the canonically-normalized covariantly-holomorphic symplectic section $\mathcal{V}^M$ does. The black-hole potential is related to the potential V appearing in the H-FGK action by

$$V_{\rm bh} = -\mathsf{W}\ V\,, \tag{1.86}$$

as a function of the variables H^M, and it is always extremized by the near-horizon value $B^M = \beta\mathcal{Q}^M$ for any proportionality constant β.

W can be derived from $\mathcal{V}^M$ as follows:

1. Introduce an auxiliary complex variable X with the same Kähler weight as $\mathcal{V}^M$. Then we can define the two Kähler-neutral real symplectic vectors $\mathcal{R}^M$ and $\mathcal{I}^M$

   $$\mathcal{V}^M/X \equiv \mathcal{R}^M + i\mathcal{I}^M\,. \tag{1.87}$$

   The components of $\mathcal{R}^M$ can be expressed in terms of those of $\mathcal{I}^M$ by solving the *stabilization equations* a.k.a. *Freudenthal duality equations* [191]. The functions $\mathcal{R}^M(\mathcal{I})$ are characteristic of each theory, but they are always homogeneous of first degree in the $\mathcal{I}^M$.

   It can be shown that

   $$X = \tfrac{1}{\sqrt{2}}e^{U+i\alpha}\,, \tag{1.88}$$

   where e^U is the metric function (see (1.70)) and α is an arbitrary τ-dependent phase which does not enter in the Lagrangian: different choices of α give different definitions of the variables H^M which, nevertheless, describe the same physical variables. This freedom gives rise to a local symmetry of the H-FGK action, known as *local Freudenthal duality* [189].

2. Given those functions, the *Hesse potential* $\mathsf{W}(\mathcal{I})$ is just

   $$\mathsf{W}(\mathcal{I}) \equiv \mathcal{R}_M(\mathcal{I})\mathcal{I}^M\,, \tag{1.89}$$

   so it is, by construction, homogeneous of second degree in $\mathcal{I}^M$.

It is customary to relabel these variables

$$H^M \equiv \mathcal{I}^M\,, \qquad \tilde{H}^M \equiv \mathcal{R}^M\,, \longrightarrow \left\{\begin{array}{rcl} \mathcal{V}^M/X &=& \tilde{H}^M + iH^M \equiv \mathcal{H}^M\,. \\ \mathsf{W}(H) &=& \tilde{H}_M(H)H^M\,. \end{array}\right. \tag{1.90}$$

The physical fields of the solution can be obtained from the H variables as

$$Z^i = \frac{\mathcal{V}^i/X}{\mathcal{V}^0/X} = \frac{\tilde{H}^i(H) + iH^i}{\tilde{H}^0(H) + iH^0}\,, \qquad e^{-2U} = \frac{1}{2|X|^2} = \tilde{H}_M(H)H^M\,. \tag{1.91}$$

The main advantages of this formalism can be enumerated as follows (for more details see chapter 2)

- The H variables transform linearly under the electric-magnetic duality group G of the theory:

$$H^{M\,\prime} = S^M{}_N H^N\,, \qquad (S^M{}_N) \in \mathrm{G} \subset \mathrm{Sp}(2n_v + 2; \mathbb{R})\,, \tag{1.92}$$

  which implies that any solution must be of the form

$$H^M(\tau) = c^\sigma(\tau)\; U_\sigma^M\,, \tag{1.93}$$

  where the functions $c^\sigma(\tau)$ are duality invariant; the symplectic vectors U_σ^M are constant vectors that may depend on the physical parameters of the theory (mass M, electric and magnetic charges $\mathcal{Q}^M$ and asymptotic values of the scalars Z^i_∞) and must be *equivariant* w.r.t. the duality group, i.e.

$$U_\sigma^M(M, Z'_\infty, Z^{*\prime}_\infty, \mathcal{Q}') = S^M{}_N U_\sigma^N(M, Z_\infty, Z^*_\infty, \mathcal{Q})\,, \tag{1.94}$$

  with

$$Z^{i\,\prime} \equiv F^i_S(Z)\,, \qquad \mathcal{Q}^{M\,\prime} = S^M{}_N \mathcal{Q}^N\,, \tag{1.95}$$

  where $F^i_S(Z)$ is the non-linear realization of the duality transformation $S^M{}_N$ on the complex scalars. In the cases in which the number of equivariant vectors of the model is smaller than the number of H^Ms, we will automatically be left with a small number of invariant functions to be determined. The identification of equivariant vectors U_σ^M in a given model allows to produce Ansätze of the form (1.93), which can be used to solve the H-FGK equations of motion and obtain the $c^\sigma(\tau)$, which would completely determine a solution to the full SUGRA equations of motion.

- Contracting (1.83) with H^M and using the homogeneity properties of the different terms and the Hamiltonian constraint Eq. (1.82) one finds

$$\tilde{H}_M\left(\ddot{H}^M - r_0^2 H^M\right) + \frac{(\dot{H}^M H_M)^2}{\mathsf{W}} = 0\,. \tag{1.96}$$

  This allows to classify all SUGRA solutions of a given model in two categories: *conventional*, and *non-conventional*. In the extremal case ($r_0 = 0$), *conventional* solutions (all supersymmetric solutions belong to this class) correspond to variables $H^M(\tau)$ that are harmonic functions, *i.e.*, they satisfy $\ddot{H}^M = 0$. The above equation implies that they also satisfy the constraint [20]

$$\dot{H}^M H_M = 0\,, \tag{1.97}$$

[20]In the supersymmetric case this implies the absence of NUT charge [42].

the converse being untrue in general: the above constraint can be satisfied for black-hole solutions which are not given by harmonic H^Ms and that we will call *unconventional*. In the non-extremal cases, conventional solutions are those satisfying

$$\ddot{H}^M = r_0^2 H^M \,, \tag{1.98}$$

which correspond to hyperbolic functions reducing to the appropriate harmonic ones in the extremal limit. Actually, it can be shown that one can impose (1.97) without loss of generality (it can be understood as a convenient gauge-fixing condition) [189], although that does not necessarily mean that (1.98) is implied by (1.96) (actually in (1.96) one has only one equation, whereas in (1.98) there are $2(n_v + 1)$ of them).

- It works: the H-FGK formalism has been indeed used to construct new extremal and non-extremal solutions of different SUGRA theories [89, 90, 95, 96, 188, 191]. In fact, the solutions presented in chapters 3 and 4 would have been extremely difficult to obtain, if not impossible, without the H-FGK formalism.

1.3 Holography

In this section[21] we enter into the holographic world. After a more than brief introduction to the AdS/CFT correspondence, we review the rôle played by higher-order gravities in the holographic avatar, introduce the concept of entanglement entropy and its importance in holography, and review some general properties of Lifshitz geometries with hyperscaling violation (hvLf).

1.3.1 The AdS/CFT correspondence

In [311], Maldacena proposed his celebrated AdS/CFT correspondence. According to it, Type-IIB ST on $\mathrm{AdS}_5\times\mathrm{S}^5$ is physically equivalent to $\mathcal{N}=4$, $d=4$ Super-Yang-Mills with gauge group SU(N). Although the conjecture has not been *proven*, the overwhelming evidence accumulated so far strongly supports it. This is especially so in the large-N (where N is the dimension of the SU(N) matrices) limit, in which the field theory becomes infinitely strongly coupled while the ST side reduces to classical 5-dimensional SO(6) gauged SUGRA.

Grosso modo, in the large-N limit of the correspondence asymptotically AdS_5 geometries correspond to states in the $\mathcal{N}=4$, $d=4$ SYM theory living in the asymptotic boundary of those (*e.g.*, the vacuum state corresponds to pure AdS_5, whereas a thermal state is given by an asymptotically AdS_5 black hole), with the nice feature that the extra (spatial) dimension encodes the renormalization group flow of the field theory.

As anticipated before, the AdS/CFT correspondence [219, 311, 423] is the first realization of the holographic principle [404, 409]. This is nothing but the observation that, in gravity theories, the entropy contained in a certain volume cannot exceed the entropy of a black hole fitting inside such a volume (you could always create a black hole by throwing more matter into the volume, which would increase the entropy) and, as already commented, the entropy of the black hole is proportional to its horizon area. This seems to suggest that the degrees of freedom of any $(d+1)$-dimensional gravity theory can be described in terms of certain degrees of freedom living in some d-dimensional surface. The reasoning is expected to hold for general gravity theories, something that, after Maldacena's discovery, motivates the broader name *gauge/gravity duality* (or *holography*), which makes reference to several more or less robustly established correspondences in the spirit of Maldacena's, between several gravity and gauge theories in various dimensions.

A particularly nice property of the gauge/gravity duality is the fact that it allows to study the strongly coupled regime of (certain) QFTs, which is inaccessible with the standard field theory techniques, using the tools of Riemannian geometry and GR. This approach has been used, for example, to study aspects of less symmetric field theories, whose dual is given by gravity theories living in non-maximally symmetric spacetimes. From the opposite perspective, the duality provides an unrivalled framework for studying the emergent quantum nature of spacetime and gravity.

[21] Please note that by the symbol d we are meaning different things depending on the chapter/section. In the previous sections, d stood for the number of spacetime dimensions. This is the notation used so far, and the one we use in chapters 2, 3, 4, 5 and 6. However, in this section as well as in chapter 9, d will stand for the number of spacetime dimensions of the dual QFT (so $d+1$ is the spacetime dimension of the gravity theory). Finally, in chapters 7 and 8, d will stand for the number of spatial dimensions of the dual QFT (so $d+2$ is the spacetime dimension of the gravity theory).

In spite of the colossal effort and the tremendous success made so far in these directions, there are still multiple formidable challenges for holography. On the one hand, one would like to extend this duality to *realistic* QFTs, such as QCD, in which a first-principles description of the phenomena observed at the strongly coupled regime is lacking (how great would it be to explain *confinement* holographically? [424]). On the other, the holographic principle should work for gravity theories in general backgrounds and not only for AdS spacetimes. However, constructing asymptotically flat or de Sitter versions of the correspondence has proven to be strikingly difficult (the fact that the AdS case is simpler is intimately related to the fact that the near horizon limit of supersymmetric black objects, which play an special rôle in ST due to the privileges this symmetry confers on them within the theory, generically contains an AdS factor). In addition, our knowledge of the AdS/CFT correspondence is fairly limited if we slightly move away from the large-N limit, not to mention if we try to consider the full Type-IIB string theory.

1.3.2 Holography and higher-order gravities

A particular issue with holography we have already mentioned arises when we try to move from the large-N limit in the dual CFT. In that situation, α' corrections appear in the effective SUGRA equations of motion in the form of higher-order terms in the Riemann tensor (and the rest of fields). For example, the first stringy corrections to the 10-dimensional Type-IIB action [56][22] appear at third order in α'

$$I_{\rm IIB} = I^{(0)}_{\rm IIB} + \alpha'^3 I^{(1)}_{\rm IIB} + ... \tag{1.99}$$

where $I^{(0)}_{\rm IIB}$ stands for the classical Type-IIB SUGRA action, and the dots refer to higher-order corrections. For gravity solutions of the form $\mathcal{A}_5 \times \mathrm{S}^5$, where $\mathcal{A}_5$ is an Einstein manifold of negative curvature, the 10-dimensional gravity sector of the theory at this order is equivalent to the following 5-dimensional action [86]

$$I = \frac{1}{16\pi G}\int d^5x\sqrt{-g}\left[R + \frac{12}{L^2} + \frac{\zeta(3)\alpha'^3}{8}W\right], \tag{1.100}$$

where L is some length parameter which coincides with the AdS radius in the absence of the cubic term, and

$$W \equiv \left[C^{\rho\mu\nu\kappa}C_{\phi\mu\nu\eta} + \frac{1}{2}C^{\rho\kappa\mu\nu}C_{\phi\eta\mu\nu}\right]C_{\rho}{}^{\iota\psi\phi}C^{\eta}{}_{\iota\psi\kappa}\,, \tag{1.101}$$

where

$$C_{\mu\nu\rho\sigma} \equiv R_{\mu\nu\rho\sigma} - \frac{2}{3}\left[g_{\mu[\rho}R_{\sigma]\nu} - g_{\nu[\rho}R_{\sigma]\mu}\right] + \frac{1}{6}Rg_{\mu[\rho}g_{\sigma]\nu} \tag{1.102}$$

is the five-dimensional Weyl tensor.

As we can see, in this case the first correction to the Einstein-Hilbert action appears at fourth order in curvature. However, in generic situations in arbitrarty dimensions one expects corrections already at second order (which would mean $\mathcal{O}(\alpha')$), *i.e.*, something

[22]Strictly speaking, Type-IIB SUGRA has no action, because of the self-duality condition of the Ramond-Ramond 4-form field strength, but you know what I mean.

like

$$I = \frac{1}{16\pi G}\int d^{(d+1)}x\sqrt{-g}\left[R+\frac{d(d-1)}{L^2}+L^2\left\{\alpha_1 R^2+\alpha_2 R_{\mu\nu}R^{\mu\nu}+\alpha_3 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right\}+...\right], \tag{1.103}$$

where the α_i would be dimensionless constants, and the dots stand for higher-order terms.

The interest in considering higher-order corrections to the SUGRA action in the context of holography is not restricted to the obvious fact that it allows us to probe the CFT in a regime which is not infinitely strongly coupled. In some sense, the situation is similar to that found (for instance) when studying the Hydrogen atom energy states: when relativistic corrections are considered, several accidental degeneracies are broken, and the structure proves to be much richer than that suggested by the non-relativistic calculation. A neat example of this analogy was found in [84], where the authors showed that considering higher-order terms in the holographic game allowed for a violation of the previously [288] presumed universal bound for the ratio of shear viscosity to volume entropy density ($\eta/s = 1/4\pi$ in natural units) obtained for AdS black hole horizons. In that case, *the degeneracy* is broken in the sense that the new higher-order terms produce corrections of the form $\eta/s = 1/4\pi\,(1 + c\alpha + ...)$, where α stands for a generic dimensionless higher-order coupling, and c is some constant, so depending on the theory, the value of the ratio will be larger or smaller than the previously claimed universal lower bound. In a similar fashion, including higher-order terms allows to study hidden connections between apparenty unrelated observables. Consider for example the thermal entropy density of an AdS black hole and the two-point function of the holographic stress tensor. In general, the first scales with the temperature through an expression of the form[23] $s \sim T^{(d-1)}$ (here $d+1$ is the number of dimensions of the bulk theory), whereas the second looks something like [169, 356] $\langle T_{ab}(x)T_{cd}(0)\rangle \sim \mathcal{I}_{ab,cd(x)}/x^{2d}$, where $\mathcal{I}_{ab,cd(x)}$ is some tensor irrelevant for our discussion here. The point is now the following: if we compute these quantities in gravity theories which do not include higher-order terms, the constants in front of these two expressions are certain dimensionless numbers which have nothing to do with each other (in particular, the second one turns out to equal the central charge of the dual CFT when $d = 4$ [85]). However, if we include a generic higher-order term with coupling constant α, the expressions will be modified to something like $s \sim (1 + c_1\alpha)T^{(d-1)}$,

$$\langle T_{ab}(x)T_{cd}(0)\rangle \sim (1 + c_2\alpha)\mathcal{I}_{ab,cd(x)}/x^{2d}, \tag{1.104}$$

so a structure of the same kind appears in both relations, which makes one wonder, for example, in which cases might $c_1 = c_2$? What would that tell us about the relation between these observables? Is there a finite number of constants c_i characterizing a higher-order theory which appear repeatedly as we compute different observables? And so on. This is the kind of questions we address in chapter 9. As a remarkable result, we will find that two well-defined and unambiguous charges κ and σ, which can be obtained from the holographic entanglement entropy corresponding to an entangling region with a geometric singularity in a CFT_3 (see below), turn out to be such that the ratios κ/C_T and σ/C_T (where C_T is the proportionality constant in the stress tensor two-point function, so $\langle T_{ab}(x)T_{cd}(0)\rangle = C_T\mathcal{I}_{ab,cd(x)}/x^{2d}$ in general) does not change when we introduce different higher-order terms. This leads us to wonder whether such ratios, which read

$$\frac{\kappa}{C_T} = \frac{\Gamma\left(\frac{3}{4}\right)^4\pi^2}{6}, \tag{1.105}$$

[23]We will see that this behaviour is changed for geometries with hyperscaling violation.

$$\frac{\sigma}{C_T} = \frac{\pi^2}{24}\,, \tag{1.106}$$

are universal. By comparing these results with the free field results which we can compute using certain computations available in the literature, we find that κ/C_T is not universal for general theories. However, very remarkably, we find that σ/C_T is equal to the holographic result both for a free scalar and a free fermion, which suggests that this ratio is indeed a universal number which might well be the same for general conformal field theories. On the one hand, this universality strongly supports the validity of the holographic prescriptions for the computation of HEE. On the other, it explains why the results for σ corresponding to the fermion and the scalar are equal (up to a factor 2), a phenomenom which had been found (only within some accuracy range) withouth explanation in [119]. Finally, we use the universality of this ratio to improve the available free field results for σ. In particular, holography remarkably allows us to compute σ_{scalar} and σ_{fermion} exactly.

A final comment about higher-order gravities (applying not only in the context of holography) is the fact that they are usually not well behaved quantum mechanically, as they suffer from certain pathologies such as ghosts. In particular, in higher-order gravities the metric contains new degrees of freedom, something related to the fact that the linearized equations of motion are no longer of second order. In order to study their behaviour, one can consider an analogy [341] consisting of a massless scalar field whose equation of motion has been corrected with a fourth-order term, just like the equation of motion for a graviton would be in a curvature-squared gravity

$$\left(\Box + \frac{\lambda}{M^2}\Box^2\right)\phi = 0\,, \tag{1.107}$$

$M^2 \sim 1/L^2$ being some high energy scale. Then, the propagator for the field will read

$$\frac{1}{q^2\left(1-\lambda q^2/M^2\right)} = \frac{1}{q^2} - \frac{1}{q^2 - M^2/\lambda}\,. \tag{1.108}$$

Thus, the $q^2 = 0$ pole will be the usual massless mode, whereas that with $q^2 = M^2/\lambda$ will be related to new massive states. However, independently of the sign of λ, these extra degrees of freedom will contribute negatively to the propagator, so they are ghosts. Fortunately, not all higher-order gravities present this behaviour, and even if they do, the masses of these modes will generically be at the Planck scale, where the low energy field theory description is not valid anymore, making their physical interpretation obscure.

We will talk again about higher-derivative gravities in the next subsection, which is devoted to the subject of holographic entanglement entropy.

1.3.3 Holographic entanglement entropy

There are many observables one can wish to compute in a QFT (specially in the perturbative regime), such as correlation functions of local operators. Non-local quantities are also important, and usually encode valuable physical information. An example of such quantities is entanglement entropy (EE). For a particular QFT and a spatial region A, the EE between de degrees of freedom in A and those in the complement $\bar{A}$ is defined as: $S = -\mathrm{Tr}\left[\rho_A \log \rho_A\right]$, where ρ_A is the reduced density matrix obtained by integrating out the degrees of freedom in $\bar{A}$.

Entanglement entropy has become an ubiquitous tool in fields as diverse as condensed matter [8,218,292,364], quantum information [345,425], string theory and quantum gravity [68, 73, 101, 236, 295, 310, 336, 347, 378, 379, 400, 415], and QFTs [100, 117, 119, 123, 275, 276, 388].

As its name suggests, EE is a measure of the degree of entanglement between the degrees of freedoms living at A and $\bar{A}$, and has some nice properties which make it so popular. To begin with, it is computable for certain QFTs, which is already a non-trivial point. Also, as opposed to thermal entropy, EE is non-vanishing at zero temperature, so it can be used to probe the quantum properties of the ground state for a given quantum system. In addition, for two-dimensional CFTs it is proportional to the central charge [238], with similar relations occuring for higher-dimensional CFTs as well [85] (so it allows for a rough estimation of the number of degrees of freedom of the theory). In addition, it captures valuable information on renormalization group flows. In particular, it is possible to construct c-functions in various dimensions using EE [341]. Also, in holography, it seems to be a key observable to consider when trying to understand the emergent nature of spacetime or the physics of black holes (see, *e.g.*, [25, 68, 310, 348]).

The ultraviolet (UV) behaviour of EE for arbitrary $(d+1)$-dimensional QFTs is expected to be [119]:

$$S = \frac{k_{d-1}}{\delta^{d-1}} + \ldots + \frac{k_1}{\delta} + \gamma \log \frac{l}{\delta} + S_0 \,, \tag{1.109}$$

δ being a short distance cutoff, S_0, γ and k_{d-i} constants, and l a characteristic length of A. The coefficient in front of the leading term is proportional to the *area* of the boundary of A ($k_{d-1} \sim l^{d-1}$), a behaviour which is argued to be caused by the entanglement between degrees of freedom living at both sides of ∂A, and which is often referred to as the *area law* [73, 400] of EE. Another interesting term is the one proportional to $\log l/\delta$: while the constants k_{d-i} are unphysical, since they are not related to well-defined quantities in the continuum[24], the coefficient γ is expected to be independent of the regularization procedure. When d is even, γ is generically non-vanishing, whereas for odd-d QFTs the logarithmic divergence disappears, and S_0 is the universal term, unless the entangling region A contains a geometric singularity. Thus, when the boundary of the entangling region presents such kind of singularity (imagine for example a corner in $d = 3$), the EE contains a *universal* term of that form, where γ is a function of the opening angle of the singularity [118,120,180,237]. In chapter 9, we study how the introduction of higher-order gravities affects this universal contribution within the context of *holographic entanglement entropy*, which we are about to introduce.

As we have explained, holography allows us to study certain QFTs in the strongly coupled regime using GR tools, which tend to be computationally simpler *by several orders of magnitude*. In the case of entanglement entropy, this is especially true, as the QFT computation using standard methods is extremely challenging even in the weakly coupled or free regimes, while the holographic prescription we will present in a second is *very simple* and computationally powerful in lots of cases.

Holographic entanglement entropy (HEE) for theories dual to Einstein gravity can be computed using the Ryu-Takayanagi prescription [379][25]. According to it, the HEE for a certain region A living in the boundary of some asymptotically AdS_{d+1} spacetime is

[24] Indeed, if we shift $\delta \to \delta\epsilon$, the coefficient k_{d-i} changes as $k_{d-i} \to k_{d-i}\epsilon^{i-d}$.

[25] This prescription has been recently proven under certain conditions in [296].

given by

$$S = \underset{m \sim A}{\text{ext}} \left[\frac{\mathcal{A}(m)}{4G} \right] , \tag{1.110}$$

where m are codimension-2 bulk surfaces homologous to A with $\partial m = \partial A$, and $\mathcal{A}(m)$ is the $(d-1)$-dimensional volume (*area*) of m. Hence, HEE in theories with an Einstein gravity dual is obtained by extremizing the area functional over all possible bulk surfaces homologous to A whose boundary coincides with ∂A.

The Ryu-Takayanagi prescription is obviously reminiscent of the Bekenstein-Hawking formula for the entropy of black holes. Indeed, since the minimal surface tends to wrap the horizon when there is one [346], formula (1.110) can be regarded as a generalization of Bekenstein and Hawking's.

Besides reproducing all the known results available from CFTs in various dimensions, (1.110) can be used for numerous purposes. To name a few:

- The possibility of proving holographic c-theorems in various dimensions using the universal terms in the HEE along renormalization group flows [341].
- Establishing seemingly deep connections between AdS/CFT and renormalization group schemes, in which the geometry of AdS can be constructed as an emergent quantity [348].
- Providing supporting evidence for certain conjectures concerning the quantum nature of gravity [68], including the ER=EPR [310] one [405].

As we have said, (1.110) can be used for holographic theories dual to Einstein gravity. When higher-order gravities are considered, this needs to be somehow generalized. In view of the analogy between the Bekenstein-Hawking formula and the Ryu-Takayanagi one, it is tempting to guess that an analogous generalization of the Wald entropy formula for black holes in higher-order gravities [420]

$$S_{\text{Wald}} = \frac{1}{4G} \int_{\text{H}} d^{d-1}x \sqrt{h_{\text{H}}} \frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\sigma}} \epsilon_{\mu\nu} \epsilon_{\rho\sigma} , \tag{1.111}$$

(where h_H stands for the pull-back metric on the horizon H, $\mathcal{L}$ is the gravity Lagrangian and $\epsilon_{\mu\nu}$ is the binormal to the horizon) should work as well for the entanglement entropy. However, in [247] this guess was proven to be incorrect, since this expression would produce wrong universal terms. Alternative expressions producing the right terms are known for certain higher-order gravities. In particular, for Lovelock [247, 256], curvature-squared [185, 336], and f(Lovelock) [92, 381]. Quite remarkably, a formula for general gravity theories involving arbitrary contractions of the Riemann tensor $\mathcal{L}(R_{\mu\nu\rho\sigma})$, which satisfies several consistency checks, has been proposed by Dong [161] (see also [65, 103]). The key point which explains why Wald's formula is not suitable for HEE is that it does not involve extrinsic curvatures of the surface (horizon), something that makes sense in that case since Killing horizons have vanishing extrinsic curvatures. However, entangling surfaces will in general have non-vanishing extrinsic curvatures, appearing in the HEE formulas for higher-order gravities.

Computing the HEE for a simple enough entangling region A in a particular higher-order gravity (assuming we know what the right functional is) entails three steps. First, we need to parametrize the family of codimension-2 surfaces susceptible of extremizing the functional. Secondly, we need to extremize the functional by solving the corresponding Euler-Lagrange equations. Finally, we need to evaluate the on-shell functional including a short-distance cut-off δ so the final result is regularized. Step two is very involved in general, since the equations of motion one has to solve are usually of order higher than two in derivatives of the functions parametrizing the surface (as opposed to the case of Einstein gravity), and expansions for small values of the gravitational couplings are often the best one can do. Naturally, one finds that the surfaces extremizing the new functional are no longer minimal-area surfaces (see, *e.g.*, [85]), although remarkable exceptions occur (see chapters 8 and 9).

1.3.4 Lifshitz spacetimes with hyperscaling violation

As we explained in the introduction, the AdS/CFT correspondence has been lately *extended* in a variety of ways in the hope of accounting for the physics of more realistic quantum field theories, such as QCD and condensed matter systems (see, *e.g.*, [1,116,229,369]).

One possibility consists of considering systems in which, albeit scaling symmetry is respected, space and time do not scale in the same manner, so conformal (and Lorentz) invariance is broken. This is the case of the so-called Lifshitz fixed points. These are characterized by a *dynamical critical exponent* z, which determines the anisotropic scaling in the time direction t

$$t \to \lambda^z t\,,\; x_i \to \lambda x_i\,,\; i = 1, ..., d-1\,, \tag{1.112}$$

being x_i the $d-1$ spatial dimensions of the d-spacetime in which the field theory under consideration is defined. The class of $(d+1)$-dimensional dual spacetime geometries with the appropriate symmetries can be written as [261, 287, 410]

$$ds^2 = -\frac{L^2}{r^{2z}}dt^2 + \frac{L^2}{r^2}\left[dr^2 + d\vec{x}^2_{(d-1)}\right]\,, \tag{1.113}$$

which clearly reduces to AdS_{d+1} in the Poincaré patch for $z = 1$.

A further generalization can be achieved by considering the following family of spacetime metrics [131]

$$ds^2 = L^2 r^{\frac{2(\theta-d+1)}{d-1}}\left[-r^{-2(z-1)}dt^2 + dr^2 + d\vec{x}^2_{(d-1)}\right]\,. \tag{1.114}$$

These geometries (which are conformally Lifshitz) include, in addition to z, another exponent, customarily named θ, and are characterized by the following behavior under rescalings of the coordinates

$$t \to \lambda^z t\,,\; x_i \to \lambda x_i\,,\; r \to \lambda r\,,\; ds^2 \to \lambda^{\frac{2\theta}{d-1}} ds^2\,. \tag{1.115}$$

A system whose thermal entropy density scales as $s \sim T^{d-1}$ is said to possess a hyperscaling behaviour. When the dynamical exponent is present, this scaling gets modified to $s \sim T^{\frac{d-1}{z}}$. In field theories with the kind of scaling defined by (7.4), the thermal entropy scales in turn as $s \sim T^{\frac{d-1-\theta}{z}}$ [213, 246], and so, from the thermodynamic point of view, $d-1-\theta$ acts as the effective number of space-like dimensions of the system [246]. The fact that s does not scale with its naive power of the temperature is a violation of the hyperscaling behaviour [178, 246] (the hyperscaling case being obviously $\theta = 0$)[26], and the above class

[26] Holographically, this would correspond to the entropy of a black brane whose spacetime metric asymptotes to one of these solutions [250].

of metrics has been consequently named *hyperscaling-violating Lifshitz metrics* (hvLf in short). Although the $r^{\frac{2\theta}{d-1}}$ factor spoils dimensional analysis in (1.114), this can be easily restored by including an additional scale r_F: $r^{\frac{2\theta}{d-1}} \to (r/r_F)^{\frac{2\theta}{d-1}}$.

In order to have a clear interpretation of a constant-r slice (with $r \to 0$) of the geometry defined by (1.114) as the boundary of the metric, it is necessary to require $\theta < d-1$. From a different perspective, $\theta > d-1$ would correspond to a negative effective number of spatial dimensions according to the previous arguments. Also, when $\theta > 0$, hvLf metrics suffer from a curvature UV-singularity in the Einstein frame: indeed, the Kretschmann invariant scales as $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \sim r^{-4\theta/(d-1)}$. In appearance, this means that hvLf metrics with $\theta < 0$ are completely reliable in the UV, whereas those with $0 < \theta < d-1$ need to be completed asymptotically, something which is usually performed through the assumption that spacetime is described by (1.114) only above some scale r_F, but asymptotes to some well-behaved solution, such as AdS_{d+1}, as $r << r_F$. As explained in [136], this statement is imprecise. The authors argue that hvLf geometries with $\theta \neq 0$ typically require a UV-divergent (linear) dilaton, which allows one to tune the curvature singularity (appearing in the cases in which $0 < \theta < d-1$) by changing to an appropriate Weyl frame, and completely absorb it in such scalar field. The linear running character of the dilaton is a characteristic feature of general hvLf backgrounds (with $\theta \neq 0$) so one needs to be careful when interpreting the UV physics from the field theory perspective not only for $\theta > 0$, but also for $\theta < 0$. The situation is similar to that found for non-conformal branes, where the dual theory is known to be SYM (with $d \neq 4$). In that case, the dilaton, which is related to the YM coupling, also runs in the UV, which means that the theory is either asymptotically free or needs a UV completion (depending on the dimension). In order to determine what the case is, one needs the exact relation between the dilaton and the coupling. When the YM coupling blows up in the UV, SUGRA is not a valid description and S-duality needs to be used. For hvLf metrics, however, the dual theory is not known and the approach taken in the literature is more phenomenological/engineering-like since the SUGRA result is taken to *define* what is meant by the dual theory.

hvLf and asymptotically-hvLf solutions have been extensively (and intensively) studied in the context of holography in *e.g.*, [4,5,162,246,349,385]. The gravity models in which solutions of this kind have been found and studied include, for example, Einstein-Maxwell-Dilaton (EMD) [6,98,131,146,165,166,179,199,250,274,360], supergravity and string theory [9,87,88,162,220,343,361] and EMD plus curvature-squared terms [203,286,350]. The inclusion of higher-curvature terms in the gravitational action is motivated in the particular case of Lifshitz and hvLf geometries by, e.g., trying to change the (θ, z) parameter space allowed by the null energy conditions (NEC) or curing the characteristic infrared (IR) divergent behaviour of the dilaton [246] appearing in EMD theories (see [350] for details on these issues).

In the first part of chapter 8, we show that hvLf geometries are much more common than previously thought. In particular, in most of the previous papers devoted to the construction of solutions of this class (or others asymptoting to it in different regimes), hvLf metrics are obtained as solutions to Einstein-Maxwell-Dilaton models (although some of these are embedded in some SUGRA or ST). We find that hvLf geometries appear in the near-horizon and near-singularity limits of generic black-hole solutions of $\mathcal{N} = 2$, $d = 4$ SUGRA (in particular, these are theories without scalar potential, as oppossed to the previous known examples). We also show how solutions of this kind can be obtained from limiting procedures of standard solutions such as Schwarzschild's black hole, or by smearing of some other previously known solutions.

In the second part of chapter 8, on the other hand, we perform a systematic study of the existence of hvLf and asymptotically-hvLf geometries for a broad class of theories suitable for accommodating a general $\mathcal{N}=2$, $d=4$ SUGRA gauged with Fayet-Iliopoulos terms, and we construct new solutions of that kind (both purely hvLf and asymptotically hvLf) for a particular model of gauged SUGRA which can be embedded in Type-IIB string theory.

Coming back to entanglement entropy, we saw before that the generic UV behaviour of that quantity for generic QFTs included a leading term scaling with the *area* of the entangling surface (8.1). When this term is absent, or there is a different leading term, the area law is *violated*. This is the case of 2D CFTs, where EE scales logarithmically with the length of A, l, and γ turns out to be proportional to the central charge of the theory [100,238]

$$S=\frac{c}{3}\log\frac{l}{\delta}\,, \tag{1.116}$$

as we said before. In higher dimensional theories, violations of the area law appear in QFTs with Fermi surfaces [326,407,426]. In such cases, S acquires a logarithmic dependence on the characteristic length of A

$$S\sim(lk_F)^{(d-2)}\log(lk_F)\,, \tag{1.117}$$

being k_F the Fermi momentum[27], and the area law is violated. It has been argued that certain QFTs with Fermi surfaces might be holographically engineered by considering the family of hvLf metrics in the case $\theta=d-2$ [162,246,349]. Indeed, using holography, one can compute the HEE for a stripe of width l and infinite length $L_S\to+\infty$ (this length plays the role of an IR cut-off) [162]

$$S=\frac{L^{d-1}L_S^{(d-2)}}{2G(d-\theta-2)}\left[\delta^{-(d-\theta-2)}-(l/2)^{(\theta-d+2)}\left[\frac{\sqrt{\pi}\,\Gamma\left(\frac{d-\theta}{2(d-\theta-1)}\right)}{\Gamma\left(\frac{1}{2(d-\theta-1)}\right)}\right]^{(d-\theta-1)}\right]\,. \tag{1.118}$$

When $\theta=0$, we recover the usual AdS_{d+1} expression [378], and when $\theta=d-2$, this expression gets modified to include a logarithmic leading term, and becomes [162]

$$S=\frac{L^{d-1}L_S^{(d-2)}}{2G}\log\frac{2l}{\delta}\,, \tag{1.119}$$

In [350] it was raised the question of whether new violations of the area law might be found for other values of θ when higher-order gravities were added to the Einstein gravity Lagrangian. We answer this question in chapter 8, where we show that the leading contribution to the HEE in this class of geometries always comes from the Einstein gravity term, although new logarithmic terms appear for generic higher-curvature gravities (of order n in curvature) for

$$\theta=\frac{(d-1)(d-2)}{(d-2n+1)}\,. \tag{1.120}$$

[27] This behaviour comes from the effective 2D CFT which governs the physics of modes at the Fermi surface [385,407].

These would always be subleading except for $n = 1$, which is the well-known case $\theta = d-2$. We also find the form of the universal term at first order in the coupling for a gravity Lagrangian with an R^2 correction, as well as the exact corrections to the area-law term for general curvature-squared gravities. This information allows us to conjecture the general form of holographic entanglement entropy for these geometries in arbitrary higher-order gravities.

2

Black holes and equivariant vectors in $\mathcal{N}=2$, $d=4$ supergravity

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The intensive search for black-hole solutions of supergravity theories over the last 25 years has been a very rewarding one in respect to the supersymmetric (also known as BPS in the literature, even if this concept is not equivalent, but wider) ones. Even though the existence of extremal non-supersymmetric black holes was discovered long time ago [273, 352] and we know that they are subject to the same attractor mechanism as the supersymmetric ones [173], only a few general families of solutions have been constructed for some classes of theories [74] and we are still far from having a complete understanding of their structure and general properties. The situation w.r.t. non-extremal solutions, studied recently, e.g., in [89, 190, 191, 321, 323] is even worse: even if all extremal black-hole solutions may be deformed (*i.e.*, heated up) to a non-extremal one, then we do not know the non-extremal deformations of many of them; in general we don't know whether there are obstructions to such a deformation and what they are. We also don't know whether, in each theory, there is only one family of non-extremal black-hole solutions from which all the extremal ones can be obtained by taken the appropriate limits, such as it happens in the few models studied so far [190, 267, 305, 321, 323]. The (stringy) non-extremal black hole landscape is a largely uncharted territory.

It is clear that to answer these questions new tools are needed since the first-order equations associated to unbroken Supersymmetry are of no help here and the second-order equations of motion of the FGK effective action [173] are still very hard to solve. Several approaches have been proposed to this end. For instance, it has been shown that in general one can construct first-order flow equations for extremal non-supersymmetric and non-extremal black holes Refs. [15, 18, 127, 132, 187, 192, 258, 328, 362] and many such equations have been constructed. From them one can extract interesting information about the near-horizon and spacelike infinity limits (whence about the entropy and mass of the solutions), but in practice these equations are obtained when the solutions are already known, which somewhat diminishes their usefulness.

The most common approach to the search of stationary black-hole solutions, pioneered in Ref. [83], consists in the dimensional reduction over the time direction. For 4-dimensional theories, this results in a 3-dimensional theory consisting of a non-linear σ-

model coupled to gravity (in 3 dimensions the vector fields can be dualized into scalars).[1] When the σ-model corresponds to a homogeneous space one can show that the system is integrable and use the standard techniques to classify and obtain explicit black-hole solutions, see *e.g.* [55, 75–77, 133, 221]. This approach has been quite a successful one, but for the moment it has not provided complete answers to the above questions.

More recently, a new approach for the 4- and 5-dimensional $\mathcal{N}=2, d=4$ supergravity theories coupled to n_v vector supermultiplets has been introduced in Ref. [324][2]. This is nothing but the H-FGK formalism, introduced in the previous section. The H variables arise naturally in the supersymmetric cases [200,320], but it has been shown that they can be used in more general (but always stationary) cases. As we stressed, the main virtue of the new variables, when compared to the scalar fields present in the FGK effective action, is that they transform linearly under the duality group (embedded in $\mathrm{Sp}(2n_v+2;\mathbb{R})$ in the $d=4$ case and in $\mathrm{SO}(n_v+1)$ in $d=5$ case).

In previous works [89,95,191,323], the description of the simplest families of solutions was investigated (that we will call *conventional* in Section 2.2) for which the H-variables are harmonic functions (in the extremal case) or linear combinations of hyperbolic sines and cosines (in the non-extremal case). Some general features of the formalism, like the invariance of the effective action under local Freudenthal duality rotations [189] have also been studied.

Our main goal in this chapter is to study the main aspect of the formalism, namely the linear equivariance under duality transformations of the charges and moduli that characterize a given solution, and show how to exploit the requirement of linear equivariance in order to find attractors and construct explicit extremal solutions in some already well-studied models: the axidilaton and the $\overline{\mathbb{CP}}^n$ models. We also want to make progress towards answering the questions posed at the beginning of this introduction using these new tools. In the *conventional* cases that we have studied so far, it is known how one can arrive at (extremal) solutions described by harmonic functions from (non-extremal) solutions described by hyperbolic sines and cosines: we will apply our new tools to a *non-conventional* (non-supersymmetric) extremal solution of the t^3 model not considered in the previous works Refs. [191, 323]. This solution, which has been known for some time [74, 187, 208, 300], is characterized by H-variables that contain anharmonic terms and its deformation into a non-supersymmetric (finite-temperature) solution has proven elusive [188]. We think that, in order to search for this non-extremal generalization (if it exists), it is necessary to know more about the structure of the extremal solution and we will show how the new tools can help us to this end.

The remainder of the chapter is organized as follows: in Section 2.1 we explain how equivariant charge vectors enter in black-hole solutions when we express them in the H-variables of this formalism. In Section 2.2 we explain when the usual harmonic ansatz becomes insufficient to write the general family of solutions associated to some attractor (expressed through an equivariant charge vector). This insufficiency indicates the need of adding anharmonic terms to the H-variables giving rise to what we have called *unconventional* black-hole solutions. Then, in Section 2.3 we give a general form for the first-order flow equations of any static black-hole solution of these theories that applies, in particular, to the unconventional solutions. In Sections 2.4 and 2.5 we review the supersymmetric and

[1] Further assumptions (staticity plus an ansatz for the 3-dimensional metric) lead to the FGK effective action, presented in the introduction, with its characteristic effective *black-hole potential* [173].

[2] A closely-related approach has been proposed in Ref. [329, 331, 332].

non-supersymmetric extremal solutions (which are completely conventional) of two simple models, studying their duality symmetries and their equivariant vectors. In Section 2.6 we turn to the t^3 model, showing how its extremal, non-supersymmetric solutions are non-conventional. We, then, construct and study this unconventional family of solutions using a basis of equivariant vectors. Our conclusions and comments on further directions of work can be found in Section 5.4.

2.1 Explicit solutions and equivariant vectors

The main advantage of the H-FGK formalism (see section 1.2.4) is the linear behavior of the variables under transformations of the electric-magnetic duality group G of the theory:

$$H^{M\,\prime} = S^{M}{}_{N} H^{N}\,, \qquad (S^{M}{}_{N}) \in \mathrm{G} \subset \mathrm{Sp}(2n_v+2;\mathbb{R})\,. \tag{2.1}$$

This linear behavior can dramatically simplify the construction of explicit solutions to theories with a non-trivial duality group as it implies that any solution must be of the form

$$H^{M}(\tau) = c^{\sigma}(\tau)\; U^{M}_{\sigma}\,, \tag{2.2}$$

where the functions $c^{\sigma}(\tau)$ are duality invariant; the symplectic vectors U^{M}_{σ} are constant vectors that may depend on the physical parameters of the theory (mass M, electric and magnetic charges $\mathcal{Q}^{M}$ and asymptotic values of the scalars Z^{i}_{∞}) and must be *equivariant* w.r.t. the duality group, *i.e.*,

$$U^{M}_{\sigma}(M, Z'_{\infty}, Z^{*\prime}_{\infty}, \mathcal{Q}') = S^{M}{}_{N} U^{N}_{\sigma}(M, Z_{\infty}, Z^{*}_{\infty}, \mathcal{Q})\,, \tag{2.3}$$

with

$$Z^{i\,\prime} \equiv F^{i}_{S}(Z)\,, \qquad \mathcal{Q}^{M\,\prime} = S^{M}{}_{N}\mathcal{Q}^{N}\,, \tag{2.4}$$

where $F^{i}_{S}(Z)$ is the non-linear realization of the duality transformation $S^{M}{}_{N}$ on the complex scalars.

In some cases, the number of equivariant vectors of the theory can be greater than[3] or equal to the number of variables H^{M}. In that case, one does not win much by using the above ansatz. In other cases, however, the number can be much smaller and we will be left with a small number of invariant functions to be determined.

In the near-horizon limit of extremal black-hole solutions, the value of the variables H^{M} will be dominated by one equivariant vector that we denote by B^{M} and that can be defined, in our conventions, by[4]

$$B^{M} \equiv \lim_{\tau\to-\infty} -\frac{\sqrt{2}H^{M}}{\tau}\,. \tag{2.5}$$

The values of the scalars on the horizon, Z^{i}_{h}, are completely determined by this equivariant vector upon use of the general expression of the scalars as functions of the variables

[3] If it is greater, we can eliminate some from the ansatz, since they will be linearly dependent on the rest.

[4] Observe that this definition is completely general: given the behavior of the 3-dimensional transverse metric in the near-horizon limit as a function of τ and the degree of homogeneity of $e^{-2U} = \mathsf{W}(H)$ as a function of the H-variables, in regular black-hole solutions the functions $H^{M}(\tau)$ are dominated by these constant vectors in the near-horizon limit.

H^M [324]

$$Z^i(H) = \frac{\tilde{H}^i + iH^i}{\tilde{H}^0 + iH^0}\,, \qquad \Rightarrow \qquad Z^i_{\rm h} = Z^i(B)\,, \tag{2.6}$$

and also extremize the black-hole potential $V_{\rm bh}(H,\mathcal{Q})$ as a function of the variables H^M:

$$\partial_M V_{\rm bh}(H,\mathcal{Q})|_{H=B} = 0\,. \tag{2.7}$$

The vectors B^M, which in this context can be called attractors, can also be written in the form

$$B^M = b^\sigma U^M_\sigma\,, \tag{2.8}$$

where the b^σ are duality-invariant constants such that the products bU^M have the same dimensions as electric and magnetic charges.

Clearly these vector attractors must contain more information than the values of the scalars on the horizon $Z^i_{\rm h}$ (the standard attractors). On the other hand, when the model has a high degree of symmetry the requirement of equivariance imposes strong constraints on the possibilities and it simplifies the task of finding the attractors B^M.

A similar discussion can be made for the values of the variables H^M at spatial infinity, which in the employed coordinate system lies at $\tau = 0$.

The amount of simplification introduced by the above observation that the variables H^M must always be of the form Eq. (2.2) depends on our ability to find a sufficient number of equivariant vectors; the Freudenthal dual of the charge vector $\tilde{\mathcal{Q}}^M$ is, by construction, a prime example of equivariant vector, but there are other systematic ways of finding them. Let us consider, first, equivariant vectors that only depend on the charges. They can be seen as an endomorphism of the $(2n_v+2)$-dimensional vector space of charges and their equivariance is equivalent to the fact that these endomorphisms commute with the duality transformations (which are also endomorphisms of charge space). Thus, linear (not necessarily symplectic) transformations that commute with G provide a second example of equivariant vectors.

To study non-linear cases, let us expand an equivariant vector and the duality transformations around the identity

$$U^M_\sigma(\mathcal{Q}) \sim \mathcal{Q}^M + \xi^M(\mathcal{Q})\,, \qquad (S\mathcal{Q})^M \sim \mathcal{Q}^M + \alpha^A {\eta_A}^M(\mathcal{Q})\,, \tag{2.9}$$

where $S \in \mathrm{G} \subset \mathrm{Sp}(2n_v+2;\mathbb{R})$ and, therefore,

$${\eta_A}^M(\mathcal{Q}) \;=\; (T_A)^M{}_N\, \mathcal{Q}^N\,, \tag{2.10}$$

where $T_A \in \mathrm{Sp}(2n_v+2;\mathbb{R})$ are the generators of the duality group; the condition of equivariance is equivalent to requiring that the Lie brackets of these two kinds of generators vanish[5]

$$[U,\eta_A] = 0\,, \quad \Rightarrow \quad (T_A)^M{}_N \mathcal{Q}^N \partial_M U^P = (T_A)^P{}_R U^R\,, \quad \text{where} \quad \partial_M U^P \equiv \frac{\partial U^P}{\partial \mathcal{Q}^M}\,. \tag{2.11}$$

On taking the derivative with respect to $\mathcal{Q}^P$ of both sides of this equation we find the integrability condition

$$(T_A)^M{}_N \mathcal{Q}^N \partial_M \mathsf{P} = 0\,, \qquad \mathsf{P} \equiv \partial_M U^M = \Omega^{MN}\partial_M U_N\,. \tag{2.12}$$

[5] Obviously, also ξ must be an equivariant vector, whence we can replace ξ by U in what follows for the purpose of writing an equation characterizing equivariant vectors.

which implies that P is an invariant function of the charges. Thus, equivariant vectors are associated to invariants by the above equation. The simplest invariant is just $\mathsf{P}=0$ and equivariant vectors such that $\partial_{[M}U_{N]}=0$ are associated to it; clearly there may be more possibilities as locally they must be of the form $U_M=\partial_M h$ for some non-vanishing invariant h (possibly up to additive numerical constants) and one can check that the equivariance condition is automatically satisfied. For instance, if we take $h=\mathsf{W}/2$, then $U_M=\tilde{\mathcal{Q}}_M$.

For equivariant vectors that depend (non-holomorphically) on the moduli Z^i_∞, the equivariance condition takes the form

$$(T_A)^M{}_N\mathcal{Q}^N\partial_M U^P + k_A{}^i\partial_i U^P + k_A{}^{*\,i^*}\partial_{i^*}U^P = (T_A)^P{}_R U^R\,, \tag{2.13}$$

where $K_A\equiv k_A{}^i(Z)\partial_i+\text{c.c.}$ are the Killing vectors that generate the action of the duality group G on the scalar manifold preserving the holomorphic and Kähler structures. Again, $\mathsf{P}\equiv\partial_M U^M$ must be an invariant and a particularly simple case is $\mathsf{P}=0$ and $U_M=\partial_M h$ where, now, h is required to be invariant only up to additive functions of the moduli. A recurring example is

$$h \;=\; \log\;(\mathcal{Z}(\mathcal{Q}))\;, \tag{2.14}$$

where $\mathcal{Z}(\mathcal{Q})$ is the central charge defined as

$$\mathcal{Z}(\mathcal{Q})\equiv\mathcal{V}_M\mathcal{Q}^M\,, \tag{2.15}$$

so using the definition of the H-variables it can be written as

$$\mathcal{Z}(\mathcal{Q})=\frac{e^{-i\alpha}}{\sqrt{2\mathsf{W}}}\mathcal{H}_M\mathcal{Q}^M\,. \tag{2.16}$$

The associated (complex) equivariant vector is

$$U_M=\frac{\partial\, h}{\partial\mathcal{Q}^M}=\frac{\mathcal{V}_M}{\mathcal{Z}(\mathcal{Q})}\,. \tag{2.17}$$

The real and imaginary parts provide two real moduli-dependent equivariant vectors. It should be obvious that one can use, instead of the central charge any fake central charge, but the result may not be a new equivariant vector.

The Lie bracket of two equivariant vectors is also an equivariant vector, so that the equivariant vectors form a Lie algebra that commutes with that of the duality group G.

Finally, in the cases that we are going to study, we will show how one can construct equivariant vectors by using other methods like solution-generating techniques.

2.2 Conventional and unconventional solutions

As explained in Ref. [324], contracting the equations of motion derived from the H-FGK action Eq. (1.81) with H^M and using the homogeneity properties of the different terms and the Hamiltonian constraint Eq. (6.6) one finds, in the extremal case $r_0=0$[6], the equation

$$\mathsf{W}\tilde{H}_M\,\ddot{H}^M+(\dot{H}^M H_M)^2=0\,. \tag{2.18}$$

[6] In this discussion we will only consider the extremal case because in the rest of the chapter we are going to restrict ourselves to it.

In what we are going to call from now on *conventional* extremal solutions (supersymmetric or not) the variables $H^M(\tau)$ are harmonic functions, *i.e.*, they satisfy $\ddot{H}^M=0$. The above equation implies that they also satisfy the constraint[7]

$$\dot{H}^M H_M = 0\,. \tag{2.19}$$

Conventional extremal solutions have been intensively studied in Ref. [191]. However, how general are these solutions? Can all the extremal black-hole solutions be written in a conventional form? (The answer in the supersymmetric case is yes.) If not, what are the limitations and how can they be overcome as to obtain the most general extremal black-hole solutions that depend on the maximal number of independent physical parameters?

To investigate these issues, it is convenient to review in detail the construction of conventional extremal black-hole solutions: extremal black-holes are associated to values of the scalar fields $Z^i_{\rm h}$ (attractors) that extremize the black-hole potential [173]. As explained in the previous section, in the H-FGK formulation attractors appear as symplectic vectors B^M that extremize the black-hole potential when written in terms of the H-variables. These attractors B^M are defined up to normalization because the black-hole potential is invariant under rescalings of the H^Ms and also up to global Freudenthal rotations. Furthermore, as functions of the charges and moduli, the attractors B^M are equivariant under duality transformations. A family of extremal black holes closed under duality will be associated to a given equivariant vector expressed as a set of functions of the charge components and moduli $B^M(\mathcal{Q}, Z_\infty, Z^*_\infty)$. We are going to focus on moduli-independent attractors, *i.e.*, the so-called *true attractors*.

The attractor B^M determines the near-horizon form of the solution. We can always construct a solution describing the AdS$^2\times$S$_2$ solution that describes the near-horizon geometry by choosing the appropriate normalization of B^M: indeed, one can check that the harmonic functions

$$H^M = -\tfrac{1}{\sqrt{2}} B^M \tau\,, \tag{2.20}$$

always satisfy the equations of motion as long as the condition

$$V_{\rm bh}(B,\mathcal{Q}) = -\tfrac{1}{2}\mathsf{W}(B)\,, \tag{2.21}$$

determining the normalization of B^M is met.

To construct a solution with the same near-horizon behavior and with an asymptotically-flat region we must add to the H^M above a constant vector A^M. The condition Eq. (2.19) and the normalization of the metric at infinity become two constraints for A^M

$$B^M A_M = 0\,, \qquad \mathsf{W}(A) = 1\,, \tag{2.22}$$

that leave $2n_v$ real constants, which is just the right amount to describe the asymptotic values of the n_v complex scalars Z^i_∞. Only if we cannot add a vector A^M satisfying these two constraints, then the most general solution associated to the attractor B^M cannot be conventional and we will have to add anharmonic terms to the H^M.

We can reformulate this question as follows: if we add to the H^M in Eq. (2.20) an infinitesimal vector ε^M satisfying $B^M\varepsilon_M=0$, do we get another solution to the Hamiltonian constraint Eq. (1.82) and equations of motion Eq. (1.83)? To first order in ε^M, the

[7]The converse is not always true: the above constraint can be satisfied for extremal black-hole solutions which are not given by harmonic H^Ms and that we will call *unconventional*.

Hamiltonian constraint will be solved by the perturbed solution

$$H'^M = H^M + \varepsilon^M\,, \qquad H^M = -\tfrac{1}{\sqrt{2}}B^M\tau\,, \qquad B^M\varepsilon_M = 0\,, \tag{2.23}$$

if

$$\varepsilon^M\left\{\tfrac{1}{2}\partial_M g_{NP}\dot{H}^N\dot{H}^P + \partial_M V(H,\mathcal{Q})\right\} = 0\,. \tag{2.24}$$

Evaluating this equation at the near-horizon solution H^M, using $V_{\rm bh}(H,\mathcal{Q}) = -\mathsf{W}(B)V(H,\mathcal{Q})$, the homogeneity properties of the different terms, the fact that $\partial_M V_{\rm bh}(B,\mathcal{Q}) = 0$ and the condition (2.21), we arrive at

$$\varepsilon^M\left\{\tfrac{1}{4}B^NB^P\partial_M\partial_N\partial_P\log\mathsf{W}(B) - \tfrac{1}{2}\partial_M\log\mathsf{W}(B)\right\} = 0\,, \tag{2.25}$$

which is an equation in the variables B^M (including the partial ∂_M derivatives, which should be understood as partial derivatives with respect to B^M) and is identically satisfied on account of the scale invariance of $\log\mathsf{W}(B)$.

The analogous condition on the equations of motion, Eqs. (1.83), reads

$$\varepsilon^M\left\{\partial_M g_{NP}\ddot{H}^P + \partial_M(\partial_P g_{QN} - \tfrac{1}{2}\partial_N g_{PQ})\dot{H}^P\dot{H}^Q + \partial_M\partial_N V(H,\mathcal{Q})\right\} = 0\,, \tag{2.26}$$

and, after evaluation on the near-horizon solution we get a homogenous equation that, again, can be read as an equation on the variables B^M. Using the same properties we used with the Hamiltonian constraint plus $B^M\varepsilon_M = 0$ we get a non-trivial equation for ε^M

$$\mathfrak{M}_{MN}\varepsilon^N = 0\,, \quad \text{with} \quad \mathfrak{M}_{MN} \equiv \mathsf{W}(B)\partial_M\partial_N\log\mathsf{W}(B) + 2\frac{\tilde{B}_M\tilde{B}_N}{\mathsf{W}(B)} - \partial_M\partial_N V_{\rm bh}(B,\mathcal{Q})\,. \tag{2.27}$$

We are interested in the number of independent solutions to this equation that satisfy the constraint $B^M\varepsilon_M = 0$, *i.e.*, in the rank of $\mathfrak{M}_{MN}$. The rank should be at most 1 as this implies a single linear constraint on the components of ε^M, which should be equivalent to $B^M\varepsilon_M = 0$. If the rank of $\mathfrak{M}_{MN}$ happens to be bigger than 1, then there are not enough unconstrained components of ε^M for the family of solutions to have arbitrary values of the moduli and the most general solution based on the chosen attractor, must necessarily contain anharmonic terms.

For cubic models, the need of anharmonic ansätze to construct the most general, generating, non-supersymmetric, extremal, black-hole solution of [300] and [208] was first observed in [187] and later confirmed in [74] and [188]. In the next sections we will see how the obstruction to the fully harmonic ansatz arises in the particular case of the t^3 model. For the non-extremal case of these theories, the situation is still unclear [188].

2.3 The general first-order flow equations

The central charge of an $\mathcal{N}=2, d=4$ SUGRA theory is defined by Eq. (2.15) and, in terms of the H-variables it takes the form of Eq. (2.16) which we copy here for convenience

$$\mathcal{Z}(\mathcal{Q}) = \frac{e^{-i\alpha}}{\sqrt{2\mathsf{W}}}(\tilde{H}_M + iH_M)\mathcal{Q}^M\,. \tag{2.28}$$

Let us consider a generalization of the central charge, denoted by $\mathcal{Z}(\phi, \sqrt{2}\mathfrak{D}H)$, in which we replace the second argument (the charge vector) by the Freudenthal-covariant derivative of H^M introduced in Ref. [189], *i.e.*,

$$\mathfrak{D}H^M \equiv \dot{H}^M + A\tilde{H}^M\,, \qquad A \equiv \frac{\dot{H}^N H_N}{\mathsf{W}}\,. \tag{2.29}$$

Since $H_M\mathfrak{D}H^M = 0$ and $\tilde{H}_M\tilde{H}^M = 0$ identically, we immediately find that

$$|\mathcal{Z}(\phi, \sqrt{2}\mathfrak{D}H)| = \pm\frac{\tilde{H}_M\dot{H}^M}{\sqrt{\mathsf{W}}} = \pm\frac{\partial_M\mathsf{W}\,\dot{H}^M}{2\sqrt{\mathsf{W}}} = \pm\frac{d\sqrt{\mathsf{W}}}{d\tau} = \pm\frac{de^{-U}}{d\tau}\,, \tag{2.30}$$

which is the first-order equation for the metric function[8]. Observe that $H_M\mathfrak{D}H^M = 0$ implies that the phase of $\mathcal{Z}(\phi, \sqrt{2}\mathfrak{D}H)$ is equal to the phase of $\pm X$. The sign must be chosen so as to make $\pm\tilde{H}_M\dot{H}^M > 0$ and, since the mass of the solution corresponding to $e^{-2U} = \mathsf{W}(H)$ is given by

$$M = -\tfrac{1}{2}\left.\frac{de^{-2U}}{d\tau}\right|_{\tau=0} = -\tfrac{1}{2}\left.\dot{\mathsf{W}}\right|_{\tau=0} = -\left.\tilde{H}_M\dot{H}^M\right|_{\tau=0}\,, \tag{2.31}$$

we find that for regular solutions (with positive mass) we must choose the lower sign:

$$\frac{de^{-U}}{d\tau} = -|\mathcal{Z}(\phi, \sqrt{2}\mathfrak{D}H)|\,. \tag{2.32}$$

From Eq. (2.8) of Ref. [355] we have that

$$\frac{dZ^i}{d\tau} = -2X\mathcal{G}^{ij^*}\mathcal{D}_{j^*}\mathcal{V}^*_M\dot{H}^M\,. \tag{2.33}$$

We can rewrite $\dot{H}^M$ as

$$\dot{H}^M = \mathfrak{D}H^M - A\tilde{H}^M = \mathfrak{D}H^M - A\left(\frac{\mathcal{V}^M}{2X} + \text{c.c.}\right)\,, \tag{2.34}$$

and plug it into the previous equation to get

$$\begin{aligned}\frac{dZ^i}{d\tau} &= -2X\mathcal{G}^{ij^*}\mathcal{D}_{j^*}\mathcal{Z}^*(\phi,\mathfrak{D}H) = 4Xe^{-i\alpha}\mathcal{G}^{ij^*}\partial_{j^*}|\mathcal{Z}^*(\phi,\mathfrak{D}H)| \\ &= 2e^U\mathcal{G}^{ij^*}\partial_{j^*}|\mathcal{Z}^*(\phi,\sqrt{2}\mathfrak{D}H)|\,,\end{aligned} \tag{2.35}$$

where we have used Eq. (1.88) and the equality of the phases of $-X$ and $|\mathcal{Z}(\phi, \sqrt{2}\mathfrak{D}H)|$. This is the second first-order equation[9].

Some remarks are in order:

1. In these derivations we have assumed neither extremality or non-extremality of the solutions nor any explicit form of the variables H^M (harmonic or hyperbolic)[10].

[8] This equation reduces to Eq. (5.9) of Ref. [187] in the extremal limit. Observe that the Freudenthal-covariant derivative corresponds to Eq. (5.6) of the same reference.

[9] Again, this equation reduces to Eq. (5.10) of Ref. [187] in the extremal limit.

[10] Actually, we have written *solutions* but we have not used at any moment the fact that the H^M solve the equations of motion. The first-order equations that we have derived are, therefore, valid for any configuration of the variables H^M, although their use is essentially limited to solutions.

Furthermore, we have not assumed the Freudenthal gauge-fixing condition $\dot{H}^N H_N = 0$. Only the properties of Special Geometry encoded in the H-FGK formalism have been used. Therefore, the first-order Eqs. (2.32) and (2.35) apply to any static black-hole solution of ungauged $\mathcal{N}=2, d=4$ supergravity coupled to vector multiplets.

2. These first-order equations reduce to those found in the literature starting from Ref. [173] in the extremal/harmonic (*i.e.*, $A = \dot{H}^N H_N = 0$) cases: if $H^M = A^M - \frac{1}{\sqrt{2}}B^M\tau$ for some constant symplectic vectors A^M (which encode the values of the scalars at spatial infinity) and the attractor B^M, then

$$|\mathcal{Z}(\phi, \sqrt{2}\mathfrak{D}H)| = |\mathcal{Z}(\phi, B)|\,, \tag{2.36}$$

which is known as *fake central charge* when $B^M \neq \mathcal{Q}^M$ and coincides with the central charge in the supersymmetric case $B^M = \mathcal{Q}^M$.

3. In the general (non-supersymmetric) case $\mathfrak{D}H$ will be τ-dependent and its near-horizon ($\tau \to -\infty$) and spatial infinity ($\tau \to 0^-$) limits, will not necessarily be equal: in the near-horizon limit $\lim_{\tau\to-\infty}\mathfrak{D}H^M \equiv -\frac{1}{\sqrt{2}}B^M$ and in the spacelike infinity limit $\lim_{\tau\to 0^-}\mathfrak{D}H^M \equiv -\frac{1}{\sqrt{2}}E^M$ and, generically, $B^M \neq E^M$.

$$M = -\lim_{\tau\to 0^-}\frac{de^{-U}}{d\tau} = |\mathcal{Z}(\phi_\infty, E)|\,, \tag{2.37}$$

$$S = \pi\left[\lim_{\tau\to-\infty}\frac{de^{-U}}{d\tau}\right]^2 = \pi|\mathcal{Z}(\phi_{\rm h}, B)|^2\,, \tag{2.38}$$

where ϕ_∞ and $\phi_{\rm h}$ are the values of the scalars at spatial infinity and on the horizon, respectively. Different fake central charges $\mathcal{Z}(\phi, E)$ and $\mathcal{Z}(\phi, B)$ drive the metric function in the spatial-infinity and near-horizon regions, respectively. This behavior is present in the non-supersymmetric extremal solutions of the cubic models studied in Refs. [52, 74, 187, 207, 300] which have anharmonic H^Ms[11].

4. In Ref. [127] and subsequent literature the first-order flow equations were given in terms of superpotential functions $W(\phi, B)$ which depend only on a constant fake charge vector B^M and which has a structure similar, but not identical, to the central charge. Those first-order equations must be completely equivalent to Eqs. (2.32,2.35), because the same variables, for the same solution, cannot obey two different sets of first-order equations. We do not know how to prove this equivalence in general, and it will have to be checked case by case.

2.4 The axidilaton model

The axidilaton model is defined by the prepotential

$$\mathcal{F} = -i\mathcal{X}^0\mathcal{X}^1\,, \tag{2.39}$$

[11] The H^Ms of those solutions do not satisfy the constraint $\dot{H}^M H_M = 0$. A change of Freudenthal gauge can bring the solutions to the $\dot{H}^M H_M = 0$ gauge but cannot make the H^M harmonic [189].

and has only one complex scalar that we will denote by λ that is given by

$$\lambda \equiv i\mathcal{X}^1/\mathcal{X}^0 \,. \tag{2.40}$$

In terms of λ and in the $\mathcal{X}^0 = i/2$ gauge, the Kähler potential and metric are

$$\mathcal{K} = -\ln \Im\mathfrak{m}\lambda \,, \qquad \mathcal{G}_{\lambda\lambda^*} = (2\Im\mathfrak{m}\lambda)^{-2} \,, \tag{2.41}$$

and therefore λ, which must take values in the upper half complex plane, parametrizes the coset space $\mathrm{Sl}(2;\mathbb{R})/\mathrm{SO}(2)$.

The canonically-normalized covariantly-holomorphic symplectic section $\mathcal{V}$ is, in the gauge in which the Kähler potential is given by Eq. (2.41),

$$\mathcal{V} = \frac{1}{2(\Im\mathrm{m}\lambda)^{1/2}} \begin{pmatrix} i \\ \lambda \\ -i\lambda \\ 1 \end{pmatrix} \,, \tag{2.42}$$

and the central charge and its holomorphic covariant derivative are

$$\begin{aligned} \mathcal{Z}(\mathcal{Q}) &= \frac{1}{2\sqrt{\Im\mathrm{m}\lambda}} \left[(p^1 - iq_0) - (q_1 + ip^0)\lambda \right] \,, \\ \mathcal{D}_\lambda \mathcal{Z} &= \frac{i}{4(\Im\mathrm{m}\lambda)^{3/2}} \left[(p^1 - iq_0) - (q_1 + ip^0)\lambda^* \right] \,. \end{aligned} \tag{2.43}$$

It is useful to define the fake charge and associated fake central charge

$$\mathcal{P} \equiv \begin{pmatrix} p^0 \\ -p^1 \\ q_0 \\ -q_1 \end{pmatrix} \,, \qquad \mathcal{Z}(\mathcal{P}) \equiv \frac{1}{2\sqrt{\Im\mathrm{m}\lambda}} \left[(-p^1 - iq_0) - (-q_1 + ip^0)\lambda \right] \,, \tag{2.44}$$

in terms of which

$$\mathcal{G}^{ij^*}\mathcal{D}_i\mathcal{Z}\mathcal{D}_{j^*}\mathcal{Z}^* = |\mathcal{Z}(\mathcal{P})|^2 \,, \tag{2.45}$$

so that the black-hole potential takes the simple form

$$-V_{\mathrm{bh}} = |\mathcal{Z}(\mathcal{Q})|^2 + |\mathcal{Z}(\mathcal{P})|^2 \,. \tag{2.46}$$

The black-hole solutions of this model have been exhaustively studied in Refs. [23, 57, 141, 190, 194, 195, 198, 205, 266, 269, 305, 351, 377, 387]. Our goal here is to illustrate the general results and methods described in the previous sections using this well-known model. First, let us recall what are the symmetries of this model in its original formulation.

2.4.1 The global symmetries of the axidilaton model

The full axidilaton model (and not just the axidilaton kinetic term) is invariant under global $\mathrm{Sl}(2;\mathbb{R})$ transformations. Let us start by describing the action of this group on the axidilaton field: parametrize a generic element of $\mathrm{Sl}(2;\mathbb{R})$ as

$$\Lambda \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,, \qquad \text{with} \quad ad - bd = 1 \,, \tag{2.47}$$

then the axidilaton transforms as

$$\lambda' = \frac{a\lambda + b}{c\lambda + d}\,. \tag{2.48}$$

The scalar manifold metric admits 3 holomorphic Killing vectors which can be taken to be

$$K_1 = \lambda\partial_\lambda + \text{c.c.}\,, \qquad K_2 = \tfrac{1}{2}(1-\lambda^2)\partial_\lambda + \text{c.c.}\,, \qquad K_3 = \tfrac{1}{2}(1+\lambda^2)\partial_\lambda + \text{c.c.}\,, \tag{2.49}$$

and satisfy the commutation relations of the Lie algebra $\mathfrak{sl}(2;\mathbb{R})$

$$[K_m, K_n] = \epsilon_{mnq}\eta^{qp}K_p\;, \quad \Rightarrow f_{mn}{}^p = -\epsilon_{mnq}\eta^{qp}, \quad (m,n,\ldots = 1,2,3)\;, \tag{2.50}$$

where $\epsilon_{123} = +1$ and $\eta = \text{diag}(++-)$; η is proportional to the Killing metric of $\mathfrak{so}(1,2) \simeq \mathfrak{sl}(2;\mathbb{R}) \simeq \mathfrak{sp}(2;\mathbb{R})$. The infinitesimal $\mathrm{Sl}(2;\mathbb{R})$ transformations of λ can be written using these Killing vectors as

$$\delta_\alpha\lambda = \alpha^m k_m{}^\lambda = \tfrac{1}{2}(\alpha^2+\alpha^3) + \alpha^1\lambda - \tfrac{1}{2}(\alpha^2-\alpha^3)\lambda^2\,. \tag{2.51}$$

The infinitesimal linear transformations associated to the above choice of Killing vectors are, in terms of the Pauli matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \mathbb{1}_{2\times 2} + \alpha^m T_m\,, \qquad T_1 = -\tfrac{1}{2}\sigma^3\,,\; T_2 = -\tfrac{1}{2}\sigma^1\,,\; T_3 = \tfrac{i}{2}\sigma^2\,, \tag{2.52}$$

and satisfy the Lie algebra

$$[T_m, T_n] = -\epsilon_{mnq}\eta^{qp}T_p\,. \tag{2.53}$$

The action of the finite $\mathrm{Sl}(2;\mathbb{R})$ transformations on the Kähler potential and on the canonical covariantly-holomorphic symplectic section $\mathcal{V}$ given in Eq. (2.42) is

$$\mathcal{K}'(\lambda) \;\equiv\; \mathcal{K}(\lambda'(\lambda)) = \mathcal{K}(\lambda) + 2\Re\mathfrak{e} f(\lambda)\,, \tag{2.54}$$

$$\mathcal{V}'^M(\lambda) \;\equiv\; \mathcal{V}^M(\lambda'(\lambda)) = e^{-i\Im\mathfrak{m} f(\lambda)}\; S^M{}_N\mathcal{V}^N\,, \tag{2.55}$$

where the holomorphic function $f(\lambda)$ of the Kähler transformation and the symplectic rotation $S^M{}_N$ are given by

$$f(\lambda) \;=\; \ln(c\lambda + d)\,, \tag{2.56}$$

$$(S^M{}_N) \;=\; \begin{pmatrix} d & & -c & \\ & a & & b \\ -b & & a & \\ & c & & d \end{pmatrix}. \tag{2.57}$$

In this 4-dimensional representation the infinitesimal generators T_m are given by

$$(T_1{}^M{}_N) = -\tfrac{1}{2}\begin{pmatrix} \sigma^3 & \\ & -\sigma^3 \end{pmatrix}, \quad (T_2{}^M{}_N) = -\tfrac{1}{2}\begin{pmatrix} & \sigma^3 \\ \sigma^3 & \end{pmatrix}, \quad (T_3{}^M{}_N) = \tfrac{1}{2}\begin{pmatrix} & \mathbb{1} \\ -\mathbb{1} & \end{pmatrix}. \tag{2.58}$$

The same transformations act on all the symplectic vectors of the theory and, in particular, on the variables H^M and the charge vectors $\mathcal{Q}^M$. In this formulation of the axidilaton system there seem to be no further symmetries[12].

[12] We will see, however, that there is an additional U(1) factor in the symmetry group that only has

Equivariant vectors of the axidilaton model

In this model there is no need to solve any equation to find 4 linearly independent equivariant vectors: observe that the symplectic vector of charges is the direct sum of two real $\mathrm{Sl}(2;\mathbb{R})$ doublets a^i and b_i $(i,j=1,2)$, namely

$$(a^i) \equiv \begin{pmatrix} p^1 \\ q_1 \end{pmatrix} , \qquad (b_i) \equiv (p^0,\, q_0)\,. \tag{2.59}$$

These doublets transform respectively contravariantly and covariantly, that is

$$a'^{\,i} = \Lambda^i{}_j\, a^j\,, \qquad b'_i = b_j\,(\Lambda^{-1})^j{}_i\,, \tag{2.60}$$

where $(\Lambda^i{}_j)$ is the matrix given in Eq. (2.47), which furthermore satisfies

$$(\Lambda^{-1})^i{}_j = \Omega^{ki}\,\Lambda^l{}_k\,\Omega_{lj}\,, \qquad (\Omega_{ij}) = (\Omega^{ij}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \tag{2.61}$$

because $\mathrm{Sl}(2;\mathbb{R}) \simeq \mathrm{Sp}(2;\mathbb{R})$. We can use the symplectic metric Ω to raise and lower doublet indices such as i and j, so $a_i \equiv \Omega_{ij}a^j$ and $b^i = b_j\Omega^{ji}$. The only non-vanishing $\mathrm{Sl}(2;\mathbb{R})$ invariant that can be built out of these two doublets is

$$a^i b_i = p^0 p^1 + q_0 q_1 \equiv \tfrac{1}{2}\mathsf{W}(\mathcal{Q})\,. \tag{2.62}$$

Let us denote by $\mathcal{Q}^M(a,b)$ the standard symplectic charge vector seen as the direct sum of the two doublets a and b. Using the two doublets we can construct three further, up to a global sign, inequivalent charge vectors that under $\mathrm{Sl}(2;\mathbb{R})$ transform in the same way as $\mathcal{Q}^M(a,b)$, *i.e.*, equivariantly; the four equivariant charge vectors are

$$\begin{aligned}
\mathcal{Q}^M(a,b) &\equiv \begin{pmatrix} p^0 \\ p^1 \\ q_0 \\ q_1 \end{pmatrix} , \qquad \mathcal{Q}^M(b,-a) \equiv \begin{pmatrix} -q_1 \\ -q_0 \\ p^1 \\ p^0 \end{pmatrix} , \\
\mathcal{Q}^M(-a,b) &\equiv \begin{pmatrix} p^0 \\ -p^1 \\ q_0 \\ -q_1 \end{pmatrix} , \quad \mathcal{Q}^M(-b,-a) \equiv \begin{pmatrix} -q_1 \\ q_0 \\ p^1 \\ -p^0 \end{pmatrix} .
\end{aligned} \tag{2.63}$$

These equivariant vectors are generically linearly independent and provide a basis of equivariant vectors; any other equivariant vector, in particular the attractors B^M, can be expanded w.r.t. this base, *e.g.*

$$B^M = b^\sigma U_\sigma^M\,, \quad \text{with} \quad \{U_\sigma\} = \{\mathcal{Q}, \tilde{\mathcal{Q}}, \mathcal{P}, \tilde{\mathcal{P}}\}\,. \tag{2.64}$$

We will plug this general ansatz into the equation $\partial_M V_{\rm bh}(H,\mathcal{Q})|_{H=B} = 0$ as to find the most general attractor of the theory in Section 2.4.4, but at this point we already know

a non-trivial action on objects with symplectic indices and that coincides with the continuous global Freudenthal duality transformation. The scalars do not transform under this symmetry. On the other hand, only this U(1) symmetry is also a local symmetry of the H-FGK formalism. We would like to thank Alessio Marrani for clarifying discussions on this point.

some general results: The standard charge vector $\mathcal{Q}^M(a,b)$ will be the supersymmetric attractor, as usual, and we are going to see, $\mathcal{Q}^M(b,-a)$ is its Freudenthal dual

$$\mathcal{Q}^M(b,-a) = \tilde{\mathcal{Q}}^M(a,b) = \tilde{\mathcal{Q}}^M\,. \tag{2.65}$$

On the other hand, $\mathcal{Q}^M(-a,b)$ is the non-supersymmetric attractor $\mathcal{P}^M$ and $\mathcal{Q}^M(b,a)$ is its Freudenthal dual

$$\mathcal{Q}^M(-a,b) = \mathcal{P}^M\,, \qquad \mathcal{Q}^M(b,a) = \tilde{\mathcal{Q}}^M(b,-a) = \tilde{\mathcal{P}}^M\,. \tag{2.66}$$

It is easy to see that

$$\mathsf{W}(\tilde{\mathcal{Q}}) = \mathsf{W}(\mathcal{Q}) = -\mathsf{W}(\mathcal{P}) = -\mathsf{W}(\tilde{\mathcal{P}})\,. \tag{2.67}$$

These four vectors are related by $\mathrm{Sp}(4;\mathbb{R})$ transformations that however do not belong to $\mathrm{Sl}(2;\mathbb{R}) \subset \mathrm{Sp}(4;\mathbb{R})$:

$$\tilde{\mathcal{Q}}^M = \mathcal{A}^M{}_N\mathcal{Q}^N\,, \qquad (\mathcal{A}^M{}_N) \equiv \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}, \tag{2.68}$$

$$\mathcal{P}^M = \mathcal{B}^M{}_N\mathcal{Q}^N\,, \qquad (\mathcal{B}^M{}_N) \equiv \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}. \tag{2.69}$$

The only non-vanishing symplectic contractions between these four vectors are

$$\tilde{\mathcal{Q}}_M\mathcal{Q}^M = -\tilde{\mathcal{P}}_M\mathcal{P}^M = \mathsf{W}(\mathcal{Q})\,. \tag{2.70}$$

Apart from these moduli-independent equivariant vectors we can construct the generic moduli-dependent ones by taking the real and imaginary parts of Eq. (2.17), in which we can replace $\mathcal{Q}$ by any of the other three equivariant vectors. Observe that when we use the Freudenthal dual charge, we obtain the same complex equivariant vector but multiplied by $-i$.

2.4.2 H-FGK formalism

The solution of the stabilization equations of this theory is

$$\mathcal{R}_M(\mathcal{I}) = \mathcal{A}_{MN}\mathcal{I}^N\,, \qquad (\mathcal{A}_{MN}) \equiv \begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix}, \tag{2.71}$$

where σ^1 is the standard Pauli matrix. $\mathcal{A} = (\mathcal{A}_{MN})$ is a symplectic matrix:

$$\mathcal{A}\Omega\mathcal{A} = \Omega\,, \tag{2.72}$$

which is not surprising since it is just $-\mathcal{M}_{MN}(\mathcal{F})$. It follows that $(\mathcal{A}^M{}_N) = (\Omega^{PM}\mathcal{A}_{PN}) = -\Omega\mathcal{A}$ is also a symplectic matrix.

By definition, the original and tilded, *i.e.*, Freudenthal dual, H-variables are related by[13]

$$\tilde{H}_M(H) = \mathcal{A}_{MN}H^N\,, \qquad \tilde{H}^M(H) = \mathcal{A}^M{}_NH^N\,. \tag{2.74}$$

[13] Explicitly, we have

$$(\tilde{H}^M) = \begin{pmatrix} -\sigma^{1\,\Lambda\Sigma}H_\Sigma \\ \sigma^1{}_{\Lambda\Sigma}H^\Sigma \end{pmatrix} = \begin{pmatrix} -H_1 \\ -H_0 \\ H^1 \\ H^0 \end{pmatrix}. \tag{2.73}$$

This vector should be compared with $\mathcal{Q}^M(b,-a)$ in Eq. (2.63).

Therefore in this simple model the Freudenthal duality transformation is linear and is, furthermore, a symplectic transformation. It is clearly a transformation that does not belong to the global symmetries that act on the axidilaton (*i.e.*, $\mathrm{Sl}(2;\mathbb{R})$ whose embedding into $\mathrm{Sp}(4;\mathbb{R})$ is given in Eq. (2.57)), but it is a symmetry transformation that acts on objects with symplectic indices such as the vector fields and as such must be considered a part of the duality group of the model[14].

As expected in Freudenthal duality

$$\mathcal{A}^{M}{}_{P}\,\mathcal{A}^{P}{}_{N}=-\delta^{M}{}_{N}\,. \tag{2.75}$$

We can extend the Freudenthal duality transformation to all symplectic vectors. The properties

$$\tilde{X}_{M}Y^{M}=\tilde{Y}_{M}X^{M}=-Y_{M}\tilde{X}^{M}\,,\quad\Rightarrow\quad \tilde{X}_{M}\tilde{Y}^{M}=X_{M}Y^{M}\,, \tag{2.76}$$

which hold in this particular model for any two symplectic vectors X^M and Y^M because Freudenthal duality is a symplectic transformation, will be used very often.

The Hesse potential is given by the $\mathrm{Sl}(2;\mathbb{R})$ invariant discussed in earlier sections

$$\mathsf{W}(H)\equiv\tilde{H}_{M}(H)H^{M}=\mathcal{A}_{MN}H^{M}H^{N}=2(H^{0}H^{1}+H_{0}H_{1})\,, \tag{2.77}$$

and in accordance with the general formalism it determines the model completely: the effective action can be constructed entirely from it and the metric function e^{-2U} and the axidilaton λ are related to the Hesse potential by

$$e^{-2U}=\mathsf{W}(H)\,,\qquad \lambda\equiv iZ=i\frac{\tilde{H}^{1}+iH^{1}}{\tilde{H}^{0}+iH^{0}}=\frac{H^{1}+iH_{0}}{H_{1}-iH^{0}}\,. \tag{2.78}$$

The metric $g_{MN}(H)$ of this system can be written in the form

$$g_{MN}=2\,\mathfrak{N}_{MNPQ}\,\frac{H^{P}H^{Q}}{\mathsf{W}^{2}}\,, \tag{2.79}$$

where we have defined the constant matrix

$$\mathfrak{N}_{MNPQ}\equiv\mathcal{A}_{MN}\mathcal{A}_{PQ}-2\mathcal{A}_{MP}\mathcal{A}_{NQ}-\Omega_{MP}\Omega_{NQ}\,. \tag{2.80}$$

Using this notation, the derivatives of the metric take the form

$$\partial_{M}g_{PQ}=-4\frac{\tilde{H}_{M}}{\mathsf{W}}g_{PQ}+4\mathfrak{N}_{PQ(MR)}\frac{H^{R}}{\mathsf{W}^{2}}\,, \tag{2.81}$$

and the Christoffel symbols of the first kind are given by[15]

$$\begin{aligned}[PQ,M] &= 2\frac{\tilde{H}_{M}g_{PQ}-\tilde{H}_{P}g_{QM}-\tilde{H}_{Q}g_{PM}}{\mathsf{W}}\\ &\quad -[6\mathcal{A}_{PQ}\mathcal{A}_{MR}-4\mathcal{A}_{M(P}\mathcal{A}_{Q)R}+4\Omega_{M(P}\Omega_{Q)R}]\frac{H^{R}}{\mathsf{W}^{2}}\,.\end{aligned} \tag{2.82}$$

[14]See footnote 12.

[15]We remind the reader that the metric $g_{MN}(H)$ is not invertible, so we cannot use the standard Christoffel symbols $\Gamma_{PQ}{}^{M}\equiv g^{NM}[PQ,M]$.

It is easy to check that $\tilde{H}^M[PQ, M] = 0$, as required by Freudenthal duality invariance.

The potential V can be written in the convenient form

$$\mathsf{W}^2 V(H, \mathcal{Q}) = -\tfrac{1}{2}\mathsf{W}(\mathcal{Q})\mathsf{W} + (H^M \tilde{\mathcal{Q}}_M)^2 + (H^M \mathcal{Q}_M)^2 \,, \tag{2.83}$$

and its derivative reads

$$\partial_M V = -4\frac{\tilde{H}_M}{\mathsf{W}}\left[V + \tfrac{1}{4}\frac{\mathsf{W}(\mathcal{Q})}{\mathsf{W}}\right] + 2(\mathcal{Q}_M\mathcal{Q}_N + \tilde{\mathcal{Q}}_M\tilde{\mathcal{Q}}_N)\frac{H^N}{\mathsf{W}^2}\,; \tag{2.84}$$

using the properties Eq. (2.76) it is easy to see that $\tilde{H}^M\partial_M V = 0$, which is the last requirement for having local Freudenthal duality [189].

Observe that, in this model, a Freudenthal duality transformation of the charge vectors *only* (that is: not of the variables H^M), not only preserves $\mathsf{W}(\mathcal{Q})$ but also the complete potential and black-hole potential, *i.e.*,

$$\mathsf{W}(\tilde{\mathcal{Q}}) = \mathsf{W}(\mathcal{Q}) \quad \Rightarrow \quad V(H, \tilde{\mathcal{Q}}) = V(H, \mathcal{Q})\,, \quad \text{and} \quad V_{\text{bh}}(H, \tilde{\mathcal{Q}}) = V_{\text{bh}}(H, \mathcal{Q})\,. \tag{2.85}$$

On the other hand, using the definition of the fake charge Eq. (2.44) one can show that for any values of H^M

$$-V_{\text{bh}}(\mathcal{Q}) \;=\; -\tfrac{1}{2}\mathsf{W}(\mathcal{Q}) + 2|\mathcal{Z}(\mathcal{Q})|^2 = -\tfrac{1}{2}\mathsf{W}(\mathcal{P}) + 2|\mathcal{Z}(\mathcal{P})|^2 = -V_{\text{bh}}(\mathcal{P})\,, \tag{2.86}$$

$$|\mathcal{Z}(\mathcal{P})|^2 \;=\; |\mathcal{Z}(\mathcal{Q})|^2 - \tfrac{1}{2}\mathsf{W}(\mathcal{Q})\,. \tag{2.87}$$

The first identity means that, if $\mathcal{Q}$ is an attractor, so will $\mathcal{P}$. The fact that it is an identity for arbitrary values of H^M means that replacing $\mathcal{Q}$ by $\mathcal{P}$ in an extremal solution gives another extremal solution with the attractor $\mathcal{P}$. The second identity is a consequence of the first and implies that

$$\begin{aligned} \mathsf{W}(\mathcal{Q}) &< 0\,, \quad \Rightarrow \quad |\mathcal{Z}(\mathcal{P})| > |\mathcal{Z}(\mathcal{Q})|\,, \\ \mathsf{W}(\mathcal{Q}) &> 0\,, \quad \Rightarrow \quad |\mathcal{Z}(\mathcal{Q})| > |\mathcal{Z}(\mathcal{P})|\,, \end{aligned} \tag{2.88}$$

for all values of H^M. The second case should correspond to the supersymmetric attractor in which the evaporation process stops when the mass equals the largest central charge, which in this case is the true one.

Finally, observe that this black-hole potential satisfies the curious interchange property

$$V_{\text{bh}}(H, \mathcal{Q}) = \frac{\mathsf{W}(H)}{\mathsf{W}(\mathcal{Q})}\, V_{\text{bh}}(\mathcal{Q}, H)\,. \tag{2.89}$$

2.4.3 The symmetries in the H-FGK formalism

In Section 2.4.1 we discussed the global symmetries of the axidilaton model (more precisely, of its scalar manifold metric) when it is described in terms of the standard fields and have studied the embedding of these symmetries into $\mathrm{Sp}(4;\mathbb{R})$. It is in this form that we expect these symmetries to be present in the H-FGK formalism. On the other hand, there may be additional non-obvious symmetries such as Freudenthal duality (which is in general non-linear) in the H-FGK formalism.

Let us consider first the kinetic term: if we consider only linear transformations of the H^M

$$\delta H^M = T^M{}_N H^N\,, \tag{2.90}$$

it is evident that they will leave the kinetic term invariant if they are symplectic transformations, *i.e.*,

$$\Omega_{P(M}T^P{}_{N)} = 0\,, \tag{2.91}$$

and are furthermore symmetries of the Hesse potential

$$\delta \mathsf{W} = 2\tilde{H}_M \delta H^M = 2\tilde{H}_M\ T^M{}_N\ H^N = 0 \quad \longrightarrow \quad [\Omega\mathcal{A}, T] = \beta \mathbb{1}_{4\times 4}\,, \tag{2.92}$$

where β is a real constant that can vanish. It is not difficult to see that for infinitesimal symplectic transformations, β must indeed vanish, and the only independent generators that solve the above equation are the three $\mathrm{Sl}(2;\mathbb{R})$ generators T_i given in Eq. (2.58) plus

$$T_4 = \tfrac{1}{2}\mathcal{A}\Omega\,, \tag{2.93}$$

which generates the Freudenthal transformations and commutes with the generators of $\mathrm{Sl}(2;\mathbb{R})$[16].

It can be checked that these symmetries leave invariant the metric g_{MN}. Actually, the metric is invariant under the constant rescalings of the H^M

$$T_5 \equiv \tfrac{1}{4}\mathbb{1}_{4\times 4}\,, \tag{2.94}$$

which are not symplectic transformations and leave the Hesse potential invariant only up to a multiplicative constant, in the same way as the Kähler potential is invariant under isometries of the Kähler metric only up to Kähler transformations.

We can study now the invariance of the potential using the expression for $\partial_M V$ given in Eq. (2.84). The first term cancels for $i = 1, 2, 3, 4$ (we do not need to check $i = 5$: the potential is homogeneous of degree -2 and $\delta_5 V = -2V \neq 0$ in general) and the rest gives

$$\delta_i V = -2H^N T_i{}^M{}_N(\mathcal{Q}_M\mathcal{Q}_N + \tilde{\mathcal{Q}}_M\tilde{\mathcal{Q}}_N)\frac{H^N}{\mathsf{W}^2}\,, \tag{2.95}$$

which vanishes only for the Freudenthal transformation $i = 4$ unless we also perform the same transformation on the charge vector: this means that $\mathrm{Sl}(2;\mathbb{R})$ is only a *pseudo-symmetry* of the system, since the constants that enter the action are rotated. The charges appear as integration constants of the solution of the equations of motion for the electrostatic and magnetostatic potentials in Ref. [173] and $\mathrm{Sl}(2;\mathbb{R})$ is probably a (standard) symmetry of the effective theory before that.

There are no conserved quantities associated to pseudo-symmetries, whence there is only one conserved current: the one associated to the Freudenthal duality. This current vanishes, however, identically, which is a generic feature of the formalism.

[16]it is not difficult to see that the Hesse potential of the axidilaton model is not determined by $\mathrm{Sl}(2;\mathbb{R})$ invariance alone: one must require invariance under Freudenthal duality.

2.4.4 Critical points

The critical points of this model are equivariant vectors B^M satisfying the equations

$$\partial_M V_{\rm bh}|_{H=B} = -2\frac{\tilde{B}_M}{\mathsf{W}(B)}\left[V_{\rm bh}(B,\mathcal{Q})-\tfrac{1}{2}\mathsf{W}(\mathcal{Q})\right]-2(\mathcal{Q}_M\mathcal{Q}_N+\tilde{\mathcal{Q}}_M\tilde{\mathcal{Q}}_N)\frac{B^N}{\mathsf{W}(B)}=0\,. \quad (2.96)$$

Using the basis of equivariant vectors $\{U_\sigma\} = \{\mathcal{Q},\tilde{\mathcal{Q}},\mathcal{P},\tilde{\mathcal{P}}\}$ constructed in Section 2.4.1, we can write any such solution as

$$B^M = a\mathcal{Q}^M + \tilde{a}\tilde{\mathcal{Q}}^M + b\mathcal{P}^M + \tilde{b}\tilde{\mathcal{P}}^M\,. \quad (2.97)$$

The only non-vanishing symplectic products of the four basis vectors are

$$\tilde{\mathcal{Q}}_M\mathcal{Q}^M = \mathsf{W}(\mathcal{Q})\,, \qquad \tilde{\mathcal{P}}_M\mathcal{P}^M = -\mathsf{W}(\mathcal{Q})\,, \quad (2.98)$$

and a very simple calculation gives

$$\begin{aligned}\partial_M V_{\rm bh}|_{H=B} \quad &= \quad \frac{-2}{(a^2+\tilde{a}^2-b^2-\tilde{b}^2)}\Big\{\tilde{a}(b^2+\tilde{b}^2)\mathcal{Q}_M - a(b^2+\tilde{b}^2)\tilde{\mathcal{Q}}_M \\ &\quad +\tilde{b}(a^2+\tilde{a}^2)\mathcal{P}_M - b(a^2+\tilde{a}^2)\tilde{\mathcal{P}}_M\Big\}=0\,,\end{aligned} \quad (2.99)$$

which only admits two non-trivial solutions: $b=\tilde{b}=0$ and $a=\tilde{a}=0$. The first solution, up to global normalization (which is undetermined in this formalism because the black-hole potential is scale-invariant), corresponds to a global Freudenthal rotation with arbitrary angle of the standard supersymmetric attractor $B^M=\mathcal{Q}^M$ and the second corresponds to a global Freudenthal rotation of the standard non-supersymmetric attractor $B^M=\mathcal{P}^M$ [190].

We obtain the following relations

$$V_{\rm bh}(\mathcal{P},\mathcal{P}) = -V_{\rm bh}(\mathcal{Q},\mathcal{P}) = V_{\rm bh}(\mathcal{P},\mathcal{Q}) = -V_{\rm bh}(\mathcal{Q},\mathcal{Q}) = \tfrac{1}{2}\mathsf{W}(\mathcal{Q})\,, \quad (2.100)$$

that are necessary to have the corresponding near-horizon solutions, see Eq. (2.21).

2.4.5 Conventional extremal solutions

As a first simple illustration of the methods proposed in the first section of this chapter, we are going to review the construction of the extremal solutions[17] performed in Ref. [191].

From the results of that paper we know that all of them (including the extremal non-supersymmetric ones) are going to be conventional, but it is important for us to understand why. Thus, we start from the near-horizon solutions given by Eq. (2.20) where B^M takes the values of the attractors found in the previous section, normalized so that (see Eq. (2.21))

$$V_{\rm bh}(B,\mathcal{Q}) = V_{\rm bh}(B,B) = -\tfrac{1}{2}\mathsf{W}(B)\,. \quad (2.101)$$

[17] The axidilaton model is a particular case ($n=1$) of the $\overline{\mathbb{CP}}^n$ model. We will construct the most general non-extremal solutions of that model (and, hence, of the axidilaton model) later.

The attractors that satisfy these conditions are global Freudenthal rotations of the standard supersymmetric attractor $\mathcal{Q}^M$ and of the non-supersymmetric one $\mathcal{P}^M$, *i.e.*,

$$\begin{aligned} \text{either} \qquad B^M &= \cos\theta\, \mathcal{Q}^M + \sin\theta\, \tilde{\mathcal{Q}}^M \,, \\ \text{or} \qquad B^M &= \cos\theta\, \mathcal{P}^M + \sin\theta\, \tilde{\mathcal{P}}^M \,. \end{aligned} \tag{2.102}$$

The results of Section (2.2) guarantee that Eq. (2.20) provides a near-horizon solution for these choices of B^M. Now, to see if we can extend these solutions to asymptotically flat solutions by adding an infinitesimal constant vector to these H^M as in Eq. (2.23), we have to compute the rank of $\mathfrak{M}_{MN}$ in Eq. (2.27) to find how many independent solutions ε^M exist.

It is enough to consider a charge configuration whose orbit covers the complete charge space (see Appendix A.1) and, therefore, we set $p^0 = p^1 = 0$, getting, for the supersymmetric $(+)$ and non-supersymmetric $(-)$ cases, the matrix

$$(\mathfrak{M}_{MN}) = \tfrac{1}{2} \begin{pmatrix} \frac{1}{q_1^2} & \pm\frac{1}{q_0 q_1} & 0 & 0 \\ \pm\frac{1}{q_0 q_1} & \frac{1}{q_0^2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{2.103}$$

This matrix has rank 1 and, furthermore, the three independent solutions to Eq. (2.27) satisfy the constraint $B^M \varepsilon_M = 0$. This means that there is no obstruction to the addition of arbitrary (up to normalization $\mathsf{W}(A) = 1$ and the condition $B^M A_M = 0$) constants A^M to the near-horizon harmonic functions, which now take the form

$$H^M = A^M - \tfrac{1}{\sqrt{2}} B^M \tau \,. \tag{2.104}$$

The two independent components of A^M describe the two real moduli of this theory $\Re\mathfrak{e}(\lambda_\infty)$, $\Im\mathfrak{m}(\lambda_\infty)$ and A^M is given by [191]

$$A^M = \sqrt{2}\, \Im\mathfrak{m} \left(\frac{\mathcal{Z}^*(\phi_\infty, B)}{|\mathcal{Z}(\phi_\infty, B)|} \mathcal{V}^M_\infty \right) . \tag{2.105}$$

To show that the equations of motion are satisfied for finite constants A^M (which is only needed in the non-supersymmetric case) we can proceed as follows: from the linearity of the H^M it is possible to show that these configurations satisfy first-order flow equations [355]. These, in turn can be shown to imply the standard second-order equations of motion if and only if the identity

$$V_{\text{bh}}(H, \mathcal{Q}) = V_{\text{bh}}(H, B) \,, \tag{2.106}$$

is satisfied for arbitrary values of H. This is evident for $B^M = \mathcal{Q}^M$ (the supersymmetric attractor) and has been shown for $B^M = \mathcal{P}^M$ (the non-supersymmetric attractor) in Eq. (2.46) and the invariance of the black-hole potential under Freudenthal transformations of the charges extends this result to the other two (physically indistinguishable) attractors and proves that these configurations are classical solutions of the model.

2.4.6 Unconventional solutions

We do not expect more extremal black-hole solutions to the axidilaton model since the solutions constructed in the previous section already have the maximal number of independent physical parameters (charges $\mathcal{Q}^M$ and moduli λ_∞) which are constrained only by the requirement that the horizon has a non-vanishing area, *i.e.*, $\mathsf{W}(B) > 0$.

On the other hand, we can rewrite these solutions in an unconventional form (*i.e.*, so that $\dot{H}^M H_M \neq 0$) by using local Freudenthal duality transformations, but in this case doing so merely complicates the form of the solution in the H-FGK formalism.

2.5 The $\overline{\mathbb{CP}}^n$ model

The prepotential of the $\overline{\mathbb{CP}}^n$ model is given by[18]

$$\mathcal{F} = -\tfrac{i}{4}\eta_{\Lambda\Sigma}\mathcal{X}^\Lambda\mathcal{X}^\Sigma\,, \qquad (\eta_{\Lambda\Sigma}) = \mathrm{diag}(+-\cdots-)\,. \tag{2.107}$$

The $\overline{\mathbb{CP}}^n$ model contains n scalar fields given by

$$Z^i \equiv \mathcal{X}^i/\mathcal{X}^0\,, \tag{2.108}$$

but it is convenient to add $Z^0 \equiv 1$ and we define

$$(Z^\Lambda) \equiv \left(\mathcal{X}^\Lambda/\mathcal{X}^0\right) = (1, Z^i)\,, \qquad (Z_\Lambda) \equiv (\eta_{\Lambda\Sigma}Z^\Sigma) = (1, Z_i) = (1, -Z^i)\,. \tag{2.109}$$

The Kähler potential, the Kähler metric, the inverse Kähler metric and the covariantly holomorphic symplectic section read

$$\begin{aligned}
\mathcal{K} &= -\log\left(Z^{*\Lambda}Z_\Lambda\right)\,,\\
\mathcal{G}_{ij^*} &= -e^{\mathcal{K}}\left(\eta_{ij^*} - e^{\mathcal{K}}Z^*_i Z_{j^*}\right)\,,\\
\mathcal{G}^{ij^*} &= -e^{-\mathcal{K}}\left(\eta^{ij^*} + Z^i Z^{*\,j^*}\right)\,,\\
\mathcal{V} &= e^{\mathcal{K}/2}\begin{pmatrix} Z^\Lambda \\ \\ -\tfrac{i}{2}Z_\Lambda \end{pmatrix}\,.
\end{aligned} \tag{2.110}$$

It is also convenient to define the following complex charge combinations

$$\Gamma_\Lambda \equiv q_\Lambda + \tfrac{i}{2}\eta_{\Lambda\Sigma}p^\Sigma\,, \tag{2.111}$$

in terms of which the central charge, its holomorphic Kähler-covariant derivative and the black-hole potential are

$$\begin{aligned}
\mathcal{Z} &= e^{\mathcal{K}/2}Z^\Lambda\Gamma_\Lambda \equiv \mathcal{Z}(\Gamma)\,,\\
\mathcal{D}_i\mathcal{Z} &= e^{3\mathcal{K}/2}Z^*_i Z^\Lambda\Gamma_\Lambda - e^{\mathcal{K}/2}\Gamma_i\,,\\
-V_{\rm bh} &= 2e^{\mathcal{K}}|Z^\Lambda\Gamma_\Lambda|^2 - \Gamma^{*\,\Lambda}\Gamma_\Lambda\,.
\end{aligned} \tag{2.112}$$

[18] The black-hole solutions of this model have been studied in [190].

We can extend this complex notation to any symplectic vector:

$$\text{if} \quad (A^M) = \begin{pmatrix} a^\Lambda \\ b_\Lambda \end{pmatrix} \quad \text{then} \quad \left\{ \begin{array}{rcl} \mathcal{A}_\Lambda & \equiv & b_\Lambda + \frac{i}{2}\eta_{\Lambda\Sigma}a^\Sigma \,, \\ \mathcal{A}^\Lambda & \equiv & \eta^{\Lambda\Sigma}\mathcal{A}_\Sigma \;=\; \eta^{\Lambda\Sigma}b_\Sigma + \frac{i}{2}a^\Lambda \,, \end{array} \right. \tag{2.113}$$

and the symplectic product of two vectors becomes

$$A_M B^M = -2\Im\mathfrak{m}(\mathcal{A}_\Lambda \mathcal{B}^{*\,\Lambda}) \,, \tag{2.114}$$

where of course $\mathcal{A}_\Lambda \mathcal{B}^{*\,\Lambda} = \mathcal{A}^\Lambda \mathcal{B}^*{}_\Lambda$. We will use both notations, based on convenience.

2.5.1 The global symmetries of the $\overline{\mathbb{CP}}^n$ model

The n complex scalars of the $\overline{\mathbb{CP}}^n$ model parametrize the symmetric coset space $\mathrm{SU}(1,n)/\mathrm{SU}(n)$, and the full theory is invariant under global $\mathrm{SU}(1,n)$ transformations[19]. If $\Lambda^\Lambda{}_\Sigma$ is a generic element in the fundamental representation of $\mathrm{SU}(1,n)$, *i.e.*, if it satisfies

$$\Lambda^{*\,\Gamma}{}_\Lambda\, \eta_{\Gamma\Delta}\, \Lambda^\Delta{}_\Sigma = \eta_{\Lambda\Sigma} \,, \qquad (\text{or} \;\; \Lambda^\dagger \eta \Lambda = \eta) \,, \qquad \det\Lambda = 1 \,, \tag{2.115}$$

then its action on the scalars is given by

$$Z'^{\,\Lambda} = \frac{\Lambda^\Lambda{}_\Sigma Z^\Sigma}{\Lambda^0{}_\Sigma Z^\Sigma} \,, \qquad Z'_{\Lambda} = \frac{\Lambda_\Lambda{}^\Sigma Z_\Sigma}{\Lambda^0{}_\Sigma Z^\Sigma} \,, \tag{2.116}$$

where we have raised and lowered the indices of the $\mathrm{SU}(1,n)$ matrix with the metric η. In the fundamental representation the $n(n+2)$ infinitesimal generators of $\mathfrak{su}(1,n)$

$$\Lambda^\Lambda{}_\Sigma \sim \delta^\Lambda{}_\Sigma + \alpha^m\, T_m{}^\Lambda{}_\Sigma \,, \tag{2.117}$$

are matrices such that $T_{m\,\Lambda\Sigma} = \eta_{\Lambda\Gamma}\, T_m{}^\Gamma{}_\Sigma$ is anti-Hermitean. Substituting the infinitesimal linear transformations in the non-linear transformation rules of the scalars, Eq. (2.116), we find that they take the form

$$Z'^{\,\Lambda} = Z^\Lambda + \alpha^m k_m{}^\Lambda(Z) \,, \tag{2.118}$$

where $k_m{}^\Lambda(Z)$, the holomorphic part of the Killing vectors K_m, is given by[20]

$$k_m{}^\Lambda(Z) = T_m{}^\Lambda{}_\Sigma\, Z^\Sigma - T_m{}^0{}_\Omega\, Z^\Omega Z^\Lambda \,. \tag{2.119}$$

The commutation relations of the generators T_m and the Lie brackets of the Killing vectors are related as usual:

$$[T_m, T_n] = f_{mn}{}^p\, T_p \,, \qquad [K_m, K_n] = -f_{mn}{}^p\, K_p \,. \tag{2.120}$$

[19] Actually, the coset space can also be described as $\mathrm{U}(1,n)/\mathrm{U}(n)$, which would imply that the global symmetry group of the model is $\mathrm{U}(1,n)$. As in the axidilaton model (the $n=1$ case), the extra $\mathrm{U}(1)$, that does not act on the scalars, is the Freudenthal duality group (see footnote 12). We thank Alessio Marrani for clarifying discussions on this point.

[20] The $\Lambda = 0$ component vanishes, as it should, but it is useful to keep it.

The action of the finite $\mathrm{SU}(1,n)$ transformations on the Kähler potential and on the canonical covariantly-holomorphic symplectic section $\mathcal{V}$ are given by the obvious generalization of Eqs. (2.54) and (2.55) where now

$$f(Z) = \log\left(\Lambda^0{}_\Sigma Z^\Sigma\right), \tag{2.121}$$

$$(S^M{}_N) = \begin{pmatrix} \Re\mathfrak{e}\Lambda^\Lambda{}_\Sigma & -2\Im\mathfrak{m}\Lambda^{\Lambda\Sigma} \\ \tfrac{1}{2}\Im\mathfrak{m}\Lambda_{\Lambda\Sigma} & \Re\mathfrak{e}\Lambda_\Lambda{}^\Sigma \end{pmatrix}, \tag{2.122}$$

where once again we have raised and lowered the indices of $\Lambda^\Lambda{}_\Sigma$ with η. The condition $\Lambda^\dagger\eta\Lambda = \eta$ implies for the real and imaginary parts of Λ

$$\Re\mathfrak{e}\Lambda_{\Delta\Lambda}\,\Im\mathfrak{m}\Lambda^\Delta{}_\Sigma = \Im\mathfrak{m}\Lambda_{\Delta\Lambda}\,\Re\mathfrak{e}\Lambda^\Delta{}_\Sigma\,, \qquad \Re\mathfrak{e}\Lambda_{\Delta\Lambda}\,\Re\mathfrak{e}\Lambda^\Delta{}_\Sigma + \Im\mathfrak{m}\Lambda_{\Delta\Lambda}\,\Im\mathfrak{m}\Lambda^\Delta{}_\Sigma = \eta_{\Lambda\Sigma}\,, \tag{2.123}$$

and implies that the matrix $(S^M{}_N)$ constructed above satisfies $S^T\Omega S = \Omega$ and therefore belongs to $\mathrm{Sp}(2n_v+2;\mathbb{R})$. The infinitesimal generators in this representation, *i.e.*, $(T_m{}^M{}_N)$, can be constructed in the same way, leading to

$$(T_m{}^M{}_N) = \begin{pmatrix} \Re\mathfrak{e}T_m{}^\Lambda{}_\Sigma & -2\Im\mathfrak{m}T_m{}^{\Lambda\Sigma} \\ \tfrac{1}{2}\Im\mathfrak{m}T_{m\,\Lambda\Sigma} & \Re\mathfrak{e}T_{m\,\Lambda}{}^\Sigma \end{pmatrix}. \tag{2.124}$$

Equivariant vectors

The search for equivariant vectors is simplified by using the complex combinations defined above: we look for vectors $\mathcal{B}^\Lambda$ behaving as Γ^Λ under duality transformations, *i.e.*, such that its complex conjugate transforms in the fundamental representation of $\mathrm{SU}(1,n)$

$$\Gamma^{*\,\prime\,\Lambda} = \Lambda^\Lambda{}_\Sigma\,\Gamma^{*\,\Sigma}\,, \quad \Rightarrow \quad \mathcal{B}^{*\,\prime\,\Lambda} = \Lambda^\Lambda{}_\Sigma\,\mathcal{B}^{*\Sigma}\,. \tag{2.125}$$

Observe that $\Gamma^{*\,\Lambda}\Gamma_\Lambda$ and $\mathcal{B}^{*\,\Lambda}\mathcal{B}_\Lambda$ are duality invariants.

The simplest equivariant vectors are, up to a complex constant, just equal to the charge vector Γ^Λ. This constant is relevant because, as we will see, the complex form of the Freudenthal dual of the charge vector

$$\tilde{\mathcal{Q}}^M = \begin{pmatrix} -2\,\eta^{\Sigma\Lambda}q_\Lambda \\ \tfrac{1}{2}\,\eta_{\Lambda\Sigma}p^\Lambda \end{pmatrix}, \tag{2.126}$$

is just $\tilde{\Gamma}^\Lambda = -i\Gamma^\Lambda$, whence the phase of the constant corresponds to a global Freudenthal duality rotation. This immediately implies that the $\mathrm{SU}(1,n)$ invariants $\Gamma^{*\,\Lambda}\Gamma_\Lambda$ and $\mathcal{B}^{*\,\Lambda}\mathcal{B}_\Lambda$ are also invariant under Freudenthal $\mathrm{U}(1)$ duality. There may be other equivariant vectors which are functions of the charges only, but we will not need them.

We can use the moduli Z^Λ_∞ in order to construct more equivariant vectors. Again, up to normalization, the only one we will need is the generic vector given in Eq. (2.17). Multiplying it by the invariant $\Gamma^{*\,\Lambda}\Gamma_\Lambda$ as to give it the right dimensions for later convenience, we have the equivariant vector

$$\Sigma^\Lambda \equiv \frac{Z^{*\,\Lambda}_\infty}{Z^{*\,\Sigma}_\infty\Gamma^*_\Sigma}\,\Gamma^{*\,\Sigma}\Gamma_\Sigma\,. \tag{2.127}$$

We will see that in order to find the most general solutions of this model, it is enough to consider complex linear combinations of the two equivariant vectors constructed thus far:

$$\mathcal{B}^\Lambda = \alpha\Gamma^\Lambda + \beta\Sigma^\Lambda\,, \tag{2.128}$$

where α and β are complex duality invariants (including pure numbers).

Using this information we can see that in this model (for generic n), in distinction to the axidilaton model, we cannot define a fake charge $\mathcal{B}^\Lambda$ and its associated fake central charge $\mathcal{Z}(\mathcal{B})$ such that

$$\mathcal{G}^{ij^*}\mathcal{D}_i\mathcal{Z}\mathcal{D}_{j^*}\mathcal{Z}^* = |\mathcal{Z}(\mathcal{B})|^2 = e^{\mathcal{K}}|Z^\Lambda\Gamma_\Lambda|^2 - \Gamma^{*\,\Lambda}\Gamma_\Lambda\,, \tag{2.129}$$

or such that

$$V_{\rm bh}(\mathcal{Q}) = V_{\rm bh}(\mathcal{B})\,, \tag{2.130}$$

for arbitrary values of the scalars. This fact has important implications for the construction of extremal non-supersymmetric solutions as the first-order equations do not imply the second order ones, which therefore have to be solved explicitly. In this chapter we are going to construct directly the general non-extremal solutions from which all the extremal ones can be obtained in the appropriate limits.

2.5.2 H-FGK formalism

The stabilization equations of this model are solved by a linear relation between $\mathcal{R}_M$ and $\mathcal{I}^M$, as in the axidilaton case:

$$\mathcal{R}_M(\mathcal{I}) = \mathcal{A}_{MN}\mathcal{I}^N\,, \qquad (\mathcal{A}_{MN}) = \begin{pmatrix} \frac{1}{2}\eta_{\Lambda\Sigma} & 0 \\ 0 & 2\eta^{\Lambda\Sigma} \end{pmatrix}\,, \tag{2.131}$$

which implies that the Freudenthal dual can be expressed as

$$\tilde{H}^M = \mathcal{A}^M{}_N H^N\,, \qquad (\mathcal{A}^M{}_N) = (\Omega^{PM}\mathcal{A}_{PN}) = \begin{pmatrix} 0 & -2\eta^{\Lambda\Sigma} \\ \frac{1}{2}\eta_{\Lambda\Sigma} & 0 \end{pmatrix}\,. \tag{2.132}$$

As in the axidilaton case, $\mathcal{A}_{MN}$ is a symplectic matrix, but, in contradistinction to that case, $\mathcal{A}^M{}_N$ is not. In terms of the complex H-variables[21]

$$\mathcal{H}_\Lambda \equiv H_\Lambda + \tfrac{i}{2}\eta_{\Lambda\Sigma}H^\Sigma\,, \tag{2.133}$$

discrete Freudenthal duality is equivalent to multiplication by a factor of $-i$.

The Hesse potential reads

$$\mathsf{W}(H) = \mathcal{A}_{MN}H^M H^N = \tfrac{1}{2}\eta_{\Lambda\Sigma}H^\Lambda H^\Sigma + 2\eta^{\Lambda\Sigma}H_\Lambda H_\Sigma = 2\mathcal{H}^{*\,\Lambda}\mathcal{H}_\Lambda\,, \tag{2.134}$$

and the metric function e^{-2U} and the scalars Z^i can be easily obtained from it as

$$e^{-2U} = \mathsf{W}(H)\,, \qquad Z^i = \frac{\tilde{H}^i + iH^i}{\tilde{H}^0 + iH^0} = \frac{H_i + \frac{i}{2}H^i}{-H_0 + \frac{i}{2}H^0} = \frac{\mathcal{H}_i^*}{\mathcal{H}_0^*}\,. \tag{2.135}$$

[21] Observe that, in his notation, $\mathcal{H}^\Lambda \equiv \eta^{\Lambda\Sigma}\mathcal{H}_\Sigma$ but $H^\Lambda \neq \eta^{\Lambda\Sigma}H_\Sigma$.

The metric $g_{MN}(H)$ and the potential $V(H)$ have the same structure as in the axidilaton case when we write them in terms of the matrix $\mathcal{A}_{MN}$ (which, evidently, is different). Then, the expressions from Eq. (2.79) to Eq. (2.84) are also valid here upon use of the new matrix $\mathcal{A}_{MN}$.

The central charge of the model, Eq. (2.112), takes in the H-FGK formalism the form

$$\mathcal{Z}(H,\mathcal{Q}) = -\frac{(H_0+\frac{i}{2}H^0)}{\left|H_0+\frac{i}{2}H^0\right|}\,\frac{(\tilde{H}_M+iH_M)\mathcal{Q}^M}{\sqrt{2\mathsf{W}(H)}}. \tag{2.136}$$

It is easy to check that, like in the axidilaton case, this black-hole potential satisfies

$$V_{\rm bh}(H,\mathcal{Q}) = \frac{\mathsf{W}(\mathcal{Q})}{\mathsf{W}(H)}\,V_{\rm bh}(\mathcal{Q},H)\,. \tag{2.137}$$

2.5.3 Critical points

Using the complex notation we can write the equation for the critical points $\mathcal{B}_\Lambda$ of the black-hole potential of this model in the form

$$\tfrac{i}{2}\mathsf{W}(\mathcal{B})\,\partial^*_\Lambda V_{\rm bh}|_{\mathcal{H}=\mathcal{B}} = \frac{\mathcal{B}^\Sigma\Gamma^*_\Sigma}{\mathsf{W}(B)}\left[\mathcal{B}^{*\,\Delta}\Gamma_\Delta\mathcal{B}_\Lambda - \mathcal{B}^{*\,\Delta}\mathcal{B}_\Delta\Gamma_\Lambda\right] = 0\,, \tag{2.138}$$

and can be solved by

$$\mathcal{B}^\Sigma\Gamma^*_\Sigma = 0\,, \quad \text{or} \quad \mathcal{B}^{*\,\Delta}\Gamma_\Delta\mathcal{B}_\Lambda - \mathcal{B}^{*\,\Delta}\mathcal{B}_\Delta\Gamma_\Lambda = 0\,. \tag{2.139}$$

Inserting the general ansatz (2.128) into the first condition we find that it is satisfied for

$$\alpha = -\beta\,, \quad \Rightarrow \quad \mathcal{B}^\Lambda = \alpha(\Gamma^\Lambda - \Sigma^\Lambda)\,, \tag{2.140}$$

which, up to normalization (which is not fixed in this approach), leaves us with one arbitrary global phase associated to Freudenthal duality: this is the moduli-dependent attractor found in Ref. [190].

Inserting our ansatz (2.128) into the second condition we get the equation

$$\beta(\alpha^*+\beta^*)\Gamma^{*\,\Delta}\Gamma_\Delta\Sigma_\Lambda - \left[2\mathfrak{Re}(\alpha\beta^*) + \frac{|\beta|^2\Gamma^{*\,\Sigma}\Gamma_\Sigma}{|\mathcal{Z}_\infty(\Gamma)|^2}\right]\Gamma^{*\,\Delta}\Gamma_\Delta\Gamma_\Lambda = 0\,. \tag{2.141}$$

The coefficients of the two equivariant vectors must vanish separately, which can only happen for $\beta=0$, whence $\mathcal{B}^\Lambda = \alpha\Gamma^\Lambda$: up to normalization and the Freudenthal duality phase, this is the supersymmetric attractor.

2.5.4 Conventional non-extremal solutions

In this section we are going to show how the knowledge of the equivariant vectors of the model simplifies the construction of solutions in the H-FGK formalism. We are going to see that the most general solution can be written as

$$\mathcal{H}^\Lambda(\tau) = a(\tau)\Gamma^\Lambda + b(\tau)\Sigma^\Lambda\,, \tag{2.142}$$

where $a(\tau)$ and $b(\tau)$ are two complex, duality-invariant functions of τ to be determined. Already, at this stage, we see that this ansatz reduces dramatically the number of real functions to be found, from $2n_v+2$ to just 4, and all of this without any loss of generality.

First of all, we are going to impose the usual Freudenthal gauge-fixing condition $\dot{H}^M H_M = 0$ [189] which in complex notation takes the form

$$\Im\mathfrak{m}(\dot{\mathcal{H}}^{*\,\Lambda}\mathcal{H}_\Lambda) = 0\,. \tag{2.143}$$

As shown in Ref. [189], assuming this condition, the contraction of the equations of motion with H^M leads to the equation

$$\tilde{H}_M\left(\ddot{H}^M - r_0^2 H^M\right) = 0\,, \tag{2.144}$$

which can always be solved by

$$\ddot{H}^M = r_0^2 H^M\,, \quad \Rightarrow \quad \ddot{\mathcal{H}}^\Lambda = r_0^2\mathcal{H}^\Lambda\,. \tag{2.145}$$

This is not necessarily the only solution of Eq. (2.144), but as we are going to see it allows us to solve the rest of the equations without imposing unnecessary constraints on the physical parameters of the solution. This equation combined with the equivariant ansatz leads to

$$\mathcal{H}^\Lambda(\tau) = \left[c_1 e^{r_0\tau} + c_3 e^{-r_0\tau}\right]\Gamma^\Lambda + \left[c_2 e^{r_0\tau} + c_4 e^{-r_0\tau}\right]\Sigma^\Lambda\,, \tag{2.146}$$

so it only remains to determine the four complex invariants c_i $(i=1,\cdots,4)$ in terms of the charges Γ_Λ, the moduli Z^Λ_∞ and the mass M (or alternatively of the non-extremality parameter r_0).

These four constants can be constrained even further by requiring that the ansatz gives the right asymptotic behavior for the physical fields in Eq. (2.135): requiring that $Z^\Lambda_\infty = \mathcal{H}^{*\,\Lambda}_\infty/\mathcal{H}^{*\,0}_\infty$ we get[22]

$$c_1 + c_3 = 0\,. \tag{2.147}$$

Asymptotic flatness requires that $\mathcal{H}^{*\Lambda}_\infty\mathcal{H}_{\Lambda,\infty} = \frac{1}{2}$ which, upon use of the above condition, gives

$$|c_2 + c_4|^2 - \frac{|\mathcal{Z}_\infty(\Gamma)|^2}{2(\Gamma^{*\,\Lambda}\Gamma_\Lambda)^2} = 0\,, \tag{2.148}$$

where $\mathcal{Z}_\infty(\Gamma)$ is the central charge at spatial infinity. The gauge-fixing condition (2.143) gives (again, upon use of Eq. (2.147))

$$\Im\mathfrak{m}\left[c_3^*(c_2+c_4)\right] + \Im\mathfrak{m}\left[c_2^* c_4\right]\frac{\Gamma^{*\,\Lambda}\Gamma_\Lambda}{|\mathcal{Z}_\infty(\Gamma)|^2} = 0\,. \tag{2.149}$$

Finally, we can still make global Freudenthal duality rotations, which are not fixed by Eq. (2.143): this freedom cannot be used to solve Eq. (2.149) but can be used to simplify it by fixing the phase of one of the constants to a convenient value.

Using the gauge-fixing condition (2.143), the Hamiltonian constraint takes the form

$$\left[\dot{\mathcal{H}}^{*\,\Lambda}\dot{\mathcal{H}}_\Lambda - \tfrac{1}{2}\Gamma^{*\,\Lambda}\Gamma_\Lambda\right]\mathcal{H}^{*\,\Sigma}\mathcal{H}_\Sigma - 2(\dot{\mathcal{H}}^{*\,\Lambda}\mathcal{H}_\Lambda)^2 + \left|\mathcal{H}^{*\,\Lambda}\Gamma_\Lambda\right|^2 - r_0^2(\mathcal{H}^{*\,\Lambda}\mathcal{H}_\Lambda)^2 = 0\,, \tag{2.150}$$

[22] In the (H-)FGK coordinate system, spatial infinity corresponds to the limit $\tau \to 0^-$.

and using the gauge-fixing condition plus Eq. (2.145) and the Hamiltonian constraint above, the equations of motion take the form

$$\mathcal{H}^*_\Lambda\left[2(\dot{\mathcal{H}}^{*\,\Sigma}\mathcal{H}_\Sigma)^2-\left|\mathcal{H}^{*\,\Sigma}\Gamma_\Sigma\right|^2\right]+\Gamma^*_\Lambda(\mathcal{H}^{*\,\Sigma}\Gamma_\Sigma)(\mathcal{H}^{*\,\Delta}\mathcal{H}_\Delta)-2\dot{\mathcal{H}}^*_\Lambda(\dot{\mathcal{H}}^{*\,\Sigma}\mathcal{H}_\Sigma)(\mathcal{H}^{*\,\Delta}\mathcal{H}_\Delta)=0\,. \tag{2.151}$$

The coefficients of the two equivariant vectors Γ_Λ and Σ_Λ must vanish independently, which implies that we must solve the following equations

$$a^*\left[2(\dot{\mathcal{H}}^{*\,\Sigma}\mathcal{H}_\Sigma)^2-\left|\mathcal{H}^{*\,\Sigma}\Gamma_\Sigma\right|^2\right]+(\mathcal{H}^{*\,\Sigma}\Gamma_\Sigma)(\mathcal{H}^{*\,\Delta}\mathcal{H}_\Delta)-2\dot{a}^*(\dot{\mathcal{H}}^{*\,\Sigma}\mathcal{H}_\Sigma)(\mathcal{H}^{*\,\Delta}\mathcal{H}_\Delta)=0\,, \tag{2.152}$$

$$b^*\left[2(\dot{\mathcal{H}}^{*\,\Sigma}\mathcal{H}_\Sigma)^2-\left|\mathcal{H}^{*\,\Sigma}\Gamma_\Sigma\right|^2\right]-2\dot{b}^*(\dot{\mathcal{H}}^{*\,\Sigma}\mathcal{H}_\Sigma)(\mathcal{H}^{*\,\Delta}\mathcal{H}_\Delta)=0\,. \tag{2.153}$$

The coefficients of b^* and $\dot{b}^*$ in the last equation are real (on account of the gauge-fixing condition) and this implies that the phases of c_2 and c_4 must be the same up to π (the global sign) so that $\Im\mathfrak{m}(c_2^*c_4)=0$. Then, Eq. (2.149) states that the phase of c_3 must be the same as that of c_2 and c_4, again up to π. We know that in the near-horizon limit (*i.e.*, $\tau\to-\infty$) of the extremal non-supersymmetric case the phases of c_3 and c_4 must differ by π and, since this difference is constant, this must always be the case. Furthermore, in the extremal non-supersymmetric case $\mathcal{Z}_\infty(\Gamma)=0$ and Eq. (2.148) implies that c_2 and c_4 must also have opposite global signs. Therefore we find

$$\arg(c_3)=\arg(c_2)=\arg(c_4)+\pi\equiv\theta\,, \tag{2.154}$$

and, by making use of the global Freudenthal duality freedom

$$|c_2|-|c_4|=-\frac{|\mathcal{Z}_\infty(\Gamma)|}{\sqrt{2\Gamma^{*\,\Lambda}\Gamma_\Lambda}}\,. \tag{2.155}$$

To simplify the calculations further, we introduce the constant A

$$|c_2|+|c_4|=-\frac{|\mathcal{Z}_\infty(\Gamma)|}{\sqrt{2\Gamma^{*\,\Lambda}\Gamma_\Lambda}}A\,, \tag{2.156}$$

which allows us to rewrite Eq. (2.146) as

$$\mathcal{H}^\Lambda(\tau)=e^{i\theta}\left\{-2|c_3|\sinh r_0\tau\Gamma^\Lambda+\frac{|\mathcal{Z}_\infty(\Gamma)|}{\sqrt{2\Gamma^{*\,\Lambda}\Gamma_\Lambda}}\left[(1+A)e^{-r_0\tau}+(1-A)e^{r_0\tau}\right]\Sigma^\Lambda\right\}\,. \tag{2.157}$$

It is now straightforward to solve the equations of motion for the three constants θ, A and $|c_3|$, for which it is convenient to express the final result using the mass M (defined in Eq. (2.31))

$$M=r_0\left[A+2\sqrt{2}|c_3||\mathcal{Z}_\infty(\Gamma)|\right]\,. \tag{2.158}$$

The final result is

$$
\begin{aligned}
|c_3| &= \frac{|\mathcal{Z}_\infty(\Gamma)|}{2\sqrt{2}Mr_0}\,, && (2.159)\\
A &= \frac{M^2-|\mathcal{Z}_\infty(\Gamma)|^2}{Mr_0}\,, && (2.160)\\
e^{i\theta} &= \pm\frac{\mathcal{Z}_\infty(\Gamma)}{|\mathcal{Z}_\infty(\Gamma)|}\,, && (2.161)\\
M^2r_0^2 &= \left[M^2-|\hat{\mathcal{Z}}_\infty|^2\right]\left[M^2-|\mathcal{Z}_\infty(\Gamma)|^2\right]\,, && (2.162)
\end{aligned}
$$

which is precisely the result obtained in Ref. [190].

We do not expect any other Freudenthal-inequivalent solutions to this model since the solutions we just found have the maximal number of independent physical parameters.

2.6 The t^3 model

The t^3-model is characterized by the prepotential

$$
\mathcal{F}(\mathcal{X}) = -\tfrac{5}{6}\frac{(\mathcal{X}^1)^3}{\mathcal{X}^0}\,. \tag{2.163}
$$

In terms of the coordinate $t=\mathcal{X}^1/\mathcal{X}^0$, the Kähler potential and the scalar-manifold metric are given by

$$
\mathcal{K} = -3\ln\Im\mathfrak{m}\,t - \ln\tfrac{20}{3}\,, \qquad \mathcal{G}_{tt^*} = \tfrac{3}{4}\,(\Im\mathfrak{m}\,t)^{-2}\;; \tag{2.164}
$$

the covariantly holomorphic symplectic section reads

$$
\mathcal{V}(t,t^*) = e^{\mathcal{K}/2}\begin{pmatrix} 1\\ t\\ \frac{5}{6}t^3\\ -\frac{5}{2}t^2\end{pmatrix}\,, \tag{2.165}
$$

and the central charge, its covariant derivative, the black-hole potential and its partial derivative read

$$
\begin{aligned}
\mathcal{Z} &\equiv e^{\frac{1}{2}\mathcal{K}}\hat{\mathcal{Z}}\,, && (2.166)\\
\mathcal{D}_t\mathcal{Z} &\equiv \tfrac{i}{2}\frac{e^{\frac{1}{2}\mathcal{K}}}{\Im\mathfrak{m}\,t}\hat{\mathcal{W}}\,, && (2.167)\\
-V_{\rm bh} &= e^{\mathcal{K}}\left[|\hat{\mathcal{Z}}|^2+\tfrac{1}{3}|\hat{\mathcal{W}}|^2\right]\,, && (2.168)\\
-\partial_t V_{\rm bh} &= \tfrac{i}{20}(\Im\mathfrak{m}\,t)^{-4}\left[(\hat{\mathcal{W}}^*)^2+3\hat{\mathcal{W}}\hat{\mathcal{Z}}^*\right]\,, && (2.169)
\end{aligned}
$$

where we have defined

$$
\begin{aligned}
\hat{\mathcal{Z}} &= \tfrac{5}{6}p^0t^3-\tfrac{5}{2}p^1t^2-q_1t-q_0\,, && (2.170)\\
\hat{\mathcal{W}} &= \tfrac{5}{2}p^0t^2t^*-\tfrac{5}{2}p^1t(t+2t^*)-q^1(2t+t^*)-3q^0\,. && (2.171)
\end{aligned}
$$

Observe that all these objects are well defined only iff $\Im\mathfrak{m}\,t>0$.

2.6.1 The global symmetries of the t^3 model

The t^3 model as a theory of $\mathcal{N}=2, d=4$ supergravity is invariant under global $\mathrm{Sl}(2;\mathbb{R})$ transformations, just like the axidilaton model, since their Kähler metrics are identical up to a numerical factor. The action of $\mathrm{Sl}(2;\mathbb{R})$ on t is identical to its action on λ, which was discussed in Section 2.4.1. The transformations of the Kähler potential and covariantly-holomorphic symplectic section Eqs. (2.54,2.55) are determined by the holomorphic function $f(t)$ and the $\mathrm{Sp}(4;\mathbb{R})$ matrix $S^M{}_N$ given by

$$f(t) \;=\; 3\ln(ct+d)\,, \tag{2.172}$$

$$(S^M{}_N) \;=\; \begin{pmatrix} d^3 & 3d^2c & \frac{6}{5}c^3 & -\frac{6}{5}dc^2 \\ bd^2 & (ad+2bc)d & \frac{6}{5}ac^2 & -\frac{2}{5}(2ad+bc)c \\ \frac{5}{6}b^3 & \frac{5}{2}ab^2 & a^3 & -a^2b \\ -\frac{5}{2}b^2d & -\frac{5}{2}(2ad+bc)b & -3a^2c & (ad+2bc)a \end{pmatrix}. \tag{2.173}$$

In this case the 4-dimensional representation of the generators T_m are given by

$$(T_1{}^M{}_N) = \begin{pmatrix} 3 & & & \\ & 1 & & \\ & & -3 & \\ & & & -1 \end{pmatrix}, \qquad (T_2{}^M{}_N) = \begin{pmatrix} & -3 & & \\ -1 & & & 4/5 \\ & & & 1 \\ & 5 & 3 & \end{pmatrix},$$

$$(T_3{}^M{}_N) = \begin{pmatrix} & -3 & & \\ 1 & & & 4/5 \\ & & & -1 \\ & -5 & 3 & \end{pmatrix}. \tag{2.174}$$

As in the axidilaton model, the same transformations act on all the symplectic vectors of the theory and, in particular on H^M and $\mathcal{Q}^M$. There are no more symmetries in this formulation of the model.

Equivariant vectors of the t^3 model

It is not difficult to see that, from the point of view of $\mathrm{Sl}(2;\mathbb{R})$, the symplectic vectors such as the charge vector $\mathcal{Q}^M$ transform as a quadruplet, *i.e.*, a fully symmetric 3-index covariant tensor $\mathcal{Q}_{ijk}=\mathcal{Q}_{(ijk)}$ (in the notation used in Section 2.4.1). The relation between the components of this tensor and those of the charge vector is

$$\mathcal{Q}_{111}=p^0\,,\qquad \mathcal{Q}_{112}=-p^1\,,\qquad \mathcal{Q}_{122}=-\tfrac{2}{5}q_1\,,\qquad \mathcal{Q}_{222}=-\tfrac{6}{5}q_0\,. \tag{2.175}$$

It is useful to observe that the contraction of two quadruplets is related to the symplectic product by

$$A_{ijk}B^{ijk}=-\tfrac{6}{5}A^M B_M\,. \tag{2.176}$$

By definition, any new $\mathrm{Sl}(2;\mathbb{R})$ quadruplet that we construct out of t_∞ and $\mathcal{Q}_{ijk}$ can be transformed according to the above rules into an equivariant symplectic vector of the

t^3-model. The $\mathrm{Sl}(2;\mathbb{R})$ index notation makes this construction easy, but, as we are going to see, insufficient.

In order to construct $\mathrm{Sl}(2;\mathbb{R})$ invariants and other quadruplets it is useful to define the matrix

$$m^i{}_j \equiv \mathcal{Q}^{ikl}\mathcal{Q}_{jkl}\,, \tag{2.177}$$

whose components take the values

$$m^1{}_1 = -m^2{}_2 = -\tfrac{2}{5}(p^1q_1+3p^0q_0)\,,\qquad m^1{}_2 = \tfrac{12}{5}p^1q_0 - \tfrac{8}{25}(q_1)^2\,,\qquad m^2{}_1 = \tfrac{4}{5}p^0q_1+2(p^1)^2\,. \tag{2.178}$$

The square of this matrix is

$$m^i{}_k\; m^k{}_j = -\tfrac{36}{25}\; J_4(\mathcal{Q})\; \delta^i{}_j\,, \tag{2.179}$$

where, since $\delta^i{}_j$ is an invariant tensor, the coefficient $J_4(\mathcal{Q})$ must be an invariant of order four in the charges; this quartic invariant is explicitly given by

$$J_4(\mathcal{Q}) \equiv \tfrac{8}{45}p^0(q_1)^3 + \tfrac{1}{3}(p^1q_1)^2 - (p^0q_0)^2 - 2p^0q_0p^1q_1 - \tfrac{10}{3}(p^1)^3q_0\,. \tag{2.180}$$

This is the only independent invariant that can be constructed from the charge alone. We can construct invariants taking traces of powers of m and taking also the determinant: the traces of odd powers vanish and those of even powers are proportional to $J_4(\mathcal{Q})$. Furthermore, the determinant is also proportional to $J_4(\mathcal{Q})$, *i.e.*,

$$\det(m) = \tfrac{36}{25}J_4(\mathcal{Q})\,. \tag{2.181}$$

The simplest quadruplet that can be built out of the original one $\mathcal{Q}_{ijk}$ is

$$\mathcal{Q}_{(ij|l}\; m^l{}_{|k)}\,. \tag{2.182}$$

This tensor is necessarily proportional to the Freudenthal dual of $\mathcal{Q}_{ijk}$ since

$$\mathcal{Q}_{(ij|l}m^l{}_{|k)} = \tfrac{1}{4}\frac{\partial \mathrm{Tr}\, m^2}{\partial \mathcal{Q}^{ijk}} = -\tfrac{18}{25}\frac{\partial J_4(\mathcal{Q})}{\partial \mathcal{Q}^{ijk}}\,. \tag{2.183}$$

Using higher powers of m does not give anything new as

$$\mathcal{Q}_{(i|lm}m^l{}_{|j}m^m{}_{k)} = \mathcal{Q}_{(ij|l}m^l{}_m m^m{}_{|k)} = -\tfrac{36}{25}\; J_4(\mathcal{Q})\; \mathcal{Q}_{ijk}\,. \tag{2.184}$$

We must use, therefore, contractions of $\mathcal{Q}_{ijk}$ such that the free indices are not those of $m^i{}_j$. At cubic order in $\mathcal{Q}_{ijk}$ there is only one possibility, which vanishes identically

$$\mathcal{Q}_{(i|lm}\mathcal{Q}_{|j|n}{}^l\mathcal{Q}_{|k)}{}^{mn} = 0\,, \tag{2.185}$$

due to the antisymmetry of the symplectic metric Ω_{ij}. At order five in $\mathcal{Q}_{ijk}$ we can consider

$$\mathcal{Q}_{i,i_1,i_2}\mathcal{Q}_{j,j_1,j_2}\mathcal{Q}_{k,k_1,k_2}\mathcal{Q}^{i_1,j_1,k_1}\mathcal{Q}^{i_2,j_2,k_2} \;=\; -\tfrac{36}{25}\; J_4(\mathcal{Q})\; \mathcal{Q}_{ijk}\,, \tag{2.186}$$

$$\mathcal{Q}_{(i|mn}\mathcal{Q}_{|j|pq}\mathcal{Q}_{|k)}{}^{mp}m^{nq} \;=\; 0\,. \tag{2.187}$$

Up to at least order 9 there are no quadruplets other than $\mathcal{Q}_{ijk}$ and its Freudenthal dual that can be constructed by these tensor methods.

To find more, we have to solve Eq. (2.11). Since this is a very complicated task, we are going to restrict ourselves to a generating charge configuration with $p^0 = q_1 = 0$, *i.e.*,

$$(\mathcal{Q}^M) = \begin{pmatrix} 0 \\ p^1 \\ q_0 \\ 0 \end{pmatrix} . \tag{2.188}$$

This subspace is preserved by the $\mathrm{Sl}(2;\mathbb{R})$ transformations with $b = c = 0$ and $d = 1/a$ (or equivalently by the infinitesimal transformations generated by T_1), to which by analogy we shall refer to as the *small group*. It is not difficult to see that by acting on this charge vector with the transformations with appropriate charge-dependent parameters $b \neq 0\,,\ c \neq 0$ (or, equivalently, by the infinitesimal transformations generated by T_2 and T_3) we can generate the complete generic charge vector with four unrestricted charge components.

It should be clear that if we construct vectors in the subspace $p^0 = q_1 = 0$ that are equivariant under the small group, then by acting on these vectors with the same transformations that generate the complete charge vector, we will obtain vectors that are equivariant under the full duality group, *i.e.*, $\mathrm{Sl}(2;\mathbb{R})$, and which reduce to the former when we set $p^0 = q_1 = 0$. Since duality transformations preserve linear independence, a base for the small-group-equivariant vectors will be transformed into a base of the duality-group-equivariant vectors; seeing this reasoning we shall refer to a small-group-equivariant vector as an *equivariant-generating vector*.

The equation that these equivariant-generating vectors have to solve is the restriction of Eq. (2.11) to just T_1 and allow for no dependence on p^0 nor q_1, *i.e.*,

$$p^1\frac{\partial U^P}{\partial p^1} - 3q_0\frac{\partial U^P}{\partial q_0} = \beta^{(P)}U^{(P)}\,, \qquad (\beta^P) = \begin{pmatrix} 3 \\ 1 \\ -3 \\ -1 \end{pmatrix} , \tag{2.189}$$

which is solved by

$$U^P = \sum_i a_i^{(P)}(p^1)^{\alpha_i^{(P)}}(q_0)^{\frac{\alpha_i^{(P)}-\beta^{(P)}}{3}}\,, \tag{2.190}$$

for arbitrary constants a_i^P, α_i^P (the parenthesis enclosing the indices P indicate that they are not summed over and the index i runs over an arbitrary number of terms). For simplicity, we can choose them to depend only on p^1 ($\alpha^P = \beta^P$) or only on q_0 ($\alpha_i^P = 0$) and take them to have only one term:

$$U^P = a^{(P)}(p^1)^{\beta^{(P)}}\,, \qquad U^P = a^{(P)}(q_0)^{-\beta^{(P)}/3}\,. \tag{2.191}$$

To avoid charges with fractional components, we choose the first option and get a basis of equivariant-generating vectors

$$U_\sigma{}^P \sim \delta_\sigma{}^{(P)}(p^1)^{\beta^{(P)}}\,. \tag{2.192}$$

We have found it convenient to normalize these vectors and give them names $\{R, S, U, V\}$

$$R \equiv \begin{pmatrix} \frac{10}{3}(p^1)^3 \\ 0 \\ 0 \\ 0 \end{pmatrix} , \quad S \equiv \begin{pmatrix} 0 \\ 0 \\ (\frac{10}{3}(p^1)^3)^{-1} \\ 0 \end{pmatrix} , \quad U \equiv \begin{pmatrix} 0 \\ p^1 \\ 0 \\ 0 \end{pmatrix} , \quad V \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1/p^1 \end{pmatrix} . \tag{2.193}$$

The only non-vanishing symplectic contractions of these four vectors are

$$R_M S^M = -1\,, \qquad U_M V^M = -1\,, \tag{2.194}$$

and they satisfy the completeness relation

$$R^M S_N - S^M R_N + U^M V_N - V^M U_N = \delta^M{}_N\,. \tag{2.195}$$

We can decompose any equivariant-generating vector, such as $\mathcal{Q}^M$ w.r.t. this basis and the expression will have the same form after acting with the duality group. For $\mathcal{Q}^M$ we find

$$R_M \mathcal{Q}^M = -\tfrac{10}{3}(p_1)^3 q_0 = J_4(\mathcal{Q})|_{p^0=q_1=0}\,, \qquad V_M \mathcal{Q}^M = 1\,, \tag{2.196}$$

from which we find that in general

$$\mathcal{Q}^M = U^M - J_4(\mathcal{Q}) S^M\,. \tag{2.197}$$

The Freudenthal dual charge vector is (using the results of the next section) given by

$$\tilde{\mathcal{Q}}^M = \frac{1}{\mathsf{W}(\mathcal{Q})} R^M + \tfrac{3}{4}\mathsf{W}(\mathcal{Q}) V^M\,, \qquad \mathsf{W}(\mathcal{Q}) = 2\sqrt{J_4(\mathcal{Q})}\,. \tag{2.198}$$

As for the moduli-dependent equivariant vectors, we can use the generic construction in Eq. (2.17) replacing $\mathcal{Q}$ with different equivariant vectors.

2.6.2 H-FGK formalism

The stabilization equations can be solved in a completely general way [389] and the result is summarized by the Hesse potential which, in terms of the quartic invariant

$$J_4(H) \equiv \tfrac{8}{45}H^0(H_1)^3 + \tfrac{1}{3}(H^1H_1)^2 - (H^0H_0)^2 - 2H^0H_0H^1H_1 - \tfrac{10}{3}(H^1)^3H_0\,, \tag{2.199}$$

can be expressed as

$$\mathsf{W}(H) = 2\sqrt{J_4(H)}\,. \tag{2.200}$$

It is convenient to introduce the fully symmetric rank-4 $\mathbb{K}$-tensor [17,315], implicitly defined by[23]

$$\mathbb{K}_{MNPQ}H^M H^N H^P H^Q \equiv J_4(H)\,. \tag{2.201}$$

Using this tensor, we can write

$$\tilde{H}_M = \frac{\partial_M J_4}{\mathsf{W}} = 4\frac{\mathbb{K}_{MNPQ}H^N H^P H^Q}{\mathsf{W}}\,, \tag{2.202}$$

$$\mathcal{M}_{MN}(\mathcal{F}) = -\frac{\partial_M\partial_N J_4}{\mathsf{W}} + 2\frac{\partial_M J_4 \partial_N J_4}{\mathsf{W}^3} = -12\frac{\mathbb{K}_{MNPQ}H^P H^Q}{\mathsf{W}} + 2\frac{\tilde{H}_M\tilde{H}_N}{\mathsf{W}}\,, \tag{2.203}$$

$$g_{MN} = 24\frac{\mathbb{K}_{MNPQ}H^P H^Q}{\mathsf{W}^2} - 8\frac{\tilde{H}_M\tilde{H}_N}{\mathsf{W}^2} - 2\frac{H_M H_N}{\mathsf{W}^2}\,, \tag{2.204}$$

[23] In most of what follows, the exact form of the $\mathbb{K}$-tensor will be irrelevant. The formulae and results obtained will, therefore, be valid for any $\mathcal{N}=2, d=4$ theory with Hesse potential of the same generic form.

and one can check (*e.g.* using a symbolic manipulation program) the following properties:

$$J_4(\tilde{H}) = J_4(H)\,, \tag{2.205}$$

$$\mathbb{K}_{MNPQ}\tilde{H}^N\tilde{H}^P\tilde{H}^Q = -\tfrac{1}{4}\mathsf{W}H_M\,, \tag{2.206}$$

$$\mathbb{K}_{MNPQ}\tilde{H}^P\tilde{H}^Q = \mathbb{K}_{MNPQ}H^PH^Q + \tfrac{1}{6}(H_MH_N - \tilde{H}_M\tilde{H}_N)\,, \tag{2.207}$$

$$\mathbb{K}_{MNPQ}H^P\tilde{H}^Q = -\tfrac{1}{6}H_{(M}\tilde{H}_{N)}\,. \tag{2.208}$$

These properties (which hold for any symplectic vector with non-vanishing quartic invariant which implies the existence of the Freudenthal dual) imply the invariance under Freudenthal duality of W, $\mathcal{M}_{MN}(\mathcal{F})$ and the potential $V(H)$; the latter can be rewritten in the manifestly Freudenthal-duality-invariant form

$$V(H) = -3\mathsf{W}^{-2}\left\{\mathbb{K}_{MNPQ}\left(H^PH^Q + \tilde{H}^P\tilde{H}^Q\right) - \tfrac{1}{2}\left(H_MH_N + \tilde{H}_M\tilde{H}_N\right)\right\}\mathcal{Q}^M\mathcal{Q}^N\,. \tag{2.209}$$

It is, however, not possible to express it in a form manifestly invariant under the Freudenthal duality transformation of the charge vector $\mathcal{Q}^M \to \tilde{\mathcal{Q}}^M$.

The physical fields are given in terms of the H-variables by the usual expressions

$$e^{-2U} = 2\mathsf{W} = 2\sqrt{J_4(H)}\,, \tag{2.210}$$

$$t = \frac{\tilde{H}^1 + iH^1}{\tilde{H}^0 + iH^0} = -\frac{3H^0H_0 + H^1H_1}{5(H^1)^2 + 2H^0H_1} + i\frac{3\mathsf{W}}{2\left[5(H^1)^2 + 2H^0H_1\right]}\,. \tag{2.211}$$

Very small vectors

The vectors R^M and S^M turn out to be very small charge vectors of this model [74, 241], owing to the following properties:

$$\mathbb{K}_{MNPQ}R^PR^Q = -\tfrac{1}{6}R_MR_N\,, \qquad \mathbb{K}_{MNPQ}S^PS^Q = -\tfrac{1}{6}S_MS_N\,, \tag{2.212}$$

that leads to (in obvious shorthand notation)

$$\mathbb{K}_MR^3 = \mathbb{K}_MS^3 = 0\,, \qquad J_4(R) = J_4(S) = 0\,. \tag{2.213}$$

On the other hand, the vectors U^M and V^M are both small vectors

$$J_4(U) = J_4(V) = 0\,. \tag{2.214}$$

2.6.3 Critical points

The complexity of this model forces us to use a symbolic manipulation program and, further, impose the restriction $p^0 = q_1 = 0$ on the charges to search for the critical points of the black-hole potential. Apart from the standard supersymmetric attractor $B^M = \mathcal{Q}^M$ we find only one physically acceptable attractor given by

$$(B^M) = \begin{pmatrix} 0 \\ p^1 \\ -q_0 \\ 0 \end{pmatrix} . \tag{2.215}$$

It is an equivariant vector and we can write it in the form

$$B^M = U^M + J_4(\mathcal{Q})S^M = \mathcal{Q}^M + 2J_4(\mathcal{Q})S^M . \tag{2.216}$$

The quartic invariant for this vector can be computed readily using Eqs. (2.212–2.214), and

$$S_M \mathcal{Q}^M = 0 , \qquad \tilde{\mathcal{Q}}_M S^M = -1/\mathsf{W}(\mathcal{Q}) , \tag{2.217}$$

and, by Eq. (2.198), it reads

$$\begin{aligned} J_4(B) &= \mathbb{K}B^4 = \mathbb{K}[\mathcal{Q} + 2J_4(\mathcal{Q})S]^4 = \mathbb{K}\mathcal{Q}^4 + 8J_4(\mathcal{Q})\mathbb{K}\mathcal{Q}^3 S \\ &= J_4(\mathcal{Q}) + 2J_4(\mathcal{Q})\mathsf{W}(\mathcal{Q})\tilde{\mathcal{Q}}_M S^M \\ &= -J_4(\mathcal{Q}) . \end{aligned} \tag{2.218}$$

2.6.4 Conventional extremal solutions

The supersymmetric solutions of this model are constructed as usual, and we will focus on the extremal non-supersymmetric ones which are associated to the attractor $B^M = U^M + J_4(\mathcal{Q})S^M$. For the near-horizon solutions, the H^M take the standard form Eq. (2.20) since Eq. (2.21) is satisfied. Now we must investigate whether we can add constant terms A^M to these harmonic functions satisfying only the normalization condition $\mathsf{W}(A) = 1$ and the constraint $B^M A_M = 0$, which is equivalent, at the infinitesimal level, to investigating the space of solutions to Eq. (2.27). For simplicity, we work with a generating charge configuration with $p^0 = q_1 = 0$. We find for the non-supersymmetric attractor

$$(\mathfrak{M}_{MN}) = \tfrac{1}{2} \begin{pmatrix} \frac{21}{20}\frac{q_0}{(p^1)^3} & 0 & 0 & -\frac{3}{20}\frac{1}{(p^1)^2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{3}{20}\frac{1}{(p^1)^2} & 0 & 0 & \frac{1}{4}\frac{1}{p^1 q_0} \end{pmatrix} , \tag{2.219}$$

whose rank is 2. The solutions to Eq. (2.27) have the form $(\varepsilon^M) = \left(\begin{smallmatrix} 0 \\ \varepsilon^1 \\ \varepsilon_0 \\ 0 \end{smallmatrix}\right)$ and satisfy $B^M \varepsilon_M = 0$ but we still have to impose the normalization condition $\mathsf{W}(A) = 1$ on the two non-vanishing components, which leaves us with only one independent solution that can only describe one independent real moduli; this modulus turns out to be $\Im\mathfrak{m}(t_\infty)$. It can be shown that the solution takes the form [188]

$$\left(H^{M}\right)=\begin{pmatrix} 0 \\ \\ s^{1}\left\{\sqrt{\frac{3}{10\Im\mathrm{m}\,t_{\infty}}}-\frac{1}{\sqrt{2}}|p^{1}|\tau\right\} \\ \\ -s_{0}\left\{\sqrt{\frac{5(\Im\mathrm{m}\,t_{\infty})^{3}}{24}}-\frac{1}{\sqrt{2}}|q_{0}|\tau\right\} \\ \\ 0 \end{pmatrix}, \tag{2.220}$$

where we have defined

$$s^{M}\equiv\mathrm{sgn}(\mathcal{Q}^{M})\,, \tag{2.221}$$

and where we have to require $s^{1}=s_{0}$ for the solution to be regular.

Having $\Re\mathrm{e}\,t_{\infty}=0$ poses a very important problem because even though the charge vector with $p^{0}=q_{1}$ can generate via $\mathrm{Sl}(2;\mathbb{R})$ duality transformations a complete charge vector with four independent charges, it cannot at the same time generate an independent $\Re\mathrm{e}\,t_{\infty}\neq 0$. In other words, this solution is not a *generating solution*; its orbit under $\mathrm{Sl}(2;\mathbb{R})$ rotations will not fully cover the space of parameters. A necessary and sufficient condition for a solution to be generating is that all the $\mathrm{Sl}(2;\mathbb{R})$ invariants of the theory are independent when evaluated on the charges and moduli of that solution [61, 62]. As we show in detail in Appendix A.1.2, the solution (2.220) does not satisfy this condition.

In order to have a generating solution for the class of extremal non-supersymmetric black-hole solutions associated to the attractor $B^{M}=U^{M}+J_{4}(\mathcal{Q})S^{M}$, we need to add $\Re\mathrm{e}\,t_{\infty}\neq 0$ to the solution and it should be clear that this cannot be done if we make a conventional, *i.e.*, harmonic, ansatz: the H^{M} must contain anharmonic terms.

For future use, it is useful to have symplectic-covariant expressions for the constraints on A^{M} imposed by the equations of motion for a harmonic ansatz:

$$A_{M}U^{M}=0\,,\qquad A_{M}S^{M}=0\,. \tag{2.222}$$

$A_{M}B^{M}=0$ only imposes the weaker condition $A_{M}(U^{M}+J_{4}(\mathcal{Q})S^{M})=0$. The above constraints imply that A^{M} has to take the form

$$A^{M}=aU^{M}+bS^{M}\,, \tag{2.223}$$

for some invariant coefficients a and b, and it cannot contain terms proportional to the vectors R^{M} and V^{M}.

2.6.5 Unconventional extremal solutions

The missing free parameter must be added to the above solution by adding anharmonic terms to the harmonic ansatz: let us don the harmonic functions of the undeformed solution with hats, so that

$$\hat{H}^{M}=A^{M}-\tfrac{1}{\sqrt{2}}B^{M}\tau\,, \tag{2.224}$$

where B^{M} is given by the attractor (2.216) and A^{M} satisfies the constraints Eqs. (2.222) but is otherwise arbitrary (up to asymptotic flatness normalization). Observe that this implies that

$$\hat{H}_{M}U^{M}=\hat{H}_{M}S^{M}=0\,,\quad\Rightarrow\quad\hat{H}=a(\tau)U^{M}+b(\tau)S^{M}\,, \tag{2.225}$$

where $a(\tau)$ and $b(\tau)$ are duality-invariant harmonic functions of τ. Terms proportional to R^M and V^M are excluded if the coefficients are harmonic functions; a term proportional to V^M can always be eliminated by a local Freudenthal duality transformation, whence we expect that it is enough to add a (necessarily anharmonic) term proportional to R^M. It turns out that such a solution [188][24] has the form[25]

$$H^M = \hat{H}^M - \frac{\chi R^M}{R_N H^N}\,, \tag{2.226}$$

where χ is another independent parameter, like A^M. The values of χ and A^M are determined by requiring that the physical fields have the right asymptotic behavior at spatial infinity ($e^{-2U} \to 1\,,\ t \to t_\infty$ when $\tau \to 0^-$) as follows: first of all, observe that as a consequence of Eq. (2.225) the property

$$H_M U^M = 0\,, \tag{2.227}$$

is satisfied everywhere and in particular at spatial infinity where

$$H^M \overset{\tau\to 0^-}{\longrightarrow} H^M_\infty = A^M - \frac{\chi R^M}{R_N A^N}\,. \tag{2.228}$$

Then, using the definition of $H^M = \mathcal{I}^M$, Eq. (1.87), in Eq. (2.227) plus Eq. (1.88) at spatial infinity we find

$$0 = H_{M\,\infty}U^M = \Im\mathfrak{m}\left(\frac{\mathcal{V}_{M\,\infty}}{X_\infty}\right)U^M = \Im\mathfrak{m}\left(\frac{\mathcal{Z}_\infty(U)}{X_\infty}\right) = \sqrt{2}\Im\mathfrak{m}\left(\frac{\mathcal{Z}_\infty(U)}{e^{i\alpha_\infty}}\right)\,. \tag{2.229}$$

This implies that

$$e^{i\alpha_\infty} = \pm\frac{\mathcal{Z}_\infty(U)}{|\mathcal{Z}_\infty(U)|}\,, \tag{2.230}$$

which can be used again in the definition of $H^M = \mathcal{I}^M$ to give

$$H^M_\infty = \pm\sqrt{2}\Im\mathfrak{m}\left(\frac{\mathcal{V}^M_\infty}{\mathcal{Z}_\infty(U)}\right)|\mathcal{Z}_\infty(U)|\,. \tag{2.231}$$

To determine the overall sign we will demand that the functions $H^M(\tau)$ never vanish for $\tau \in [-\infty, 0)$, a condition that is usually related to the positivity of the mass. Contracting the above result with S^M and using Eq. (2.225) we get

$$\frac{\chi}{R_N A^N} = \pm\sqrt{2}\Im\mathfrak{m}\left(\frac{\mathcal{Z}_\infty(S)}{\mathcal{Z}_\infty(U)}\right)|\mathcal{Z}_\infty(U)|\,, \tag{2.232}$$

which, after substitution in Eq. (2.228) gives the value of the constants A^M, satisfying Eqs. (2.222), as an equivariant symplectic vector, function of the physical parameters of the solution

$$A^M = \pm\sqrt{2}(\delta^M{}_N - R^M S_N)\Im\mathfrak{m}\left(\frac{\mathcal{V}^M_\infty}{\mathcal{Z}_\infty(U)}\right)|\mathcal{Z}_\infty(U)|\,. \tag{2.233}$$

[24] This solution can be obtained by truncation from the STU-model solution in Ref. [208] and is also a particular case of the general extremal non-supersymmetric solutions of cubic models of Ref. [74]. It has also been obtained by using integrability methods in the action that one obtains in the approach of Ref. [83] (see also [132]): its derivation can be found in Section 9.4 (page 76) of Ref. [182]. The solution belongs to the orbit $\mathcal{O}^3_{22}$ in the classification of Ref. [181] (see Table 2 of that reference).

[25] This definition is not recursive because $R_N H^N = R_N \hat{H}^N$.

With this information we can compute $R_N A^N$ to find, from Eq. (2.232) the value of the invariant parameter χ as a function of the physical parameters of the solution[26]

$$\chi = -2\Im\mathfrak{m}\left(\frac{\mathcal{Z}_\infty(R)}{\mathcal{Z}_\infty(U)}\right)\Im\mathfrak{m}\left(\frac{\mathcal{Z}_\infty(S)}{\mathcal{Z}_\infty(U)}\right)|\mathcal{Z}_\infty(U)|^2\,. \tag{2.235}$$

For $p^0 = q_1 = 0$, the solution takes the explicit (but not manifestly equivariant) form

$$\left(H^M\right) = \begin{pmatrix} -\frac{1}{2}\frac{\Re\mathfrak{e}\,t_\infty}{\Im\mathfrak{m}\,t_\infty}\dfrac{1}{H_0}\,, \\ \\ s^1\left\{\sqrt{\frac{3}{10\Im\mathfrak{m}\,t_\infty}} - \frac{1}{\sqrt{2}}|p^1|\tau\right\} \\ \\ -s_0\left(\frac{|t_\infty|}{\Im\mathfrak{m}\,t_\infty}\right)^2\left\{\sqrt{\frac{5\Im\mathfrak{m}\,t_\infty}{24}} - \frac{1}{\sqrt{2}}|q_0|\tau\right\} \\ \\ 0 \end{pmatrix}. \tag{2.236}$$

The mass of this solution can be computed using the general formula Eq. (2.31). From the definition of $\tilde{H}_M$ we have

$$\tilde{H}_M(0) = \pm\sqrt{2}\Re\mathfrak{e}\left(\frac{\mathcal{V}_{\infty\,M}}{\mathcal{Z}_\infty(U)}\right)|\mathcal{Z}_\infty(U)|\,, \tag{2.237}$$

and

$$\dot{H}^M(0) = -\frac{1}{\sqrt{2}}\left[B^M - \frac{\chi J_4(\mathcal{Q})}{(RA)^2}R^M\right]\,, \tag{2.238}$$

from which we get the covariant expression

$$M = \pm|\mathcal{Z}_\infty(U)|\left\{1 - \frac{1}{3}J_4(\mathcal{Q})\Im\mathfrak{m}\left(\frac{\mathcal{Z}_\infty(V)}{\mathcal{Z}_\infty(U)}\right)\left[\Im\mathfrak{m}\left(\frac{\mathcal{Z}_\infty(R)}{\mathcal{Z}_\infty(U)}\right)\right]^{-1}\right\}\,. \tag{2.239}$$

This last expression reduces for $p^0 = q_1 = 0$ (selecting the upper sign in Eq. (2.231)) to

$$M = e^{\mathcal{K}_\infty/2}\left(|q_0| + \tfrac{5}{2}|t_\infty|^2|p^1|\right)\,. \tag{2.240}$$

Observe that the value of the mass differs from the absolute value of the associated fake central charge B^M:

$$M \neq |\mathcal{Z}(\phi_\infty, B)|\,. \tag{2.241}$$

The above result should be compared to the mass of the supersymmetric black hole which is given by the standard formula $M = |\mathcal{Z}_\infty(\mathcal{Q})|$ and reduces for $p^0 = q_1 = 0$ to[27] the following expression,

$$M = e^{\mathcal{K}_\infty/2}\sqrt{\left[|q_0| - \tfrac{5}{2}(\Re\mathfrak{e}t_\infty)^2|p^1|\right]^2 + \tfrac{25}{4}(\Im\mathfrak{m}t_\infty)^4|p^1|^2 + 5(\Im\mathfrak{m}t_\infty)^2|q_0p^1|}\,, \tag{2.242}$$

[26] In terms of the invariants $i_1, \cdots, i_5$ of the theory given in Eqs. (A.1)-(A.5)

$$\chi = \tfrac{1}{4}(-J_4(\mathcal{Q}))^{-1/6}\left\{\left(i_1 + i_2 - \frac{(i_1 - i_2/3)^3}{J_4(\mathcal{Q})} - \frac{4\,i_3}{\sqrt{-J_4(\mathcal{Q})}}\right)^{1/3} - \left(i_1 + i_2 - \frac{(i_1 - i_2/3)^3}{J_4(\mathcal{Q})} + \frac{4\,i_3}{\sqrt{-J_4(\mathcal{Q})}}\right)^{1/3}\right\}\,. \tag{2.234}$$

[27] We have used that $p^1q_0 > 0$ for the non-supersymmetric case and $p^1q_0 < 0$ for the supersymmetric one.

which can be rewritten in the equivalent form

$$M = e^{\mathcal{K}_\infty/2}\sqrt{\left[|q_0| + \tfrac{5}{2}|t_\infty|^2|p^1|\right]^2 - 10(\Re\mathfrak{e}t_\infty)^2|q_0p^1|}\,, \tag{2.243}$$

which shows that the mass of the supersymmetric black hole is always smaller than the mass of the non-supersymmetric one with charges of equal absolute value.

The entropy is given by the square of the fake central charge at the horizon

$$S = \pi|\mathcal{Z}(\phi_h, B)|^2 = \pi\mathsf{W}(B)/2 = \pi\sqrt{-J_4(\mathcal{Q})}\,. \tag{2.244}$$

As discussed in Section 2.3, an interesting characteristic of the unconventional solutions is that, in distinction to what happens for the conventional ones, the flow of the black-hole metric function e^{-U} from infinity to the horizon is not governed by a simple fake central charge $\mathcal{Z}(\phi, B)$ since the near-horizon limit of the metric is related to $\mathcal{Z}(\phi_{\rm h}, B)$ but the spacelike infinity limit is not related to $\mathcal{Z}(\phi_\infty, B)$. The first-order flow equations for these black holes can be written in terms of a superpotential $W(\phi, B)$ or, equivalently, in terms of the "fake central charge" $\mathcal{Z}(\phi, \sqrt{2}\mathfrak{D}H)$ defined in Section 2.3.

It is possible to prove analytically that the general configuration Eq. (2.226) solves the equations of motion by using the duality-invariant properties of the equivariant vectors A^M, B^M and R^M that appear in its definition (that is: not reducing the equations to the $p^0 = q_1$ case) and the properties of the $\mathbb{K}$-tensor of this model, see Eqs. (2.218). As an intermediate step, we derive the following relations, which are valid only for the H^Ms of our ansatz:

$$\begin{aligned}
\mathbb{K}_{MN}\hat{H}^2 &= \tfrac{1}{2}(VH)^2R_{(M}V_{N)} + \tfrac{1}{2}(VH)(RH)V_MV_N + \tfrac{1}{18}(VH)^2U_MU_N \\
&\quad -\tfrac{1}{3}(VH)(RH)U_{(M}S_{N)} - \tfrac{1}{6}(RH)^2S_MS_N\,,
\end{aligned} \tag{2.245}$$

$$\begin{aligned}
\mathbb{K}_{MN}\hat{H}\mathcal{Q} &= \tfrac{1}{2}(VH)R_{(M}V_{N)} + \tfrac{1}{4}[J_4(\mathcal{Q})(VH) + (RH)]V_MV_N + \tfrac{1}{18}(VH)U_MU_N \\
&\quad -\tfrac{1}{6}[J_4(\mathcal{Q})(VH) + (RH)]U_{(M}S_{N)} - \tfrac{1}{6}J_4(\mathcal{Q})(RH)S_MS_N\,,
\end{aligned} \tag{2.246}$$

$$\mathbb{K}_{MN}\hat{H}R = -\tfrac{1}{3}(RH)R_{(M}S_{N)} - \tfrac{1}{6}(RH)U_{(M}V_{N)} - \tfrac{1}{6}(VH)R_{(M}U_{N)}\,. \tag{2.247}$$

Using these identities it is easy to show, for instance, that

$$J_4(H) = J_4(\hat{H}) - \chi^2\,, \qquad J_4(\hat{H}) = (VH)^3\,(RH)\,. \tag{2.248}$$

2.7 Conclusions

In this chapter we have shown how the equivariance of the H variables under duality transformations translates into equivariance of the constant symplectic vectors that occur in their explicit expressions. Using the H-FGK formalism we have studied under what conditions the extremal solutions associated to a given attractor can be described, for all values of the charges and moduli, by harmonic Hs alone and when it is necessary to add anharmonic terms to them. We have called these two kinds of solutions conventional, respectively unconventional.

As mentioned in the introduction, it is not known how unconventional extremal solutions (which are necessarily non-supersymmetric, since we know that all the supersymmetric ones are conventional) can be deformed into non-extremal solutions, with non-zero temperature but the same values of the charges and moduli. The H-FGK formalism and the use of equivariant vectors can help us to solve this problem and, as a first step, we have shown how to apply these methods to well-known examples of theories with conventional and unconventional solutions.

In the case of the unconventional extremal solutions of the t^3-model we have shown, first of all, how the criterion found in Section 2.2 indicates the need for anharmonic terms and which equivariant vectors these terms should depend on. We have then described the solution entirely in terms of these objects and we have computed the general form of the mass and the entropy. The second has a well-known form in terms of the near-horizon limit $\mathcal{Z}(\phi_{\rm h}, B)$ of a *fake central charge*, $\mathcal{Z}(\phi, B)$, constructed from what we have called (in the context of the H-FGK formalism) attractor B^M. The mass instead is not given by the spacelike infinity limit of this fake central charge $M = |\mathcal{Z}(\phi_\infty, B)|$ but rather by the spacelike infinity of a different one $\mathcal{Z}(\phi, E)$ with $E^M \neq B^M$. The first-order flow equations that govern the system (which have been given in Refs. [74, 187]) are written in term of non-standard fake central charge $\mathcal{Z}(\phi, \sqrt{2}\mathcal{D}H)$ whose second argument is τ-dependent and correctly interpolates between B^M (on the horizon) and E^M (at spacelike infinity).

The behavior of the metric function in the unconventional solutions gets modified in the asymptotic region but remains unchanged in the near-horizon region, where it is still governed by the attractor mechanism. This behavior is reminiscent, but opposite, to that of the colored non-Abelian supersymmetric black holes of Refs. [319] in which the near-horizon geometry is modified by the non-Abelian effects while the asymptotic one is unchanged by them.

The formalism and the methods presented in this chapter can be applied to the problem of finding the non-extremal generalization of the unconventional solutions studied in this chapter.

3

Stringy black holes in Type-IIA string theory

This chapter is based on

Pablo Bueno, Rhys Davies and C. S. Shahbazi,
"Quantum black holes in Type-IIA string theory",
JHEP ***1301*** *(2013) 089.* [arXiv:1210.2817 *[hep-th]*] *[89]*.

Pablo Bueno and C. S. Shahbazi,
"Non-perturbative black holes in Type-IIA string theory vs. the no-hair conjecture",
Class. Quantum Grav. ***31*** *(2013) 015023.* [arXiv:1304.8079 *[hep-th]*] *[95]*.

Supergravity solutions have played, and continue to play, a prominent rôle in the new developments of string theory. The body of literature about black hole solutions (and p-branes) that has been accumulated during the past thirty years is enormous, but only recently the issue of non-extremality was systematically investigated, and by now, it could be said that we have at our disposal more or less well-established methods to deal with non-extremal solutions [132, 137, 153, 190, 284, 323, 324, 329, 331] in SUGRA. However, explicit non-extremal solutions to supergravity models with perturbative quantum (*stringy*) corrections (see below for further explanation) are yet to be constructed. These kind of solutions may be relevant in order to understand how the deformation of the scalar geometry modifies the solutions of the theory, and also in order to relate the macroscopic computation of the entropy with the microscopic calculation in a string theory set-up, once sub-leading corrections to the prepotential are taken into account [34, 36]. These kinds of corrections differ from the higher-order corrections, which, together with the corresponding microscopic string theory computation, have been already studied in the literature [302] (for a very nice review about this and related topics, as well as for further references, see [330]).

In the first part of this chapter we are going to use the H-FGK formalism [190, 323, 324] in order to take a small step in the study of non-extremal black holes in supergravity in the presence of quantum corrections. Through a consistent truncation, we are going to define a particular class of black holes, which is characterized by existing only when the quantum perturbative corrections are included in the action. These kinds of solutions, which we have chosen to call *quantum black holes*[1], display a remarkable behavior: the so called *large-volume limit* $\Im\mathrm{m}z^i \to \infty$ is in fact not a large volume limit of the Calabi-Yau (C.Y.) manifold, whose volume remains constant and fixed by topological data. In addition, the regularity conditions of the black hole solutions impose the topological restriction

[1] It is worth pointing out again that the term *quantum* does not refer to space-time but to world-sheet properties in this context [330]. In this respect, although such denomination is widely spread in the literature, the adjective *stringy* is probably more acqurate.

$h^{1,1} > h^{2,1}$ in the compactification C.Y. For small $h^{1,1}$ the condition is particularly restrictive, and since this case is the most manageable one from the point of view of black hole solutions, we prove the existence of C.Y. manifolds obeying $h^{1,1} > h^{2,1}$ by explicit construction for the $h^{1,1} = 3$ case. These C.Y. manifolds are new in the literature.

The perturbative corrections, encoded in a single term $i\frac{c}{2}$ in the prepotential, introduce a highly non-trivial difficulty in the model, which makes almost hopeless the resolution of the equations of the theory. Surprisingly enough, we are able to find a black hole solution with non-constant scalars, similar to the $D0-D4-D4-D4$ black hole solution of the STU model, and which can be used as a toy model to study the microscopic description of black holes in the presence of quantum perturbative corrections and away from extremality.

3.1 Type-IIA string theory on a Calabi–Yau manifold

Type-IIA string theory compactified to 4D on a C.Y. three-fold, with Hodge numbers $(h^{1,1}, h^{2,1})$, is described, up to two derivatives, by a $\mathcal{N} = 2, d = 4$ supergravity whose prepotential is given in terms of an infinite series around $\Im\text{m} z^i \to \infty$ [109–111]

$$\mathcal{F} = -\frac{1}{3!}\kappa^0_{ijk} z^i z^j z^k + \frac{ic}{2} + \frac{i}{(2\pi)^3} \sum_{\{d_i\}} n_{\{d_i\}} Li_3\left(e^{2\pi i d_i z^i}\right) \quad , \tag{3.1}$$

where z^i, $i = 1, ..., n_v + 1 = h^{1,1}$, are the scalars in the vector multiplets. There are also $h^{2,1} + 1$ hypermultiplets in the theory. However, they can be consistently set to a constant value [386]. $c = \frac{\chi\zeta(3)}{(2\pi)^3}$ is a model-dependent number, being χ the Euler characteristic, which for C.Y. three-folds is given by $\chi = 2(h^{1,1} - h^{2,1})$. κ^0_{ijk} are the classical intersection numbers, $d_i \in \mathbb{Z}^+$ is a $h^{1,1}$-dimensional summation index and $Li_3(x)$ is the third polylogarithmic function. The first two terms in the prepotential correspond to tree level and higher-order perturbative contributions in the α'-expansion, respectively

$$\mathcal{F}_{\text{P}} = -\frac{1}{3!}\kappa^0_{ijk} z^i z^j z^k + \frac{ic}{2} \quad , \tag{3.2}$$

whereas the third term accounts for non-perturbative corrections produced by world-sheet instantons. These configurations get produced by (non-trivial) embeddings of the world-sheet into the C.Y. three-fold. The holomorphic mappings of the genus 0[2] string world-sheet onto the $h^{1,1}$ two-cycles of the C.Y. three-fold are classified by the nubers d_i, which count the number of wrappings of the world-sheet around the i−th generator of the integer homology group $H_2(\text{C.Y.}, \mathbb{Z})$. The number of different mappings for each set of $\{d_i\}$ $(\equiv \{d_1, ..., d_{h^{1,1}}\})$ or, in other words, the number of genus 0 instantons is denoted by $n_{\{d_i\}}$[3]

$$\mathcal{F}_{\text{NP}} = \frac{i}{(2\pi)^3} \sum_{\{d_i\}} n_{\{d_i\}} Li_3\left(e^{2\pi i d_i z^i}\right) \quad . \tag{3.3}$$

[2] Genus ≥ 1 instantons contribute with higher-derivative corrections.

[3] See, e.g. [330] for more details on the stringy origin of the prepotential.

The full prepotential can be rewritten in homogeneous coordinates $\mathcal{X}^\Lambda$, $\Lambda = (0, i)$ as

$$F(\mathcal{X}) = -\frac{1}{3!}\kappa^0_{ijk}\frac{\mathcal{X}^i\mathcal{X}^j\mathcal{X}^k}{\mathcal{X}^0} + \frac{ic(\mathcal{X}^0)^2}{2} + \frac{i(\mathcal{X}^0)^2}{(2\pi)^3}\sum_{\{d_i\}} n_{\{d_i\}} Li_3\left(e^{2\pi i d_i \frac{\mathcal{X}^i}{\mathcal{X}^0}}\right) \quad , \tag{3.4}$$

with the scalars z^i given by

$$z^i = \frac{\mathcal{X}^i}{\mathcal{X}^0} \quad . \tag{3.5}$$

Therefore, this coordinate system is only valid away from the locus $\mathcal{X}^0 = 0$.

The non-perturbative corrections (3.3) are exponentially suppressed and therefore can be safely ignored going to the large volume limit. Therefore our starting point is going to be Eq. (3.2), which in homogeneous coordinates $\mathcal{X}^\Lambda$, $\Lambda = (0, i)$, can be written as

$$F(\mathcal{X}) = -\frac{1}{3!}\kappa^0_{ijk}\frac{\mathcal{X}^i\mathcal{X}^j\mathcal{X}^k}{\mathcal{X}^0} + \frac{ic}{2}\left(\mathcal{X}^0\right)^2 \quad . \tag{3.6}$$

The scalar geometry defined by (3.6) is the so called *quantum corrected* d-SK geometry[4] [156], [157]. In this scenario, the classical case is modified and the scalar manifold, due to the correction encoded in c, is no longer homogeneous, and therefore, the geometry has been *corrected* by stringy effects.

We are interested in constructing spherically symmetric, static, black hole solutions of the theory defined by Eq. (3.6).This is the subject of the next section.

3.1.1 A *quantum* class of black holes

For (3.2), the general form of the Hesse potential $\mathsf{W}(H)$ in the H-FGK formalism (see 1.2.4) is an extremely involved function, and one cannot expect to solve in full generality the corresponding differential equations of motion, or even the associated algebraic equations of motion obtained by making use of the hyperbolic Ansatz for the H^M. Therefore, we are going to consider a particular truncation, which will give us the desired *quantum* black holes

$$H^0 = H_0 = H_i = 0, \quad p^0 = p_0 = q_i = 0\,. \tag{3.7}$$

Eq. (3.7) implies

$$\mathsf{W}(H) = \alpha\left|\kappa^0_{ijk}H^iH^jH^k\right|^{2/3} , \tag{3.8}$$

where $\alpha = \frac{(3!c)^{1/3}}{2}$ must be positive in order to have a non-singular metric. Hence $c > 0$ is a necessary condition in order to obtain a regular solution and a consistent truncation. The corresponding *black hole potential* reads

$$V_{\text{bh}} = \frac{\mathsf{W}(H)}{4}\partial_{ij}\log\mathsf{W}(H)\mathcal{Q}^i\mathcal{Q}^j\,. \tag{3.9}$$

The scalar fields, purely imaginary, are given by

$$z^i = i\,(3!c)^{1/3}\,\frac{H^i}{\left(\kappa^0_{ijk}H^iH^jH^k\right)^{1/3}} , \tag{3.10}$$

[4]The attractor points of this model have been extensively studied in [50]. Related works can be found in [44] [45].

and are subject to the following constraint, which ensures the regularity of the Kähler potential ($\mathcal{X}^0 = 1$ gauge)

$$\kappa^0_{ijk}\Im\mathrm{m}z^i\Im\mathrm{m}z^j\Im\mathrm{m}z^k > \frac{3c}{2}\,. \tag{3.11}$$

Substituting Eq. (3.10) into Eq. (3.11), we obtain

$$c > \frac{c}{4}\ , \tag{3.12}$$

which is an identity (assuming $c > 0$) and therefore imposes no constraints on the scalars. This phenomenon can be traced back to the fact that the the Kähler potential is constant when evaluated on the solution, and given by

$$e^{-\mathcal{K}} = 6c\ , \tag{3.13}$$

which is well defined, again, if $c > 0$. Since the volume of the C.Y. manifold is proportional to $e^{-\mathcal{K}}$, Eq. (3.13) implies that such volume remains constant and, in particular, that the limit $\Im\mathrm{m}z^i \to \infty$ does not imply a large volume limit of the compactification C.Y. manifold, a remarkable fact that can be seen as a purely stringy characteristic of our solution[5]. Notice that it is also possible to obtain the classical limit $\Im\mathrm{m}z^i \gg 1$ taking $c \gg 1$, that is, choosing a Calabi-Yau manifold with large enough c. In this case we would have also a truly large volume limit.

We have seen that, in order to obtain a consistent truncation, a necessary condition is $c > 0$, which implies that $\mathsf{W}(H)$ is well defined. We can go even further and argue that this is a sufficient condition by studying the equations of motion of the H-FGK formalism.

A consistent truncation requires that the equation of motion of the truncated field is identically solved for the *truncation value* of the field. First, notice that the set of solutions of Eqs. (1.83) and (1.82), taking into account (4.1), is non-empty, since there is a *model-independent* solution, given by

$$H^i = a^i - \frac{p^i}{\sqrt{2}}\tau, \qquad r_0 = 0\,, \tag{3.14}$$

which corresponds to a supersymmetric black hole. However, the equations of motion don't know about supersymmetry: it is system of differential equations whose solution can be written as

$$H^M = H^M(a, b)\ , \tag{3.15}$$

where we have made explicit the dependence in $2n_v + 2$ integration constants. When the solution (3.15) is plugged into (1.82) is when we impose, through r_0, a particular condition about the extremality of the black hole. If $r_0 = 0$ the integration constants are fixed such as the solution is extremal. In general there is not a unique way of doing it, one of the possibilities being always the supersymmetric one. Therefore, given that for our particular truncation the supersymmetric solution always exists, we can expect the existence also of the corresponding solution (3.15) of the equations of motion, from which the supersymmetric solution may be obtained through a particular choice of the integration constants that make (3.15) fulfilling (1.82) for $r_0 = 0$. We conclude, hence, that

$$\left\{H^P = 0, \mathcal{Q}^P = 0\right\} \quad \Rightarrow \mathcal{E}_P = 0\,, \tag{3.16}$$

[5]Notice that in order to consistently discard the non-perturbative terms in Eq. (3.1) we only need to take the limit $\Im\mathrm{m}z^i \to \infty$. Therefore, the behavior of the C.Y. volume in such limit plays no role.

(where $\mathcal{E}_P$ stands for the corresponding equation of motion) and therefore the truncation of as many H's as we want, together with the correspondet $\mathcal{Q}$'s, is consistent as long as $\mathsf{W}(H)$ remains well defined, something that in our case is assured if $c > 0$. From Eq. (3.1) it can be checked that the case $c = 0$, that is $h^{1,1} = h^{2,1}$, can in principle be cured by non-perturbative effects.

It is easy to see that the truncation is not consistent in the classical limit, and therefore, we can conclude that the corresponding solutions are *genuinely* quantum (stringy) solutions, which only exist when perturbative quantum effects are incorporated into the action.

Hence, we can conclude that if we require our theory to contain regular *quantum* black holes there is a topological restriction on the Calabi-Yau manifolds that we can choose to compactify Type-IIA string theory. The condition can be expressed as

$$c > 0 \quad \Rightarrow \quad h^{1,1} > h^{2,1} \,. \tag{3.17}$$

Eq. (3.17) is a stringent condition on the compactification C.Y. manifolds, in particular for small $h^{1,1}$. In fact, for small enough $h^{1,1}$ it could be even possible that no Calabi–Yau manifold existed such Eq. (3.17) is fulfilled. We will investigate this issue for $h^{1,1} = 3$, explicitly constructing the corresponding C.Y. manifolds and finding also particular *quantum* black hole solutions, in the next section[6].

3.2 New Calabi–Yau manifolds

In this section we will present the construction of new Calabi–Yau manifolds which satisfy $h^{1,1} = 3$ and $h^{2,1} < 3$, as required for the truncation presented in the previous section.

Calabi–Yau threefolds with both Hodge numbers small are relatively rare; two large and useful databases are the complete intersections in products of projective spaces (CICY's) [104], and hypersurfaces in toric fourfolds [32, 289], but the manifolds in these lists all satisfy the inequality $h^{1,1} + h^{2,1} > 21$. Smaller Hodge numbers can be found by taking quotients by groups which have a free holomorphic action on one of these manifolds (see, e.g., [78, 108, 152] and references therein), but none of the known spaces constructed this way satisfy our requirements.

Our technique here will be to begin with known manifolds with $h^{1,1} < 3$, $h^{2,1} < 4$, and a non-trivial fundamental group, and find hyperconifold transitions [150, 151] to new manifolds with the required Hodge numbers. Briefly, these transitions occur because a generically-free group action on a Calabi–Yau will develop fixed points on certain codimension-one loci in the moduli space. The fixed points are necessarily singular, and typically nodes [150], so the quotient space develops a point-like singularity which is a quotient of the conifold — a *hyperconifold*. These singularities can be resolved to give a new smooth Calabi–Yau. If the subgroup which develops a fixed point is $\mathbb{Z}_N$, then the change in Hodge numbers for one of these transitions is $\delta(h^{1,1}, h^{2,1}) = (N-1, -1)$.

Interestingly, there are examples which one might naïvely believe would lead to

[6]It is known in the literature the existence of the so called *rigid* C.Y. manifolds [105, 106, 401], which obey $h^{1,1} > 0, h^{2,1} = 0$, being therefore admissible compactification spaces. However, in order to have a tractable theory, we need a small enough $h^{1,1}$, yet not too small to yield a trivial theory. The choice $h^{1,1} = 3$ fulfills both conditions.

manifolds with $(h^{1,1},\, h^{2,1}) = (3,0)$ and $(h^{1,1},\, h^{2,1}) = (3,2)$, but none of these work out;[7] instead, we have two examples with $(h^{1,1},\, h^{2,1}) = (3,1)$, but with different intersection forms and Kähler cones.

3.2.1 $(h^{1,1},\, h^{2,1}) = (3,1)$ and diagonal intersection form

For the first example, we start with a manifold $X^{1,3}$, where the superscripts are the Hodge numbers $(h^{1,1},\, h^{2,1})$, and fundamental group $\mathbb{Z}_5\times\mathbb{Z}_2\times\mathbb{Z}_2 \cong \mathbb{Z}_{10}\times\mathbb{Z}_2$. It was first discovered in [108], and we briefly review the construction here. The manifold is obtained as a free quotient of a CICY $X^{5,45}$ that is given by the vanishing of two multilinear polynomials in a product of five $\mathbb{P}^1$'s; the configuration matrix [104] is

$$\begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{array}\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{3.18}$$

Let us call the two polynomials p_1, p_2, and take homogeneous coordinates $t_{i,a}$ on the ambient space, where $i = 0,1,2,3,4$ is understood mod 5, and $a = 0,1$ is understood mod 2. Then the action of the quotient group is generated by

$$\begin{aligned} g_{10} &: \; t_{i,a} \to t_{i+1,a+1} \quad ; \quad p_1 \leftrightarrow p_2 \;, \\ g_2 &: \; t_{i,a} \to (-1)^a t_{i,a} \;\; ; \quad p_1 \to p_1 \;,\; p_2 \to -p_2 \;. \end{aligned}$$

Note that these commute only up to projective equivalence, but this is sufficient. To define polynomials which transform appropriately, we start with the following quantities:

$$m_{abcde} = \sum_{i=0}^{4} t_{i,a} t_{i+1,b} t_{i+2,c} t_{i+3,d} t_{i+4,e} \;. \tag{3.19}$$

Then it is easily checked that the following are the most general polynomials which transform correctly:

$$\begin{aligned} p_1 &= \frac{A_0}{5} m_{00000} + A_1 m_{00011} + A_2 m_{00101} + A_3 m_{01111} \;, \\ p_2 &= \frac{A_0}{5} m_{11111} + A_1 m_{11100} + A_2 m_{11010} + A_3 m_{10000} \;, \end{aligned}$$

where the A_α are arbitrary complex constants. For generic values of the coefficients, these polynomials define a smooth manifold on which the group $\mathbb{Z}_{10}\times\mathbb{Z}_2$ acts freely; in this way we find a smooth quotient family $X^{1,3} = X^{5,45}/\mathbb{Z}_{10}\times\mathbb{Z}_2$.

We now need to specialise to a sub-family of $X^{5,45}$ which *does* have fixed points of the group generator g_2. Specifically, consider the point given by

$$t_{0,1} = t_{1,1} = t_{2,1} = t_{3,0} = t_{4,0} = 0, \tag{3.20}$$

[7] In each case, the spaces have unavoidable symmetries which make it impossible to create just a single hyperconifold singularity. Resolving the extra singularities pushes $h^{1,1}$ higher.

which is fixed by the action of g_2. Substituting the above into the polynomials gives the values $p_1 = A_1,\ p_2 = 0$, so if we set $A_1 = 0$, $X^{5,45}$ will contain this point. The argument of [150] guarantees that it will be a singularity, and one can check that for general values of the other coefficients, it is a node, so the quotient space $X^{1,3}$ develops a $\mathbb{Z}_2$-hyperconifold singularity. In fact, there are nine other points related to the above by the action of the other group generator g_{10}, so the covering space $X^{5,45}$ actually has ten nodes. Since these are all identified by the group action, $X^{1,3}$ develops only a single $\mathbb{Z}_2$-hyperconifold. This can be resolved by a single blow-up, and we obtain a new manifold with Hodge numbers $(h^{1,1},\, h^{2,1}) = (2, 2)$, and fundamental group $\mathbb{Z}_{10}$.

To get all the way to $X^{3,1}$, we need to go through another $\mathbb{Z}_2$-hyperconifold transition. If we also set $A_2 = 0$, then $X^{5,45}$ also passes through another fixed point of g_2, given by

$$t_{0,1} = t_{1,1} = t_{2,0} = t_{3,1} = t_{4,0} = 0\ , \tag{3.21}$$

as well as the nine points related to this by the action of g_{10}.

It can be checked that when $A_1 = A_2 = 0$, $X^{5,45}$ has exactly twenty nodes, at the points described above, and is smooth elsewhere. Therefore $X^{1,3}$ has precisely two $\mathbb{Z}_2$-hyperconifold singularities, which we can resolve independently to obtain a new smooth Calabi–Yau manifold $X^{3,1}$.

The intersection form and Kähler cone

To find the supergravity theory coming from compactification on $X^{3,1}$, we need to calculate its triple intersection form, and for this we need a basis for $H^2(X^{3,1}, \mathbb{Z})$ (throughout, we will implicitly talk about only the torsion-free part of the cohomology). There is a natural basis which consists of one divisor class inherited from $X^{1,3}$, and the two exceptional divisor classes coming from the two blow-ups.

First, let us find an integral generator of $H^2(X^{1,3}, \mathbb{Z})$. On the covering space $X^{5,45}$, let H_i be the divisor class given by the pullback of the hyperplane class from the i^{th} $\mathbb{P}^1$. Then the invariant divisor classes are multiples of $H \equiv H_0 + H_1 + H_2 + H_3 + H_4$. However, H itself, although an invariant *class*, does not have an invariant *representative*. The class $2H$ does, however; an example of an invariant divisor in $2H$ is the surface given by the vanishing of

$$f = (m_{00011})^2 + (m_{11100})^2 + (m_{00101})^2 + (m_{11010})^2\ . \tag{3.22}$$

This is a particularly convenient choice, as it gives a smooth divisor even on the singular family of threefolds given by $A_1 = A_2 = 0$, which misses the singular points.

Let D_1 be the divisor given by setting $f = 0$ and then taking the quotient, and let D_2 and D_3 be the two exceptional divisors. Then, since $f \neq 0$ on the fixed points of the group action, we immediately see that D_1, D_2, and D_3 are all disjoint, and the only intersection numbers which might be non-zero are D_1^3, D_2^3, and D_3^3.

For D_2^3 and D_3^3, we can use an easy general argument. For any smooth surface S in a Calabi–Yau threefold, the adjunction formula gives $S\big|_S \sim K_S$, where K_S is the canonical divisor class. The triple intersection number S^3 is therefore equal to K_S^2. Each of D_2 and D_3 is isomorphic to $\mathbb{P}^1\times\mathbb{P}^1$, so we find $D_2^3 = D_3^3 = 8$.

To calculate D_1^3, we note that D_1 descends from the divisor class $2H$ on $X^{5,45}$. Since this is embedded in a product of projective spaces, we can calculate intersection numbers purely from degrees; it is easy to check that on $X^{5,45}$, $(2H)^3 = 960$. We divide by a freely-acting group of order twenty, so on the quotient space we find $D_1^3 = \frac{960}{20} = 48$.

To summarise, the non-vanishing triple intersection numbers of $X^{3,1}$, in the basis D_1, D_2, D_3, are

$$\kappa^0_{111} = 48 \ , \quad \kappa^0_{222} = \kappa^0_{333} = 8 \ . \tag{3.23}$$

We can also say something about the Kähler cone. Certainly D_1 is positive everywhere except on the exceptional divisors, where it is trivial. On the other hand, each exceptional divisor D contains curves C for which $D \cdot C = -1$. From this information, we can glean that the Kähler cone is some sub-cone of $t_1 > 0, t_2 < 0, t_3 < 0$, and certainly includes the region where t_1 is much larger than $|t_2|$ and $|t_3|$.

3.2.2 $(h^{1,1},\, h^{2,1}) = (3,1)$ and non-diagonal intersection form

For our second example, we will again start with a free quotient of a CICY manifold, with configuration matrix

$$\begin{array}{c c}
\begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{array} &
\left[\begin{array}{cccc} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{array}\right] \\
 & \begin{array}{cccc} p & q & r_1 & r_2 \end{array}
\end{array}$$

where the labels on the columns denote the respective polynomials. This manifold has Euler number zero, and a series of splittings and contractions (explained in [104,107,108]) establishes that it is in fact isomorphic to the 'split bicubic' or Schoen manifold, with Hodge numbers $(h^{1,1},\, h^{2,1}) = (19, 19)$.

Let us take homogeneous coordinates σ_a on the first $\mathbb{P}^1$, $s_{i,a}$ on the next three, and $t_{i,a}$ on the last three, where $i = 0, 1, 2$, and $a = 0, 1$ are understood mod 3 and mod 2 respectively. The quotient group of interest is the dicyclic group $\mathrm{Dic}_3 \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_4$, which is the only non-trivial semi-direct product of $\mathbb{Z}_3$ and $\mathbb{Z}_4$. It is generated by two elements g_3 and g_4, of orders given by their subscripts, with the relation $g_4 g_3 g_4^{-1} = g_3^2$, and acts on the ambient space and polynomials as follows:

$$\begin{aligned}
g_3 &: \ \sigma_a \to \sigma_a \ , \ s_{i,a} \to s_{i+1,a} \ , \ t_{i,a} \to t_{i+1,a} \ ; \text{ all polynomials invariant} \ , \\
g_4 &: \ \sigma_a \to (-1)^a \sigma_a \ , \ s_{i,a} \to (-1)^{a+1} t_{-i,a} \ , \ t_{i,a} \to s_{-i,a} \ ; \ p \to -q \ , \ q \to p \ , \ r_1 \leftrightarrow r_2 \ .
\end{aligned}$$

In order to write down polynomials which transform appropriately, let us first define the g_3-invariant quantities

$$m_{abc} = \sum_i s_{i,a} s_{i+1,b} s_{i+2,c} \ \ , \ n_{abc} = \sum_i t_{i,a} t_{i+1,b} t_{i+2,c} \ . \tag{3.24}$$

Then by choice of coordinates (consistent with the above action), we can take the polynomials to be

$$p = \frac{1}{3}m_{000} + m_{011} \quad , \quad q = \frac{1}{3}n_{000} + n_{011} \qquad ,$$

$$r_1 = (a_0 m_{001} + \frac{1}{3}a_1 m_{111})\sigma_0 + (a_0 m_{001} + \frac{1}{3}a_2 m_{111})\sigma_1 \quad ,$$

$$r_2 = (a_0 n_{001} + \frac{1}{3}a_1 n_{111})\sigma_0 - (a_0 n_{001} + \frac{1}{3}a_2 n_{111})\sigma_1 \quad ,$$

where a_0, a_1, a_2 are arbitrary complex coefficients, defined only up to overall scale. It can be checked that for generic values of these coefficients, the corresponding manifold is smooth, and the group acts on it without fixed points. We therefore obtain a smooth quotient manifold $X^{2,2}$, where the value $h^{2,1} = 2$ corresponds to the two free coefficients (once we factor out overall scale) in the above polynomials.[8]

We will now show that there is a $\mathbb{Z}_2$-hyperconifold transition from $X^{2,2}$ to a manifold with $(h^{1,1},\, h^{2,1}) = (3,1)$. To do this, we need to arrange for the unique order-two element, g_4^2, to develop a fixed point. Consider the point in the ambient space given by

$$\frac{\sigma_1}{\sigma_0} = -1 \ , \ s_{0,1} = s_{1,1} = s_{2,0} = t_{0,0} = t_{1,0} = t_{2,0} = 0 \ . \tag{3.25}$$

This is fixed by g_4^2, but the other elements of the group permute this and five other g_4^2-fixed points. If we evaluate the polynomials at the point above, we find

$$p = q = r_1 = 0 \ , \ r_2 = a_1 + a_2 \ , \tag{3.26}$$

and their values at the other five fixed points are related by the group action to the ones above. So if $a_1 + a_2 = 0$, the Calabi–Yau will intersect these fixed points. By expanding the polynomials around any one of these points, we find that it has a node at each of them, so on the quotient space, we obtain a single $\mathbb{Z}_2$-hyperconifold singularity. Resolving this takes us to a new smooth manifold $Y^{3,1}$ (we use the letter Y to distinguish this from the other $(3,1)$ manifold we constructed). Its fundamental group is $\mathrm{Dic}_3/\langle g_4^2\rangle \cong S_3$, the symmetric group on three letters (the behaviour of fundamental groups under hyperconifold transitions such as this one is described in [152]).

The intersection form and Kähler cone

To calculate the intersection form of $Y^{3,1}$, we start with $X^{2,2}$ and its covering space $X^{19,19}$. Part of $H^{1,1}(X^{19,19}, \mathbb{Z})$ is generated by the pullbacks of the hyperplane classes of the $\mathbb{P}^1$ spaces. We will denote these by[9] $H_0, H_1, \ldots, H_6$. Looking at the group action, we can see that there are exactly two invariant divisor classes constructed from these: H_0 and $H_1 + H_2 + \ldots + H_6$. In contrast to the last example, each of these actually contains an invariant representative, and we get a basis $\{D_1, D_2\}$ for $H^{1,1}(X^{2,2}, \mathbb{Z})$ by simply taking the two invariant classes above and quotienting.

On the covering space, we can calculate intersection numbers simply by counting degrees, and we find that

$$H_0(H_1 + H_2 + \ldots + H_6)^2 = 72 \ , \ (H_1 + H_2 + \ldots + H_6)^3 = 216 \ , \tag{3.27}$$

[8] Counting independent coefficients does not always give the value of $h^{2,1}$, but in this case it does; perhaps the most direct way to obtain this is to notice that the manifold is obtained via a conifold transition on a codimension two locus in the moduli space of a manifold $X^{1,4}$, which was described at length in [79].

[9] Do not confuse these with the H variables of the H-FGK formalism.

and all others vanish. Dividing by the order of the group, we see that on the quotient space

$$D_1 D_2^2 = 6 \ , \ D_2^3 = 18 \ , \tag{3.28}$$

and the other triple intersections are zero.

Finally, we perform the transition to $Y^{3,1}$; denote the class of the exceptional divisor by D_3. It is easy enough to check that D_1 and D_2 have representatives which miss the singularity, so their pullbacks to $Y^{3,1}$ are disjoint from the exceptional divisor, and we get $D_1 \cdot D_3 = D_2 \cdot D_3 = 0$. Once again, the exceptional divisor is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, so by the argument of the last section, $D_3^3 = 8$.

Summarising, the non-zero intersection numbers on $Y^{3,1}$ are

$$\kappa^0_{122} = 6 \ , \ \kappa^0_{222} = 18 \ , \ \kappa^0_{333} = 8 \ . \tag{3.29}$$

By similar reasoning to the last case, we can say that the Kähler cone is some sub-cone of $t_1 > 0, t_2 > 0, t_3 < 0$, and includes the region where $|t_3|$ is sufficiently small compared to t_1 and t_2.

3.3 Quantum black hole solutions with $h^{1,1} = 3$

In section 3.1 we have presented a particular truncation of the equations of motion of $\mathcal{N} = 2, d = 4$ ungauged supergravity in a static, spherically symmetric background, which turned out to be consistent only for positive values of the quantum perturbative coefficient c (3.17). In the next two sections we are going to explicitly construct regular non-extremal (and therefore non-supersymmetric) black hole solutions to the truncated theory. In particular, we will start studying the cases where the C.Y. manifold is of the type constructed in section 3.2, to wit:

$$X^{3,1} \Rightarrow \kappa^0_{111} = 48 \ , \ \kappa^0_{222} = \kappa^0_{333} = 8 \, , \tag{3.30}$$

$$Y^{3,1} \Rightarrow \kappa^0_{122} = 6 \ , \ \kappa^0_{222} = 18 \ , \ \kappa^0_{333} = 8 \, . \tag{3.31}$$

For these two sets of intersection numbers, Eq. (3.8) becomes, respectively

$$\mathsf{W}(H) = \alpha \left| 48 \left(H^1\right)^3 + 8 \left[\left(H^2\right)^3 + \left(H^3\right)^3 \right] \right|^{2/3} \ , \tag{3.32}$$

$$\mathsf{W}(H) = \alpha \left| 18 \left(H^2\right)^2 \left[H^1 + H^2\right] + 8 \left(H^3\right)^3 \right|^{2/3} \ . \tag{3.33}$$

For simplicity we take $H^1 = s_2 H^2 = s_3 H^3 \equiv H$ ($s_{2,3} = \pm 1$), $p^1 = p^2 = p^3 \equiv p$. For this particular configuration we find a non-extremal solution for each set of intersection numbers given by

$$H = a \cosh(r_0 \tau) + \frac{b}{r_0} \sinh(r_0 \tau), \qquad b = s_b \sqrt{r_0^2 a^2 + \frac{p^2}{2}} \ , \tag{3.34}$$

where, now and henceforth, $s_b = \pm 1$. The scalars, which turn out to be constant, read

$$z^1 = i (3! c)^{1/3} \lambda^{-1/3} = s_{2,3} z^{2,3} \, , \tag{3.35}$$

where

$$\lambda = [48 + 8(s_2 + s_3)] \quad \text{for} \quad X^{3,1}, \qquad (3.36)$$
$$\lambda = [18 + 18s_2 + 8s_3] \quad \text{for} \quad Y^{3,1} .$$

The explicit form of the metric reads, in turn,

$$ds^2 = \left[\frac{1}{2}(3!c)^{1/3}\left[a\cosh(r_0\tau) + \frac{b}{r_0}\sinh(r_0\tau)\right]^2\right]^{-1} dt^2 \qquad (3.37)$$
$$-\frac{1}{2}(3!c)^{1/3}\left[a\cosh(r_0\tau) + \frac{b}{r_0}\sinh(r_0\tau)\right]^2 \left[\frac{r_0^4}{\sinh^4 r_0\tau}d\tau^2 + \frac{r_0^2}{\sinh^2 r_0\tau}d\Omega^2_{(2)}\right] . \qquad (3.38)$$

Since the scalars are constant and don't depend on the charges, we cannot perform the $\Im\mathrm{m} z^i \to \infty$ limit that fully suppress the non-perturbative corrections. Still, the exponent in Eq. (3.3) is, in both cases, of order

$$2\pi i d_i z^i \sim -\frac{1}{3}\sum_{i=1}^{3} d_i, \quad d_i \geq 1 , \qquad (3.39)$$

and therefore we find small non-perturbative corrections, in particular one order smaller than the perturbative part of the prepotential $\mathcal{F}_{Pert} \sim 10 \cdot \mathcal{F}_{Non-Pert}$. The solution lies inside the Kähler cone when

$$s_2 = s_3 = -1, \quad \text{for } X^{3,1} , \qquad (3.40)$$

$$s_2 = -s_3 = 1, \quad \text{for } Y^{3,1} . \qquad (3.41)$$

This can be verified by explicitly checking the positive-definiteness of the Kähler metric

$$\mathcal{G}_{ij^*} = \partial_i \partial_{j^*}\mathcal{K} \qquad (3.42)$$

evaluated on the solution. It turns out that the only sets of $\{s_2, s_3\}$ which give rise to positive-definite Kähler metrics (and, as a consequence, to solutions lying inside the Kähler cone) are the ones shown above. These conditions on the signs of the scalar fields are in full agreement with those obtained in subsections (3.2.2) and (3.2.1), since $\Im m z^i = t^i$ [110].

Imposing asymptotic flatness, the constant a gets fixed to

$$a = -s_b \frac{\Im m z^1_\infty}{\sqrt{3c}} . \qquad (3.43)$$

It is now easy to compute the mass and the entropy of the outer/inner horizon

$$M = r_0\sqrt{1 + \frac{3cp^2}{2r_0^2(\Im\mathrm{m} z^1_\infty)^2}} , \qquad (3.44)$$

$$S_\pm = r_0^2\pi\left(\sqrt{1 + \frac{3cp^2}{2r_0^2(\Im\mathrm{m} z^1_\infty)^2}} \pm 1\right)^2 . \qquad (3.45)$$

This implies that the product of both entropies only depends on the charge

$$S_+S_- = \frac{\pi^2\alpha^2}{4}p^4\lambda^{4/3}\,, \tag{3.46}$$

so it does not depend on the asymptotic value of the moduli $\Im\mathrm{m} z^1_\infty$.

It is worth stressing that the Ansatz $H^i = a^i + b^i\tau$ in the extremal ($r_0 = 0$) case was successfully used to obtain solutions with constant scalars but different critical points, in some cases particularly involved. However, presumably due to the complexity of the calculations, we have not been able to find a solution with non-constant scalars for any of the two models analyzed in this section. This may suggest also a more stabilized behavior for the scalars in the presence of perturbative quantum corrections.

3.4 Quantum corrected STU model

In this section we consider a very special case, the so-called STU model, in the presence of perturbative quantum corrections, obtaining the first non-extremal solution of this kind with non-constant scalars. In order to do so, we set $n_v = 3$, $\kappa^0_{123} = 1$. From (3.8) we obtain[10]

$$\mathsf{W}(H) = \alpha\left|H^1H^2H^3\right|^{2/3}\quad, \tag{3.47}$$

where $\alpha = 3c^{1/3}$. The scalar fields are given by

$$z^i = ic^{1/3}\frac{H^i}{(H^1H^2H^3)^{1/3}}\quad, \tag{3.48}$$

The τ-dependence of the H^M can be found by solving the equations of motion plus Hamiltonian constraint of the H-FGK formalism, Eqs. (1.83) and (1.82). We find the solution

$$H^i = a^i\cosh(r_0\tau) + \frac{b^i}{r_0}\sinh(r_0\tau)\,,\qquad b^i = s^i_b\sqrt{r_0^2(a^i)^2 + \frac{(p^i)^2}{2}}\quad. \tag{3.49}$$

The three constants a^i can be fixed relating them to the value of the scalars at infinity and imposing asymptotic flatness. We have, hence, four conditions for three parameters and therefore one would expect a relation among the $\Im\mathrm{m} z^i_\infty$, leaving c undetermined. However, the explicit calculation shows that the fourth relation is compatible with the others, and therefore no extra constraint is necessary. The a^i are given by

$$a^i = -s^i_b\frac{\Im\mathrm{m} z^i_\infty}{\sqrt{3c}}\quad. \tag{3.50}$$

The mass and the entropy, in turn, read

$$M = \frac{r_0}{3}\sum_i\sqrt{1 + \frac{3c(p^i)^2}{2r_0^2(\Im\mathrm{m} z^i_\infty)^2}}\quad, \tag{3.51}$$

$$S_\pm = r_0^2\pi\prod_i\left(\sqrt{1 + \frac{3c(p^i)^2}{2r_0^2(\Im\mathrm{m} z^i_\infty)^2}} \pm 1\right)^{2/3}\quad, \tag{3.52}$$

[10]We have to stress that we haven't been able to construct an explicit C.Y. manifold with $\kappa^0_{123} = 1$ and $h^{2,1} < 3$.

and therefore the product of the inner and outer entropy only depends on the charges again

$$S_+S_- = \frac{\pi^2\alpha^2}{4}\prod_i \left(p^i\right)^{4/3} \quad . \tag{3.53}$$

In the extremal limit we obtain the supersymmetric as well as the non-supersymmetric extremal solutions, depending on the sign chosen for the charges.

It is important to point out that the black holes presented in this section have been succesfully uplifted to M-theory in the supersymmetric limit in [54]. There, the authors show that they can be interpreted as arising from three stacks of M2 branes on a conical singularity. This allows them to relate them to a system of D3 branes carrying momentum and to give a microscopic interpretation of their entropy. As we have seen (the macroscopic entropy of the supersymmetric black holes can be obtained by taking the square root of Eq. (3.53)), our solutions present the particularity that, in contradistinction to the previously studied cases [260, 312, 313, 403], their entropy does not scale with the square-root of the product of their charges (which microscopically comes from using Cardy's formula for the count of states in a certain 1+1 dimensional system of branes and strings), but with a power 2/3. The authors are able to microscopically reproduce the exact expression of the entropy

$$S_{\text{SUSY}} = \frac{3c^{\frac{1}{3}}}{2}\pi \left|p^1p^2p^3\right|^{2/3} , \tag{3.54}$$

up to a global factor. The result is such that if the entropies match, some dependence of the entropy on trascendental numbers dissapears, which is argued to be a nontrivial check for theirs to be the right microscopic description of our *quantum* black holes.

3.5 Non-perturbative α'-corrected solutions

In the previous sections, we have used the H-FGK formalism (see section 1.2.4), to defined a new class of black holes for Type-IIA C.Y. compactifications. These have the property that they exist only when the perturbative corrections to the prepotential are included and that no classical limit can be assigned to them (because the truncation itself becomes inconsistent in that limit). They were called, in consequence, *quantum* or *stringy* black holes. Therefore, for self-mirror C.Y. manifolds such black holes do not exist, since in that case the perturbative corrections exactly vanish. However, the situation can be changed if we add non-perturbative corrections to the prepotential. That is the case we are going to consider now. We will obtain the first explicit black hole solution of Type-IIA string theory compactified on a self-mirror Calabi-Yau threefold in the presence of non-perturbative corrections, proving at the same time that these non-perturbative corrections *lift* the singular behaviour of the quantum black holes to a regular one.

Somewhat surprisingly, we will obtain a class of solutions which involves Lambert's W function [291], which is multi-valued in a certain real domain. We will explain how this fact seems to provide an appropriate scenario for a potential new kind of violation of the (corresponding uniqueness conjecture, and as a consequence of the) no-hair conjecture in four dimensions. It turns out that, in our set-up, string theory forbids the use of the Lambert function to that end. However, the possibility remains open in an exclusive supergravity set-up (not necessarily embedded in string theory), and there does not seem to be a reason to discard it right away. Actually, we will see in the next chapter how this can be actually achieved.

In order to tackle the construction of these new black hole solutions of (3.1), we are going to consider the same truncation as in Eq. (3.7), namely

$$H^0 = H_0 = H_i = 0, \quad p^0 = q_0 = q_i = 0\,. \tag{3.55}$$

Under this assumption, the stabilization equations take the form

$$\begin{pmatrix} iH^i \\ \tilde{H}_i \end{pmatrix} = \frac{e^{\mathcal{K}/2}}{X}\begin{pmatrix} \mathcal{X}^i \\ \frac{\partial F(\mathcal{X})}{\partial \mathcal{X}^i} \end{pmatrix}, \begin{pmatrix} \tilde{H}^0 \\ 0 \end{pmatrix} = \frac{e^{\mathcal{K}/2}}{X}\begin{pmatrix} \mathcal{X}^0 \\ \frac{\partial F(\mathcal{X})}{\partial \mathcal{X}^0} \end{pmatrix}. \tag{3.56}$$

The physical fields can be obtained from the H^i as

$$e^{-2U} = \tilde{H}_i H^i\,, \quad z^i = i\frac{H^i}{\tilde{H}^0}\,, \tag{3.57}$$

as soon as $\tilde{H}^0$ and $\tilde{H}^i$ are determined. In order to obtain $\tilde{H}^0$ as a function of H^i, we need to solve the highly involved equation

$$\frac{\partial F(H)}{\partial \tilde{H}^0} = 0\,, \tag{3.58}$$

where $F(H)$ stands for the prepotential expressed in terms of the H^i

$$F(H) = \frac{i}{3!}\kappa^0_{ijk}\frac{H^iH^jH^k}{\tilde{H}^0} + \frac{ic(\tilde{H}^0)^2}{2} + \frac{i(\tilde{H}^0)^2}{(2\pi)^3}\sum_{\{d_i\}} n_{\{d_i\}} Li_3\left(e^{-2\pi d_i \frac{H^i}{\tilde{H}^0}}\right). \tag{3.59}$$

Once this is done, it is not difficult to express $\tilde{H}^i$ in terms of H^i. Indeed, from (3.56) we simply have

$$\tilde{H}_i = -i\frac{\partial F(H)}{\partial H^i}\,. \tag{3.60}$$

If we expand (3.58), we find

$$-\frac{1}{3!}\kappa^0_{ijk}\frac{H^iH^jH^k}{(\tilde{H}^0)^3} + c + \frac{1}{4\pi^3}\sum_{\{d_i\}} n_{\{d_i\}}\left[Li_3\left(e^{-2\pi d_i\frac{H^i}{\tilde{H}^0}}\right) + Li_2\left(e^{-2\pi d_i\frac{H^i}{\tilde{H}^0}}\right)\left[\frac{\pi d_i H^i}{\tilde{H}^0}\right]\right] = 0\,. \tag{3.61}$$

Solving (3.61) for $\tilde{H}^0$ in full generality seems to be an extremely difficult task. However, if we go to the large volume compactification limit ($\Im\mathrm{m} z^i >> 1$), we can make use of the following property of the polylogarithmic functions

$$\lim_{|w|\to 0} Li_s(w) = w\,, \forall s \in \mathbb{N}\,, \tag{3.62}$$

since, in our case, $w = e^{-2\pi d_i \Im m z^i}\,, \forall\,\{d_i\} \in (\mathbb{Z}^+)^{h^{1,1}}$. Eq. (3.62) enables us to rewrite (3.61) as

$$-\frac{1}{3!}\kappa^0_{ijk}\frac{H^iH^jH^k}{(\tilde{H}^0)^3} + c + \frac{1}{4\pi^3}\sum_{\{d_i\}} n_{\{d_i\}}\left[e^{-2\pi d_i\frac{H^i}{\tilde{H}^0}} + e^{-2\pi d_i\frac{H^i}{\tilde{H}^0}}\left[\frac{\pi d_i H^i}{\tilde{H}^0}\right]\right] = 0\,, \quad \Im\mathrm{m} z^i >> 1\,. \tag{3.63}$$

The dominant contribution in this regime, aside from the cubic one, is given by c. In the previous subsections and [191], the first non-extremal black hole solutions (with constant and non-constant scalars) of (3.1) were obtained ignoring the non-perturbative corrections. In particular, the solutions of [89] turned out to be purely *quantum*, in the sense that not only the classical limit $c \to 0$ was ill-defined, but also the truncated theory became inconsistent and therefore no classical limit could be assigned to such solutions. An interesting question to ask now is whether the non-perturbative contributions could actually be able to cure or at least improve this behaviour. On the other hand, it is also interesting *per se* to explore the existence of black hole solutions when the subleading contribution to the prepotential is not given by c, but has a non-perturbative origin. In order to tackle these two questions, let us restrict ourselves to C.Y. three-folds with vanishing Euler characteristic ($c = 0$), the so-called self-mirror C.Y. three-folds. Under this assumption, and considering only the subleading contribution in (3.61), which is now given by the fourth term in (3.63), such equation becomes[11]

$$-\frac{1}{3!}\kappa^0_{ijk}\frac{H^iH^jH^k}{(\tilde{H}^0)^3} + \frac{1}{4\pi^3}\sum_{\{d_i\}} n_{\{d_i\}} e^{-2\pi d_i \frac{H^i}{\tilde{H}^0}} \left[\frac{\pi d_i H^i}{\tilde{H}^0}\right] = 0\,. \tag{3.64}$$

The sum over $\{d_i\}$ in (3.64) will be dominated in each case by a certain term corresponding to a particular vector $\left\{\hat{d}_i\right\}$ (and, as a consequence, to a particular $n_{\hat{d}_i} \equiv \hat{n}$), which, since we are assuming $\Im\mathrm{m} z^i >> 1$, is the only one that we need to consider. That is, $\left\{\hat{d}_i\right\}$ corresponds to the set of d_i that labels the most relevant term in the infinite sum present in (3.64). Hence, this equation becomes

$$-\frac{1}{3!}\kappa^0_{ijk}\frac{H^iH^jH^k}{(\tilde{H}^0)^3} + \frac{\hat{n}}{4\pi^3} e^{-2\pi \hat{d}_i \frac{H^i}{\tilde{H}^0}} \left[\frac{\pi \hat{d}_i H^i}{\tilde{H}^0}\right] = 0\,, \quad \Im\mathrm{m} z^i >> 1\,. \tag{3.65}$$

This is solved by[12]

$$\tilde{H}^0 = \frac{\pi \hat{d}_l H^l}{W_a\left(s_a\sqrt{\frac{3\hat{n}(\hat{d}_n H^n)^3}{2\kappa^0_{ijk}H^iH^jH^k}}\right)}\,, \tag{3.66}$$

where $W_a(x)$, $(a = 0, -1)$ stands for (any of the two real branches of) the *Lambert W function*[13] (also known as *product logarithm*), and $s_a = \pm 1$. Using now Eq. (3.60) we can obtain $\tilde{H}^i$. The result is

$$\tilde{H}_i = \frac{1}{2}\kappa^0_{ijk}\frac{H^jH^k}{\pi\hat{d}_l H^l} W_a\left(s_a\sqrt{\frac{3\hat{n}(\hat{d}_m H^m)^3}{2\kappa^0_{pqr}H^pH^qH^r}}\right)\,. \tag{3.67}$$

The physical fields can now be written as a function of the H^i as

$$e^{-2U} = \mathsf{W}(H) = \frac{\kappa^0_{ijk}H^iH^jH^k}{2\pi\hat{d}_m H^m} W_a\left(s_a\sqrt{\frac{3\hat{n}(\hat{d}_l H^l)^3}{2\kappa^0_{pqr}H^pH^qH^r}}\right)\,, \tag{3.68}$$

[11] $e^{2\pi i d_i z^i} << \pi d_i \Im\mathrm{m} z^i e^{2\pi i d_i z^i}$ for $\Im\mathrm{m} z^i >> 1$.

[12] Henceforth we will be using W for the Hessian potential, and W for the Lambert function. We hope this is not a source of confusion.

[13] See the Appendix B.1 for more details.

$$z^i = i\frac{H^i}{\pi \hat{d}_m H^m} W_a \left(s_a \sqrt{\frac{3\hat{n}(\hat{d}_l H^l)^3}{2\kappa^0_{pqr} H^p H^q H^r}} \right) . \tag{3.69}$$

In order to have a regular solution, we need to have a positive definite metric warp factor e^{-2U}. Since, as explained in Appendix B.1, $\text{sign}\,[W_a(x)] = \text{sign}\,[x]\,,\ a = 0, -1,\ x \in D^a_{\mathbb{R}}$, we have to require that

$$s_0 \equiv sign\left[\kappa^0_{ijk}\frac{H^i H^j H^k}{\hat{d}_m H^m}\right] , \tag{3.70}$$

$$s_{-1} \equiv -1\,. \tag{3.71}$$

On the other hand, since $W_0(x) = 0$ for $x = 0$ and $W_{-1}(x)$ is a real function only when $x \in \left[-\frac{1}{e}, 0\right)$, we have to impose that the argument x of W_a lies entirely either in $\left[-\frac{1}{e}, 0\right)$ or in $(0, \infty)$ for all $\tau \in (-\infty, 0)$, since e^{2U} cannot be zero in a regular black hole solution for any $\tau \in (-\infty, 0)$. This condition must be imposed in a case by case basis, since it depends on the specific form of the symplectic vector $H^M = H^M(\tau)$ as a function of τ. Notice that if $x \in \left[-\frac{1}{e}, 0\right) \quad \forall \ \tau \in (-\infty, 0)$ we can in principle[14] choose either W_0 or W_{-1} to build the solution, whereas if $x \in (0, +\infty) \quad \forall \ \tau \in (-\infty, 0)$, only W_0 is available.

Needless to say, in order to construct actual solutions, we have to solve the H-FGK equations of motion (1.83) (plus hamiltonian constraint (1.82)) using the Hessian potential given by (3.68). Fortunately, such equations admit a *model-independent* solution which is obtained choosing the H^i to be harmonic functions in the flat transverse space, with one of the poles given in terms of the corresponding charge

$$H^i = a^i - \frac{p^i}{\sqrt{2}}\tau, \qquad r_0 = 0\,. \tag{3.72}$$

In fact, it is a virtue of the H-FGK formalism to make explicit how every $\mathcal{N} = 2$, $d = 4$ supergravity theory admits a solution of the form

$$H^M = a^M - \frac{\mathcal{Q}^M}{\sqrt{2}}\tau, \qquad r_0 = 0\,, \qquad a^M \mathcal{Q}_M = 0\,, \tag{3.73}$$

where the last equation encodes the absence of Taub-NUT charge. It can be easily verified that Eq. (3.73) does indeed satisfy Eqs. (1.83) and (1.82) independently of the model. This corresponds to a supersymmetric black hole solution [37, 320, 411].

3.6 The general supersymmetric solution

As we have said, plugging (3.72) into (3.69) and (3.68) provides us with a supersymmetric solution without solving any further equation. The entropy of such a solution reads

$$S = \frac{1}{2}\kappa^0_{ijk}\frac{p^i p^j p^k}{\hat{d}_m p^m} W_a\left(s_a \beta\right) , \tag{3.74}$$

$$\beta \equiv \sqrt{\frac{3\hat{n}(\hat{d}_l p^l)^3}{2\kappa^0_{pqr} p^p p^q p^r}} ,$$

[14] As we will see in section 4.2, the possibility $s_0 = s_{-1} = -1$ will not be consistent with the large volume approximation we are considering.

and the mass is given by

$$M = \dot{U}(0) = \frac{1}{2\sqrt{2}}\left[\frac{3\kappa^0_{ijk}p^i a^j a^k}{\kappa^0_{pqr}a^p a^q a^r}\left[1 - \frac{1}{1+W_a(s_a\alpha)}\right] - \frac{d_l p^l}{d_n a^n}\left[1 - \frac{3}{2\left(1+W_a(s_a\alpha)\right)}\right]\right], \tag{3.75}$$

$$\alpha \equiv \sqrt{\frac{3\hat{n}(\hat{d}_l a^l)^3}{2\kappa^0_{pqr}a^p a^q a^r}}\,. \tag{3.76}$$

In the approximation under consideration, we are neglecting terms $\sim e^{-2\pi d_i \Im m z^i}$ with respect to those going as $\sim \pi d_i \Im m z^i e^{-2\pi d_i \Im m z^i}$. Taking into account (3.69), this assumption is translated into the condition

$$W_a(x)e^{-2W_a(x)} >> e^{-2W_a(x)}\,. \tag{3.77}$$

It is clear that this condition is satisfied for $a = 0$ if $x \in [\alpha, \beta]$ for positive and suficiently large values of α and β. Constructing a solution such that the values of p^i and a^i correspond to large enough α and β may or may not be possible depending on the compactification data. For example, if $\hat{d}_i = (1, 0, ..., 0)$ and $\kappa^0_{iii} = 0\,\forall\, i = 1, ..., n_v$, it is clear that taking $a^1 >> 1$ and $p^1 >> 1$ satisfies the corresponding condition.

It is also clear, however, that (3.77) is not satisfied at all for $x \in [-\frac{1}{e}, 0)$, which is the range for which both branches of the Lambert function are available.

If we assume $x \in [\alpha, \beta]$ for suficiently large $\alpha, \beta \in \mathbb{R}^+$, $a = 0$ and W_0 is the only real branch of the Lambert function. In that case, $s = s_0 = 1$, and we have

$$e^{-2U} = \frac{\kappa^0_{ijk}H^i H^j H^k}{2\pi\hat{d}_m H^m}W_0\left(\sqrt{\frac{3\hat{n}(\hat{d}_l H^l)^3}{2\kappa^0_{pqr}H^p H^q H^r}}\right), \tag{3.78}$$

$$z^i = i\frac{H^i}{\pi\hat{d}_m H^m}W_0\left(\sqrt{\frac{3\hat{n}(\hat{d}_l H^l)^3}{2\kappa^0_{pqr}H^p H^q H^r}}\right). \tag{3.79}$$

In the conformastatic coordinates we are working with, the metric warp factor e^{-2U} is expected to diverge at the event horizon ($\tau \to -\infty$) as τ^2. In addition, we have to require $e^{-2U} > 0\ \forall\tau \in (-\infty, 0]$, and impose asymptotic flatness $e^{-2U(\tau=0)} = 1$. The last two conditions read

$$\frac{\kappa^0_{ijk}H^i H^j H^k}{2\pi\hat{d}_n H^n} > 0\quad \forall\tau \in (-\infty, 0]\,, \tag{3.80}$$

$$\frac{\kappa^0_{ijk}a^i a^j a^k}{2\pi\hat{d}_m a^m}W_0\left(\alpha\right) = 1\,, \tag{3.81}$$

whereas the first one turns out to hold, since

$$e^{-2U} \overset{\tau\to-\infty}{\longrightarrow} \frac{\kappa^0_{ijk}p^i p^j p^k}{8\pi\hat{d}_m p^m}W_0\left(\beta\right)\tau^2\,. \tag{3.82}$$

(3.80) and (3.81) can in general be safely imposed in any particular model we consider. Finally, the condition for a well-defined and positive mass $M > 0$ can be read off from (3.75).

3.7 Multivalued functions and the no-hair conjecture

As we explained in the previous section, our approximation is not consistent with a solution such that $x \in [-\frac{1}{e}, 0)$. This forbids the domain in which $W(x)$ is a multivalued function (both W_0 and W_{-1} are real there). However, it seems legitimate to ask what the consequences of having two different branches would have been, had this constraint not been present. In principle, we could have tried to assign the asymptotic ($\tau \to 0$) and near horizon ($\tau \to -\infty$) limits to any particular pair of values of the arguments of W_0 and W_{-1} (x_0 and x_{-1} respectively) through a suitable election of the parameters available in the solution. In particular, had we chosen $x_0|_{\tau=0} = x_{-1}|_{\tau=0} = -1/e$ and $x_0|_{\tau\to-\infty} = x_{-1}|_{\tau\to-\infty} = \beta$, $\beta \in (-1/e, 0)$, both solutions would have had exactly the same asymptotic behavior (and therefore the scalars of both solutions would have coincided at spatial infinity), and we would have been dealing with two completely different regular solutions with the same mass[15], charges and asymptotic values of the scalar fields, in contradiction[16] with the corresponding black hole uniqueness conjecture (and, as a consequence, with the no-hair conjecture). At this point, and provided that our approximation is not consistent with such presumable two-branched solution, the feasibility of this reasoning in a different context can only be catalogued as *speculative* at the very least. However, a violation of the no-hair conjecture in four-dimensional supergravity would have interesting consequences independently of whether the solution is embedded in string theory or not. In this regard, the very possibility that the stabilization equations may admit (for certain more or less complicated prepotentials) solutions depending on multivalued functions seems to open up a window for possible violations of the no-hair conjecture in the context of $\mathcal{N}=2$ $d=4$ supergravity. The question (whose answer is widely assumed to be "no") is now: is it possible to find a four-dimensional (super)gravity theory admitting more than one stable black hole solution with the same mass, electric, magnetic and scalar charges? We devote the next chapter to prove the answer to this question to be yes.

[15] Although $W'_{0,-1}(x)$ are divergent at $x = -1/e$ (as explained in the Appendix B.1), and the definition of M would involve derivatives of the Lambert function at that point, it would not be difficult to cure this behaviour and get a positive (and finite) mass by imposing $\dot{x}(\tau) \overset{\tau\to 0}{\longrightarrow} 0$ faster than $|W'_{0,-1}(x)| \overset{x\to -1/e}{\longrightarrow} \infty$.

[16] Up to possible stability issues, which should be carefully studied.

4

The violation of the no-hair conjecture in $d = 4$ ungauged supegravity

This chapter is based on

Pablo Bueno and C. S. Shahbazi,
"The violation of the no-hair Conjecture in four-dimensional ungauged Supergravity",
Class. Quantum Grav. ***31*** *(2014) 145005. [*arXiv:1310.6379 *[hep-th]] [96].*

An interesting feature of black holes comes from their exclusiveness. Indeed, it has been known for a long time now that all the stationary, asymptotically flat, black hole solutions to the Einstein-Maxwell theory, in a sort of general relativistic version of the Gauss law, are uniquely determined by a few parameters: their mass, their angular momentum and their electric and magnetic charges [69, 115, 253, 259, 317] [1].

The possible generalizations of these uniqueness (or *no-hair*) theorems to systems with more fields (such as scalars or non-Abelian vectors) has been an active area of research [38, 41, 143] since the proofs of the theorems for the simplest cases were carried out. On the other hand, the seek for counterexamples to the corresponding conjectured uniqueness theorems in such scenarios has also attracted a lot of attention, and produced some interesting results. In particular, it is now known that the no-hair conjecture can be violated or circumvented in certain Einstein-Yang-Mills-Higgs systems (See [10–13, 196, 216, 282, 422] and references therein) and in higher-curvature theories of gravity [271, 316].

In this chapter we are going to construct a particular $\mathcal{N} = 2$, $d = 4$ ungauged Supergravity model admitting pairs of supersymmetric, static, spherically symmetric and asymptotically flat black hole solutions sharing the same mass, charges and asymptotic values of the scalar fields, providing, to the best of our knowledge, the first counterexample to the corresponding uniqueness conjecture in the context of an ungauged Supergravity theory, and one of the first (some previous examples can be found in [12]) for a system without scalar potential, non-Abelian vector fields or higher-order curvature corrections.

In the previous chapter, we obtained for the first time black hole solutions to a Type-IIA String Theory compactification on a Calabi-Yau manifold in the presence of non-perturbative corrections to the Special Kähler geometry of the vector multiplet sector. These black holes were given in terms of harmonic functions on euclidean $\mathbb{R}^3$, as it must be for supersymmetric black hole solutions of ungauged four dimensional Supergravity

[1] Thus, the only possible solution for a stationary, axisymmetric and electrovacuum black hole is given by the well-known Kerr-Newmann spacetime [318].

[243, 325], but they also contained a special function called the *Lambert function*[2]. As we argued in [95], the fact that the Lambert function is multivalued opened up the possibility of using its different branches to build inequivalent black hole solutions with the same conserved charges at infinity. However, such possibility was forbidden by the large volume compactification limit we assumed to hold through all the calculations: that limit only allowed us to consider solutions such that the argument of the Lambert function lied into a set of values for which the function was uniquely valued.

Inspired by this result, we are going to construct a particular Supergravity model that can be analitically solved, and such that its supersymmetric black hole solutions share some of the characteristics of those found in the previous chapter, but without any approximation involved. In particular, we will be able to construct solutions whose metric and scalars will depend on the Lambert function. In this case both branches will be available, and we will show how to construct a family of pairs of inequivalent solutions, providing a violation of the conjecture.

In order to illustrate the result, we will show an explicit example for a model with two scalar fields. We will find that both solutions are regular, in the sense that the only physical singularity of the space-time will be hidden by an event horizon of non-zero positive area (for each solution in the pair). However, we will also see that the Special Kähler metric will not be positive definite (just like happens in other counterexamples to the conjecture [316]) when evaluated on the solutions we have constructed explicitly or, equivalently, that the energy-momentum tensor of at least one of the scalars will not satisfy in general the null energy condition (NEC). In this respect, and although the no-hair conjecture does not make in principle reference to stability issues, it is fair to say that the spirit of the conjecture remains partially alive.

4.1 A stringy motivation for the model

The purpose of this chapter is to study the supersymmetric black hole solutions of a particular $\mathcal{N} = 2$ four dimensional ungauged Supergravity coupled to vector multiplets, which we will find to *violate* the folk uniqueness theorems that are supposed to hold in unaguged four-dimensional Supergravity. Of course, such model did not appear out of the blue, but it has his seed and motivation in the results found in the previous chapter. In section 4.2 a new class of supersymmetric black hole solutions of type-IIA String Theory compactified to four dimensions on a Calabi-Yau manifold in the presence of non-perturbative stringy corrections was obtained. In order to solve the involved stabilization equations, we were forced to consider the large volume limit $\Im m z^i \to \infty$ of the compactification, where certain simplifications could be made. As a consequence, the approximation $\Im m z^i \to \infty$ had to be also imposed on the solution. As explained in section 3.7, only one of the two real branches of the W function (the one with $a = 0$) was consistent with such condition, which also implied the argument of $W_0(x(\tau))$ to be positive. We argued how, had not this condition been present, we could have tried to build two different solutions solving the same equations of motion, by choosing $W_0(x(\tau))$ or $W_{-1}(x(\tau))$. In fact, we could have assigned, through a suitable election of the parameters available in the solution, the near horizon ($\tau \to -\infty$) and asymptotic ($\tau \to 0$) limits of the argument $x(\tau)$ of $W_0(x(\tau))$ and $W_{-1}(x(\tau))$ to any pair of values chosen at will. In particular, we could have selected

[2] See appendix B.1.

$x(\tau = 0) = -1/e$ and $\lim_{\tau\to-\infty} x(\tau) = \beta$, $\beta \in (-1/e, 0)$, and then the solution built with $W_0(x(\tau))$ and the one constructed with $W_{-1}(x(\tau))$ would have had exactly the same asymptotic behaviour, but different profiles away from infinity (note also (B.1) that W_0 and W_{-1} are not even symmetric, in contradistinction to the branches of other real multi-valued functions such as the inverse trigonometric functions). That is, we would have been dealing with two completely different regular solutions with the same mass M, charges and asymptotic values of the scalar fields, in contradiction with the aforementioned conjecture. Let us state that when we write *regular*, we mean a black hole solution with positive mass M such that there is a unique physical singularity in the space-time and it is hidden by an event horizon with non-zero, positive area.

In order to accomplish the construction of our solutions, we are going to somewhat forget about String Theory and propose a prepotential which we can solve exactly, and such that the corresponding supersymmetric solutions enjoy the same desirable properties as the String-Theory-forbidden ones of [95]. In particular, we will use the same truncation in the H-variables, to wit

$$H^0 = H_0 = H_i = 0, \quad p^0 = q_0 = q_i = 0\,. \tag{4.1}$$

In addition, we want the Lambert function to appear when solving the corresponding 0-electric component of the stabilization equations. We have found that the following prepotential fulfils the required conditions

$$F(\mathcal{X}) = n\,[d_n\mathcal{X}^n]\left[\mathcal{X}^0 e^{2id_l\frac{\mathcal{X}^l}{\mathcal{X}^0}} - 2i\,[d_m\mathcal{X}^m]\,Ei\left[2id_l\frac{\mathcal{X}^l}{\mathcal{X}^0}\right]\right] - d_{ijk}\frac{\mathcal{X}^i\mathcal{X}^j\mathcal{X}^k}{\mathcal{X}^0}\,, \tag{4.2}$$

where $Ei(z)$ is the *exponential integral function*[3], and $d_{ijk} = d_{(ijk)}$, n and d_i[4] are now arbitrary constants not constrained by any String Theory requirement, since we are considering a purely Supergravity model.

In the next section we are going to obtain the supersymmetric black hole solutions corresponding to the four dimensional $\mathcal{N} = 2$ Supergravity theory defined by (4.2), assuming the truncation (4.1).

4.2 The supersymmetric solution

As we have already stressed, in the H-FGK formalism, it is trivial to see that any $\mathcal{N} = 2$, $d = 4$ Supergravity model admits a solution for the H^M variables given by

$$H^M = A^M - \frac{\mathcal{Q}^M}{\sqrt{2}}\tau\,, \tag{4.3}$$

which turns out to correspond to a supersymmetric black hole [201, 243, 320, 411]. Using the truncation (4.1) we have

$$H^i = a^i - \frac{p^i}{\sqrt{2}}\tau\,, \qquad H^M = 0\,, \;\; M \neq i\,. \tag{4.4}$$

[3] See appendix B.2.

[4] The indices $i, j, k, l\ldots$ run from 1 to a fixed arbitrary positive integer n_v.

For the prepotential under consideration (4.2), and the truncation (4.1), it is easy to see that the corresponding stabilization equation for $\tilde{H}^0$ is solved by

$$\tilde{H}^0 = \frac{d_l H^l}{W_a\left(s_a\sqrt{\frac{n(d_n H^n)^3}{d_{ijk}H^i H^j H^k}}\right)}\,. \tag{4.5}$$

This is precisely the same result that we found for the $\tilde{H}^0$ of the solution in the String Theory case Eq. (3.66), and which incorporates the Lambert function, as we wanted. The remaining stabilization equation is solved by

$$\tilde{H}_i = \frac{3d_{ijk}H^j H^k}{\tilde{H}^0} + nd_i\left[e^{-\frac{2d_l H^l}{\tilde{H}^0}}\tilde{H}^0 + \left[4d_m H^m\right] Ei\left[-\frac{2d_q H^q}{\tilde{H}^0}\right]\right]\,. \tag{4.6}$$

$\tilde{H}_i$ becomes an explicit function of the H^i once we substitute (4.5) into (4.6). In any case the result is different from the corresponding one in the String Theory solution, which is to be expected since the model, although sharing some general characteristics, is different. The metric warp factor is hence given by

$$e^{-2U} = n\left[d_n H^n\right]\left[e^{-\frac{2d_l H^l}{\tilde{H}^0}}\tilde{H}^0 + \left[4d_m H^m\right] Ei\left[-\frac{2d_q H^q}{\tilde{H}^0}\right]\right] + \frac{3d_{ijk}H^i H^j H^k}{\tilde{H}^0}\,, \tag{4.7}$$

whereas the scalars read

$$z^i = \frac{\mathcal{X}^i}{\mathcal{X}^0} = i\frac{H^i}{d_l H^l}W_a\left(s_a\sqrt{\frac{n(d_n H^n)^3}{d_{ijk}H^i H^j H^k}}\right)\,. \tag{4.8}$$

This completes the general construction of the supersymmetric solution. Of course, now we have to require, in order to have a regular solution, several conditions which will now be studied.

4.2.1 Regularity conditions

In order to have a regular solution the following requirements have to be satisfied:

1. The warp factor must be non zero, namely

$$e^{2U} > 0\,, \qquad \forall\,\tau \in (-\infty, 0)\,. \tag{4.9}$$

2. The mass M of the solution must be positive and finite

$$M \equiv \dot{U}(\tau \to 0) > 0\,. \tag{4.10}$$

   This requires a bit more explanation. Indeed, it turns out that the definition of the black hole mass involves derivatives of the Lambert function evaluated at $x(\tau = 0)$, which will appear multiplicatively in the different factors of $\dot{U}(\tau)$. As we have sketched already and will explain in the next section, in order to jeopardize the no-hair conjecture we want to fix the parameters of our solution in a way such that the argument of the Lambert function evaluated at spatial infinity ($\tau = 0$) takes the value $-1/e$, where the two branches of W make contact. However, it turns out that $W'(x)$ diverges as $x \to -1/e$ (as explained in the appendix B.1). Fortunately, it is not difficult to cure this behaviour and get a positive (and finite) mass by choosing the parameters of the solution to be such that $\dot{x}(\tau) \overset{\tau\to 0}{\longrightarrow} 0$ faster than $|W'_{0,-1}(x)| \overset{x\to -1/e}{\longrightarrow} \pm\infty$. For instance, we can impose that the coefficient of order τ^0 in the numerator of $\dot{x}(\tau)$ vanishes. As we will see in the explicit examples of section 4.4, this suffices to obtain a finite and positive mass for our pairs of inequivalent black holes.

3. The Kähler potential must be consistently defined. That is

$$e^{-\mathcal{K}} = i\,\Omega_M \bar{\Omega}^M \tag{4.11}$$

must be positive. For the prepotential (4.2) the Kähler potential is given by

$$\begin{aligned} e^{-\mathcal{K}} &= i d_{ijk}(z-\bar{z})^i(z-\bar{z})^j(z-\bar{z})^k + i n d_i\,(z+\bar{z})^i \left(e^{2id_l z^l} - e^{-2id_l \bar{z}^l}\right) \\ &+ 4n|d_i z^i|^2 \left(Ei\left[2id_i z^i\right] + Ei\left[-2id_i \bar{z}^i\right]\right) . \end{aligned} \tag{4.12}$$

Since the supersymmetric solution that we have constructed has purely imaginary scalars, we can use $\bar{z}^i = -z^i$ to simplify this expression

$$\frac{e^{-\mathcal{K}}}{8} = i d_{ijk} z^i z^j z^k + n|d_i z^i|^2 Ei\left[2id_i z^i\right] . \tag{4.13}$$

To summarize, if we obtain a solution such that the metric factor, the Kähler potential, and the mass are definite positive, we will have a regular black hole solution with a physical singularity hidden by an event horizon, and no other space-time singularities.

4.3 The violation of the no-hair conjecture

The resolution of the stabilization equations given in section (4.2) gives us the opportunity to build the supersymmetric solution either using W_0 (solution which we will denote by S_0) or W_{-1} (solution which we will denote by S_{-1}). Therefore, in order to prepare the set up for the violation of the uniqueness conjecture, we need to construct a solution such that the argument of $W_a\left(s_a\sqrt{\frac{n(d_n H^n)^3}{d_{ijk}H^iH^jH^k}}\right)$, which we denote by $x(\tau)$, lies entirely in the interval $(-1/e, 0)$, only *touching* the value $-1/e$ when $\tau = 0$, that is, at spatial infinity. Notice that if we want the argument $x(\tau)$ to be negative we have to chose $s_0 = s_{-1} = -1$, which we will assume henceforth. This way, we will be able to construct two different black hole solutions that solve the same equations of motion, and have the same mass, charges and moduli at infinity, but however are different, since the profiles of W_0 and W_{-1} are different (and asymmetric) when evaluated in $(-1/e, 0)$. Hence, we need to impose

$$x(0) = -\sqrt{\frac{n(d_n a^n)^3}{d_{ijk}a^ia^ja^k}} = -\frac{1}{e}\,, \tag{4.14}$$

and

$$x(\tau) \in (\frac{-1}{e}, 0)\,,\ \forall\,\tau \in (-\infty, 0)\,. \tag{4.15}$$

Of course, as explained in the previous section, in order to have a regular solution we need to impose $M > 0$ and $e^{-2U}, e^{-\mathcal{K}} > 0$ for $\tau \in (-\infty, 0)$. Assuming that (4.10), (and the discussion under it) and (4.14) hold, the value of the scalars at infinity as well as the mass, for both solutions S_0 and S_{-1} will be given by

$$M = \dot{U}(\tau \to 0), \qquad \Im z^i_\infty = -\frac{a^i}{d_l a^l}. \tag{4.16}$$

In order to show that it is indeed possible (and actually easy) to choose the parameters available in the model in a way such that we can obey all the conditions (regularity plus (4.14) and (4.15)), in the next section we will explicitly construct a pair of solutions satisfying the required properties for a particular model with two scalar fields.

Another issue, related to the stability of the solution, is the positive definiteness of the scalar metric $\mathcal{G}_{i\bar{j}}$ evaluated on the solution. Such a condition, which is related to the fulfilment of the NEC associated to the energy-momentum tensor of all the scalar fields in our solution, turns out to be difficult to satisfy. In particular, for the simple models in which we have worked out the explicit construction of pairs of solutions with the same masses, charges and asymptotic values of the moduli (like the one in section 4.4), the scalar metric turns out to have both positive and negative eigenvalues (for both solutions in each pair), meaning that some of the scalars in our solutions fail to satisfy the NEC (just like in other counterexamples to the no-hair conjecture [316]). At this point it is not clear to us whether this is a feature shared by all the possible solutions eluding the no-hair conjecture susceptible of being constructed in our model (for any number of scalar fields), or not. This is an open question which could be addressed from different approaches. On the one hand, one could always try to map (brute force-wise) the parameter space for models with different numbers of scalars, looking for a solution satisfying all the requirements but with a positive definite scalar metric. It would also be possible to consider other prepotentials giving rise to stabilization equations whose solutions involve multivalued functions, and study the situation therein. On the other hand, it might just be that our procedure of *placing the spatial infinity* at the branch point of the Lambert function necessarily implies some unstable behaviour for the corresponding solutions, not incompatible with their regularity. This could be related to the structure of the attractor flows associated to each pair of solutions. Let us see how this works.

4.3.1 Attractors

Although both solutions S_0 and S_{-1} have exactly the same asymptotic limit $\tau \to 0$, since the flow is different, one should expect that the corresponding attractors z_0 and z_{-1} are different. This is indeed the case; they are given by

$$z^i_a = \frac{p^i}{d_l p^l} W_a \left(-\sqrt{\frac{n(d_l p^l)^3}{d_{ijk} p^i p^j p^k}} \right). \tag{4.17}$$

This can be understood in the context of the basins of attractions [268]. Let us suppose that we impose

$$x(0) = \alpha, \qquad \alpha \in (-\frac{1}{e}, \beta), \tag{4.18}$$

instead of $x(0) = -\frac{1}{e}$. Then S_0 and S_{-1} have different asymptotic limits at spatial infinity. In particular, the asymptotic value of the scalars at infinity is different for S_0

and S_{-1}. Therefore, we have two basins of attraction B_0 and B_1 such that the solution S_0 corresponds to B_0 and S_{-1} corresponds to B_{-1}. What happens when we impose

$$x(0) = -\frac{1}{e} \tag{4.19}$$

is that we precisely choose a point which lies in B_0 and B_{-1}, that is, we choose a point in the common border of the two basins of attraction. As a result, we end up with two different solutions, with different attractors, which however have the same mass, charges and asymptotic values of the scalares at infinity.

This standpoint suggests that there could be, in fact, some instability associated to our election of the Lambert's function argument at the branch point. If this were the case, it would simply mean that, just like appears to happen in other counterexamples available in the literature (but those usually in theories with scalar potential, gaugings or higher order curvature corrections), the no-hair Conjecture remains robust when stability issues are considered.

4.4 An explicit example

Let us consider a model with two scalar fields z^1 and z^2. The warp factor of the spacetime metric and the scalars can be read off directly from (4.7) and (4.8) with $H^1 = a^1 - \frac{p^1}{\sqrt{2}}\tau$, $H^2 = a^2 - \frac{p^2}{\sqrt{2}}\tau$. Imposing the regularity conditions, the correct asymptotic behaviour of the metric ($e^{2U} \overset{\tau \to 0}{\Longrightarrow} 1$) and choosing the parameters in the argument of the two branches of the Lambert function in the way explained in the previous section (and such that (4.15) and (4.14) hold), it is not difficult to construct solutions with the required properties (and which, in all the examples constructed automatically satisfy the condition $e^{-\mathcal{K}} > 0 \ \forall \tau \in (-\infty, 0)$). Let us choose a particular model with $d_1 = d_2 = 1$, $d_{122} = 0$, $d_{222} \simeq -0,270$, $d_{211} \simeq 0,320$, $d_{111} \simeq -2,040$, $n \simeq -0,011$, and with the following constants for our solutions: $\Im m z^1_\infty = -1/3$, $\Im m z^2_\infty = -2/3$, $p^1 = p^2 = 1$. The explicit dependence on τ of the warp factor and the imaginary parts of our scalars for the examples at hand is in general very messy, so instead of reproducing it here, let us have a look at the corresponding plots for this particular example, for which the mass turns out to be $M = 2/3$

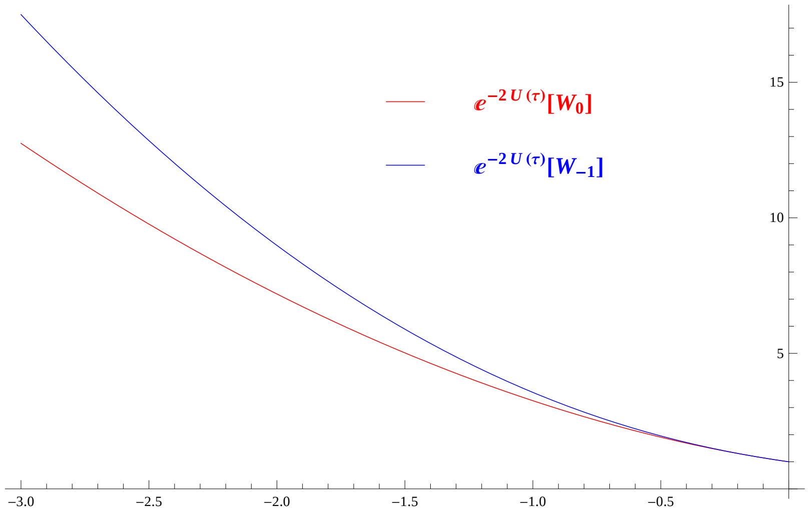

Figure 4.1: The profiles of the metric warp factors corresponding to the two solutions outside the event horizon $\tau \in (-\infty, 0)$. Both metrics asymptote to Minkowski spacetime at spatial infinity.

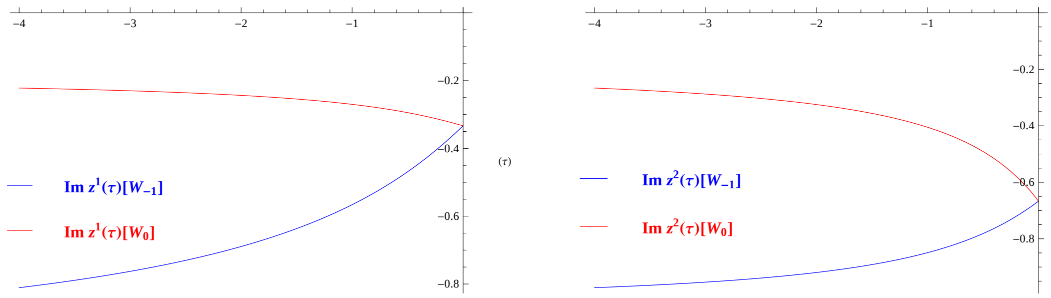

Figure 4.2: The profiles of the imaginary parts of the scalar fields outside the event horizon $\tau \in (-\infty, 0)$. As we can see, their asymptotic values $\Im m z^1_\infty$ and $\Im m z^2_\infty$ coincide for both solutions (recall that spatial infinity is at $\tau \to 0^-$).

As we can see, both solutions are completely regular, and share the same mass, charges, and asymptotic values of the scalars.

5

$\mathcal{N} = 2$ Einstein-Yang-Mills' static two-center solutions

This chapter is based on

Pablo Bueno, Patrick Meessen, Tomas Ortín and Pedro F. Ramírez
"$\mathcal{N} = 2$ Einstein-Yang-Mills' static two-center solutions",
JHEP ***1412*** *(2014) 093.* [arXiv:1410.4160 *[hep-th]*] [91].

Contrary to what one might think, multi-black hole solutions need not be related to supersymmetry or, like in the case of Kastor and Traschen's solution in Ref. [272], fake-supersymmetry. Proof of this is given by various solutions *e.g.* the ones presented in Refs. [53] and [139]. The benefit of using supersymmetry, however, is that after a few decades' worth of investigations there are workable recipes for creating supersymmetric solutions, which greatly facilitates the construction and study of multi-black hole solutions.

The construction is particularly straightforward in ungauged $\mathcal{N} = 2$, $d = 4$ supergravity coupled to vector multiplets where there are clear-cut rules for a supersymmetric multi-object solution to give rise to a well-defined multi-black hole solution [42, 140, 159, 228, 252, 309, 358, 359]: i) positive mass of the constituents, ii) the near-horizon limit has to have definite entropy, iii) the 2^{nd} law of thermodynamics must hold in the coalescence of constituents, and iv) the Denef constraints [159] must be satisfied. Depending on the charges the latter may constrain the distance between the constituents but it always implies the absence of NUT charge.

The oft forgotten case of ungauged $\mathcal{N} = 2$, $d = 4$ supergravity coupled to non-Abelian vector multiplets, which we refer to as $\mathcal{N} = 2$ Einstein-Yang-Mills, is similar to the Abelian case in that there is a well-defined recipe for constructing supersymmetric solutions [244, 245]. However, the construction of supersymmetric solutions is greatly hindered not only by the fact that not every Abelian theory can be non-Abelianized, but doubly more so by the fact that the supersymmetric recipe requires the use of solutions of the (non-Abelian) Bogomol'nyi equation on $\mathbb{R}^3$ [72]. Our lack of knowledge of the space of all solutions to this equation is a serious limitation to the application of the supersymmetric recipe: there exists a vast literature on single monopole solutions, *i.e.* regular single-center solutions to the Bogomol'nyi equation (see *e.g.* Refs. [406]). Depending on the chosen $\mathcal{N} = 2$, $d = 4$ model, these can be used to construct globally regular supergravity solutions known as *global monopoles*. A lot less is known about the singular solutions to the Bogomol'nyi equation which are the ones which give rise to black holes with different degrees of non-Abelian hair [244, 245, 319]. Finally, even less is known about multi-center solutions to the Bogomol'nyi equation. These are the ones we need in order to to apply the supersymmetric recipe to the construction of multi-center supergravity solutions, with centers that correspond to global monopoles or black holes.

Luckily enough, some explicit solutions are known.[1] In this chapter we are going to use the solutions of the SU(2) Bogomol'nyi equation found by Cherkis and Durcan [138] and Blair and Cherkis [71] (which we will generalize by adding *Protogenov hair* [319]). These solutions describe an 't Hooft-Polyakov (-Protogenov) monopole in the presence of an arbitrary number of Dirac monopoles embedded in SU(2), all having charge opposite to the one carried by the former. These solutions can (in principle) give rise to supergravity solutions describing black holes in the presence of a global monopole. The construction of these solutions is, at the same time, our main goal and our main result.

Before we start constructing multi-black hole solutions, however, it is worth reviewing briefly some of the previous work on solutions of YM theories coupled to gravity[2]. Most of the previous work on this topic was focused on pure Einstein-Yang-Mills (EYM) theories, (the minimal non-Abelian extension of the Einstein-Maxwell theory), ignoring the possible existence of unbroken supersymmetry which is, however, one of our main concerns here.

Soon after the discovery of the 't Hooft-Polyakov monopole [370, 408] several groups found solutions to the pure EYM theory [429] whose SU(2) gauge field is that of the Wu–Yang SU(2) monopole [428]. The metric of all these solutions is that of the (dS or AdS) non-extremal Reissner-Nordström black hole and the singularity in the gauge field (generically expected for static YM solutions [160]) is covered by an event horizon.

This coincidence of the metrics is due to the relation between the Wu–Yang SU(2) monopole and the non-Abelian embedding of the Dirac monopole Eq. (C.15): they are related by a singular gauge transformation and therefore give rise to exactly the same energy-momentum tensor as it is gauge invariant whether the gauge transformation is singular or not. For this reason, these solutions have been regarded as not truly non-Abelian, even though there are potentially measurable differences, see *e.g.* Refs. [113, 232].

Finding less trivial ("genuinely or essentially non-Abelian") solutions proved much more difficult and a *non-Abelian baldness theorem* stating that the only black-hole solutions of the EYM SU(2) theory with a regular horizon and non-vanishing magnetic charge had to be non-Abelian embeddings of the Reissner–Nordström solution was proven in [193]. This theorem was subsequently generalized to prove the absence of regular monopole or dyon solutions to the EYM theory in Refs. [70, 171].

An "essentially non-Abelian" solution, globally regular [396] to EYM theory had already been found: the Bartnik-McKinnon particle [30]. The Bartnik-McKinnon particle and its black hole-type generalizations [418], are in fact families of unstable solutions indexed by a discrete parameter and evade the non-Abelian baldness theorem by being bald, *i.e.* they have no asymptotic charge. It is worth pointing out that even though these solutions are only known numerically, they have been proven to exist [394].

The classification of the possible EYM solutions for the gauge group SU(2) [395] suggests that one has to add more fields to the theory in order to get "essentially non-Abelian" black-hole or gravitating monopole solutions with non-vanishing charges. Investigations of solutions to the EYM theory coupled to a Higgs field in the adjoint representation [293]

[1] Finite-energy, multi-center solutions of the Yang-Mills or Yang-Mills-Higgs system which do not satisfy the Bogomol'nyi equation like those in Refs. [278, 280, 283] are also known.

[2] For more comprehensive reviews see *e.g.* Refs. [417].

in the BPS-limit, a theory that is closer to the one we are going to study than EYM, shows that a globally well-defined 't Hooft-Polyakov monopole exists and furthermore the existence of other Bartnik-McKinnon-like solutions.

As far as 4-dimensional supergravity is concerned we have the (supersymmetric) Harvey-Liu [231] and the Chamseddine-Volkov [130] regular gravitating monopole solutions to gauged $\mathcal{N}=4$, $d=4$ supergravity; in $\mathcal{N}=2$, $d=4$ theories there are analytical solutions describing global monopole solutions and non-Abelian black hole solutions with and without asymptotic magnetic charge. Needless to say, all the solutions mentioned in this little historical *exposé* describe the fields corresponding to a single object. To our knowledge, there are no known, essentially non-Abelian multi-object analytic[3] solutions and this article intends to fill this gap by constructing static solutions describing the interplay between an 't Hooft-Polyakov monopole and a Dirac monopole of opposite charge in two generic classes of gauged $\mathcal{N}=2$, $d=4$ models.

As we stressed in the introduction, in the theories we have called $\mathcal{N}=2$, $d=4$ SEYM the gauge group does not contain any part of the R-symmetry group. Indeed, in general (ungauged) $\mathcal{N}=2$, $d=4$ theories, the global symmetry group G can be written as

$$\mathrm{G}=\mathrm{G}_{\mathrm{V}}\times\mathrm{G}_{\mathrm{hyper}}\times\mathrm{SU}(2)_{\mathrm{R}}\times\mathrm{U}(1)_{\mathrm{R}}\,, \tag{5.1}$$

where G_{V} and $\mathrm{G}_{\mathrm{hyper}}$ stand for the isometry groups of the special and quaternionic Kähler manifolds respectively. When a (necessarily non-Abelian) subgroup of G_{V} is gauged (as in $\mathcal{N}=2$, $d=4$ SEYM theories) the scalar potential is positive semidefinite, which allows for asymptotically De-Sitter and asymptotically flat solutions (such as the ones we construct in this paper). This is in contradistinction to theories in which a subgroup of $\mathrm{SU}(2)_{\mathrm{R}}$ (or the complete $\mathrm{SU}(2)_{\mathrm{R}}$) is gauged via Fayet-Iliopoulos terms[4] in whose case the scalar potential becomes negative definite, the solutions thus being asymptotically anti-De Sitter. Lately, an intense effort has been devoted to the construction of black-hole solutions of theories with Abelian gaugings (that is, theories in which a subgroup $\mathrm{U}(1)\in\mathrm{SU}(2)_{\mathrm{R}}$ has been gauged); see, for instance, Refs. [97, 209, 224, 242, 285, 412] and references therein. The case in which the full $\mathrm{SU}(2)_{\mathrm{R}}$ has been gauged remains as unexplored as challenging, even though the general form of the timelike supersymmetric solutions of this theory has been given in Ref. [322].

This chapter is organized as follows: in section 5.1 we review the theories we are going to work with ($\mathcal{N}=2$, $d=4$ Super-Einstein-Yang-Mills theories) and the recipe for constructing timelike supersymmetric solutions (black holes, in particular). In section 5.2 we apply that recipe to construct single, static supersymmetric black-hole and monopole solutions of two particular examples of SU(2)-gauged $\mathcal{N}=2$, $d=4$ SEYM: the $\overline{\mathbb{CP}}^3$ model (quadratic) (5.2.2) and the ST[2, 4] model (cubic) (5.2.3). We use as seeds for these solutions the single-center solutions of the Bogomol'nyi equations reviewed in section 5.2.1. In section 5.3 we construct multi-black-hole solutions for the same models using the multi-center solutions of the Bogomol'nyi equations reviewed in section 5.3.1. Our conclusions are contained in section 5.4. In the Appendices we review a particularly interesting single-

[3] Numerical, multi-center solutions have been found previously, though. See, *e.g.* Refs. [279, 281]. Some of those solutions can be embedded in $\mathcal{N}=1, d=4$ supergravity. However, representing massive objects, they can never be supersymmetric in that theory. The embedding in higher-$\mathcal{N}$ supergravities is much more difficult (if possible at all). We thank J. Kunz for pointing these works to us.

[4] The overall $\mathrm{U}(1)_R$ group cannot be gauged in this way. The Abelian gaugings discussed in the literature deal with a subgroup $\mathrm{U}(1)\in\mathrm{SU}(2)_{\mathrm{R}}$.

center solution of the SU(2) Bogomol'nyi equations which appears in different guises: as a "Lorentzian meron" (Appendix C.1), as the Wu-Yang monopole (Appendix C.2) or as a solution of the Skyrme model (Appendix C.3). A higher-charge generalization of this solution is reviewed in Appendix C.4.

5.1 $\mathcal{N}=2$, $d=4$ SEYM and its supersymmetric black-hole solutions (SBHSs)

In this section we are going to introduce the class of theories that we have called $\mathcal{N}=2$, $d=4$ SEYM theories and we are going to review the recipe to construct all their timelike supersymmetric solutions, presented in Ref. [245]. We shall be extremely brief. The interested reader can find more details in Refs. [183, 244, 353]; our conventions are those of Refs. [244, 245, 353].

5.1.1 The theory

$\mathcal{N}=2$, $d=4$ SEYM theories can be seen as the simplest $\mathcal{N}=2$ supersymmetrization of the Einstein-Yang-Mills (EYM) theories. They are nothing but theories of $\mathcal{N}=2$, $d=4$ supergravity coupled to n vector multiplets in which a (necessarily non-Abelian)[5] subgroup of the isometry group of the (Special Kähler) scalar manifold has been gauged using some of the vector fields of the theory as gauge fields[6].

We will only be concerned with the bosonic sector of the theory, which consists on the metric $g_{\mu\nu}$, the vector fields $A^{\Lambda}{}_{\mu}$ ($\Lambda=0,1,\cdots,n$) and the complex scalars Z^i ($i=1,\cdots,n$). The action of the bosonic sector reads

$$\begin{aligned} S[g_{\mu\nu}, A^{\Lambda}{}_{\mu}, Z^i] \;=\; & \int d^4x\sqrt{|g|}\,\big[R + 2\mathcal{G}_{ij^*}\mathfrak{D}_{\mu}Z^i\mathfrak{D}^{\mu}Z^{*\,j^*} + 2\Im\mathfrak{m}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,\mu\nu}F^{\Sigma}{}_{\mu\nu} \\ & -2\Re\mathfrak{e}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,\mu\nu}\star F^{\Sigma}{}_{\mu\nu} - V(Z,Z^*)\big]\,. \end{aligned} \tag{5.2}$$

In this expression, $\mathcal{G}_{ij^*}$ is the Kähler metric, $\mathfrak{D}_{\mu}Z^i$ is the gauge-covariant derivative

$$\mathfrak{D}_{\mu}Z^i = \partial_{\mu}Z^i + gA^{\Lambda}{}_{\mu}k_{\Lambda}{}^{i}\,, \tag{5.3}$$

$F^{\Lambda}{}_{\mu\nu}$ is the vector field strength

$$F^{\Lambda}{}_{\mu\nu} = 2\partial_{[\mu}A^{\Lambda}{}_{\nu]} - gf_{\Sigma\Gamma}{}^{\Lambda}A^{\Sigma}{}_{\mu}A^{\Gamma}{}_{\nu}\,, \tag{5.4}$$

$\mathcal{N}_{\Lambda\Sigma}$ is the period matrix and, finally, $V(Z,Z^*)$ is the scalar potential

$$V(Z,Z^*) = -\tfrac{1}{4}g^2\Im\mathfrak{m}\mathcal{N}^{\Lambda\Sigma}\mathcal{P}_{\Lambda}\mathcal{P}_{\Sigma}\,. \tag{5.5}$$

Since the imaginary part of the period matrix is negative definite, the scalar potential is positive semidefinite, which leads to asymptotically-flat or -De Sitter solutions.

[5] The theory becomes identical to the ungauged one when the gauge group is Abelian.

[6] A global symmetry group can be gauged if it acts on the vector fields in the adjoint representation. Furthermore, it is required to be a symmetry of the prepotential; see *e.g.* ref. [245] for more details.

In the above equations, $k_\Lambda{}^i(Z)$ are the holomorphic Killing vectors of the isometries that have been gauged[7] and $\mathcal{P}_\Lambda(Z,Z^*)$ the corresponding momentum maps, which are related to the Killing vectors and to the Kähler potential $\mathcal{K}$ by

$$i\mathcal{P}_\Lambda \;=\; k_\Lambda{}^i\partial_i\mathcal{K}-\lambda_\Lambda\,, \tag{5.6}$$

$$k_{\Lambda\, i^*} \;=\; i\partial_{i^*}\mathcal{P}_\Lambda\,, \tag{5.7}$$

for some holomorphic functions $\lambda_\Lambda(Z)$. Furthermore, the holomorphic Killing vectors and the generators T_Λ of the gauge group satisfy the Lie algebras

$$[k_\Lambda,k_\Sigma]=-f_{\Lambda\Sigma}{}^\Gamma k_\Gamma\,,\qquad [T_\Lambda,T_\Sigma]=+f_{\Lambda\Sigma}{}^\Gamma T_\Gamma\,. \tag{5.8}$$

For the gauge group SU(2), which is the only one we are going to consider, we use lowercase indices[8] $a,b,c=1,2,3$ and the structure constants are $f_{ab}{}^c=-\varepsilon_{abc}$, so

$$[k_a,k_b]=+\varepsilon_{abc}k_c\,,\qquad [T_a,T_b]=-\varepsilon_{abc}T_c\,. \tag{5.9}$$

We will use the fundamental representation, in which the generators are proportional to the standard Pauli matrices[9] σ^a

$$T_a\equiv+\tfrac{i}{2}\sigma^a\,,\quad\Rightarrow\quad \mathrm{Tr}(T_aT_b)=-\tfrac{1}{2}\delta_{ab}\,. \tag{5.11}$$

The equations of motion of the theory can be written in the following form:

$$\begin{aligned}G_{\mu\nu}+2\mathcal{G}_{ij^*}[\mathfrak{D}_{(\mu}Z^i\mathfrak{D}_{\nu)}Z^{*\,j^*}-\tfrac{1}{2}g_{\mu\nu}\mathfrak{D}_\rho Z^i\mathfrak{D}^\rho Z^{*\,j^*}]&\\ +4\mathcal{M}_{MN}\mathcal{F}^M{}_\mu{}^\rho\mathcal{F}^N{}_{\nu\rho}+\tfrac{1}{2}g_{\mu\nu}V(Z,Z^*)&=0,\end{aligned} \tag{5.12}$$

$$\mathfrak{D}^2Z^i+\partial^iG_{\Lambda\,\mu\nu}\star F^{\Lambda\,\mu\nu}+\tfrac{1}{2}\partial^iV(Z,Z^*)=0, \tag{5.13}$$

$$\mathfrak{D}_\nu\star G_\Lambda{}^{\nu\mu}+\tfrac{1}{4}g\left(k_{\Lambda\,i^*}\mathfrak{D}_\mu Z^{*i^*}+k^*_{\Lambda\,i}\mathfrak{D}_\mu Z^i\right)=0\,, \tag{5.14}$$

where $G_{\Lambda\,\mu\nu}$ is the dual vector field strength

$$G_\Lambda\equiv\mathfrak{Re}\mathcal{N}_{\Lambda\Sigma}F^\Sigma+\mathfrak{Im}\mathcal{N}_{\Lambda\Sigma}\star F^\Sigma\,, \tag{5.15}$$

$\mathcal{F}^M{}_{\mu\nu}$ is the symplectic vector of vector field strengths

$$(\mathcal{F}^M)\equiv\begin{pmatrix}F^\Lambda\\ G_\Lambda\end{pmatrix}\,, \tag{5.16}$$

[7] The employed notation associates a Killing vector to each value of the index Λ in order to avoid the introduction of yet another class of indices and the embedding tensor (See *e.g.* the reviews [413]); it is understood that not all the k_Λ need to be non-vanishing.

[8] These will be a certain subset of those represented by $\Lambda,\Sigma,\ldots$.

[9] These are

$$\sigma^1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma^2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad \sigma^3=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\qquad \sigma^a\sigma^b=\delta^{ab}+i\varepsilon^{abc}\sigma^c\,. \tag{5.10}$$

$\mathcal{M}_{MN}$ is the symmetric $2(n+1)\times 2(n+1)$ matrix defined by

$$(\mathcal{M}_{MN}) \equiv \begin{pmatrix} \Im\mathfrak{m}\mathcal{N}_{\Lambda\Sigma} + R_{\Lambda\Gamma}\Im\mathfrak{m}\mathcal{N}^{-1|\Gamma\Omega}R_{\Omega\Sigma} & -R_{\Lambda\Gamma}\Im\mathfrak{m}\mathcal{N}^{-1|\Gamma\Sigma} \\ \\ -\Im\mathfrak{m}\mathcal{N}^{-1|\Lambda\Omega}R_{\Omega\Sigma} & \Im\mathfrak{m}\mathcal{N}^{-1|\Lambda\Sigma} \end{pmatrix}, \tag{5.17}$$

and

$$\mathfrak{D}_\nu \star G_\Lambda{}^{\nu\mu} = \partial_\nu \star G_\Lambda{}^{\nu\mu} + g f_{\Lambda\Sigma}{}^{\Gamma} A^{\Sigma}{}_\nu \star G_\Lambda{}^{\nu\mu}\,. \tag{5.18}$$

Most of the literature and earlier work on non-Abelian black-hole and monopole solutions has been carried out in the context of the Einstein-Yang-Mills (EYM) and Einstein-Yang-Mills-Higgs (EYMH) theories. Before closing this introduction, it is worth discussing the relation between those and the theories we are considering here. The main differences of the latter w.r.t. the former are the complexification of the Higgs field and the presence of a non-trivial period matrix. A further difference is the possibility of having more general scalar manifolds, which is reflected in the expressions of the gauge-covariant derivatives of the scalar fields. Solutions to the $\mathcal{N}=2$, $d=4$ SEYM theory have a chance of being also solutions of the EYMH theory if they have covariantly-constant scalars with identical phases (*e.g.* all of them purely imaginary). Then, if the scalar potential vanishes on the solutions, they also have a chance of being solutions to the EYM system as well; as we are going to see, some of the solutions found in Refs. [244, 245] are also solutions of the EYM theory and have the same metric as the EYM solutions of Refs. [113, 429].

5.1.2 The recipe to construct SBHSs of $\mathcal{N}=2$, $d=4$ SEYM

To construct timelike supersymmetric solutions of the $\mathcal{N}=2$, $d=4$ SEYM theory, it suffices to follow this recipe [244, 245] to find the elementary building blocks of the solutions, which are the $2(n+1)$ time-independent functions $(\mathcal{I}^M) = \begin{pmatrix}\mathcal{I}^\Lambda \\ \mathcal{I}_\Lambda\end{pmatrix}$:

1. Take a solution of the Bogomol'nyi equations
$$\tilde{F}^{\Lambda}{}_{\underline{mn}} = -\tfrac{1}{\sqrt{2}}\varepsilon_{mnp}\tilde{\mathfrak{D}}_{\underline{p}}\mathcal{I}^\Lambda, \tag{5.19}$$
for a gauge field $\tilde{A}^{\Lambda}{}_{\underline{m}}$ ($\underline{m}=1,2,3$ labels the 3 spatial coordinates) and a real "Higgs" field $\mathcal{I}^\Lambda$. $\tilde{\mathfrak{D}}_{\underline{p}}\mathcal{I}^\Lambda$ is the covariant derivative in the adjoint representation with gauge field $\tilde{A}^{\Lambda}{}_{\underline{m}}$. Observe that this equation has to be solved in the gauged (non-Abelian) and ungauged (Abelian) directions. The integrability condition in the Abelian directions is the familiar requirement that the $\mathcal{I}^\Lambda$ be harmonic functions on $\mathbb{R}^3$.

2. Find the functions $\mathcal{I}_\Lambda$ by solving these equations:
$$\tilde{\mathfrak{D}}_{\underline{m}}\tilde{\mathfrak{D}}_{\underline{m}}\mathcal{I}_\Lambda = \tfrac{1}{2}g^2\left[f_{\Lambda(\Sigma}{}^{\Gamma} f_{\Delta)\Gamma}{}^{\Omega}\,\mathcal{I}^\Sigma\mathcal{I}^\Delta\right]\mathcal{I}_\Omega\,. \tag{5.20}$$
In the non-Abelian directions these equations can, in many cases, be solved by taking $\mathcal{I}_\Lambda \propto \mathcal{I}^\Lambda$, but currently we only know how to generate non-trivial solutions to them in the cases where the gauge doublet $(\tilde{A}^\Lambda, \mathcal{I}^\Lambda)$ describes a non-Abelian Wu-Yang monopole; Observe that $\mathcal{I}_\Lambda = 0$ is always a solution, but the physical fields may be singular in some models.

   In the Abelian directions, the $\mathcal{I}_\Lambda$ are just independent harmonic functions on $\mathbb{R}^3$.

3. Given the functions $\mathcal{I}^M$, we must find the 1-form on $\mathbb{R}^3$ $\omega_{\underline{m}}$ by solving the following equation:

$$\partial_{[\underline{m}}\omega_{\underline{n}]} = \varepsilon_{mnp}\mathcal{I}_M\tilde{\mathfrak{D}}_{\underline{p}}\mathcal{I}^M = \varepsilon_{mnp}\left(\mathcal{I}_\Lambda\tilde{\mathfrak{D}}_{\underline{p}}\mathcal{I}^\Lambda - \mathcal{I}^\Lambda\tilde{\mathfrak{D}}_{\underline{p}}\mathcal{I}_\Lambda\right) . \tag{5.21}$$

   The integrability conditions of this equation impose constraints on the integration constants of the functions $\mathcal{I}^M$ in exactly the same manner as in the ungauged case [31, 159].

   In the case of static solutions, *i.e.* when $\omega = 0$, the above equation becomes a constraint on the integration constants of the functions $\mathcal{I}^M$ that will have to be solved. Observe, however, that this constraint is independent of the specific $\mathcal{N}=2$, $d=4$ model and only depends on the choice of gauge group; possible restrictions on the solution to said constraint can come from the desired behaviour of the physical fields in the full solution.

4. To reconstruct the physical fields from the functions $\mathcal{I}^M$ we need to solve the stabilization equations, a.k.a. *Freudenthal duality equations*, which give the components of the Freudenthal dual[10] $\tilde{\mathcal{I}}^M(\mathcal{I})$ in terms of the functions $\mathcal{I}^M$ [189]; These relations completely characterize the model of $\mathcal{N}=2$, $d=4$ supergravity.

   Equivalently, the $\tilde{\mathcal{I}}$ can be derived from a homogeneous function of degree 2 $W(\mathcal{I})$ called the *Hesse potential* as [31, 324, 331]

$$\tilde{\mathcal{I}}_M \;=\; \tfrac{1}{2}\frac{\partial W}{\partial \mathcal{I}^M} \quad\longrightarrow\quad W(\mathcal{I}) \;=\; \tilde{\mathcal{I}}_M\mathcal{I}^M \,. \tag{5.22}$$

5. The metric takes the form

$$ds^2 = e^{2U}(dt+\omega)^2 - e^{-2U}dx^m dx^m \,, \tag{5.23}$$

   where $\omega = \omega_{\underline{m}}dx^m$ is the above spatial 1-form and the metric function e^{-2U} is given by

$$e^{-2U} = \tilde{\mathcal{I}}_M(\mathcal{I})\mathcal{I}^M = W(\mathcal{I}) \,. \tag{5.24}$$

6. The scalar fields are given by

$$Z^i = \frac{\tilde{\mathcal{I}}^i + i\mathcal{I}^i}{\tilde{\mathcal{I}}^0 + i\mathcal{I}^0} \,. \tag{5.25}$$

7. The components of the vector fields are given by

$$A^\Lambda{}_t \;=\; -\tfrac{1}{\sqrt{2}}e^{2U}\tilde{\mathcal{I}}^\Lambda \,, \tag{5.26}$$

$$A^\Lambda{}_{\underline{m}} \;=\; \tilde{A}^\Lambda{}_{\underline{m}} + \omega_{\underline{m}}\, A^\Lambda{}_t \,. \tag{5.27}$$

After having gone through the steps of the recipe, one ends up with a supersymmetric solution to a chosen $\mathcal{N}=2$, $d=4$ EYM theory and what remains to be done is to analyze the constraints coming from imposing appropriate regularity conditions such as the absence of naked singularities.

[10] In Refs. [244, 245, 320] the components of the Freudenthal dual are denoted by $\mathcal{R}^M$.

5.2 Static, single-SBHSs of SU(2) $\mathcal{N}=2$, $d=4$ SEYM and pure EYM

Following the recipe given in section 5.1.2, we are going to construct static, single-center SBHSs of SU(2) $\mathcal{N}=2$, $d=4$ SEYM. Some of the solutions will simultaneously solve the equations of motion of the EYM and EYMH theories.

The first step consists in finding a solution $\tilde{A}^{\Lambda}{}_{\underline{m}}, \mathcal{I}^{\Lambda}$ of the SU(2) Bogomol'nyi equations in $\mathbb{R}^3$ Eqs. (5.19).

5.2.1 Single-center solutions of the SU(2) Bogomol'nyi equations in $\mathbb{R}^3$

Before we search for solutions of the Bogomol'nyi equations it is worth reviewing the origin and meaning of those equations in the context of the SU(2) Yang-Mills-Higgs theory (in the Bogomol'nyi-Prasad-Sommerfield (BPS) limit in which the Higgs potential vanishes).

The SU(2) Yang-Mills-Higgs system

With the normalization in Eq. (5.11) and writing $F \equiv F^a T_a, \Phi \equiv \Phi^a T_a$, the action of the YMH theory in our conventions reads

$$S_{\text{YMH}} = -2 \int d^4x \, \text{Tr} \left\{ \tfrac{1}{2} \mathfrak{D}_\mu \Phi \mathfrak{D}^\mu \Phi - \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} \right\} , \tag{5.28}$$

and the corresponding equations of motion are

$$\mathfrak{D}_\mu F^{\mu\nu} = g[\Phi, \mathfrak{D}^\nu \Phi] , \tag{5.29}$$

$$\mathfrak{D}^2 \Phi = 0 . \tag{5.30}$$

For static configurations $F_{t\underline{m}} = \mathfrak{D}_t \Phi = 0$, the action can be written, up to a total derivative, in the manifestly positive form

$$S_{\text{YMH}} = -2 \int d^4x \, \text{Tr} \left\{ -\tfrac{1}{4} \left(F_{\underline{mn}} \mp \varepsilon_{mnp} \mathfrak{D}_{\underline{p}} \Phi \right) \left(F_{\underline{mn}} \mp \varepsilon_{mnp} \mathfrak{D}_{\underline{p}} \Phi \right) \right\} , \tag{5.31}$$

which leads to the conclusion that static field configurations satisfying the first-order *Bogomol'nyi equations* [72]

$$F_{\underline{mn}} = \pm \varepsilon_{mnp} \mathfrak{D}_{\underline{p}} \Phi , \tag{5.32}$$

extremize the action Eq. (5.28) and are solutions of the full Yang-Mills-Higgs equations. Indeed, if we act with $\mathfrak{D}_{\underline{m}}$ on both sides of the equation and use the Ricci identity and the Bogomol'nyi equation we get the Yang-Mills equation:

$$\mathfrak{D}_{\underline{m}} F_{\underline{mn}} = \mp \varepsilon_{nmp} \mathfrak{D}_{\underline{m}} \mathfrak{D}_{\underline{p}} \Phi = \mp \tfrac{1}{2} g \varepsilon_{nmp} [F_{\underline{mp}}, \Phi] = -g[\mathfrak{D}_{\underline{n}} \Phi, \Phi] . \tag{5.33}$$

If, instead, we act with $\varepsilon_{pmn} \mathfrak{D}_{\underline{p}}$ and use the Bianchi identity, we get the Higgs equation:

$$0 = \varepsilon_{pmn} \mathfrak{D}_{\underline{p}} F_{\underline{mn}} = \pm \mathfrak{D}_{\underline{p}} \mathfrak{D}_{\underline{p}} \Phi . \tag{5.34}$$

Observe that the source of the Yang-Mills field, the *Higgs current* $g[\Phi, \mathfrak{D}\Phi]$, not only vanishes when the Higgs field is covariantly constant $\mathfrak{D}\Phi = 0$ but also when Φ and $\mathfrak{D}\Phi$ are parallel in $\mathfrak{su}(2)$.

Eqs. (5.32) are identical to the ones that arise in $\mathcal{N}=2$, $d=4$ SEYM theory, (5.19) upon the identification of the vector fields and

$$\tfrac{1}{\sqrt{2}}\mathcal{I}^a = \mp\Phi^a\,. \tag{5.35}$$

The hedgehog ansatz

In order to construct static, single-center black-hole-type solutions, it is natural to look for spherically symmetric solutions of Eqs. (5.32). Substituting the *hedgehog ansatz*

$$\mp\Phi^a = \delta^a{}_m f(r) x^m\,, \qquad A^a{}_{\underline{m}} = -\varepsilon^a{}_{mn} x^n h(r) \tag{5.36}$$

in the Bogomol'nyi Eqs. (5.32) we get an equivalent system of differential equations for $f(r)$ and $h(r)$:

$$\begin{aligned} r\partial_r h + 2h - f(1+gr^2 h) &= 0\,,\\ r\partial_r(h+f) - gr^2 h(h+f) &= 0\,. \end{aligned} \tag{5.37}$$

After Prasad and Sommerfield [373] found the solution describing the 't Hooft-Polyakov monopole in the BPS limit, Protogenov [374] classified all spherically symmetric solutions to the SU(2) Bogomol'nyi equations: the ones that can be used to generate BH-like spacetimes are a 2-parameter family $(f_{\mu,s}, h_{\mu,s})$ plus a 1-parameter family (f_λ, h_λ) given by

$$\begin{aligned} rf_{\mu,s} &= \frac{1}{gr}\left[1-\mu r\coth(\mu r+s)\right]\,, \quad & rh_{\mu,s} &= \frac{1}{gr}\left[\frac{\mu r}{\sinh(\mu r+s)}-1\right]\,,\\ rf_\lambda &= \frac{1}{gr}\left[\frac{1}{1+\lambda^2 r}\right]\,, & rh_\lambda &= -rf_\lambda\,. \end{aligned} \tag{5.38}$$

The parameter s is known in the black-hole context as the *Protogenov hair* parameter [319]. The BPS 't Hooft-Polyakov monopole [373] is the only globally regular solution of this family (which explains why it is the only one usually considered in the monopole literature[11]) and corresponds to $s=0$. In the $s\to\infty$ limit we get

$$-rf_{\mu,\infty} = \frac{\mu}{g} - \frac{1}{gr}\,, \qquad rh_{\mu,\infty} = -\frac{1}{gr}\,, \tag{5.39}$$

which, for $\mu=0$, coincides with the Wu-Yang monopole [428] given in Eq. (C.15), and is a solution of the pure Yang-Mills theory. This is possible because the Higgs current $g[\Phi,\mathfrak{D}\Phi]$ vanishes even though Φ is neither zero nor covariantly constant[12]. With a non-trivial Higgs field, though, we can assign a well-defined monopole charge to it: for any μ and s

$$\frac{1}{4\pi}\int_{S^2_\infty}\mathrm{Tr}(\hat{\Phi}F) = \frac{1}{g}\,, \qquad \hat{\Phi} \equiv \frac{\Phi}{\sqrt{|\mathrm{Tr}(\Phi^2)|}}\,. \tag{5.40}$$

The same field configuration can be seen as a Lorentzian meron (see Appendix C.1) and as a solution to the Skyrme model (see Appendix C.3), and, crucially, it is related

[11] After coupling the system to gravity, the singularities of the other solutions may become "harmless" if they can be covered by regular event horizons.

[12] Actually, the only field configuration in this ansatz with a vanishing Higgs current is this one.

to the SU(2)-embedded Dirac monopole by a singular gauge transformation (see Appendix C.2). Since the metric is oblivious to gauge transformations, singular or not, the Wu-Yang monopole gives rise to solutions whose metric is identical to that of Abelian case.[13] The same applies to the higher-charge generalizations of the Lorentzian meron/Wu-Yang monopole reviewed in Appendix C.4.

If fact, this mechanism can be used to generate Wu-Yang monopoles of higher charge from the well-known Dirac monopole solutions of charge higher than 1 embedded in SU(2), as reviewed in Appendix C.4. The metric cannot see the difference between the non-Abelian and the Abelian fields given in Eq. (5.39).

The 1-parameter family is singular for all values of the parameter λ, which also appears in black-hole solutions as hair. The magnetic charge measured at spatial infinity vanishes according to the above definition. However, it can be argued that these solutions do describe a magnetic monopole placed at the origin whose charge is screened: the entropy of black hole associated to this field has the same form as that of the black hole associated to the Wu-Yang monopole. Observe that, for $\lambda = 0$, the solution is identical to the Wu-Yang monopole with $\mu = 0$, Eqs. (5.39).

The Protogenov trick

As it turns out, many regular monopole solutions can be deformed by adding a parameter s to the argument μr, generating a family of solutions that contains the original one ($s = 0$) and, typically, a new and simpler solution in the $s \to \infty$ limit. We will refer to this procedure as *the Protogenov trick* and it can be justified as follows: let us consider, for instance, the 't Hooft-Polyakov monopole. Since the Bogomol'nyi equation is polynomial, having elementary functions such as hyperbolic functions in the solution means that they must cancel amongst themselves and that only their derivatives contribute to the polynomial part of the solution. This means that one should be able to deform the dependency of the elementary functions introducing a shift s of the radial coordinate and still solve the Bogomol'nyi equations.

Of course, the cancellations necessary for having a regular solution will not work out anymore (assuming they did work for $s = 0$) and one will end up with a family of singular solutions. We will use this trick later.

5.2.2 Embedding in the SU(2)-gauged $\overline{\mathbb{CP}}^3$ model

The $\overline{\mathbb{CP}}^3$ model

As we already explained, the $\overline{\mathbb{CP}}^n$ models have n vector supermultiplets and are defined by the quadratic prepotentials

$$\mathcal{F} = -\tfrac{i}{4}\eta_{\Lambda\Sigma}\mathcal{X}^{\Lambda}\mathcal{X}^{\Sigma}\,, \qquad (\eta_{\Lambda\Sigma}) = \mathrm{diag}(+-\cdots-)\,. \tag{5.41}$$

The n physical scalar fields can be defined as

$$Z^i \equiv \mathcal{X}^i/\mathcal{X}^0\,, \tag{5.42}$$

[13] Of course there are measurable differences between these two situations, see *e.g.* Refs. [113, 232].

and they parametrize the symmetric space $\mathrm{U}(1,n)/(\mathrm{U}(1)\times\mathrm{U}(n))$. It is convenient to define $Z^0\equiv 1$, $Z^\Lambda \equiv \mathcal{X}^\Lambda/\mathcal{X}^0$ and $Z_\Lambda \equiv \eta_{\Lambda\Sigma}Z^\Sigma$. In the $\mathcal{X}^0=1$ gauge, the Kähler potential and the Kähler metric are given by

$$\mathcal{K} = -\log\left(Z^{*\Lambda}Z_\Lambda\right), \quad \mathcal{G}_{ij^*} = -e^{\mathcal{K}}\left(\eta_{ij^*} - e^{\mathcal{K}}Z^*_i Z_{j^*}\right), \quad \Rightarrow \quad 0 \leq \sum_i |Z^i|^2 < 1\,. \tag{5.43}$$

The above metric is the standard (Bergman) metric for the $\mathrm{U}(1,n)/(\mathrm{U}(1)\times\mathrm{U}(n))$ symmetric spaces [63]. The covariantly holomorphic symplectic section $\mathcal{V}$ and the period matrix $\mathcal{N}_{\Lambda\Sigma}$ are given by

$$\mathcal{V} = e^{\mathcal{K}/2}\begin{pmatrix} Z^\Lambda \\ \\ -\frac{i}{2}Z_\Lambda\end{pmatrix}, \qquad \mathcal{N}_{\Lambda\Sigma} = \tfrac{i}{2}\left[\eta_{\Lambda\Sigma} - 2\frac{Z_\Lambda Z_\Sigma}{Z^\Gamma Z_\Gamma}\right]. \tag{5.44}$$

The isometry subgroup $\mathrm{SU}(1,n)$ acts linearly, in the fundamental representation, on the coordinates $\mathcal{X}^\Lambda$

$$\mathcal{X}'^\Lambda = \Lambda^\Lambda{}_\Sigma\mathcal{X}^\Sigma\,, \quad \text{with} \quad \Lambda^\dagger\eta\Lambda = \eta\,, \quad \text{and} \quad \det\Lambda = 1\,. \tag{5.45}$$

This linear action induces a non-linear action on the special coordinates:

$$Z'^\Lambda = \frac{\Lambda^\Lambda{}_\Sigma Z^\Sigma}{\Lambda^0{}_\Sigma Z^\Sigma}\,. \tag{5.46}$$

The Kähler potential is invariant under these transformations up to Kähler transformations $\mathcal{K}' = \mathcal{K} + f + f^*$ with

$$f(Z) = \log\left(\Lambda^0{}_\Sigma Z^\Sigma\right). \tag{5.47}$$

The $n(n+2)$ infinitesimal generators T_m of $\mathfrak{su}(1,n)$ in the fundamental representation are defined by

$$\Lambda^\Lambda{}_\Sigma \sim \delta^\Lambda{}_\Sigma + \alpha^m\, T_m{}^\Lambda{}_\Sigma\,, \quad \text{with} \quad \eta T^\dagger_m\eta = -T_m\,, \quad \text{and} \quad T_m{}^\Lambda{}_\Lambda = 0\,. \tag{5.48}$$

Substituting this definition into Eq. (5.46) we find an expression for the holomorphic Killing vectors[14].

$$Z'^\Lambda = Z^\Lambda + \alpha^m k_m{}^\Lambda(Z)\,, \qquad k_m{}^\Lambda(Z) = T_m{}^\Lambda{}_\Sigma\, Z^\Sigma - T_m{}^0{}_\Omega\, Z^\Omega Z^\Lambda\,, \tag{5.49}$$

and, from this expression, we also find explicit expressions for the holomorphic functions $\lambda_m(Z)$ and the momentum maps

$$\lambda_m = T_m{}^0{}_\Sigma Z^\Sigma\,, \qquad \mathcal{P}_m = ie^{\mathcal{K}}T_m{}^\Lambda{}_\Sigma Z^\Sigma Z^*_\Lambda = ie^{\mathcal{K}}\eta_{\Lambda\Omega}T_m{}^\Lambda{}_\Sigma Z^\Sigma Z^{*\,\Omega}\,. \tag{5.50}$$

Although the theory is invariant under the whole $\mathrm{SU}(1,n)$ group, the prepotential is invariant only under the subgroup of $\mathrm{SU}(1,n)$ with real matrices, $\mathrm{SO}(1,n)$, which is the largest group that we could eventually gauge. However, the requirements that the vectors must transform in the adjoint representation restricts the possibilities to either $\mathrm{SO}(1,2)$ or $\mathrm{SO}(3)$ (if $n\geq 2$ or $n\geq 3$, respectively); we are going to consider the latter case embedded into the minimal model admitting this gauge group, namely $\overline{\mathbb{CP}}^3$.

[14]The $k_m{}^0(Z)$ component vanishes identically, as it must, but it is convenient to keep it.

In this model, the adjoint indices $a, b, c, \ldots$ and the fundamental indices $i, j, k, \ldots$ take the same values $1, 2, 3$ and we will only use the latter. The infinitesimal transformations of the scalars are

$$\delta_\alpha Z^i = \alpha^j T_j{}^i{}_k Z^k \,, \quad \text{where} T_j{}^i{}_k = f_{jk}{}^i = -\epsilon_{jki} \,, \tag{5.51}$$

and the momentum maps, holomorphic Killing vectors etc. take the values

$$\mathcal{P}_i = -ie^{\mathcal{K}} \epsilon_{ijk} Z^j Z^{*\,k} \,, \qquad k_i{}^j = \epsilon_{ijk} Z^k \,, \qquad \lambda_i = 0 \,. \tag{5.52}$$

This means that the gauge-covariant derivative of the scalars is just that of a complex adjoint SO(3) scalar

$$\mathfrak{D}_\mu Z^i = \partial_\mu Z^i - g\epsilon_{ijk} A^j{}_\mu Z^k, \tag{5.53}$$

and that the scalar potential takes the form

$$V(Z, Z^*) \;=\; -\tfrac{1}{2} g^2 e^{\mathcal{K}} \epsilon_{ijk} \epsilon_{imn} Z^j Z^{*k^*} Z^m Z^{*n^*} \;=\; \tfrac{1}{2} g^2 \left| \vec{Z} \times \vec{Z}^* \right|^2 \,. \tag{5.54}$$

The solutions

To construct the solutions of this model[15] we just have to follow the recipe spelled out in section 5.1.2. We will only consider static solutions (so $\omega = 0$ and $\tilde{A}^\Lambda{}_{\underline{m}} = A^\Lambda{}_{\underline{m}}$). First of all, we need a solution of the Bogomol'nyi Eqs. (5.19). These equations split into an Abelian part (the 0th component) and the non-Abelian part (the $i = 1, 2, 3$ components):

$$F^0{}_{\underline{mn}} \;=\; -\tfrac{1}{\sqrt{2}} \epsilon_{mnp} \partial_{\underline{p}} \mathcal{I}^0 \,, \tag{5.55}$$

$$F^i{}_{\underline{mn}} \;=\; -\tfrac{1}{\sqrt{2}} \epsilon_{mnp} \mathfrak{D}_{\underline{p}} \mathcal{I}^i \,. \tag{5.56}$$

The Abelian equation is solved by

$$\mathcal{I}^0 = A^0 + \frac{p^0/\sqrt{2}}{r} \,, \tag{5.57}$$

where A^0 is an integration constant and p^0 is the normalized Abelian magnetic charge. The non-Abelian set of equations can be solved making the identification Eq. (5.35) and using Protogenov's solutions Eqs. (5.38).

The second step in the recipe (finding solutions $\mathcal{I}_\Lambda$ to Eqs. (5.20)) will be solved, for the sake of simplicity, by choosing another harmonic function in the Abelian direction and vanishing functions in the rest:

$$\mathcal{I}_0 = A_0 + \frac{q_0/\sqrt{2}}{r} \,, \qquad \mathcal{I}_i = 0 \,. \tag{5.58}$$

The third point in the recipe, combined with the staticity of the solutions implies the following constraint on the integration constants:

$$A^0 q_0 - A_0 p^0 = 0 \,. \tag{5.59}$$

[15] All these solutions have already been presented in Refs. [244, 245, 319]. We review them here for pedagogical reasons and also for the sake of making easier the comparison with the solutions of other models.

Another constraint will arise from the normalization of the metric at infinity. The solution is completely determined and, now, we only have to write the physical fields and make, if necessary, sensible choices of the values of the physical parameters to make the solutions regular.

In order to write the physical fields we need the solutions of the Freudenthal duality equations of this model. These are given by (see, *e.g.* Ref. [90])

$$(\tilde{\mathcal{I}}^M) = \begin{pmatrix} \tilde{\mathcal{I}}^\Lambda \\ \tilde{\mathcal{I}}_\Lambda \end{pmatrix} = \begin{pmatrix} -2\eta^{\Lambda\Sigma}\mathcal{I}_\Sigma \\ \frac{1}{2}\eta_{\Lambda\Sigma}\mathcal{I}^\Sigma \end{pmatrix}, \quad \Rightarrow \quad e^{-2U} = \tfrac{1}{2}\eta_{\Lambda\Sigma}\mathcal{I}^\Lambda\mathcal{I}^\Sigma + 2\eta^{\Lambda\Sigma}\mathcal{I}_\Lambda\mathcal{I}_\Sigma\,, \tag{5.60}$$

and the metric function and the physical scalars are given by

$$e^{-2U} = \tfrac{1}{2}(\mathcal{I}^0)^2 + 2(\mathcal{I}_0)^2 - (rf)^2, \tag{5.61}$$

$$Z^i = \frac{\sqrt{2}rf}{\mathcal{I}^0 + 2i\mathcal{I}_0}\delta^i{}_m\frac{x^m}{r}. \tag{5.62}$$

At least one of the two functions $\mathcal{I}^0, \mathcal{I}_0$ must be different from zero for the metric function to be positive. Then, there are two possible cases, depending on the vanishing of the Abelian charges p^0, q_0:

I. $p^0 = q_0 = 0$ The only regular solution is the one with $s = 0$ (the 't Hooft-Polyakov monopole). In this solution, the integration constants satisfy the normalization condition

$$\tfrac{1}{2}(A^0)^2 + 2(A_0)^2 = 1 + (\mu/g)^2\,. \tag{5.63}$$

They are also related to the asymptotic values of the scalars. These cannot be constant, in general, because the scalars transform under local SU(2) transformations, but they are covariantly constant and their asymptotic values are determined by a single gauge-invariant complex parameter that we call Z_∞:[16]

$$Z^i \sim Z_\infty\delta^i{}_m\frac{x^m}{r}\,, \qquad Z_\infty \equiv \frac{\mu/g}{1+(\mu/g)^2}\left(\tfrac{1}{\sqrt{2}}A^0 - \sqrt{2}iA_0\right)\,, \qquad 0 \leq |Z_\infty|^2 < 1\,. \tag{5.64}$$

These expressions lead to the following identification of the integration constant μ in terms of the physical parameters:

$$\mu^2 = \frac{|Z_\infty|^2}{1-|Z_\infty|^2}g^2\,, \tag{5.65}$$

and to the following expression for the mass of the solution

$$M_{\text{monopole}} = \sqrt{\frac{|Z_\infty|^2}{1-|Z_\infty|^2}}\frac{1}{g}\,. \tag{5.66}$$

This asymptotically flat solution has no singularities nor horizons (one finds a Minkowski spacetime in the $r \to 0$ limit, hence zero entropy and temperature). Globally-regular solutions of this kind [130,231] are sometimes called *global monopoles*.

[16] Observe that the scalar potential of this theory, Eq. (5.54), vanishes at infinity for those solutions, which is why they are asymptotically flat.

Observe that a solution of the ungauged theory with

$$H^0 = A^0\,, \qquad H_0 = A_0\,, \qquad H^1 = A^1 + \frac{\sqrt{2}}{gr}\,, \tag{5.67}$$

in which the non-Abelian monopole is replaced by an Abelian monopole with the same charge, would have the same asymptotic behavior but it would mean having a naked singularity at some value of $r > 0$.

II. $p^0 q_0 \neq 0$ [17] Solving Eq. (5.59) the metric can be written in the form

$$e^{-2U} = \frac{1}{1-|Z_\infty|^2} H^2 - (rf)^2, \tag{5.68}$$

$$Z^i = \frac{2\beta}{p^0 + 2iq_0} \frac{rf}{H} \delta^i{}_m \frac{x^m}{r}\,, \tag{5.69}$$

where H is the harmonic function

$$H \equiv 1 + \frac{\beta}{r}\,, \qquad \beta^2 = (1-|Z_\infty|^2) W_{\rm RN}(\mathcal{Q})/2\,, \qquad W_{\rm RN}(\mathcal{Q}) \equiv \tfrac{1}{2}(p^0)^2 + 2(q_0)^2\,, \tag{5.70}$$

and the integration constant μ is still given by Eq. (5.65). We have expressed all the constants (except for Protogenov's hair parameter s and λ) in terms of physical constants. Observe that the isolated solution f_* has $\mu = 0$ and corresponds to $Z_\infty = 0$. These identifications allow us to compute the mass and entropy of all the possible solutions in terms of the physical parameters. We get a completely general mass formula and two formulae for the entropy, one for the $s \neq 0$ solutions and another one for the $s = 0$ and the isolated solutions (which corresponds to $Z_\infty = 0$):

$$M = \sqrt{\tfrac{1}{2}\frac{W_{RN}(\mathcal{Q})}{1-|Z_\infty|^2}} + M_{\rm monopole}, \tag{5.71}$$

$$S/\pi = \tfrac{1}{2} W_{\rm RN}(\mathcal{Q}) - \frac{1}{g^2}\,, \quad \text{for} \quad s \neq 0 \;\; \text{and} \;\; Z_\infty = 0, \tag{5.72}$$

$$S/\pi = \tfrac{1}{2} W_{\rm RN}(\mathcal{Q}), \quad \text{for} \quad s = 0\,, \tag{5.73}$$

where $M_{\rm monopole}$ is given by Eq. (5.66).

The entropy is moduli-independent as in the ungauged case and both the entropy and the mass are independent of the hair parameters s and λ.

Observe that the charge of the BPS 't Hooft-Polyakov monopole $s = 0$ does not contribute to the entropy which suggests that it must be associated to a pure state in the quantum theory. The non-Abelian field of the isolated solution does not contribute to the mass at infinity ($M_{\rm monopole} = 0$ for $Z_\infty = 0$) but there is a magnetic-charge contribution to the entropy, which suggests that there really is a magnetic charge but it is screened at infinity. Further support for this interpretation comes

[17] It is easier to work with both charges non-vanishing. The results will still be valid when we set one of them to zero.

from the near-horizon limit of the scalars, which is the covariantly-constant function of the charges

$$Z^i_{\rm h} = \frac{1/g}{\frac{1}{2}p^0 + iq_0}\, \delta^i{}_m \frac{x^m}{r}. \tag{5.74}$$

even for the isolated case, when no magnetic charge is observed at infinity.

In the case of the 1-parameter (λ) family, neither the mass nor the entropy depend on λ.

Some of the solutions in this family can also be seen as solutions of the pure EYM theory. They are identical to those obtained in Refs. [113, 429]. As discussed at the end of section 5.1.1, we need to tune the parameters of the solutions so as to get covariantly constant scalars which do not contribute to the energy-momentum tensor This is only possible for the $s \to \infty$ solutions (Wu–Yang monopoles) for which rf is a harmonic function. In that case

$$Z^i = Z\, \delta^i{}_m \frac{x^m}{r}\,, \qquad Z = \frac{1/g}{\frac{1}{2}p^0 + iq_0} = Z_\infty\,. \tag{5.75}$$

The metric is identical to that of a Reissner-Nordström black hole. These solutions were called *black hedgehogs* in Ref. [245] and *black merons* in Ref. [113] because the gauge field of the Wu–Yang monopole can also be understood as Lorentzian meron solution.

A closely related solution with non-covariantly constant scalars was obtained in a different context in Ref. [270].

5.2.3 Embedding in SU(2)-gauged ST$[2, n]$ models

The $ST[2, n]$ models

The $ST[2, n]$ models are cubic models with $n_V = n+1$ vector supermultiplets and as many complex scalars and, as all other cubic models, they can be embedded in type II String Theory compactified Calabi-Yau 3-folds and then uplifted to M-theory. They can also be obtained from corresponding models of $N = 1$, $d = 5$ supergravity compactified on S^1.

A generic cubic model is defined by the prepotential

$$\mathcal{F} \;=\; -\frac{1}{3!}\, d_{ijk}\, \frac{\mathcal{X}^i \mathcal{X}^j \mathcal{X}^k}{\mathcal{X}^0}\,, \tag{5.76}$$

where d is completely symmetric in its indices; the $ST[2, n]$ models are characterized by d-tensors with non-vanishing components $d_{1\alpha\beta} = \eta_{\alpha\beta}$ where $(\eta_{\alpha\beta}) = \mathrm{diag}(+ - \cdots -)$ and where the indices α, β take n values between 2 and $n + 1$.

The scalar $Z^1 = \mathcal{X}^1/\mathcal{X}^0$ plays a special role and parametrizes a $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ coset space. For this and other reasons, it is called axidilaton and we will denote it by τ. The other n scalars parametrize a $\mathrm{SO}(2, n)/(\mathrm{SO}(2)\times\mathrm{SO}(n))$ coset space and will be denoted by $Z^\alpha = \mathcal{X}^\alpha/\mathcal{X}^0$ $(\alpha = 2, \cdots, n)$. The Kähler metric and 1-form connection are the products of those of the two spaces.

Using this notation and using the gauge $\mathcal{X}^0 = 1$, the canonical symplectic section Ω, the Kähler potential $\mathcal{K}$ and the components of Kähler 1-form $\mathcal{Q}_i$ and of the Kähler metric

$\mathcal{G}_{ij^*}$ are given by

$$\Omega = \begin{pmatrix} 1 \\ \tau \\ Z^\alpha \\ \frac{1}{2}\tau\eta_{\alpha\beta}Z^\alpha Z^\beta \\ -\frac{1}{2}\eta_{\alpha\beta}Z^\alpha Z^\beta \\ -\tau\eta_{\alpha\beta}Z^\beta \end{pmatrix}, \quad e^{-\mathcal{K}} = 4\Im\mathfrak{m}\,\tau\ \eta_{\alpha\beta}\Im\mathfrak{m}\,Z^\alpha\ \Im\mathfrak{m}\,Z^\beta,$$

$$\mathcal{Q}_\tau = \frac{1}{4\Im\mathfrak{m}\,\tau}, \qquad \mathcal{Q}_\alpha = \frac{\eta_{\alpha\beta}\Im\mathfrak{m}\,Z^\beta}{2\eta_{\gamma\delta}\Im\mathfrak{m}\,Z^\gamma\ \Im\mathfrak{m}\,Z^\delta},$$

$$\mathcal{G}_{\tau\tau^*} = \frac{1}{4(\Im\mathfrak{m}\,\tau)^2}, \qquad \mathcal{G}_{\alpha\beta^*} = \frac{\eta_{\alpha\gamma}\Im\mathfrak{m}\,Z^\gamma\ \eta_{\beta\delta}\Im\mathfrak{m}\,Z^\delta}{[\eta_{\epsilon\varphi}\Im\mathfrak{m}\,Z^\epsilon\ \Im\mathfrak{m}\,Z^\varphi]^2} - \frac{\eta_{\alpha\beta}}{2\eta_{\epsilon\varphi}\Im\mathfrak{m}\,Z^\epsilon\ \Im\mathfrak{m}\,Z^\varphi}. \tag{5.77}$$

The reality of the Kähler potential constrains the values of the scalars. The model has two branches characterized by

$$\Im\mathfrak{m}\,\tau > 0, \qquad \eta_{\alpha\beta}\Im\mathfrak{m}\,Z^\alpha\ \Im\mathfrak{m}\,Z^\beta > 0, \tag{5.78}$$

and

$$\Im\mathfrak{m}\,\tau\ < 0, \qquad \eta_{\alpha\beta}\Im\mathfrak{m}\,Z^\alpha\ \Im\mathfrak{m}\,Z^\beta < 0, \tag{5.79}$$

that will be distinguished where required by $+$ and $-$ indices, respectively.

Only the subgroup $\mathrm{SO}(1,n) \subset \mathrm{SO}(2,n)$ acts linearly (in the fundamental representation) on the special coordinates Z^α and the group $\mathrm{SO}(3)$ acts in the adjoint (for instance) on the coordinates $\alpha = 3,4,5$ if $n \geq 4$. We take $n = 4$ for simplicity and denote the $\alpha = 3,4,5$ indices by $a, b, \cdots = 1,2,3$. For the sake of simplicity we will write Z^a instead of Z^{a+2} for Z^3, Z^4, Z^5 etc. The generators and structure constants of $\mathfrak{so}(3)$ and their action on the scalars are the same as in the $\overline{\mathbb{CP}}^3$ model with obvious changes of notation:

$$(T_a)^b{}_c = f_{ac}{}^b = -\varepsilon_{acb}, \qquad \delta_\alpha Z^a = \alpha^b (T_b)^a{}_c Z^c = -\epsilon_{abc}\alpha^b Z^c = \alpha^b k_b{}^a(Z), \tag{5.80}$$

(τ and Z^2 are inert) so the holomorphic Killing vectors and the momentum maps are

$$k_a{}^b(Z) = \epsilon_{abc}Z^c, \qquad \mathcal{P}_a = -\tfrac{i}{2}\frac{\epsilon_{abc}Z^b Z^{*\,c^*}}{\eta_{\alpha\beta}\Im\mathfrak{m}\,Z^\alpha\ \Im\mathfrak{m}\,Z^\beta}. \tag{5.81}$$

The scalar potential has a structure similar to that of the $\overline{\mathbb{CP}}^3$ model, but more complicated. We will not give it here since it is not needed anyway.

The solutions

To find solutions to this non-Abelian model we just need to follow the recipe. First, we find the functions $\mathcal{I}^\Lambda$ and the spatial components of the vector fields $A^\Lambda{}_{\underline{m}}$ by solving the Bogomol'nyi equations

$$F^\Lambda{}_{\underline{mn}} = -\tfrac{1}{\sqrt{2}}\epsilon_{mnp}\partial_{\underline{p}}\mathcal{I}^\Lambda, \quad I = 0,1,2, \tag{5.82}$$

$$F^{a+2}{}_{\underline{mn}} = -\tfrac{1}{\sqrt{2}}\epsilon_{mnp}\mathfrak{D}_{\underline{p}}\mathcal{I}^{a+2}, \quad a = 1,2,3, \tag{5.83}$$

(we will suppress the $+2$ in the non-Abelian indices in most places). The Abelian equations are solved by harmonic functions and the non-Abelian ones by making the identification Eq. (5.35) with the Higgs field and using Protogenov's solutions Eqs. (5.38), as we did in the $\overline{\mathbb{CP}}^3$ model.

Next, we have to find the functions $\mathcal{I}_\Lambda$ by solving Eqs. (5.20). In the Abelian directions $\Lambda = 0, 1, 2$ we can simply choose harmonic functions and in the non-Abelian ones we take $\mathcal{I}_a = 0$. This choice gives non-singular solutions, as we are going to see. We will also set some of the harmonic functions to zero for simplicity.

The Hesse potential defined in Eq. (5.22) can be found from Shmakova's solution of the stabilization (or Freudenthal duality) equations for cubic models [389]; it can be written as

$$\mathsf{W}(\mathcal{I}) = 2\sqrt{J_4(\mathcal{I})}\,, \tag{5.84}$$

with the quartic invariant $J_4(\mathcal{I})$ given by

$$J_4(\mathcal{I}) \equiv (\mathcal{I}^\alpha\mathcal{I}^\beta\eta_{\alpha\beta} + 2\mathcal{I}^0\mathcal{I}_1)(\mathcal{I}_\alpha\mathcal{I}_\beta\eta^{\alpha\beta} - 2\mathcal{I}^1\mathcal{I}_0) - (\mathcal{I}^0\mathcal{I}_0 - \mathcal{I}^1\mathcal{I}_1 + \mathcal{I}^\alpha\mathcal{I}_\alpha)^2\,. \tag{5.85}$$

This potential does not vanish for the choice $\mathcal{I}_a = 0$, as we advanced and it will remain non-singular if we set $\mathcal{I}^0 = \mathcal{I}_1 = \mathcal{I}_2 = 0$. In other words: the only non-trivial components of $\mathcal{I}^M$ are $\mathcal{I}^1, \mathcal{I}^2, \mathcal{I}^{a+2}, \mathcal{I}_0$. With this choice the metric function is given by

$$e^{-2U} = \mathsf{W}(\mathcal{I}) = 2\sqrt{-2\mathcal{I}^1\mathcal{I}_0\,\eta_{\alpha\beta}\mathcal{I}^\alpha\mathcal{I}^\beta} = 2\sqrt{-2\mathcal{I}^1\mathcal{I}_0[(\mathcal{I}^2)^2 - \mathcal{I}^a\mathcal{I}^a]}\,. \tag{5.86}$$

As instructed by the recipe in section (5.1.2), we can calculate the $\tilde{\mathcal{I}}$ from Eq. (5.22), which for our choice of non-trivial components of $\mathcal{I}^M$ means that $\tilde{\mathcal{I}}^i = 0$ $(i = 1, \cdots, 5)$; this implies that all the scalars are purely imaginary and given by

$$Z^i = i\frac{\mathcal{I}^i}{\tilde{\mathcal{I}}^0}\,, \qquad \text{where} \quad \tilde{\mathcal{I}}^0 = \frac{2\mathcal{I}^1\eta_{\alpha\beta}\mathcal{I}^\alpha\mathcal{I}^\beta}{\mathsf{W}(\mathcal{I})}\,. \tag{5.87}$$

It is convenient to write all of them in terms of $\tau = Z^1$

$$Z^\alpha = \frac{\mathcal{I}^\alpha}{\mathcal{I}^1}\tau\,, \qquad \tau = i\frac{e^{-2U}}{2\eta_{\alpha\beta}\mathcal{I}^\alpha\mathcal{I}^\beta}\,. \tag{5.88}$$

In the two ($+$ and $-$) branches of the model corresponding, respectively, to the upper and lower signs $\pm\Im\mathfrak{m}\,\tau_{(\pm)} > 0$ and, since $e^{-2U} > 0$, we must choose the functions $\mathcal{I}^\alpha_{(\pm)}$ so that

$$\pm\,\eta_{\alpha\beta}\mathcal{I}^\alpha_{(\pm)}\mathcal{I}^\beta_{(\pm)} = \pm\left[(\mathcal{I}^2_{(\pm)})^2 - \mathcal{I}^a_{(\pm)}\mathcal{I}^a_{(\pm)}\right] > 0\,. \tag{5.89}$$

In order for $\mathsf{W}(\mathcal{I})$ to be real the $\mathcal{I}_{(\pm)\,0}$ and $\mathcal{I}^1_{(\pm)}$ must be chosen so as to satisfy

$$\pm\,\mathcal{I}^1_{(\pm)}\mathcal{I}_{(\pm)\,0} < 0\,. \tag{5.90}$$

(We will suppress the $\pm$ subindices in what follows, to simplify the notation, except where this may lead to confusion.)

Observe that with our choice of non-vanishing components of $\mathcal{I}^M$ the r.h.s. of Eq. (5.21) vanishes automatically, whence the staticity condition $\omega = 0$ does not impose any constraint.

According to the preceding discussions, the non-vanishing components of $\mathcal{I}^M$ will be assumed to take the form

$$\begin{aligned}\mathcal{I}^1 &= A^1 + \frac{p^1/\sqrt{2}}{r}\,, \quad \mathcal{I}^2 = A^2 + \frac{p^2/\sqrt{2}}{r}\,, \quad \mathcal{I}^a = \sqrt{2}\,\delta^a{}_m x^m f(r)\,, \\ \mathcal{I}_0 &= A_0 + \frac{q_0/\sqrt{2}}{r}\,,\end{aligned} \tag{5.91}$$

where $f(r)$ is $f_{\mu,s}$ or f_λ in Eqs. (5.38), p^1, p^2, q_0 are magnetic and electric charges and A^1, A^2, A_0 are integration constants to be determined in terms of the asymptotic values of the scalars and the metric. These constants must have the same sign as the corresponding charges

$$\mathrm{sign}(A^{1,2}) = \mathrm{sign}(p^{1,2})\,, \qquad \mathrm{sign}(A_0) = \mathrm{sign}(q_0)\,, \tag{5.92}$$

as the functions $\mathcal{I}^1$, $\mathcal{I}^2$ and $\mathcal{I}_0$ are required to have no zeroes on the interval $r \in (0, +\infty)$ in order to avoid naked singularities there. Then, the above constraint on the signs of $\mathcal{I}^1$ and $\mathcal{I}_0$ translates into the following constraints on the signs of the charges in the two branches:

$$\mathrm{sign}(p^1)\mathrm{sign}(q_0) = \mp 1\,. \tag{5.93}$$

Defining as in the $\overline{\mathbb{CP}}^3$ case the asymptotic value Z_∞ of the adjoint scalars by

$$Z^a_\infty \equiv Z_\infty\, \delta^a{}_m \frac{x^m}{r}\,, \tag{5.94}$$

and imposing the normalization of the metric at infinity it is not hard to express the integration constants μ, A^1, A^2, A_0 in terms of the moduli (the asymptotic values of the scalars $\Im\mathfrak{m}\tau_\infty, \Im\mathfrak{m}\, Z^2_\infty$ and $\Im\mathfrak{m}\, Z_\infty$) and the coupling constant g

$$\begin{aligned} A^1 &= \frac{\mathrm{sign}(p^1)|\Im\mathfrak{m}\tau_\infty|}{\sqrt{2}\chi_\infty}\,, \\ A^2 &= \frac{\mathrm{sign}(p^2)|\Im\mathfrak{m}\, Z^2_\infty|}{\sqrt{2}\chi_\infty}\,, \\ \mu &= \frac{g|\Im\mathfrak{m}\, Z_\infty|}{2\chi_\infty}\,, \\ A_0 &= \tfrac{1}{2\sqrt{2}}\mathrm{sign}(q_0)\chi_\infty\,, \end{aligned} \tag{5.95}$$

where we have defined the combination (real in both branches of the theory)

$$\chi_\infty \equiv \sqrt{\Im\mathfrak{m}\tau_\infty\,[(\Im\mathfrak{m}\, Z^2_\infty)^2 - (\Im\mathfrak{m}\, Z_\infty)^2]}\,. \tag{5.96}$$

The mass of the solutions in terms of the moduli and the charges is

$$M = \tfrac{1}{4}\frac{\chi_\infty}{|\Im\mathfrak{m}\tau_\infty|}|p^1| + \frac{1}{2\chi_\infty}|q_0| \pm \tfrac{1}{2}\frac{|\Im\mathfrak{m}\tau_\infty \Im\mathfrak{m}\, Z^2_\infty|}{\chi_\infty}|p^2| \pm \frac{|\Im\mathfrak{m}\tau_\infty \Im\mathfrak{m}\, Z_\infty|}{\chi_\infty}\frac{1}{g}\,. \tag{5.97}$$

In the above expressions we have used two consistency conditions:

$$\mathrm{sign}(\Im\mathfrak{m}\, Z_\infty) = \mp\mathrm{sign}(p^1)\,, \qquad \mathrm{sign}(\Im\mathfrak{m}\, Z^2_\infty) = \pm\mathrm{sign}(p^1)\mathrm{sign}(p^2)\,. \tag{5.98}$$

These expressions for the integration constants and the mass are valid both for the 2- and 1-parameter families, the latter being recovered by setting $\Im\mathfrak{m}\, Z_\infty = 0$ everywhere. The contribution of the monopole charge $1/g$ to the mass disappears because it is screened.

Observe that the positivity of the mass is not guaranteed in the $-$ branch for arbitrary values of the charges and moduli: it has to be imposed by hand.

Let us now study the behavior of the solution in the near-horizon limit $r \to 0$. For $f_{\mu,s\neq 0}$ and f_λ the metric function behaves as

$$e^{-2U} \sim \sqrt{-2p^1 q_0\left[(p^2)^2 - (2/g)^2\right]}\,\frac{1}{r^2}\,, \tag{5.99}$$

which corresponds to a regular horizon in both branches. The solutions will describe regular black holes if the charges and moduli are such that $M > 0$. Observe that in the $-$ branch it is possible to chose those such that $M = 0$ with a non-vanishing entropy.

In the $f_{\mu,s=0}$ case with $p^2 \neq 0$ the solution is only well defined in the $+$ branch because there is no $1/r$ contribution from the monopole in the $r \to 0$ limit and it is impossible to satisfy the inequality $-\eta_{\alpha\beta}\mathcal{I}^\alpha\mathcal{I}^\beta > 0$ in that limit. In this case (the $+$ branch with $p^2 \neq 0$) we have

$$e^{-2U} \sim \sqrt{-2p^1 q_0 (p^2)^2}\,\frac{1}{r^2}\,, \tag{5.100}$$

which corresponds to a regular horizon.

In the $f_{\mu,s=0}$ case with $p^2 = 0$ there are two possibilities:

1. We can set $p^1 = q_0 = 0$. Then, in the $r \to 0$ limit, e^{-2U} is the moduli-dependent constant $2\sqrt{-2A^1 A_0 (A^2)^2}$. There is neither horizon nor singularity and the solution, which is a global monopole, belongs to the $+$ branch (this also guarantees that the mass is positive).

2. We can keep both $p^1 \neq 0$ and $q_0 \neq 0$, setting $A^2 = 0$ and profit from the fact that, in this limit $\Phi^a\Phi^a$ goes to zero as r^2. The solution is only well defined in the $-$ branch. The metric function takes the constant value

$$e^{-2U} \sim \sqrt{+p^1 q_0 \frac{\mu^4}{g^2}}\,, \tag{5.101}$$

We have, as far as the metric is concerned, a global monopole solution (as long as $M > 0$), but since we need two Abelian charges switched on, namely p^1 and q_0, the scalar fields and the gauge fields are singular at $r = 0$. As before, it is possible to tune the moduli and charges so that $M = 0$.

The near-horizon limits of the scalars are, in the $f_{\mu,s\neq 0}$ and f_λ cases

$$\begin{aligned}
\Im\mathfrak{m}\tau_{\mathrm{h}} &= \frac{\sqrt{-2p^1 q_0\left[(p^2)^2 - (2/g)^2\right]}}{2\left[(p^2)^2 - (2/g)^2\right]}\,,\\
\Im\mathfrak{m}\, Z^2_{\mathrm{h}} &= \frac{p^2}{p^1}\Im\mathfrak{m}\tau_{\mathrm{h}}\,,\\
\Im\mathfrak{m}\, Z^a_{\mathrm{h}} &= \frac{2\Im\mathfrak{m}\tau_{\mathrm{h}}}{gp^1}\delta^a{}_m\frac{x^m}{r}\,,
\end{aligned} \tag{5.102}$$

and, in the $f_{\mu,s=0}$ case with $p^2 \neq 0$, we get the same results up to the contribution of the monopole which disappears (formally, $1/g = 0$).

5.2.4 Embedding in pure SU(2) EYM

The scalars can only be trivialized for the Wu-Yang monopole $s = \infty$. In that case, it is easy to construct a double-extremal black hole with constant scalars and the metric is, as usual, Reissner-Nordström's.

5.3 Multi-center SBHSs

To construct multi-center SBHSs we can use the same recipe as in the single-center case but we need multi-center solutions of the Bogomol'nyi equations. We start by discussing these.

5.3.1 Multi-center solutions of the SU(2) Bogomol'nyi equations on $\mathbb{R}^3$

In the Abelian case, the multicenter solutions of the Bogomol'nyi equations are associated to harmonic functions with isolated point-like singularities. They are the seed solutions of the multi-black-hole solutions of the Einstein-Maxwell theory [140, 228, 252, 309, 358, 359] and $\mathcal{N}=2$, $d=4$ supergravities [31, 37, 42, 159]. In the non-Abelian case, the hedgehog ansatz is clearly inappropriate and more sophisticated methods need to be used. Only a few explicit solutions are known, even though solutions describing several BPS objects in equilibrium are, on general grounds, expected to exist. For instance, there is no explicit solution describing two BPS 't Hooft-Polyakov monopoles in equilibrium (see however Ref. [357]).

Perhaps not surprisingly, the only general families of explicit solutions available involve an arbitrary number of Wu-Yang or Dirac monopoles embedded in SU(2). The simplest of these only involve Wu-Yang monopoles and formally, it can be obtained from solutions describing Dirac monopoles embedded in SU(2) via singular gauge transformations [371], generalizing the constructions reviewed in Appendices C.2 (minimal charge) and C.4 (higher charge). As we have explained at length in the preceding sections, the metric is completely oblivious to these gauge transformations and takes the same form as in the Abelian cases. We will not study such solutions in this section.

In Refs. [138], using the Nahm equations [342], Cherkis and Durcan found new solutions describing one or two, charge 1, Wu-Yang monopoles embedded in SU(2) in the background of a single BPS 't Hooft-Polyakov monopole.[18] We are going to use the first of them to construct multi-center solutions of the $\overline{\mathbb{CP}}^3$ and $ST[2,4]$ models of $\mathcal{N}=2$, $d=4$ SEYM. Let us review the Cherkis-Durcan solution first: take the BPS 't Hooft-Polyakov monopole to be located at $x^n = x^n_0$ and the Wu-Yang monopole at $x^m = x^m_1$. We define

[18] In Ref. [71] Blair and Cherkis generated a solution describing an arbitrary number of charge 1 Wu-Yang monopoles in the presence of an 't Hooft-Polyakov monopole; one can easily generalize this solution to one describing an arbitrary number of charge $n(>0)$ Wu-Yang monopoles in the background of an 't Hooft-Polyakov monopole, by coalescing n charge 1 Wu-Yang monopoles. Needless to say, the Protogenov trick works as expected. For the sake of simplicity of exposition, we will not consider this more general solution in this article.

the coordinates relative to each of those centers and the relative position by

$$r^m \equiv x^m - x_0^m\,, \qquad u^m \equiv x^m - x_1^m\,, \qquad d^m \equiv u^m - r^m = x_0^m - x_1^m\,, \tag{5.103}$$

and their norms by respectively, r, u and d. The Higgs field and gauge potential of this solution (adapted to our conventions) are given by [138]

$$\pm\Phi^a = \frac{1}{g}\delta^a{}_m \left\{\left[\frac{1}{r} - \left(\mu + \frac{1}{u}\right)\frac{K}{L}\right]\frac{r^m}{r} + \frac{2r}{uL}\left(\delta^{mn} - \frac{r^m r^n}{r^2}\right)d^n\right\}, \tag{5.104}$$

$$\begin{aligned} A^a &= -\frac{1}{g}\left[\frac{1}{r} - \frac{\mu \mathrm{D} + 2d + 2u}{\mathrm{L}}\right]\frac{\varepsilon^a{}_{mn} r^m dx^n}{r} + 2\frac{\mathrm{K}}{\mathrm{L}}\frac{\varepsilon_{npq} d^n u^p dx^q}{u\mathrm{D}}\,\delta^a{}_m \frac{r^m}{r} \\ &\quad - \frac{2r}{u\mathrm{L}}\,\delta^a{}_m\left(\delta^{mn} - \frac{r^m r^n}{r^2}\right)\varepsilon_{npq} u^p dx^q\,, \end{aligned} \tag{5.105}$$

where the functions K, L, D of u and r are defined by

$$K \equiv \left[(u+d)^2 + r^2\right]\cosh \mu r + 2r(u+d)\sinh \mu r\,, \tag{5.106}$$

$$L \equiv \left[(u+d)^2 + r^2\right]\sinh \mu r + 2r(u+d)\cosh \mu r\,, \tag{5.107}$$

$$\mathrm{D} = 2\,(ud + u^m d^m) = (d+u)^2 - r^2\,. \tag{5.108}$$

The function D is clearly zero along the direction[19] $u^m/u = -d^m/d$ signaling the possible presence of a Dirac string in Eq. (5.105); that this is however not the case is demonstrated in Ref. [71].

In the models that we are going to study, the Higgs field enters the metric in the combination $\Phi^a\Phi^a$, which takes the value

$$\Phi^a\Phi^a = \frac{1}{g^2}\left\{\left[\frac{1}{r} - \left(\mu + \frac{1}{u}\right)\frac{K}{L}\right]^2 + \frac{4|\vec{r}\times\vec{d}|^2}{u^2L^2}\right\}. \tag{5.109}$$

To better understand this solution one will consider several limits:

1. The limit in which we take the BPS 't Hooft-Polyakov anti-monopole infinitely far away, keeping the Dirac monopole at x_1^m: in this limit $d \to \infty$, $r^m \sim -d^m$ while u remains finite. The Higgs and gauge fields take the form

$$\pm\Phi^a \sim -\frac{1}{g}\delta^a{}_m\left(\mu + \frac{1}{u}\right)\frac{d^m}{d}\,, \tag{5.110}$$

$$A^a \sim -\frac{1}{g}\left(1 + \frac{d^m}{d}\frac{u^m}{u}\right)^{-1}\varepsilon_{mnp}\frac{d^m}{d}\frac{u^m}{u}d\frac{u^p}{u}\,. \tag{5.111}$$

The gauge field should be compared with the embedding of a Dirac monopole with a string in the direction $-d^m$ into the direction $\delta^a{}_m d^m T^a$ of the gauge group, Eqs. (C.11) and (C.17) with $s^m = -d^m$.

[19] This is the half of the line that joins $r=0$ to $u=0$ that stretches from the Dirac monopole $u=0$ to infinity in the direction opposite to the 't Hooft-Polyakov monopole at $r=0$

2. The limit in which we take the Dirac monopole infinitely away, keeping the BPS 't Hooft-Polyakov anti-monopole at x_0^m: In this limit $d \to \infty$, $u^m \sim d^m$ while r remains finite. The Higgs and gauge fields become those of a single BPS 't Hooft-Polyakov anti-monopole at x_0^m.

3. In the limit in which we are infinitely far away from both monopoles ($r \to \infty$, $u \to \infty$), which remain at a finite relative distance, the Higgs and gauge fields take the form

$$\pm\Phi^a = -\left[\frac{\mu}{g} + \mathcal{O}(|x|^{-2})\right] \delta^a{}_m \frac{x^m}{|x|}\,, \tag{5.112}$$

$$A^a = -\frac{1}{g}\varepsilon^a{}_{mn}\frac{x^m dx^n}{|x|^2} + \frac{1}{2g}\delta^a{}_m \frac{x^m}{|x|}\left(\frac{\varepsilon_{npq}d^n x^p dx^q}{|x|^2}\right). \tag{5.113}$$

The first term in the gauge potential is identical to that of a Wu-Yang anti-monopole (compare with Eq. (C.2)). This is also the asymptotic behavior of the BPS 't Hooft-Polyakov monopole. The Higgs field is asymptotically covariantly constant and, in particular

$$\Phi^a\Phi^a \sim \frac{\mu^2}{g^2} + \mathcal{O}(\frac{1}{|x|^2})\,. \tag{5.114}$$

4. The limit in which we approach the center of the BPS 't Hooft-Polyakov anti-monopole $r^m \to 0$, $u^m \to d^m$

$$\Phi^a\Phi^a \sim \frac{1}{4g^2d^2(1+\mu d)^2} + \mathcal{O}(r)\,. \tag{5.115}$$

This limit is finite and only vanishes when the Dirac monopole is taken to infinity $d \to \infty$.

For finite values of d, Eq. (5.109) says that $\Phi^a\Phi^a$ can only vanish along the line that stretches from $r=0$ to $u=0$ so $\vec{r}\times\vec{d}=0$. Substituting $r^m = \alpha d^m$ in $\Phi^a\Phi^a$ we get a function of α and of the parameter μd. Plotting the functions of α for different values of μd we find that they have a single zero, which is also a local minimum. At this minimum the second derivative does not vanish, and therefore, there, $\Phi^a\Phi^a \sim \mathcal{O}(r^2)$, as in the single-monopole case. However, the value of this second derivative depends on the direction.

5. The limit in which we approach the singularity of the Dirac monopole $u^m \to 0$, $r^m \to -d^m$

$$\Phi^a\Phi^a \to \frac{1}{g^2}\left\{\frac{1}{u^2} + \left(\frac{1}{d} - \mu\right)\frac{1}{u}\right\} + \mathcal{O}(1)\,. \tag{5.116}$$

Growing Protogenov hair

As we have argued in section (5.2.1) we can add a Protogenov hair parameter s to the Cherkis & Durcan solution by simply replacing the argument μr of the hyperbolic sines and cosines in the functions K and L by the shifted on $\mu r + s$. We do not need to write explicitly the solution, but we do need to reconsider the different limits studied for the $s=0$ case:

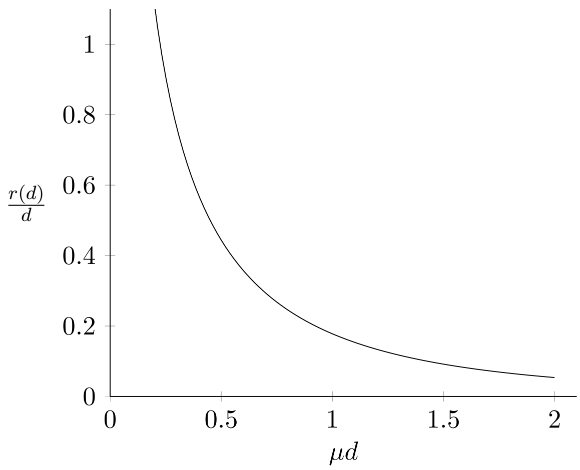

Figure 5.1: The zeros of the Higgs density as measured by r as a function of the dimensionless separation μd.

1. In the limit in which we take the BPS 't Hooft-Polyakov-Protogenov anti-monopole infinitely away, keeping the Dirac monopole at x_1^m the Higgs and gauge fields become, to leading order, those of the Dirac monopole with the Dirac string in the direction $-d^m$, as in the $s=0$ case (See Eqs. (5.110) and (5.105)).

2. In the limit in which we take the Dirac monopole infinitely away, keeping the BPS 't Hooft-Polyakov-Protogenov anti-monopole at x_0^m the Higgs and gauge fields become those of a single BPS 't Hooft-Polyakov-Protogenov anti-monopole at $x^m = x_0^m$ (the first two equations (5.38)).

3. In the limit in which we are infinitely far away from both monopoles ($r \to \infty$, $u \to \infty$), which remain at a finite relative distance, the Higgs and gauge fields take the same form as in the $s=0$ case, Eqs. (5.112-5.114).

4. The limit in which we approach the singularity of the BPS 't Hooft-Polyakov-Protogenov anti-monopole $r^m \to 0$, $u^m \to d^m$ (for $s \neq 0$)

$$\pm\Phi^a \sim \frac{1}{g}\delta^a{}_m\left[\frac{1}{r} - \left(\mu + \frac{1}{d}\right)\coth s + \mathcal{O}(r)\right]\frac{r^m}{r}\,, \tag{5.117}$$

$$\Rightarrow \Phi^a\Phi^a \sim \frac{1}{g^2r^2} + \mathcal{O}\left(\frac{1}{r}\right)\,, \tag{5.118}$$

which is similar to the behaviour near the Dirac monopole as in Eq. (5.116) (with u replaced by r).

5. The limit in which we approach the singularity of the Dirac monopole $u^m \to 0$, $r^m \to -d^m$ we have the same behavior as in the $s=0$ case Eq. (5.116).

The solutions with Protogenov hair have another limit, namely the one in which $s \to \infty$; this case will be studied separately.

The $s \to \infty$ limit solution

In this limit we get a solution that describes the same Dirac monopole together with a ($\mu \neq 0$) Wu-Yang anti-monopole:[20]

$$\pm\Phi^a = \frac{1}{g}\delta^a{}_m \left[-\mu + \frac{1}{r} - \frac{1}{u}\right]\frac{r^m}{r}\,, \tag{5.119}$$

$$A^a = \frac{1}{g}\frac{\varepsilon^a{}_{mn} r^m dx^n}{r^2} + \frac{1}{g}\frac{\varepsilon_{npq} d^n u^p du^q}{u(ud + u^r d^r)}\delta^a{}_m \frac{r^m}{r}\,. \tag{5.120}$$

This solution is a particular example of a more general family describing an arbitrary number of Dirac monopoles in the background of a Wu-Yang anti-monopole. These solutions can be obtained from a solution describing only Dirac monopoles embedded in SU(2) via a singular gauge transformation that only removes the Dirac string of one of them, which becomes the Wu-Yang anti-monopole. The general family of solutions can be written in the form:

$$\Phi = \Phi_{\text{WY}} + HU\,, \qquad A = A_{\text{WY}} + CU\,, \tag{5.121}$$

where U is the SU(2) (and $\mathfrak{su}(2)$) matrix defined in Eq. (C.1) and where Φ_{WY} and A_{WY} are the Higgs and Yang-Mills fields of a Wu-Yang monopole, given, respectively, by

$$\mp\Phi_{\text{WY}} = \frac{1}{2g}\left[-\mu + \frac{1}{r}\right]U\,, \tag{5.122}$$

and by Eq. (C.2) and where H is a function and C a 1-form on $\mathbb{R}^3$. If we substitute into the Bogomol'nyi equations (5.32) and use, on the one hand, that they are satisfied by the pair $A_{\text{WY}}, \Phi_{\text{WY}}$, and, on the other hand, that U is covariantly constant with the connection A_{WY} we arrive at the Dirac monopole equation

$$dC = \star_{(3)} dH\,. \tag{5.123}$$

The integrability condition of this equation is $d\star_{(3)}dH = 0$ so H is any harmonic function. We can choose it to have isolated poles at the points $x^m = x_i^m$ $i = 1, \cdots, N$

$$H = \sum_i \frac{p_i}{2u_i}\,, \qquad u_i^m \equiv x^m - x_i^m\,, \tag{5.124}$$

in which case C is the 1-form potential of N Dirac monopoles with charges p_i which can be constructed by summing over the potentials of each individual monopole:

$$C = \sum C_i\,, \qquad dC_i = \star_{(3)} d\frac{p_i}{2u_i}\,. \tag{5.125}$$

The expression for each of the C_i is of the form Eq. (C.11) where we can, in principle, choose the direction s_i^m of each Dirac string independently:

[20] One can see fairly easily that in the limiting solution one can, as far as the Bogomol'nyi equations are concerned, allow for μ to be negative; for finite values of s this is impossible.

$$C_i = \frac{p_i}{2}\left(1-\frac{s_i^m}{s_i}\frac{u_i^m}{u_i}\right)^{-1}\varepsilon_{mnp}\frac{s_i^m}{s_i}\frac{u_i^m}{u_i}d\frac{u_i^p}{u_i}\,, \qquad \text{(no sum over } i). \tag{5.126}$$

This solution of the Yang-Mills-Higgs system shares two important properties with the original Wu-Yang monopole and which are related to the fact that they are related to Abelian embeddings by singular gauge transformations:

1. Both Φ and $D\Phi$ are proportional to U:

$$\Phi = \left(-\frac{\mu}{2g}+\frac{1}{2gr}+H\right)U\,, \qquad D\Phi = d\left(-\frac{\mu}{2g}+\frac{1}{2gr}+H\right)U\,, \tag{5.127}$$

   and, therefore, commute with each other, so the Higgs current vanishes and the gauge field is, by itself, a solution of the pure Yang-Mills theory.

2. The gauge field strength is also proportional to U, the coefficient being the field strength of an Abelian gauge field:

$$F(A) = d(B+C)U\,, \tag{5.128}$$

   which implies that the energy-momentum tensors are related as in the single-center case.

These solutions can be generalized even further, by allowing the the charge of the "original" Wu-Yang monopole at $r=0$ to be n/g (that is: using the generalization of the Wu-Yang monopole due to Bais [22] which is studied in Appendix C.4). If we now substitute into the Bogomol'nyi equations (5.32) the ansatz

$$\Phi = \Phi_{(n)} + HU_{(n)}\,, \qquad A = A_{(n)} + CU_{(n)}\,, \tag{5.129}$$

where $U_{(n)}, A_{(n)}$ and $\Phi_{(n)}$ are given, respectively, in Eqs. (C.28),(C.29) and (C.34), H is a function and C a 1-form on $\mathbb{R}^3$, and use that they are satisfied by the pair $A_{(n)}, \Phi_{(n)}$ and that $U_{(n)}$ is covariantly constant with the connection $A_{(n)}$, we arrive again at the Dirac monopole equation (5.123).

Since all these solutions are related to Abelian embeddings, they contribute to the black-hole solutions as the Abelian solutions. We will not consider them in what follows.

5.3.2 Embedding in the SU(2)-gauged $\overline{\mathbb{CP}}^3$ model

We can use the Cherkis & Durcan solution of the SU(2) Bogomol'nyi equations reviewed in the previous section as a seed solution for a multicenter solution of $\mathcal{N}=2$, $d=4$ SEYM, adding the same harmonic functions as in the single-center case $(\mathcal{I}^0, \mathcal{I}_0)$ or a generalization with poles at the locations of the monopoles $r=0$[21] and $u=0$. More explicitly, we take

$$\begin{aligned}
\mathcal{I}^0 &= A^0 + \frac{p^0_r/\sqrt{2}}{r} + \frac{p^0_u/\sqrt{2}}{u}\,,\\
\mathcal{I}_0 &= A_0 + \frac{q_{r,0}/\sqrt{2}}{r} + \frac{q_{u,0}/\sqrt{2}}{u}\,,\\
\mathcal{I}^i &= \mp\sqrt{2}\Phi^i(r,u)\,,\\
\mathcal{I}_i &= 0\,,
\end{aligned} \tag{5.130}$$

where $\Phi^i(r,u)$ is the Higgs field of the Cherkis & Durcan solution. The metric and scalar fields take the form

$$e^{-2U} = \tfrac{1}{2}(\mathcal{I}^0)^2 + 2(\mathcal{I}_0)^2 - \Phi^i\Phi^i\,, \tag{5.131}$$

$$Z^i = \frac{\mp\sqrt{2}\Phi^i}{\mathcal{I}^0 + 2i\mathcal{I}_0}\,. \tag{5.132}$$

The normalization of the metric and scalars at infinity leads to the same relations between the integration constants A^0, A_0, μ and the physical constants Z_∞, g as in the single-center case, namely

$$\tfrac{1}{\sqrt{2}}A^0 + \sqrt{2}iA_0 = \frac{Z^*_\infty}{|Z_\infty|}\frac{1}{\sqrt{1-|Z_\infty|^2}}\,, \qquad \mu = \frac{|Z_\infty|}{\sqrt{1-|Z_\infty|^2}}g\,. \tag{5.133}$$

The integrability conditions of Eq. (5.21) are, in this case,

$$\mathcal{I}_0\partial_{\underline{m}}\partial_{\underline{m}}\mathcal{I}^0 - \mathcal{I}^0\partial_{\underline{m}}\partial_{\underline{m}}\mathcal{I}_0 = 0\,, \tag{5.134}$$

and lead to the following relations between the integration constants:

$$A^0(q_{r,0}+q_{u,0}) - A_0(p^0_r + p^0_u) = 0\,, \tag{5.135}$$

$$J - \tfrac{1}{\sqrt{2}}d(A^0 q_{u,0} - A_0 p^0_u) = 0\,, \tag{5.136}$$

where we have defined the constant

$$J \equiv p^0_r q_{u,0} - q_{r,0}p^0_u\,. \tag{5.137}$$

The first equation is equivalent to Eq. (5.59) for the total charges and the second equation determines the relative distance d in terms of J and $A^0 q_{u,0} - A_0 p^0_u$ provided that $J \neq 0$. When that is the case, the solution is not static and has an angular momentum J directed along the line that joins the monopoles $J^m = Jd^m/d$. The corresponding 1-form

[21] The location of the BPS 't Hooft-Polyakov anti-monopole is not completely clear: it is sometimes argued that the center of the monopole is the point at which the Higgs vanishes and the full gauge symmetry is restored. As we have discussed, that point is not $r=0$. We could try to place the poles of the harmonic functions at that point, but, given that its location is not known analytically and the expansion of $\Phi^a\Phi^a$ around it is difficult to compute, we will not try to do that here.

ω can be constructed by the standard procedure of the Abelian case. However, since this complicates the analysis of the regularity of the solutions, we will stick to the static case and require $J=0$.

In order to have regular solutions, the charges at each center must be chosen as in the corresponding single-center case: since there is an Abelian monopole at $u=0$, we must switch on either p_u^0 or $q_{u,0}$ to have a regular horizon there. We can treat them both as non-vanishing with no loss of generality. Then, there are two possibilities:

I. $p_r^0 = q_{r,0} = 0$**:** Only for $s=0$ ('t Hooft-Polyakov anti-monopole at $r=0$) has the solution a chance of being regular at $r=0$. Solving Eq. (5.135) the solution can be written in the form

$$e^{-2U} = \frac{1}{1-|Z_\infty|^2} H^2 - \Phi^i\Phi^i \,, \tag{5.138}$$

$$Z^i = \frac{2\beta}{p^0+2iq_0}\frac{\Phi^i}{H} \,, \tag{5.139}$$

where H is the harmonic function

$$H \equiv 1 + \frac{\beta}{u} \,, \qquad \beta^2 = (1-|Z_\infty|^2) W_{\mathrm{RN}}(\mathcal{Q}_u)/2 \,, \qquad W_{\mathrm{RN}}(\mathcal{Q}_u) \equiv \tfrac{1}{2}(p_u^0)^2 + 2(q_{u,0})^2 \,. \tag{5.140}$$

The free parameters of this solution are the charges p_u^0, $q_{u,0}$ and the single modulus $|Z_\infty|$.

Studying the $u\to 0$ limit we find a black hole with entropy

$$S_u/\pi = \tfrac{1}{2} W_{\mathrm{RN}}(\mathcal{Q}_u) - \frac{1}{g^2} \,, \tag{5.141}$$

as in the corresponding single-center case.

In the $r \to 0$ limit e^{-2U} is constant. The positivity of the constant is guaranteed if S_u is positive. The total entropy of the solution is just the entropy of the black hole at $u=0$ and the Dirac monopole does contribute to it.

The mass of the solution, expressed in terms of the independent parameters of the solution, p_u^0, $q_{u,0}$ and $|Z_\infty|$ takes the form

$$M = M_r + M_u \,, \tag{5.142}$$

$$M_r = -M_{\mathrm{monopole}} \,, \tag{5.143}$$

$$M_u = \sqrt{\tfrac{1}{2}\frac{W_{RN}(\mathcal{Q}_u)}{1-|Z_\infty|^2}} + M_{\mathrm{monopole}} \,, \tag{5.144}$$

where M_{monopole} is given by Eq. (5.66). The contributions of the monopole and the 't Hooft-Polyakov monopole to the mass cancel each other.

II. p_r^0 **or** $q_{r,0} \neq 0$ We can treat both charges as non-vanishing with no loss of generality. Solving Eqs. (5.135) and (5.137), we can write the solution as in Eqs. (5.138) and

(5.139) where, now,

$$\begin{aligned} H &\equiv 1+\frac{\beta_r}{r}+\frac{\beta_u}{u}\,, \qquad \beta^2_{r,u}=(1-|Z_\infty|^2)W_{\rm RN}(\mathcal{Q}_{r,u})/2\,, \\ W_{\rm RN}(\mathcal{Q}_{r,u}) &\equiv \tfrac{1}{2}(p^0_{r,u})^2+2(q_{r,u,0})^2\,. \end{aligned} \tag{5.145}$$

The free parameters of this solution are the charges p^0_u, $q_{u,0}$ and $|Z_\infty|$ and either p^0_r or $q_{r,0}$, since they must be proportional to those of the other center. The areas of each of the horizons are as in the single-center case. In particular, the BPS 't Hooft-Polyakov monopole ($s=0$) does not contribute to the entropy of the $r=0$ center. The mass is given by

$$M = M_r+M_u\,, \tag{5.146}$$

$$M_r = \sqrt{\tfrac{1}{2}\frac{W_{RN}(\mathcal{Q}_r)}{1-|Z_\infty|^2}}-M_{\rm monopole}\,, \tag{5.147}$$

$$M_u = \sqrt{\tfrac{1}{2}\frac{W_{RN}(\mathcal{Q}_u)}{1-|Z_\infty|^2}}+M_{\rm monopole}\,, \tag{5.148}$$

and the contributions of the monopole and anti-monopole cancel each other. In the $s\to\infty$ limit it can be easily seen that the solution is completely regular everywhere (e^{-2U} only vanishes at $r=0$ and $u=0$) if the Abelian charges as chosen so that the horizons are regular. This guarantees that all the terms in e^{-2U} are positive. For finite s this is more difficult to proof analytically, but, since the Higgs field has a better behavior than in the $s\to\infty$ case, it is reasonable to expect that it will also be true. We have checked numerically that this is so in several examples.

5.3.3 Embedding in the SU(2)-gauged ST[2,4] model

The metric and scalar fields of the solution are now given by

$$e^{-2U} = 2\sqrt{-2\mathcal{I}^1\mathcal{I}_0[(\mathcal{I}^2)^2-2\Phi^a\Phi^a]}\,, \tag{5.149}$$

$$Z^1 \equiv \tau = i\frac{e^{-2U}}{2[(\mathcal{I}^2)^2-2\Phi^a\Phi^a]}\,, \qquad Z^2=\frac{\mathcal{I}^2}{\mathcal{I}^1}\tau\,, \qquad Z^a=\frac{\sqrt{2}\Phi^a}{\mathcal{I}^1}\tau\,, \tag{5.150}$$

where Φ^a is the Higgs field of the Cherkis & Durcan solution (deformed with the Protogenov hair parameter s) and where the harmonic functions $\mathcal{I}^1$, $\mathcal{I}^2$ and $\mathcal{I}_0$ are allowed to have poles at $r=0$ and $u=0$:

$$\begin{aligned} \mathcal{I}^1 &= A^1+\frac{p^1_r/\sqrt{2}}{r}+\frac{p^1_u/\sqrt{2}}{u}\,, \qquad \mathcal{I}^2 = A^2+\frac{p^2_r/\sqrt{2}}{r}+\frac{p^2_u/\sqrt{2}}{u}\,, \\ \mathcal{I}_0 &= A_0+\frac{q_{r,0}/\sqrt{2}}{r}+\frac{q_{u,0}/\sqrt{2}}{u}\,. \end{aligned} \tag{5.151}$$

As in the $\overline{\mathbb{CP}}^3$ case, the Abelian charges at each center must be chosen with the same criteria as in the corresponding single-center case. This means, in particular, that the Abelian charges at $u=0$, $p_u^1, q_{u,0}$ must be non-vanishing. p_u^2 may need to be activated, depending on the branch we are considering. At $r=0$, for $s\neq 0$ we get exactly the same possibilities, but, for $s=0$ there are two possibilities:

1. p_r^1, $q_{r,0}, p_r^2$ non-vanishing. We find a black hole at $r=0$ in the + branch.
2. $p_r^1 = q_{r,0} = p_r^2 = 0$. e^{-2U} is a complicated d-dependent constant in the $r=0$ limit and we get a global monopole.

Here we find an important difference with the single-center case, due to the fact that $\Phi^a\Phi^a$ is a finite constant in the $r\to 0$ limit instead of going to zero as r^2: there is no solution with $p_r^1 q_{r,0}\neq 0$ and $p_r^2=0$. In order to have such a global monopole solution with $p^1 q_0 \neq 0$ and $p^2=0$ in equilibrium with the monopole at $u=0$ one may try to place those charges at the point at which $\Phi^a\Phi^a=0$, but the resulting solution may not be well defined there because the limit of the metric function depends on the direction from which we approach that point.

The entropy of the solution is the sum of the entropies of both centers (vanishing for global monopoles). As in the $\overline{\mathbb{CP}}^3$ case, the monopole at each center does contribute to the center entropy (except for global monopoles). The contributions of the monopole and anti-monopole to the mass cancel each other:

$$M = \tfrac{1}{4}\frac{\chi_\infty}{|\Im\mathfrak{m}\tau_\infty|}|p_u^1+p_r^1| + \frac{1}{2\chi_\infty}|q_{u,0}+q_{r,0}| \pm \tfrac{1}{2}\frac{|\Im\mathfrak{m}\tau_\infty \Im\mathfrak{m}\, Z_\infty^2|}{\chi_\infty}|p_u^2+p_r^2|\,. \tag{5.152}$$

5.4 Conclusions

In this chapter we have discussed the construction of supersymmetric multi-object solutions in $\mathcal{N}=2$, $d=4$ EYM theories, specifically in the so-called $\overline{\mathbb{CP}}^{n\geq 3}$ and ST$[2,n]$ models. These models were chosen due to their workability, the fact that they allow for a SU(2) gauging and (in the second case) for their stringy origin. Starting with a deformation of the solutions to the SU(2) Bogomol'nyi equation found by Cherkis and Durcan that adds to the 't Hooft-Polyakov monopole *Protogenov hair*, we have been able to construct *bona fide* two-center solutions. These solutions describe a Dirac monopole embedded in SU(2) in the presence of either a global monopole (the supergravity solution corresponding to the 't Hooft-Polyakov monopole) or a non-Abelian black hole (a supergravity solution with an 't Hooft-Polyakov-Protogenov monopole). In order to make the comparison with the single-object case easier, we included a detailed discussion of the embeddings of the spherically symmetric solutions to the SU(2) Bogomol'nyi equations into the two models, and expressed the whole solution in terms of charges and moduli of the physical fields.

The constructed solutions are all static. It would be very interesting to study dyonic solutions and to see how this interplays with the Denef constraint; the stumbling block in this respect is not so much the Bogomol'nyi equation as the equation (5.20); for the moment the only general solution we know of is to take $\mathcal{I}_\Lambda \sim \mathcal{I}^\Lambda$ in the gauged directions, but this automatically solves the Denef constraint. The only case for which we can find non-trivial dyonic solutions is for the multi-Wu-Yang solutions, or if you like the $s\to\infty$ limit of the deformed Cherkis and Durcan's solution; we refrain from discussing these solutions here as, due to gauge invariance, even taking into account the singular gauge transformation, the restriction coming from the Denef constraint is basically the one corresponding to the Abelian theory.

A natural question that follows from the results presented here and in Refs. [244, 245, 319] is whether we could use a charge k SU(2) monopole to construct globally regular solutions; the answer is yes: observe that the construction of globally regular solutions in section (5.2) hinges exclusively but crucially on the fact that the used monopole solution is regular and is such that $\Phi^a\Phi^a \leq \lim_{|\vec{x}|\to\infty}\Phi^a\Phi^a$. A charge-$k$ monopole may be rather difficult to construct but the regularity is guaranteed and also the last needed ingredient is known to be satisfied: indeed, using the Bogomol'nyi equation (5.32) one can show that

$$\partial_{\underline{m}}\partial_{\underline{m}}\ \Phi^a\Phi^a \ = \ F^a{}_{\underline{mm}}F^a{}_{\underline{mm}} \ \geq \ 0 \ . \tag{5.153}$$

This equation together with the Hopf maximum principle and the regularity, implies that the function $\Phi^a\Phi^a$ is bounded from above by its value on the sphere at infinity, which is exactly what one needs.

As was said in the introduction, the creation and study of non-Abelian solutions to $d=4$ supergravity theories is in its infancy and this holds doubly so for the higher dimensional theories. One possible reason is that the structure of supersymmetric solutions to higher supergravities (see *e.g.* Refs. [43, 114]) is more entangled than the one given in the recipe in section 5.1.2. For example, naively one would expect that Kronheimer's link of monopoles on $\mathbb{R}^3$ to instantons on GH-spaces, would carry over to the supersymmetric solutions as in $d=4$ the base space is $\mathbb{R}^3$ and that in $d=5$ must be hyper-Kähler; *i.e.* one would expect the instanton equation to show up in the recipe for cooking up 5-dimensional supersymmetric solutions. Perhaps it does, but it definitely is not obvious where and how it is making its appearance in such a clear-cut manner as in $d=4$.

The 4- and 5-dimensional EYMH theories are, however, related by dimensional reduction/oxidation, whence the solutions to the cubic models presented in here could be oxidized to 5-dimensions and can be studied with the hope of unraveling the structure of 5-dimensional supersymmetric solutions.

6

Non-extremal branes

This chapter is based on

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Supergravity branes have played a rôle of outermost importance in String Theory since they were discovered to be the macroscopic counterparts of many String Theory microscopic extended objects, during the second String Revolution [367]. However, strictly speaking, this correspondence is limited to the extremal cases, which have been thoroughly studied in the literature. Much less attention has been paid to non-extremal Supergravity branes (which are regular in general, in contrast to the extremal ones), since they do not obey first order differential equations and its String Theory interpretation is less clear. In this chapter we are interested in further understanding the structure of general non-extremal Supergravity branes and its behaviour under *electric-magnetic* duality.

In reference [153], a generalization of the FGK-formalism [173] to an arbitrary number of space-time dimensions d and worldvolume dimensions (p+1) was presented. The d-dimensional class of theories considered in [153] describes gravity coupled to a given number of scalars $\phi^i\,, i = 1, \ldots, n_\phi$, and (p+1)-forms $A^\Lambda_{(\mathrm{p}+1)}\,, \Lambda = 1, \ldots, n_A$, and are given by the following, two-derivative, action

$$S = \int d^dx\sqrt{|\mathrm{g}|}\left\{R + \mathcal{G}_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j + 4\tfrac{(-1)^{\mathrm{p}}}{(\mathrm{p}+2)!}I_{\Lambda\Omega}(\phi)F^\Lambda_{(\mathrm{p}+2)}\cdot F^\Omega_{(\mathrm{p}+2)}\right\}\,, \tag{6.1}$$

where $F^\Lambda_{(\mathrm{p}+2)} = (\mathrm{p}+2)dA^\Lambda_{(\mathrm{p}+1)}$ are the (p+2)-form field strengths and the scalar dependent, negative definite, matrix $I_{\Lambda\Omega}(\phi)$ describes the couplings of scalars ϕ^i to the $(\mathrm{p}+1)$-forms $A^\Lambda_{(\mathrm{p}+1)}$. The generic space-time metric considered in [153] was

$$ds^2_{(d)} = e^{\frac{2}{\mathrm{p}+1}U}\left[W^{\frac{\mathrm{p}}{\mathrm{p}+1}}dt^2 - W^{-\frac{1}{\mathrm{p}+1}}d\vec{z}^{\,2}_{(\mathrm{p})}\right] - e^{-\frac{2}{\tilde{\mathrm{p}}+1}U}\gamma_{(\tilde{\mathrm{p}}+3)}\,, \tag{6.2}$$

$$\gamma_{(\tilde{\mathrm{p}}+3)} = \mathcal{X}^{\frac{2}{\tilde{\mathrm{p}}+1}}\left[\mathcal{X}^2\frac{d\rho^2}{(\tilde{\mathrm{p}}+1)^2} + d\Omega^2_{(\tilde{\mathrm{p}}+2)}\right]\,, \tag{6.3}$$

where $\mathcal{X} \equiv \left(\frac{\omega/2}{\sinh\left(\frac{\omega}{2}\rho\right)}\right)$, $\vec{z}_{(\mathrm{p})} \equiv \left(z^1, \ldots, z^{\mathrm{p}}\right)$ are spatial worldvolume coordinates and $d = \mathrm{p} + \tilde{\mathrm{p}} + 4$ so $\tilde{\mathrm{p}}$ is the number of spatial dimensions of the dual brane. $d\Omega^2_{(\tilde{\mathrm{p}}+2)}$ stands for the round metric on the $(\tilde{\mathrm{p}}+2)$-sphere of unit radius, and ω is a constant that corresponds to the *non-extremality* parameter of the black-brane solution. In other words, the black-brane is extremal if and only if $\omega = 0$.

Assuming the space-time background (6.2) and that all the fields of the theory depend exclusively on the radial coordinate ρ, the equations of motion of (6.1) are equivalent to the following set of ordinary differential equations [153]

$$\ddot{U} + e^{2U} V_{\rm BB} \;=\; 0\,, \tag{6.4}$$

$$\ddot{\phi}^i + \Gamma_{jk}{}^i \dot{\phi}^j \dot{\phi}^k + \tfrac{d-2}{2(\tilde{\rm p}+1)({\rm p}+1)} e^{2U} \partial^i V_{\rm BB} \;=\; 0\,, \tag{6.5}$$

$$(\dot{U})^2 + \tfrac{({\rm p}+1)(\tilde{\rm p}+1)}{d-2} \mathcal{G}_{ij} \dot{\phi}^i \dot{\phi}^j + e^{2U} V_{\rm BB} \;=\; c^2\,, \tag{6.6}$$

where $V_{\rm BB}$ stands for the so-called *black-brane* potential

$$V_{\rm BB}(\phi, q) \equiv 2\alpha^2 \frac{2({\rm p}+1)(\tilde{\rm p}+1)}{(d-2)} \left(I^{-1}\right)^{\Lambda\Omega} q_\Lambda q_\Omega\,, \tag{6.7}$$

and c^2 is a real semi-definite positive constant given by

$$c^2 \equiv \frac{({\rm p}+1)(\tilde{\rm p}+2)}{4(d-2)}\omega^2 - \frac{(\tilde{\rm p}+1){\rm p}}{4(d-2)}\gamma^2\,, \tag{6.8}$$

and γ is another constant whose origin will be clear in a moment. Notice that the system of differential equations above only involves the metric factor U and the scalar fields ϕ^i, since the (p+1)-forms can be eliminated in terms of the corresponding charges $q_\Lambda\,, \Lambda = 1, \ldots, n_A$, by explicitly integrating the Maxwell equations.

Remarkably enough, it turns out that W can also be explicitly integrated yielding

$$W = e^{\gamma\rho}\,, \tag{6.9}$$

where γ is the (integration) constant which appears in (6.8).

In [153] it was argued that in order to have a regular black-brane solution, we must have [1] $\gamma = \omega$ and therefore $c^2 = \frac{\omega^2}{4}$.

To sum up, in reference [153] it was found that the above ansatz corresponds to a black-brane solution (not necessarily regular) of the theories defined by the generic action (6.1) if equations (7.68), (7.72) and (6.6) are satisfied.

6.1 Electromagnetic dual solutions

It can be seen that the FGK system of equations is completely fixed once we know the following data: the Riemannian metric $\mathcal{G}_{ij}$ of the non-linear sigma model, the number p of spatial dimensions of the brane and the matrix $I_{\Lambda\Omega}$ describing the couplings of the scalars and the (p+1)-forms. Actually, the FGK-system is invariant under the interchange

$${\rm p} \leftrightarrow \tilde{\rm p}\,, \tag{6.10}$$

which however does not leave invariant the space-time metric, which represents now the metric of a $\tilde{\rm p}$ brane. A $\tilde{\rm p}$ brane naturally couples to a $(\tilde{\rm p}+1)$-form, that is, to the

[1] In the ansatz at hand, the event horizon (if any) will correspond to $\rho \to +\infty$, whereas spatial infinity will be at $\rho \to 0^+$. In order for the worldvolume metric to be regular in the near horizon limit, $e^U \propto e^{\frac{\omega\rho}{2}}$ and $W \sim e^{\omega\rho}$, which fixes $\gamma = \omega$.

magnetic duals of the electric (p+1)-forms $A^{\Lambda}_{(\mathrm{p}+1)}$. Therefore, in order to properly perform the interchange (6.10) we also have to change the electric matrix I_{el} of couplings to the magnetic I_{mag} one. Schematically the transformation is

$$\mathrm{p} \leftrightarrow \tilde{\mathrm{p}}\,, \qquad I_{el} \leftrightarrow I_{mag}\,. \tag{6.11}$$

The only term in the FGK-system that depends on $I_{\Lambda\Omega}$ is the black-brane potential V_{BB}. Therefore, if

$$\left(I^{-1}\right)^{\Lambda\Omega}_{el} q_{\Lambda} q_{\Omega} = \left(I^{-1}\right)^{\Lambda\Omega}_{mag} q'_{\Lambda} q'_{\Omega}\,, \tag{6.12}$$

where $q'_{\Lambda} = A^{\Omega}_{\Lambda} q_{\Omega},\ A \in \mathrm{Gl}(n_A, \mathbb{R})$, then the FGK-system is invariant under the transformation (6.11), up to a redefinition of the charges, and therefore with the same solution of the FGK-system we can construct two space-time solutions, the electric-brane solution and the magnetic-brane solution. In order to see when condition (6.12) holds, we have to change from electric variables $A^{\Lambda}_{(\mathrm{p}+1)}$ to the magnetic ones $\tilde{A}_{(\tilde{\mathrm{p}}+1)\Lambda}$ in the action (6.1). The equations of motion and the Bianchi identities for the electric fields $A^{\Lambda}_{(\mathrm{p}+1)}$ are

$$d\left(I_{\Lambda\Omega} * F^{\Omega}_{(\mathrm{p}+2)}\right) = 0\,, \qquad dF^{\Lambda}_{(\mathrm{p}+2)} = 0\,. \tag{6.13}$$

Now we define

$$G_{(\tilde{\mathrm{p}}+2)\Lambda} = I_{\Lambda\Omega} * F^{\Omega}_{(\mathrm{p}+2)}\,. \tag{6.14}$$

and thus the equations of motion for the electric vector fields can be written as a Bianchi identity for $G_{(\tilde{\mathrm{p}}+2)\Lambda}$

$$dG_{(\tilde{\mathrm{p}}+2)\Lambda} = 0 \Rightarrow G_{(\tilde{\mathrm{p}}+2)\Lambda} = d\tilde{A}_{(\tilde{\mathrm{p}}+1)\Lambda} \ \text{ locally}\,. \tag{6.15}$$

Equation (6.14) can be inverted as follows

$$F^{\Lambda}_{(\mathrm{p}+2)} = (-1)^{(d-1)+(\mathrm{p}+2)(\tilde{\mathrm{p}}+2)} \left(I^{-1}\right)^{\Lambda\Omega} * G_{(\tilde{\mathrm{p}}+2)\Omega} \tag{6.16}$$

Substituting equation (6.16) in equation (6.1), we deduce that

$$I_{mag} = I^{-1}_{el}\,. \tag{6.17}$$

Given equation (6.17) and equation (6.12) we obtain that a sufficient condition to obtain the same FGK-system for electric and magnetic branes is that there exists a matrix $A \in \mathrm{Gl}(n_A, \mathbb{R})$ such that the following *self-duality* condition holds

$$I^{-1} = A I A^T\,. \tag{6.18}$$

Without invoking supersymmetry we can say little more beyond equation (6.18), since the couplings in the action (6.1) are in principle arbitrary aside from some regularity conditions. Supersymmetry, however, constrains the couplings and therefore it is easier to analyze when equation (6.18) is satisfied.

Supergravity non-linear sigma models are constrainted by supersymmetry and related to the couplings of the (p+1)-forms and the scalars of the theory. Let us now consider the general situation of an extended ungauged Supergravity, where the scalar manifold is a homogeneous space of the form

$$\mathcal{M}_S = \frac{G}{H}\,, \tag{6.19}$$

and the matrix I of the couplings between the (p+1)-forms and the scalars is a coset representative, namely $I \in \frac{G}{H}$. The coset element I must be taken in a particular representation, namely I is in the representation $R(G)$ that acts on the charges of the corresponding electric p-forms of the theory. This is the standard situation happening in an extended Supergravity in diverse dimensions. From the self-duality condition (6.18) we are interested in coset representatives I such that there exists a matrix $A \in Gl(n_A, \mathbb{R})$ satisfying

$$I^{-1} = AIA^T \,. \tag{6.20}$$

There is a sufficient condition on G such that the self-duality condition (6.20) is implied. Let us assume that the Lie group leaves invariant a bilinear form $\mathcal{B} \in V^* \otimes V^*$, where V is the n_A-dimensional representation vector space of G, or in other words, $q_\Lambda \in V$. The condition of G leaving invariant $\mathcal{B}$ can be rewritten as follows

$$R^T \mathcal{B} R = \mathcal{B} \,, \qquad R \in R(G) \,, \tag{6.21}$$

where $R(G)$ is the corresponding representation of G as automorphisms of V. Now, the self-duality condition does not have to be satisfied by an arbitrary element in G but for an element in G/H which, in the representation $R(G)$ must be symmetric in order to be an admissible I [2]. Assuming then that $R^T = R$ we can rewrite (6.21) as follows

$$R^{-1} = \mathcal{B}^{-1} R \mathcal{B} \,, \qquad R \in R(G) \,, \tag{6.22}$$

and therefore if

$$\mathcal{B}^T = \mathcal{B}^{-1} \,, \tag{6.23}$$

then equation (6.20) is satisfied and the corresponding FGK model is self-dual, meaning that the system of differential equations to be solved for the electric p-brane and the corresponding magnetic $\tilde{\text{p}}$-brane is exactly the same.

There are several Supergravities where condition (6.23) holds. Just to name a few: Type-IIB Supergravity, where $G = \mathrm{Sl}(2, \mathbb{R})$, $H = \mathrm{SO}(2)$ so $\mathcal{B} = \mathrm{antidiag}(1, -1)$; nine-dimensional $\mathcal{N} = 2$ Supergravity, where $G = \mathrm{Sl}(2, \mathbb{R}) \times \mathrm{O}(1,1)$ and $H = \mathrm{O}(2)$, quotienting only the first factor and $\mathcal{B} = \mathrm{antidiag}(1, -1) \times \mathrm{diag}(1, -1)$; four-dimensional $\mathcal{N} = 8$ Supergravity, where $G = \mathrm{E}_{7(7)}$ acting on the **56** irrep. on the charges, $H = \mathrm{SU}(8)/\mathbb{Z}_2$ and $\mathcal{B}$ is the symplectic form in the **56**-dimensional vector space; four-dimensional $\mathcal{N} = 6$ Supergravity, with $G = \mathrm{SO}^*(12)$, $H = \mathrm{U}(6)$ and $\mathcal{B}$ is the identity matrix, etc.

6.2 The (p, q)-black-string of Type-IIB Supergravity

Let us see how this works in an particular example, namely the (p, q)-black-strings and (p, q)-5-black branes of Type-IIB Supergravity. First, we will use the effective FGK variables to construct the non-extremal (p, q)-black-string, new in the literature, and then, we will show how in the FGK framework this solution is actually the same as the non-extremal (p, q)-5-black-brane, also new. Before getting started, let us review the basic properties of the extremal (p, q)-string of Schwarz [382].

From the stringy perspective, a (extremal) (p, q)-string is a bound state of Type-IIB String Theory composed of p *D-strings* (*D1s*), charged under the RR two-form $C_{(2)}$,

[2] Even if it is non-symmetric, when contracting with $F^{\Lambda}_{(\text{p}+2)}$ in (6.1) only the symmetric part survives.

and q *fundamental strings* (*F1s*), with charge under the NS-NS two-form B. Type-IIB Supergravity is invariant under a global $\mathrm{SL}(2,\mathbb{R})$ symmetry, so all the states of the theory are accomodated in multiplets of such group. In particular any state can be generated from another one living in the same multiplet by applying a $\mathrm{SL}(2,\mathbb{R})$ transformation. This is the case for the $D1$ and $F1$ solutions, which are related to each other via this IIB S-duality. Similarly, we can generate a (p,q)-string starting from one of them, and performing a general enough $\mathrm{SL}(2,\mathbb{R})$ transformation. This was done for the first time by Schwarz [382], who also gave the corresponding Supergravity version of the solution. In fact, from the Supergravity perspective, all these states correspond to extremal black strings charged under one or both two-forms. All these solutions are nevertheless singular, given that the corresponding black-string singularities are naked. As we will see, this behavior is cured in the non-extremal case, and we will be able to construct a regular non-extremal (p,q)-black-string solution.

The relevant truncated Type-IIB Supergravity Lagrangian is

$$S=\int d^{10}x\sqrt{|\mathrm{g}|}\left[R+\frac{1}{2}\frac{\partial_\mu\tau\partial^\mu\bar{\tau}}{(\Im\mathrm{m}\tau)^2}+\frac{1}{2\cdot 3!}\mathcal{H}^T\mathcal{M}^{-1}\mathcal{H}\right] \tag{6.24}$$

where $\mathcal{H}\equiv d\mathfrak{B}$, with $\mathfrak{B}^T\equiv\left(C_{(2)},B\right)$ and $\mathcal{M}\equiv\frac{1}{\Im\mathfrak{m}\tau}\begin{pmatrix}|\tau|^2 & \Re\mathfrak{e}\tau\\ \Re\mathfrak{e}\tau & 1\end{pmatrix}$ with $\Im\mathfrak{m}\tau>0$ is the coset representative of the space $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ parametrized by the axidilaton $\tau\equiv C_{(0)}+ie^{-\Phi}$. Since black strings in ten dimensions have $\mathrm{p}=1$ and $\tilde{\mathrm{p}}=5$, let us set [3]

$$d=10\,,\quad \mathrm{p}=1\,,\quad \tilde{\mathrm{p}}=5 \tag{6.25}$$

in the FGK effective action (6.1). Now, the key point to notice is that the action (6.24) is a particular case of (6.1), by taking $n_\phi=2\,,n_A=2$ and making the following identifications

$$\phi^1=C_{(0)}\,,\;\phi^2=e^{-\Phi}\,,\;\mathcal{G}_{ij}=e^{2\Phi}\frac{\delta_{ij}}{2}\,,\;I(\phi)\equiv-\frac{1}{8}\mathcal{M}^{-1}\,, \tag{6.26}$$

where $i,j=1,2$ and $\tau=C_{(0)}+ie^{-\Phi}$. We thus obtain that the black-brane potential for this truncation of Type-IIB Supergravity is given by

$$-V_{\mathrm{BB}}(\phi,q)=\mathcal{M}^{\Lambda\Omega}q_\Lambda q_\Omega=e^{\Phi}\left(|\tau|^2p^2+q^2+2pqC_{(0)}\right)\,, \tag{6.27}$$

where $\Lambda,\Omega=1,2$ and we have defined $\alpha^2=\frac{1}{2^4\cdot 3}$ and $q_1\equiv p$, $q_2\equiv q$. Therefore, in order to obtain the black-string solutions of the theory (6.24) we *just* have to solve the system of ordinary differential equations given by (7.68), (7.72) and (6.6) assuming equations (6.25), (6.26) and (6.27). Notice that $\mathcal{M}$ is definite positive and therefore $V_{\mathrm{BB}}(\phi,q)$ in (6.27) is negative definite.

In reference [153], it was shown that for regular extremal black-brane solutions, the value ϕ_H of the scalars at the black-brane horizon obeys

$$\partial_i V_{\mathrm{BB}}(\phi_H,q)=0\,,\qquad i=1,\ldots,n_\phi\,. \tag{6.28}$$

The solutions ϕ_H of equation (6.28) are the so-called black-brane attractors, and generalize to black-brane solutions the popular concept of black-hole attractor. Notice that equation

[3]There should be no confusion about the p that denotes the number of spatial dimensions of a given brane and the p in the (p,q)-strings, which corresponds to its charge under $C_{(2)}$.

(6.28) completely fixes the value of the scalars at the horizon in terms of the charges, as long as there are no *flat directions*. Taking the black-brane potential as in (6.27), one easily finds that (6.28) has no solutions for the (p,q)-black-string system, meaning that there does not exist any extremal regular black-string solution of Type-IIB Supergravity with non-trivial scalars.

The most general extremal solution of this kind was constructed by Schwarz in [382]. It is given, in standard coordinates by

$$\begin{aligned} ds_E^2 &= H^{-\frac{3}{4}}\left[dt^2 - dz^2\right] - H^{\frac{1}{4}} d\vec{x}^2\,, \\ \mathfrak{B}_{tz} &= \mathfrak{a}\left(H^{-1}-1\right)\,,\; \mathcal{M} = \mathfrak{a}\mathfrak{a}^T H^{-\frac{1}{2}} + \mathfrak{b}\mathfrak{b}^T H^{\frac{1}{2}}\,, \end{aligned} \tag{6.29}$$

where

$$H = 1 + \frac{h}{r^6}\,, \tag{6.30}$$

$r^2 \equiv \vec{x}^2$ and $\mathfrak{a}^T = (a_1, a_2)$ and $\mathfrak{b}^T = (b_1, b_2)$ are two constant vectors to be expressed in terms of the physical parameters of the solution and subject to the constraint $\mathfrak{a}^T \eta \mathfrak{b} = a_1 b_2 - a_2 b_1 = 1$. The relation between $\mathcal{M}$ and H can be inverted to obtain the expression for the axidilaton, which reads

$$\tau = \frac{a_1 a_2 + b_1 b_2 H}{a_2^2 + b_2^2 H} + \frac{i\sqrt{H}}{a_2^2 + b_2^2 H}\,. \tag{6.31}$$

It is not difficult to recover the $D1$ and $F1$ solutions from the (p,q)-black-string one by setting $C_{(0)} = 0$ and $q = 0$ or $p = 0$ respectively in each case.

The standard coordinates can be related to the FGK ones through the change $r = \rho^{-\frac{1}{6}}$. It is straightforward to check that equations (7.68), (7.72) and (6.6) with $c = 0$ are satisfied by Schwarz's (p,q)-black-string (6.29) [4]. We find that the singular extremal (p,q)-black-string can be generalized to a regular non-extremal solution, given by

$$\begin{aligned} ds_E^2 &= H^{-\frac{3}{4}}\left[W dt^2 - dz^2\right] - H^{\frac{1}{4}}\left[W^{-1} dr^2 + r^2 d\Omega_{(7)}^2\right]\,, \\ \mathfrak{B}_{tz} &= \pm\mathfrak{a}\left(H^{-1}-1\right)\,,\quad \tau = \frac{a_1 a_2 + b_1 b_2 H}{a_2^2 + b_2^2 H} + \frac{i\sqrt{H}}{a_2^2 + b_2^2 H}\,, \\ H &= 1 + \frac{h}{r^6}\,,\; W = 1 + \frac{2c}{r^6}\,,\; h = c + \frac{2}{\sqrt{3}}\sqrt{|V_{\mathrm{BB}\infty}| + \frac{3c^2}{4}}\,, \end{aligned} \tag{6.32}$$

$$\begin{aligned} a_1 &= \frac{\left(q\, C_{(0)\infty} + p|\tau_\infty|^2\right) e^{\Phi_\infty}}{\sqrt{|V_{\mathrm{BB}\infty}|}}\,,\; b_1 = -\frac{q}{\sqrt{|V_{\mathrm{BB}\infty}|}}\,, \\ a_2 &= \frac{\left(q + p\, C_{(0)\infty}\right) e^{\Phi_\infty}}{\sqrt{|V_{\mathrm{BB}\infty}|}}\,,\; b_2 = \frac{p}{\sqrt{|V_{\mathrm{BB}\infty}|}}\,, \\ V_{\mathrm{BB}\infty} &\equiv -e^{\Phi_\infty}\left(q^2 + 2pqC_{(0)\infty} + p^2|\tau_\infty|^2\right)\,, \end{aligned}$$

where we have expressed all the parameters of the solution in terms of the corresponding physical quantities (charges $\mathfrak{q}$ and asymptotic values of the axion and dilaton). The FGK

[4] In particular, the relation between $U(\rho)$ and $H(r)$ is given by $H(r)^{-\frac{3}{4}} = e^{U(\rho)}$.

variables in which this solution was obtained are related to the standard ones by the change of variables

$$r^6 = \frac{2c}{e^{2c\rho}-1}\,, \quad H(r)^{-3/4} = e^{U(\rho)}e^{-c\rho}\,. \tag{6.33}$$

It can be easily seen that the general non-extremal solution we have found reduces to all the known solutions, namely, the non-extremal $D1$-brane by taking $C_{(0)} = 0$, $q = 0$; the non-extremal $F1$-string by setting $C_{(0)} = 0$, $p = 0$; and Schwarz's extremal (p, q)-string by taking the $c \to 0$ limit. This non-extremal (p, q)-black-string posseses the same metric as the non-extremal $D1$ and $F1$, and an axidilaton with both real an imaginary parts having the same expression as Schwarz's extremal (p, q)-string (6.29) (although everything depends now also on the non-extremality parameter $c = \omega/2$).

As we explained before, the FGK equations (7.68), (7.72) and (6.6) are blind under electric-magnetic duality for a broad class of bosonic actions. That is indeed the case of the action (6.24). Indeed, all the equations of motion of the FGK-formalism coming from (6.24) are invariant under the interchange $\mathrm{p} \leftrightarrow \tilde{\mathrm{p}}$, $I_{el} \leftrightarrow I_{mag}$. The only subtlety appears in the black-brane potential. Since $\mathcal{M}^{-1} = \eta^T \mathcal{M}\,\eta$, this goes from

$$-V_{BB}^{(C_{(2)},B)} = \mathfrak{q}^T\mathcal{M}\mathfrak{q} = e^{\Phi}\left(|\tau|^2 p^2 + q^2 + 2pqC_{(0)}\right)\,, \tag{6.34}$$

in the *electric* version of the action, to

$$-V_{BB}^{(C_{(6)},B^{(6)})} = \mathfrak{q}_5^T\mathcal{M}\mathfrak{q}_5 = e^{\Phi}\left(|\tau|^2 p_5^2 + q_5^2 + 2p_5q_5C_{(0)}\right)\,, \tag{6.35}$$

in the *magnetic* one, provided that we define the charges $\mathfrak{q}_5$ as

$$\mathfrak{q}_5 = (p_5, q_5)^T \equiv \eta\mathfrak{q} = (q, -p)^T\,, \quad \eta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,. \tag{6.36}$$

Hence, in the effective FGK variables, pairs consisting of a black string and a 5-black-brane solving the equations of motion of the corresponding ten-dimensional action appear as a single solution. This corresponds in general to a black string of charges (p, q) under $(C_{(2)}, B)$ and a 5-black-brane with charges $(q, -p)$ under $(C_{(6)}, B^{(6)})$. Also, the fact that both black-brane potentials are equivalent implies that no regular 5-black-brane extremal objects exist.

The known 5-brane solutions of Type-IIB Supergravity correspond to the non-extremal *D5-brane*, the non-extremal *S5* and the analogous of Schwarz's extremal black-string, the *(p, q)-5-brane* of Lu and Roy [307]. Using the very same solution of the FGK system (6.32) it is straightforward to construct the non-extremal *(p, q)-5-brane*, which can be easily seen to reduce to the known cases just mentioned.

6.3 Double-extremal black-branes

As we have explained, there is a black-brane attractor mechanism at work for extremal black-branes ($\omega = 0$), which fixes the scalars at the horizon as the critical points ϕ_H of the black-brane potential. Indeed, assuming regularity of the scalars at the horizon as well as a regular Riemannian scalar metric, the value of the scalars at the horizon ϕ_H for an extremal black-brane solution satisfies (6.28). We will use now the FGK-formalism for black-branes to prove the existence of a universal [5] black-brane solution with constant scalars, and a

[5] In the sense that it will have the same expression for any theory of the form (6.1).

universal near-horizon behaviour, if condition (6.28) is satisfied. In this case, however, such condition will appear as a constraint from imposing the scalars to be constant (often refered to as *double-extremality*) and not from requiring the non-extremality parameter c to vanish. Indeed, for constant scalars, the FGK system of equations reduces to

$$\ddot{U} + e^{2U} V_{\rm BB} = 0 \,, \tag{6.37}$$
$$\partial_i V_{\rm BB} = 0 \,, \tag{6.38}$$
$$(\dot{U})^2 + e^{2U} V_{\rm BB} = c^2 \,. \tag{6.39}$$

Note that equations (6.37), (6.38) and (6.39) do not depend on the number p of spatial dimensions of the brane. Notice also that $V_{\rm BB}(q)$ will be now a constant constructed from the product of the constant $n_A \times n_A$ kinetic matrix $\left(I^{-1}\right)^{\Lambda\Omega}$ and the charge vectors q_Λ, see (6.7). Thus, a double-extremal black brane will in general be charged under the n_A (p+1)-forms $A^{\Lambda}_{(\rm p+1)}$ present in the theory.

Equation (6.38) can be automatically solved if the black-brane potential has at least one critical point, something that must be analyzed in a case by case basis and that we will assume henceforth. Equation (6.37) is the derivative of equation (6.39), and thus we are left with a single equation. This was to be expected, provided there is only one variable left to be integrated, namely U. Equation (6.39) can be explicitly integrated and the solution is given by

$$e^{-2U} = \frac{|V_{\rm BB}| \sinh^2 (c\rho + s)}{c^2} \,, \tag{6.40}$$

where s is an integration constant. Normalizing the metric to obtain Minkowski space-time at spatial infinity fixes s to be given by

$$s = \operatorname{arcsinh}\left(\frac{c}{\sqrt{|V_{\rm BB}|}}\right) \,. \tag{6.41}$$

Therefore, inserting equation (6.40) into the general metric (6.2) we obtain a complete $(p_1, p_2, ..., p_{n_A})$-p-black-brane solution with constant scalars which solves the theory (6.1). The metric factor e^{-2U} is well defined for $\rho \in [0, +\infty)$ and therefore the solution contains a horizon at $\rho \to +\infty$ and is regular. Taking the extremal limit $c \to 0$ we obtain

$$e^{-2U} = \left(1 + \sqrt{|V_{\rm BB}|}\rho\right)^2 \,, \tag{6.42}$$

which corresponds to a regular extremal universal black-brane solution. We can obtain now the near-horizon geometry of the extremal solution simply by taking the limit $\rho \to +\infty$ in the general extremal metric where now U is given by equation (6.42). Making the change of coordinates $\rho = r^{\rm p+1}$ and relabeling $\vec{z}$ and t we can rewrite the final result as follows

$$\lim_{\rho\to\infty} ds^2_{(d)} = |V_{\rm BB}|^{\frac{1}{\tilde{\rm p}+1}} \left[\frac{(\rm p+1)^2}{(\tilde{\rm p}+1)^2}\frac{1}{r^2}\left[dt^2 - d\vec{z}^{\,2}_{(\rm p)} - dr^2\right] + d\Omega^2_{(\tilde{\rm p}+2)}\right] \,, \tag{6.43}$$

which corresponds to the space $\mathrm{AdS}_{(2+\rm p)} \times S^{\tilde{\rm p}+2}$. Notice that the near-horizon geometry (6.43) is itself a solution of the equations of motion, and corresponds again to a universal solution with constant scalars. Let us remind the reader that in order for both the universal black-brane solution, or the near-horizon solution to exist, the only requirement is that the n_ϕ scalars present in the theory can be consistently chosen to be constant. This is equivalent to requiring the black-brane potential to have a critical point.

A simple case in which we can easily construct the double-extremal solution corresponds to $\mathcal{N}=2$, $d=5$ supergravity coupled to one vector multiplet. A model of this theory gets completely determined by specifying a completely symmetric tensor C_{IJK} (see, e.g. [153], for details), which in this case reads $C_{011}=1/3$. The black-brane potential of the model reads

$$-V_{\rm BB}=\frac{1}{3}\left[\left(p^0\right)^2 e^{-2\sqrt{\frac{2}{3}}\phi}+2\left(p^1\right)^2 e^{\sqrt{\frac{2}{3}}\phi}\right]\,, \tag{6.44}$$

being ϕ the only scalar of the theory, and p^0, p^1 the charges under the 2-forms $B_{0\mu\nu}$ and $B_{1\mu\nu}$ dual to the graviphoton and the 1-form of the vector multiplet respectively [153]. Now, (6.44) has a critical point for

$$\phi_h=\sqrt{\frac{2}{3}}\log\left(\left|\frac{p^0}{p^1}\right|\right)\,, \tag{6.45}$$

at which

$$-V_{\rm BB}(\phi_h,p)=\left[|p^0|(p^1)^2\right]^{2/3}\,. \tag{6.46}$$

Therefore, the double-extremal black string of this model is given by

$$e^{-2U}=\frac{\left[|p^0|(p^1)^2\right]^{2/3}\sinh^2\left(c\rho+s\right)}{c^2}\,, \tag{6.47}$$

with

$$s=\operatorname{arcsinh}\left(\frac{c}{\left[|p^0|(p^1)^2\right]^{1/3}}\right)\,. \tag{6.48}$$

7

Lifshitz-like solutions with hyperscaling violation in supergravity

This chapter is based on

Pablo Bueno, Wissam Chemissany, Patrick Meessen, Tomas Ortín and C. S. Shahbazi,
"Lifshitz-like solutions with hyperscaling violation in ungauged supergravity",
JHEP ***1301*** *(2013) 189.* [arXiv:1209.4047 [hep-th]]. [88].

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Eur. Phys. J. C (2014) ***74:2684****.* [arXiv:1212.4826 [hep-th]]. [87].

7.1 HvLf solutions in ungauged supergravity

The gauge/gravity duality [219, 311, 423] has proven to be an outstandingly successful and fruitful framework for probing the physics of strongly coupled field theories. The paradigmatic AdS/CFT correspondence, which established the physical equivalence between $d = 4$, $\mathcal{N} = 4$ Super-Yang-Mills and type-IIB String Theory on $\mathrm{AdS}_5\times\mathrm{S}^5$ [311] has been extended over the years in a variety of ways in the hope of accounting for the physics of more realistic quantum field theories, such as QCD and condensed matter systems (see, e.g., [1, 116, 229, 369] for reviews on these subjects).

One such extension consists of considering systems in which, albeit scaling symmetry is respected, space and time do not scale in the same way, so conformal (and Lorentz) invariance is broken. This is the case of the so-called Lifshitz fixed points, characterized by a *dynamical critical exponent* z, which determines the anisotropic scaling in the time direction t

$$t \to \lambda^z t\,,\; x_i \to \lambda x_i\,,\; i = 1, ..., d\,, \tag{7.1}$$

being x_i the d spatial dimensions of the $(d+1)$-spacetime in which the field theory under consideration is defined. The class of $(d+2)$-dimensional dual spacetime geometries with the appropriate symmetries can be written, in some coordinate system, as [261, 287, 410]

$$ds^2 = -\frac{L^2}{r^{2z}}dt^2 + \frac{L^2}{r^2}\left[dr^2 + d\vec{x}^2_{(d)}\right]\,, \tag{7.2}$$

which reduces to AdS_{d+2} in the Poincaré patch for $z = 1$. Embedding solutions of this kind (and others which asymptote to them) into gravity and String Theory models and studying their properties in the holographic framework has been subject of study in numerous previous works (see, e.g. [24, 60, 125, 134, 163, 213, 217, 225, 230]), and remains an active area of research.

A further generalization can be achieved by considering the following family of spacetime metrics [131]

$$ds^2 = L^2 r^{\frac{2(\theta-d)}{d}} \left[-r^{-2(z-1)}dt^2 + dr^2 + d\vec{x}^2_{(d)}\right] . \tag{7.3}$$

These geometries (which are conformally Lifshitz) include, in addition to z, another exponent, customarily named θ, and are characterized by the following transformation rules under rescalings of the coordinates

$$t \to \lambda^z t\,,\; x_i \to \lambda x_i\,,\; r \to \lambda r\,,\; ds^2 \to \lambda^{\frac{2\theta}{d}} ds^2\,. \tag{7.4}$$

A system whose thermal entropy scales as $S_{\text{th.}} \sim T^d$ is said to possess a hyperscaling behaviour. When the dynamical exponent is present, this scaling gets modified to $S_{\text{th.}} \sim T^{\frac{d}{z}}$. It can be seen that in field theories with the kind of scaling defined by (7.4), thermal entropy scales in turn as $S_{\text{th.}} \sim T^{\frac{d-\theta}{z}}$ [213, 246], and so, from the thermodynamic point of view, $d-\theta$ acts as the effective number of space-like dimensions of the system [246]. The fact that $S_{\text{th.}}$ does not scale with its naive power of the temperature corresponds therefore to a violation of the hyperscaling behaviour [178, 246] (the hyperscaling case being obviously $\theta = 0$)[1], and the above class of metrics has been consequently named *hyperscaling-violating Lifshitz metrics* (hvLf in short). Although the $r^{\frac{2\theta}{d}}$ factor spoils dimensional analysis in (7.3), this can be easily restored by including an additional scale r_F: $r^{\frac{2\theta}{d}} \to (r/r_F)^{\frac{2\theta}{d}}$, which we will often fix to 1 henceforth.

So far, hvLf metrics (7.3) with $\theta \neq 0$ have only been found in solutions to Einstein-Maxwell-dilaton-type effective actions of the form [6, 98, 131, 146, 165, 166, 179, 199, 203, 250, 274, 286, 350, 360][2],

$$S = \frac{1}{16\pi G_N}\int \sqrt{|g|}\left\{R + \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - Z(\phi)F^{\mu\nu}F_{\mu\nu} - 2\Lambda - V(\phi)\right\} . \tag{7.5}$$

In the first part of this chapter, we are going to show how to construct systematically solutions of ungauged supergravity (theories which do not fit, in general, in the action (7.5)) whose metrics are, or approach in certain limits, hvLf metrics with certain values of z and θ. The first of our constructions makes use of the FGK formalism originally developed to study static, spherically symmetric, asymptotically flat, black hole solutions of 4-dimensional ungauged supergravity theories [173], and we start by reviewing this formalism in Section 7.2. We will then generalize the FGK formalism to metrics which are not spherically symmetric. The main result is that there are (at least) two cases in which the equations of motion of the metric function and scalar fields are identical to those of the spherically symmetric one. Thus, one can use the solutions of the standard black hole case and construct solutions with entirely different spacetime metrics.

In section 7.3 we study the behaviour of the new solutions in the neighborhood of the values of the radial coordinate corresponding, in the original solution, to the inner and outer horizons, spatial infinity and the curvature singularity. We will find hvLf metrics in

[1] From the holographic perspective, this would correspond to the entropy of a black brane whose spacetime metric asymptotes to one of these solutions [250].

[2] Although in some cases these actions are embedded in string theory or supergravity [9, 162, 220, 343, 361]

some of these limits. In Section 7.4 we investigate how hvLf metrics arise in other limits of more standard metrics and propose other procedures to construct, in particular, supersymmetric hvLf spacetimes by *smearing* extremal supersymmetric black hole solutions of 4-dimensional $\mathcal{N} = 2$ supergravity. In Section 7.3 we briefly discuss the generalization of these results to higher dimensions. A brief discussion of our results can be found in Section **??** and the appendix contains a summary of properties of hvLf metrics.

7.2 The generalized FGK formalism

Following Ref. [173] we consider the action

$$I = \int d^4x \sqrt{|g|}\, \{R + \mathcal{G}_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j + +2\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda}{}_{\mu\nu}F^{\Sigma\,\mu\nu} - 2\Re\mathrm{e}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda}{}_{\mu\nu}\star F^{\Sigma\,\mu\nu}\}\,, \tag{7.6}$$

where $\mathcal{N}_{\Lambda\Sigma}$ is the complex scalar-dependent (*period*) matrix. The bosonic sector of any ungauged supergravity theory in 4 dimensions can be put in this form. The number of scalars labeled by $i, j, \cdots$ and of vector field labeled by $\Lambda, \Sigma, \cdots$, the scalar metric $\mathcal{G}_{ij}$ and the period matrix $\mathcal{N}_{\Lambda\Sigma}$ depend on the particular theory under consideration.

Since we want to obtain static solutions, we consider the metric

$$ds^2 = e^{2U}dt^2 - e^{-2U}\gamma_{\underline{mn}}dx^{\underline{m}}dx^{\underline{n}}\,, \tag{7.7}$$

where $\gamma_{\underline{mn}}$ is a 3-dimensional (*transverse*) Riemannian metric to be specified later. Using Eq. (7.7) and the assumption of staticity of all the fields, we perform a dimensional reduction over time in the equations of motion that follow from the above general action. We obtain a set of reduced equations of motion that we can write in the form[3]

$$\nabla_{\underline{m}}\left(\mathcal{G}_{AB}\partial^{\underline{m}}\tilde{\phi}^B\right) - \tfrac{1}{2}\partial_A\mathcal{G}_{BC}\partial_{\underline{m}}\tilde{\phi}^B\partial^{\underline{m}}\tilde{\phi}^C = 0\,. \tag{7.8}$$

$$R_{\underline{mn}} + \mathcal{G}_{AB}\partial_{\underline{m}}\tilde{\phi}^A\partial_{\underline{n}}\tilde{\phi}^B = 0\,. \tag{7.9}$$

$$\partial_{[\underline{m}}\psi^\Lambda\partial_{\underline{n}]}\chi_\Lambda = 0\,, \tag{7.10}$$

where all the tensor quantities refer to the 3-dimensional metric $\gamma_{\underline{mn}}$ and we have defined the metric $\mathcal{G}_{AB}$

$$\mathcal{G}_{AB} \equiv \begin{pmatrix} 2 & & \\ & \mathcal{G}_{ij} & \\ & & 4e^{-2U}\mathcal{M}_{MN}\end{pmatrix}\,, \tag{7.11}$$

in the *extended* manifold of coordinates $\tilde{\phi}^A = \left(U, \phi^i, \psi^\Lambda, \chi_\Lambda\right)$, where

$$(\mathcal{M}_{MN}) \equiv \begin{pmatrix} (\mathfrak{I} + \mathfrak{R}\mathfrak{I}^{-1}\mathfrak{R})_{\Lambda\Sigma} & -(\mathfrak{R}\mathfrak{I}^{-1})_{\Lambda}{}^{\Sigma} \\ -(\mathfrak{I}^{-1}\mathfrak{R})^{\Lambda}{}_{\Sigma} & (\mathfrak{I}^{-1})^{\Lambda\Sigma}\end{pmatrix}\,, \quad \mathfrak{R}_{\Lambda\Sigma} \equiv \Re\mathrm{e}\mathcal{N}_{\Lambda\Sigma}\,, \quad \mathfrak{I}_{\Lambda\Sigma} \equiv \Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}\,. \tag{7.12}$$

Eqs. (7.8) and (7.9) can be obtained from a three-dimensional effective action

$$I = \int d^3x\sqrt{|\gamma|}\left\{R + \mathcal{G}_{AB}\partial_{\underline{m}}\tilde{\phi}^A\partial^{\underline{m}}\tilde{\phi}^B\right\}\,, \tag{7.13}$$

[3] See Ref. [173] for more details on this reduction.

but we still need to add the constraint Eq. (7.10).

If we now decide to consider spherically-symmetric transverse metrics only, as it is appropriate to describe single, static black holes, we can choose, as in Ref. [173]

$$\gamma_{\underline{mn}}dx^{\underline{m}}dx^{\underline{n}} = \frac{d\tau^2}{W_{-1}^4} + \frac{d\Omega_{-1}^2}{W_{-1}^2}\,, \tag{7.14}$$

where W_{-1} is a function of the (inverse) radial coordinate τ to be determined and

$$d\Omega_{-1}^2 \equiv d\theta^2 + \sin^2\theta\, d\phi^2\,, \tag{7.15}$$

is the metric of the round 2-sphere of unit radius. With this choice, Eq. (7.10) is automatically solved, the equation of $W_{-1}(\tau)$ can be integrated completely, giving

$$W_{-1}(\tau) = \frac{\sinh(r_0\tau)}{r_0}\,, \tag{7.16}$$

and we are left with just

$$\frac{d}{d\tau}\left(\mathcal{G}_{AB}\frac{d\tilde{\phi}^B}{d\tau}\right) - \tfrac{1}{2}\partial_A\mathcal{G}_{BC}\frac{d\tilde{\phi}^B}{d\tau}\frac{d\tilde{\phi}^C}{d\tau} = 0\,, \tag{7.17}$$

$$\mathcal{G}_{BC}\frac{d\tilde{\phi}^B}{d\tau}\frac{d\tilde{\phi}^C}{d\tau} = 2r_0^2\,. \tag{7.18}$$

The integration constant r_0 is the *non-extremality parameter*: when r_0 vanishes, the metric describes extremal black holes (if the solution satisfies the necessary regularity conditions).

The electrostatic and magnetostatic potentials $\psi^\Lambda, \chi_\Lambda$ only appear through their τ-derivatives. The associated conserved quantities are the magnetic and electric charges p^Λ, q_Λ and can be used to eliminate completely the potentials. The remaining equations of motion can be put in the convenient form

$$U'' + e^{2U}V_{\rm bh} = 0\,, \tag{7.19}$$

$$(U')^2 + \tfrac{1}{2}\mathcal{G}_{ij}\phi^{i\,\prime}\phi^{j\,\prime} + e^{2U}V_{\rm bh} = r_0^2\,, \tag{7.20}$$

$$(\mathcal{G}_{ij}\phi^{j\,\prime})' - \tfrac{1}{2}\partial_i\mathcal{G}_{jk}\phi^{j\,\prime}\phi^{k\,\prime} + e^{2U}\partial_i V_{\rm bh} = 0\,, \tag{7.21}$$

in which the primes indicate differentiation with respect to τ and the so-called *black-hole potential* $V_{\rm bh}$ is given by[4]

$$-V_{\rm bh}(\phi,\mathcal{Q}) \equiv -\tfrac{1}{2}\mathcal{Q}^M\mathcal{Q}^N\mathcal{M}_{MN}\,, \qquad (\mathcal{Q}^M) \equiv \begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix}\,. \tag{7.22}$$

Eqs. (7.19) and (7.21) can be derived from the effective action

$$I_{\rm eff}[U,\phi^i] = \int d\tau\left\{(U')^2 + \tfrac{1}{2}\mathcal{G}_{ij}\phi^{i\,\prime}\phi^{j\,\prime} - e^{2U}V_{\rm bh}\right\}\,. \tag{7.23}$$

[4]As in Ref. [190], we adopt the sign of the black-hole potential opposite to most of the literature on black-hole attractors, conforming instead to the conventions of Lagrangian mechanics.

Eq. (7.20) is nothing but the conservation of the Hamiltonian (due to absence of explicit τ-dependence of the Lagrangian) but with a particular value of the integration constant (r_0^2).

A fair number of solutions of this system for different theories of 4-dimensional $\mathcal{N}=2$ supergravity coupled to vector supermultiplets are known (see *e.g.* Ref. [190, 331]). They describe single, charged, static, spherically-symmetric, asymptotically-flat, non-extremal black holes which generalize the Reissner-Nordström solution and have two horizons that coincide when the non-extremality parameter r_0 vanishes. The metric covers the exterior of the outer (event) horizon when the (*inverse*) radial coordinate[5] τ takes values in the interval $(-\infty, 0)$, whose limits are, respectively, the event horizon and spatial infinity. The interior of the inner (Cauchy) horizon corresponds to the interval $(\tau_s, +\infty)$, whose limits are, respectively, the singularity and the inner horizon.

We may also be interested in spacetime metrics which are not spherically symmetric, in which case we have to use a different transverse metric. In principle, these metrics are not appropriate to describe isolated, static black holes but here we are ultimately interested in Lifshitz metrics with a transverse metric invariant under the 2-dimensional Euclidean group, Thus, we can take, for instance, the following simple generalization of the spherically-symmetric transverse metric Eq. (7.14):

$$\gamma_{\underline{mn}} dx^{\underline{m}} dx^{\underline{n}} = \frac{d\tau^2}{W_\kappa^4} + \frac{d\Omega_\kappa^2}{W_\kappa^2}\,, \tag{7.24}$$

where W_κ is a function of τ and $d\Omega_\kappa^2$ is the metric of the 2-dimensional symmetric space of curvature κ and unit radius:

$$d\Omega_{-1}^2 \equiv d\theta^2 + \sin^2\theta\, d\phi^2\,, \tag{7.25}$$

$$d\Omega_{+1}^2 \equiv d\theta^2 + \sinh^2\theta\, d\phi^2\,, \tag{7.26}$$

$$d\Omega_0^2 \equiv d\theta^2 + d\phi^2\,. \tag{7.27}$$

In these three cases the equation for $W_\kappa(\tau)$ can be integrated and the results are

$$W_{-1} = \frac{\sinh r_0\tau}{r_0}\,, \tag{7.28}$$

$$W_1 = \frac{\cosh r_0\tau}{r_0}\,, \tag{7.29}$$

$$W_0^{\pm} = a e^{\mp r_0\tau}\,, \tag{7.30}$$

where a is a real arbitrary constant with dimensions of inverse length.

It turns out that if we follow now for the $\kappa = 0, +1$ cases the procedure described above for the $\kappa = -1$ case we arrive to exactly the same system of equations (7.19)-(7.21) and, therefore, to the same effective action Eq. (7.83). It follows that all the solutions for $\left(U, \phi^i\right)$ obtained in the spherically-symmetric case $\kappa = -1$ are also solutions for the

[5] Observe that τ has dimensions of inverse length, since r_0 has, conventionally, dimensions of length.

$\kappa = 0, +1$ cases as well. In other words: every solution of the system of equations (7.19)-(7.21) provides us with four different solutions of the original theory, by simply using the four different transverse metrics.

Since, as mentioned above, there exists a number of solutions of those equations that describe single, static, asymptotically-flat non-extremal black holes when we take $\kappa = -1$ [190, 324, 331], we can simply take those solutions and study them setting $\kappa = 0$ or $+1$ in the transverse metric. Observe that one integration constant has been fixed to normalized the metric at spatial infinity, something we may not need to do in the $\kappa = 0, +1$ cases, but the normalization could be changed at any moment, if necessary.

In what follows we are going to study the asymptotic behaviour of generic solutions (U, ϕ^i), normalized to describe single, static, asymptotically-flat non-extremal black holes for $\kappa = -1$ when we take the transverse metric with $\kappa = 0$.

7.3 Solutions with Lifshitz-like asymptotics

Since we are going to use the metric functions e^{-2U} corresponding to charged, spherically-symmetric, asymptotically-flat, non-extremal black-hole solutions, we start by reviewing their asymptotic behaviors at the outer $(+)$ and inner $(-)$ horizons[6] (placed, respectively, at $\tau = -\infty$ and $\tau = +\infty$) and at spatial infinity $\tau = 0$.

- The standard normalization of these asymptotically-flat black holes requires that
$$\lim_{\tau \to 0^-} e^{-2U} = 1 \, . \tag{7.31}$$
- When τ approaches the two horizons, $\tau \to \mp\infty$, the metric function behaves as
$$e^{-2U} \sim \frac{S_\pm}{4\pi r_0^2} e^{\mp 2r_0\tau} \, , \tag{7.32}$$
where S_+ (resp. S_-) is the entropy of the outer (resp. inner) horizon, which is assumed to be non-vanishing (which is equivalent to require regularity of the black-hole solution). If we use the spherically-symmetric transverse metric the spacetime metric approaches in these limits a product of a Rindler metric and a 2-sphere of area $4S_\pm$. Studying the Rindler metric by conventional methods one finds that the temperatures of the horizons $T_\pm$ obey the Smarr-like relation [204]
$$r_0 = 2S_\pm T_\pm \, . \tag{7.33}$$
- e^{-2U} vanishes for some value of $\tau_s \in (0, +\infty)$ at which the physical singularity of the black-hole spacetime lies. We may also want to study the behaviour of e^{-2U} near this value of τ but we do not know of any general result on this respect. We will have to study each particular case separately.

To find new solutions, we are going to plug black-hole metric functions in the general static metric Eq. (7.7) with the transverse metric Eq. (7.24) with $\kappa = 0$, *i.e.* with Eq. (7.27) and Eq. (7.30). It is convenient to set $a = 1/r_0$ so no new length scale is introduced in the metric, which takes two possible forms:

$$ds^2_{(\pm)} = e^{2U} dt^2 - e^{-2U} \left[e^{\pm 4r_0\tau} r_0^4 d\tau^2 + e^{\pm 2r_0\tau} r_0^2 \left(d\theta^2 + d\phi^2 \right) \right] \, . \tag{7.34}$$

[6] Uncharged, static black holes only have outer horizon. The discussion of the behaviour of the metric function in the interior of the inner horizon does not apply to them.

Asymptotic behaviour of $ds^2_{(-)}$: Using the general properties of the metric function e^{-2U} described above it is easy to see that in the limit $\tau \to -\infty$ this metric behaves as

$$ds^2_{(-)} \sim \frac{4\pi r_0^2}{S_+} e^{2r_0\tau} dt^2 - \frac{S_+}{4\pi r_0^2} e^{-2r_0\tau} \left[e^{-4r_0\tau} r_0^4 d\tau^2 + e^{-2r_0\tau} r_0^2 \left(d\theta^2 + d\phi^2\right)\right] . \tag{7.35}$$

The change of coordinates

$$r \equiv e^{-r_0\tau} , \qquad \tilde{t} \equiv \frac{4\pi r_0^2}{S_+} t/r_0 , \qquad x^1 \equiv \theta , \qquad x^2 \equiv \phi , \tag{7.36}$$

brings the metric to the form

$$ds^2_{(-)} \sim \frac{S_+}{4\pi} r^4 \left[r^{-6} d\tilde{t}^2 - dr^2 - dx^i dx^i\right] , \tag{7.37}$$

which is a hvLf metric of the form Eq. (7.3) with $z = 4$, $\theta = 6$ and radius

$$\ell \sim r_0 , \tag{7.38}$$

up to dimensionless factors (functions of the quotient S_+/r_0^2); observe that this asymptotic hvLf space lies in the class of Ricci flat hvLf spaces in Eq. (E.8).

The metric $ds^2_{(-)}$ is regular at $\tau = 0$. Spatial infinity is not there because the radial distance between points with $\tau = 0$ and points with $\tau < 0$ is finite and not infinite, as in the black-hole case. For τ equal to a certain τ_s, $e^{-2U} = 0$ and the metric will be singular, as in the black-hole case. Finally, in the $\tau \to +\infty$ limit the metric is the product of Rindler spacetime times $\mathbb{R}^2$, which can be understood as a flat event horizon with the same temperature as that of the inner horizon of the associated black-hole solution.

Asymptotic behaviour of $ds^2_{(+)}$: The analysis is completely analogous to the previous case: in the limit $\tau \to -\infty$ we find a flat event horizon whose temperature is that of the outer horizon of the associated black hole, there is a singularity at $\tau = \tau_s$ and a hyperscaling Lifshitz metric in the $\tau \to +\infty$ limit. The Lifshitz radius is, once again, $\ell = r_0$.

7.3.1 Examples

The Schwarzschild black hole: This is the only uncharged, static, spherically-symmetric, black-hole solution of the class of theories we are considering and has only one horizon (the event horizon) at (conventionally) $\tau \to -\infty$ in these coordinates, which only cover the exterior. The metric function for the Schwarzschild black hole in these coordinates is

$$e^{-2U} = e^{2M\tau} , \tag{7.39}$$

The spacetime metric $ds^2_{(-)}$ constructed with the Schwarzschild metric function takes the explicit form

$$ds^2_{(-)} = e^{-2M\tau} dt^2 - e^{-2M\tau} M^4 d\tau^2 - M^2 \left(d\theta^2 + d\phi^2\right) . \tag{7.40}$$

In the coordinates

$$e^{-M\tau} M \equiv r , \tag{7.41}$$

it reads

$$ds^2_{(-)} = r^2 d(t/M)^2 - dr^2 - M^2\left(d\theta^2 + d\phi^2\right) . \tag{7.42}$$

which is the product of a 2-dimensional Rindler spacetime ($\mathcal{R}i^2$) with $\mathbb{R}^2$. The temperature of the flat horizon would be that of the Schwarzschild black hole $T \sim M^{-1}$. Observe that this is not just the asymptotic behaviour of the metric: the metric is everywhere identically $\mathcal{R}i^2 \times \mathbb{R}^2$. As is well-known, this metric is just a wedge of the Minkowski spacetime which can be recovered by analytical extension of this metric.

Observe that the above metric makes sense for $\tau \in (-\infty, +\infty)$ or $r \in (0, +\infty)$ since as discussed above, there is not spatial infinity at $\tau = 0$.

The metric $ds^2_{(+)}$ is in this case

$$ds^2_{(+)} = e^{-2M\tau} dt^2 - e^{6M\tau} M^4 d\tau^2 - e^{4M\tau} M^2\left(d\theta^2 + d\phi^2\right) , \tag{7.43}$$

and, in the coordinates

$$e^{M\tau} \equiv r\,, \tag{7.44}$$

it takes the form

$$ds^2_{(+)} = r^{-2} d(t/M)^2 - r^4 M^2 dr^2 - r^4 M^2\left(d\theta^2 + d\phi^2\right) = M^2 r^4\left\{r^{-6} dt^2 - dr^2 - d\theta^2 - d\phi^2\right\} , \tag{7.45}$$

which is the $z = 4, \theta = 6, \ell \sim M$ hvLf metric everywhere in the spacetime, and not just asymptotically. Yet again, this metric is defined for all values of τ or for all $r \in (0, +\infty)$.

The Reissner-Nordström black hole: The embedding of the Reissner-Nordström black hole in pure 4-dimensional $\mathcal{N} = 2$ supergravity (the supersymmetrization of the Einstein-Maxwell theory). The metric function of this solution in the τ coordinates is [190]

$$e^{-2U} = \left[\cosh r_0\tau - \frac{M}{r_0}\sinh r_0\tau\right]^2 , \qquad r_0^2 \equiv M^2 - |\mathcal{Z}|^2 , \tag{7.46}$$

where

$$\mathcal{Z} = \tfrac{1}{2}p - iq\,, \tag{7.47}$$

is the central charge of pure 4-dimensional $\mathcal{N} = 2$ supergravity in the chosen conventions.

It is evident that the asymptotic behaviour of the metrics $ds^2_{(\pm)}$ fits in the general case discussed above. Having the explicit form of the metric, we can also study the behaviour of the spacetime metric near the singularity at τ_s at which $e^{-2U}(\tau_s) = 0$. It is, however, easier to do it in the coordinates in which the metric function has the standard form

$$e^{-2U} = \frac{r^2}{(r - r_+)(r - r_-)} , \qquad r_\pm = M \pm r_0\,. \tag{7.48}$$

The coordinate transformation that relates these two forms of the metric function is

$$r = -\left[\cosh r_0\tau - \frac{M}{r_0}\sinh r_0\tau\right]\left[\frac{\sinh r_0\tau}{r_0}\right]^{-1} . \tag{7.49}$$

If we make this coordinate transformation in the full $ds^2_{(\pm)}$ metrics, they take the form

$$ds^2_{(\pm)} = \frac{(r - r_+)(r - r_-)}{r^2} dt^2 - \frac{r_0^4 r^2}{(r - r_\pm)(r - r_\mp)^5} dr^2 - \frac{r_0^2 r^2}{(r - r_\mp)^2}(d\theta^2 + d\phi^2) . \tag{7.50}$$

According to the general discussion, we should find the singularity in the extension of the metric beyond $\tau = 0$ to positive values of τ. This corresponds in these coordinates to values of r "beyond $r = +\infty$". Thus, we define the coordinate $u \equiv 1/r$ which overlaps with r for $u > 0$ and extends the metric for $u \leq 0$. In these coordinates the metric takes the form

$$ds^2_{(\pm)} = (1-r_+u)(1-r_-u)dt^2 - \frac{r_0^4}{(1-r_\pm u)(1-r_\mp u)^5}dr^2 - \frac{r_0^2}{(1-r_\mp u)^2}(d\theta^2 + d\phi^2)\,, \quad (7.51)$$

and, in the $u \to -\infty$ limit it approaches the metric

$$ds^2_{(\pm)} = r_+r_-u^2dt^2 - \frac{r_0^4}{r_\pm r_\mp^5 u^6}du^2 - \frac{r_0^2}{r_\mp^2 u^2}(d\theta^2 + d\phi^2)\,, \quad (7.52)$$

which can be put in the hvLf form with $z = 3, \theta = 4$ (which implies $C(\theta, z) = 0$) with the coordinate change $r' \equiv 1/u$ using rescaled the coordinates $\tilde{t} \equiv r_\pm t/r_0^2$, $\rho \equiv r'/r_\mp$, $x^1 \equiv \sqrt{r_+r_-}/r_0\theta$, $x^2 \equiv \sqrt{r_+r_-}/r_0\phi$.

Observe that the two consecutive coordinate changes $r = 1/u$, $u = 1/r'$ mean that we can get the same result taking the limit of the metric when r approaches $r = 0$ (which corresponds to the value $\tau = \tau_s$) "from the left". In fact, the same result is obtained if we take the near-singularity limit from the right.

Summarizing, the interior of the inner horizon region $r < r_-$ has, therefore, two boundaries, at $r = r_-$ and at $r = 0$. When the metric approaches $r = r_-$ from the left, the metric $ds^2_{(+)}$ approaches a hvLf metric with $z = 4$ and $\theta = 6$ and the metric $ds^2_{(-)}$ approaches $\mathcal{R}i^2 \times \mathbb{R}^2$, as we have seen before. When r approaches $r = 0$ the metric approaches a hvLf metric with $z = 3, \theta = 4$.

The fact that a hvLf metric can describe the near-singularity limit of a metric that has been obtained as a deformation of a regular black-hole metric is very suggestive. Observe that the deformed metric Eq. (7.50) differs from the standard Reissner-Nordström metric in factors of $(r - r_\pm)$, which are irrelevant in the $r \to 0$ limit, and in the 2-dimensional metric $d\Omega^2_\kappa$ which has $\kappa = -1$ for the standard, spherically symmetric Reissner-Nordström black hole. In the next section we are going to see that there is a limit of the Reissner-Nordström black hole in which $d\Omega^2_{-1}$ approaches $d\Omega^2_0$. The near-singularity limit of this Reissner-Nordström black hole will be described by a hvLf metric with $z = 3, \theta = 4$.

7.4 More hvLf metrics

In this subsection we want to discuss some other ways of obtaining hvLf spacetimes.

7.4.1 hvLf spaces from limiting procedures

A 2-sphere looks locally (in small enough patches) like a plane. Thus, we can flatten $d\Omega^2_{-1}$ by looking at a small neighborhood of $\theta = \pi/2$ and we can study near-horizon and near-singularity limits of standard, spherically-symmetric, black-hole solutions. The near-horizon limits will give, obviously, $\mathcal{R}i^2 \times \mathbb{R}^2$ metrics (or $AdS_2 \times \mathbb{R}^2$ metrics in the extremal cases).

Let us consider the near-singularity limit of the Reissner-Nordström black hole in a small patch around $\theta = \pi/2$:

$$\begin{aligned} ds^2 &= \frac{(r-r_+)(r-r_-)}{r^2}dt^2 - \frac{r^2}{(r-r_+)(r-r_-)}dr^2 - r^2(d\theta^2+d\phi^2) \\ &\sim \frac{r_+r_-}{r^2}dt^2 - \frac{r^2}{r_+r_-}dr^2 - r^2(d\theta^2+d\phi^2)\,, \end{aligned} \tag{7.53}$$

which can can be put in the hvLf form with $z = 3$, $\theta = 4$ and $\ell = \sqrt{r_+r_-}$ with the coordinate change $r/\sqrt{r_+r_-} \to r$, $t/\sqrt{r_+r_-} \to t$.

We can also take the near-singularity limit of the Schwarzschild metric with negative mass in a neighborhood of $\theta = \pi/2$

$$\begin{aligned} ds^2 &= \left(1+\frac{2|M|}{r}\right)dt^2 - \left(1+\frac{2|M|}{r}\right)^{-1}dr^2 - r^2(d\theta^2+d\phi^2) \\ &\sim \frac{2|M|}{r}dt^2 - \frac{r}{2|M|}dr^2 - r^2(d\theta^2+d\phi^2)\,, \end{aligned} \tag{7.54}$$

which can be put in the hvLf form with $z = 4$, $\theta = 6$ and $\ell = |M|/2$ with the coordinate change $2r/|M| \to \rho^2$, $4t/|M| \to t$.

7.4.2 Supersymmetric hvLf spaces from smearing

As was mentioned briefly in Section 7.2, the extremal limit ($r_0 \to 0$) of the 4-dimensional metric describes a single static black hole and the natural question, one we have been ignoring, is what happens in the case $\kappa = 0$.

The first thing that changes is the asymptotic behaviour of e^{-2U}, which for an extremal black hole reads

$$\lim_{\tau\to-\infty} e^{-2U} = \frac{S}{\pi}\,\tau^2\,, \tag{7.55}$$

where S is the entropy of the black hole. The second thing is that the extremal limit of $W^{\pm}_{\kappa}$ is just the constant a which has the dimension of inverse length, whence the 4-dimensional metric becomes

$$ds^2_0 = e^{2U}\,dt^2 - e^{-2U}\,\left[d(a^{-2}\tau)^2 + d\vec{x}^2\,\right]\,, \tag{7.56}$$

where we have defined $x^1 = \theta/a$ and $x^2 = \phi/a$. It is straightforward to see that in the region $\tau \to -\infty$ this leads to a hvLf space with $\theta = 4$ and $z = 3$. Similarly to what happens in the Schwarzschild black hole case in Section 7.3.1, one can see that the extremal RN black hole of electrical charge q, which has $e^{-U} = 1 - \frac{|q|}{\sqrt{2}}\tau$, is this asymptotic hvLf.

Now we are going to see that this solution is just a particular case of a very wide class of solutions with hvLf asymptotics.

One of the most interesting features of the extremal RN black hole is that it is supersymmetric in pure 4-dimensional $\mathcal{N} = 2$ supergravity. As is well known, the most general supersymmetric static solution of this theory can be written, using Cartesian coordinates in the transverse space $\vec{y}_3 \equiv (y^1, y^2, y^3)$ as

$$ds^2_{susy} = e^{2U}dt^2 - e^{-2U}\,d\vec{y}_3^{\,2}\,, \tag{7.57}$$

where the metric function has the form[7]

$$e^{-2U} = \tfrac{1}{2}(H^0)^2 + 2(H_0)^2\,, \tag{7.58}$$

where H_0 and H^0 are two real harmonic functions in the flat transverse space which satisfy the staticity constraint

$$H^0\partial_m H_0 - H_0\partial_m H^0 = 0\,, m = 1,2,3\,. \tag{7.59}$$

In these coordinates, the standard, spherically symmetric ($\kappa = -1$), purely electric extremal RN black hole corresponds to the choice of harmonic functions

$$H^0 = 0\,, \qquad H_0 = 1 + \tfrac{1}{\sqrt{2}}\frac{|q|}{|\vec{y}_3|}\,. \tag{7.60}$$

However, other choices (usually discarded when one is only interested in black holes) are possible and are also supersymmetric. For instance, one can consider harmonic functions that depend on only one of the transverse coordinates, say $y^3 \equiv \rho$. This corresponds, physically, to the smearing of the spherically-symmetric solution in the (y^1, y^2) plane and, mathematically, to the substitution of the factor $1/r$ by ρ in all the harmonic functions of the spherically-symmetric solution. The staticity constraint Eq. (7.59) is automatically satisfied is it was in the spherically-symmetric solution.

From the the extremal RN solution, this choice gives the new *smeared* solution

$$ds^2 = \tfrac{1}{2}(H_0)^{-2}dt^2 - 2(H_0)^2\ \left[d\rho^2 + dy^i dy^i\right]\ , \qquad H_0 = 1 + \tfrac{1}{\sqrt{2}}|q|\rho\,, \tag{7.61}$$

and this solution is identical to the $\kappa = 0$ solution in Eq. (7.56)[8] with $\tau = -\rho$. Furthermore, the $z \to \infty$ limit, which gives the $\theta = 3$, $z = 4$ hvLf space corresponds to the choice

$$H_0 = \tfrac{1}{\sqrt{2}}|q|\rho\,, \tag{7.62}$$

and, therefore, it is an exact, supersymmetric solution.

Once this connection between hvLf metrics and smeared supersymmetric black holes of 4-dimensional $\mathcal{N} = 2$ supergravity has been established, we can systematically construct supersymmetric hvLf metrics using the well-known systematic procedure to construct all the supersymmetric black hole solutions of any 4-dimensional $\mathcal{N} = 2$ supergravity coupled to vector supermultiplets [37, 159, 301, 320] and choosing harmonic functions that depend on only one coordinate in transverse space. The $\rho \to \infty$ limit is the same in all the cases (namely a $\theta = 3$, $z = 4$ hvLf spacetime), provided that the original, spherically-symmetric solution has a regular near-horizon limit. The scalar fields, which have non-trivial profiles in the smeared solutions, become constant in the $\rho \to \infty$ limits, just as they do in the black-hole near-horizon limits.

There are, however, more possibilities, if we start from supersymmetric black holes with singular horizon. A good example is provided by the supersymmetric D0-D4 black

[7] We use the conventions of Ref. [190].

[8] Observe that, in the non-extremal case, we cannot view the $\kappa = 0$ solutions as the smearing of $\kappa = -1$ solutions.

holes embedded in the *STU* model [35, 49, 164][9]. After the smearing, the three complex scalars Z^i, $i = 1, 2, 3$ and metric function of these solutions are given by

$$Z^i = -4ie^{2U} H_0 H^i \,, \tag{7.63}$$

$$e^{-2U} = 4\sqrt{H_0 H^1 H^2 H^3} \,, \tag{7.64}$$

where the four harmonic functions H_0, H^1, H^2, H^3 are

$$\begin{aligned} H_0 &= s_0 \left\{ a_0 + \tfrac{1}{\sqrt{2}} \frac{|q_0|}{|\vec{y}_3|} \right\} \,, \\ H^i &= s^{(i)} \left\{ a^{(i)} + \tfrac{1}{\sqrt{2}} \frac{|p^{(i)}|}{|\vec{y}_3|} \right\} \,, \end{aligned} \tag{7.65}$$

where a_0, a^i are constants related to the asymptotic ($r \to \infty$) values of the scalars, q_0, p^i are electric and magnetic charges and s_0, s^i are the signs of those charges. Only two sets of signs of charges lead to supersymmetric and regular black holes: all charges positive or negative. In particular, none of these charges can vanish.

The associated smeared solutions are given by Eqs. (7.63) and (7.64) with the harmonic functions given by

$$\begin{aligned} H_0 &= s_0 \left\{ a_0 + b_0 \rho \right\} \,, \\ H^i &= s^{(i)} \left\{ a^{(i)} + b^{(i)} \rho \right\} \,. \end{aligned} \tag{7.66}$$

The constants b_0, b^i, which we can take to be positive, are related to electric and magnetic fluxes. The staticity condition Eq. (7.59) is satisfied for any values of the constants and, in particular, we can take any number of them to vanish.

When all the b_0, b^i constants are different from zero, we can take all the a_0, a^i to vanish or take the $\rho \to \infty$ limit. In both cases $e^{-2U} = 4\sqrt{b_0 b^1 b^2 b^3}\rho^2$ and we get a $\theta = 3$, $z = 4$ hvLf spacetime with constant scalars.

When one of them (b_0, for instance) vanishes we must keep $a_0 \neq 0$, and we get

$$Z^i = -i\frac{a_0 b^i}{\sqrt{a_0 b^1 b^2 b^3}} \rho^{-1/2} \,, \qquad e^{-2U} = 4\sqrt{a_0 b^1 b^2 b^3}\rho^{7/2} \tag{7.67}$$

which is a $\theta = 7/2$, $z = 5/2$ hvLf spacetime, now with non-trivial scalars. Other choices of vanishing constant b lead to different scalar profiles by the same θ and z.

It is easy to see that, for $n = 0, \cdots, 4$ non-vanishing constants b, one gets a hvLf spacetime with $\theta = 2 + n/2$ and $z = 1 + n/2$ and various scalar profiles. Perhaps not surprisingly $C_{(\theta,z)} = (4 - n)/n$ and only vanishes for $n = 4$.

7.4.3 Higher dimensional generalization

In ref. [321] the FGK formalism was Generalised to higher dimensional cases, and it is only natural to consider the higher dimensional generalizations of the results presented in the

[9] Again, we use the notation and conventions of Ref. [190] where the details can be found.

foregoing sections, starting off by the ones in Section 7.2: the D-dimensional generalization of the FGK metric reads

$$ds^2 = e^{2U}dt^2 - e^{-\frac{2}{d-1}U}\left[\frac{d\rho^2}{(d-1)^2\, W_\kappa^{2d/(d-1)}} + \frac{\mathsf{h}_{ij}\, dx^i dx^j}{W_\kappa^{2/(d-1)}}\right] , \tag{7.68}$$

where h is the metric of a d-dimensional Riemannian Einstein space; this metric is normalized such that

$$R(\mathsf{h})_{ij} \;=\; (d-1)\,\kappa\,\mathsf{h}_{ij}\;. \tag{7.69}$$

The normalization is such that a d-sphere with the round metric has $\kappa = -1$.

A so-so calculation then shows that the conditions for the resulting FGK equations of motion to be κ as well as W_κ independent, are

$$W_\kappa\,\ddot{W}_\kappa \;-\; \dot{W}_\kappa^2 \;=\; \kappa \quad\text{ and }\quad \ddot{W}_\kappa \;=\; \mathcal{B}^2\, W_\kappa\;; \tag{7.70}$$

$\mathcal{B}$ plays the rôle of the D-dimensional non-extremality constant which on dimensional grounds can be written as r_0^{d-1}. The solutions to the conditions (7.70) are

$$W_{-1} \;=\; \frac{\sinh(\mathcal{B}\rho)}{\mathcal{B}} \quad ,\quad W_0^{\pm} \;=\; a\,e^{\mp\mathcal{B}\rho} \quad\text{ and }\quad W_1 \;=\; \frac{\cosh(\mathcal{B}\rho)}{\mathcal{B}}\;. \tag{7.71}$$

By looking at the, in general, fractional powers of W that appear in the metric (7.68), we see that in contradistinction to the 4-dimensional case, the putative horizon lies at $\rho\to\infty$ which means that the near-horizon behaviour for a non-extremal black hole implies

$$\lim_{\rho\to\infty} e^U \;\sim\; e^{-\mathcal{B}\rho}\;. \tag{7.72}$$

The above means that given a solution to a D-dimensional FGK system, we can as before deform the $\kappa = -1$ solution as in Section 7.2, and obtain new solutions with properties similar to the ones encountered in the foregoing sections. For example, concerning the $\rho\to\infty$ behaviour of the metric we see that

The W_0^+ case: In this case the $\rho\to\infty$ spacetime is hvLf with

$$\theta \;=\; \frac{d(d+1)}{d-1} \quad\text{ and }\quad z \;=\; \frac{2d}{d-1}\;, \tag{7.73}$$

which, as one can see from Eq. (E.8), corresponds to the Ricci flat hvLf spaces.

The W_0^- case: Together with the condition (7.72), we see that the resulting $\rho\to\infty$ spacetime is a Rindler wedge times $\mathbb{R}^d$.

In the $\kappa = -1$ case, the validity of the ρ-coordinate, *i.e.* $\rho\in[0,\infty)$ is principally determined by W_{-1} and one imposes conditions on e^U in order to obtain metrics describing the spacetime outside the outer horizon. In particular, the zero of W_{-1} at $\rho = 0$, together with the regularity of e^U there, allows for the identification of $\rho = 0$ with asymptotic spacetime. W_0 is, however, an all-together different beast and the naive validity of the coordinate, *i.e.* $\rho\in[0,\infty)$, can be extended till one encounters a zero or a pole in e^U; the former signaling a horizon, the latter a singularity. Let us illustrate this point with

The 5-dimensional STU model: The FGK equations for the 5-dimensional STU model are completely separable, whence the full analytical solution is known. The general solution satisfying Eq. (7.72) and having constant scalars in the limit $\rho \to \infty$ is given by (see *e.g.* [323, 331])

$$e^{-3U} = \frac{|\mathsf{q}_1\mathsf{q}_2\mathsf{q}_3|}{\mathcal{B}^3} \sinh(\alpha_1 + \mathcal{B}\rho) \sinh(\alpha_2 + \mathcal{B}\rho) \sinh(\alpha_3 + \mathcal{B}\rho) , \tag{7.74}$$

where the q are the electrical charges and the α's are some real constants; in the $\kappa = -1$ case they are chosen such that $U(\rho = 0) = 1$ and one obtains a Minkowski space with the regular normalization. In the $\kappa = 0$ case, however, the point $\rho = 0$ is not asymptotic and there is therefore no need to impose said condition on the α's. In fact, let $0 < \alpha_1 \leq \alpha_2 \leq \alpha_3$, then we can extend the definition of ρ to the point $\rho_s = -\alpha_1/\mathcal{B}$, where we have added a subscript to highlight the fact that at that point we're facing a curvature singularity.

As in Section 7.3.1 we can consider the near-singularity metric: in general we will find a hvLf space and the characteristic parameters (θ, z) will depend on the order of the zero of e^{-3U} in Eq. (7.74). Denoting this number by p, whence $\mathsf{p} = 1, 2$ or 3,[10] we see that the near-singularity hvLf is given by

$$\theta = 3 + \frac{\mathsf{p}}{2} \; , \quad z = 1 + \frac{\mathsf{p}}{2} \quad \text{whence } C_{(\theta,z)} = \frac{2(3-\mathsf{p})}{\mathsf{p}} \geq 0 \, , \tag{7.75}$$

and, furthermore, the null energy condition (E.7) is always satisfied.

In higher dimensions we can also construct hvLf solutions by smearing extremal, supersymmetric black-hole solutions. The procedure is entirely similar to the one followed in four dimensions.

In a higher-dimensional context, it is natural to consider the following brane-like generalization of the hvLf metric (7.3)

$$ds^2_{d+2} = \ell^2 r^{-2(d-\theta)/d} \left[r^{-2(z-1)} \left(dt^2 - dy^a dy^a \right) - dr^2 - dx^i dx^i \right] , \quad a = 1, \cdots, p \, , \;\; i = 1, \cdots d \, . \tag{7.76}$$

The $p = 0$ case is the original hvLf metric and a metric with $d = 0$ and $p =\neq 0$ can be rewritten as a $p = 0, d \neq 0$ by a coordinate change.

It should come as no surprise that we can obtain metrics of this kind by smearing extremal supersymmetric p-brane metrics. As an example, consider the 10-dimensional Dp-brane solutions in the Einstein frame

$$\begin{aligned} ds^2 &= H^{\frac{p-7}{8}} [dt^2 - d\vec{y}_p^{\,2}] - H^{\frac{p+1}{8}} d\vec{x}^{\,2}_{8-p} \, , \\ C_{(p+1)\, ty^1 \cdots y^p} &= \pm e^{-\phi_0} (H^{-1} - 1) \, , \\ e^{-2\phi} &= e^{-2\phi_0} H^{\frac{p-3}{2}} \, . \end{aligned} \tag{7.77}$$

In all cases, we can take[11]

$$H \sim \rho \, , \tag{7.78}$$

[10] To wit: $\mathsf{p} = 1$ implies $\alpha_2 > \alpha_1$, $\mathsf{p} = 2$ means $\alpha_3 > \alpha_2 = \alpha_1$ and $\mathsf{p} = 3$ means $\alpha_3 = \alpha_2 = \alpha_1$. Let us in passing observe that the case $\mathsf{p} = 3$ corresponds to the deformation of the 5-dimensional Reissner-Nordström black hole.

[11] In the $p = 8$ case there is no smearing involved, since there is only one transverse dimension.

and put the metric in the form

$$
\begin{aligned}
ds^2 &\sim \rho^{\frac{p+1}{8}} \left\{\rho^{-1}[dt^2 - d\vec{y}_p^{\,2}] - d\rho^2 - d\vec{x}_{8-p}^{\,2}\right\} , \\
C_{(p+1)\,ty^1\cdots y^p} &= \sim \rho^{-1} , \\
e^{-2\phi} &\sim \rho^{\frac{p-3}{2}} ,
\end{aligned}
\tag{7.79}
$$

which is of the above form with $p = p$, $z = 3/2$ and $\theta = (8-p)(p+17)/16$ for $p < 8$. The case $p = 0$ (the D0-brane) is a standard hvLf metric with $d = 8$, $\theta = 8.5$ and $z = 3/2$, which satisfies the null energy condition (E.7) but does not avoid the null curvature singularity in the IR region ($\rho \to \infty$). The string coupling constant reads $e^{\phi} = r^{3/4}$, which goes to zero in the UV. The case $p = 8$, after a change of coordinates $\varrho \equiv \rho^{3/2}$ is also a standard hvLf metric ($p = 0$) with $d = 8$, $\theta = 25/3$ and $z = 1$ which also satisfies the null energy condition (E.7) but is singular in the IR region ($r \to \infty$).

7.5 HvLf solutions in gauged supergravity

Let us now extend the previous analysis to a general class of gravity theories coupled to scalars and vectors, up to two derivatives, in the presence of a scalar potential, in principle arbitrary. We will focus later on 4-dimensional $\mathcal{N} = 2$ Supergravity in the presence of Fayet-Iliopoulos terms.

In section (7.6) we dimensionally reduce the general action of gravity coupled to an arbitrary number of scalars and vectors in the presence of a scalar potential assuming a general static background which naturally fits the anisotropic scaling properties which correspond to *hvLf*-like solutions. In section (7.7) we adapt the general formalism to the Einstein-Maxwell-Dilaton system. In section (7.8) we focus on 4-dimensional $\mathcal{N} = 2$ Supergravity in the presence of Fayet-Iliopoulos terms (which produce the appearence of a scalar potential in the corresponding supergravity), were we exploit the symplectic structure of the theory in order to obtain further results. We also embed a particular truncation of the t^3-model in Type-IIB String Theory compactified on a Sasaki-Einstein manifold times S^1. In section (7.9) we perform an analysis of the properties of purely *hvLf* solutions for the general class of theories considered. In addition, we provide a general recip to obtain *hvLf*-like solutions of a particular class of Einstein-Maxwell-Dilaton systems, reducing the problem to the resolution of an algebraic equation. We apply the procedure to obtain explicit solutions, some of them embedded in String Theory.

7.6 The general theory

We are interested in Lifshitz-like solutions with hyperscaling violation (*hvLf* of the four-dimensional action

$$
S = \int d^4x \sqrt{|g|}\, \left\{R + \mathcal{G}_{ij}\partial_\mu\phi^i\partial^\mu\phi^j + 2I_{\Lambda\Sigma}F^{\Lambda}{}_{\mu\nu}F^{\Sigma\,\mu\nu} - 2R_{\Lambda\Sigma}F^{\Lambda}{}_{\mu\nu}\star F^{\Sigma\,\mu\nu} - V(\phi)\right\} ,
\tag{7.80}
$$

that generalizes the action considered in Ref. [173, 190] by including a generic scalar potential $V(\phi)$. We will take care of the constraints imposed by $\mathcal{N} = 2$ supersymmetry on the field content, the kinetic matrices ($I_{\Lambda\Sigma}(\phi) < 0,\ R_{\Lambda\Sigma}(\phi)$), the scalar metric $\mathcal{G}_{ij}(\phi)$ and the scalar potential $V(\phi)$ later on.

The idea now is to dimensionally reduce the action (7.80) using an appropriate ansatz for the metric. Since *hvLf* solutions are in particular static, a first step is to constrain the form of the metric to be

$$ds^2 = e^{2U}dt^2 - e^{-2U}\gamma_{\underline{m}\,\underline{n}}dx^{\underline{m}}dx^{\underline{m}}\,, \quad \underline{m}, \underline{n} = 1, \ldots, 3\,, \tag{7.81}$$

A sensible choice for γ, that fits the anisotropic scaling properties that we look for in a *hvLf* solution, is given by

$$\gamma = \gamma_{\underline{m}\,\underline{n}}dx^{\underline{m}}dx^{\underline{m}} = e^{2W}\left(dr^2 + \delta_{ab}dx^a dx^b\right)\,, \quad a, b = 1, 2\,, \tag{7.82}$$

where e^W is an undetermined function of the "radial" coordinate r. We now proceed to dimensionally reduce the lagrangian (7.80) with the choice of metric given by Eqs. (7.81) and (7.82).

Assuming that all the fields are static, only depend on r, and following the same steps as in Refs. [173, 321][12], one arrives to a set of equations of motion for the variables $U(r),\ W(r),\ \phi^i(r)$ that can be derived from the following effective action $\left(' = \frac{d}{dr}\right)$

$$S = \int dr\, e^W \left\{2U'^2 - 2W'^2 + \mathcal{G}_{ij}\phi^{i\,\prime}\phi^{j\,\prime} - 2e^{2(U-W)}V_{\rm bh} + e^{-2(U-W)}V\right\}\,, \tag{7.83}$$

if we set the value of the Hamiltonian (which is conserved, due to the lack of explicit r-dependence of the Lagrangian) to zero, that is:

$$2U'^2 - 2W'^2 + \mathcal{G}_{ij}\phi^{i\,\prime}\phi^{j\,\prime} + 2e^{2(U-W)}V_{\rm bh} - e^{-2(U-W)}V = 0\,. \tag{7.84}$$

The one-dimensional effective equations of motion are given by

$$\begin{aligned}
e^{-W}\left[e^W U'\right]' + e^{2(U-W)}V_{\rm bh} + \frac{1}{2}e^{-2(U-W)}V &= 0\,, && (7.85)\\
e^{-W}\left[e^W\right]'' + e^{-2(U-W)}V &= 0\,, && (7.86)\\
e^{-W}\left[e^W\mathcal{G}_{ij}\phi^{j\,\prime}\right]' - \frac{1}{2}\partial_i\mathcal{G}_{jk}\phi^{j\,\prime}\phi^{k\,\prime} + e^{2(U-W)}\partial_i V_{\rm bh} - \frac{1}{2}e^{-2(U-W)}\partial_i V &= 0\,, && (7.87)
\end{aligned}$$

to which we have to add the Hamiltonian constraint (7.84). The kinetic term for the scalars, as well as the scalar potential $V(\phi)$ and the black hole potential $V_{\rm bh}(\phi, \mathcal{Q})$, can be solely expressed in terms of U and W, i.e.,

$$\begin{aligned}
V &= -e^{2U-2W}\left[W'' + W'^2\right]\,,\\
V_{\rm bh}(\phi, \mathcal{Q}) &= -\frac{1}{2}e^{2W-2U}\left[2U'' + 2U'W' - W'' - W'^2\right]\,,\\
\mathcal{G}_{ij}\phi^{i\,\prime}\phi^{j\,\prime} &= -2\left[-U'' - U'W' + U'^2 + W''\right]\,.
\end{aligned} \tag{7.88}$$

Eqs. (7.88) are useful in order to obtain, given a particular metric, the behavior of different quantities, like $V(\phi)$ and $V_{\rm bh}(\phi, \mathcal{Q})$, or ϕ^i for models with small enough number of scalars, in terms of the coordinate r. Of course, only metrics compatible with the equations of motion will yield consistent results.

[12] A related procedure, used to obtain non-extremal AdS_4 black hole solutions can be found in [284] and [285]. For related Refs. about solutions in gauged Supergravity see [28, 29].

7.6.1 Constant scalars: generalities

For constant scalars ϕ^i, the potential $V(\phi)$ and the black hole potential $V_{\text{bh}}(\phi, \mathcal{Q})$ become constant quantities, the former playing the role of a cosmological constant and the latter of a *generalized* squared charge, magnetic and electric. In the case of constant scalars, Eq. (7.87) is not identically satisfied, but it becomes the following constraint

$$e^{4(U-W)}\partial_i V_{\text{bh}} = \frac{1}{2}\partial_i V \,. \tag{7.89}$$

We have two different options in order to fulfil Eq. (7.89).

Constant scalars as *double* critical points: $\partial_i V_{\text{bh}} = 0, \quad \partial_i V = 0$. Of course, the system of equations given by

$$\partial_i V_{\text{bh}} = 0, \quad \partial_i V = 0 \,, \tag{7.90}$$

is overdetermined. However, let's assume that a consistent solution to (7.90) exists and is given by

$$\phi^i = \phi^i_c\left(\mathcal{Q}, \phi_\infty\right) \,, \tag{7.91}$$

i.e., the values of the scalars are fixed in terms of the electric and magnetic charges, and we have included a dependence on ϕ_∞ to formally consider the existence of flat directions. We will see later on that, in fact, Eq. (7.90) happens in 4-dimensional $\mathcal{N} = 2$ Supergravity. The equations of motion reduce to

$$\begin{aligned} e^{-W}\left[e^{W}U'\right]' + e^{2(U-W)}V_{\text{bh}} + \frac{1}{2}e^{-2(U-W)}V &= 0, & (7.92)\\ e^{-W}\left[e^{W}\right]'' + e^{-2(U-W)}V &= 0\,, & (7.93)\end{aligned}$$

together with the hamiltonian constraint

$$2U'^2 - 2W'^2 + 2e^{2(U-W)}V_{\text{bh}} - e^{-2(U-W)}V = 0\,. \tag{7.94}$$

Metric functions identified: $e^U = \beta e^W, \quad \beta \in \mathbb{R}^+$ and $2\beta^4\partial_i V_{\text{bh}} = \partial_i V(\phi)$. In this case, the equations of motion imply

$$2\beta^4\partial_i V_{\text{bh}} = \partial_i V\,, \quad 2\beta^4 V_{\text{bh}} = V\,. \tag{7.95}$$

Assuming Eqs. (7.95), there is a unique solution, which is precisely $AdS_2\times\mathbb{R}^2$. Eqs. (7.95) can be understood as necessary and sufficient conditions for a gravity theory coupled to scalars and vector fields, up to two derivatives, to contain an $AdS_2 \times \mathbb{R}^2$ solution. Therefore, given a particular theory of such kind, with a specific potential $V(\phi)$ and black hole potential $V_{\text{bh}}(\phi)$, one only has to impose Eqs. (7.95) in order to check the existence of an $AdS_2 \times \mathbb{R}^2$ solution. The parameter β can be always found to be

$$\beta^4 = \frac{V}{2V_{\text{bh}}}\,, \tag{7.96}$$

and we are left with

$$\frac{1}{2}\partial_i \log V_{\text{bh}} = \partial_i \log V\,. \tag{7.97}$$

Eq. (7.97) is a system of n_v equations for at least n_v variables (the n_v constant scalars), and hence in general it will be compatible and the theory will contain an $AdS_2 \times \mathbb{R}^2$ solution. Only in pathological cases the system (7.97) will be incompatible and the theory will fail to contain an $AdS_2 \times \mathbb{R}^2$ solution.

7.7 The Einstein-Maxwell-Dilaton model

Before we discuss the possible embeddings of Eq. (7.80) in gauged Supergravity and String Theory, let's consider the Einstein-Maxwell-Dilaton (E.M.D.) system, whose action is characterized by the following choices, to be made in Eq. (7.80)

$$F^{\Lambda\,\mu\nu} = F^{\mu\nu}, \qquad I_{\Lambda\Sigma} = I = \frac{Z(\phi)}{2} < 0, \qquad R_{\Lambda\Sigma} = R = 0, \qquad \phi^i = \phi, \qquad G_{ij} = \frac{1}{2}. \tag{7.98}$$

Hence, the E.M.D. action reads

$$S_{EMD} = \int d^4x\sqrt{|g|}\left\{R + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + Z(\phi)F^2 - V(\phi)\right\}\,, \tag{7.99}$$

i.e., we consider a single vector field and a single scalar field. Moreover, the coupling given by R is taken to be zero, which greatly simplifies the black hole potential $V_{\rm bh}(\phi, \mathcal{Q})$, which is therefore given by

$$V_{\rm bh}(\phi, \mathcal{Q}) = \frac{1}{4}\left[Z(\phi)p^2 + Z(\phi)^{-1}q^2\right]\,, \tag{7.100}$$

where q and p are the electric and magnetic charges, respectively. The equations of motion take the form

$$\begin{aligned} e^{-W}\left[e^W U'\right]' + e^{2(U-W)}\frac{1}{4}\left[Zp^2 + Z^{-1}q^2\right] + \frac{1}{2}e^{-2(U-W)}V &= 0\,, & (7.101)\\ e^{-W}\left[e^W\right]'' + e^{-2(U-W)}V &= 0\,, & (7.102)\\ e^{-W}\left[e^W\phi'\right]' + e^{2(U-W)}\frac{\partial_\phi Z}{2}\left[p^2 - \frac{q^2}{Z^2}\right] - e^{-2(U-W)}\partial_\phi V &= 0\,, & (7.103)\end{aligned}$$

and the hamiltonian constraint reads

$$2U'^{\,2} - 2W'^{\,2} + \frac{1}{2}\phi'^{\,2} + \frac{e^{2(U-W)}}{2}\left[Zp^2 + Z^{-1}q^2\right] - e^{-2(U-W)}V = 0\,. \tag{7.104}$$

For non-constant scalars, Eq. (7.103) is automatically satisfied if

$$\begin{aligned} V &= -e^{2(U-W)}\left[W'^2 + W''\right]\,, & (7.105)\\ \phi'^2 &= 4\left[-U'^2 + U'W' + U'' - W''\right]\,, & (7.106)\end{aligned}$$

and Z is such that

$$\begin{aligned} Z &= \frac{1}{p^2}\left[\Upsilon \pm \sqrt{\Upsilon^2 - p^2q^2}\right]\,,\ \text{if } p,\ q \neq 0, & (7.107)\\ Z &= \frac{2\Upsilon}{p^2}\ \text{if } q = 0,\ p \neq 0 & (7.108)\\ Z &= \frac{q^2}{2\Upsilon}\ \text{if } p = 0, & (7.109)\end{aligned}$$

where

$$\Upsilon = 2V_{\rm bh} = e^{2(W-U)}\left[-2U'W' + W'^2 - 2U'' + W''\right]. \tag{7.110}$$

Theories with conventional and sensible matter have to satisfy the null-energy condition (NEC) $n_\mu n_\nu T^{\mu\nu} \geq 0$, where n_μ is an arbitrary null vector and $T^{\mu\nu}$ is the correspondent energy-momentum tensor. This condition translates, for the E.M.D. case, into the following constraints

$$\Upsilon \leq 0, \;\; \phi'^2 \geq 0. \tag{7.111}$$

Hence, it is equivalent to the requirement of a semi-negative definite black hole potential, and a semi-positive definite kinetic term for the scalar field, compatible with the condition $Z(\phi)$.

Another coordinate system: A-B-f coordinates.

There is another system of coordinates which we will use along these sections, and that will be useful for different purposes. It is related to the U-W system of coordinates by the following identifications:

$$\left(\frac{dr}{d\tilde{r}}\right)^2 = f^{-1}(\tilde{r}), \qquad e^{2U} = e^{2(A(\tilde{r})+B(\tilde{r}))} f(\tilde{r}), \qquad e^{2W} = e^{4A(\tilde{r})+2B(\tilde{r})} f(\tilde{r})\,, \tag{7.112}$$

giving rise to the metric

$$ds_f^2 = \ell^2 e^{2A(\tilde{r})}\left[e^{2B(\tilde{r})} f(\tilde{r}) dt^2 - \frac{d\tilde{r}^2}{f(\tilde{r})} - \delta_{ij} dx^i dx^j\right]\,, \tag{7.113}$$

which has proven to be useful (see e.g. [250], [162]) in order to obtain solutions exhibiting $hvLf$ asymptotics when $f(\tilde{r})$ is a function of $\tilde{r}$ that obeys

$$f(\tilde{r}_h) = 0, \;\; \tilde{r}_h \in \mathbb{R}^+ \qquad \lim_{\tilde{r}\to\tilde{r}_0} f(\tilde{r}) = 1\,. \tag{7.114}$$

The equations of motion (7.101), (7.102) and (7.103) can be rewritten accordingly as[13]

$$e^{-2A-B}\left[e^{2A+B} f \left[A' + B' + \frac{f'}{2f}\right]\right]' + e^{-2A}\frac{1}{4}\left[Zp^2 + Z^{-1}q^2\right] + \frac{1}{2}e^{2A}V(\phi) = 0 \tag{7.115}$$

$$e^{-2A-B}\left[f^{1/2}\left[e^{2A+B} f^{1/2}\right]'\right]' + e^{2A}V(\phi) = 0 \tag{7.116}$$

$$e^{-2A-B}\left[e^{2A+B} f\phi'\right]' + e^{-2A}\frac{\partial_\phi Z}{2}\left[p^2 - Z^{-2}q^2\right] - e^{2A}\partial_\phi V(\phi) = 0, \tag{7.117}$$

where $' = \frac{d}{d\tilde{r}}$. The Hamiltonian constraint is given by

$$-2f\left[3A'^2 + 2A'\left[B' + \frac{f'}{2f}\right]\right] + \frac{f}{2}\phi'^2 + \frac{e^{-2A}}{2}\left[Zp^2 + Z^{-1}q^2\right] - e^{2A}V(\phi) = 0\,. \tag{7.118}$$

Again, for non-constant dilaton this set of equations is equivalent to[14]

$$V = \frac{e^{-2A}}{2}\left[-3f'[2A'+B'] - 2f\left[2A'' + \left[2A'+B'\right]^2 + B''\right] - f''\right], \tag{7.119}$$

$$\phi'^2 = 4\left[-A'' + A'B' + A'^2\right], \tag{7.120}$$

$$\Upsilon = -\frac{e^{2A}}{2}\left[f'\left[2A'+3B'\right] + 2f\left[2A'B' + B'' + B'^2\right] + f''\right]. \tag{7.121}$$

[13] From now on, we will use always the symbol "r" to denote the "radial" coordinate, independently of which coordinate system we use, which will be specified by other means.

[14] Eqs. (7.107), (7.108), (7.109) hold.

7.8 $\mathcal{N}=2$ Supergravity with F.I. terms

The action (7.80) has great generality and basically covers any possible theory of gravity coupled to abelian vector fields and scalars up to two derivatives. However, in order to embed our results in String Theory, it is convenient to focus on the bosonic sector of 4-dimensional $\mathcal{N}=2$ Supergravity, which is a particular case of (7.80). More precisely, we are going to consider gauged 4-dimensional $\mathcal{N}=2$ in the presence of n_v abelian vector multiplets, where the gauge group is contained in the R-symmetry group of automorphisms of the supersymmetry algebra. Normally one refers to this theory as 4-dimensional $\mathcal{N}=2$ Supergravity with Fayet-Iliopoulos terms (from now on, $\mathcal{N}=2$ FI to abridge) [14]. The general lagrangian of $\mathcal{N}=2$ FI is given by

$$\begin{aligned} S &= \int d^4x\sqrt{|g|}\,\{R + 2\mathcal{G}_{ij^*}\partial_\mu z^i\partial^\mu z^{*\,j^*} + 2\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,\mu\nu}F^{\Sigma}{}_{\mu\nu} \\ &\qquad -2\Re\mathrm{e}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,\mu\nu\star}F^{\Sigma}{}_{\mu\nu} - V_{\mathrm{fi}}\,(z,z^*)\}\,. \end{aligned} \tag{7.122}$$

The indices $i,j,\ldots = 1,\ldots,n_v$ run over the scalar fields and the indices $\Lambda,\Sigma,\ldots = 0,\ldots,n_v$ over the 1-form fields. The scalar potential generated by the F.I. terms reads

$$V_{\mathrm{fi}}\,(z,z^*) = -3|\mathcal{Z}_g|^2 + \mathcal{G}^{ij^*}\mathfrak{D}_i\mathcal{Z}_g\mathfrak{D}_{j^*}\mathcal{Z}_g^*, \qquad \mathfrak{D}_i\mathcal{Z}_g = \partial_i\mathcal{Z}_g + \frac{1}{2}\partial_i\mathcal{K}\mathcal{Z}_g\,, \tag{7.123}$$

where $\mathcal{K}$ is the Kähler potential, $\mathcal{Z}_g$ is given by[15]

$$\mathcal{Z}_g \equiv \mathcal{Z}_g\,(z,z^*) = g_M\mathcal{V}^M = \mathcal{V}^M g^N\Omega_{MN} = -g^\Lambda\mathcal{M}_\Lambda + g_\Lambda\mathcal{L}^\Lambda\,, \tag{7.124}$$

and the g^M is a symplectic vector related to the embedding tensor θ_M, that selects the combination of vectors that gauges $U(1)\subset R$-symmetry group, as follows[16]

$$g_M = g\theta_M\,, \tag{7.125}$$

g being the gauge coupling constant. The corresponding one-dimensional effective action and the hamiltonian constraint are given, respectively, by

$$S = \int dr\, e^W\left\{U'^{\,2} - W'^{\,2} + \mathcal{G}_{ij^*}z^{i\,\prime}z^{j^*\,\prime} - e^{2(U-W)}V_{\mathrm{bh}} + \frac{1}{2}e^{-2(U-W)}V_{\mathrm{fi}}\right\}\,, \tag{7.126}$$

$$U'^{\,2} - W'^{\,2} + \mathcal{G}_{ij^*}z^{i\,\prime}z^{j^*\,\prime} + e^{2(U-W)}V_{\mathrm{bh}} - \frac{1}{2}e^{-2(U-W)}V_{\mathrm{fi}} = 0\,. \tag{7.127}$$

The black-hole potential takes the simple form

$$-V_{\mathrm{bh}}(z,z^*,\mathcal{Q}) = |\mathcal{Z}|^2 + \mathcal{G}^{ij^*}\mathfrak{D}_i\mathcal{Z}\mathfrak{D}_{j^*}\mathcal{Z}^*\,, \tag{7.128}$$

where

$$\mathcal{Z} = \mathcal{Z}(z,z^*,\mathcal{Q}) \equiv \langle\mathcal{V}\mid\mathcal{Q}\rangle = -\mathcal{V}^M\mathcal{Q}^N\Omega_{MN} = p^\Lambda\mathcal{M}_\Lambda - q_\Lambda\mathcal{L}^\Lambda\,, \tag{7.129}$$

is the *central charge* of the theory.

[15] We assume the conventions of [320].

[16] Supergravity gaugings are originally electric, breaking therefore the symplectic covariance present in the ungauged case. The embedding tensor formalism allows to formally keep the theory simplectically covariant by introducing magnetic and electric gaugings.

Constant scalars and supersymmetric attractors. In section (7.6) we studied the case of constant scalars in the general theory (7.80). We found that, besides the solution $AdS_2 \times \mathbb{R}^2$, there was another possible solution, if Eq. (7.90) holds. We will see now how this is always possible in $\mathcal{N} = 2$ FI. The general theory of the attractor mechanism in ungauged 4-dimensional Supergravity proves that, for extremal black holes, the value of the scalars at the horizon is fixed in terms of the charges $\mathcal{Q}^M$, and given by the so called *critical points* or *attractors*, *i.e.*, solutions to the system

$$\partial_i V_{\rm bh}\left(\mathcal{Q},\phi\right)_{|_{\phi_c}} = 0\,. \tag{7.130}$$

There might be some residual dependence in the value at infinity if the potential has flat directions. If the scalars are constant, they have to be given again by (7.130) in the extremal as well as in the non-extremal case. It can be proven that there is always a class of attractors, called supersymmetric, which obey

$$\partial_i \left|\mathcal{Z}\right|_{|_{\phi_c}} = 0\,, \quad \text{and} \quad \mathfrak{D}_i \mathcal{Z}_{|_{\phi_c}} = 0\,, \tag{7.131}$$

and therefore, given the definitions (7.123) and (7.128), they also obey (7.90) if $\mathcal{Q}^M \sim g^M$. Hence, setting the scalars to constant values given by the supersymmetric attractor points of the black hole potential is always a consistent truncation, provided that g^M is identified with $\mathcal{Q}^M$, which besides fixes the value of the black hole potential and the scalar potential exclusively in terms of the charges.

7.8.1 The t^3-model

In this section we consider a particular $\mathcal{N} = 2$ FI model which can be embedded in String Theory. In particular we start from Type-IIB String Theory compactified on a Sasaki-Einstein manifold to five dimensions. This theory can be consistently truncated as to yield pure $\mathcal{N} = 1,\ d = 5$ Supergravity with Fayet-Iliopoulos terms, which, due to the absence of scalars, introduce a cosmological constant. Further compactification on S^1 gives us the desired four dimensional theory, which is defined by [124, 125, 202, 299]

$$n_v = 1, \quad F\left(\mathcal{X}\right) = -\frac{\left(\mathcal{X}^1\right)^3}{\mathcal{X}^0}, \quad g^0 = g^1 = g_0 = 0 \Rightarrow V_{\rm fi}\left(t,t^*\right) = \frac{-\beta^2}{\Im{\rm m} t}\,, \tag{7.132}$$

where $\beta^2 = g_1^2/3$, and we have defined the inhomogeneous coordinate on the Special Kähler manifold SU(1,1)/U(1), by

$$t = \frac{\mathcal{X}^1}{\mathcal{X}^0}\,. \tag{7.133}$$

This theory is known as the t^3-model, and although the String Theory embedding requires the gauging specified in Eq. (7.132), we are going to study it in full generality, particularizing only at the end.

The canonically normalized symplectic section $\mathcal{V}$ is, in a certain gauge,

$$\mathcal{V} = e^{\mathcal{K}/2}\begin{pmatrix} 1 \\ t \\ t^3 \\ -3t^2 \end{pmatrix}, \tag{7.134}$$

where the Kähler potential is

$$\mathcal{K} = -\log\left[i\,(t-t^*)^3\right]. \tag{7.135}$$

As a consequence, the Kähler metric reads

$$\mathcal{G}_{tt^*} = \frac{3}{4}\frac{1}{(\Im\mathrm{m}t)^2}, \tag{7.136}$$

and the central charge

$$\mathcal{Z} = \frac{p^0 t^3 - 3t^2 p^1 - q_0 - q_1 t}{2\sqrt{2\Im m t^3}}. \tag{7.137}$$

The period matrix $\mathcal{N}_{IJ}$ is, in turn, given by

$$\mathrm{Re}\mathcal{N}_{IJ} = \begin{pmatrix} -2\Re^3 & 3\Re^2 \\ 3\Re^2 & -6\Re \end{pmatrix}, \qquad \mathrm{Im}\mathcal{N}_{IJ} = \begin{pmatrix} -(\Im^3+3\Re^2\Im) & 3\Re\Im \\ 3\Re\Im & -3\Im \end{pmatrix}, \tag{7.138}$$

where we use the notation: $\Re \equiv \Re\mathrm{e}t$, $\Im \equiv \Im\mathrm{m}t$. The general expressions of V_{bh} and V_{fi}, which can be obtained using Eqs. (7.123), (7.124), (7.128) and (7.129) read

$$\begin{aligned} V_{\mathrm{bh}} = &-\frac{1}{6\Im^3}\Big[3\Im^6 {p^0}^2 + 9\Im^4\left[p^1 - p^0\Re\right]^2 + \Im^2\left[q_1 + 6p^1\Re - 3p^0\Re^2\right]^2 \\ &+3\left[q_0 + \Re\left[q_1 + 3p^1\Re - p^0\Re^2\right]\right]^2\Big], \end{aligned} \tag{7.139}$$

$$V_{\mathrm{fi}} = -\frac{1}{3\Im}\left[g_1^2 + 3g_1\left[g^1\Re + g^0\left[\Im^2+\Re^2\right]\right] + 9\left[g_0\left[-g^1 + g^0\Re\right] + {g^1}^2\left[\Im^2+\Re^2\right]\right]\right]. \tag{7.140}$$

Let's consider the truncation $\Re\mathrm{e}t = 0$. In order to satisfy all the original equations of motion (those with $\Re\mathrm{e}t$ arbitrary) in such a case, we must impose the additional constraints

$$\partial_\Re V_{\mathrm{bh}}(\Re = 0) = \partial_\Re V_{\mathrm{fi}}(\Re = 0) = 0. \tag{7.141}$$

These conditions explicitly read

$$3\Im p^0 p^1 - 2\frac{p^1 q_1}{\Im} - \frac{q_0 q_1}{\Im^3} = 0, \tag{7.142}$$

$$3g_0 g^0 + g_1 g^1 = 0, \tag{7.143}$$

and are satisfied (without loss of generality in the functional form of the potentials) if we make

$$p^1 = q_1 = 0; \quad g_0 = g^1 = 0 \vee g^0 = g_1 = 0. \tag{7.144}$$

Thus, setting $\Re\mathrm{e}t$ to zero in a consistent manner notably simplifies the expressions for the potentials

$$V_{\mathrm{bh}} = -\frac{1}{2}\left[\frac{q_0^2}{\Im^3} + {p^0}^2\Im^3\right], \tag{7.145}$$

$$V_{\mathrm{fi}}^{I} = -\left[\frac{g_1^2}{3\Im} + g_1 g^0\Im\right], \; V_{\mathrm{fi}}^{II} = -\left[-\frac{3g_0 g^1}{\Im} + 3{g^1}^2\Im\right] \tag{7.146}$$

The action is, making the redefinition $t \equiv \Re + ie^{-\frac{\phi}{\sqrt{3}}}$, given by

$$S^{I}_{\Re=0} = \int d^4x\sqrt{|g|}\left\{R + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - 2e^{-\sqrt{3}\phi}\left(F^0\right)^2 + \frac{g_1^2}{3}e^{\frac{\phi}{\sqrt{3}}} + g_1 g^0 e^{-\frac{\phi}{\sqrt{3}}}\right\}, \quad (7.147)$$

$$S^{II}_{\Re=0} = \int d^4x\sqrt{|g|}\left\{R + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - 2e^{-\sqrt{3}\phi}\left(F^0\right)^2 - 3g_0 g^1 e^{\frac{\phi}{\sqrt{3}}} + 3{g^1}^2 e^{-\frac{\phi}{\sqrt{3}}}\right\}, \quad (7.148)$$

where we have already set A^1_μ to zero, in order to make the truncation consistent with the corresponding equation of motion.

Embedding the t^3-model system in the E.M.D. As it can be trivally verified, we have just obtained the action (7.99) with

$$Z(\phi) = -2e^{-\sqrt{3}\phi}, \quad q^2 = 4q_0^2, \quad p^2 = {p^0}^2, \quad (7.149)$$

and the scalar potential of the E.M.D. system (Eq. (7.182)) given by

$$V(\phi) = c_1 e^{-\frac{\phi}{\sqrt{3}}} + c_2 e^{+\frac{\phi}{\sqrt{3}}}; \quad c_1^I = -g_1 g^0, \quad c_1^{II} = -3{g^1}^2, \quad c_2^I = -\frac{g_1^2}{3}, \quad c_2^{II} = 3g_0 g^1. \quad (7.150)$$

Hence, we find that our axion-free t^3-system with those particular choices of Z and V gets embedded in the E.M.D. model and, for $g^0 = g^1 = g_0 = 0$, also in String Theory in the way explained at the beginning of this section. In such a case, Eq. (7.147) clearly becomes

$$S_{ST} = \int d^4x\sqrt{|g|}\left\{R + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - 2e^{-\sqrt{3}\phi}\left(F^0\right)^2 + \frac{g_1^2}{3}e^{\frac{\phi}{\sqrt{3}}}\right\}. \quad (7.151)$$

7.9 *hvLf* solutions

In this section we are going to construct (purely and asymptotically) *hvLf* solutions to Eq. (7.80). After establishing some results on the properties of the solutions corresponding to the pure *hvLf* case in the general set-up of Eq. (7.88), we focus on the E.M.D. system, obtaining the *hvLf* solutions allowed by the embedding of our *axion-free* Supergravity model in this system. Then, we provide a recipe to construct asymptotically *hvLf* solutions to these theories in the presence of constant and non-constant dilaton fields.

7.9.1 Purely *hvLf* solutions: general remarks

The *hvLf* metric in four dimensions, given by

$$ds^2 = \ell^2 r^{\theta-2}\left(r^{-2(z-1)}dt^2 - dr^2 - \delta_{ij}dx^i dx^j\right), \quad (7.152)$$

is recovered in our set-up for specific values of $U(r)$ and $W(r)$, namely

$$e^{2U(r)} = \ell^2 r^{\theta-2z}, \quad e^{2W(r)} = \ell^4 r^{2(\theta-z-1)}. \quad (7.153)$$

For purely *hvLf* solutions, the equations of motion can be further simplified by direct substitution of (7.153)

$$(\theta-2z)(\theta-z-2) + 2r^{4-\theta}\ell^{-2}V_{\rm bh} + r^\theta\ell^2 V = (7.154)$$

$$(\theta-z-1)(\theta-z-2) + r^\theta\ell^2 V = (7.155)$$

$$r^{-2(\theta-z-1)}\left(r^{2(\theta-z-1)}\mathcal{G}_{ij}\phi^{j\,\prime}\right)' - \frac{1}{2}\partial_i\mathcal{G}_{jk}\phi^{j\,\prime}\phi^{k\,\prime} + r^{2-\theta}\ell^{-2}\partial_i V_{\rm bh} - \frac{1}{2}r^{\theta-2}\ell^2\partial_i V = (7.156)$$

The Hamiltonian constraint reads

$$(2-\theta)(3\theta-4z-2)+2r^2\mathcal{G}_{ij}\phi^{i\prime}\phi^{j\prime}+4r^{4-\theta}\ell^{-2}V_{\rm bh}-2r^{\theta}\ell^2 V=0\,. \tag{7.157}$$

Eqs. (7.88) can be also adapted to the purely *hvLf* case. We find

$$V=-\ell^{-2}\mathcal{X}_{(\theta,z)}r^{-\theta},\qquad V_{\rm bh}(\phi,\mathcal{Q})=\frac{1}{2}\ell^2\mathcal{Y}_{(\theta,z)}r^{\theta-4},\qquad \mathcal{G}_{ij}\dot{\phi}^i\dot{\phi}^j=\frac{1}{2}\mathcal{W}_{(\theta,z)}r^{-2}\,, \tag{7.158}$$

where

$$\mathcal{X}_{(\theta,z)} = (\theta-z-2)(\theta-z-1), \tag{7.159}$$
$$\mathcal{Y}_{(\theta,z)} = (\theta-z-2)(z-1), \tag{7.160}$$
$$\mathcal{W}_{(\theta,z)} = (\theta-2)(\theta-2z+2)\,. \tag{7.161}$$

Eqs. (7.154)-(7.157) are the general equations of motion that need to be solved in order to find a *hvLf* solution to any theory that belongs to the class defined by Eq. (7.80). Likewise, Eq. (7.158) provides the behaviour of the black hole potential and the scalar potential, in terms of the variable r, for any *hvLf* solution consistent with the equations of motion. $\mathcal{G}_{ij}$ is positive-definite, therefore

$$\mathcal{G}_{ij}\phi^{i\prime}\phi^{j\prime}\geq 0\Leftrightarrow \mathcal{W}_{(\theta,z)}\geq 0,\qquad \mathcal{G}_{ij}\phi^{i\prime}\phi^{j\prime}=0\ \Leftrightarrow\ \phi^{i\prime}=0\ \forall i\,, \tag{7.162}$$

and hence we can establish the following result: all the scalar fields of any purely *hvLf* solution of any theory describable by Eq. (7.80) are constant *iff* $\theta=2$, or $z=1+\theta/2$. In addition, $V_{\rm bh}$ is, in our conventions, a negative definite function, hence $V_{\rm bh}\leq 0\Leftrightarrow\mathcal{Y}_{(\theta,z)}\leq 0$. These two conditions on the sign of $\mathcal{W}_{(\theta,z)}$ and $\mathcal{Y}_{(\theta,z)}$ are equivalent to imposing the null-energy condition (NEC) to our purely *hvLf* solutions, as we commented before, and define a region of acceptable solutions in the (θ,z)-plane, as we shall see.

It is possible to stablish some general results for the *hvLf* solutions of any theory describable by Eq. (7.80) attending to the vanishing of V, $V_{\rm bh}$ and/or $\mathcal{G}_{ij}\dot{\phi}^i\dot{\phi}^j$. Let's proceed.

1. $\theta=2$
   In this situation $\mathcal{G}_{ij}\dot{\phi}^i\dot{\phi}^j=0$, and

$$V = -\ell^{-2}z(z-1)r^{-2}, \tag{7.163}$$
$$V_{\rm bh} = -\frac{1}{2}\ell^2 z(z-1)r^{-2}. \tag{7.164}$$

   The NEC imposes $z\in(-\infty,0]\cup[1,+\infty)$, and we have the two special cases: $\theta=2,\ z=0$ (which corresponds to Rindler spacetime) and $\theta=2,\ z=1$ (which is Minkowski space-time) for which $V=V_{\rm bh}=0$ as well.

2. $z=1+\frac{\theta}{2}$
   We have again $\mathcal{G}_{ij}\dot{\phi}^i\dot{\phi}^j=0$, and

$$V = -\ell^{-2}\left(\frac{\theta}{2}-3\right)\left(\frac{\theta}{2}-2\right)r^{-\theta}, \tag{7.165}$$
$$V_{\rm bh} = -\frac{1}{2}\ell^2\left(\frac{\theta}{2}-3\right)\frac{\theta}{2}r^{\theta-4}. \tag{7.166}$$

The NEC translates into $\theta \in [0,6]$, and we have three more special cases: the Ricci flat one: $\theta = 6,\ z = 4$ corresponding to $V = V_{\rm bh} = 0$ (this is a particular case of the general formalism developed in [88] for ungauged 4-dimensional $\mathcal{N} = 2$ Supergravity); $\theta = 4,\ z = 3$, which corresponds to $V = 0$, $V_{\rm bh} = -\ell^2$ (also in agreement with the results of [88]); and $\theta = 0,\ z = 1$, which is nothing but the AdS_4 space-time in a conformally flat representation, and the only solution with vanishing black hole potential, and constant (non-zero) scalar potential compatible with the equations: $V_{\rm bh} = 0$, $V \equiv \Lambda = -\ell^{-2}6$.

3. $z = 1,\ \theta \neq 2,\ z \neq 1 + \frac{\theta}{2}$
   We have $V_{\rm bh} = 0$, whereas

$$V = -\ell^{-2}\,(\theta-3)\,(\theta-2)\,r^{-\theta}, \tag{7.167}$$

$$\mathcal{G}_{ij}\dot{\phi}^i\dot{\phi}^j = \frac{1}{2}(\theta-2)\theta\; r^{-2}. \tag{7.168}$$

   The NEC becomes now $\theta \in (-\infty, 0] \cup [2, \infty)$, and we have the limit case $\theta = 3,\ z = 1$ which will be a particular case of the family considered in the next paragraph.

4. $z = \theta - 2,\ \theta \neq 2,\ z \neq 1 + \frac{\theta}{2}$
   This situation imposes $V = V_{\rm bh} = 0$, whereas

$$\mathcal{G}_{ij}\dot{\phi}^i\dot{\phi}^j = \frac{1}{2}(\theta-2)(6-\theta)\; r^{-2}. \tag{7.169}$$

   The NEC reads $\theta \in [2,6]$. These will be solutions of the Einstein-Dilaton system for $\mathcal{G}_{ij} = \frac{1}{2}\delta_{ij},\ i = 1$, and

$$\phi = \phi_0 + \sqrt{(\theta-2)(6-\theta)}\,\log r. \tag{7.170}$$

5. $z = \theta - 1,\ \theta \neq 2,\ z \neq 1 + \frac{\theta}{2}$
   We have now $V = 0$, while

$$V_{\rm bh} = -\frac{1}{2}\ell^2\,(\theta-2)\,\frac{\theta}{2}r^{\theta-4}. \tag{7.171}$$

$$\mathcal{G}_{ij}\dot{\phi}^i\dot{\phi}^j = \frac{1}{2}(\theta-2)(4-\theta)\; r^{-2}, \tag{7.172}$$

   and the NEC becomes $\theta \in [2,4]$.

Another particularly interesting case corresponds to the Einstein-Maxwell system with a cosmological constant: $\mathcal{G}_{ij}\dot{\phi}^i\dot{\phi}^j = 0$, $V \equiv \Lambda$. However, this could only be realized for $\theta = 0,\ z = 1$, which imposes the vanishing of $V_{\rm bh}$. Hence, there is no purely *hvLf* solution (for non-vanishing vector fields) for such model.

Purely *hvLf* in the E.M.D.

If we particularize now to the E.M.D. system, we find

$$V = -\ell^{-2}\mathcal{X}_{(\theta,z)}r^{-\theta}, \tag{7.173}$$

$$\Upsilon = 2V_{\rm bh} = \ell^2\mathcal{Y}_{(z,\theta)}r^{\theta-4}, \tag{7.174}$$

$$\phi = \phi_0 + \sqrt{\mathcal{W}_{(z,\theta)}}\,\log(r) \;\Rightarrow\; r = e^{\frac{\phi}{\sqrt{z}}}\,. \tag{7.175}$$

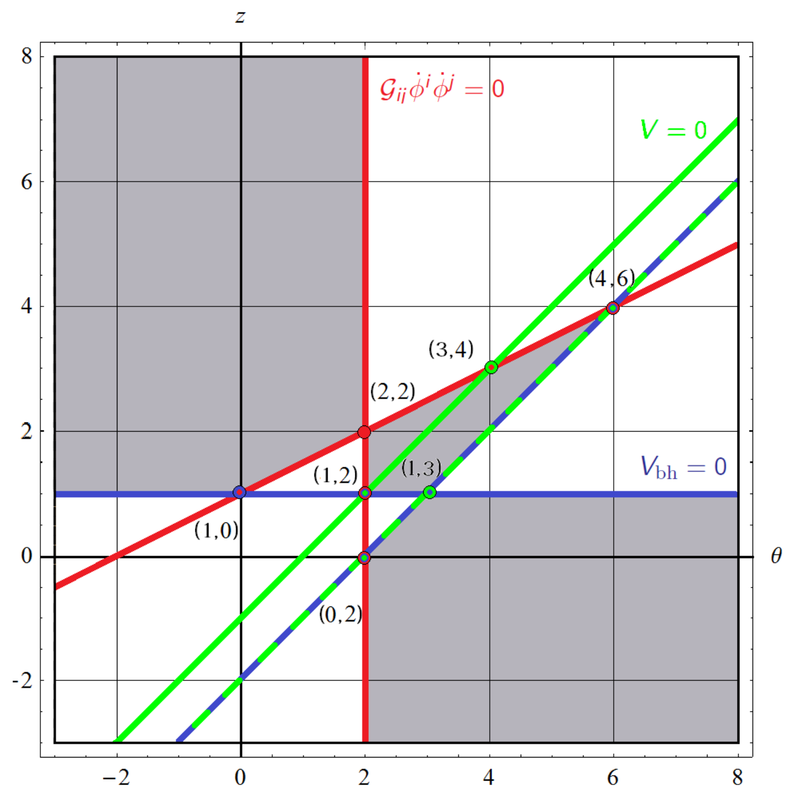

Figure 7.1: *Purely hvLf (θ, z) plane. Red lines correspond to $\mathcal{G}_{ij}\dot{\phi}^i\dot{\phi}^j = 0$, the blue ones to $V_{\rm bh} = 0$, and those in green to $V = 0$. The shaded regions represent solutions which satisfy the NEC.*

Therefore, V and $V_{\rm bh}$ written as functions of ϕ, must take the form

$$V(\phi) = -\ell^2 \mathcal{X} e^{-\frac{\theta\phi}{\sqrt{z}}}, \tag{7.176}$$

$$V_{\rm bh}(\phi) = \frac{1}{2}\ell^2 \mathcal{Y} e^{\frac{(\theta-4)\phi}{\sqrt{z}}}. \tag{7.177}$$

This means, on the one hand, that any E.M.D. theory susceptible of containing *hvLf* solutions has a scalar potential which depends on ϕ through one single exponential (becoming a constant when $\theta = 0$, $\theta = 2$ or $z = 1 + \theta/2$ ($\phi = \phi_0$ in the last two cases)) [64]. On the other hand, the gauge coupling function is constant for $\theta = 4$, and again if $\phi = \phi_0$.

t^3-model

Let's see now what the situation is for the truncation of the t^3-model considered in the previous section. In this case, $V^{I,II} = c_1 e^{-\phi/\sqrt{3}} + c_2 e^{\phi/\sqrt{3}}$ with $c_2 = 0 \;\Rightarrow\; c_1 = 0$ in the case I, and $c_1 = 0 \;\Rightarrow\; c_2 = 0$ in the case II. Since we can only keep one of the exponentials (in order to match V with Eq. (7.176)), the only possibility is setting $g^0 = 0$ ($c_1 = 0$) in the case I (which leaves us with the String Theory embedded model), and g_0 ($c_2 = 0$) in the case II. In both situations, $Z(\phi) = -2\, e^{-\sqrt{3}\phi}$. In I there exists one only solution, which is magnetic, and corresponds to $\theta = -2$, $z = 3/2$, $g_1^2 = 297/(4\ell^2)$ and $p^2 = 11\ell^2/4$. On the other hand, case II admits one only solution (magnetic as well) for $\theta = 1$, $z = 3$, $g^{1^2} = 4/\ell^2$, and $p^2 = 8\ell^2$. Both solutions satisfy the NEC, as it was desirable, and have a running dilaton given by Eq. (7.175) with $\mathcal{Z} = 12$ and $\mathcal{Z} = 3$ respectively.

7.9.2 Asymptotically *hvLf* in the E.M.D.

Non-constant scalar field

In order to construct new solutions with *hvLf* asymptotics, we switch now to A-B-f variables. The required form for A and B is

$$e^{2A} = r^{\theta-2}, \quad e^{2B} = r^{-2(z-1)}. \tag{7.178}$$

With this election, Eq. (7.120) can be directly integrated, yielding

$$\phi = \phi_0 + \sqrt{(\theta-2)(\theta+2-2z)}\,\log(r). \tag{7.179}$$

Υ and V, in turn, become[17]

$$V = \frac{1}{2} r^{-\theta}\left[\left[1-\theta+z\right]\left[2\left[\theta-2-z\right]f + 3rf'\right] - r^2 f''\right], \tag{7.180}$$

$$\Upsilon = r^{\theta-4}\left[f\left[(\theta-2-z)(z-1)\right] - \frac{r}{2}\left[(1+\theta-3z)f' + rf''\right]\right]. \tag{7.181}$$

In order to tackle the problem of constructing asymptotically *hvLf* metrics, and taking into account the form of $V(\phi)$ and $Z(\phi)$ for our *axion-free* model (and others present in the literature), we can start by considering these functions to have the generic form

$$V(\phi) = c_1 e^{-s_1\phi} + c_2 e^{s_2\phi} + c_3, \tag{7.182}$$

$$Z(\phi) = d_1 e^{-t_1\phi} + d_2 e^{t_2\phi} + d_3. \tag{7.183}$$

The form of $V(\phi)$ is motivated by the expression of V_{fi} appearing in the *axion-free* t^3 model, as well as in other String Theory truncations present in the literature (see, e.g. [148], [213]). On the other hand, additional terms to the single-exponential gauge coupling have been introduced to mimic the quantum corrections appearing from String Theory (see, e.g. [227]), in an attempt to cure the logarithmic behavior of the dilaton, which blows up in the deep IR, pointing out the non-negligibility of quantum corrections in this regime. The expressions for $V(\phi)$ and $Z(\phi)$ can be introduced in Eqs. (7.180) and (7.107), (7.108) or (7.109) (depending on whether we are searching for electric, magnetic or dyonic solutions) using Eq. (7.181). Once this is done, we are left with two second-order differential equations for $f(r)$ which can in general be converted into a first order equation plus a constraint that remains to be fulfilled. Obtaining the general solution in the presence of so many arbitrary parameters (c_1, c_2, c_3, d_1, d_2, d_3, s_1, s_2, t_1, t_2, z and θ) seems not to be possible and therefore we are forced to consider further simplifications, keeping in mind that the procedure does work for other set-ups in which $Z(\phi)$ and $V(\phi)$ are given by a different choice of the parameters in (7.182) and (7.183). Taking into account the form

[17] Recall that Z is given in term of Υ in Eqs. (7.107), (7.108) and (7.109) depending on the case

of the potentials obtained in the *axion-free* t^3 model, let's assume $s_1 = s_2$, $d_2 = d_3 = 0$ (we allow t_1 to be positive or negative)

$$V(\phi) = c_1 e^{-s_1\phi} + c_2 e^{s_1\phi} + c_3, \tag{7.184}$$

$$Z(\phi) = d_1 e^{-t_1\phi}. \tag{7.185}$$

The general form of the blackening factor, valid in all cases (electric, magnetic and dyonic), reads

$$f(r) = \frac{c_3 r^{\theta}}{D_3} + \frac{c_2 r^{\theta+s_1\Delta}}{D_2} + \frac{c_1 r^{\theta-s_1\Delta}}{D_1} + \frac{d_1 p^2 r^{4-\theta-t_1\Delta}}{2D_p} + \frac{q^2 r^{4-\theta+t_1\Delta}}{2d_1 D_q} + K r^{2-\theta+z}, \tag{7.186}$$

where $\Delta = \sqrt{(\theta-2)(\theta-2z+2)}$, K is an integration constant, and

$$D_1 = (\theta-2)(2-2\theta+s_1\Delta+z), \tag{7.187}$$

$$D_2 = (\theta-2)(2-2\theta-s_1\Delta+z), \tag{7.188}$$

$$D_3 = (\theta-2)(2-2\theta+z), \tag{7.189}$$

$$D_p = (\theta-2)(2-t_1\Delta-z), \tag{7.190}$$

$$D_q = (\theta-2)(2+t_1\Delta-z). \tag{7.191}$$

As we said, there is an additional (non trivial) constraint to be satisfied

$$f''(r) - 2r^{\theta-2}\left[-c_3 - c_1 r^{-s1\Delta} - c_2 r^{s_1\Delta} - \frac{1}{2}r^{-\theta}(\theta-z-1)\left[2(-2+\theta-z)f(r) + 3rf'(r)\right]\right] = 0. \tag{7.192}$$

At this point, there are several ways to construct solutions. On the one hand, it is possible to impose values to z and θ and find the corresponding potentials admitting solutions for particular blackening factors. On the other hand, it is possible to fix the coefficients in the exponents of Z and V and find the blackening factors allowed by Eq. (7.192). We will proceed along the lines of the second possibility, looking for solutions embedded in the Supergravity t^3 model. Before doing so, let's consider the general case in which the exponents in $Z(\phi)$ and $V(\phi)$ are such that $s_1 = \theta/\Delta$, $t_1 = (4-\theta)/\Delta$, and $c_2 = q = 0$. The result is a family of solutions for arbitrary values of z and θ determined by

$$c_1 = \frac{d_1 p^2(\theta-z-1)}{2(1-z)}, \tag{7.193}$$

$$f(r) = \frac{d_1 p^2}{2(1-z)(z-\theta+2)}\left[1 - Kr^{2+z-\theta}\right], \tag{7.194}$$

which is well known (see, e.g. [162], [213], [131])

$$f(r) \sim 1 - Kr^{2+z-\theta}. \tag{7.195}$$

The same family can also be found for electric solutions setting $s_1 = \theta/\Delta$, $t_1 = (\theta-4)/\Delta$, and $c_1 = p = 0$. In that case, the solution is given by

$$c_1 = \frac{q^2(\theta-1-z)}{2d_1(1-z)}, \tag{7.196}$$

$$f(r) = \frac{q^2}{2d_1(1-z)(z-\theta+2)}\left[1 - Kr^{2+z-\theta}\right]. \tag{7.197}$$

t^3-model

1. **Magnetic solutions.** As we saw, a consistent truncation of the t^3-model can be embedded in the E.M.D. system for $s_1 = 1/\sqrt{3}$, $t_1 = \sqrt{3}$, $c_3 = 0$. It turns out that setting $q = 0$, it is possible to construct two families of solutions which, in the apropriate cases, asymptote to the purely *hvLf* ones constructed in the previous subsection. The first one is determined by

$$c_1 = 0,\ \theta = 2\left(1 - \frac{\Delta}{\sqrt{3}}\right),\ c_2 = Ap^2, \tag{7.198}$$

where A is a constant depending on z and θ. The blackening factor is given by

$$f(r) = Cp^2 r^{\left(2-\frac{\Delta}{\sqrt{3}}\right)} + Kr^{\left(\frac{2\Delta}{\sqrt{3}}+z\right)}, \tag{7.199}$$

where C is another z, θ-dependent constant. Needless to say, the metric will not, in general, asymptote to a *hvLf* (with exponents z, θ) as $r \to 0$ except for particular values of θ and z. However, if we choose $\theta = -2$, $z = 3/2$, $c_2 = -9p^2$, we find

$$f(r) = \frac{4p^2}{11}\left[1 - Kr^{\frac{11}{2}}\right]. \tag{7.200}$$

The second family is characterized by

$$c_2 = 0,\ \theta = \left(2 - \frac{\Delta}{\sqrt{3}}\right),\ c1 = Ap^2, \tag{7.201}$$

where A is another constant, and the blackening factor reads

$$f(r) = Cp^2 r^{\left(2-\frac{2\Delta}{\sqrt{3}}\right)} + Kr^{\left(\frac{\Delta}{\sqrt{3}}+z\right)}. \tag{7.202}$$

If we set $\theta = 1$, $z = 3$, it becomes

$$f(r) = \frac{p^2}{8}\left[1 - Kr^4\right] \tag{7.203}$$

which, as we will see in a moment, is a particular a case of a dyonic solution admitted by the model.

2. **Electric solutions.** Similarly, we can construct two families of electric solutions. The first one is characterized by

$$c_1 = 0,\ \theta = \left(2 + \frac{\Delta}{\sqrt{3}}\right),\ c_2 = Aq^2, \tag{7.204}$$

where, once more, A is a constant depending on z and θ. The blackening factor is given by

$$f(r) = Cq^2 r^{\left(2+\frac{2\Delta}{\sqrt{3}}\right)} + Kr^{\left(-\frac{\Delta}{\sqrt{3}}+z\right)}, \tag{7.205}$$

whereas for the second

$$c_1 = Aq^2,\ \theta = 2\left(1 + \frac{\Delta}{\sqrt{3}}\right),\ c_2 = 0, \tag{7.206}$$

$$f(r) = Cq^2 r^{\left(2+\frac{\Delta}{\sqrt{3}}\right)} + Kr^{\left(-\frac{2\Delta}{\sqrt{3}}+z\right)}. \tag{7.207}$$

In contradistinction to the magnetic cases, for no values of (θ, z) the above solutions take the form of Eq. (7.195). This is obviously connected to the fact that no purely *hvLf* electric solutions exist in this model for non-constant dilaton and scalar potential, as we saw before.

3. **Dyonic solutions.** It is possible to show that a dyonic solution does exist for $\theta = 1$, $z = 3$, $c_2 = 0$, and $c_1 = -3p^2/2$, with a blackening factor given by

$$f(r) = \frac{p^2}{8}\left[1 - Kr^4 + \frac{q^2}{p^2}r^6\right]. \tag{7.208}$$

The corresponding metric Eq.(7.113) reads (after the redefinitions $dR^2 = 8dr^2/p^2$, $dT^2 = 8dt^2/p^2$)

$$ds_f^2 = \frac{L^2}{R}\left\{\left[1 - KR^4 + \frac{p^4q^2}{512}R^6\right]\frac{dT^2}{R^4} - \frac{dR^2}{\left[1 - KR^4 + \frac{p^4q^2}{512}R^6\right]} - d\vec{x}^2\right\}. \tag{7.209}$$

It asymptotes to a *hvLf* as $R \to 0$ with $\theta = 1,\ z = 3$, and to a different one as $R \to \infty$ with $\theta = 5/2$, $z = 3/2$ as it can be seen by taking the limit in the previous expression, and defining $\rho \sim R^{-2}$

$$ds_f^2 \overset{R\to+\infty}{\sim} \frac{L^2}{R}\left[R^2 dT^2 - \frac{dR^2}{R^6} - d\vec{x}^2\right], \tag{7.210}$$

$$ds_f^2 \overset{[R\to+\infty,\ R^{-2}=\rho]}{\sim} L^2\rho^{1/2}\left[\frac{dT^2}{\rho} - d\rho^2 - d\vec{x}^2\right], \tag{7.211}$$

which corresponds to $\theta = 5/2,\ z = 3/2$. The value of K can be fixed in a way such that $\exists\ R_h \in \mathbb{R}^+\ /\ f(R_h) = 0$, or chosen to get a positive-definite metric in the whole spacetime.

In the previous section, we constructed two consistent truncations of this model (which we called "I" and "II"). The first one is such that $c_2 = 0 \Rightarrow c_1 = 0$, and hence the solution can be embedded in that model only for a vanishing V_{fi} and magnetic charge. For the second, in turn, we get the conditions $g_0 = 0$, $\left(g^1\right)^2 = p^2/2$. It is interesting to investigate how the solution gets modified by turning off the electric or the magnetic charge. Obviously, setting $q = 0$ does not change the $R \to 0$ behavior, but does change the $R \to +\infty$ one. In such a case, the metric becomes

$$ds_f^2 \overset{[R\to+\infty, R^{-1}=\rho]}{\sim} \rho\left[dT^2 - d\rho^2 - d\vec{x}^2\right], \tag{7.212}$$

which is conformal to Minkowski, and corresponds to a *hvLf* with $\theta = 3,\ z = 1$. On the other hand, restoring q and setting $p = 0$, imposes the vanishing of V_{fi}, and the solution is $\theta = 3,\ z = 1$ as $R \to 0$, and again $\theta = 5/2,\ z = 3/2$ as $R \to +\infty$.

It turns out that there exists another dyonic solution for $\theta = 5/2,\ z = 3/2$[18]. This is somehow "dual" to the previous one, as it presents the same IR and UV behavior but with both regimes interchanged. It is characterized by $c_1 = 0$, $c_2 = -\frac{3q^2}{8}$, and

$$f(r) = 2p^2\left[1 - Kr + \frac{q^2}{16p^2}r^3\right]. \tag{7.213}$$

In our "I" truncation, $c_2^I = -g_1^2/3 \Rightarrow g_1^2 = 9q^2/8$. Making the redefinitions $dR^2 = dr^2/(2p^2)$, $dT^2 = \sqrt{2}pdt^2$, it reads

$$ds_f^2 = L^2R^{1/2}\left\{\left[1 - KR + \frac{pq^2}{4\sqrt{2}}R^3\right]\frac{dT^2}{R} - \frac{dR^2}{\left[1 - KR + \frac{pq^2}{4\sqrt{2}}R^3\right]} - d\vec{x}^2\right\}. \tag{7.214}$$

As $R \to +\infty$, this becomes

$$ds_f^2 \overset{[UV,\ R=\rho^{-2}]}{\sim} \frac{L^2}{\rho}\left[\frac{dT^2}{\rho^4} - d\rho^2 - d\vec{x}^2\right], \tag{7.215}$$

up to constants, which corresponds to a *hvLf* with $\theta = 1,\ z = 3$.

Constant scalar field

Let's consider now the case of a constant scalar field, $\phi' = 0$. As explained in section 7.6, we consider

$$\partial_\phi V_{\text{bh}} = \partial_\phi V = 0. \tag{7.216}$$

In this case, the potential and the coupling become constant and we can write $V = \Lambda$, $Z = -Z_0^2$. When Z and V are given by Eqs. (7.183) and (7.182), Eq. (7.216) translates into

$$\partial_\phi V_{\text{bh}}(\phi = 0) = \partial_\phi Z\left(p^2 - \frac{q^2}{Z^2}\right)|_{\phi=0} = (-t_1d_1 + t_2d_2)\left(p^2 - \frac{q^2}{(d_1+d_2)^2}\right) = 0, \tag{7.217}$$

$$\partial_\phi V(\phi = 0) = (s_2c_2 - s_1c_1) = 0, \tag{7.218}$$

where we have imposed $\phi = 0$ to be a critical point of the potentials. We choose to fulfill the first condition demanding $(d_1 + d_2)^2 = q^2/p^2$ which, when $d_1 = 0$, reads $d_2 = -|q/p|$. On the other hand, the second condition is $s_1c_1 = s_2c_2$, that becomes $c_1 = c_2$ when both exponents (s_1 and s_2) coincide. After imposing these constraints, V and Z become

$$V = c_2\left(\frac{s_2}{s_1} + 1\right) + c_3 \equiv \Lambda (= 2c_2 + c_3 \text{ if } s_2 = s_1), \tag{7.219}$$

$$Z = -\left|\frac{q}{p}\right| \equiv -Z_0^2. \tag{7.220}$$

We have two cases: $z = 1 + \theta/2$ and $\theta = 2$ (and the one in the intersection: $z = 2$, $\theta = 2$).

[18]One may wonder why we did not find a purely *hvLf* for these values of the exponents in the previous subsection. The reason is that for $\theta = 5/2,\ z = 3/2$ we have $\mathcal{X}_{(\theta,z)} = 0$, which implies the vanishing of V_{fi} in the purely *hvLf* case. In fact, to recover the pure solution, we have to set $K = q = c_2 = 0$, and since we have already set $c_1 = 0$, this would make $V_{\text{fi}} = 0$.

1. $z = 1 + \frac{\theta}{2},\ \theta \neq 2$. In this situation, it is possible to find a solution which imposes no further constraints on V and $V_{\rm bh}$. This reads

$$f(r) = -Kr^{3-\theta/2} + \frac{\left[12Z_0^2 r^{4-\theta} - 2\Lambda r^{\theta}\right]}{3(\theta-2)^2}. \tag{7.221}$$

The case $z = 1, \theta = 0$, in which we expect to recover AdS_4 asymptotically is a particularization of this. The blackening factor reads then

$$f(r) = -\frac{\Lambda}{6} - Kr^3 + Z_0^2 r^4, \tag{7.222}$$

Assuming a negative cosmological constant, $\Lambda = -|\Lambda|$, this can be rewritten as

$$f(r) = \frac{|\Lambda|}{6}\left[1 - Kr^3 + \frac{6Z_0^2}{|\Lambda|}r^4\right]. \tag{7.223}$$

If we define $dT^2 = |\Lambda|dt^2/6$, $dR^2 = 6dr^2/|\Lambda|$, the metric Eq. (7.113) becomes

$$ds_f^2 = \frac{L^2}{R^2}\left\{\left[1 - KR^3 + \frac{|\Lambda|Z_0^2}{6}R^4\right]dT^2 - \frac{dR^2}{\left[1 - KR^3 + \frac{|\Lambda|Z_0^2}{6}R^4\right]} - d\vec{x}^2\right\}, \tag{7.224}$$

which, of course, asymptotes to AdS_4 as $R \to 0$, and is such that $\exists\, R_h \in \mathbb{R}^+ \,/\, f(R_h) = 0$ for $K > 0$. Similarly, the metric blows up as $R \to \infty$, behaving as a *hvLf* with $\theta = 4,\ z = 3$. Indeed,

$$ds_f^2 \overset{R\to\infty}{\sim} \frac{L^2}{R^2}\left[R^4 dT^2 - \frac{dR^2}{R^4} - d\vec{x}^2\right] \tag{7.225}$$

up to constants; if we make now the change $\rho \sim 1/R$

$$ds_f^2 \overset{\rho\to 0}{\sim} L'^2\rho^2\left[\frac{dT^2}{\rho^4} - d\rho^2 - d\vec{x}^2\right], \tag{7.226}$$

we find a *hvLf* metric with $\theta = 4,\ z = 3$ as we have said. If we plug these values $\theta = 4,\ z = 3$ in Eq. (7.221) we find a new solution, which behaves asymptotically as this one (with the IR and UV regions interchanged). Indeed, its blackening factor reads

$$f(r) = Z_0^2\left[1 - Kr + \frac{|\Lambda|}{6Z_0^2}r^4\right], \tag{7.227}$$

and with the redefinitions $dR^2 = dr^2/Z_0^2,\ dT^2 = dt^2/Z_0^2$

$$ds_f^2 = L^2R^2\left\{\left[1 - KR + \frac{|\Lambda|Z_0^2}{6}R^4\right]\frac{dT^2}{R^4} - \frac{dR^2}{\left[1 - KR + \frac{|\Lambda|Z_0^2}{6}R^4\right]} - d\vec{x}^2\right\}. \tag{7.228}$$

As $R \to 0$, it becomes a *hvLf* with $\theta = 4,\ z = 3$, and as $R \to \infty$,

$$ds_f^2 = L^2R^2\left[dT^2 - \frac{dR^2}{R^4} - d\vec{x}^2\right], \tag{7.229}$$

which we can rewrite as ($\rho = 1/R$)

$$ds_f^2 = \frac{L'^2}{\rho^2}\left[dT^2 - d\rho^2 - d\vec{x}^2\right], \tag{7.230}$$

which is AdS_4.

2. $\theta = 2$. This case imposes the constraint $Z_0^2 = -\frac{\Lambda}{2}$, and can be solved for any value of z. The general form of $f(r)$, which applies for $z \neq 2$ is now

$$f(r) = \frac{2Z_0^2 r^2}{(z-2)^2} + r^z K_1 + r^{2(z-1)} K_2, \tag{7.231}$$

whereas for $z = 2$ we have

$$f(r) = 2r^2 \log(r)\left[K_2 + Z_0^2 \log(r)\right] + K_1 r^2. \tag{7.232}$$

For example, if we consider the case $\theta = 2$, $z = 1$, we inmediatly find the asymptotically flat metric (as $r \to 0$)

$$f(r) = 1 - Kr + 2Z_0^2 r^2, \tag{7.233}$$

$$ds_f^2 = l^2\left\{dt^2\left[1 - Kr + 2Z_0^2 r^2\right] - \frac{dr^2}{\left[1 - Kr + 2Z_0^2 r^2\right]} - d\vec{x}^2\right\}, \tag{7.234}$$

for which, once more $\exists\, r_h \in \mathbb{R}^+ \;/\; f(r_h) = 0$ for $K > 0$. As $r \to \infty$, up to constants, it behaves as

$$ds_f^2 \overset{[R\to+\infty]}{\sim} l'^2\left[e^{2R}dt^2 - dR^2 - d\vec{x}^2\right], \tag{7.235}$$

where we defined $R = \log r$. This is nothing but $AdS_2 \times \mathbb{R}_2$. On the other hand, if we set $\theta = 2$, $z = 2$, from Eq. (7.232) we find

$$f(r) = 2r^2 \log(r)\left[-K + Z_0^2 \log(r)\right] + r^2 \overset{[R=\log r]}{=} e^{2R}\left[1 - KR + 2Z_0^2 R^2\right], \tag{7.236}$$

$$ds_f^2 = l^2\left\{dt^2\left[1 - KR + 2Z_0^2 R^2\right] - \frac{dR^2}{\left[1 - KR + 2Z_0^2 R^2\right]} - d\vec{x}^2\right\} \tag{7.237}$$

which is nothing but Eq. (7.234).

7.10 Conclusions

We have started by describing several procedures to construct, from known black-hole and black-brane solutions of any ungauged supergravity theory, non-trivial gravitational solutions whose "near-horizon" and "near-singularity" limits are Lifshitz-like metrics with hyperscaling violation which depend on the physical parameters of the original black-hole solution.

In particular, this shows that hvLf metrics are very generic, and not restricted to particular EMD models as frequently assumed in the previous literature.

Since the new solutions can be constructed from any black-hole solution of any ungauged supergravity, many of them can be easily embedded in String Theory.

In the last sections, we have studied purely *hvLf* and *hvLf*-like solutions of the general class of theories defined by the Lagrangian (7.80), which covers any theory of gravity coupled to an arbitrary number of scalars and vector fields up to two derivatives. We have obtained the general effective one-dimensional equations of motion that need to be solved in order to obtain *hvLf*-like solutions. The general analysis is intended to complete the case-by-case results present in the literature in a unified framework: given a particular kinetic matrix $(I_{\Lambda\Sigma}(\phi),\ R_{\Lambda\Sigma}(\phi))$, a scalar metric $\mathcal{G}_{ij}(\phi)$ and a scalar potential $V(\phi)$, the equations of motion of the theory follow trivially by plugging them into (7.85)-(7.87) and the Hamiltonian constraint (7.84). For this broad family of theories, we have discussed the existence and properties of purely *hvLf* solutions attending to the presence (or absence) of non-constant scalar fields and non-vanishing black hole and scalar potentials.

In the context of $\mathcal{N}=2$ FI Supergravity, we have studied the t^3-model, for which we have explicitly constructed two consistent *axion-free* embeddings in the E.M.D. system, one of which is, in turn, embedded in Type-IIB String Theory for a particular choice of embedding tensor θ_M.

In addition, we obtained the general form of the $f(r)$ function (for the set of metrics determined by Eqs. (7.178) and (7.113)), up to a constraint, for a rather general family of (Supergravity inspired) scalar and black-hole potentials, and explicitly constructed some dyonic solutions for the t^3 truncations considered. We have provided a straightforward procedure to construct asymptotically *hvLf* solutions covered by Eqs. (7.178) and (7.113) for the family of theories specified by Eqs. (7.184), reducing the task to solving a single algebraic constraint given by Eq. (7.192).

We have avoided, on purpose, the term *black hole* to denote the *hvLf*-like solutions obtained in this chapter. The reason is that, although may of the solutions *look like* black holes, a proof (for example by constructing the corresponding Penrose-Carter diagram) is still missing. Therefore, any results obtained from them implicitly assuming that they do represent a black hole must be interpreted carefully, knowing that those would be yet to be proven statements.

8

Holographic entanglement entropy in hvLf geometries

This chapter is based on

Pablo Bueno and Pedro F. Ramírez,
"Higher-curvature corrections to holographic entanglement entropy in geometries with hyperscaling violation",
JHEP ***1412*** *(2014) 078. [*`arXiv:1408.6380` *[hep-th]] [94].*

Let us consider again the familiy of Lifshitz-like metrics with hyperscaling violation defined in the previous chapter. In order to have a clear interpretation of a constant r slice (with $r \to 0$) of the geometry defined by (7.3) as the boundary of the metric, let us consider $\theta < d$ metrics from now on[1]. $\theta > d$ would correspond to a negative effective number of spatial dimensions according to the arguments previously explained. Also, when $\theta > 0$, hvLf metrics suffer from a curvature UV-singularity in the Einstein frame: indeed, the Kretschmann invariant scales as $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \sim r^{-4\theta/d}$. In appearance, this means that hvLf metrics with $\theta < 0$ are completely reliable in the UV, whereas those with $0 < \theta < d$ need to be completed asymptotically, something which is usually performed through the assumption that spacetime is described by (7.3) only above some scale r_F, but asymptotes to some well-behaved solution, such as AdS_{d+2}, as $r << r_F$. As explained in [136], this statement is imprecise. The authors argue that hvLf geometries with $\theta \neq 0$ typically require a UV-divergent (linear) dilaton, which allows one to tune the curvature singularity (appearing in the cases in which $0 < \theta < d$) by changing to an appropriate Weyl frame, and completely absorb it in such scalar field. The linear running character of the dilaton is a characteristic feature of general hvLf backgrounds (with $\theta \neq 0$) so one needs to be careful when interpreting the UV physics from the field theory perspective not only for $\theta > 0$, but also for $\theta < 0$[2]. We will come back to this in the discussion section.

There are several ways in which holography allows us to study the properties of the dual quantum field theories (QFTs). A prominent example is the computation of *entanglement entropy* (EE), which will be the subject of this chapter.

Entanglement entropy has indeed become an essential tool in fields as diverse as condensed matter [8, 218, 292, 364], quantum information [345, 425], String Theory and quantum gravity [68, 73, 101, 236, 295, 310, 336, 347, 378, 379, 400, 415], and QFT [100, 117, 119, 123, 275, 276, 388].

[1]The formulation of the holographic dictionary for hvLf geometries has been addressed in [135, 136].
[2]We thank Robert C. Myers and Ioannis Papadimitriou for their comments and explanations about this point

As explained in the introduction, the ultraviolet (UV) behaviour of the EE for general $(d+1)$-dimensional QFTs is expected to be [119]:

$$S = \frac{k_{d-1}}{\delta^{d-1}} + \ldots + \frac{k_1}{\delta} + k_0 \log \frac{l}{\delta} + S_0 \,, \tag{8.1}$$

where δ is a short distance cutoff, S_0, k_0 and k_i constants, and l is a characteristic length of A. The coefficient of the leading term is proportional to the *area* of the boundary of A ($k_{d-1} \sim l^{d-1}$), a behaviour which is usually argued to be caused by the entanglement between degrees of freedom living at both sides of ∂A. This is the so-called *area law* [73, 400] of entanglement entropy. When the leading term in EE depends on the characteristic length of A in a different fashion, we speak about a violation of this law. One such kind of violation occurs when the leading contribution to S contains a factor which scales logarithmically with the characteristic length of A (see below). Another example of this happens when the leading term scales with a power of l different from the dimension of ∂A (see, e.g. [333]).

An interesting point to notice is the fact that k_0 is universal in the following sense: if we shift $\delta \to \delta\epsilon$, the coefficients k_i are shifted by $k_i \to k_i \epsilon^{-i}$, whereas k_0 remains the same by virtue of the properties of the logarithm (the shift is absorbed in S_0). As a consequence, k_0 is independent of the regularization prescription (and usually related to the central charge of the underlying QFT in the case of CFTs).

As we have said, although the area law turns out to hold for a vast range of systems, it is well-known that this is not always the case. A paradigmatic example is given by 2D CFTs, where EE scales logarithmically with the length of A, l, and k_0 turns out to be proportional to the central charge of the theory [100, 238]

$$S = \frac{c}{3} \log \frac{l}{\delta} \,. \tag{8.2}$$

In higher dimensional theories, violations of the area law appear in QFTs with Fermi surfaces [326, 407, 426]. In such cases, S acquires a logarithmic dependence on the characteristic length of A

$$S \sim (l k_F)^{(d-1)} \log(l k_F) \,, \tag{8.3}$$

being k_F the Fermi momentum[3], and the area law is violated. It has been argued that certain QFTs with Fermi surfaces might be holographically engineered by considering the family of hvLf metrics in the case $\theta = d-1$ [162, 246, 349], as we will review in section 8.2; indeed in these cases, the HEE exhibits a logarithmic violation of the area law (note that the case $\theta = 0$ precisely corresponds to AdS_3). Also, as observed in [162], the leading term in the HEE expression will not respect this *law* for any value of $(d-1) \leq \theta \leq d$.

In the context of holography, EE for theories dual to Einstein gravity can be computed through the Ryu-Takayanagi prescription [379][4]. According to this, the holographic entanglement entropy (HEE) for a certain region A living in the boundary of some asymptotically AdS_{d+2} spacetime is given by

$$S_{EG} = \underset{m \sim V}{\operatorname{ext}} \left[\frac{\mathcal{A}(m)}{4G} \right] , \tag{8.4}$$

[3]Such behaviour comes from the effective 2D CFT which governs the physics of modes at the Fermi surface [385, 407]

[4]Remarkably, this prescription has been recently proven under certain conditions in [296].

where m are codimension-2 bulk surfaces homologous to A with $\partial m = \partial A$, and $\mathcal{A}(m)$ is the d-dimensional volume (*area*) of m. Hence, HEE in theories with an Einstein gravity dual is obtained by extremizing the area functional over all possible bulk surfaces homologous to A whose boundary coincides with ∂A.

The situation changes when we start considering higher-curvature terms in the bulk Lagrangian. In such cases, the Ryu-Takayanagi prescription does not produce the correct answer for the HEE. Actually, (9.8) might be somehow regarded as a generalization of the Bekenstein-Hawking formula for the entropy of black holes [27,39,233], which suggests that the expression for the EE in the presence of higher-derivative gravities might be obtained by applying the same generalization to Wald's formula, which gives the black hole entropy in this class of theories [420][5]

$$S_{\rm Wald} = \frac{1}{4G}\int_{\rm H} d^2y\sqrt{h_{\rm H}}\frac{\partial\mathcal{L}}{\partial R_{\mu\nu\rho\sigma}}\epsilon_{\mu\nu}\epsilon_{\rho\sigma}\,. \tag{8.5}$$

However, in [247] this was shown to be wrong, since this expression would produce incorrect universal terms. Alternative expressions yielding the right terms are known for Lovelock gravities [247, 256, 381] as well as for curvature-squared theories [185, 336]. Remarkably enough, a general formula for any theory involving arbitrary contractions of the Riemann tensor $\mathcal{L}(R_{\mu\nu\rho\sigma})$, which seems to satisfy several consistency checks, has been recently proposed by Dong [161] (see also, e.g. [65, 66, 103, 170])). The corresponding expressions would contain a Wald-like term as well as additional terms involving contractions of extrinsic curvatures (which vanish in the case of a Killing horizon) with second derivatives of the Lagrangian with respect to the Riemann tensor.

In this chapter we are going to study the effects of including higher-order curvature terms in the gravity Lagrangian on the HEE formula for hvLf geometries. The motivation for this study is manyfold. On the one hand, studying higher-order gravity Lagrangians in the holographic context is intrinsically interesting, given that such terms generically appear as α' corrections in the appropriate String Theory embedding, corresponding to moving away from the infinitely coupled regime in the dual field theory. Secondly, as we have explained, hvLf geometries have been shown to provide interesting violations of the area law of EE for certain values of θ and, particularly interestingly, logarithmic terms for $\theta = d-1$, in whose case they have been argued to be intimately related to certain condensed matter systems. A natural question to ask is how the inclusion of higher-curvature terms will alter the structure of the HEE and whether these modifications can lead to new logarithmic terms, which might contain universal information about the dual theory (see the discussion about the UV interpretation of hvLf metrics in section 8.3). Also, the expressions for HEE in higher-order Lagrangians which are known at present are restricted to a handful of theories, as explained before, and have not been *proven* in general. This makes interesting to check how they perform in different situations, probing whether they produce sensible results in the different cases. An example of this is given by Gauss-Bonnet gravity in $d = 2$ (4-dimensional spacetime). In such case, the HEE (which can be obtained using the so-called Jacobson-Myers (JM) functional [256])[6] should not change with respect to the Einstein gravity case, since the equations of motion are unchanged in this case, and any remainder of λ_{GB} should be completely removed by including the boundary term prescribed in the JM functional.

[5]In (8.5), $\mathcal{L}$ is the gravity Lagrangian, H stands for the horizon, $h_{\rm H}$ is the induced metric on it and $\epsilon_{\mu\nu}$ is a binormal to H.

[6]See section 8.1.

In the next section we study the structure of divergences of HEE for a stripe in the boundary of hvLf metrics when $\theta \leq 0$, for higher-order gravities. We start with curvature-squared, for which the HEE functional is known [185], dealing with the cases of R^2, Gauss-Bonnet and Ricci^2. We will find that a single new divergence appears in all cases, and how it cannot become logarithmic for any value of θ except for $\theta = 0$, $d = 1$, corresponding to the well-known AdS$_3$ case. However, extending the analysis to higher-curvature (nth-order) gravities we will find that new logarithmic divergences will show up for

$$\theta = \frac{d(d-1)}{d-2(n-1)}\,, \tag{8.6}$$

provided $d < 2(n-1)$. We will therefore find that an infinite family of hvLf geometries produces new logarithmic contributions to the HEE formula when these geometries are embedded in higher-curvature gravities. For R^2 gravity we will be able to compute the $\mathcal{O}(\lambda_1)$ correction to the universal constant term as well. Also, in the section devoted to Gauss-Bonnet gravity, we show explicitly that the boundary term in the JM functional exactly cancels the bulk surface contribution when $d = 2$, as expected.

In section 8.2 we study the case $0 < \theta < d$, for which we consider a UV AdS-completion of the geometry, following the steps of [349]. We will find that (8.6) holds for the appearance of logarithmic contributions to the HEE, with the difference that now $d > 2(n-1)$. However, both conditions together will turn out to restrict the allowed values of $\theta > 0$ to the well-known case of $\theta = d-1$ [162, 246, 349], corresponding to Einstein gravity.

In section 8.3 we summarize our findings, comment on possible extensions and conclude.

Finally, in appendix D we consider the case in which the anisotropic scaling occurs along a spatial direction instead of time, which can be understood as a double Wick rotation of the standard hvLf geometry [7,154], and analyze how this changes the discussion of the previous sections. New logarithmic terms are found here for some combinations of z, θ and d.

8.1 HEE for hvLf geometries in higher-curvature gravities I: $\theta \leq 0$

∘ *Einstein gravity.*

Before considering higher-curvature corrections, let us start reviewing the Einstein gravity result for the HEE of hvLf geometries. We do so here for the class of metrics with $\theta \leq 0$, which we study in this section. Along this chapter we will consider an entangling region A consisting of a multi-dimensional infinite strip s of width l and infinite length $L_S \to +\infty$ (this length plays the role of an IR cut-off), $s = \{(t_E, r, x_1, x_2, ..., x_d)$ s.t., $t_E = 0$, $x_1 \in [-l/2, l/2]$, $x_{2,...,d} \in (-L_S/2, +L_S/2)\}$. As explained in the introduction, HEE for field theories dual to Einstein gravities[7] can be computed using the Ryu-Takayanagi prescription [379]

$$S_{EG} = \frac{1}{4G}\int_m d^d x\sqrt{g_m}\,, \tag{8.7}$$

[7]By this we mean theories with Lagrangians given by $\mathcal{L} = R - 2\Lambda + \mathcal{L}_{\text{other fields}}$.

where m is the bulk surface homologous to A, with $\partial m = \partial A$, which extremizes the above functional, and g_m is the determinant of the induced metric on m.

The translational symmetry of the strip along the directions $2, ..., d$ allows us to parametrize the entangling surface m as $r = h(x_1)$. For our hvLf geometry (1.114), the induced metric on such a surface reads

$$ds_m^2 = L^2 h^{\frac{2(\theta-d)}{d}} \left[\left[1+\dot{h}^2\right] dx_1^2 + d\vec{x}^2_{(d-1)}\right] , \tag{8.8}$$

where $d\vec{x}^2_{(d-1)} \equiv dx_2^2 + ... + dx_d^2$. Using this expression and the fact that m must be mirror symmetric with respect to the plane $x_1 = 0$, we find

$$S_{EG} = \frac{L^d L_S^{(d-1)}}{2G} \int_0^{l/2} dx_1 \, h^{(\theta-d)} \sqrt{1+\dot{h}^2} \, , \tag{8.9}$$

The *Lagrangian* does not depend explicitly on x_1, so we have a conserved quantity

$$h_*^{(\theta-d)} = \frac{h^{(\theta-d)}}{\sqrt{1+\dot{h}^2}} \, , \tag{8.10}$$

where h_* is the turning point of the surface, in which $\dot{h}|_{h_*} = 0$. Substituting this expression in (8.9), we find

$$S_{EG} = \frac{L^d L_S^{(d-1)} h_*^{(\theta-d+1)}}{2G} \int_{\delta/h_*}^{1} \frac{u^{(\theta-d)} du}{\sqrt{1-u^{2(d-\theta)}}} \, , \tag{8.11}$$

where we made the change of variable $u = h/h_*$ and introduced the UV cut-off $(h(x_1) \to \delta) \leftrightarrow (x_1 \to \pm l/2)$. The turning point is related to the strip width through

$$\frac{l}{2} = \int_0^{l/2} dx_1 = h_* \int_0^1 \frac{u^{(d-\theta)} \, du}{\sqrt{1-u^{2(d-\theta)}}} = h_* \frac{\sqrt{\pi}\Gamma\left(\frac{1+d-\theta}{2(d-\theta)}\right)}{\Gamma\left(\frac{1}{2(d-\theta)}\right)} \, . \tag{8.12}$$

These two integrals allow us to obtain the final expression for the entanglement entropy of the strip

$$S_{EG} = \frac{L^d L_S^{(d-1)}}{2G(d-\theta-1)} \left[\delta^{-(d-\theta-1)} - (l/2)^{(\theta-d+1)} \left[\frac{\sqrt{\pi}\Gamma\left(\frac{1+d-\theta}{2(d-\theta)}\right)}{\Gamma\left(\frac{1}{2(d-\theta)}\right)}\right]^{(d-\theta)}\right] . \tag{8.13}$$

This is the beautiful formula found in [162]. As we can see, the scaling behavior of the HEE gets modified with respect to the AdS$_{d+2}$ case [378] by factors with dimensions of (length)$^\theta$. In particular, we find a corrected exponent for the divergent term of order

$$\mathfrak{B}_0 \equiv d - \theta - 1 \, . \tag{8.14}$$

Of course, $\mathfrak{B}_0$ is always positive for $\theta < 0$. One can introduce an intermediate scale r_F as explained in the introduction, which would modify the factors $\delta^\theta \to (\delta/r_F)^\theta$ and $(l/2)^\theta \to (l/(2r_F))^\theta$. When $\theta = 0$, we recover the usual AdS$_{d+2}$ expression [378]

$$S_{EG} = \frac{L^d L_S^{(d-1)}}{2G(d-1)} \left[\delta^{-(d-1)} - (l/2)^{(1-d)} \left[\frac{\sqrt{\pi}\Gamma\left(\frac{1+d}{2d}\right)}{\Gamma\left(\frac{1}{2d}\right)}\right]^d\right] , \tag{8.15}$$

which in the limit case of $d = 1$, corresponding to AdS$_3$, yields a logarithmic divergence

$$S_{EG} = \frac{L}{2G} \log \left[\frac{l}{\delta} \right] . \tag{8.16}$$

It is well-known that hvLf geometries can produce logarithmic terms in the HEE for $\theta = d-1$. However, given that these cases correspond to metrics with $0 < \theta < d$ for $d \geq 2$, we will review them in section 8.2, along with the corresponding new higher-order terms.

∘ *Higher-curvature corrections to HEE.*
We are interested now in considering higher-order curvature corrections to the bulk action and see how they affect the HEE expression for hvLf geometries. In general, the gravitational action will be given by Einstein's gravity plus an (infinite) sum of higher-curvature terms with small coupling constants (otherwise, the semiclassical approximation would not make sense)

$$\mathcal{I}_g = \frac{1}{16\pi G} \int d^{d+2}x\sqrt{g} \left[R + \frac{d(d+1)}{\tilde{L}^2} + \tilde{L}^2 \left[\lambda_1 R^2 + \lambda_2 R_{\mu\nu}R^{\mu\nu} + \lambda_3 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \right] + \tilde{L}^4 \mathcal{O}(R^3) \right] , \tag{8.17}$$

being $\tilde{L}$ a length scale which would coincide with the AdS$_{d+2}$ radius L for Einstein gravity, but would be different in general otherwise, and $\lambda_{1,2,3,...}$ dimensionless couplings.

The next step would correspond now to choose some *matter* content and solve the equations of motion for the corresponding fields trying to determine if our hvLf family of metrics (7.3) can be embedded into the theory. The case of curvature-squared gravity was studied in [350], where the authors consider an EMD system with general curvature-squared corrections. For our purposes, it suffices to recall the fact that hvLf geometries are indeed solutions of the corresponding equations of motion, and are expected to appear as well as solutions to similar EMD gravities with even higher-curvature corrections. Another interesting piece of information we can extract from [350] is the fact that the NEC arising in a general EMD curvature-squared gravity reduces in general to a pair of conditions on (z, θ) and the couplings of the new terms, plus the well-known NEC of the Einstein gravity case [162]

$$(z-1)(z-\theta+d) \geq 0 \, , \tag{8.18}$$
$$(d-\theta)(d(z-1)-\theta) \geq 0 \, , \tag{8.19}$$

which in the case under consideration in this chapter, i.e., $d > \theta$, reduces to the condition $z \geq 1$. From now on, we restrict ourselves to this case, although as we will see, our results would not get modified for $z < 1$ since z will not appear in the exponents of the different terms in the HEE expressions for our hvLf geometries[8].

Unfortunately, computing HEE in general higher-curvature gravities is a very hard task at present because Dong's recipe [161] turns out to be difficult to apply in most cases, with some exceptions such as Lovelock [247, 256], curvature-squared [185] and $f(R)$ gravities [161, 420]. Nevertheless, making use of the results found in curvature-squared gravity plus some general arguments, which we will discuss in a moment, we will to try to say something about the structure of divergences of the HEE in any higher-curvature gravity for our hvLf geometries.

[8]The situation will change in appendix D, where we will consider a doubly Wick-rotated version of (1.114).

There are two steps one needs to take in order to successfully obtain the HEE expression in any higher-curvature gravity for any background, assuming the HEE functional is known. The first is extremizing such a functional, whereas the second corresponds to evaluating the on-shell integral. The first one is undeniably harder in general, since the equations of motion we pretend to solve will usually be of high order in derivatives, and very non-linear. However, we can note the following: in the HEE expression we will find in general a sum of divergent terms coming from the on-shell evaluation of the integral near the boundary, plus a constant term related to the bulk contribution. In geometries in which the higher the order of the curvature term the faster it goes to zero in the UV, we will find an expression consisting of a leading Einstein gravity divergence plus possible subleading divergences coming from the higher-order terms, plus a constant term. The question is now how the fact that the entangling surface is different in higher-order gravities with respect to the Einstein gravity case affects the HEE expression, given that the functional we need to extremize is different. We expect the surface to be significantly different away from the UV, where the new terms become large, producing therefore new corrected constant terms. However, as we approach the boundary, where the divergences are to appear, the higher-order terms will die out, and the shape of the entangling surface should not differ much from the Einstein gravity one. This is analogous to computing the area for different surfaces sharing boundary with the extremal area one, m. The result will of course differ, but the order of the divergences will be the same as the one found for m. Thus, it is reasonable to expect that the new divergent terms (if any) appearing in the HEE expression for higher-curvature terms will be produced from the evaluation of the on-shell integral using the surface which extremizes the area functional of Einstein gravity, without having to find the surface which extremizes the new functional. In other words, the new entangling surface should not change the structure of divergences with respect to the one with extremal area and this has two interesting consequences. First, we can identify the order of the divergences of higher-order gravity terms using the extremal area surface, and second, every new divergence will appear at order $\mathcal{O}(\lambda)$ in the corresponding gravitational coupling. Therefore, any term of order $\mathcal{O}(\lambda^2)$ or higher will appear next to a constant, arising from the bulk contribution to the integral.

At this point it is convenient to stress that the study of the structure of divergences of the HEE is physically motivated by the fact that it allows us to determine the dependence of the different terms with the size of the entangling region. In particular, we can use this to check if the area law holds, unveil the presence of universal terms, etc.

Let us now turn to the real calculations. We are going to study in full detail the case of R^2 gravity, in which we will be able to compute the corrected extremal surface. This will allow us to illustrate how the above argument works, and use it to compute the structure of divergences for general curvature-squared gravities, including the more involved cases of Gauss-Bonnet and Ricci^2 gravities. We will finish this section showing how the results found for these theories allow us to conjecture the form of all divergences in any higher-order curvature gravity for our hvLf metrics. Let us start with curvature-squared gravities.

8.1.1 R^2 gravity

The most general curvature-squared gravity action can be written in terms of three contractions involving the Riemann tensor. These can be chosen to be

$$\mathcal{I}_{\text{curv}^2} = \frac{1}{16\pi G}\int d^{d+2}x\sqrt{g}\left[R + \frac{d(d+1)}{\tilde{L}^2} + \tilde{L}^2\left[\lambda_1 R^2 + \lambda_2 R_{\mu\nu}R^{\mu\nu} + \lambda_{GB}\mathcal{X}_4\right]\right], \quad (8.20)$$

where $\mathcal{X}_4 = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ is the Gauss-Bonnet term, which in four bulk dimensions corresponds to the Euler density of the spacetime manifold.

In the case of R^2 gravity, the HEE functional[9] is given by [185]

$$S_{R^2} = \frac{1}{4G}\int_m d^d x\sqrt{g_m}\left[1 + 2\lambda_1\tilde{L}^2 R\right]. \quad (8.21)$$

For our hvLf metrics (1.114) the Ricci scalar reads

$$R = \kappa\frac{r^{-2\theta/d}}{\tilde{L}^2}, \quad (8.22)$$

where we have defined the constant

$$\kappa \equiv -\frac{2\tilde{L}^2}{L^2}\left[z^2 + zd + \frac{d+1}{2}\left[d - 2\theta - \frac{\theta}{d}(2z-\theta)\right]\right]. \quad (8.23)$$

As a curiosity, there are certain combinations of (z,θ) for which κ vanishes, meaning that the R^2 contribution identically vanishes, and does not produce any correction at all with respect to the Einstein gravity result. The corresponding curves for which this happens are shown in Figure 8.1. Leaving this case aside, the expression for the entanglement

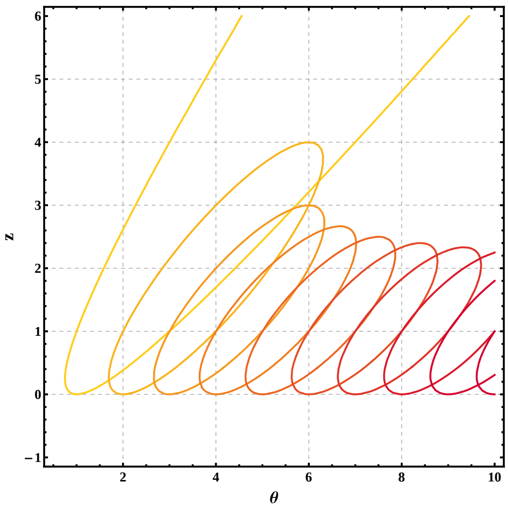

Figure 8.1: Curves (θ, z) for which the Ricci scalar of hvLf metrics vanishes. $d = 1$ is depicted in yellow, whereas darker lines correspond to $d = 2, 3, \ldots$

[9]The functional proposed by [185] for the HEE of curvature-squared gravities has been used in several works, including [2, 3, 67].

entropy of the strip becomes, using (9.60)

$$S_{R^2} = \frac{L^d L_S^{(d-1)}}{2G} \int_0^{l/2} dx_1\, h^{(\theta-d)} \sqrt{1+\dot{h}^2} \left[1 + 2\kappa\lambda_1 h^{-2\theta/d}\right] . \tag{8.24}$$

Since the functional does not depend on x_1 explicitly, there is again a first integral which we can use to write the expression for $\dot{h}$ in terms of h. We have

$$\sqrt{1+\dot{h}^2} = \frac{f(h)h^{(\theta-d)}}{f(h_*)h_*^{(\theta-d)}} , \quad \text{with} \;\; f(x) \equiv \left[1 + 2\kappa\lambda_1 x^{-2\theta/d}\right] , \tag{8.25}$$

where h_* is again the turning point of the surface, characterized by $\dot{h}|_{h_*} = 0$. We can use this relation to rewrite (8.24) in terms of $u \equiv h/h_*$ as

$$S_{R^2} = \frac{L^d L_S^{(d-1)} h_*^{\theta-d+1}}{2G} \int_{\delta/h_*}^{1} du \, \frac{u^{(\theta-d)} f(uh_*)}{\sqrt{1 - u^{2(d-\theta)} \frac{f(h_*)^2}{f(uh_*)^2}}} , \tag{8.26}$$

where we have introduced again an ultraviolet cut-off $h \to \delta$ to account for the divergent terms. Note that despite the intricated appearance of the integrand it is already possible at this level to keep track of those divergences. Indeed we can study its behaviour in the limit $u \to 0$

$$\lim_{u\to 0} \frac{u^{(\theta-d)} f(uh_*)}{\sqrt{1 - u^{2(d-\theta)} \frac{f(h_*)^2}{f(uh_*)^2}}} = u^{(\theta-d)} \left[1 + 2\kappa\lambda_1 (uh_*)^{-2\theta/d}\right] \left[1 + \mathcal{O}\left(u^{2(d-\theta)}\right)\right] , \tag{8.27}$$

so the terms with a negative power in u, and therefore those resulting into divergences, arise from the product $u^{(\theta-d)} \left[1 + 2\kappa\lambda_1 (uh_*)^{-2\theta/d}\right]$. This agrees with what we anticipated in our previous discussion: had we taken the Einstein gravity surface (9.60), and computed the HEE integral (8.24), we would have found the same divergent terms. It is also important to stress that this expression is valid for any value of the coupling λ_1, so if we expanded in powers of λ_1, the only divergence would appear at order $\mathcal{O}(\lambda_1)$, as anticipated. Taking into account (8.27) we find that the entanglement entropy is of the form

$$S_{R^2} = \frac{L^d L_S^{(d-1)}}{2G} \left[\frac{1}{\mathfrak{B}_0}\delta^{-\mathfrak{B}_0} + \frac{2\kappa\lambda_1}{\mathfrak{B}_1}\delta^{-\mathfrak{B}_1}\right] + S_0 , \tag{8.28}$$

with

$$\mathfrak{B}_0 \equiv d - \theta - 1 , \tag{8.29}$$

$$\mathfrak{B}_1 \equiv \mathfrak{B}_0 + \frac{2\theta}{d} , \tag{8.30}$$

and S_0 being a constant term which we will discuss later. As we can see, the inclusion of the R^2 term introduces a new divergence in the HEE. This contribution is not dominant, and the leading divergence is again the Einstein gravity, one as expected. It is also impossible to produce a logarithmic divergence from this term, since this would correspond to $\theta = \frac{d(d-1)}{(d-2)}$, which is larger than 0 for any $d > 1$. An exception is $d = 1$, $\theta = 0$, which would correspond to AdS_3, for which both $\mathfrak{B}_0$ and $\mathfrak{B}_1$ would be logarithmic. In the special case of Lifshitz geometries, $\theta = 0$, the Ricci scalar is constant and the entanglement entropy diverges as

$$S_{R^2}|_{\theta=0} = (1 + 2\kappa|_{\theta=0}\lambda_1)\, S_{EG}|_{\theta=0} , \tag{8.31}$$

where $S_{EG}|_{\theta=0}$ is just the HEE for a strip in AdS$_{d+2}$ (recall that, although $z \neq 1$ in general, the dynamical exponent does not enter into the HEE expression for Einstein gravity), which can be read from (8.13), and

$$\kappa|_{\theta=0} = -\frac{2\tilde{L}^2}{L^2}\left[z^2 + zd + \frac{d(d+1)}{2}\right] . \tag{8.32}$$

As we can see, the dynamical exponent does appear in the HEE formula (through κ) when we consider this curvature-squared contribution, as opposed to the Einstein gravity case[10]. However, it does not contribute to the exponents of the divergences, and it will not do so for any higher-curvature gravity, simply because the induced metric on any entangling surface extremizing the corresponding functional will not depend on z in general, given that it only appears in the g_{tt} component of the hvLf metric (1.114). In order to make z appear in the exponents of the HEE terms, we need to consider an anisotropic scaling of a spatial coordinate instead of time. This will be studied in appendix D. The appearance of the new divergence $\delta^{-\mathfrak{B}_1}$ is a distinctive feature of hvLf geometries: for AdS or even Lifshitz geometries, the inclusion of additional higher-curvature terms in the bulk action just shifts the coefficient in front of $\delta^{-\mathfrak{B}_0}$, without producing any new divergent term.

Coming back to R^2 gravity, in order to extract information about the finite term S_0 in (8.28) we are going to consider the case $\lambda_1 << 1$ (which is a reasonable assumption as we are considering the higher-curvature terms to be corrections to the leading Einstein gravity action), so we can Taylor-expand around $\lambda_1 = 0$. We do so in the expression for the entanglement entropy up to order λ_1 and perform the integration afterwards. The result reads

$$S_0 = -\frac{L^d L_S^{(d-1)}}{2G}\left\{\frac{G_0 h_*^{-\mathfrak{B}_0}}{\mathfrak{B}_0} + 2\kappa\lambda_1 h_*^{-\mathfrak{B}_1}\left[\frac{G_0}{(\mathfrak{B}_0+1)} + G_1\left[\frac{1}{\mathfrak{B}_1} - \frac{1}{(\mathfrak{B}_0+1)}\right]\right]\right\} + \mathcal{O}(\lambda_1^2) , \tag{8.33}$$

where we defined the constants

$$G_0 \equiv \frac{\sqrt{\pi}\,\Gamma\left(\frac{\mathfrak{B}_0+2}{2(\mathfrak{B}_0+1)}\right)}{\Gamma\left(\frac{1}{2(\mathfrak{B}_0+1)}\right)} , \quad G_1 \equiv \frac{\sqrt{\pi}\,\Gamma\left(\frac{2+2\mathfrak{B}_0-\mathfrak{B}_1}{2(\mathfrak{B}_0+1)}\right)}{\Gamma\left(\frac{1+\mathfrak{B}_0-\mathfrak{B}_1}{2(\mathfrak{B}_0+1)}\right)} . \tag{8.34}$$

The turning point h_* is in this case related to the strip width through

$$\frac{l}{2} = \int_0^{l/2} dx_1 = h_* \int_0^1 \frac{f(h_*)u^{(d-\theta)}\,du}{f(uh_*)\sqrt{1 - u^{2(d-\theta)}\frac{f(h_*)^2}{f(uh_*)^2}}} . \tag{8.35}$$

At first order in λ_1, we can perform the integral and invert the expression to find

$$h_* = \frac{l/2}{G_0}\left[1 + \frac{2\kappa\lambda_1}{(\mathfrak{B}_0+1)}\left[\frac{l/2}{G_0}\right]^{(\mathfrak{B}_0-\mathfrak{B}_1)}\left[1 - \frac{G_1}{G_0}\right]\right] . \tag{8.36}$$

Substitution into (8.33) leads to a kind simplification, and the full entanglement entropy expression at this order is finally given by

$$S_{R^2} = \frac{L^d L_S^{(d-1)}}{2G}\left\{\frac{\delta^{-\mathfrak{B}_0}}{\mathfrak{B}_0} - \frac{(l/2)^{-\mathfrak{B}_0}G_0^{\mathfrak{B}_0}G_0}{\mathfrak{B}_0} + 2\kappa\lambda_1\left[\frac{\delta^{-\mathfrak{B}_1}}{\mathfrak{B}_1} - \frac{(l/2)^{-\mathfrak{B}_1}G_0^{\mathfrak{B}_1}G_1}{\mathfrak{B}_1}\right]\right\} + \mathcal{O}(\lambda_1^2) . \tag{8.37}$$

[10]The fact that a Lifshitz geometry ($\theta = 0$) produced an unaltered HEE with respect to the AdS case for Einstein gravity was first observed in [397].

This expression is exact at linear order in λ_1. The Einstein gravity result, given by the first two terms, is corrected by a divergent plus a constant term at first order, plus a constant contribution of order $\mathcal{O}(\lambda_1^2)$.

8.1.2 Gauss-Bonnet gravity

Let us now turn to the case of Gauss-Bonnet gravity. The HEE functional for this theory was proposed in [247] and, as we mentioned, corresponds to a particular case of the JM functional, suitable for Lovelock gravities. Including the boundary term, which we will make use of for $d = 2$, the expression reads

$$S_{GB} = \frac{1}{4G}\int_m d^d x\sqrt{g_m}\left[1 + 2\lambda_{GB}\tilde{L}^2\mathcal{R}_m\right] + \frac{\lambda_{GB}\tilde{L}^2}{G}\int_{\partial m} d^{d-1}y\sqrt{g_{\partial m}}\mathcal{K}\,, \tag{8.38}$$

where $\mathcal{R}_m$ is the Ricci scalar of m, ∂m is the $(d-1)$-dimensional boundary of m, $h_{\partial m}$ stands for the determinant of the induced metric on ∂m, and $\mathcal{K}$ is the trace of its extrinsic curvature.

In the case of our hvLf geometries, the Ricci scalar of the induced metric on m (9.60) reads

$$\mathcal{R}_m = \frac{(d-1)(d-\theta)h^{-2\theta/d}}{(1+\dot{h}^2)^2 L^2}\left[\left(\dot{h}^2 + \dot{h}^4\right)\left(\frac{(d-2)\theta}{d^2} - 1\right) + \frac{2h\ddot{h}}{d}\right]. \tag{8.39}$$

As we can see, it identically vanishes for $d = 1$, which was expectable since the Gauss-Bonnet term $\mathcal{X}_4$ is identically zero in 3D gravity[11].

The way to proceed now is again trying to extremize (8.38) and evaluate the *on-shell* integral. The simplest case and, at the same time, one of singular interest, is given by $d = 2$. There, the Gauss-Bonnet contribution reduces to a boundary term, and does not modify the gravitational equations of motion. From the HEE perspective, the integral of the Ricci scalar of a 2D surface embedded in a certain manifold (which is precisely the expression we have here) is proportional to its Euler characteristic, which is a topological quantity, independent of the geometry of m. Therefore, when $d = 2$ we expect the entangling surface to be the same as in Einstein gravity and the Gauss-Bonnet bulk contribution $\propto \int \mathcal{R}_m$ to be cancelled by the boundary term involving the integral of the extrinsic curvature of ∂m. Let us explicitly show that this is indeed the case for hvLf geometries.

It is straightforward to check that the equations of motion for $h(x_1)$ do not get modified, and we have the very same first integral as in the Einstein gravity case (8.10), which we rewrite here for convenience

$$h_*^{(\theta-2)} = \frac{h^{(\theta-2)}}{\sqrt{1+\dot{h}^2}}\,. \tag{8.40}$$

The Ricci scalar on m simplifies to

$$\mathcal{R}_m = \frac{(\theta-2)}{h_*^\theta L^2}\left[u^{-\theta} - (\theta-1)u^{(4-3\theta)}\right], \tag{8.41}$$

[11] The same would occur for $d = \theta$, so no corrections to HEE are produced by this term in such a limit case.

where we have used again $u \equiv h/h_*$. We can now compute the integral involving the bulk terms in (8.38). The result is a sum of the Einstein gravity term (8.13) and the following divergence

$$\frac{1}{4G}\int_m d^d y\sqrt{g_m}\left[2\lambda_{GB}\tilde{L}^2\mathcal{R}_m\right] = \frac{(2-\theta)\tilde{L}^2 L_S\lambda_{GB}}{2G}\frac{1}{\delta}\,. \tag{8.42}$$

Interestingly, the exponent of the divergence does not depend on θ. In order to verify the cancellation of this term with the boundary one, we need to compute the metric induced on ∂m, and the trace of the extrinsic curvature of such boundary understood as an embedding on m. ∂m is characterized by $h \to \delta$, $x_1 = \text{const}$. We find, after some algebra

$$\begin{aligned}\sqrt{g_{\partial m}} &= L\delta^{\left(\frac{\theta-2}{2}\right)}\,, \\ \mathcal{K}_{\partial m} &= \frac{(\theta-2)}{2}\frac{\delta^{-\frac{\theta}{2}}}{L}\,,\end{aligned} \tag{8.43}$$

and hence

$$\frac{\lambda_{GB}\tilde{L}^2}{G}\int_0^{L_S} dx_2\sqrt{g_{\partial m}}\mathcal{K} = \frac{(\theta-2)\tilde{L}^2 L_S\lambda_{GB}}{2G}\frac{1}{\delta}\,. \tag{8.44}$$

As we can see, this contribution exactly cancels the intrinsic curvature contribution of (8.42), as expected.

In the case $d > 2$ things get much more involved. The functional we pretend to extremize contains derivatives of $h(x_1)$ up to order two, so no first integral is available now. Similarly, although the equations of motion are second-order as well, and not fourth-order as one would expect for a random second-order gravity[12], they turn out to be impossible to treat analytically. However, as we argued before we do not need to obtain the surface extremizing (8.38) in order to obtain the divergent terms in the HEE expression (although we would if we wanted to provide the corresponding corrected constant terms). Indeed, let us use (8.10) to compute the divergences produced by the bulk integral in (8.38). Following the same steps as for R^2 gravity we find[13]

$$S_{GB} = \frac{L^d L_S^{(d-1)}}{2G}\left\{\frac{\delta^{-\mathfrak{B}_0}}{\mathfrak{B}_0} - \frac{(l/2)^{-\mathfrak{B}_0}G_0^{\mathfrak{B}_0}G_0}{\mathfrak{B}_0} + \xi\lambda_{GB}\left[\frac{\delta^{-\mathfrak{B}_1}}{\mathfrak{B}_1} + c_{1,GB}\right]\right\} + \mathcal{O}(\lambda_{GB}^2)\,, \tag{8.45}$$

where now

$$\xi \equiv \frac{\tilde{L}^2}{L^2}(d-1)(d-\theta)\,, \tag{8.46}$$

and $c_{1,GB}$ is a constant term that should be computed using the entangling surface extremizing (8.38). As we can see, the expression is completely analogous to the one found for R^2 gravity (8.37): added to the Einstein gravity contribution we find a single divergence of the same order as the one encountered in that case plus a constant correcting the universal term. The fact that the divergences produced by R^2 and Gauss-Bonnet gravities match is not trivial, given that in the first case we are simply adding a term scaling as $\sim u^{-2\theta/d}$ (see (8.24)) to the "1" of Einstein gravity in the HEE integral, whereas for Gauss-Bonnet we find two terms when we substitute $\dot{h}(h)$ and $\ddot{h}(h)$ in (8.39) and (8.38)): one scaling like the R^2 one, plus another one going as $\sim u^{-2\theta/d+2(d-\theta)}$ which, however,

[12]Recall Gauss-Bonnet is a particular Lovelock gravity, which is the most general family of higher-order gravity theories in any dimension with second-order equations of motion.

[13]For the case $d = 3$, the appearance of $\mathfrak{B}_1$ in Gauss-Bonnet was anticipated in [290].

does not produce divergences when $\theta \leq 0$. In this case, the dynamical exponent does not appear in the curvature-squared contribution, simply because it does not appear in the pull-back metric on m and, as a consequence, in $\mathcal{R}_m$. Let us see what happens for our last curvature-squared theory: Ricci-squared gravity.

8.1.3 $R_{\mu\nu}R^{\mu\nu}$ gravity

For this theory, the entanglement entropy functional reads [185]

$$S_{\text{Ricci}^2} = \frac{1}{4G}\int_m d^dx\sqrt{g_m}\left[1+\lambda_2\tilde{L}^2\left(R_{(\hat{a})}{}^{(\hat{a})} - \frac{1}{2}K^{(\hat{a})\,2}\right)\right]. \tag{8.47}$$

In this expression, the first term stands for the contraction of the Ricci tensor associated to the spacetime metric with the two mutually orthogonal unit vectors normal to the entangling surface m, $n_{(\hat{a})}$, $\hat{a}=1,2$ according to

$$R_{(\hat{a})}{}^{(\hat{a})} \equiv R_{\mu\nu}n^{\mu}_{(\hat{a})}n^{\nu}_{(\hat{b})}\delta^{(\hat{a})(\hat{b})}. \tag{8.48}$$

The second term is the sum of the squares of the two extrinsic curvatures of m

$$K^{(\hat{a})}_{\mu\nu} = \nabla_\mu n^{(\hat{a})}_\nu, \tag{8.49}$$

associated to those two vectors

$$K^{(\hat{a})\,2} \equiv g^{\mu\nu}g^{\rho\sigma}K^{(\hat{a})}_{\mu\nu}K^{(\hat{b})}_{\rho\sigma}\delta_{(\hat{a})(\hat{b})}. \tag{8.50}$$

For the hvLf metrics (1.114), the two vectors normal to the entangling surface m associated to our strip are given by

$$n_{(1)} = \frac{r^{z-\theta/d}}{L}\partial_t, \quad n_{(2)} = \frac{r^{1-\theta/d}}{L\sqrt{1+\dot{h}^2}}\left(\partial_r - \dot{h}\partial_{x_1}\right). \tag{8.51}$$

Making use of this we can evaluate the above expressions to get

$$\begin{aligned} R_{(\hat{a})}{}^{(\hat{a})} - \frac{1}{2}K^{(\hat{a})\,2} &= \frac{h^{-2\theta/d}}{d^2L^2}\Bigg[d(d+dz-2\theta)(\theta-d-z) + \frac{d\left[\theta^2+d((1-z)z-\theta)\right]}{1+\dot{h}^2} \\ &\quad - \frac{\left[(\theta(d+1)-d(d+z))(1+\dot{h}^2)+dh\ddot{h}\right]^2}{2\left[1+\dot{h}^2\right]^3}\Bigg]. \end{aligned} \tag{8.52}$$

Following our previous steps, we can make use of (8.10) to determine the divergences in the HEE for this theory. The result is

$$S_{\text{Ricci}^2} = \frac{L^dL_S^{(d-1)}}{2G}\left\{\frac{\delta^{-\mathfrak{B}_0}}{\mathfrak{B}_0} - \frac{(l/2)^{-\mathfrak{B}_0}G_0^{\mathfrak{B}_0}G_0}{\mathfrak{B}_0} + \gamma\lambda_2\left[\frac{\delta^{-\mathfrak{B}_1}}{\mathfrak{B}_1} + c_{1,\text{Ricci}^2}\right]\right\} + \mathcal{O}(\lambda_2^2), \tag{8.53}$$

where now

$$\gamma \equiv \frac{\tilde{L}^2}{L^2}\frac{(d+dz-2\theta)(\theta-d-z)}{d}, \tag{8.54}$$

and c_{1,Ricci^2} is the correction to the constant term at first order in λ_2. Again, we find the same kind of term as in the two previous cases. In light of this, we conclude that $\mathfrak{B}_1 = 2\theta/d + d - \theta - 1$ is the only new divergent term produced at the level of curvature-squared gravities when $\theta < 0$. As we already said, this means that no additional logarithmic divergences can appear at this order of curvature for this class of metrics.

8.1.4 Higher-curvature gravities and new logarithmic terms

In the previous subsections we have studied the structure of terms of HEE for general curvature-squared gravities in the case of an entangling region A consisting of a strip in the boundary of hvLf metrics with $\theta \leq 0$. The result is that, in spite of the different terms appearing for the distinct HEE functionals in the various curvature-squared theories, we find that one single additional divergent term appears. This might suggest that if we moved on and considered even higher curvature gravities, one single additional divergence would appear at each order in curvature (this would mean, e.g., that the 10 independent curvature-cubed gravities [158], with their different corresponding functionals would give rise to the same single divergent term, and so on). Although this conjecture seems to ask for stronger evidence, it is important to notice that at the curvature-squared gravities level we are already considering the two kinds of terms that are expected to appear in the HEE functional at all orders in curvature [161], namely: contractions of curvature bulk tensors with normal vectors to the entangling surface m, and contractions of extrinsic curvatures of m with bulk tensors. If our conjecture was right, we could extract the divergent term common to all theories at each order in curvature by computing the HEE expression for the simplest higher-order gravity in each order. This is, of course, R^n gravity.

For an R^n gravity or, more in general, for an $f(R)$ gravity

$$I_{f(R)} = \frac{1}{16\pi G}\int d^{d+2}x\sqrt{g}\left[R + \frac{d(d+1)}{\tilde{L}^2} + \lambda_{f(R)} f(R)\right] , \tag{8.55}$$

(where $\lambda_{f(R)}$ is now a dimensionful coupling), the HEE functional is known to be [161]

$$S_{f(R)} = \frac{1}{4G}\int_m d^2x\sqrt{g_m}\left[1 + \lambda_{f(R)}\frac{df(R)}{dR}\right] , \tag{8.56}$$

and so for $f(R) = R^n$, $\lambda_{f(R)} = \lambda_{R^n}\tilde{L}^{2(n-1)}$ and

$$S_{R^n} = \frac{1}{4G}\int_m d^2x\sqrt{g_m}\left[1 + n\lambda_{R^n}\tilde{L}^{2(n-1)}R^{(n-1)}\right] . \tag{8.57}$$

We can actually extremize this functional and find the HEE expressions following exactly the same steps as in the case of R^2. The result is

$$S_{R^n} = \frac{L^d L_S^{(d-1)}}{2G}\left[\frac{\delta^{-\mathfrak{B}_0}}{\mathfrak{B}_0} - \frac{(l/2)^{-\mathfrak{B}_0}G_0^{\mathfrak{B}_0}G_0}{\mathfrak{B}_0} + n\kappa^{(n-1)}\lambda_{R^n}\left[\frac{\delta^{-\mathfrak{B}_1}}{\mathfrak{B}_1} - \frac{(l/2)^{-\mathfrak{B}_1}G_0^{\mathfrak{B}_1}G_1}{\mathfrak{B}_1}\right]\right] + \mathcal{O}(\lambda_{R^n}^2) , \tag{8.58}$$

where $\mathfrak{B}_1$ is now given by

$$\mathfrak{B}_1 = \frac{2(n-1)\theta}{d} + d - \theta - 1 . \tag{8.59}$$

G_0 and G_1 are again given by (8.34) taking the new value of $\mathfrak{B}_1$. As we can see, (8.58) includes the $\mathcal{O}(\lambda_{R^n})$ correction to the universal term as well as a divergence of order $\mathfrak{B}_1$. This is always subleading with respect to $\mathfrak{B}_0$ and, interestingly, it becomes logarithmic when

$$\theta = \frac{d(d-1)}{d-2(n-1)} , \tag{8.60}$$

provided that $2(n-1) > d$. This value of θ resembles the $\theta = d-1$ famous result for which a logarithmic divergence is found in the HEE for Einstein gravity ($n=1$), as we will review in a moment. However, this new set of divergences is found for $\theta < 0$, whereas the other occurs with $\theta = d-1 \geq 0$. Obviously, when $n=2$, the only possibility is $d=1$, which makes $\theta = 0$ and reduces to the AdS$_3$ case already studied at the beginning of the section. For $n > 2$, however, the situation is much richer, and we find a plethora of new logarithmic divergences (see Figure 8.2).

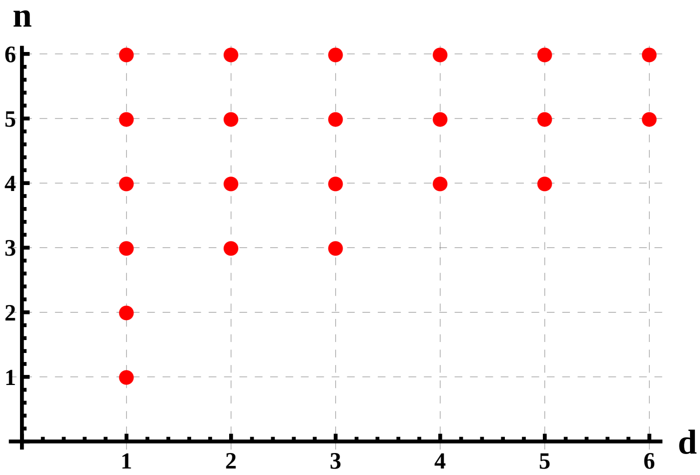

Figure 8.2: Values of n and d for which the corresponding R^n gravities produce terms including a logarithmic dependence on l for certain values of $\theta \leq 0$. The graph extends to the $n > 6$, $d > 6$ region in an obvious way.

When (8.60) is satisfied and $2(n-1) > d$, the HEE expression becomes

$$S_{R^n} = \frac{L^d L_S^{(d-1)}}{2G}\left[\frac{\delta^{-\mathfrak{B}_0}}{\mathfrak{B}_0} - \frac{(l/2)^{-\mathfrak{B}_0} G_0^{\mathfrak{B}_0} G_0}{\mathfrak{B}_0} + n\kappa^{(n-1)}\lambda_{R^n}\left[\log\left[\frac{l}{\delta}\right] + c_{R^n}\right]\right], \tag{8.61}$$

where now

$$\mathfrak{B}_0 = \frac{2(n-1)(d-1)}{2(n-1)-d}, \tag{8.62}$$

and c_{R^n} is a constant correcting the universal term. Therefore, we see that starting from curvature-cubed gravities, introducing higher-order terms in the gravitational action allows one to find new logarithmic contributions to the HEE for hvLf geometries. In both (8.58) and (8.61) we find a leading divergence whose coefficient scales with the area of the boundary of our entangling region. However, while in (8.58) the coefficient of the subleading term is also proportional to ∂A, in (8.61) we find a different scaling, provided there appears a factor which depends logarithmically on the width of the stripe l.

If our guess is right, (8.58) (and (8.61) when it applies) would be the right expression (swapping κ, λ_{R^n} and so on for the corresponding parameters) for the HEE of a strip in the boundary of a hvLf geometry with $\theta \leq 0$ for any higher-order gravity of n-th order in the Riemann tensor.

8.2 HEE for hvLf geometries in higher-curvature gravities II: $0 < \theta < d$

In this section we turn to the case of $0 < \theta < d$, corresponding to hvLf metrics whose curvature invariants diverge in the UV (as $r \to 0$). In order to do so, we follow the steps of [349] and consider these hvLf metrics to be completed asymptotically by an AdS geometry[14]. Hence, we will assume them to hold only above certain scale r_F.

Again, HEE for this class of hvLf spacetimes was studied for Einstein gravity, e.g., in [349] and [162]. In order to be consistent with the conventions in [349], whose results we plan to generalize here, let us make a change of coordinates in (1.114)

$$r = R^{\frac{d}{(d-\theta)}}\,, \tag{8.63}$$

and let us relabel $R \to r$ so there is no confusion between the radial coordinate and the Ricci scalar. Our hvLf geometries read now

$$ds^2 = \frac{L^2}{r^2}\left[-\frac{dt^2}{r^{\frac{2d(z-1)}{d-\theta}}} + r^{\frac{2\theta}{d-\theta}}dr^2 + d\vec{x}^2_{(d)}\right]. \tag{8.64}$$

The idea is to start with a metric of the form

$$ds^2 = \frac{L^2}{r^2}\left[-f(r)dt^2 + g(r)dr^2 + d\vec{x}^2_{(d)}\right], \tag{8.65}$$

and require it to be asymptotically AdS_{d+2} while assuming it to posses some intermediate hvLf-like behaviour

$$\begin{aligned} g(r) &\simeq \left[\frac{r}{r_F}\right]^{\frac{2\theta}{d-\theta}}, \quad (r >> r_F)\,, \\ g(r) &\simeq 1\,, \quad (r << r_F)\,, \end{aligned} \tag{8.66}$$

$$\begin{aligned} f(r) &\simeq \left[\frac{r}{r_F}\right]^{\frac{2d(1-z)}{d-\theta}}, \quad (r >> r_F)\,, \\ f(r) &\simeq 1\,, \quad (r << r_F)\,. \end{aligned}$$

Now, if we parametrize the entangling surface as $x_1 = F(r)$, computing the induced metric to obtain the area-functional is straightforward, and the result reads [349]

$$S_{EG} = \frac{L^d L_S^{d-1}}{2G}\int_\delta^{r_*}\frac{dr}{r^d}\sqrt{g(r) + \dot{F}(r)^2}\,. \tag{8.67}$$

r_* is the turning point now, where $\dot{F}(r)$ diverges. For this functional there is a first integral given by

$$\dot{F} = \frac{r^d}{r_*^d}\sqrt{\frac{g(r)}{1 - r^{2d}/r_*^{2d}}}\,, \tag{8.68}$$

so in the end we find

$$S_{EG} = \frac{L^d L_S^{d-1}}{2G}\int_\delta^{r_*}\frac{dr}{r^d}\sqrt{\frac{g(r)}{1 - r^{2d}/r_*^{2d}}}\,. \tag{8.69}$$

[14] See [162] for a different approach, analogous to the one we follow in the previous section.

The turning point is related to the strip width through

$$\frac{l}{2} = \int_0^{r_*} dr \frac{r^d}{r_*^d} \sqrt{\frac{g(r)}{1 - r^{2d}/r_*^{2d}}} \,. \tag{8.70}$$

In order to compute these integrals, we need to specify what the exact functional form of $g(r)$ is. However, we can simplify the issue by assuming the entangling surface to probe deep into the IR, so $r_* >> r_F$ [349]. In such a case, (8.69) and (8.70) can be estimated making use of (8.66), and the result is [349]

$$S_{EG} = \frac{L^d L_S^{d-1}}{2G} \left[\frac{\delta^{-(d-1)}}{(d-1)} + \frac{c}{r_F^{d-1}} \frac{l^{-\mathfrak{B}_0}}{r_F^{-\mathfrak{B}_0}} + \ldots \right] , \tag{8.71}$$

where c is a numerical constant and the dots refer to subleading contributions which we are neglecting in the limit $r_* >> r_F$. Therefore, we find an area-law term, plus a term which depends on the intermediate scale r_F. When $\theta = d-1$, (8.71) produces a logarithmic dependence on r_F [349],

$$S_{EG} = \frac{L^d L_S^{d-1}}{2G} \left[\frac{\delta^{-(d-1)}}{(d-1)} + \frac{c}{r_F^{d-1}} \log \left[\frac{l}{r_F} \right] + \ldots \right] . \tag{8.72}$$

This expression resembles the EE expression expected for a QFT with a Fermi surface [407, 426]

$$S = \alpha \frac{L_S^{d-1}}{\delta^{d-1}} + \beta L_S^{d-1} k_F^{d-1} \log(l k_F) + \ldots , \tag{8.73}$$

being k_F de Fermi momentum and α, β numerical positive constants. We see that the parameter r_F can be thus interpreted as the Fermi surface scale $r_F \sim k_F^{-1}$.

In order to study the effect of higher-curvature gravities we should repeat the analysis of section 8.1 and start considering curvature-squared gravities one by one. However, taking into account that our approach relies on approximating the spacetime geometry by two different metrics, namely AdS in the UV and hvLf above some scale r_F without specificating its exact form, the calculations for the Gauss-Bonnet and Ricci^2 terms become rather filthy and obscure the main goal of this section, which is nothing but studying the kind of terms that one should expect from general higher-order gravities. Therefore, let us stick to R^n gravity, for which we can find the surface extremizing the HEE functional for the general metric (8.65) and make a treatment as rigorous as the one performed in [349] for Einstein gravity. Following previous steps we find the expression for the HEE functional to be

$$S_{R^n} = \frac{L^d L_S^{d-1}}{2G} \int_\delta^{r_*} \frac{dr}{r^d} T(r) \sqrt{\frac{g(r)}{1 - \frac{T(r_*)^2}{T(r)^2} \frac{r^{2d}}{r_*^{2d}}}} \,, \tag{8.74}$$

where

$$T(x) \equiv \left[1 + n \lambda_{R^n} \tilde{L}^{2(n-1)} R^{(n-1)}(x) \right] , \tag{8.75}$$

with the turning point being related to $l/2$ by

$$\frac{l}{2} = \int_0^{r_*} dr \frac{r^d}{r_*^d} T(r) \sqrt{\frac{g(r)}{1 - \frac{T(r_*)^2}{T(r)^2} \frac{r^{2d}}{r_*^{2d}}}} \,. \tag{8.76}$$

It is a tedious but otherwise straightforward calculation to perform the previous on-shell integral and rewrite it in terms of l at order $\mathcal{O}(\lambda_{R^n})$[15]. The final result is

$$S_{R^n} = \frac{L^d L_S^{d-1}}{2G}\left[\frac{\delta^{-(d-1)}}{(d-1)}(1+\lambda_{R^n}c_0) + \frac{c}{r_F^{d-1}}\frac{l^{-\mathfrak{B}_0}}{r_F^{-\mathfrak{B}_0}} + \frac{c_1\lambda_{R^n}}{r_F^{d-1}}\frac{l^{-\mathfrak{B}_1}}{r_F^{-\mathfrak{B}_1}} + \mathcal{O}(\lambda_{R^n}^2)\right], \tag{8.77}$$

where, just as in the $\theta \leq 0$ case

$$\mathfrak{B}_0 \equiv d-\theta-1\,, \tag{8.78}$$

$$\mathfrak{B}_1 \equiv \mathfrak{B}_0 + \frac{2\theta(n-1)}{d}\,, \tag{8.79}$$

and c_0, c_1 are numerical constants. As we can see, the kind of terms appearing here resembles those found for $\theta \leq 0$ geometries. In particular, the term with the power $\mathfrak{B}_1$ produces a logarithmic term when

$$\theta = \frac{d(d-1)}{d-2(n-1)}\,, \tag{8.80}$$

as long as $d > 2(n-1)$ and $\theta < d$. This seems to generalize the case $\theta = d-1$ to R^n gravities for positive values of the hyperscaling violation exponent. However, $\theta < d$ imposes the following constraint on the order of the gravitational theory admitting such a term

$$3-2n > 0\,, \tag{8.81}$$

which of course is only satisfied for $n = 1$. This reduces to the well-known case of Einstein gravity corresponding to $\theta = d-1$. Therefore, as opposed to the $\theta \leq 0$ case, we do not find additional logarithmic terms in this case for any higher-curvature gravity. Nevertheless, it is not clear that $\mathfrak{B}_1$ is the only new contribution susceptible of arising in this case for general nth-order gravities. Further study in this direction would be desirable.

8.3 Discussion and perspectives

In this chapter we have considered the effects of higher-order gravity Lagrangians on the HEE expression for geometries with hyperscaling violation. Although the cut-off dependence of the HEE In section 8.1 we have argued that for $\theta \leq 0$, in order to extract the structure of terms for general higher-curvature gravities, it suffices to evaluate the corresponding on-shell functionals on the extremal area surface, without having to obtain the new surfaces extremizing those functionals, something that would be nevertheless necessary for obtaining the right corrected constant terms. This argument is explicitly illustrated for R^2 gravity, for which we can actually extremize the new functional and find the first-order correction to the universal term of the HEE. Our results show that for a general curvature-squared gravity, in addition to the Einstein gravity divergence ($\delta^{-\mathfrak{B}_0}$, with $\mathfrak{B}_0 = d-\theta-1$), there appears a single new one, at order $\mathcal{O}(\lambda)$ in the gravitational coupling of the form $\delta^{-\mathfrak{B}_1}$, with $\mathfrak{B}_1 = 2\theta/d + d - \theta - 1$.

[15] It is interesting to note that expanding in powers of λ_{R^n} and neglecting higher order contributions is right in this case because the term which goes with the coupling in $T(r)$ scales as $\sim 1/r^{2\theta(n-1)/d}$, with a positive exponent for $\theta > 0$, so when we evaluate the integral at $r \to r_* >> r_F$, the term involving λ_{R^n} is small, and the expansion makes sense.

The fact that, in spite of the different structure of the corresponding HEE functionals for R^2 (8.37), Gauss-Bonnet (8.45) and Ricci^2 (8.53) gravities, we find only one divergence of the same order in all cases led us to conjecture that this result extends to arbitrary nth-order gravities, so the divergent term found for R^n, $\mathfrak{B}_1 = 2(n-1)\theta/d + d - \theta - 1$, would be the only one appearing for any other theory of that order in curvature when $\theta \leq 0$. It might be that the result does not extend to $n \geq 3$ and that new divergent terms appear when those nth-order Lagrangians differ from the simple R^n case. Even if that were the case, that would imply that we are forgetting new contributions, not that $\mathfrak{B}_1$ gets substituted by them. Indeed, the on-shell evaluation of the Wald-like term [161]

$$\frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\sigma}} \epsilon_{\mu\nu}\epsilon_{\rho\sigma}\,, \text{ with } \epsilon_{\mu\nu} = n_\mu^{(\hat{a})} n_\nu^{(\hat{b})} \epsilon_{(\hat{a})(\hat{b})}\,, \tag{8.82}$$

will always contain at least one term scaling with the $(n-1)$th power of the Ricci scalar, which is precisely the one giving rise to $\mathfrak{B}_1$. Therefore, $\mathfrak{B}_1$ will always be there for nth-order gravities, although in some cases it might be followed by other divergences appearing for $n \geq 3$.

We have observed that the behaviour arising from Einstein gravity gets corrected for higher-order gravities (at least) by the addition of a new divergent term in which the cut-off scales with a different power, depending on θ, but which is also proportional to the area of the entangling region boundary. Area-law usually tells us about local correlations amongst UV degrees of freedom in the boundary theory. Our findings seem to be suggesting that such correlations get significantly modified when the higher-order couplings are turned on, something which happens to be distinctive of general hvLf geometries with respect to the cases of AdS or Lifshitz without hyperscaling violation, for which the structure of divergences remains unchanged ($\theta = 0$ and so $\mathfrak{B}_0 = \mathfrak{B}_1 = d-1$) and the only difference produced by the inclusion of such terms is a shift on the coefficient in front of $\delta^{-(d-1)}$ (see (8.31) [397]). Nevertheless, it is important to note that, as explained in the introduction, hvLf backgrounds with $\theta \neq 0$ generically suffer from a linearly divergent dilaton in the UV. This obscures the interpretation of the structure of divergences found in the HEE expression in terms of the degrees of freedom of the dual theory (which, to the best of our knowledge, is not known at present for general hvLf backgrounds). The situation is similar to that found for non-conformal branes, where the dual theory is known to be SYM (with $d \neq 4$). In that case, the dilaton, which is related to the YM coupling, also runs in the UV, which means that the theory is either asymptotically free or it needs a UV completion (depending on the dimension). In order to determine what the case is, one needs the exact relation between the dilaton and the coupling. When the YM coupling blows up in the UV, supergravity is not a valid description and S-duality needs to be used. For hvLf metrics, however, the dual theory is not known and the approach taken in the literature is more phenomenological/engineering-like since the supergravity result is taken to *define* what is meant by the dual theory[16]. Either way, comparing the results found in sections 8.1 and 8.2, we see that, regardless of the approach we take in computing HEE for these geometries, to wit: either assuming them to be *valid descriptions* in the UV (as in [162]), or considering some AdS completion (as in [349]), we find that the structure of the result does not change, and the novelty is always related to the appearance of a new term $\Lambda^{-\mathfrak{B}_1}$, being Λ the scale at which the hvLf geometry is reliable.

[16]We thank again Robert C. Myers and Ioannis Papadimitriou for the explanations appearing in this paragraph.

Coming back to our results, as we saw, the new term found becomes logarithmic when $d < 2(n-1)$ for hvLf geometries with

$$\theta = \frac{d(d-1)}{d-2(n-1)}, \tag{8.83}$$

which extends the famous result of $\theta = d-1$ valid for Einstein gravity to negative values of θ. For Einstein gravity ($n=1$) $\mathfrak{B}_0 = \mathfrak{B}_1$ and this becomes the leading divergence, whereas in the rest of cases ($n > 1$) we have an area-law-like term with the cut-off scaling as $\delta^{-\mathfrak{B}_0}$ plus the subleading logarithmic term.

Trying to extend this also to the $0 < \theta < d$ range, we considered the hvLf geometry to be UV-completed by AdS_{d+2}, arising the former above some scale r_F and computed HEE in that case for R^n gravity. We found that $\mathfrak{B}_1$ was the only new contribution again. However, for $0 < \theta < d$ we saw that this exponent could not vanish for any n except $n=1$, reducing to the well-known case $\theta = d-1$. In our computation we assume the turning point to probe the IR region, $r_* >> r_F$, in order to be able to approximate the on-shell integrals. It could be that an exact calculation making also use of an exact geometry interpolating between hvLf and AdS in the UV such as the one proposed in [349] gives rise to additional contributions to the HEE when embedded in higher-curvature gravities (and possibly including new logarithmic terms in some cases). Clarifying this possibility and, in general, proving (or refuting) our conjecture on the presence of $\mathfrak{B}_1$ as the only new divergence for general gravities would be interesting. Of course, this looks like a hard task at present.

As we have seen, the fact that all contributions coming from higher-curvature terms are subleading with respect to the Einstein gravity ones forbids these to produce violations of the area law, although we have shown that in certain cases they would yield universal terms which contain factors scaling logarithmically with the stripe width. Therefore, according to our results, only in the exotic case in which the considered gravitational theories did not include the Einstein gravity term could the HEE exhibit new violations of the area law.

In Figure 8.3 we show the values of n and θ for which R^n (and general nth-order gravities) introduce logarithmic terms for different values of d. The points on the horizontal line $n=1$ as well as those on the axis $\theta = 0$ correspond, respectively, to the cases already known in the literature, namely: hvLf with $\theta = d-1$ and AdS_3, whereas those in the quadrant $n > 0$, $\theta < 0$ are the new ones (extending infinitely for larger values of n and $-\theta$).

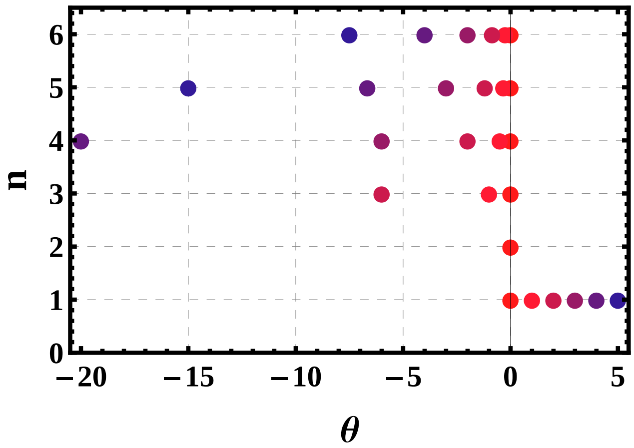

Figure 8.3: Values of n and θ for which R^n gravities produce logarithmic divergences for different values of d. Orange dots correspond to $d = 1$ and those in blue to $d = 6$.

Finally, the results obtained here should be extendable to other entangling regions different from the strip, such as cylinders, m-spheres and, ideally, arbitrary entangling regions. In principle, we expect subleading divergences to appear when more *complicated* entangling surfaces are considered. These would be produced by geometric integrals along the entangling surface (see [378] for an account of this for pure AdS_{d+2}). It would be of most interest to investigate how these divergences get modified in hvLf backgrounds. For n-spheres, for example, this has not been accomplished yet (to the best of our knowledge); not even in the simplest case of Einstein gravity.

9

Corner contributions to holographic entanglement entropy

This chapter is based on

Pablo Bueno and Robert C. Myers,
"Corner contributions to holographic entanglement entropy",
in preparation.

Pablo Bueno, Robert C. Myers and William Witczak-Krempa
"Universality of corner entanglement in gapless quantum matter",
in preparation.

Entanglement entropy (EE) has emerged as a useful tool in a variety of research areas, including condensed matter physics [8,218,292,364], quantum information [345,425], quantum field theory (QFT) [100,117,119,123,275,276,388] and quantum gravity [68,73,101,295,310,347,378,379,400,415]. In the context of quantum field theory, we define the EE for a spatial region V as: $S = -\mathrm{Tr}\,(\rho_V \log \rho_V)$, where ρ_V is the reduced density matrix computed by integrating out the degrees of freedom in the complementary region $\overline{V}$. The focus of the discussion in this chapter comes from considering the EE for a three-dimensional conformal field theory (CFT), which will have an expansion of the form

$$S_{EE} = c_1 \frac{\mathcal{A}}{\delta} - q\, \log\left(H/\delta\right) - 2\pi c_0 + \mathcal{O}\left(\delta/H\right)\,, \tag{9.1}$$

where $\mathcal{A}$, H and δ are, respectively, the perimeter of the entangling surface, some macroscopic length characteristic of the geometry (*e.g.,* we could choose $H = \mathcal{A}$) and a short-distance cut-off needed to regulate the calculation. Of course, the first term in this expansion is the celebrated 'area law' contribution to the EE [73,400]. However, the dimensionless coefficient c_1 of this linear divergence depends on the details of the regulator and so cannot be used to characterize the underlying CFT. In contrast, in the absence of the logarithmic term (see below), the constant c_0 is a universal constant intrinsic to the CFT and also the geometry of the (smooth) entangling surface.[1] For example, when the latter is a circle, c_0 plays the role of a 'central charge' in the F-theorem [121,257,277,340,341].

Another universal contribution in eq. (9.1) is the one proportional to $\log(H/\delta)$, which arises when the entangling surface (the boundary of the region V) contains cor-

[1]Of course, in gapped systems with topological order, this finite contribution would correspond to the topological entanglement entropy [226,275,294].

ners [118, 120, 180, 237][2,3] — see figure 9.1. Hence the dimensionless coefficient q is a function of the opening angle, *i.e.,* $q = q(\Omega)$. In our discussion, we focus on the contribution of a single corner in the entangling surface. If several corners were present, the coefficient of logarithmic contribution to the EE would simply involve the sum of independent contributions $q(\Omega_i)$ where Ω_i is the opening angle of the i'th corner. The form of the function $q(\Omega)$ is constrained by various properties of entanglement entropy [118, 120, 237]: for pure states, the fact that $S_{EE}(V) = S_{EE}(\overline{V})$ requires that $q(\Omega) = q(2\pi - \Omega)$. Further, strong subadditivity imposes

$$q(\Omega) \geq 0\,, \qquad \partial_\Omega q(\Omega) \leq 0 \qquad \text{and} \qquad \partial^2_\Omega q(\Omega) \geq \frac{|\partial_\Omega q(\Omega)|}{\sin\Omega} \qquad \text{for} \qquad \Omega \leq \pi\,, \tag{9.2}$$

i.e., $q(\Omega)$ is a positive convex function on the range $0 \leq \Omega \leq \pi$.

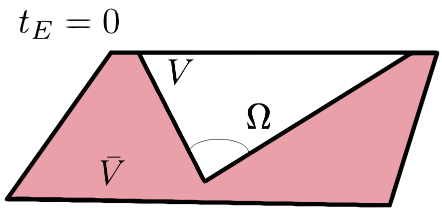

Figure 9.1: (Colour online) A corner in the entangling surface with opening angle Ω.

In fact, the functional form of $q(\Omega)$ is precisely constrained in particular limits. For small opening angles, the function has a pole with

$$\lim_{\Omega\to 0} q(\Omega) \equiv \frac{\kappa}{\Omega} + \cdots\,. \tag{9.3}$$

As we will review in appendix F, this form for small angles can be fixed by using a conformal mapping to relate the universal corner contribution to the EE corner to the universal contribution for a narrow strip. Of course, $q(\Omega)$ vanishes when the entangling surface becomes smooth, *i.e.,* $q(\pi) = 0$. Further, we can expect that $q(\Omega)$ is smooth in the vicinity of $\Omega = \pi$ and hence the constraint $q(\Omega) = q(2\pi - \Omega)$ (for pure states) requires that to leading order,

$$q(\Omega) \simeq \sigma\,(\pi - \Omega)^2 + \cdots\,, \tag{9.4}$$

for $\Omega \sim \pi$. In fact, this constraint will require that $q(\Omega)$ can be represented in a Taylor series with only even powers of $(\pi - \Omega)$ [118]. Hence we may use $q(\Omega)$ in the limits $\Omega \to 0$ and $\Omega \to \pi$ to define two interesting coefficients, κ and σ, which characterize the underlying CFT.

The corner contribution to the entanglement entropy has been studied in a variety of systems: free scalar and fermion field theories [118–120], calculations at a quantum critical point [19], numerical simulations in interacting lattice models [251, 263, 264, 391], interacting scalar field theories [262] and also holographic calculations with Einstein gravity in the bulk [237]. The results obtained in the literature suggest that $q(\Omega)$ contains

[2] Our discussion focuses on three-dimensional CFT's, however, similar logarithmic contributions may appear in theories which break conformal symmetry [94, 162, 246, 248, 349].

[3] The generalization to higher-dimensional singular surfaces was performed in [339].

interesting and unambiguous information about the underlying quantum field theory. In particular, it appears to be an interesting measure of the number of degrees of freedom — see, *e.g.,* [118, 120, 262]. By the latter proposition, we would expect that the coefficients, κ and σ, will themselves characterize the number of degrees of freedom in the underlying CFT.[4] Motivated by this idea, we will take the liberty to refer to these coefficients as 'central charges,' in a certain abuse of notation.

In this chapter, we will study the universal term arising from the presence of corners in the entangling surface for three-dimensional holographic conformal field theories. One of our objectives is to study if the corner charges above have any simple relation to any other known constants, which provide a similar counting of degrees of freedom and might be accessed with more conventional probes of the theory, or if κ and σ are really distinct quantities. As we will discuss below, we can not make a meaningful comparison if the bulk theory corresponds to Einstein gravity. Hence our approach will be to study the corner contributions for a family of extended holographic models which include higher curvature interactions in the bulk gravity theory. Generally, any quantities in the corresponding dual boundary theories, *e.g.,* the corner term, will now depend on the new (dimensionless) gravitational couplings for these higher order terms. This additional dependence on the new couplings allows us to make a nontrivial comparison of κ and σ with various other constants in the boundary CFT's. In particular, we will compare with the coefficients appearing in the universal terms in the EE of a strip and of a disk, in the thermal entropy density, and in the two-point function of the holographic stress tensor.

In fact, beyond the corner charges, the entire functional form of $q(\Omega)$ is characteristic of the underlying CFT. Hence another interesting question to consider is how this function changes with the inclusion of higher curvature interactions in the bulk. In this case, we find that for all of the holographic models studied here, $q(\Omega)$ is only modified by an overall factor but the functional dependence on Ω is not modified by the new gravitational interactions. However, as discussed in section 9.3.1, we do not believe that this behaviour is universal and that the functional form of $q(\Omega)$ will be modified with sufficiently general higher curvature theories in the bulk. One simple consequence of $q(\Omega)$ not being changed here is that the two corner charges are simply related in all of our holographic models, *i.e.,* we will see that $\kappa/\sigma = 4\,\Gamma(3/4)^4$. Hence we focus most of our discussions on the small angle charge κ in the following.

A final question, which we consider below, is whether our holographic analysis can reveal any features of the corner contribution which are universal to all three-dimensional CFT's. We examine this question briefly in section 9.3.2 by comparing our holographic results with the corner terms in the free QFT's with a conformal scalar and with a massless fermion, as were calculated in [118–120].

Let us now summarize our key results:

The results for the ratios of the corner charge κ with other various coefficients in the dual boundary theory are given in Table 9.1. The most interesting ratio is κ/C_T, the corner charge over the central charge appearing in two-point function of the stress tensor (9.100), which is independent of all of the gravitational couplings. Hence this ratio is universal for the broad class of holographic CFT's studied here.

In fact, as we noted above, the functional form of $q(\Omega)$ is not modified by any of

[4] Refs. [119, 120] discussed σ for this purpose in the context of free field theories.

the higher curvature interactions, except for an overall factor. Given the above result, the entire function $q(\Omega)/C_T$ is universal for the broad class of holographic CFT's studied here. This normalization then provides an interesting way to compare the corner contribution between any general three-dimensional CFT's. Comparing our holographic result with the corresponding free field results,[5] we see that the free field curves agree with the holographic result remarkably well — see figure 9.6. The free fermion and scalar curves deviate for the holographic result by at most 2.4% and 11%, respectively. Hence we suggest that the holographic expression for $q(\Omega)/C_T$, which is easily evaluated across the full range of $\Omega = 0$ to π, provides a good bench mark with which to compare the analogous results for general three-dimensional CFT's.

The maximum discrepancy between the holographic and free field results for $q(\Omega)/C_T$ occurs as $\Omega \to 0$ but somewhat surprisingly they agree perfectly in the limit $\Omega \to \pi$. That is, the holographic CFT's and the two free field theories exhibit the same ratio

$$\frac{\sigma}{C_T} = \frac{\pi^2}{24}\,. \tag{9.5}$$

Hence we are lead to conjecture that this ratio is in fact a universal constant for general conformal field theories in three dimensions.

The remainder of the chapter is organized as follows: In section 9.1, we first review the holographic calculation of the entanglement entropy for a corner in the boundary of AdS_4 with Einstein gravity in the bulk. Then in section 9.1.1, we study the effects of adding various higher curvature interactions to the bulk gravity theory on the universal corner term. In doing so, we show that the functional form of $q_E(\Omega)$ is universal to all of the theories considered here and evaluate the corner charge κ appearing in each case. In section 9.2, we compare this corner charge in the higher curvature theories with similar quantities appearing in other physical observables, *i.e.,* the coefficients appearing in the universal contribution in the entanglement entropy of a strip and of a disk, in the thermal entropy density and in the two-point correlator of the stress tensor. In section 9.3, we summarize our results. We also discuss the possibility of modifying the shape of the extremal surface in the holographic entanglement entropy in more general higher curvature theories of gravity, and hence modifying the functional form of $q(\Omega)$ in the dual boundary theories. We also comment on the relation between our holographic results and the analogous results obtained for free field theories. In appendix F, we explain the conformal mapping which relates the corner charge κ with the coefficient of the universal term in the entanglement entropy of a strip. In appendix G, we compute the corner contribution for a general $f(R)$ theory and explain in some detail the linearized equations of motion used to compute the two-point function of the stress tensor. Finally in appendix H, we present the integrals used in [118–120] to evaluate the coefficient σ for the free massless scalar and fermion theories and show that when evaluated with sufficient precision that they yield the simple rational values predicted by our conjecture (9.5).

9.1 Corner term in holographic entanglement entropy

In this section we study the corner contribution to the entanglement entropy for holographic CFT's dual to higher curvature theories of gravity. In particular, we will consider

[5] Similar comparisons were made in [120], but without normalizing by the central charge C_T.

bulk actions which contain general curvature-squared interactions and which are functions of Lovelock densities [381]. However, we begin by reviewing the calculation of the corner contribution to holographic entanglement entropy with just Einstein gravity in the bulk, which was originally performed in [237].

The bulk geometry will be four-dimensional Euclidean anti-de Sitter space in Poincaré coordinates[6]

$$ds^2 = \frac{\tilde{L}^2}{z^2}\left(dz^2 + dt_{\rm E}^2 + d\rho^2 + \rho^2 d\theta^2\right) , \tag{9.6}$$

which is a solution for Einstein gravity coupled to a negative cosmological constant

$$I_0 = \frac{1}{16\pi G}\int d^4x\sqrt{g}\left[\frac{6}{L^2} + R\right] \tag{9.7}$$

as long as we set $\tilde{L} = L$. The dual boundary theory then lives in the flat three-dimensional geometry with metric $d\tilde{s}^2 = dt_{\rm E}^2 + d\rho^2 + \rho^2 d\theta^2$. The region for which we calculate the entanglement entropy will be defined as $V = \{t_{\rm E} = 0, \rho > 0, |\theta| \leq \Omega/2\}$, as illustrated in figure 9.1. Hence the entangling surface ∂V has a corner with opening angle Ω at the origin. Note that in the following, at as well as the usual short-distance cut-off δ, we will also introduce an infrared regulator scale, *i.e.*, $\rho_{max} = H$, to ensure that the entanglement entropy does not diverge.

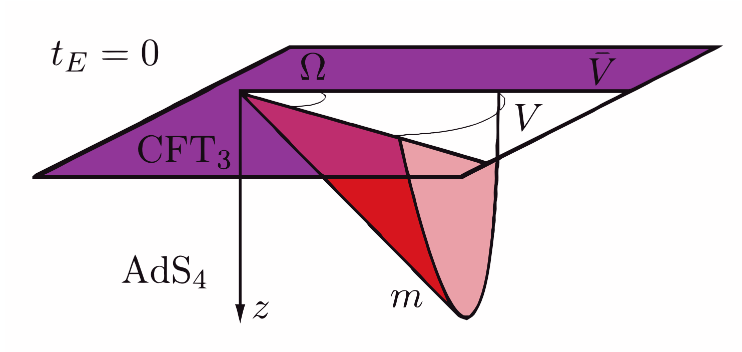

Figure 9.2: (Colour online) A kink in a constant Euclidean time slice $t_E = 0$ in the boundary of AdS_4.

Now, the corresponding holographic entanglement entropy (HEE) is computed using the Ryu-Takayanagi prescription for the entanglement entropy of conformal field theories dual to Einstein gravity [378, 379].[7] According to this, the entanglement entropy of a certain region V in our four-dimensional boundary theory is given by

$$S_{EE}(V) = \underset{m\sim V}{\rm ext}\left[\frac{\mathcal{A}(m)}{4G}\right] , \tag{9.8}$$

where m are codimension-2 bulk surfaces which are *homologous* to V in the boundary (and in particular $\partial m = \partial V$), and $\mathcal{A}(m)$ denotes the area of m. Figure 9.2 illustrates the extremal bulk surface for the region V defined above.

[6]See appendix C for conventions.
[7]This prescription has been recently proven under certain conditions in [296].

Now following [237], we parametrize the bulk surfaces m as $z = z(\rho, \theta)$ for the present case of corner region V. Further, the scaling symmetry of AdS, along with the fact that there is no other scale in the problem, allow us to limit the ansatz for the extremal surface to $z = \rho\, h(\theta)$, where $h(\theta)$ is a function satisfying $h \to 0$ as $\theta \to \pm\Omega/2$. With this ansatz, the induced metric on the surface becomes

$$ds_m^2 = \frac{\tilde{L}^2}{\rho^2}\left(1 + \frac{1}{h^2}\right) d\rho^2 + \frac{\tilde{L}^2}{h^2}\left(1 + \dot{h}^2\right) d\theta^2 + \frac{2\tilde{L}^2 \dot{h}}{\rho\, h} d\rho\, d\theta \,. \tag{9.9}$$

where $\dot{h} \equiv dh/d\theta$. The entanglement entropy functional becomes then

$$S_{EE} = \frac{1}{4G}\int_m d\theta\, d\rho\, \sqrt{\gamma} = \frac{\tilde{L}^2}{2G}\int_{\delta/h_0}^{H} \frac{d\rho}{\rho}\int_0^{\Omega/2-\epsilon} d\theta\, \frac{\sqrt{1 + h^2 + \dot{h}^2}}{h^2}\,, \tag{9.10}$$

where γ denotes the determinant of the induced metric (9.128), we have introduced a UV cut-off at $z = \delta$ and $h_0 \equiv h(0)$, which will be the maximum value of $h(\theta)$. As we already mentioned above, the ρ integral is also cut-off as some large distance H. Finally, the angular cut-off ϵ is defined in such that at $z = \delta$, $\rho\, h(\Omega/2 - \epsilon) = \delta$. Extremizing the above expression yields the equation of motion for $h(\theta)$, which reads

$$\ddot{h}(h + h^3) + h^4 + 3h^2 + 2(\dot{h}^2 + 1) = 0\,. \tag{9.11}$$

However, the corresponding 'Hamiltonian' is a conserved quantity, since there is no explicit θ dependence in eq. (9.10). Therefore we find the following first integral

$$\frac{1 + h^2}{h^2\sqrt{1 + h^2 + \dot{h}^2}} = \frac{\sqrt{1 + h_0^2}}{h_0^2}\,, \tag{9.12}$$

where we used $\dot{h}(0) = 0$. We can use eq. (9.12) to replace $\dot{h}$ in terms of h and trade the integral over θ for one over h. After some algebra, eq. (9.10) becomes

$$S_{EE} \;=\; \frac{\tilde{L}^2}{2G}\int_{\delta/h_0}^{H} \frac{d\rho}{\rho}\int_0^{\sqrt{(\rho/\delta)^2 - 1/h_0^2}} dy\, \sqrt{\frac{1 + h_0^2(1 + y^2)}{2 + h_0^2(1 + y^2)}}\,, \tag{9.13}$$

where we have also substituted $y = \sqrt{1/h^2 - 1/h_0^2}$. Near the boundary ($y \to \infty$), the integrand behaves as

$$\sqrt{\frac{1 + h_0^2(1 + y^2)}{2 + h_0^2(1 + y^2)}} \sim 1 + \mathcal{O}\left(\frac{1}{y^2}\right)\,. \tag{9.14}$$

Therefore, the y integration diverges in the limit that $\delta \to 0$. However, we can isolate this divergence by adding and subtracting one to the integrand. Hence we recast eq. (9.13) as

$$S_{EE} = \frac{\tilde{L}^2}{2G}\int_{\delta/h_0}^{H} \frac{d\rho}{\rho}\int_0^{\infty} dy\left[\sqrt{\frac{1 + h_0^2(1 + y^2)}{2 + h_0^2(1 + y^2)}} - 1\right] + \frac{\tilde{L}^2}{2G}\int_{\delta/h_0}^{H}\frac{d\rho}{\rho}\sqrt{\frac{\rho^2}{\delta^2} - \frac{1}{h_0^2}}\,. \tag{9.15}$$

In the limit that $\delta \to 0$, this expression can be further simplified to produce the final result

$$S_{EE} = \frac{\tilde{L}^2}{2G}\frac{H}{\delta} - q(\Omega)\, \log\left(\frac{H}{\delta}\right) - \left(\frac{\pi\tilde{L}^2}{4Gh_0} + q(\Omega)\log(h_0)\right) + \mathcal{O}(\frac{\delta}{H})\,, \tag{9.16}$$

where the function $q(\Omega)$ is given by

$$q_E(\Omega) = \frac{\tilde{L}^2}{2G}\int_0^\infty dy \left[1 - \sqrt{\frac{1+h_0^2(1+y^2)}{2+h_0^2(1+y^2)}}\right]. \tag{9.17}$$

The result in eq. (9.16) has precisely the expected form given in eq. (9.1), *i.e.*, the first term in eq. (9.16) is, of course, the area law contribution, whereas the second is the universal contribution associated with the corner. The last one is the constant term, which does not have a universal character in the present situation.

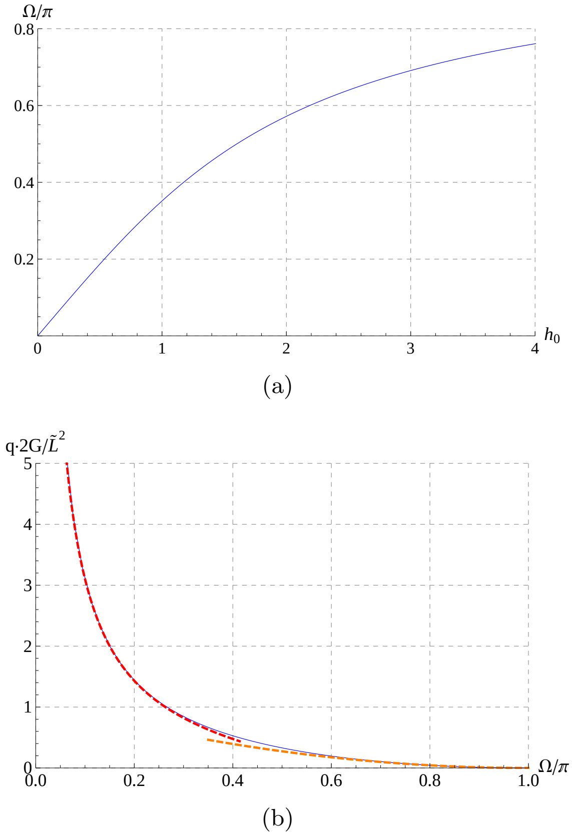

Figure 9.3: (Colour online) (a) Ω/π as a function of h_0 and (b) $\frac{2G}{\tilde{L}^2}q$ as a function of Ω/π. In the second panel, the dashed lines correspond to the approximate expressions derived in eqs. (9.20) and (9.23) for small opening angles (red) and the smooth limit (orange), respectively.

In eq. (9.17), we have added a subscript 'E' to denote this function as the corner contribution with Einstein gravity in the bulk. The dependence of $q_E(\Omega)$ on the opening angle is implicit on the right-hand side of eq. (9.17) through the dependence of h_0 on Ω. The latter can be determined by evaluating

$$\Omega = \int_{-\Omega/2}^{+\Omega/2} d\theta = \int_0^{h_0} dh \, \frac{2h^2\sqrt{1+h_0^2}}{\sqrt{1+h^2}\sqrt{(h_0^2-h^2)(h_0^2+(1+h_0^2)h^2)}} \tag{9.18}$$

and the result is shown in figure 9.3(a). The coefficient of the corner term is then plotted in figure 9.3(b) and we can see that $q_E(\Omega)$ does indeed satisfy all the various constraints explained in the introduction, *e.g.*, see eq. (9.2). For small values of the opening angle, *i.e.*, $\Omega \to 0$, we find

$$\Omega = \frac{2\sqrt{\pi}\,\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} h_0 - \frac{\left[3\Gamma\left(\frac{3}{4}\right)^2 - \Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right)\right]}{6\sqrt{2\pi}} h_0^3 + \mathcal{O}(h_0^5)\,, \tag{9.19}$$

$$q_E(\Omega) = \frac{\tilde{L}^2}{2\pi G}\Gamma\left(\tfrac{3}{4}\right)^4 \frac{1}{\Omega} - \frac{\tilde{L}^2}{G}\frac{\pi\,\Gamma\left(\frac{1}{4}\right)}{48\sqrt{2}\,\Gamma\left(\frac{3}{4}\right)^3}\,\Omega + \mathcal{O}(\Omega^3)\,, \tag{9.20}$$

which is shown as the dashed red line in figure 9.3(b)[8]. Comparing the latter with eq. (9.3), we see that in this holographic model, the universal 'central charge' associated with the small angle limit of the corner contribution is

$$\text{Einstein gravity :}\quad \kappa_E = \frac{\tilde{L}^2}{2\pi G}\Gamma\left(\tfrac{3}{4}\right)^4 . \tag{9.21}$$

Considering the limit of a smooth entangling surface, *i.e.*, $\Omega \to \pi - \varepsilon$, we have

$$\varepsilon = \frac{\pi}{h_0} + \mathcal{O}(h_0)\,, \tag{9.22}$$

$$q_E(\pi - \varepsilon) = \frac{\tilde{L}^2}{8\pi G}\,\varepsilon^2 + \mathcal{O}(\varepsilon^4)\,, \tag{9.23}$$

which is shown as the dashed orange line in figure 9.3(b). Comparing this result with eq. (9.4), we see that the universal 'central charge' associated with the limit of a nearly smooth entangling surface in this holographic model is

$$\text{Einstein gravity :}\quad \sigma_E = \frac{\tilde{L}^2}{8\pi G}\,. \tag{9.24}$$

Another interesting case to consider is a right-angled corner, *i.e.*, $\Omega = \pi/2$, for which we find

$$q_E\,(\pi/2) \simeq 0.11823\,\frac{\tilde{L}^2}{G} \simeq 0.32944\,\kappa_E \simeq 2.9714\,\sigma_E\,. \tag{9.25}$$

This case naturally arises in numerical calculations of entanglement entropy, *e.g.*, [262].

9.1.1 Higher curvature gravity

Having reviewed the calculation for Einstein gravity in the bulk, we now turn to considering the effect of higher curvature interactions in the bulk theory. For such cases, the Ryu-Takayanagi prescription must be generalized, as was first considered in [154, 184, 247]. In particular, the Bekenstein-Hawking formula on the right-hand side of eq. (9.8) must be replaced by a new entropy functional which accounts for the new gravitational interactions. Hence eq. (9.8) is replaced by

$$S_{EE}(V) = \underset{m\sim V}{\text{ext}}\, S_{\text{grav}}(m)\,, \tag{9.26}$$

where the entropy functional S_{grav} depends on the details of the gravitational theory. This is a familiar idea in the context of black hole entropy where the Wald entropy formula

[8] Notice that eq. (9.20) fits the exact $q_E(\Omega)$ curve remarkably well for not so small angles.

[254, 255, 420] extends $\mathcal{A}/(4G)$ with higher curvature corrections. A natural suggestion would be that the HEE should be calculated by extremizing the Wald entropy evaluated on the bulk surfaces m, however, it was shown that this approach would be incorrect since it fails to produce the proper universal contributions to the entanglement entropy [247]. The latter universal terms are properly reproduced in the special case of Lovelock gravity [154, 247] using an alternative entropy functional [256] — see below. More generally the appropriate entropy functional is the Wald entropy plus additional terms involving the extrinsic curvature, which would vanish if evaluated on the Killing horizon of a stationary black hole [103,161,185,336]. There has been an effort to extend the derivation [296] of the Ryu-Takayanagi prescription to higher curvature theories of gravity [21,65,66,103,161,327] and a general formula was proposed for theories involving interactions with contractions of arbitrary powers of the Riemann tensor (but no derivatives of the curvature). While this general expression was shown to satisfy several consistency checks [161], it seems that it must still be further refined for general theories involving cubic and higher powers of the curvature [21, 66, 327]. In any event, the correct entropy functional is known for general curvature-squared gravity in the bulk and we will use this to determine the modifications to the corner contribution in HEE for these theories in section 9.1.1.

To go beyond curvature squared gravity, we turn to the generalized Lovelock theories considered by [381]. In these theories, the Lagrangian is given by an arbitrary functional of extended 'topological' densities, *i.e.,* scalars constructed from the curvature tensor which if integrated over a manifold of the appropriate dimension would yield the Euler characteristic. Hence Lovelock gravity [303, 304] would be the simplest example in which the Lagrangian is a linear functional of these topological densities. Another well-known class of theories which take this form would be $f(R)$ gravity [399] since the Ricci scalar corresponds to the Euler density for two-dimensional manifolds. In studying these generalized Lovelock theories, [381] proposed a formula for the gravitational entropy which satisfied a classical increase theorem for linearized perturbations of Killing horizons. We interpret the fact that their definition applies for at least small deviations away from a Killing horizon, as evidence that it will yield the correct gravitational entropy in the more general context of evaluating HEE. Then applying this prescription allows us to evaluate the modifications to the corner contribution in HEE for a certain class of theories involving cubic and higher powers of the curvature in section 9.1.1.

It is worth stressing that most often we will be working perturbatively in the gravitational couplings for the higher curvature interactions. We will try to make clear when our expressions are valid for general values of the new couplings and when they correspond to perturbative approximations.

Curvature-squared gravity

The bulk action of the most general curvature-squared gravity can be written as

$$I_2 = \frac{1}{16\pi G}\int d^4x\sqrt{g}\left[\frac{6}{L^2} + R + \lambda_1 L^2 R^2 + \lambda_2 L^2 R_{\mu\nu}R^{\mu\nu} + \lambda_{\rm GB} L^2 \mathcal{X}_4\right], \tag{9.27}$$

where

$$\mathcal{X}_4 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2 \tag{9.28}$$

is the Gauss-Bonnet term, *i.e.,* the Euler density for four-dimensional manifolds. Hence the last interaction does not effect the gravitational equations of motion since we are working with four bulk dimensions. However, as we will see, this term still contributes a topological term to the entropy functional. The AdS_4 metric in eq. (9.6) is still a solution of the full equations of motion for any value of λ_1 and λ_2 provided $\tilde{L} = L$.[9]

The expression for the entanglement entropy in this family of theories is given by eq. (9.26) where $S_{\rm grav}$ takes the form [161, 185, 247, 336]

$$S_2 = \frac{\mathcal{A}(m)}{4G} + \frac{L^2}{4G}\int_m d^2y\,\sqrt{\gamma}\left[2\lambda_1 R + \lambda_2\left(R^{\hat{a}}{}_{\hat{a}} - \frac{1}{2}K^{\hat{a}}K_{\hat{a}}\right) + 2\lambda_{\rm GB}\mathcal{R}\right], \tag{9.29}$$

where γ_{ij}, $K^{\hat{a}}_{ij}$ and $\mathcal{R}$ are, respectively, the induced metric, the second fundamental form and the intrinsic Ricci scalar of the bulk surface m — see appendix C for a complete description of our conventions.[10] Before proceeding with detailed calculations of HEE, let us make some general observations about the expected results.

First, it is worthwhile to note that the gravitational action (9.27) would also include various boundary terms, *e.g.,* see [168, 334], and that similar boundary terms should be expected to appear in the entropy functional (9.29). However, while the addition of such boundary terms may effect the coefficient in the area law contribution to the entanglement entropy (9.1) in the boundary theory, one can infer from the local geometric form of these boundary terms that they will not modify the logarithmic contribution to S_{EE} [247]. Again, the robustness of the logarithmic term here is a reflection of the fact that it is a universal contribution whose value is independent of the precise details of the UV regulator. Of course, since our interest lies in determining the universal corner term $q(\Omega)$, we will ignore any boundary terms that might be added to eq. (9.29).

Next, let us examine the form of the entropy functional in eq. (9.29). The λ_1 and λ_2 terms both contain contributions involving the curvature of the background spacetime geometry. However, since we are evaluating the HEE in empty AdS_4, the latter terms are just constants, *e.g.,* $R = -12/\tilde{L}^2$. Hence the entropy functional is not modified by these terms except for a shift in the overall factor multiplying the area of bulk surface.[11]

We also note that any surface which extremizes the area, as in eq. (9.8), will satisfy $K^{\hat{a}} = \gamma^{ij}K^{\hat{a}}_{ij} = 0$. Now looking at eq. (9.29), we see that the λ_2 contribution includes a term that is quadratic in $K^{\hat{a}}$. Hence an extremal area surface will also be a saddle point of this term. That is, if we deform away from the extremal area surface by some deformation parameterized by a small parameter ε, then we will have $K^{\hat{a}} \sim \mathcal{O}(\varepsilon)$ and $K^{\hat{a}}K_{\hat{a}} \sim \mathcal{O}(\varepsilon^2)$. Therefore extremal area surfaces will also extremize the new contribution (or any other contribution) to the HEE functional that is quadratic in the trace of the extrinsic curvature.[12]

Lastly since we are working with a four-dimensional bulk spacetime, m will be a two-dimensional manifold and hence $\int_m\sqrt{\gamma}\,\mathcal{R}$, appearing as $\lambda_{\rm GB}$ contribution in eq. (9.29), will be proportional to a topological invariant (namely, the Euler characteristic) of m, up to

[9] This result is special to four dimensions. With a higher dimensional bulk, one would generally find $\tilde{L}^2 = L^2/f_\infty$ where f_∞ is a function of all three of the dimensionless couplings, λ_1, λ_2 and $\lambda_{\rm GB}$.

[10] The last term in eq. (9.29) corresponds to a particular case of the Jacobson-Myers entropy functional for Lovelock gravities [256].

[11] This simple shift may not arise when we are evaluating HEE in more general backgrounds, but this is a general result for backgrounds which are Einstein geometries, *i.e.,* $R_{\mu\nu} = -3/\tilde{L}^2\, g_{\mu\nu}$.

[12] The full equations arising from extremizing the new functional will be very non-linear in general and so there may be other saddle points for which $K^{\hat{a}} \neq 0$. However, we will also demand that the bulk surfaces reduce to the corresponding extremal area surfaces in the limit that $\lambda_i \to 0$. Therefore, these new non-linear solutions (if they exist at all) would be discarded since they would not satisfy this condition.

boundary terms. Therefore just as the corresponding interaction in the bulk action (9.27) does not modify the gravitational equations of motion, this term in the HEE functional will not contribute to the equations determining the bulk surface which extremizes eq. (9.29).

Given the above discussion, we conclude that the extremal area surface for any given entangling region in the boundary of pure AdS_4 will also extremize the HEE functional (9.29) for the same calculation of entanglement entropy in the boundary theory dual to curvature-squared gravity. The only effect of the 'higher curvature' corrections in eq. (9.29) will be to change the final entanglement entropy by an overall factor depending on the new couplings, λ_1, λ_2 and $\lambda_{\rm GB}$. In the problem of interest, this indicates that the corner coefficient $q(\Omega)$ will only be changed by this same overall factor. Hence the charge (9.21) is multiplied by an overall factor but the functional dependence of $q(\Omega)$ on the opening angle is precisely the same as compared to Einstein gravity. We note that the above observations actually have broader applicability and that this result will apply to a wide class of theories beyond the special case of curvature-squared gravity — we return to a discussion of this point in section 9.3. Let us now turn to the detailed calculations to see how the different contributions in eq. (9.29) affect the universal corner term in HEE.

R^2 gravity

If we focus on the simplest case of R^2 gravity, *i.e.,* set $\lambda_2 = 0 = \lambda_{\rm GB}$, the gravitational entropy functional reduces to

$$S_2 = \frac{\mathcal{A}(m)}{4G} + \frac{L^2\lambda_1}{2G}\int_m d^2y\,\sqrt{\gamma}\,R = \frac{\mathcal{A}(m)}{4G}\,(1-24\lambda_1)\,, \tag{9.30}$$

where we substituted $R = -12/L^2$ to produce the last expression. Therefore, as discussed above, the corresponding corner coefficient is simply multiplied by an overall factor relative to the Einstein gravity[13]

$$q(\Omega) = (1-24\lambda_1)\;q_E(\Omega) \tag{9.31}$$

and the corresponding charge becomes

$$\kappa = (1-24\lambda_1)\;\kappa_E\,. \tag{9.32}$$

$R_{\mu\nu}R^{\mu\nu}$ gravity

In the case of $R_{\mu\nu}R^{\mu\nu}$ gravity, the HEE functional becomes

$$\begin{aligned} S_2 &= \frac{\mathcal{A}(m)}{4G} + \frac{L^2\lambda_2}{4G}\int_m d^2y\,\sqrt{\gamma}\left(R^{\hat a}{}_{\hat a} - \frac{1}{2}K^{\hat a}K_{\hat a}\right) \qquad (9.33)\\ &= \frac{\mathcal{A}(m)}{4G}\,(1-6\lambda_2) - \frac{\lambda_2}{8G}\int_m d^2y\sqrt{\gamma}\,\frac{\left[2(1+\dot h^2)+3h^2+h^4+(h+h^3)\ddot h\right]^2}{\left(1+h^2+\dot h^2\right)^3}\,. \end{aligned}$$

where the expression in the second line was produced by first substituting $R^{\hat a}{}_{\hat a} = g^{\perp\mu\nu}R_{\mu\nu} = -6/L^2$ and by evaluating $K^{\hat a}K_{\hat a}$ for the bulk surface defined by $z = \rho\,h(\theta)$ — see appendix

[13] As we describe in appendix G, these results can be straightforwardly extended to the case of a general $f(R)$ gravity.

C for details. Varying the above expression will produce a nonlinear differential equation for $h(\theta)$ which, because of the last term, involves third and fourth order derivatives, as well as first and second order derivatives. However, as we explained above, the solution should still be the same extremal area surface which we found with Einstein gravity. The latter occurs because the geometric form of the equation determining the extremal area surface is precisely $K^{\hat{a}} = 0$. Indeed comparing with eq. (9.11), we see that the factor in the numerator of the last term above is precisely the equation determining the profile $h(\theta)$ with Einstein gravity. Because this factor is squared, the profile satisfying eq. (9.11) will also satisfy the full equation of motion coming from eq. (9.33) and further, in evaluating the HEE, the last term will not contribute because this factor simply vanishes. Hence the HEE and in particular, the corner coefficient, is determined by the Bekenstein-Hawking term, as with Einstein gravity but now multiplied by an additional factor. Therefore the charge defined by the corner term as in eq. (9.3) becomes simply

$$\kappa = (1 - 6\lambda_2)\, \kappa_E \,. \tag{9.34}$$

Gauss-Bonnet gravity

For pure Gauss-Bonnet gravity, eq. (9.29) reduces to

$$S_2 = \frac{\mathcal{A}(m)}{4G} + \frac{L^2 \lambda_{\rm GB}}{2G} \int_m d^2 y \sqrt{\gamma}\, \mathcal{R}\,. \tag{9.35}$$

Above, we argued that the second term would not affect the profile of the bulk surface nor contribute to the universal corner contribution. With the bulk profile $z = \rho\, h(\theta)$, it is not difficult to show that the combination $\sqrt{\gamma}\mathcal{R}$ can be written as a total derivative (see appendix C for details)

$$\sqrt{\gamma}\, \mathcal{R} = \frac{d}{d\theta}\left[\frac{2}{\rho}\, \frac{\dot{h}}{h\sqrt{1+h^2+\dot{h}^2}}\right]. \tag{9.36}$$

In fact, this is sufficient to conclude that the universal corner contribution will be identical to that in eq. (9.17), as expected.

However, let us examine the contribution of the Gauss-Bonnet term to the HEE in more detail. Using eq. (9.36), this contribution can be written now as

$$\Delta S_{\rm GB} = \frac{L^2 \lambda_{\rm GB}}{2G} \int_m d^2 y \sqrt{\gamma}\, \mathcal{R} = -\frac{L^2 \lambda_{\rm GB}}{G} \int_{\delta/h_0}^{H} \frac{d\rho}{\rho} \left[\frac{\dot{h}}{h\sqrt{1+h^2+\dot{h}^2}}\right]_{\theta=0}^{\theta=\Omega/2-\epsilon} . \tag{9.37}$$

We can make use of eq. (9.12) to replace $\dot{h}$ in terms of h. By doing so, and recalling that $h(\Omega/2-\epsilon) = \delta/\rho$ and $h(0) = h_0$, the above expression reduces to

$$\Delta S_{\rm GB} = -\frac{L^2 \lambda_{\rm GB}}{G}\frac{H}{\delta} + \mathcal{O}(1)\,. \tag{9.38}$$

Hence, including the Einstein gravity, the final result for the HEE in this case becomes

$$S_{EE} = \frac{L^2}{2G}\frac{H}{\delta}\,(1 - 2\lambda_{\rm GB}) - q_E(\Omega)\, \log\!\left(\frac{H}{\delta}\right) + \mathcal{O}(1)\,. \tag{9.39}$$

Hence the (nonuniversal) coefficient of the area law term has be modified here but the corner contribution is precisely the same as with just Einstein gravity in the bulk.

It was commented above that the entropy functional (9.29) might be supplemented by boundary terms but that the logarithmic term in the HEE, *i.e.,* the corner contribution, is unaffected by such terms [247]. Gauss-Bonnet gravity provides an illustrative exercise since there is a natural boundary term to be added gravitational entropy functional [247]

$$S_2 = \frac{\mathcal{A}(m)}{4G} + \frac{L^2\lambda_{\rm GB}}{2G}\int_m d^2y\,\sqrt{\gamma}\,\mathcal{R} + \frac{L^2\lambda_{\rm GB}}{G}\int_{\partial m} dy\sqrt{\tilde{\gamma}}\,\mathcal{K}\,, \tag{9.40}$$

where ∂m is the one-dimensional boundary of m at the cut-off surface $z=\delta$. Further $\tilde{\gamma}$ and $\mathcal{K}$ denote the determinant of the induced metric and the trace of the extrinsic curvature, respectively, on this boundary. It is straightforward to evaluate these quantities and to produce the result

$$\Delta S'_{\rm GB} = \frac{L^2\lambda_{\rm GB}}{G}\int_{\partial m} dy\sqrt{\tilde{\gamma}}\,\mathcal{K} = \frac{L^2\lambda_{\rm GB}}{G}\int_{\delta/h_0}^{H}\frac{d\rho}{\delta} = \frac{L^2\lambda_{\rm GB}}{G}\frac{H}{\delta} + \mathcal{O}(1)\,. \tag{9.41}$$

Adding this contribution to eq. (9.39) leaves

$$S_{EE} = \frac{L^2}{2G}\frac{H}{\delta} - q_E(\Omega)\,\log\left(\frac{H}{\delta}\right) + \mathcal{O}(1) \tag{9.42}$$

and we see that with the additional boundary term in eq. (9.40), there is no $\lambda_{\rm GB}$ dependence in either the area law term or the logarithmic contribution in the entanglement entropy. The latter reflects the fact that with the additional boundary term, the Gauss-Bonnet contribution in eq. (9.40) is a purely topological contribution. In any event, as expected, the universal corner contribution remains unaffected by the addition of this boundary term, which implicitly represents a modification of the regulator used to define the entanglement entropy in the dual QFT.

To summarize our results for curvature-squared gravity (9.27) in the bulk, we found that the functional form of $q(\Omega)$ is not modified. Rather the holographic expression only differs from that in the Einstein gravity by some overall factor. Hence the charge defined by the small Ω limit, as in eq. (9.3), becomes

$$\kappa \;=\; (1-24\lambda_1-6\lambda_2)\;\kappa_E\,, \tag{9.43}$$

where the Einstein charge κ_E is given eq. (9.21).

Generalized Lovelock gravity

Recall that Lovelock gravities [303, 304] are the most general higher curvature gravity theories with second-order equations of motion. The corresponding action can be written as

$$I_{\rm LL} = \frac{1}{16\pi G}\int d^{d+1}x\,\sqrt{g}\left[\frac{d(d-1)}{L^2} + R + \sum_{p=2}^{\lfloor\frac{d+1}{2}\rfloor}\lambda_p L^{2p-2}\mathcal{L}_{2p}(R)\right]\,, \tag{9.44}$$

where λ_p are dimensionless couplings and $\mathcal{L}_{2p}$ correspond to the dimensionally extended $2p$-dimensional Euler densities

$$\mathcal{L}_{2p}(R) \equiv \frac{1}{2^p}\delta^{\nu_1\nu_2\cdots\nu_{2p-1}\nu_{2p}}_{\mu_1\mu_2\cdots\mu_{2p-1}\mu_{2p}}\,R^{\mu_1\mu_2}{}_{\nu_1\nu_2}\cdots R^{\mu_{2p-1}\mu_{2p-2}}{}_{\nu_{2p-1}\nu_{2p-2}}\,. \tag{9.45}$$

Here $\delta^{\nu_1\nu_2\cdots\nu_{2p-1}\nu_{2p}}_{\mu_1\mu_2\cdots\mu_{2p-1}\mu_{2p}}$ denotes a totally antisymmetric product of $2p$ Kronecker deltas. Hence when $p = (d+1)/2$, $\mathcal{L}_{2p}$ is topological and when $p > (d+1)/2$, $\mathcal{L}_{2p}$ simply vanishes. Of course, the cosmological constant and Einstein terms in eq. (9.44) could be incorporated into the sum as $\mathcal{L}_0$ and $\mathcal{L}_2$, respectively. Recently, there has been renewed interest in these theories in the context of the AdS/CFT correspondence where these theories provide toy models of holographic CFT's in which the central charges differ from one another, *e.g.,* see [?] and the references therein. For this class of theories (9.44), HEE is evaluated with eq. (9.26) using the following entropy functional [154, 247]

$$S_{\rm JM} = \frac{\mathcal{A}(m)}{4G} + \frac{1}{4G}\int_m d^{d-1}y\,\sqrt{\gamma}\,\sum_{p=2}^{\lfloor\frac{d+1}{2}\rfloor} p\,\lambda_p\,L^{2p-2}\mathcal{L}_{2p-2}(\mathcal{R})\,, \tag{9.46}$$

where now $\mathcal{L}_{2p-2}(\mathcal{R})$ is constructed with the intrinsic curvature tensor of the induced metric on m.

Recently, Sarkar and Wall proposed a generalization of the Lovelock theories with an action of the form [381]

$$I_{\rm SW} = \frac{1}{16\pi G}\int d^{d+1}x\,\sqrt{g}\,f(\mathcal{L}_0,\mathcal{L}_2,\mathcal{L}_4,\cdots,\mathcal{L}_{2k})\,, \tag{9.47}$$

where f is some general function of the extended Euler densities up to $k = \lfloor(d+1)/2\rfloor$ — we will assume that f is a polynomial. Hence these new generalized Lovelock theories might also be seen as an extension of $f(R)$ gravity [399]. In general, the gravitational equations of motion will involve fourth order derivatives of the metric in these new theories. However, the motivation to considering these theories is to examine the second law of black hole thermodynamics in higher curvature theories. In fact, [381] found an expression for the gravitational entropy which satisfies a classical increase theorem for linearized perturbations of Killing horizons

$$S_{\rm SW} = \frac{1}{4G}\int d^{d-1}y\,\sqrt{\gamma}\,\sum_{p=1}^{\lfloor\frac{d+1}{2}\rfloor} p\,\frac{\partial f}{\partial\mathcal{L}_{2p}(R)}\,\mathcal{L}_{2p-2}(\mathcal{R}_m)\,. \tag{9.48}$$

Certainly, this expression also reduces to that in eq. (9.46) when f is linear and the action (9.47) is simply the Lovelock action (9.44). We take these facts, in particular, that eq. (9.48) applies for (at least small) deviations away from a Killing horizon, as evidence that it will yield the correct gravitational entropy in the more general context of using eq. (9.26) to evaluate HEE.

Hence we will use the generalized Lovelock theories (9.47) as framework to examine the corner contribution in HEE. Since we are working in a four-dimensional bulk spacetime, all of the $\mathcal{L}_{2p}$ with $p = 3, 4, ...$ will vanish identically. Therefore, we can only construct the new gravity action with powers of the Ricci scalar R and the four-dimensional Euler density $\mathcal{X}_4$, given in eq. (9.28). Hence we consider supplementing the standard cosmological constant and Einstein terms in eq. (9.7) with higher curvature interactions of the form

$$\triangle I_{v,w} = \frac{\lambda_{v,w}}{16\pi G}\int d^4x\,\sqrt{g}\,L^{2v+4w-2}\,R^v\,\mathcal{X}_4^w\,, \tag{9.49}$$

with integers $v, w \geq 1$. Then using eq. (9.48), the corresponding entropy functional becomes

$$\triangle S_{v,w} = \frac{\lambda_{v,w}}{4G}\int_m d^2y\,\sqrt{\gamma}\,L^{2v+4w-2}\,\left[v\,R^{v-1}\mathcal{X}_4^w + 2w\,R^v\mathcal{X}_4^{w-1}\,\mathcal{R}\right]\,. \tag{9.50}$$

Now we are evaluating this expression in a pure AdS_4 background (9.6) and so it may be simplified by substituting $R = -12/\tilde{L}^2$ and $\mathcal{X}_4 = 24/\tilde{L}^4$ to yield

$$\triangle S_{v,w} = (-1)^{v-1}\, 2^{2v+3w-4}\, 3^{v+w-1}\, \frac{\lambda_{v,w}}{G} \int_m d^2y\sqrt{\gamma}\, \left[v - wL^2\,\mathcal{R}\right]\, f_\infty^{v+2w-1}\,. \tag{9.51}$$

Note the power of $f_\infty = L^2/\tilde{L}^2$ appearing in the integrand above. We have kept this factor here to indicate that in general after solving the gravitational equations, one finds that the curvature scale $\tilde{L}$ no longer coincides with the scale L set by the cosmological constant. In particular, we find

$$1 - f_\infty + (-1)^v 2^{2v+3w-2} 3^{v+w-1}\,(2 - v - 2w)\,\lambda_{v,w}\, f_\infty^{v+2w} = 0\,. \tag{9.52}$$

However, note that if we are working perturbatively in the coupling, we have $f_\infty = 1 + \mathcal{O}(\lambda_{v,w})$.

With the simplifications produced by working in AdS_4, the modifications to the entropy functional have reduced to a term proportional to the area of the bulk surface and another involving an integral of the intrinsic Ricci scalar over m. Hence at this point, we can turn to our results from the previous subsection where both terms were encountered before. In particular, neither term modifies the profile of the extremal surface in the bulk and further the area term only changes the corner contribution by an overall factor while the term involving $\mathcal{R}$ does not contribute to this universal term at all. More precisely, given the precise results in eq. (9.51), we find that the charge associated with the corner term becomes

$$\kappa = \left[1 - (-1)^v\, 2^{2v+3w-2}\, 3^{v+w-1}\, v\,\lambda_{v,w} + \mathcal{O}(\lambda_{v,w}^2)\right] \kappa_{E}\,, \tag{9.53}$$

where the result is expressed to leading order in the perturbative expansion in the coupling.

To make this analysis more concrete, let us extend the general curvature-squared theory (9.27) with the generalized Lovelock interactions which are third- and fourth-order in the curvature

$$\begin{aligned} I = \frac{1}{16\pi G} \int d^4x\, \sqrt{g} \bigg[& \frac{6}{L^2} + R + L^2 \left(\lambda_1 R^2 + \lambda_2 R_{\mu\nu}R^{\mu\nu} + \lambda_{\mathrm{GB}}\mathcal{X}_4\right) \\ & + L^4 \left(\lambda_{3,0} R^3 + \lambda_{1,1} R\mathcal{X}_4\right) + L^6 \left(\lambda_{4,0} R^4 + \lambda_{2,1} R^2\mathcal{X}_4 + \lambda_{0,2}\mathcal{X}_4^2\right) \bigg]\,. \end{aligned} \tag{9.54}$$

Then the final expression of the corner coefficient and the corresponding charge take the simple form

$$q(\Omega) = \alpha\, q_E(\Omega) \quad \text{and} \quad \kappa = \alpha\, \kappa_E \tag{9.55}$$

where to leading order in the dimensionless couplings, the overall coefficient is given by

$$\alpha = 1 - 24\lambda_1 - 6\lambda_2 + 432\lambda_{3,0} + 24\lambda_{1,1} - 6912\lambda_{4,0} - 576\lambda_{2,1} + \mathcal{O}(\lambda^2)\,. \tag{9.56}$$

Of course, $q_E(\Omega)$ and κ_E are the corresponding quantities evaluated for Einstein gravity, as given in eqs. (9.17) and (9.21), respectively. The fact that the functional form of $q(\Omega)$ is unchanged results because the higher curvature contributions to the entropy functional studied here do not modify the profile of the extremal surface in the bulk. We do not expect that this behaviour is completely universal and it may be modified in theories with even more general higher curvature interactions. We will come back to this point in the discussion section.

9.2 Comparison with other charges

By considering the limit of a small opening angle in eq. (9.3), we identified a 'central charge' that appears in the entanglement entropy of regions where boundary has corners. When evaluated for holographic CFT's dual to Einstein gravity, the result (9.21) is proportional to the ratio $\tilde{L}^2/G \sim \tilde{L}^2/\ell^2_{Planck}$. The latter ratio is well known to be indicative of the number of degrees of freedom in the boundary theory. However, for Einstein gravity, the same ratio is ubiquitous for physical quantities involving a similar count of degrees of freedom, *e.g.,* the entropy density of a thermal bath. The pervasiveness of $\tilde{L}^2/G$ arises since this is the only dimensionless parameter that is intrinsic to the bulk theory with Einstein gravity. By considering higher curvature theories for the bulk gravity, as in the previous section, we are introducing more dimensionless couplings and we can begin to distinguish the various charges in the boundary theory, *e.g.,* see [85, 247, 335]. Our objective here is to use our holographic results to determine if the corner charge κ should be considered a new and distinct charge or if it is proportional to charges already appearing in other physical quantities. In particular, in the following, we compare κ to the analogous charges appearing in: 1) the entanglement entropy of an infinite strip; 2) the entanglement entropy of a disk; 3) the entropy density of a thermal bath and 4) the two-point function of the stress tensor. Again, with Einstein gravity in the bulk, all of these quantities are proportional to $\tilde{L}^2/G$. While the same is true (with our conventions) with the higher curvature theories, the additional dimensionless couplings also give each a unique signature, as we will see in the following.

9.2.1 Entanglement entropy for a strip

We begin with the entanglement entropy of an infinite strip. For a general three-dimensional CFT, the result will take the form [119, 338]

$$S_{EE} = c_1\,\frac{2H}{\delta} - \tilde{a}\,\frac{H}{\ell} + \mathcal{O}(\delta) \tag{9.57}$$

where ℓ is the width of the strip and H is a long distance scale introduced to regulate the length of the strip, *i.e.,* the area of the entangling surface is $2H$. The universal coefficient $\tilde{a}$ can be isolated with

$$\tilde{a} = \frac{\ell^2}{H}\,\frac{\partial S_{EE}}{\partial \ell}\,. \tag{9.58}$$

We will find that $\tilde{a} = \kappa$ in our HEE calculations below. In fact, this result holds for general three-dimensional CFT's and has a simple explanation since there is a conformal transformation that (essentially) relates the corresponding entanglement geometries — see appendix F.

Holographic calculations of the entanglement entropy of a strip were first carried out in [378, 379] with Einstein gravity in the bulk. To start, we write AdS$_4$ metric as

$$ds^2 = \frac{\tilde{L}^2}{z^2}\left(dz^2 + dt_{\rm E}^2 + dx_1^2 + dx_2^2\right)\,. \tag{9.59}$$

Let us parameterize the strip in the boundary as the region B= $\{t_{\rm E} = 0,\, x_1 \in [-\ell/2, \ell/2]\}$. As noted above, we also introduce an IR regulator by, *e.g.,* making the x_2 direction periodic with period $\triangle x_2 = H$ and with $H \gg \ell$. The translational symmetry along x_2 allows us to parametrize the entangling surface m as $z = h(x_1)$, so the induced metric on the surface becomes

$$ds_m^2 = \frac{\tilde{L}^2}{h^2}\left(\left[1+\dot{h}^2\right]dx_1^2 + dx_2^2\right)\,, \tag{9.60}$$

where $\dot{h} = \partial_{x_1} h$. Focusing on Einstein gravity [237, 378, 379], we look for surfaces m extremizing the area functional, which in this case is given by

$$S_B = \frac{\tilde{L}^2}{4G} H \int_{-\ell/2}^{\ell/2} dx_1\,\frac{1}{h^2}\sqrt{1+\dot{h}^2}\,. \tag{9.61}$$

Since the integrand does not depend on x_1 explicitly, there is conserved first integral which can be used to write

$$\dot{h} = -\frac{\sqrt{z_*^4 - h^4}}{h^2}\,, \tag{9.62}$$

where z_* is the maximal value of z reached by the extremal surface. The latter can be identified in terms of ℓ through

$$\ell = 2\int_0^{\ell/2} dx_1 = 2\int_0^{z_*}\frac{h^2\,dh}{\sqrt{z_*^4 - h^4}} = \frac{\sqrt{2}}{\sqrt{\pi}}\,\Gamma\left(\tfrac{3}{4}\right)^2 z_*\,. \tag{9.63}$$

The final result for the entanglement entropy with Einstein gravity in the bulk is

$$S_B = \frac{\tilde{L}^2}{2G}\frac{H}{\delta} - \frac{\tilde{L}^2}{2\pi G}\,\Gamma\left(\tfrac{3}{4}\right)^4\frac{H}{\ell}\,, \tag{9.64}$$

Hence the corresponding universal coefficient is

$$\tilde{a}_E = \frac{\tilde{L}^2}{2\pi G}\,\Gamma\left(\tfrac{3}{4}\right)^4\,, \tag{9.65}$$

which exhibits the expected factor of $\tilde{L}^2/G$, and further comparing with eq. (9.21), we see that $\tilde{a}_E = \kappa_E$.

This calculation of HEE is easily extended to the higher curvature theories considered in section 9.1.1, taking into account the general remarks made there. We use the prescription (9.26) with the generalized entropy functionals for those theories given in eqs. (9.29) and (9.50). However, as we found before, the terms involving the trace of the extrinsic curvature do not contribute, those with the intrinsic Ricci scalar only contribute boundary terms and those involving bulk curvatures only modify the Einstein result by an overall factor. It is straightforward to verify these expectations with explicit calculations and the final result is

$$S_B = \beta\frac{\tilde{L}^2}{2G}\frac{H}{\delta} - \alpha\,\frac{\tilde{L}^2}{2\pi G}\,\Gamma\left(\tfrac{3}{4}\right)^4\frac{H}{\ell}\,, \tag{9.66}$$

where α is precisely the same factor given in eq. (9.56). The coefficient β appearing in the area law term is another function of the couplings λ_i, which is not needed here but does not coincide with α in general.[14] Hence the final result for the universal coefficient is

$$\tilde{a} = \alpha\,\tilde{a}_E\,, \tag{9.67}$$

and so we find that $\tilde{a} = \kappa$ in all of these examples. As noted above, this is in fact a general result for three-dimensional CFT's.

[14] In fact, the same factor β appears below in the HEE calculation for a disk.

9.2.2 Entanglement entropy for a disk

For a general three-dimensional CFT, the entanglement entropy of a disk will take the form [122, 222]

$$S_{EE} = c_1 \frac{2\pi R}{\delta} - 2\pi\, c_0 + \mathcal{O}(\delta) \tag{9.68}$$

where R is the radius of the disk. The universal coefficient c_0 can be isolated here by evaluating [297]

$$\tilde{a} = \frac{1}{2\pi}\left(R\,\frac{\partial S_{EE}}{\partial R} - S_{EE}\right). \tag{9.69}$$

Of course, in this case, the universal constant c_0 plays the an important role as the central charge in the F-theorem, *i.e.,* it decreases monotonically in renormalization group flows [121, 257, 277, 340, 341].

The HEE for a disk was first calculated for Einstein gravity using eq. (9.8) in [378, 379]. However, this calculation was later extended to general higher curvature theories of gravity in the bulk [122, 341]. Making use of a conformal transformation in the boundary CFT, the problem of calculating the entanglement entropy for a disk can be mapped to the question of evaluating the thermal entropy of the CFT in a particular curved background. The latter can then be evaluated as the Wald entropy of the corresponding horizon in bulk spacetime with a general gravitational theory in the bulk. The horizon actually appears as an 'observer' horizon upon transforming the bulk AdS geometry to AdS-Rindler coordinates and the extremal area surface in the standard calculation coincides with the bifurcation surface of this horizon, *e.g.,* see [172].

Our calculations of HEE for the disk followed the prescription outlined in section 9.1.1, using eq. (9.26) with the entropy functionals in eqs. (9.29) and (9.50). Using the AdS_4 metric in eq. (9.6), let us parameterize the disk in the boundary as the region $D = \{t_{\mathrm{E}} = 0,\ \rho \le R\}$. We write the profile of the bulk surface m as $z = h(\rho)$ with no dependence on θ because of the rotational symmetry of the disk. The induced metric on m then becomes

$$ds^2_m = \frac{\tilde{L}^2}{h^2}\left(\left[1+\dot{h}^2\right]d\rho^2 + \rho^2 d\theta^2\right), \tag{9.70}$$

where $\dot{h} = \partial_\theta h$. The extremal area surface becomes the hemisphere [378, 379]

$$\rho^2 + z^2 = R^2 \quad \text{with } z \ge 0\,. \tag{9.71}$$

Now in general, the entropy functional for higher curvature theories can be written as the Wald entropy plus terms which are at least quadratic in the extrinsic curvature [103, 161]. However, one can readily verify that the extrinsic curvature of the above bulk surface (9.71) vanishes and hence any extrinsic curvature terms will vanish to first order if we make variations of this surface. Since the Wald entropy only involves bulk curvatures, this entropy reduces to the area functional multiplied by an extra overall factor, as in the previous section. Hence eq. (9.71) still remains the extremal surface when calculating the HEE of a disk for any general higher curvature theory in the bulk. Hence with eqs. (9.29) and (9.50) for the theories in section 9.1.1, evaluating the HEE yields

$$S_D = \beta\,\frac{\tilde{L}^2}{2G}\,\frac{R}{\delta} - \beta\,\frac{\tilde{L}^2}{2G}\,, \tag{9.72}$$

where [15]

$$\beta = 1-24\lambda_1-6\lambda_2-2\lambda_{\rm GB}+432\lambda_{3,0}+48\lambda_{1,1}-6912\lambda_{4,0}-864\lambda_{2,1}-96\lambda_{0,2}+\mathcal{O}(\lambda^2)\,. \tag{9.73}$$

Hence the universal charge for the corresponding holographic CFT's becomes

$$c_0 = \beta\, c_{0,E} = \beta\,\frac{\tilde{L}^2}{4\pi G}\,, \tag{9.74}$$

where $c_{0,E}$ denotes the result for Einstein gravity, *i.e.,* $c_{0,E} = \tilde{L}^2/(4\pi G)$. Note that with Einstein gravity, the ratio of the universal charges for the corner and the disk is relatively simple, *i.e.,*

$$\frac{\kappa_E}{c_{0,E}} = 2\Gamma\left(\tfrac{3}{4}\right)^4\,. \tag{9.75}$$

However, comparing eqs. (9.67) and (9.74), as well as eqs. (9.56) and (9.73), we see that there is no simple relation between κ and c_0 in the general theories. In particular, we have

$$\frac{\kappa}{c_0} = 2\,\Gamma\left(\tfrac{3}{4}\right)^4\,\left(1-2\lambda_{\rm GB}-24\lambda_{1,1}+288\lambda_{2,1}+96\lambda_{0,2}+\mathcal{O}(\lambda^2)\right) \tag{9.76}$$

and so this ratio depends on the precise value of the gravitational couplings in the higher curvature theories.

9.2.3 Thermal entropy

Another quantity which might be used to characterize the number of degrees of freedom in a system is the thermal entropy. For a three-dimensional CFT, the thermal entropy density takes the form

$$s = c_S\,T^2\,. \tag{9.77}$$

The coefficient c_S is another interesting 'central charge' which is readily calculable in a holographic setting. Of course, the thermal bath in the boundary theory is dual to a planar AdS_4 black hole and we need only calculate the entropy density of the event horizon. For Einstein gravity, the black hole solution can be written as

$$ds^2 = \frac{\tilde{L}^2}{z^2}\left(\frac{dz^2}{f(z)} - f(z)dt^2 + dx_1^2 + dx_2^2\right)\quad \text{with } f(z)\equiv 1-\frac{z^3}{z_{\rm H}^3}\,, \tag{9.78}$$

where $z = z_{\rm H}$ is the position of the event horizon. The Hawking temperature is given by $T = 3/(4\pi z_{\rm H})$ and the horizon entropy is given by the Bekenstein-Hawking formula, which yields

$$S_{\rm thermal} = \frac{1}{4G}\int_{z=z_{\rm H}} \sqrt{h}\,d^2x = \frac{\tilde{L}^2}{4G\,z_{\rm H}^2}\,V_2\,, \tag{9.79}$$

where $V_2 \equiv \int dx_1 dx_2$. Now dividing by the spatial volume V_2 yields the entropy density and substituting the temperature for $z_{\rm H}$ produces an expression of the expected form given in eq. (9.77). The corresponding central charge is

$$c_{S,E} = \frac{4\pi^2}{9}\,\frac{\tilde{L}^2}{G}\,. \tag{9.80}$$

[15] For a general theory with action (9.49), the corresponding expression is

$$\beta = 1+(v+w)(-1)^{v-1}2^{2v+3w-2}3^{v+w-1}\lambda_{v,w}+\mathcal{O}(\lambda_{v,w}^2)\,.$$

Here again, we see the ubiquitous factor of $\tilde{L}^2/G$ and hence the ratio with the corner charge yields a fixed numerical factor, *i.e.*,

$$\frac{\kappa_E}{c_{S,E}} = \frac{9}{8\pi^3}\,\Gamma\big(\tfrac{3}{4}\big)^4\,. \tag{9.81}$$

Curvature-squared gravity

Just as with empty AdS_4, the black hole metric (9.78) is also a solution of the general curvature-squared gravity for any value of the parameters λ_1, λ_2 and $\lambda_{\rm GB}$ provided $\tilde{L}^2 = L^2$. Hence the only difference from the above calculations is that the horizon entropy is now given by the Wald entropy formula [254, 255, 420]. Alternatively, we can use the generalized entropy functional in eq. (9.29) since the two expressions only differ by terms involving the extrinsic curvature and the latter vanishes on the event horizon of the AdS_4 black hole. We find, in agreement with [393]

$$s = (1-24\lambda_1-6\lambda_2)\,\frac{4\pi^2\tilde{L}^2}{9\,G}\,T^2 \tag{9.82}$$

and therefore the corresponding central charge becomes

$$c_S = \gamma_2\, c_{S,E} \qquad \text{with}\;\; \gamma_2 = 1-24\lambda_1-6\lambda_2\,. \tag{9.83}$$

Comparing to eq. (9.43), we see that for curvature-squared gravity, the thermal entropy charge is modified by the same overall factor that appears in the corresponding corner charge. Hence for this family of holographic theories, the ratio of these two charges remains unchanged from the numerical factor (9.81) that appears with Einstein gravity.

Generalized Lovelock gravity

The black hole metric in eq. (9.78) is no longer a solution of the equations of motion for general theories of the form (9.49). Hence in order to explore how the thermal entropy gets modified here, we must first correct the black hole solution to linear order in the coupling $\lambda_{v,w}$. We parametrize the modified solution as

$$ds^2 = \frac{\tilde{L}^2}{z^2}\left(\frac{dz^2}{f(z)\,[1+\lambda_{v,w}f_2(z)]} - f(z)\,[1+\lambda_{v,w}f_1(z)]\,dt^2 + dx_1^2 + dx_2^2\right)\,, \tag{9.84}$$

where $f_1(z)$ and $f_2(z)$ are two nonsingular functions to be determined. This ansatz was chosen so that the position of the horizon remains at $z = z_{\rm H}$. In order to obtain $f_1(z)$ and $f_2(z)$, we substitute the above metric into the Einstein action (9.7) modified by the addition of a higher curvature interaction as in eq. (9.49) and expand to second order in the coupling $\lambda_{v,w}$.[16] From the second order action, we determine the linearized equations of motion for $f_1(z)$ and $f_2(z)$ and then solve them with the boundary conditions that both functions decay as $z \to 0$ and remain nonsingular at $z = z_{\rm H}$. Below we describe the solution and the results for the thermal entropy for each of the generalized Lovelock interactions up to quartic order in the curvatures, shown in eq. (9.54).

[16] Since we are working perturbatively in $\lambda_{v,w}$, it is sufficient to consider each higher curvature interaction (9.49) separately. Of course, the first order variations by $f_1(z)$ and $f_2(z)$ vanish identically here because to leading order, the metric solves the Einstein equations of motion.

In general, the Hawking temperature of the solution will be given by

$$T = \frac{3}{4\pi z_{\rm H}}\left(1+\frac{f_1(z_{\rm H})+f_2(z_{\rm H})}{2}\lambda_{v,w}+\mathcal{O}(\lambda_{v,w}^2)\right)\,, \tag{9.85}$$

as one can easily check.

a) R^3 and R^4 gravity

For these two particular theories, as well as any theory with only R^v interactions (*i.e.*, $w=0$), the original AdS$_4$ black hole solution (9.78) does not get corrected at any order in the couplings $\lambda_{v,0}$, *i.e.*, $f_1(z)=f_2(z)=0$. The uncorrected black hole solves the equations of motion of these theories provided the curvature scale satisfies eq. (9.52), which was also required for the pure AdS$_4$ metric (9.6) to be a solution in the new theory. Note that for $v=3$ and 4, we find the constraints $1-f_\infty+144\lambda_{3,0}f_\infty^3=0$ and $1-f_\infty-3456\lambda_{4,0}f_\infty^4=0$, respectively.

The horizon entropy is computed using the expression in eq. (9.48). However, since the Ricci scalar of the Schwarzschild-AdS$_4$ background equals that of the pure AdS$_4$ solution, the corrected thermal entropy for these theories differs from the Einstein gravity result by just a overall constant factor which is precisely the same as the $\lambda_{3,0}$ and $\lambda_{4,0}$ contributions to α in eq. (9.56). That is, we find

$$s=\gamma_a\, c_{S,E}\, T^2 \qquad \text{with} \quad \gamma_a = 1+432\lambda_{3,0}-6912\lambda_{4,0}+\mathcal{O}(\lambda^2)\,. \tag{9.86}$$

b) $R\,\mathcal{X}_4$ gravity

For this theory, the AdS curvature is given by $1-f_\infty+24\lambda_{1,1}f_\infty^3=0$ — recall that $f_\infty\equiv L^2/\tilde{L}^2$. The planar black hole (9.78) no longer solves the equations of motion and so we proceed as described above to find the corrected solution to first order in the coupling. The two functions f_1 and f_2 are

$$\begin{aligned} f_1(z) &= -\frac{18z^3(z^3+z_{\rm H}^3)}{z_{\rm H}^6}\,, \\ f_2(z) &= \frac{6z^3(11z^3-3z_{\rm H}^3)}{z_{\rm H}^6}\,. \end{aligned} \tag{9.87}$$

With the new metric, the Hawking temperature becomes

$$T_{1,1}=\frac{3}{4\pi z_{\rm H}}\left(1+6\lambda_{1,1}+\mathcal{O}(\lambda_{1,1}^2)\right)\,. \tag{9.88}$$

Using eq. (9.50), the thermal entropy then becomes

$$s=\gamma_b\, c_{S,E}\, T^2 \qquad \text{with} \quad \gamma_b = 1+24\lambda_{1,1}+\mathcal{O}(\lambda_{1,1}^2)\,. \tag{9.89}$$

We note that γ_b again agrees with the analogous factor appearing in the corner coefficient (9.53) for $v=1=w$.

We stress that, as opposed to the theories with $w=0$, the on-shell Gauss-Bonnet term $\mathcal{X}_4$ is no longer the same in the black hole background as in the pure AdS$_4$ solution (hence eq. (9.50) no longer reduces down to eq. (9.51)). Computing the horizon entropy as a function of the horizon position yields

$$s = (1 + 36\lambda_{1,1} + \mathcal{O}(\lambda_{1,1}^2))\frac{\tilde{L}^2}{4\pi G z_{\rm H}^2}\,. \tag{9.90}$$

It is only when we express the entropy density as a function of the physical temperature (9.88) that we cover the factor γ_b in eq. (9.89). Actually, it is possible to show that different parametrizations of the corrected solution give rise to different expressions for $s(z_{\rm H})$ and $T(z_{\rm H})$, which nevertheless conspire to produce the same physical result when the entropy density is written in terms of the temperature.

c) $R^2\mathcal{X}_4$ gravity

In this case, the curvature scale is determined by $1-f_\infty-576\lambda_{2,1}f_\infty^4=0$, and the functions parameterizing the corrected black hole (9.84) are

$$\begin{aligned} f_1(z) &= \frac{432z^3(z^3+z_{\rm H}^3)}{z_{\rm H}^6}\,, \\ f_2(z) &= \frac{-144z^3(11z^3-3z_{\rm H}^3)}{z_{\rm H}^6}\,. \end{aligned} \tag{9.91}$$

Further, the Hawking temperature becomes

$$T = \frac{3}{4\pi z_{\rm H}}\left(1 - 144\lambda_{2,1} + \mathcal{O}(\lambda_{2,1}^2)\right)\,, \tag{9.92}$$

while the entropy density is given by

$$s = \gamma_c\, c_{S,E}\, T^2 \qquad \text{with} \quad \gamma_c = 1 - 576\lambda_{2,1} + \mathcal{O}(\lambda_{2,1}^2)\,. \tag{9.93}$$

Here again, γ_c agrees with the analogous factor appearing in the corner coefficient (9.53) for $v=2$ and $w=1$.

d) $\mathcal{X}_4^2$ gravity

The last nontrivial interaction at fourth order in curvature corresponds to the square of the Gauss-Bonnet density, $\mathcal{X}_4^2$. To begin, let us note that interactions of the form $\mathcal{X}_4^w$ with $w\geq 2$ are not topological and do modify the gravitational equations of motion in four dimensions. It is only the linear term, *i.e.*, $w=1$ (and $v=0$), which leaves the equations of motion unchanged.

Now in this case, we have $1-f_\infty-96\lambda_{2,1}f_\infty^4=0$ and

$$\begin{aligned} f_1(z) &= \frac{8z^3(11z^6+z^3z_{\rm H}^3+z_{\rm H}^6)}{z_{\rm H}^9}\,, \\ f_2(z) &= \frac{8z^3(67z^6-83z^3z_{\rm H}^3+z_{\rm H}^6)}{z_{\rm H}^9}\,. \end{aligned} \tag{9.94}$$

The Hawking temperature is given by

$$T = \frac{3}{4\pi z_{\rm H}}\left(1 - 8\lambda_{0,2} + \mathcal{O}(\lambda_{0,2}^2)\right)\,, \tag{9.95}$$

and the thermal entropy density becomes

$$s = \gamma_d \, c_{S,E} \, T^2 \,, \qquad \text{with} \qquad \gamma_d = 1 + 16\lambda_{0,2} + \mathcal{O}(\lambda_{0,2}^2)\,. \tag{9.96}$$

Here, the factor γ_d receives a correction which is first order in $\lambda_{0,2}$ while the corresponding factor in the corner coefficient does not, *e.g.,* see eq. (9.56). Hence, we have found the first example for which the agreement is broken between the charges defined by the thermal entropy density and by the corner contribution of the entanglement entropy.

Gathering together all of the first order contributions from the new interactions appearing in the fourth-order action (9.54), we have that the thermal entropy density in the dual boundary theory takes the expected form (9.77) where the corresponding charge takes the form

$$c_S = \gamma \; c_{S,E} \tag{9.97}$$

where the Einstein result $c_{S,E}$ is given in eq. (9.80) and

$$\gamma = 1 - 24\lambda_1 - 6\lambda_2 + 432\lambda_{3,0} + 24\lambda_{1,1} - 6912\lambda_{4,0} - 576\lambda_{2,1} + 16\lambda_{0,2} + \mathcal{O}(\lambda^2) \,. \tag{9.98}$$

Comparing with eqs. (9.55) and (9.56) for the corner contribution of the entanglement entropy in the same theories, we see

$$\frac{\kappa}{c_S} = \frac{9}{8\pi^3}\,\Gamma\!\left(\tfrac{3}{4}\right)^4 \left(1 - 16\lambda_{0,2} + \mathcal{O}(\lambda^2)\right) \,. \tag{9.99}$$

That is, the ratio κ/c_S is independent of most of the additional dimensionless couplings in eq. (9.54) and it would still be given by the same numerical factor found for Einstein gravity in eq. (9.81) for the class of theories with $\lambda_{0,2} = 0$.

9.2.4 Stress tensor two-point function

Let us now turn to the two-point function for the stress tensor, which is particularly interesting since it defines a central charge for CFT's in any spacetime dimension. Evaluated in the vacuum, the functional form of this two-point correlator is completely fixed by conformal symmetry and energy conservation and for a d-dimensional CFT, it takes the form [169, 356][17]

$$\langle \, T_{ab}(x) \, T_{cd}(0) \, \rangle = \frac{C_T}{x^{2d}} \, \mathcal{I}_{ab,cd}(x) \,, \tag{9.100}$$

where

$$\mathcal{I}_{ab,cd}(x) \equiv \frac{1}{2}\left(I_{ac}(x)\, I_{db}(x) + I_{ad}(x)\, I_{cb}(x)\right) - \frac{1}{d}\,\delta_{ab}\,\delta_{cd} \tag{9.101}$$

and

$$I_{ab}(x) \equiv \delta_{ab} - 2\frac{x_a \, x_b}{x^2} \,. \tag{9.102}$$

Below we will focus on $d = 3$ but as remarked above, the above expressions provide a definition of C_T for CFT's in any spacetime dimension. In particular, eq. (9.100) is the standard definition of the central charge c in two-dimensional CFT's, *i.e.,* $C_T = c$, while

[17] Note that in this section unhatted indices from the beginning of the Latin alphabet run over the d-dimensional boundary of AdS_{d+1}.

for four dimensions, $C_T = 40\,c/\pi^4$ where c is the coefficient of the Weyl-squared term in the trace anomaly.

Of course, in a holographic framework, the stress tensor is dual to the normalizable mode of the metric [219, 423] and so evaluating eq. (9.100) requires determining the two-point boundary correlator of the gravitons in the AdS vacuum. This is a standard calculation in the context of Einstein gravity [85, 298] and one finds for three boundary dimensions

$$C_{T,E} = \frac{3}{\pi^3}\frac{\tilde{L}^2}{G}\,. \tag{9.103}$$

Once again, we see the ubiquitous factor of $\tilde{L}^2/G$ and comparing with the corner coefficient (9.21), we have

$$\frac{\kappa_E}{C_{T,E}} = \frac{\pi^2}{6}\,\Gamma\left(\tfrac{3}{4}\right)^4\,. \tag{9.104}$$

In order to investigate how the two-point function (or equivalently the graviton propagator) is modified by the introduction of higher curvature terms in the bulk, let us first recall that generically these new interactions will result in the appearance of higher-order derivatives in the gravitiational equations of motion. Hence the metric will contain additional propagating degrees of freedom beyond the usual massless spin-two graviton. Therefore in a holographic context, the metric will also couple both to the stress tensor and some new tensor operator, which is generically *nonunitary*.[18] We can understand the latter, *i.e.,* that generically the new operator generates negative norm states in the boundary CFT, with the following analogy from [341]: Consider a massless scalar field in flat space whose equation of motion has been corrected with a fourth-order term,

$$\left(\Box + \frac{\lambda}{M^2}\,\Box^2\right)\phi = 0\,, \tag{9.105}$$

where M^2 is some high energy scale and λ, the dimensionless coupling of the higher derivative interaction in the action. Then, the propagator for this field will read

$$\frac{1}{q^2 - \lambda\, q^4/M^2} = \frac{1}{q^2} - \frac{1}{q^2 - M^2/\lambda}\,. \tag{9.106}$$

Here the $q^2 = 0$ pole will correspond to the usual massless mode, whereas that at $q^2 = M^2/\lambda$ is related to a new massive degree of freedom. Regardless of the sign of λ, the sign of the second term in the propagator above will be negative and so the extra mode is a ghost. Of course, if we are working perturbatively in λ, these new degrees of freedom appear at very high energy scales. Hence if we should restrict our attention to energies much less than $M/\lambda^{1/2}$, the new scalar ghost will not go on-shell. In the holographic context, the additional ghost modes create negative norm states in the bulk theory and so they must be dual to new nonunitary operators in the boundary theory. Further, let us note that the curvature scale plays the role of the mass above, *i.e.,* $L^2 \sim 1/M^2$, and so we can expect that the conformal dimension of these operators to be set by the inverse of the gravitational couplings, *i.e.,* $\Delta^2 \sim 1/\lambda$. Hence if we consider the CFT on the background $R \times S^{d-1}$, the new operator would again be associated with high energy states.

[18] Of course, this is a typical feature of holographic theories with higher curvature interactions in the bulk, but it can be evaded in special cases. For example, $f(R)$ gravity can be re-expressed as Einstein gravity coupled to a scalar field [399]. Hence in this case, the additional CFT operator will be a scalar, which can be unitary in the appropriate circumstances.

The above example also highlights that in a perturbative framework, the extra degrees of freedom are highly in the vicinity of the physical pole. Hence our strategy in studying the graviton propagator will be to organize the linearized gravitational equations of motion which make this suppression manifest and allow us to easily identify the proper kinetic term of the physical modes. In general, writing out the linearized equations of motion for the graviton would be a very complex task but it can simplified here in two ways, as discussed in [337]. First, we are interested in the holographic version of eq. (9.100) which is evaluated in the vacuum and so we need only study the metric fluctuations in the AdS_4 background. That is, we consider a perturbed metric: $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, where $\bar{g}_{\mu\nu}$ is the AdS_4 metric (and $h_{\mu\nu} \ll 1$ for all $\mu, \nu = 0, 1, 2, 3$). In particular then, the background curvature tensor takes the form $\bar{R}^{\mu\nu}{}_{\sigma\rho} = -1/\tilde{L}^2\,(\delta^\mu{}_\sigma\,\delta^\nu{}_\rho - \delta^\mu{}_\rho\,\delta^\nu{}_\sigma)$, which greatly simplifies the form of the linearized equations of motion. That is, they can be expressed entirely in terms of covariant derivatives acting on $h_{\mu\nu}$. In order to further simplify the resulting expressions, which are still rather involved in general, we can use diffeomorphism invariance to choose a convenient gauge. In the following, we restrict ourselves to a transverse traceless gauge,[19] *i.e.,* $\bar{\nabla}^\mu h_{\mu\nu} = 0$ and $\bar{g}^{\mu\nu} h_{\mu\nu} = 0$.

With these choices, the linearized Einstein equations become

$$G^L_{\mu\nu} = -\frac{1}{2}\left[\bar{\Box} + \frac{2}{\tilde{L}^2}\right] h_{\mu\nu} = 8\pi G\, T_{\mu\nu}\,, \tag{9.107}$$

where $G^L_{\mu\nu}$ denotes the linearized Einstein tensor. We have included the stress tensor $T_{\mu\nu}$ for some additional matter fields to the right-hand side because in the following, it will be important to establish the normalization of Newton's constant, or alternatively of the graviton kinetic term. The linearized equation which results from our complete fourth-order gravity (9.54) turns out to read[20]

$$-\frac{\alpha}{2}\left[\bar{\Box} + \frac{2}{\tilde{L}^2}\right] h_{\mu\nu} - \frac{\lambda_2 L^2}{2}\left[\bar{\Box} + \frac{2}{\tilde{L}^2}\right]^2 h_{\mu\nu} = 8\pi G\, T_{\mu\nu}\,, \tag{9.108}$$

where α is precisely the constant given by eq. (9.56). Interestingly, none of the higher-order terms considered, except for the $R_{\mu\nu}R^{\mu\nu}$ interaction, produce fourth-order derivatives contributions to the linearized equation for the physical graviton $h_{\mu\nu}$ in the AdS_4 background in this gauge, which is a rather striking phenomenom.[21] It would be certainly interesting to classify the families of higher-order gravities for which this behavior is encountered at each order in curvature. We will not pursue such a goal here.

The left-hand side of eq. (9.108) is organized in a way which makes obvious the suppression of the second term in the vicinity of the physical pole, *i.e.,* for $(\bar{\Box}+2/\tilde{L}^2)h_{\mu\nu} \simeq 0$. However, the higher curvature terms still make their presence felt through the appearance of α which modifies the coefficient of the leading Einstein-like term. As commented above, one can interpret this new coefficient as modifying the normalization of Newton's constant,

[19] Let us comment that in the perturbative framework discussed here, the physical degrees of freedom still correspond to a massless spin-two graviton and so this gauge can still be applied here. Note that the traceless condition eliminates the possibility of identifying new scalar degrees of freedom, *e.g.,* as appear in $f(R)$ gravity — see footnote 18. However, these modes are regarded as unphysical with our current perturbative perspective.

[20] This result agrees with that found in [393] for four-dimensional curvature-squared gravities.

[21] In appendix G, we perform the detailed calculation in a general gauge for $f(R)$ gravity, in which the same structure is found, and show how the fourth-order terms go away in this gauge.

i.e., $G_{\rm eff} = G/\alpha$ or as having modified the normalization of the graviton kinetic term. In any event, the net effect is to modify the previous holographic calculation of the two-point correlator for Einstein gravity by an overall factor of α. Hence in the higher curvature theory (9.54), we reproduce the desired expression in eq. (9.100) where the central charge is now given by

$$C_T = \alpha\, C_{T,E} = \alpha\, \frac{3}{\pi^3}\, \frac{\tilde{L}^2}{G}\,, \tag{9.109}$$

where again, α is precisely the same constant given by eq. (9.56). Of course, we could also write this expression as $C_T = 3\tilde{L}^2/(\pi^2 G_{\rm eff})$, *i.e.,* the general result has the same form as that for the Einstein theory except that G is replaced by $G_{\rm eff}$. Therefore, the correction to the central charge appearing in the two-point correlator of the stress tensor (9.100) matches that appearing in the universal corner term. Hence all of the higher curvature theories considered here yield the same ratio (9.104) as in the Einstein theory, *i.e.,*

$$\frac{\kappa}{C_T} = \frac{\pi^2}{6}\, \Gamma\big(\tfrac{3}{4}\big)^4 \simeq 3.7092\,. \tag{9.110}$$

One might hope that this is a universal result extending beyond holography. However, in the discussion section below, we will show that this result does not hold in simple free field theories.

9.3 Discussion

In this chapter, we have studied the universal term arising from the presence of corners in the entangling surface for three-dimensional holographic conformal field theories. In general, this coefficient of the logarithmic term in eq. (9.1) is a function of the opening angle at the corner $q(\Omega)$. As we will discuss below, the precise form of this function depends on the details of the underlying CFT, however, as explained in the introduction, this function is constrained to behave as $q(\Omega) \simeq \kappa/\Omega$ in the limit of small opening angles and as $q(\Omega) \simeq \sigma\,(\Omega-\pi)^2$ in the limit of a nearly smooth entangling surface. Hence, eqs. (9.3) and (9.4) define two coefficients, κ and σ, which can be used to characterize different CFT's. Motivated by the idea that the corner contribution provides a useful measure of the number of degrees of freedom in the underlying theory, we referred to these constants as 'central charges.' In our holographic calculations, we found that the overall form of $q(\Omega)$ did not change and so the two charges were simply related in all of holographic models, *i.e.,* $\kappa/\sigma = 4\,\Gamma(3/4)^4$. Hence we focus on the small angle charge κ in the following discussion. In particular, one goal was to see if this corner charge had a simple relation to any other known 'charges,' which provide a similar counting of degrees of freedom and might be accessed with more conventional probes of the theory, or if κ is really a distinct quantity.

Our approach was to study κ for an extended holographic model involving higher curvature interactions in the bulk gravity theory, as described in section 9.1. In particular, we evaluated the corner term for an entangling surface with a sharp corner on the boundary of AdS_4, using holographic entanglement entropy (9.26). The final result,

$$\kappa = \alpha\,\kappa_E\,, \quad \text{with} \quad \alpha = 1-24\lambda_1-6\lambda_2+432\lambda_{3,0}+24\lambda_{1,1}-6912\lambda_{4,0}-576\lambda_{2,1}+\mathcal{O}(\lambda^2)\,, \tag{9.111}$$

and

$$\kappa_E = \frac{\tilde{L}^2}{2\pi G}\, \Gamma\big(\tfrac{3}{4}\big)^4\,, \tag{9.112}$$

gives κ for the broad class of gravitational theories described by the action (9.54). Our general result is proportional to $\tilde{L}^2/G$ (*i.e.,* the AdS scale squared over Newton's constant) but it is also a function of the eight dimensionless couplings appearing in the action (9.54). Next, in section 9.2, we evaluated several charges appearing in different physical quantities within the same holographic framework. In particular, we studied the analogous charges appearing in the universal terms in the EE of a strip and of a disk, in the thermal entropy density, and in the two-point function of the holographic stress tensor. All of these charges, as well as κ, are simply proportional to $\tilde{L}^2/G$ with Einstein gravity in the bulk and so they can not be distinguished from one another in the corresponding holographic CFT's. However, these charges also become dependent on the additional gravitational couplings with higher curvature gravity in the bulk. Our calculations were perturbative in the λ_i and hence the results are only linear in these couplings. However, this still allowed us to distinguish the various different charges in the boundary CFT. Hence, this extended holographic model provides an interesting framework to investigate our goal stated above, namely, to determine if the corner charge can be considered distinct or if it has a simple relation to another known central charge.

Of course, we do not have a top-down construction where the action (9.54) emerges as the low energy effective action for, *e.g.,* some string theory compactification. Rather our perspective is that such extended holographic models provide an interesting framework to test general properties of CFT's, *i.e.,* if there are certain properties common to all CFT's then they should be satisfied by the holographic CFT's defined by these models. This approach has found success in a number of interesting contexts, such as the discovery of the F-theorem [340, 341]. Below, we also look to test a simple conjecture, motivated by our holographic results, with calculations for free massless quantum field theories.

Another caveat in our analysis is that for the generalized Lovelock theories (9.47), the appropriate gravitational entropy functional to use in evaluating the HEE (9.26) is given by eq. (9.48). Recall that present evidence [21, 66, 327] suggests that the general formula for the entropy functional proposed in [161] must be further refined for higher curvature theories involving cubic and higher powers of the curvature. However, we argued that the use of eq. (9.48) is well motivated by the somewhat complementary analysis of [381] examining the second law of black hole thermodynamics in these higher curvature theories. However, it would be useful to verify this more directly when a fuller understanding of HEE in higher curvature theories emerges.

A summary of the ratios corresponding to the different charges computed in this chapter with respect to κ can be found in Table 9.1.

Constant	Ratio
Strip HEE	$\kappa/\tilde{a} = 1$
Disk HEE	$\kappa/c_0 = 2\,\Gamma\left(\frac{3}{4}\right)^4 \left(1 - 2\lambda_{\rm GB} - 24\lambda_{1,1} + 288\lambda_{2,1} + 96\lambda_{0,2} + \mathcal{O}(\lambda^2)\right)$
Thermal entropy	$\kappa/c_S = \frac{9}{8\pi^3}\,\Gamma\left(\frac{3}{4}\right)^4 \left(1 - 16\lambda_{0,2} + \mathcal{O}(\lambda^2)\right)$
$\langle T_{ab}(x)T_{cd}(0)\rangle$	$\kappa/C_T = \frac{\pi^2}{6}\,\Gamma\left(\frac{3}{4}\right)^4$

Table 9.1: Ratios comparing the corner charge κ with similar physical coefficients.

We have seen that our holographic calculations yield $\kappa = \tilde{a}$, where the latter is the coefficient of the universal term in the EE of a strip, as defined in eq. (9.58). However, this is a universal result that is expected to hold for any CFT on the basis of a conformal mapping which relates the two entanglement entropy calculations — see appendix F. Hence this result can be considered a check of our holographic calculations.

On the other hand, the charge c_0 corresponding to the universal constant in the EE of a disk is a distinct charge. Of course, the latter is the central charge which decreases monotonically in RG flows, according to the F-theorem [121,257,277,340,341]. The independence of these two charges is illustrated by eq. (9.76), which shows that the ratio κ/c_0 depends on $\lambda_{\rm GB}$, $\lambda_{1,1}$, $\lambda_{2,1}$ and $\lambda_{0,2}$. Hence these two charges depend on the details of the corresponding boundary theories in different ways. Alternatively, the ratio is independent of the remaining four gravitational couplings, λ_1, λ_2, $\lambda_{3,0}$ and $\lambda_{4,0}$. Hence there are also broad classes of theories with the same ratio κ/c_0 but it is not a universal feature common to all CFT's.

The thermal entropy density for the holographic theories was calculated as the entropy density of the corresponding AdS$_4$ planar black hole. In this case, eq. (9.99) shows that κ/c_S is not universal but only depends on $\lambda_{0,2}$, the coupling for the $(\mathcal{X}_4)^2$ interaction in eq. (9.54). However, the fact that this particular example produces a mismatch suggests that this ratio will also depend on other new couplings for more general higher curvature theories. In fact, our findings seem to suggest that the generalized Lovelock theories with $w = 0$ or 1 and arbitrary v will respect the agreement between the charges, whereas those with $w \geq 2$ will not. We have explicitly verified that this is the case for $v = 1$ and $w = 2$.[22]

Eq. (9.110) shows that the ratio κ/C_T is the same for all of the holographic theories which we studied, where C_T is the central charge appearing in the two-point function (9.100) of the stress tensor. Hence eq. (9.110) matches the result (9.104) for Einstein gravity with $\kappa/C_T = \pi^2\Gamma\left(\frac{3}{4}\right)^4/6$, at least to first order in the gravitational couplings. It is natural to conjecture that this ratio is a universal quantity for all CFT's, even beyond holography. Some further suggestive results can be found in [339], which studied singular entangling surfaces in holographic models in higher dimensions. In particular, the holographic model examined there had Gauss-Bonnet gravity in the bullk and it was found that for an entangling surface with a conical singularity, C_T controls the coefficient for the universal contribution in the limit of a small opening angle. We will test this simple conjecture below with massless free field theories finding that this result does not hold in those simple field theories.

To close let us observe that a consequence of our results is that C_T and c_S are found to disagree in general for holographic CFT's. Supporting evidence of this disagreement for general holographic theories can be found in [341], where it was shown that these two

[22] That is, κ/c_S depends on $\lambda_{1,2}$ but κ/C_T still takes the standard Einstein value (9.104).

charges are not the same for quasi-topological gravity [335, 337].

9.3.1 Shape of the extremal surface

In our holographic investigation of the corner contribution, we found that none of the higher curvature interactions which we studied led to any modification in the functional form of $q(\Omega)$. Rather it remained exactly the same as in Einstein gravity, *i.e.,* $q(\Omega) = \alpha\, q_E(\Omega)$ where the constant α is given in eq. (9.56). This result is related to the fact that all of the corresponding entropy functionals were extremized by extremal area surfaces in the AdS$_4$ background, just as in Einstein gravity. Further our discussion in section 9.1.1 suggests that this result is not simply a consequence of working to first order in a perturbative treatment of the gravitational couplings. Hence one may wonder whether this is a general feature of HEE in the AdS$_4$ vacuum for any higher curvature theory of gravity in the bulk. However, we argue that the latter is, in fact, *not* a universal result.

First we observe that the curvature tensor takes the simple form

$$R_{\mu\nu\rho\sigma} = -\frac{1}{\tilde{L}^2}\left(g_{\mu\rho}\, g_{\nu\sigma} - g_{\mu\sigma}\, g_{\nu\rho}\right) \tag{9.113}$$

in the AdS$_4$ background. Hence as in the examples in section 9.1.1, the terms in the entropy functional constructed with background curvatures will reduce to an integral of some constant over the bulk surface m, *i.e.,* they multiply the Bekenstein-Hawking contribution by some constant factor. Similarly, any terms involving a mixture of background curvatures and extrinsic curvatures will reduce to an integral of some scalar constructed purely from extrinsic curvatures (and possibly derivatives of the extrinsic curvatures). Therefore, we should consider whether in general such extrinsic curvature terms can lead to modifications in the shape of m — and functional corrections to $q(\Omega)$, as a consequence. Of course, the intuitive answer, which we confirm below, is that a sufficiently complicated contraction of extrinsic curvatures will have a nontrivial effect on the shape of m.

Following the discussion in section 9.1.1, we first observe that any term which contains two or more factors of the trace of the extrinsic curvature, *e.g.,* $K^{\hat{a}} K^{\hat{a}ij} K^{\hat{b}}_{ij} K^{\hat{b}}$, will always leave the extremal area surface unchanged. The reason is simply that $K^{\hat{a}} = 0$ is the equation of motion determining the profile on an extremal area surface. Hence, the variation of a term with two or more factors of $K^{\hat{a}}$ will produce terms which still contain this factor and so will vanish on any extremal area surface. On the other hand, one might guess that if the term $K^{\hat{a}}_{ij} K^{\hat{a}ij}$ appears in the entropy functional that it will modify the shape of the bulk surface, but we argue that in fact it also leaves the extremal area surface unchanged. This term is actually produced by a curvature squared interaction of the form $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ [185, 336]. However, this term can be easily rewritten as a linear combination of R^2, $R_{\mu\nu}R^{\mu\nu}$ and $\mathcal{X}_4$ interactions, *i.e.,* see eq. (9.28). For a pure AdS$_4$ background, we have argued in section 9.1.1 that extremal area surfaces always extremize the entropy functionals corresponding to each of these three interactions, so the same must be true with $K^{\hat{a}}_{ij} K^{\hat{a}ij}$. However, we find that terms of the form $\left(K^{\hat{a}}_{ij} K^{\hat{a}ij}\right)^n$ with $n \geq 2$ are not extremized by the extremal area surface and so we expect contributions of this kind (if they appear in the HEE formula) will modify the functional form of $q_E(\Omega)$. Similarly, cyclic contractions of extrinsic curvatures, *e.g.,* $K^{\hat{a}}_{k_1 k_2} K^{\hat{a}k_2 k_3} \ldots K^{\hat{d}k_{n-1}k_n} K^{\hat{d}\,k_1}_{k_n}$, would also modify the profile of the bulk surface so they would also change the functional form of the corner function.

Hence it is relatively simple to find terms which, if they appear in the gravitational entropy functional, would modify the profile of the bulk surface in the calculation of HEE. Hence the general expression for the universal corner term for arbitrary high curvature theories might be expected to take the form

$$S_{\text{corner}} = -q(\Omega)\log\left(\frac{H}{\delta}\right)\,, \text{ where } q(\Omega) = \alpha\, q_E(\Omega) + r(\Omega)\,, \tag{9.114}$$

where $r(\Omega)$ would be a new function of the opening angle which would depend on some gravitational couplings. If we consider the higher curvature terms as small corrections to Einstein gravity, as for the perturbative calculations in this chapter, it should be clear that $r(\Omega)$ would be highly suppressed with respect to the $q_E(\Omega)$ contribution, since it would only start appearing with interactions that are cubic or higher-order in the curvature. On the other hand, as explained in appendix F, even if such functions correct the functional form of $q_E(\Omega)$ for certain higher-order gravities, the small angle behavior of $q(\Omega)$ is still constrained to take the form

$$\lim_{\Omega\to 0} q(\Omega) = \lim_{\Omega\to 0}\left(\alpha\, q_E(\Omega) + r(\Omega)\right) = \frac{\kappa}{\Omega} + \cdots\,. \tag{9.115}$$

Further, as explained in the introduction, we will have

$$\lim_{\Omega\to \pi} q(\Omega) = \lim_{\Omega\to \pi}\left(\alpha\, q_E(\Omega) + r(\Omega)\right) = \sigma\,(\pi-\Omega)^2 + \cdots\,, \tag{9.116}$$

in the limit of a nearly smooth entangling curve. That is, eqs. (9.3) and (9.4) will still define the universal corner charges, κ and σ, for any holographic theory irrespective of the details of the entropy functional. However, let us note that for Einstein gravity and for all of the holographic theories studied here, these charges are simply related by $\kappa_E/\sigma_E = 4\,\Gamma(3/4)^4$. In general high curvature theories where the corner term is modified as in eq. (9.114), there will be no reason to expect that this simple relation still holds for these two charges.

Of course, we are not at present able to provide an explicit example of a higher curvature interaction which contributes such an 'interesting' extrinsic curvature term to the graviational entropy functional. However, in this regard, we are simply restricted by the current limitations in understanding how to construct the entropy functional given a particular interaction in the bulk action [21, 66, 327]. Still we do see no reason why these more complicated extrinsic curvature terms can not be produced by sufficiently complicated higher curvature interactions.

9.3.2 Comparison with QFT calculations

The holographic calculations performed here are expected to produce $q(\Omega)$ for certain strongly coupled three-dimensional CFT's dual to our bulk gravity theories. On the other hand, similar field theoretical results are also available for a wide range values of Ω in the case of a free scalar and a free fermion [118–120].[23] Further, it was argued [118, 346] that the holographic result for the corner contribution $q_E(\Omega)$ with Einstein gravity qualitatively agrees with these free field results. Given how dissimilar the underlying field theories are in this comparison, even a qualitative agreement may seem somewhat surprising. However,

[23] Most other results in the literature, *e.g.,* see [251, 262–264, 391], are given only for a particular value of the opening angle, *i.e.,* for $\Omega = \pi/2$ which is easily studied on a square lattice.

recall that the behaviour of $q(\Omega)$ is fixed on general grounds both for small angles and for $\Omega \simeq \pi$, *i.e.,* see eqs. (9.3) and (9.4), respectively. Further, given the universal form of $q_E(\Omega)$ at least for the broad range of holographic theories considered in this chapter, we find it interesting here to make a quantitative comparison of $q(\Omega)$ for the holographic and free field theories. In order to make such a comparison, we must start by normalizing $q(\Omega)$ for the various theories. A convenient choice is to consider $q(\Omega)/\kappa$ which will then approach $1/\Omega$ for small angles for any field theory. For all of the holographic theories which we studied, we will have $q_E(\Omega)/\kappa_E$ since the common factor of α in eq. (9.55) cancels in the ratio. Of course, $q_E(\Omega)$ is determined numerically by evaluating the integrals in eqs. (9.17) and (9.18), while κ_E is given by eq. (9.21). The corresponding charges for the free field theories were determined in [118–120] as

$$\kappa_{\text{scalar}} \simeq 0.0397 \qquad \text{and} \qquad \kappa_{\text{fermion}} \simeq 0.0722\,. \tag{9.117}$$

Now the free field results shown in figures 9.4 and 9.5 represent Taylor expansions of $q(\Omega)$ around $\Omega = \pi$ to fourteenth order, which were obtained in [119, 120]. These expansions give a reliable enough approximation for values of the opening angle which are not too small. In particular, the figures also show the lattice results obtained for $q(\Omega)$ at $\Omega = \pi/4$, $\pi/2$ and $3\pi/4$ in [118] using the numerical method developed in [363].

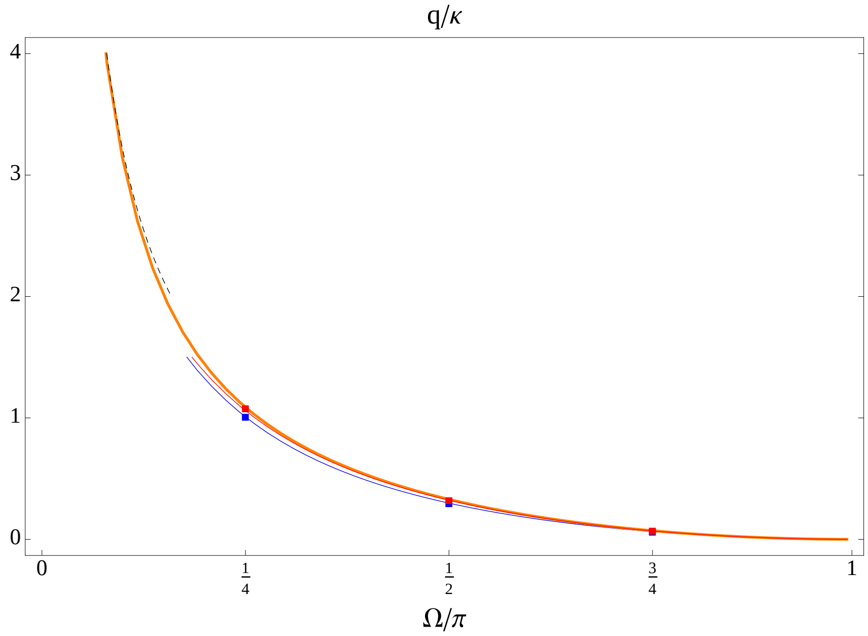

Figure 9.4: (Colour online) We show q/κ for AdS/CFT (orange), a free scalar (blue), a free fermion (red) and the lattice points (squares) obtained numerically for three values of Ω [118]. We also include the black dashed curve giving the $1/\Omega$ behavior which all of the functions will approach for small angles.

In figures 9.4 and 9.5 we see, first of all, how the Taylor expansions for the free theories are in good agreement with the corresponding lattice results. Hence the red and blue lines in these figures can be reasonably trusted at least for angles larger than $\pi/4$. As we see in figure 9.5, the holographic function $q_E(\Omega)/\kappa_E$ turns out to agree with the corresponding free fermion result within a 2% over this whole range where the results are

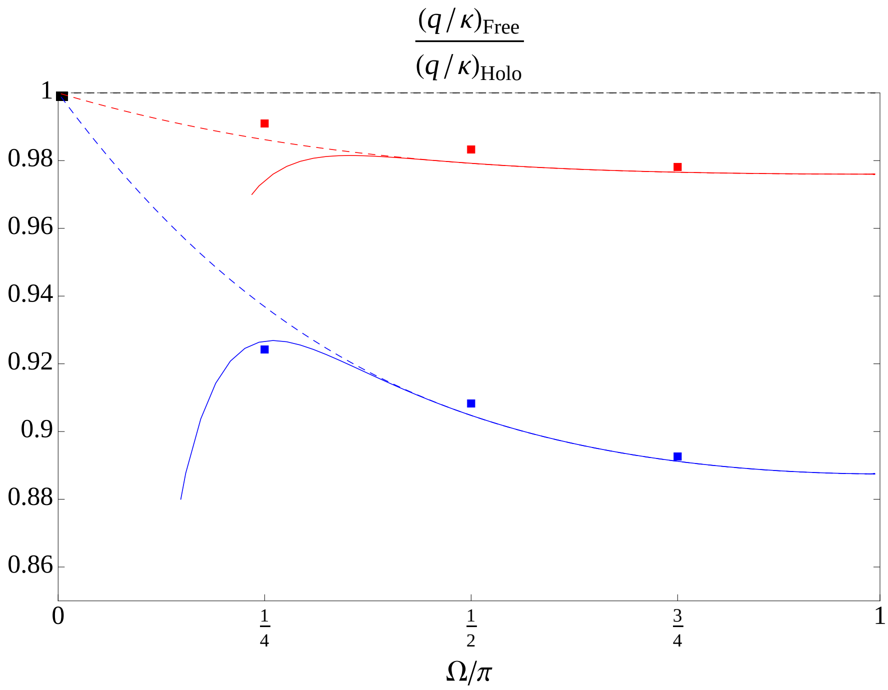

Figure 9.5: (Colour online) We show $(q/\kappa)_{\text{free}}/(q/\kappa)_{\text{holo}}$ both for the free scalar (blue), the free fermion (red) and the corresponding lattice results (squares). We also show the interpolated curves obtained using the 14 coefficients of the Taylor expansions around $\Omega = \pi$ as well as the coefficients κ in the small opening angle expansions (dashed blue and red). The black dashed line would correspond to the value for which the ratios are equal. Both theories will in fact approach the black square at the end of this line, *i.e.*, at $\Omega = 0$.

reliable. Similarly, the function for the free scalar deviates from the holographic result by no more than 11% in this range. In the small angle region, the three corner contributions normalized by κ in figure 9.4 will all approach $1/\Omega$ (shown as the black dashed line). Of course, we only see the latter behaviour is realized for the holographic result, for which we have the exact function over the whole range of Ω. The exact curves for the free scalar (fermion) would lie somewhere in between the black and the blue (red) curves in the intermediate region and so these curves will tend to lie slightly above those obtained with the Taylor series expansion around $\Omega = \pi$. Hence the exact results for the free fields would be in even better agreement with the holographic curve than we have estimated above. Figure 9.5 is also useful to determine a better estimate of where the Taylor expansions stop being reliable. Focusing on the lattice results in this figure, one might expect that the ratios $(q/\kappa)_{\text{free}}/(q/\kappa)_{\text{holo}}$ for both the scalar and the fermion will decrease monotonically for increasing Ω over the full range from $\Omega = 0$ to π. This would indicate that the expansions are starting to fail in the vicinity where their slopes become zero, *i.e.*, around $\Omega/\pi \sim 0.35$ for the fermion and $\Omega/\pi \sim 0.27$ in the case of the scalar.

As we have seen, the ratio κ/C_T equals the Einstein gravity result (9.104) for all the higher curvature theories considered here — at least, for perturbative calculations to linear order in the additional gravitational couplings. However, we might ask if this result applies quite generally for any three-dimensional CFT. Given that for the free field theories, we have at our disposal the values of κ in eq. (9.117), it is interesting to compare these corner charges to the corresponding values of C_T, which can be found in [356]:

$$C_{T,\,\text{scalar}} = \frac{3}{32\pi^2}\,, \qquad C_{T,\,\text{fermion}} = \frac{3}{16\pi^2}\,. \tag{9.118}$$

Hence the ratios become:

$$\frac{\kappa}{C_T}\Big|_{\text{holo}} \simeq 3.7092\,, \qquad \frac{\kappa}{C_T}\Big|_{\text{scalar}} \simeq 4.17945\,, \qquad \frac{\kappa}{C_T}\Big|_{\text{fermion}} \simeq 3.8005\,. \tag{9.119}$$

All of these ratios are rather close to each other but we do not have precise agreement. In particular, the fermion result differs from the holographic one by approximately 2.4% whereas the scalar ratio is off by approximately 11%. Of course, an open question which remains is whether this ratio is a universal quantity for all holographic theories, however, we can only begin to address this question when a better understanding is established for holographic entanglement entropy in general higher curvature theories.

In fact, it was not only the ratio κ/C_T but rather the entire function $q(\Omega)/C_T$ which was universal for all our higher curvature theories. Hence, even though we found that the universality of κ/C_T did not extend beyond holographic CFT's, we may ask more broadly if there are any features of the corner contribution which are universal for general three-dimensional CFT's. Hence in figure 9.6, we plot $(q(\Omega)/C_T)_{\text{free}}/(q(\Omega)/C_T)_{\text{holo}}$ for the free scalar (blue) and the free fermion (red). The figure also includes the corresponding lattice points[24] as well as the points at $\Omega/\pi = 0$, which correspond to $(\kappa/C_T)_{\text{free}}/(\kappa/C_T)_{\text{holo}}$. As can be expected from figures 9.4 and 9.5, we see that in general the corner contribution evolves slightly differently for the three cases as Ω runs from 0 to π. The ratios plotted in figure 9.6 are essentially the same in figure 9.5 except that we have changed the normalization by considering $q(\Omega)/C_T$ rather than $q(\Omega)/\kappa$. Hence again, the both ratios in the new figure seem to be monotionically decreasing starting from $(\kappa/C_T)_{\text{free}}/(\kappa/C_T)_{\text{holo}}$ at $\Omega = 0$ — see eq. (9.119). The remarkable feature in figure 9.6 is that both curves seem to reach precisely 1 at $\Omega = \pi$. That is, it appears that the ratio σ/C_T is equal for the two field theories and for our holographic theories!

Recall that we argued the behavior of $q(\Omega)$ was constrained for general CFT's near $\Omega = \pi$ and eq. (9.4) defined the charge σ with $q(\Omega) \simeq \sigma\,(\pi - \Omega)^2 + \cdots$. In particular, we found in eq. (9.23) that for Einstein gravity

$$\sigma_{\text{E}} = \frac{\tilde{L}^2}{8\pi G}\,, \tag{9.120}$$

and so given the universal form of $q(\Omega)$ for all our holographic theories in eq. (9.55), we have

$$\sigma = \alpha\,\sigma_E\,, \tag{9.121}$$

with α given again by eq. (9.56). Further in all of our holographic theories, we also have a fixed ratio:

$$\frac{\sigma}{C_T} = \frac{\sigma_E}{C_{T,E}} = \frac{\pi^2}{24} \simeq 0.411234\ . \tag{9.122}$$

We can easily compare this result with the ratio σ/C_T for the free conformal scalar and the massless fermion, since σ is simply the first coefficient in the Taylor expansions presented

[24]Although, the Taylor expansions and the lattice points seem to differ here, they are actually in good agreement and it is just that the vertical scale has been expanded here. In particular, the disagreement is less than approximately 2.5% in all cases.

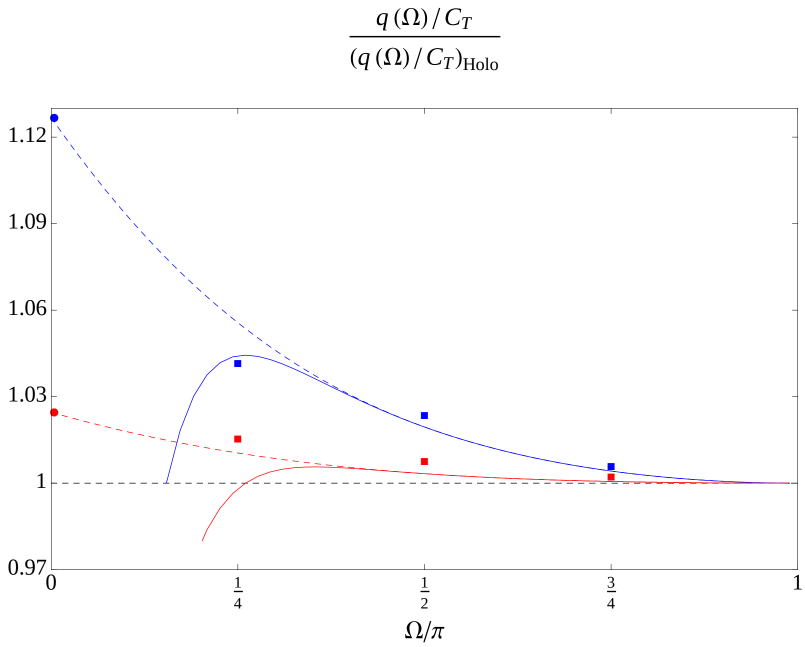

Figure 9.6: (Colour online) We show $(q/C_T)_{\rm free}/(q/C_T)_{\rm holo}$ both for the free scalar (blue), the free fermion (red) and the lattice points (squares). We also include the interpolated curves obtained using the 14 coefficients of the Taylor expansions around $\Omega=\pi$ as well as the coefficients κ in the small opening angle expansions (dashed blue and red). The black dashed line would correspond to the value for which the ratios equal 1. The dots in blue and red at $\Omega=0$ correspond to the small angle values of the ratios, namely $(\kappa/C_T)_{\rm free}/(\kappa/C_T)_{\rm holo}$.

in [118–120], and the corresponding values are

$$\sigma_{\rm scalar} \simeq 0.0039063\,, \qquad \text{and} \qquad \sigma_{\rm fermion} \simeq 0.0078125\,. \tag{9.123}$$

Hence using the values of C_T given in eq. (9.118), the desired ratios become

$$\left.\frac{\sigma}{C_T}\right|_{\rm scalar} \simeq 0.411235\,, \qquad \text{and} \qquad \left.\frac{\sigma}{C_T}\right|_{\rm fermion} \simeq 0.411235\,. \tag{9.124}$$

Hence as expected from figure 9.6, the free field ratios show a striking agreement with the holographic result, *i.e.,* they agree with a precision of at least 0.0003%! We might keep in mind that while the free field values for C_T in eq. (9.118) are exact, the corresponding values of σ in eq. (9.123) are only the approximate results of a numerical computation [118–120]. Hence the precision of the agreement between eqs. (9.122) and (9.124) is as good as could be expected.

We are emboldened then to conjecture that the ratio σ/C_T is in fact a universal constant for all three-dimensional CFT's, *i.e.,*

$$\frac{\sigma}{C_T} = \frac{\pi^2}{24} \tag{9.125}$$

for general conformal field theories in three dimensions. This conjecture can be used to predict the exact values of $\sigma_{\rm scalar}$ and $\sigma_{\rm fermion}$,

$$\sigma_{\rm scalar} = \frac{1}{256}\,, \qquad \text{and} \qquad \sigma_{\rm fermion} = \frac{1}{128}\,. \tag{9.126}$$

Of course, these values match the results shown in eq. (9.117) within the accuracy limits set by the calculations in [118–120]. However, we can do even better by going back to the original free field computations and evaluating the required integrals with an improved accuracy. The required calculations are described in appendix H and we find that the agreement between our prediction for $\sigma_{\rm scalar}$ and $\sigma_{\rm fermion}$, given by eq. (9.126), and the previous calculations for the free field results can be extended to an accuracy of one part in 10^{12}. We emphasize the required integrals (H.2) and (H.3) are extremely complicated and they are not even similar. Yet they seem to conspire to produce the simple rational numbers (9.126) predicted by holography. We feel this is striking evidence in favour of our new conjecture above!

9.4 Conventions and notation

In this chapter, Greek indices run over the entire AdS$_4$ background, whereas Latin letters from the *second* half of the alphabet $i, j, \ldots$ represent directions along the entangling surface m. m is a (co)dimension-two bulk surface with a pair of independent orthonormal vectors orthogonal to it $n^\mu_{\hat a}$ ($\hat a = \hat 1, \hat 2$), where the hatted indices from the beginning of the Latin alphabet denote tangent indices in the transverse space, so that $\delta_{\hat a \hat b} = n^\mu_{\hat a} n^\nu_{\hat b} g_{\mu\nu}$. Tangent vectors to m are defined in the usual way as $t^\mu_i \equiv \partial x^\mu / \partial y^i$, being x^μ and y^i coordinates in the full AdS$_4$ background and along the surface, respectively. The corresponding induced metric on the surface is thus given by $\gamma_{ij} \equiv t^\mu_i t^\nu_j g_{\mu\nu}$ (and its determinant $\det \gamma_{ij} \equiv \gamma$), whereas the extrinsic curvatures associated to the two normal vectors $n^\mu_{\hat a}$ read $K^{\hat a}_{ij} \equiv t^\mu_i t^\nu_j \nabla_\mu n^{\hat a}_\nu$, being ∇_μ the covariant derivative compatible with $g_{\mu\nu}$. Also, we will denote by $K^{\hat a}$ the trace of each extrinsic curvature defined through $K^{\hat a} \equiv \gamma^{ij} K^{\hat a}_{ij}$. Finally, by $K^{\hat a 2}$ we mean the sum of the squares of the two extrinsic curvatures: $K^{\hat a 2} \equiv K^{\hat a} K^{\hat b} \delta_{\hat a \hat b}$. The transverse metric can be defined as $g^\perp_{\mu\nu} \equiv n^\mu_{\hat a} n^\nu_{\hat b} \delta^{\hat a \hat b}$, and allows us to project bulk tensors in the transverse directions, *e.g.*, $R^{\hat a}{}_{\hat a} \equiv g^{\perp \mu\nu} R_{\mu\nu}$.

We write Euclidean AdS$_4$ in Poincaré coordinates as

$$ds^2 = \frac{\tilde L^2}{z^2}\left(dz^2 + dt_{\rm E}^2 + d\rho^2 + \rho^2 d\theta^2\right) . \tag{9.127}$$

The induced metric on surfaces m parametrized as $t_E = 0$, $z - \rho h(\theta) = 0$, such as the ones suitable for the kink, reads

$$ds^2_m = \frac{\tilde L^2}{\rho^2}\left(1 + \frac{1}{h^2}\right) d\rho^2 + \frac{\tilde L^2}{h^2}\left(1 + \dot h^2\right) d\theta^2 + \frac{2 \tilde L^2 \dot h}{\rho\, h} d\rho\, d\theta \, , \tag{9.128}$$

where $\dot h(\theta) \equiv \partial_\theta h$. From this one finds

$$\sqrt{\gamma} = \frac{\tilde L^2}{\rho h^2}\sqrt{1 + h^2 + \dot h^2}\, . \tag{9.129}$$

The resulting orthonormal vectors orthogonal to the surface read

$$n_{\hat 1} \;=\; \frac{z}{\tilde L}\,\partial_t \, , \tag{9.130}$$

$$n_{\hat 2} \;=\; \frac{z}{\tilde L \sqrt{1 + h^2 + \dot h^2}}\left(\partial_z - h\,\partial_\rho - \frac{\dot h}{\rho}\,\partial_\theta\right) . \tag{9.131}$$

For our pure AdS$_4$ background, we find the following expression for the projection of the Ricci tensor appearing in eq. (9.29)

$$R^{\hat{a}}{}_{\hat{a}} = g^{\perp\mu\nu} R_{\mu\nu} = -6/\tilde{L}^2\,. \tag{9.132}$$

The extrinsic curvature associated to $n_{\hat{1}}$ vanishes, whereas the one corresponding to $n_{\hat{2}}$ turns out to read

$$K^{\hat{2}}_{ij} = \begin{pmatrix} -\frac{\tilde{L}(h^2+1)}{\rho^2 h^2\sqrt{h^2+\dot{h}^2+1}} & -\frac{\tilde{L}\dot{h}}{\rho h\sqrt{h^2+\dot{h}^2+1}} \\ -\frac{\tilde{L}\dot{h}}{\rho h\sqrt{h^2+\dot{h}^2+1}} & -\frac{\tilde{L}(h^2+\ddot{h}h+\dot{h}^2+1)}{h^2\sqrt{h(\theta)^2+\dot{h}^2+1}} \end{pmatrix}. \tag{9.133}$$

From this we can easily obtain the contraction appearing in eq. (9.29)

$$K^{\hat{a}\,2} = \frac{\left[2+3h^2+h^4+2\dot{h}^2+h(1+h^2)\ddot{h}\right]^2}{\tilde{L}^2\left(1+h^2+\dot{h}^2\right)^3}\,. \tag{9.134}$$

Finally, the Ricci scalar of the entangling surface reads

$$\mathcal{R} = \frac{2(-(1+2h^2)\dot{h}^2-\dot{h}^4+(h+h^3)\ddot{h})}{\tilde{L}^2(1+h^2+\dot{h}^2)^2}\,. \tag{9.135}$$

From this and eq. (9.129), it is not difficult to check that the product $\sqrt{\gamma}\mathcal{R}$ is a total derivative. Indeed, we find

$$\sqrt{\gamma}\mathcal{R} = \frac{2(-(1+2h^2)\dot{h}^2-\dot{h}^4+(h+h^3)\ddot{h})}{\rho h^2(1+h^2+\dot{h}^2)^{3/2}} = \frac{d}{d\theta}\left[\frac{2}{\rho}\,\frac{\dot{h}}{h\sqrt{1+h^2+\dot{h}^2}}\right]. \tag{9.136}$$

A

Generating solutions of cubic models

A.1 Generating new solutions via duality

As mentioned in Section 2.6.5, a necessary and sufficient condition for a solution to be generating is that all the $\mathrm{Sl}(2;\mathbb{R})$ invariants of the theory are independent when evaluated on the charges and moduli of that solution [16,61,62,149]. In this appendix we are going to study whether or not and why the solution considered in that section is a generating one. We start by stating some general properties which we, then, apply to the (toy) axidilaton model and then to the t^3 model.

There are in general 5 independent invariants that characterize each $\mathcal{N}=2$ symmetric supergravity model. They are [126]:

$$i_1 = |\mathcal{Z}|^2\,, \tag{A.1}$$

$$i_2 = \mathcal{G}^{ij^*}\mathcal{Z}_i\mathcal{Z}^*_{j^*}\,, \tag{A.2}$$

$$i_3 = -\tfrac{1}{3}\Re\mathfrak{e}\left[\mathcal{Z}\mathcal{N}_3(\mathcal{Z}^*)\right]\,, \tag{A.3}$$

$$i_4 = \tfrac{1}{3}\Im\mathfrak{m}\left[\mathcal{Z}\mathcal{N}_3(\mathcal{Z}^*)\right]\,, \tag{A.4}$$

$$i_5 = \mathcal{G}^{ij^*}\mathcal{C}_{ijk}\mathcal{C}^*_{i^*j^*k^*}\mathcal{G}^{jl^*}\mathcal{G}^{km^*}\mathcal{G}^{j^*l}\mathcal{G}^{k^*m}\mathcal{Z}^*_{l^*}\mathcal{Z}^*_{m^*}\mathcal{Z}_l\mathcal{Z}_m\,, \tag{A.5}$$

where $\mathcal{Z}$ is the central charge, $\mathcal{G}^{ij^*}$ the inverse Kähler metric,

$$\mathcal{Z}_i \equiv \mathcal{D}_i\mathcal{Z}\,, \tag{A.6}$$

are the "matter" central charges,

$$\mathcal{C}_{ijk} \equiv \mathcal{D}_i\mathcal{V}_M\mathcal{D}_j\mathcal{D}_k\mathcal{V}^M\,, \tag{A.7}$$

and

$$\mathcal{N}_3(\mathcal{Z}^*) \equiv \mathcal{C}_{ijk}\mathcal{G}^{il^*}\mathcal{G}^{jm^*}\mathcal{G}^{kn^*}\mathcal{Z}^*_{l^*}\mathcal{Z}^*_{m^*}\mathcal{Z}^*_{n^*}\,. \tag{A.8}$$

All these invariants are function of the charges and the scalars but their combination

$$J_4(\mathcal{Q}) = (i_1 - i_2)^2 + 4i_4 - i_5\,, \tag{A.9}$$

depends quartically on the charges only. Sometimes it is advantageous to work with $J_4(\mathcal{Q})$ instead of i_5.

A.1.1 2-charge generating solutions of the axidilaton model

The minimal number of non-vanishing charges that are necessary for an extremal, supersymmetric[1], black hole of axidilaton theory to be regular is two. Taking into account the form of the Hesse potential Eq. (2.77) and of the axidilaton Eq. (2.78), it is easy to see that there are only two possible non-singular 2-charge configurations, namely $(p^0, p^1, 0, 0)^T$ and $(0, 0, q_0, q_1)^T$.

In this model, the tensor $\mathcal{C}_{ijk}$ vanishes identically, and so does $\mathcal{N}_3(\mathcal{Z}^*)$ and the invariants i_3, i_4, i_5. The model is characterized by the two invariants i_1 and i_2, which are, respectively, the squares of the absolute values of the true and fake central charges at infinity

$$i_1 = |\mathcal{Z}(\lambda_\infty, \mathcal{Q})|^2\,, \qquad i_2 = |\hat{\mathcal{Z}}(\lambda_\infty, \mathcal{Q})|^2\,, \tag{A.10}$$

and both are independent for any 2-charge solution (for $\mathfrak{Re}\,\lambda_\infty = 0$ or not) and, in principle, it should be a generating solution. However, depending on our choice of harmonic functions, the regular solutions with two charges may have a vanishing $\mathfrak{Re}\,\lambda_\infty$ and the subgroup of $\mathrm{Sl}(2;\mathbb{R})$ that generates a non-vanishing $\mathfrak{Re}\,\lambda_\infty$, which consists of matrices of the form $\left(\begin{smallmatrix} 1 & \beta \\ 0 & 1 \end{smallmatrix}\right)$ do not leave invariant the 2-charge configurations. Therefore, the $\mathrm{Sl}(2;\mathbb{R})$ orbit of the regular 2-charge configurations may not cover the full parameter space.

It is interesting to see how the impossibility of generating a solution containing the maximal number of independent parameters arises in practice in this simple case, starting from a configuration characterized by the charges $(0, 0, \hat{q}_0, \hat{q}_1)^T$ and the moduli $\hat{\lambda}_\infty = i\mathfrak{Im}\,\hat{\lambda}_\infty$ (we reserve the unhatted symbols for the final charges and moduli). This solution is determined by two harmonic functions:

$$(\hat{H}^M) = \begin{pmatrix} 0 \\ \\ 0 \\ \\ \frac{s}{\sqrt{2}}\left\{(\mathfrak{Im}\,\hat{\lambda}_\infty)^{1/2} - |\hat{q}_0|\tau\right\} \\ \\ \frac{s}{\sqrt{2}}\left\{(\mathfrak{Im}\,\hat{\lambda}_\infty)^{-1/2} - |\hat{q}_1|\tau\right\} \end{pmatrix}\,, \tag{A.11}$$

where

$$s \equiv \mathrm{sgn}(\hat{q}_0) = \mathrm{sgn}(\hat{q}_1)\,. \tag{A.12}$$

[1]The discussion can also be held for the non-supersymmetric solutions to this model, reaching the same conclusions.

The Sl(2; $\mathbb{R}$) rotated solution will depend on the original physical parameters $\hat{q}_0, \hat{q}_1, \Im\mathfrak{m}\, \hat{\lambda}_\infty$ plus the parameters of the Sl(2; $\mathbb{R}$) transformation a, b, c, d (only 3 of which are independent). We have to determine $\hat{q}_0, \hat{q}_1, \Im\mathfrak{m}\, \hat{\lambda}_\infty, a, b, c, d$ in terms of the final physical parameters to write the rotated solution in terms of its own physical parameters only.

Sl(2; $\mathbb{R}$) acts on the charge vector through the matrix Eq. (2.57) so

$$\begin{pmatrix} p^0 \\ p^1 \\ q_0 \\ q_1 \end{pmatrix} = \begin{pmatrix} d & & -c & \\ & a & & b \\ -b & & a & \\ & c & & d \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \hat{q}_0 \\ \hat{q}_1 \end{pmatrix} = \begin{pmatrix} -c\hat{q}_0 \\ b\hat{q}_1 \\ a\hat{q}_0 \\ d\hat{q}_1 \end{pmatrix} . \tag{A.13}$$

From these relations we determine a, b, c, d in terms of the final and original charges:

$$a = q_0/\hat{q}_0 \,, \qquad b = p^1/\hat{q}_1 \,, \qquad c = -p^0/\hat{q}_0 \,, \qquad d = q_1/\hat{q}_1 \,. \tag{A.14}$$

On the other hand, from the transformation rule Eq. (2.48) we get

$$\Re\mathfrak{e}\, \lambda_\infty = \frac{bd + ac(\Im\mathfrak{m}\, \hat{\lambda}_\infty)^2}{d^2 + c^2(\Im\mathfrak{m}\, \hat{\lambda}_\infty)^2} \,, \qquad \Im\mathfrak{m}\, \lambda_\infty = \frac{\Im\mathfrak{m}\, \hat{\lambda}_\infty}{d^2 + c^2(\Im\mathfrak{m}\, \hat{\lambda}_\infty)^2} \,, \tag{A.15}$$

and replacing in these relations the transformation parameters a, b, c, d by the values in Eq. (A.14), we get 2 equations that relate the 3 original to the 6 final physical parameters:

$$p^0 q_0 (\hat{q}_1)^2 (\Im\mathfrak{m}\, \hat{\lambda}_\infty)^2 + \frac{\Re\mathfrak{e}\, \lambda_\infty}{\Im\mathfrak{m}\, \lambda_\infty} (\hat{q}_0 \hat{q}_1)^2 \Im\mathfrak{m}\, \hat{\lambda}_\infty - p^1 q_1 (\hat{q}_0)^2 = 0 \,, \tag{A.16}$$

$$\Im\mathfrak{m}\, \lambda_\infty (p^0)^2 (\hat{q}_1)^2 (\Im\mathfrak{m}\, \hat{\lambda}_\infty)^2 - (\hat{q}_0 \hat{q}_1)^2 \Im\mathfrak{m}\, \hat{\lambda}_\infty + \Im\mathfrak{m}\, \lambda_\infty (q_1)^2 (\hat{q}_0)^2 = 0 \,. \tag{A.17}$$

The invariance of W implies that

$$\hat{q}_0 \hat{q}_1 = p^0 p^1 + q_0 q_1 \,, \tag{A.18}$$

and allows us to eliminate $\hat{q}_1$ from the above two equations. We can solve (A.16) and (A.17) for $\Im\mathfrak{m}\, \hat{\lambda}_\infty$ as a function of the 6 final physical parameters and $\hat{q}_0$ and, for both equations, we find $\Im\mathfrak{m}\, \hat{\lambda}_\infty \hat{q}_0^{-2}$ as a function of those 6 parameters:

$$\Im\mathfrak{m}\, \hat{\lambda}_\infty \hat{q}_0^{-2} = f_1(\mathcal{Q}, \lambda_\infty) \,, \qquad \Im\mathfrak{m}\, \hat{\lambda}_\infty \hat{q}_0^{-2} = f_2(\mathcal{Q}, \lambda_\infty) \,. \tag{A.19}$$

The consistency condition $f_1(\mathcal{Q}, \lambda_\infty) = f_2(\mathcal{Q}, \lambda_\infty)$ determines one of the two final real moduli as a complicated function of the final charges. In other words: the final solution cannot have 6 independent physical parameters, which implies that the original solution is not a generating solution.

On top of this, there seems to be another problem: we cannot solve separately the 3 original physical parameters in terms of the 6 final ones. “Fortunately” only the combination $\Im\mathfrak{m}\, \hat{\lambda}_\infty \hat{q}_0^{-2}$ appears in the rotated solution or, equivalently, in the H^M variables. Using Eqs. (A.13,A.14) and (A.18) we find the these are given by

$$
H^M = A^M - \tfrac{1}{\sqrt{2}}\mathcal{Q}^M\tau\,, \qquad \begin{pmatrix} A^0 \\ A^1 \\ A_0 \\ A_1 \end{pmatrix} = \begin{pmatrix} \frac{s}{\sqrt{2}}p^0(\Im\mathfrak{m}\,\hat{\lambda}_\infty\hat{q}_0^{-2})^{1/2} \\ \frac{s}{\sqrt{2}}p^1(p^0p^1+q_0q_1)^{-1}(\Im\mathfrak{m}\,\hat{\lambda}_\infty\hat{q}_0^{-2})^{-1/2} \\ \frac{s}{\sqrt{2}}q_0(\Im\mathfrak{m}\,\hat{\lambda}_\infty\hat{q}_0^{-2})^{1/2} \\ \frac{s}{\sqrt{2}}q_1(p^0p^1+q_0q_1)^{-1}(\Im\mathfrak{m}\,\hat{\lambda}_\infty\hat{q}_0^{-2})^{-1/2} \end{pmatrix}, \tag{A.20}
$$

In the supersymmetric case we know that we can construct a new solution which has, on top of the two non-trivial harmonic functions, two constant ones. If we write all of them in the form

$$
\hat{H}^M = \hat{A}^M - \tfrac{1}{\sqrt{2}}\hat{\mathcal{Q}}^M\tau\,, \tag{A.21}
$$

then $(\hat{\mathcal{Q}}^M)^T = (0,0,\hat{q}_0,\hat{q}_1)^T$ and, according to the general results of Ref. [191],

$$
(\hat{A}^M) = \frac{1}{\sqrt{2\Im\mathfrak{m}\,\hat{\lambda}_\infty}}\Im\mathfrak{m}\left\{\frac{\hat{q}_1\hat{\lambda}^*_\infty - i\hat{q}_0}{|\hat{q}_1\hat{\lambda}^*_\infty - i\hat{q}_0|}\begin{pmatrix} i \\ \hat{\lambda}_\infty \\ -i\hat{\lambda}_\infty \\ 1 \end{pmatrix}\right\}. \tag{A.22}
$$

This solution has two independent charges at any generic point in moduli space and should be a generating solution. The difference with the previous case is that, instead of the Eqs. (A.15), we can invert (2.48) and use Eqs. (A.14) and (A.18) to get two independent real equations that do not lead to constraints in the final physical parameters:

$$
\hat{\lambda}_\infty\hat{q}_0^{-2} = \frac{1}{(p^0p^1+q_0q_1)}\frac{q_1\lambda_\infty - p^1}{p^0\lambda_\infty + q_0}\,. \tag{A.23}
$$

The only combinations of the 4 original physical parameters that appear in the rotated solution are precisely the real and imaginary parts of $\hat{\lambda}_\infty\hat{q}_0^{-2}$ and we obtain a solution with 6 independent physical parameters.

A.1.2 2-charge solutions of the t^3 model

Again, the minimal number of non-vanishing charges that a regular, extremal, black hole of this model can have is two. A choice of charge vector that leads to regular supersymmetric and non-supersymmetric black holes is $(0,p^1,q_0,0)^T$. In the supersymmetric case, the coefficient of $-\frac{1}{\sqrt{2}}\tau$ in H^M (that we call attractor in the context of this formalism) is given by

$$
(B^M) = (\mathcal{Q}^M) = \begin{pmatrix} 0 \\ p^1 \\ q_0 \\ 0 \end{pmatrix}, \tag{A.24}
$$

and in the non-supersymmetric one, by

$$(B^M) = \begin{pmatrix} 0 \\ p^1 \\ -q_0 \\ 0 \end{pmatrix} . \tag{A.25}$$

In order to see if these charge configurations lead to generating solutions, we study the values of the invariants. For cubic models with prepotential of the form

$$\mathcal{F} = \tfrac{1}{3!} d_{ijk} \frac{\mathcal{X}^i \mathcal{X}^j \mathcal{X}^k}{\mathcal{X}^0} , \tag{A.26}$$

one has $\mathcal{C}_{ijk} = e^{\mathcal{K}} d_{ijk}$. The prepotential of the t^3 model is given in Eq. (2.163) and has $d_{111} = -5$ so $\mathcal{C}_{ttt} = \frac{3}{4}(\Im\mathfrak{m} t)^{-3}$. For this model it can be proven that only three invariants are independent and that the other two can be written as a their combination. Specifically, one finds that [128]

$$\begin{aligned} i_4 &= -\sqrt{\tfrac{4}{27} i_2^3 i_1 - i_3^2} \,, && \text{(A.27)} \\ i_5 &= \tfrac{3}{4} i_2^2 \,, && \text{(A.28)} \end{aligned}$$

and we can take, as independent basis of invariants i_1, i_2 and i_3 (which we can replace by J_4).

Now let us evaluate these invariants for the solutions with charge vector $(0, p^1, q_0, 0)^T$. The result is

$$\begin{aligned} i_1 &= \frac{3}{20(\Im\mathfrak{m}\, t_\infty)^3} \left| -\tfrac{5}{2} p^1 t_\infty^2 - q_0 \right|^2 , && \text{(A.29)} \\ i_2 &= \frac{1}{20(\Im\mathfrak{m}\, t_\infty)^3} \left| -\tfrac{5}{2} p^1 t_\infty (t_\infty + 2t_\infty^*) - 3q_0 \right|^2 , && \text{(A.30)} \\ i_3 &= -\frac{1}{75(\Im\mathfrak{m}\, t_\infty)^6} \Re\mathfrak{e} \left\{ -\tfrac{i}{8} \left(-\tfrac{5}{2} p^1 t_\infty^2 - q_0 \right) \left[-\tfrac{5}{2} p^1 t_\infty (t_\infty + 2t_\infty^*) - 3q_0 \right]^3 \right\} , && \text{(A.31)} \end{aligned}$$

and it is easy to see that if $\Re\mathfrak{e}\, t_\infty = 0$ (the *axion-free* case) they simplify to

$$\begin{aligned} i_1 &= \frac{3}{20(\Im\mathfrak{m}\, t_\infty)^3} \left[\tfrac{5}{2} p^1 (\Im\mathfrak{m}\, t_\infty)^2 - q_0 \right]^2 , && \text{(A.32)} \\ i_2 &= \frac{1}{20(\Im\mathfrak{m}\, t_\infty)^3} \left[\tfrac{5}{2} p^1 (\Im\mathfrak{m}\, t_\infty)^2 + 3q_0 \right]^2 , && \text{(A.33)} \\ i_3 &= 0 \,, && \text{(A.34)} \end{aligned}$$

We see then that in the axion-free case only two invariant are independent and according to the argument in [61] the solutions cannot be seed (generating) solutions.

It is necessary to have $\Re\mathfrak{e}\ t \neq 0$ for the the three invariants $i_1, i_2, i_3 \neq 0$ to be independent from each other and the two-charge solution to be a generating solution.

B

Special functions

B.1 The Lambert W function

The *Lambert W function* $W(z)$ was firstly introduced by Johann Heinrich Lambert in 1758 [291]. Along its history, it has found numerous applications in different areas of physics (mostly during the 20th century) [20, 51, 99, 144, 147, 197, 308, 314, 344, 383, 398].

$W(z)$ is defined (implicitly) through the equation

$$z = W(z)e^{W(z)}, \quad \forall z \in \mathbb{C}. \tag{B.1}$$

Since $f(z) = ze^z$ is not injective, $W(z)$ is not uniquely defined, and $W(z)$ stands for the whole set of branches solving (B.1). For $W : \mathbb{R} \to \mathbb{R}$, $W(x)$ has two branches $W_0(x)$ and $W_{-1}(x)$ defined respectively in the intervals $x \in [-1/e, +\infty)$ and $x \in [-1/e, 0)$ (See Figure 3). Both functions coincide in the branching point $x = -1/e$, where $W_0(-1/e) = W_{-1}(-1/e) = -1$. Therefore, the defining equation $x = W(x)e^{W(x)}$ admits two different solutions in the interval $x \in [-1/e, 0)$.

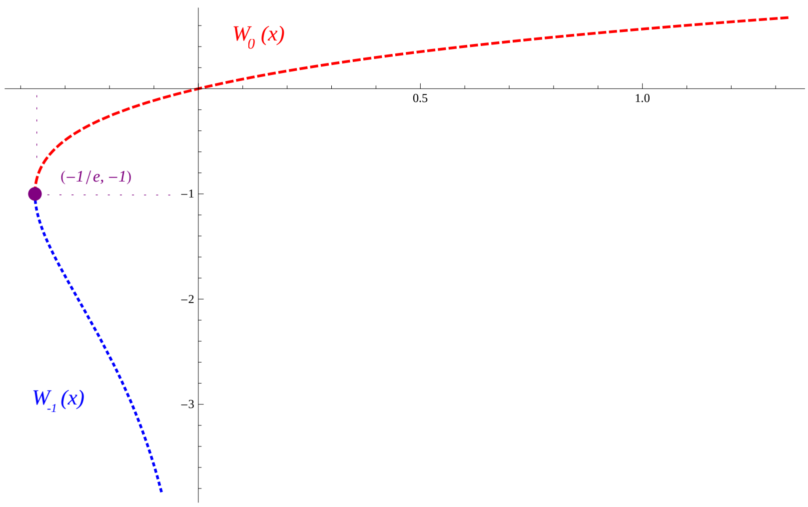

Figure B.1: The two real branches of $W(x)$.

The derivative of $W(z)$ reads

$$\frac{dW(z)}{dz} = \frac{W(z)}{z(1+W(z))}, \quad \forall z \notin \{0, -1/e\}\,; \quad \left.\frac{dW(z)}{dz}\right|_{z=0} = 1\,, \tag{B.2}$$

and is not defined for $z = -1/e$ (the function is not differentiable there). At that point one finds

$$\lim_{x\to -1/e} \frac{dW_0(x)}{dx} = \infty, \quad \lim_{x\to -1/e} \frac{dW_{-1}(x)}{dx} = -\infty\,. \tag{B.3}$$

B.2 The Exponential Integral function

The Exponential Integral $Ei\,[z]$, $z \in \mathbb{C}$ is a special function on the complex plane. For real non-zero values x it is defined as follows

$$Ei(x) = -\int_{-x}^{\infty} \frac{e^{-t}}{t} dt\,. \tag{B.4}$$

We only need the Exponential Integral function evaluated in the real numbers since in our solutions it appears only with a real argument, although in the definition of the prepotential (4.2) it appears with an argument that can be in general complex.

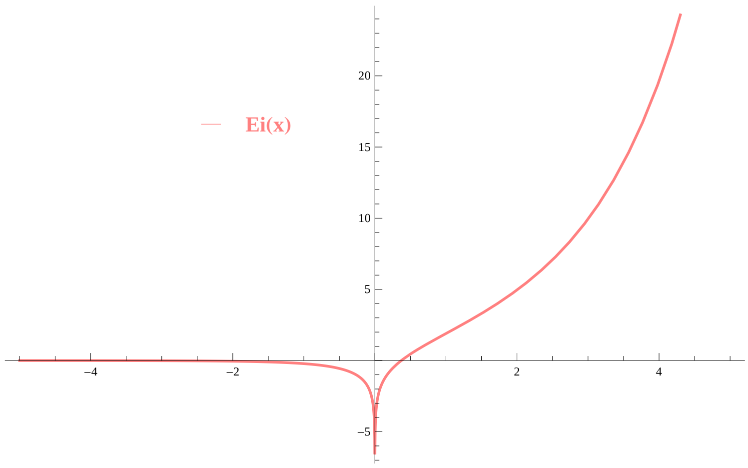

Figure B.2: The Exponential Integral function on the real axis.

$Ei(x)$ is negative for $x \in (-\infty, c)$, where $c \sim 0,375$, zero in $x = c$ and positive for $x > c$. In addition, $\lim_{x\to 0} Ei(x) = -\infty$.

C
SU(2) Yang-Mills solutions

C.1 The SU(2) Lorentzian meron

A Lorentzian meron is a classical solution to the pure SU(2) (Lorentzian) Yang-Mills theory such that the 1-form gauge field A defining it, is proportional to a pure-gauge configuration, which in our conventions would be $\frac{1}{g}dUU^{-1}$ where $U(x) \in \mathrm{SU}(2)$. In Ref. [113] $U(x)$ was chosen to be of the *hedgehog form*

$$U \equiv 2\frac{x^m}{r}\delta^a_m T_a\,, \qquad U^\dagger = U^{-1} = -U\,, \quad \Rightarrow U^2 = -\mathbb{1}_{2\times 2}\,. \tag{C.1}$$

and it was shown that A solves the Yang-Mills equations if the proportionality coefficient is 1/2, that is

$$A = \frac{1}{2g}dUU^{-1} = -\frac{1}{gr^2}\varepsilon^a{}_{mn}x^m dx^n T_a\,. \tag{C.2}$$

As we will see, this gauge field is nothing but the gauge field of the Wu-Yang SU(2) monopole given in Eq. (C.15).

Since the field strength of a pure gauge configuration vanishes, we find that $F(A)$ can be written in these two specially simple ways which we will use in Appendix C.3:

$$F(A) = \tfrac{1}{2}dA = g[A,A] = \star_{(3)}d\frac{1}{2gr}\,U\,, \tag{C.3}$$

Now we can write the non-Abelian field strength $F(A)$ in terms of $F(B)$, where $F(B)$ is the field strengths of the Dirac monopole of unit charge Eq. (C.6) that we will review in the next section

$$F(A) = F(B)\,U\,, \qquad F(B) = \star_{(3)}d\frac{1}{2gr}\,, \tag{C.4}$$

and the energy-momentum tensor of A in terms of that of B

$$T_{\mu\nu}(A) = -\tfrac{1}{2}\mathrm{Tr}[F_{\mu\rho}(A)F_\nu{}^\rho(A) - \tfrac{1}{4}\eta_{\mu\nu}F^2(A)] = F_{\mu\rho}(B)F_\nu{}^\rho(B) - \tfrac{1}{4}\eta_{\mu\nu}F^2(B) = T_{\mu\nu}(B)\,. \tag{C.5}$$

C.2 The Wu-Yang SU(2) monopole

The Wu-Yang SU(2) monopole [428] is a solution of the SU(2) Yang-Mills theory that can be obtained from the embedding of the Dirac monopole in SU(2) via a singular gauge transformation (see, *e.g.* Ref. [390] and references therein). To fix our conventions, it is convenient to start by reviewing the Wu-Yang construction of the Dirac monopole [427].

C.2.1 The Dirac monopole

The U(1) field of the Dirac monopole, that we will denote by B is defined to satisfy the Dirac monopole equation[1], which can be written in several forms:

$$F(B) \equiv dB = \star_{(3)} d\frac{1}{2gr} = -\frac{1}{2g} d\Omega^2 \,, \qquad 2\partial_{[m}B_{n]} = -\frac{1}{2g}\varepsilon_{mnp}\frac{x^p}{r^3}\,, \tag{C.6}$$

where $d\Omega^2$ is the volume 2-form of the round 2-sphere of unit radius

$$d\Omega^2 = -\tfrac{1}{2}\varepsilon_{mnp}\frac{x^m}{r} d\frac{x^n}{r} \wedge d\frac{x^p}{r} = \sin\theta d\theta \wedge d\varphi \,. \tag{C.7}$$

The value of the magnetic charge has been set to g^{-1} and it is the minimal charge allowed if the unit of electric charge is g.

The above equation does not admit a global regular solution.

$$B^{(\pm)} = -\frac{1}{2g}(\cos\theta \mp 1) d\varphi \,, \tag{C.8}$$

are local solutions regular everywhere except on the negative (resp. positive) z axis (the Dirac strings). A globally regular solution can be constructed by using $B^{\pm}$ in the upper (lower) hemisphere and using the gauge transformation

$$B^{(+)} - B^{(-)} = -d\left(\frac{1}{g}\varphi\right) \,, \tag{C.9}$$

to relate them in the overlap region. If the gauge group is U(1) where the radius of the circle is the inverse coupling constant $1/g$, the gauge transformation parameter can have a periodicity $2\pi n/g$ with $n \in \mathbb{N}$. This is the well-known Abelian Wu-Yang monopole construction [427]. In our case, since the period of φ is 2π, we get $2\pi/g$, which is the smallest value allowed $p = 1/g$. The solution that describes the monopole of charge n times the minimum is n times this one $p = n/g$.

It is useful to have the expression of $B^{(\pm)}$ in Cartesian coordinates:

$$B^{(\pm)} = \frac{1}{2g}\frac{[(0,0,\mp 1) \times (x^1, x^2, x^3)] \cdot d\vec{x}}{r^2(r \pm x^3)} \,, \tag{C.10}$$

in which the singularity at $r = \mp x^3$ becomes evident. In this form, one can easily change the position of the monopole from the origin to some other point x_0^m and the position of the Dirac string from the half line that starts from the origin in the direction $-(0,0,\mp 1)$ to

[1] This equation is just the Abelian version of the Bogomol'nyi equation.

the half line that starts at the monopole's position x_0^m hand has the direction s^m relative to that point:

$$B^{(s)} = \frac{1}{2g}\left(1 - \frac{s^m}{s}\frac{u^m}{u}\right)^{-1} \varepsilon_{mnp}\frac{s^m}{s}\frac{u^n}{u}d\frac{u^p}{u}\,, \tag{C.11}$$

with

$$u^m \equiv x^m - x_0^m\,, \qquad u^2 \equiv u^m u^m\,, \qquad s^2 \equiv s^m s^m\,. \tag{C.12}$$

C.2.2 From the Dirac monopole to the Wu-Yang SU(2) monopole

Let us consider the Abelian $B^{(+)}$ solution in Eq. (C.8) and let us embed it in SU(2) as the 3rd component of the gauge field

$$A^{(+)} \equiv 2B^{(+)}T_3\,, \qquad F(A^{(+)}) = 2F(B)T_3\,. \tag{C.13}$$

The SU(2) gauge transformation (which is evidently singular along the negative z axis and makes the whole Dirac string singularity, but the endpoint at the coordinate origin, disappear)

$$U^{(+)} \equiv \frac{1}{\sqrt{2(1+\frac{z}{r})}}\left[1 + \frac{z}{r} + 2\left(\frac{x}{r}T_2 - \frac{y}{r}T_1\right)\right]\,, \tag{C.14}$$

relates the gauged field $A^{(+)}$ to

$$A = \frac{1}{g}\varepsilon^a{}_{mn}dx^m\frac{x^n}{r^2}T_a\,, \qquad A^{(+)} = U^{(+)}A(U^{(+)})^{-1} + \frac{1}{g}dU^{(+)}(U^{(+)})^{-1}\,, \tag{C.15}$$

which is the gauge field of the Wu-Yang SU(2) monopole. As we have mentioned in the previous appendix, this is also the gauge field of the Lorentzian meron Eq. (C.2). The gauge transformation also relates T_3 to $\mathcal{U}$ in Eq. (C.1) and the Abelian vector

$$U^{(+)}\mathcal{U}(U^{(+)})^{-1} = 2T_3\,. \tag{C.16}$$

The fact that the Lorentzian meron is the Wu-Yang monopole, which is related by a gauge transformation to the Dirac monopole makes the relation Eq. (C.5) trivial.

This construction can be generalized to more general positions of the Dirac string: if we consider embedding of the Dirac monopole solution $B^{(s)}$ in Eq. (C.11) into SU(2)

$$A^{(s)} \equiv -2B^{(s)}\frac{s^m}{s}\delta_m{}^aT_a\,, \tag{C.17}$$

it is easy to see that the gauge transformation

$$U^{(s)} \equiv \frac{1}{\sqrt{2\left(1 - \frac{s^m}{s}\frac{u^m}{u}\right)}}\left[1 - \frac{s^m}{s}\frac{u^m}{u} - 2\varepsilon_{mn}{}^a\frac{s^m}{s}\frac{u^n}{u}T_a\right]\,, \tag{C.18}$$

relates it to the same Wu-Yang monopole field Eq. (C.15)

$$A^{(s)} = U^{(s)} A (U^{(s)})^{-1} + \frac{1}{g} dU^{(s)} (U^{(s)})^{-1} \,. \tag{C.19}$$

C.3 The SU(2) Skyrme model

In this appendix we are going to show that the Lorentzian meron (Wu-Yang monopole) is also associated to a solution of the equations of motion of the SU(2) Skyrme model [392] written in the form [112]

$$S_{\text{Skyrme}} = -\tfrac{1}{2} \int d^4x \left\{ \tfrac{1}{2} R_\mu R^\mu + \frac{\lambda}{16} S_{\mu\nu} S^{\mu\nu} \right\} , \tag{C.20}$$

where

$$R_\mu \equiv V^{-1} \partial_\mu V \,, \qquad S_{\mu\nu} \equiv [R_\mu, R_\nu] \,, \qquad V(x) \in \mathrm{SU}(2) \,. \tag{C.21}$$

The equations of motion are

$$\partial_\mu R^\mu + \frac{\lambda}{4} \partial_\mu [R_\nu, F^{\mu\nu}] = 0 \,. \tag{C.22}$$

If we take $V = U^{-1}$ (U given by Eq. (C.1)), then we can write $R = 2gA$ where A is Lorentzian meron's gauge field Eq. (C.2) and

$$\begin{aligned} \partial_\mu R^{i\,\mu} &= -2g \partial_m A^i{}_m = 0 \,, \\ \partial_\mu [R_\nu, F^{\mu\nu}]^i &\sim \partial_m \left(\frac{A^i{}_m}{r^2} \right) = 0 \,. \end{aligned} \tag{C.23}$$

C.4 Higher-charge Lorentzian merons and Wu-Yang monopoles

The construction of a Lorentzian meron can be generalized by using a generalization of the unit outward-pointing vector x^m/r denoted by ξ^m and defined by [22]

$$(\xi^m) \equiv \frac{1}{r} \left(\frac{\mathfrak{Im}(x^2 + ix^1)^n}{\rho^{n-1}}, \frac{\mathfrak{Re}(x^2 + ix^1)^n}{\rho^{n-1}}, x^3 \right) , \qquad \rho^2 \equiv (x^1)^2 + (x^2)^2 \,, \tag{C.24}$$

or, in spherical coordinates,

$$(\xi^m) \equiv (\sin\theta \sin n\varphi, \sin\theta \cos n\varphi, \cos\theta) \,, \tag{C.25}$$

and which reduces to x^m/r for $n = 1$. The essential properties of ξ^m are

$$d\xi^m \wedge d\xi^n = -n \varepsilon_{mnp} \xi^p d\Omega^2 \,, \tag{C.26}$$

$$-\tfrac{1}{2} \varepsilon_{mnp} \xi^m d\xi^n \wedge d\xi^p = n d\Omega^2 = \star_{(3)} d\frac{n}{r} \,, \tag{C.27}$$

The generalization of the meron solution is constructed in terms of the generalization SU(2) matrix in Eq. (C.1)

$$U_{(n)} \equiv 2\xi^m \delta_m^a T_a \,, \qquad U^{\dagger}_{(n)} = U^{-1}_{(n)} = -U_{(n)} \,, \tag{C.28}$$

and takes the form

$$A \equiv \frac{1}{2g} dU_{(n)} U^{-1}_{(n)} \,. \tag{C.29}$$

The field strength is given by

$$F(A_{(n)}) = \tfrac{1}{2} dA = g[A, A] = \star_{(3)} d\frac{n}{2gr} U_{(n)} \,, \tag{C.30}$$

and can be related to that of a Dirac monopole of charge $p = n/g$

$$F(B_{(n)}) = \star_{(3)} d\frac{n}{2gr} \,, \quad F(A_{(n)}) = F(B_{(n)}) U_{(n)} \,, \tag{C.31}$$

which is given by the expressions studied at the beginning. The energy-momentum tensor of A is also equal to that of the Abelian monopole of charge n/g B. These fields can also be related to the embedding of the charge n/g Dirac monopole into SU(2) with a generalization of the gauge transformation Eq. (C.18)

$$U^{(s)}_{(n)} \equiv \frac{1}{\sqrt{2\left(1 - \frac{s^m}{s}\xi^m\right)}} \left[1 - \frac{s^m}{s}\xi^m - 2\varepsilon_{mn}{}^a \frac{s^m}{s}\xi^n T_a\right] \,, \tag{C.32}$$

relates it to the meron gauge field:

$$U^{(s)}_{(n)} U_{(n)} (U^{(s)}_{(n)})^{-1} = -2\frac{s^m}{s}\delta_m{}^a T_a \,, \qquad U^{(s)}_{(n)} A_{(n)} (U^{(s)}_{(n)})^{-1} + \frac{1}{g} dU^{(s)}_{(n)} (U^{(s)}_{(n)})^{-1} = nB^{(s)}_{(n)} 2\frac{s^m}{s}\delta_m{}^a T_a \,. \tag{C.33}$$

To check that this gauge field solves the Yang-Mills equations of motion we first stress that, with the above connection, $U_{(n)}$ is a covariantly-constant adjoint field. Then, auxiliary the adjoint Higgs field

$$\Phi_{(n)} \equiv \left(-\frac{\mu}{2g} + \frac{n}{2gr}\right) U_{(n)} \,, \tag{C.34}$$

satisfies

$$D\Phi_{(n)} = d\frac{n}{2gr} U_{(n)} \,, \tag{C.35}$$

and the pair $A_{(n)}, \Phi_{(n)}$ satisfies the Bogomol'nyi equations (5.32) and, as a consequence the equations of motion of the Yang-Mills-Higgs system. The last equation implies that $\Phi_{(n)}$ and $D\Phi_{(n)}$ commute so the Higgs current vanishes and $A_{(n)}$ also solves the sourceless Yang-Mills equations.

D

HEE for a doubly-Wick-rotated hvLf geometry

In this appendix we study HEE for a class of geometries for which the anisotropic scaling occurs along one of the spatial dimensions instead of time [7,154]

$$ds^2 = L^2 r^{\frac{2\theta}{d}} \left(-\frac{dt^2}{r^2} + \frac{dr^2}{r^2} + \frac{d\vec{x}^2_{(d-1)}}{r^2} + \frac{dy^2}{r^{2z}} \right) . \tag{D.1}$$

This can be understood as obtained through a double Wick rotation of the usual hvLf metric (7.3). Indeed we just have to apply the following transformation to it

$$t \to iy \, , \, x_d \to it \, , \tag{D.2}$$

where x_d stands for the dth spatial coordinate. This makes the geometry covariant under the following transformations

$$y \to \lambda^z t \, , \, t \to \lambda t \, , \, x_i \to \lambda x_i \, , \, i = 1, ..., d-1 \, . \tag{D.3}$$

HEE in the framework of Einstein gravity has been already studied for this geometry in [7,154]. Here we are going to extend the study to the case of R^n gravity to illustrate how the result changes with respect to the usual hvLf case. The motivation to consider such a perversion is to make the dynamical exponent z appear in the exponents of the divergent terms in the HEE expression. This indeed results in the production of new divergences, which become logarithmic in a certain subset of the parameter space.

The region at the boundary for which we compute the entanglement entropy is the same as in the rest of the article, with the particularity that now we have anisotropic spatial scaling. We consider the strip to extend infinitely (up to the IR cut-off $L_S \to \infty$) along the special scaling coordinate, so $s = \{(t_E, r, x_1, x_2, ..., x_{d-1}, y) \text{ s.t., } t_E = 0, \;\; x_{d-1} \in [-l/2, l/2], \;\; x_{1,...,d-2} \in (-L_S/2, +L_S/2), \;\; y \in (-L_S/2, +L_S/2)\}$. The procedure used here is the same as that of section (8.1), so we will skip redundant discussions.

The HEE functional is

$$S_{R^n} = \frac{1}{4G} \int_m d^2 y \sqrt{g_m} \left[1 + n \lambda_{R^n} \tilde{L}^{2(n-1)} R^{n-1} \right] . \tag{D.4}$$

The Ricci scalar for (D.1) is the same as that for (7.3), that is, $R = \kappa r^{-2\theta/d} / \tilde{L}^2$. We can parametrize the entangling surface m as $x_{d-1} = h(r)$, so that the metric induced in such surface is

$$ds^2_m = L^2 r^{\frac{2\theta}{d}} \left[\frac{dy^2}{r^{2z}} + \left(1 + \dot{h}^2 \right) \frac{dr^2}{r^2} + \frac{d\vec{x}^2_{d-2}}{r^2} \right] , \tag{D.5}$$

The expression for the entanglement entropy becomes

$$S_{R^n} = \frac{L^d L_S^{(d-1)}}{2G} \int_\delta^{r_*} dr \sqrt{1+\dot{h}^2} f(r) r^{(\theta-d-z+1)}\,, \text{ with } f(x) \equiv \left[1 + n\kappa^{(n-1)}\lambda_{R^n} x^{-2\theta(n-1)/d}\right]\,, \tag{D.6}$$

r_* being the turning point of the surface, where $\dot{h}|_{r_*} = \infty$. The functional has a first integral associated to h, so we can express $\dot{h}$ in terms of h. By doing so and after some rearrangement we find

$$S_{R^n} = \frac{L^d L_S^{(d-1)} r_*^{\theta-d-z+2}}{2G} \int_{\delta/r_*}^1 du \frac{u^{(\theta-d-z+1)} f(uh_*)}{\sqrt{1-u^{2(d-\theta+z-1)}\frac{f(r_*)^2}{f(ur_*)^2}}}\,. \tag{D.7}$$

We need $d-\theta+z-1 > 0$ for the perturbative analysis to be consistent. Under this condition the expression looks exactly like the one in section 8.1 after promoting $(d-\theta) \to (d-\theta+z-1)$. This implies the following result for the HEE

$$S_{R^n} = \frac{L^d L_S^{(d-1)}}{2G} \left\{ \frac{\delta^{-\mathfrak{B}_0}}{\mathfrak{B}_0} - \frac{(l/2)^{-\mathfrak{B}_0} G_0^{\mathfrak{B}_0} G_0}{\mathfrak{B}_0} + n\kappa^{(n-1)}\lambda_{R^n} \left[\frac{\delta^{-\mathfrak{B}_1}}{\mathfrak{B}_1} - \frac{(l/2)^{-\mathfrak{B}_1} G_0^{\mathfrak{B}_1} G_1}{\mathfrak{B}_1} \right] \right\} + \mathcal{O}(\lambda_{R^n}^2)\,, \tag{D.8}$$

with

$$\mathfrak{B}_0 \equiv d-\theta+z-2\,, \tag{D.9}$$

$$\mathfrak{B}_1 \equiv \mathfrak{B}_0 + \frac{2\theta(n-1)}{d}\,, \tag{D.10}$$

$$G_0 \equiv \frac{\sqrt{\pi}\,\Gamma\left(\frac{\mathfrak{B}_0+2}{2(\mathfrak{B}_0+1)}\right)}{\Gamma\left(\frac{1}{2(\mathfrak{B}_0+1)}\right)}\,, \quad G_1 \equiv \frac{\sqrt{\pi}\,\Gamma\left(\frac{2+2\mathfrak{B}_0-\mathfrak{B}_1}{2(\mathfrak{B}_0+1)}\right)}{\Gamma\left(\frac{1+\mathfrak{B}_0-\mathfrak{B}_1}{2(\mathfrak{B}_0+1)}\right)}\,. \tag{D.11}$$

The divergence with $\mathfrak{B}_1$ becomes logarithmic when

$$\theta = \frac{d(d+z-2)}{d-2(n-1)}\,, \tag{D.12}$$

which gives a broad range of possibilities. However, we still need to take into account the NEC, which are different with respect to those for the standard hvLf case. For Einstein gravity, this is computed as $G_{\mu\nu}N^\mu N^\nu \geq 0$, N^μ being appropriate null vectors and $G_{\mu\nu}$ the Einstein tensor. For higher-curvature gravities, we will find additional conditions involving the couplings of the theory, which we assume to be susceptible of being satisfied by tuning those. For this metric a convenient null vector is

$$N^r = \frac{s_r}{L} r^{1-\theta/d}\,, \; N^i = \frac{s_i}{L} r^{1-\theta/d}\,, \; N^y = \frac{s_y}{L} r^{z-\theta/d}\,, \tag{D.13}$$

$$N^t = \frac{\sqrt{\sum s_i^2 + s_r^2 + s_y^2}}{L} r^{1-\theta/d}\,. \tag{D.14}$$

with the s_μ being positive constants. The NEC produces two inequalities

$$d(z-1)z + \theta(d-\theta) \leq 0\,, \tag{D.15}$$

$$(z-1)(z+d-\theta) \leq 0\,. \tag{D.16}$$

After some algebra, one can see that these limit the allowed values of z to lie in the interval

$$\frac{1-\sqrt{1+4\theta\frac{\theta-d}{d}}}{2} \leq z \leq 1\,. \tag{D.17}$$

So for each dimension d and each order in curvature n, any metric with z satisfying (D.17) will give rise to a logarithmic contribution as long as (D.12) is satisfied.

E
Some properties of the hvLf metrics

The hvLf metric (7.3) is spatially homogeneous and covariant under the scale transformations

$$x_i \to \lambda x_i \ , \ t \to \lambda^z t \ , \ r \to \lambda r \ , \ ds^2_{d+2} \to \lambda^{2\theta/d} ds^2_{d+2} \, , \tag{E.1}$$

where λ is a dimensionless parameter. Observe that this means that the Lifshitz radius ℓ is only defined up to dimensionless factors. The Ricci tensors of metrics (7.3) are given by

$$R_{tt} = \frac{(dz-\theta)(d+z-\theta)}{d\, r^{2z}} \tag{E.2}$$

$$R_{rr} = \frac{(d+z)\theta - d(z^2+d)}{d\, r^2} \tag{E.3}$$

$$R_{ij} = \frac{(\theta-d)(d+z-\theta)}{d\, r^2}\delta_{ij}. \tag{E.4}$$

This geometry generically suffers from a null curvature singularity at $r=\infty$ except for a specific set of parameter values. The singularity exists even though all curvature invariants remain finite. The tidal forces diverge as [385]

$$C_{(\theta,z)} r^{2C_{(\theta,z)}+d} \, , \qquad C_{(\theta,z)} = \frac{d(z-1)-\theta}{d-\theta} \tag{E.5}$$

where we have restricted to $C_{(\theta,z)} > 0$ for which the singularity is a null curvature singularity as surfaces of constant r become null as r goes to infinity. We distinguish several cases:

- For $\theta = 0$ we simply get the result in [240] which is appropriate for Lifshitz scaling. Ways for resolving the null curvature singularities have been presented in [26, 227].
- The case of $\theta = 0$ and $z = 1$ is the non-singular result of pure AdS.
- There are non-singular results for

$$C_{(\theta,z)} = 0 \, , \quad \text{or} \quad C_{(\theta,z)} + 1 \leq 0 \, . \tag{E.6}$$

The null energy condition in the bulk gives the conditions

$$C_{(\theta,z)} \geq 0, \qquad (z-1)(d+z-\theta) \geq 0, \tag{E.7}$$

which rules out the non-singular condition $C_{(\theta,z)}+1 \leq 0$ and leaves the condition $C_{(\theta,z)} = 0$.

There is a class of Ricci-flat hvLf spaces: they are characterized by

$$\theta \;=\; \frac{d(d+1)}{d-1} \quad \text{and} \quad z \;=\; \frac{2d}{d-1} \quad \longrightarrow \quad C_{(\theta,z)} \;=\; 0\,. \tag{E.8}$$

These spaces always solve the null energy condition and are regular in the IR interior ($r \to \infty$).

F
From the kink to the strip

As we used in the main text, the small angle limit of $q(\Omega)$ defines a universal charge κ, which can be used to distinguish different CFT's. The form of eq. (9.3) is fixed for general theories due to the existence of a conformal map relating the corner geometry to a strip. This mapping is discussed in detail in Appendix A of [339] and we only review the salient points here. As a consequence of this mapping, the expressions for the universal terms in the entanglement entropy match for both geometries, at least in the limit of small Ω or a narrow strip width. However, as we will see below, this mapping does not fix the form of $q(\Omega)$ over the entire range of the opening angle.

Let us now describe the conformal mapping: Let the CFT be defined in the background geometry which is simply $\mathbb{R}^3$, with the coordinates used in section 9.1,

$$ds^2 = dt_E^2 + d\rho^2 + \rho^2 d\theta^2 \,. \tag{F.1}$$

If we make the coordinate transformation, $t_E = r\cos\xi$ and $\rho = r\sin\xi$, the line element above becomes

$$ds^2 = dr^2 + r^2\left(d\xi^2 + \sin^2\!\xi\, d\theta^2\right) \,. \tag{F.2}$$

Next we make the coordinate change $r = L\, e^{Y/L}$ and remove the overall factor $e^{2Y/L}$ with a Weyl transformation, to find the geometry

$$ds^2 = dY^2 + L^2\left(d\xi^2 + \sin^2\!\xi\, d\theta^2\right) \,, \tag{F.3}$$

with $Y \in (-\infty, +\infty)$. Of course, this conformal transformation is the usual exponential map which takes $\mathbb{R}^3$ to $\mathbb{R}\times S^2$.

The corner region for which we calculated the entanglement entropy in section 9.1 was defined in the original coordinates (F.1) as $V = \{t_{\rm E} = 0, \rho > 0, |\theta| \le \Omega/2\}$ and so in terms of the polar coordinates (F.2), this region becomes $V = \{r > 0, \xi = \pi/2, |\theta| \le \Omega/2\}$. Finally in the cylindrical background (F.3), the corner region is mapped to an infinite strip: $V = \{Y \in (-\infty, +\infty)\,, \xi = \pi/2, |\theta| \le \Omega/2\}$. In this geometry, the density matrix would be represented by a path integral of the CFT over the cylinder with open boundary conditions imposed along the strip, *i.e.,* on surfaces just above and below $\xi = \pi/2$, along the entire length of Y and in the range $|\theta| \le \Omega/2$. Hence the entire entanglement entropy (9.1), including both the universal and nonuniversal contributions, for the corner geometry in $\mathbb{R}^3$ is readily related to that for the strip in the cylinder geometry $\mathbb{R}\times S^2$, as discussed in [339]. However, we would like instead to related the entanglement entropy of the corner region to that of a strip in flat space $\mathbb{R}^3$, as was discussed in section 9.2.1. This is where the limit of small opening angle becomes important. When $\Omega \ll 1$, the separation between both sides of the strip is much smaller than the size of the sphere and the local radius of curvature, *i.e.*, $\ell \equiv L\,\Omega \ll L$. Hence the latter scale is negligible and to leading order the entanglement entropy resembles that for a strip in flat space, *i.e.*,

$$S_{EE} = c_1\,\frac{2(Y_+ - Y_-)}{\delta} - \tilde{a}\,\frac{Y_+ - Y_-}{\ell} + \mathcal{O}(\delta/L, \ell/L) \tag{F.4}$$

where Y_+ and Y_- is regulator scales introduced to cut-off the length of the strip in the positive and negative Y directions [339] — compare to eq. (9.57). Given the preceding transformations, we see that the universal contribution (proportional to $\tilde{a}$) is mapped to

$$S_{\rm univ} = -\frac{\tilde{a}}{\Omega}\,\log\left(\frac{\rho_{max}}{\rho_{min}}\right) = -\frac{\tilde{a}}{\Omega}\,\log\left(\frac{H}{\delta}\right)\,, \tag{F.5}$$

where we have made the natural substitutions: $\rho_{max} = H$ and $\rho_{min} = \delta$. We emphasize that this expression only applies for $\Omega \ll 1$ and hence we have recovered eq. (9.3) for the corner contribution with $\kappa = \tilde{a}$.

Let us add that the coordinate transformation in the bulk geometry implementing the conformal mapping between the two boundary metrics (F.1) and (F.3) can be found as follows: The AdS$_4$ geometry can be described as a hyperbola embedded in the five-dimensional Minkowski space

$$ds^2 = -dU^2 + dV^2 + dR^2 + R^2\,d\Omega_2^2\,. \tag{F.6}$$

AdS$_4$ is defined now as the subspace

$$-\,U^2 + V^2 + R^2 = -L^2\,. \tag{F.7}$$

This constraint can be solved writing $R = rL/z$, $U + V = L^2/z$, $U - V = z + r^2/z$, and the induced metric on the hyperbola reduces to the Poincaré coordinates on AdS$_4$, given in eq. (9.6). On the other hand, the constraint (F.7) is also satisfied by $U = \sqrt{R^2 + L^2}\,\cosh(Y/L)$, $V = \sqrt{R^2+L^2}\,\sinh(Y/L)$, in which case the induced metric becomes

$$ds^2 = \frac{dR^2}{1+\frac{R^2}{L^2}} + \left(1 + \frac{R^2}{L^2}\right)dY^2 + R^2\left(d\xi^2 + \sin^2\!\xi\,d\theta^2\right)\,, \tag{F.8}$$

which is the AdS$_4$ geometry in global coordinates. Stripping off a scale factor of R^2/L^2 at large radius, the resulting boundary metric matches that in eq. (F.3). These bulk coordinates can be used to compute the HEE for the kink in essentially the same way as the calculation of section 9.1.

G
Central charges in $f(R)$ gravity

We parmeterize our general $f(R)$ gravity action [399] as

$$I_{f(R)} = \frac{1}{16\pi G}\int d^4x\sqrt{g}\left[\frac{6}{L^2} + R + \hat{\lambda}\, f(R)\right], \tag{G.1}$$

where we have made the cosmological constant and the Einstein term explict. We have also introduced a dimensionless coupling $\hat{\lambda}$ as a useful device to indicate the combined strength of the higher curvature contributions in the following. The function $f(R)$ can be a general function of the Ricci scalar, which has a Taylor series expansion beginning at order R^2 or higher. Our perspective is that $f(R)$ is parameterized by various dimensionless couplings and the necessary dimensions are provided by the cosmological constant scale L. For example, we would incorporate the three Ricci scalar terms in the action (9.54) as

$$\hat{\lambda}\, f(R) = L^2\,\lambda_1\,R^2 + L^4\,\lambda_{3,0}\,R^3 + L^6\,\lambda_{4,0}\,R^4\,. \tag{G.2}$$

In this simple class of theories, the gravitational entropy functional is simply given by the Wald entropy [372, 381], *i.e.,*

$$S_{f(R)} \;=\; \frac{1}{4G}\int_m d^2y\,\sqrt{\gamma}\,\left[1 + \hat{\lambda} f'(R)\right], \tag{G.3}$$

where $f'(R) = \partial f(R)/\partial R$. For our pure AdS$_4$ background, $f'(R)$ will be just a constant, with $R = \bar{R} = -12/\tilde{L}^2$ where

$$\frac{1}{L^2} = \frac{1}{\tilde{L}^2}\left[1 - \hat{\lambda}\, f'\left(-12/\tilde{L}^2\right)\right] - \frac{\hat{\lambda}}{6}\, f\left(-12/\tilde{L}^2\right)\,. \tag{G.4}$$

Hence determining the HEE will amount to finding the extremal area surface and evaluating eq. (9.8) with an additional overall coefficient of

$$\hat{\alpha} = 1 + \hat{\lambda}\, f'(\bar{R})\,. \tag{G.5}$$

Hence with a corner in the boundary entangling surface, the expression for the HEE will be a trivial generalization of eq. (9.16) with

$$S_{f(R)} = \hat{\alpha}\,\left[\frac{\tilde{L}^2}{2G}\frac{H}{\delta} - q_E(\Omega)\log\left(\frac{H}{\delta}\right) + \mathcal{O}(1)\right]. \tag{G.6}$$

However, we emphasize that the same overall factor (G.5) will appear in front of the entanglement entropy for any entangling surface and, in particular, for the circle. Further, it can be shown that the planar black hole solution (9.78) to the (four-dimensional) Einstein equations will also be a solution of the $f(R)$ Lagrangian. Hence the thermal entropy, which is computed by evaluating the horizon entropy using the same Wald formula (G.6), will produce the Einstein gravity result (9.79) up to an overall factor given precisely by $\alpha_{f(R)}$. Hence for this class of theories, the ratios κ/c_0 and κ/c_S will match those in Einstein gravity, as given in eqs. (9.75) and (9.81), respectively. Note that these results apply even when the strength of the gravitational couplings is large, *i.e.*, the fact that these ratios do not change is not restricted to linear order in perturbative calculations.

In order to see what happens with the two-point function (9.100) of the stress tensor, we can follow the steps of section (9.2.4) in order to find the linearized equations of motion for the massless spin-two graviton in the AdS$_4$ background. A remarkable fact about our previous linearized equations (9.108) was that none of the theories considered except that with an $R_{\mu\nu}R^{\mu\nu}$ interaction produced terms involving higher-order derivatives acting on $h_{\mu\nu}$ after we imposed the transverse traceless gauge. That is, in general, these theories do produce fourth-order derivatives of $h_{\mu\nu}$ in the linearized equations, but nevertheless these contributions all vanish, with the exception of the λ_2 term, after we set $\bar\nabla^\mu h_{\mu\nu} = 0 = h \equiv \bar g^{\mu\nu} h_{\mu\nu}$. As an illustrative exercise, we explicitly demonstrate how this works in the case of $f(R)$ gravity, where the same behavior is encountered. The full linearized equations arising from eq. (G.1) read

$$R^L_{\mu\nu} - \frac{1}{2}\bar g_{\mu\nu} R^L + \left[\frac{6}{\tilde L^2} - \frac{3}{L^2}\right] h_{\mu\nu} + \hat\lambda\, \mathcal{E}_{\mu\nu} = 0\,, \tag{G.7}$$

where

$$\mathcal{E}_{\mu\nu} \equiv f'(\bar R) R^L_{\mu\nu} - \frac{1}{2} f(\bar R) h_{\mu\nu} + f''(\bar R)\left[\bar g_{\mu\nu}\bar\Box - \bar\nabla_\mu\bar\nabla_\nu - \frac{3}{\tilde L^2}\bar g_{\mu\nu}\right] R^L - \frac{1}{2} f'(\bar R)\bar g_{\mu\nu} R^L\,, \tag{G.8}$$

and where the linearized Ricci tensor and Ricci scalar can be written as

$$\begin{aligned} R^L_{\mu\nu} &= -\frac{4}{\tilde L^2} h_{\mu\nu} + \frac{1}{\tilde L^2}\bar g_{\mu\nu} h + \frac{1}{2}\left(\bar\nabla_\mu\bar\nabla_\sigma h_\nu{}^\sigma + \bar\nabla_\nu\bar\nabla_\sigma h_\mu{}^\sigma - \bar\Box h_{\mu\nu} - \bar\nabla_\mu\bar\nabla_\nu h\right)\,, \\ R^L &\equiv \bar g^{\mu\nu} R^L_{\mu\nu} - h^{\mu\nu}\bar R_{\mu\nu} = \bar\nabla^\mu\bar\nabla^\nu h_{\mu\nu} - \bar\Box h + \frac{3}{\tilde L^2} h\,. \end{aligned} \tag{G.9}$$

As we can see, these equations involve fourth-order derivatives of the perturbation and its trace. However, in the transverse traceless gauge, it is straightforward to see that R^L vanishes and hence the fourth-order terms, which all appear in $\mathcal{E}_{\mu\nu}$, also vanish. The equations (G.7) are then notably simpler and after some massaging,[1] they yield the result:

$$-\frac{\hat\alpha}{2}\left[\bar\Box + \frac{2}{\tilde L^2}\right] h_{\mu\nu} = \hat\alpha\, G^L_{\mu\nu} = 0\,, \tag{G.10}$$

where $G^L_{\mu\nu}$ is again the linearized Einstein tensor in this gauge, as in eq. (9.107), and $\hat\alpha$ is defined in eq. (G.5). Hence with this exercise, we see all the fourth-order terms explicitly disappear from the linearized equations. Further, we can see the same overall constant (G.5) will appear here in C_T, as appeared in the corner contribution above. Therefore the ratio κ/C_T is again unchanged from the Einstein value (9.104) in the holographic CFT's dual to $f(R)$ gravity.

[1] In particular, one needs to use eq. (G.4) in order to obtain eq. (G.10).

When we impose the transverse traceless gauge, we are implicitly eliminating any new degrees of freedom and focusing entirely on the physical spin-two graviton. We can relax this condition here to see that $f(R)$ gravity introduces an additional scalar degree of freedom. In particular, the trace of the metric perturbation becomes a propagating massive scalar field. In order to find its equation, we can take the trace of the full linearized equations (G.7) without any gauge fixing. The result is

$$\begin{aligned} &-\left[\hat{\alpha}+\frac{12\hat{\lambda}}{\tilde{L}^2}f''(\bar{R})\right]\left[\bar{\nabla}^\mu\bar{\nabla}^\nu h_{\mu\nu}+\frac{3h}{\tilde{L}^2}\right] \\ &+\bar{\Box}h\left[\hat{\alpha}+\frac{21\hat{\lambda}}{\tilde{L}^2}f''(\bar{R})\right]+3\hat{\lambda}\,f''(\bar{R})\left[\bar{\Box}\bar{\nabla}^\mu\bar{\nabla}^\nu h_{\mu\nu}-\bar{\Box}^2h\right]=0\,. \end{aligned} \tag{G.11}$$

At this point, it is convenient to choose the gauge condition,

$$\bar{\nabla}^\mu h_{\mu\nu}=\bar{\nabla}_\nu h\,, \tag{G.12}$$

because this choice actually eliminates the fourth-order derivatives in the previous equation. The remaining second-order equation then simplifies to

$$3\hat{\lambda}\,f''(\bar{R})\,\bar{\Box}h-\left[\hat{\alpha}+\frac{12\hat{\lambda}}{\tilde{L}^2}f''(\bar{R})\right]h=0\,, \tag{G.13}$$

which corresponds to the equation of motion for a massive scalar field, as long as $f''(\bar{R})\neq 0$. That is, the trace h has become a dynamical degree of freedom in this case. On the other hand, if $f''(\bar{R})=0$ (*e.g.*, as in Einstein gravity), the above equation is not dynamical and would simply impose the tracelessness condition $h=0$. That is, the spin-two graviton would be the only propagating degree of freedome in this case.

We should also consider the traceless part of the metric perturbation, *i.e.*,

$$\hat{h}_{\mu\nu}=h_{\mu\nu}-\frac{1}{4}\bar{g}_{\mu\nu}\,h\,. \tag{G.14}$$

with the gauge condition (G.12). Combining this choice of gauge with eqs. (G.7) and (G.13), we get

$$\frac{1}{2}\left[\bar{\nabla}_\mu\bar{\nabla}_\nu h-\frac{1}{4}\bar{g}_{\mu\nu}\bar{\Box}h\right]\left[\hat{\alpha}-\frac{6\hat{\lambda}}{\tilde{L}^2}f''(\bar{R})\right]-\frac{\hat{\alpha}}{2}\left[\bar{\Box}+\frac{2}{\tilde{L}^2}\right]\hat{h}_{\mu\nu}=0\,. \tag{G.15}$$

which is an equation for the massive spin-2 field $\hat{h}_{\mu\nu}$ alone. However, this is not an homogenous equation. We can nevertheless define a new traceless tensor satisfying an equation of the type of eq. (G.10). This is given by[2]

$$t_{\mu\nu}\equiv\hat{h}_{\mu\nu}-\left[\frac{3\hat{\lambda}f''(\bar{R})}{\hat{\alpha}}\right]\left[\bar{\nabla}_\mu\bar{\nabla}_\nu h-\frac{1}{4}\bar{g}_{\mu\nu}\bar{\Box}h\right]\,. \tag{G.16}$$

Indeed, using (G.20) and (G.14), it can be shown that this tensor satisfies

$$-\frac{\hat{\alpha}}{2}\left[\bar{\Box}+\frac{2}{\tilde{L}^2}\right]t_{\mu\nu}=0\,. \tag{G.17}$$

[2]The procedure followed here for determining the physical spin-2 field closely follows [393], where the analysis was done for curvature-squared gravities.

So $t_{\mu\nu}$ represents the physical massless spin-2 graviton coupling to the holographic stress tensor. Note that eq. (G.21) is trivial whenever $f''(\bar{R}) = 0$ (like in Einstein gravity), so in that case, the traceless part of $\hat{h}_{\mu\nu}$ already corresponds to the massless mode.

It is interesting to consider the scalar degree of freedom more explicitly. Hence let us consider the case of R^2 gravity, for which we write $\hat{\lambda}\, f(R) = \lambda_1 L^2 R^2$. Hence we have $\hat{\lambda}\, f'(\bar{R}) = 2\lambda_1 L^2 \bar{R} = -24\lambda_1 L^2/\tilde{L}^2$ and $\hat{\lambda}\, f''(\bar{R}) = 2\lambda_1 L^2$. Further, as noted in section 9.1.1, the solution of eq. (G.4) is simply $\tilde{L} = L$. Combining these expressions in eq. (G.13) then yields

$$\left[\lambda_1 \bar{\Box} - \frac{1}{6L^2}\right] h = 0\,, \tag{G.18}$$

and hence h obeys the standard equation of motion for a free scalar with mass: $m^2 = 1/(6\lambda_1 L^2)$. Using the standard holographic dictionary [219, 311, 423], h is dual to a scalar operator in the three-dimensional boundary CFT with

$$\Delta = \frac{3}{2} + \sqrt{\frac{9}{4} + \frac{1}{6\lambda_1}}\,. \tag{G.19}$$

Hence if λ_1 is small and positive, h corresponds to a highly irrelevant operator with $\Delta \simeq 1/\sqrt{6\lambda_1}$ and with positive norm. If λ_1 is small and negative, Δ becomes imaginary indicating that the standard AdS/CFT dictionary is breaking down. In this limit, h is a ghost-like scalar with a tachyonic mass which exceeds the Breitenloner-Freedman bound [82, 82]. Hence the bulk theory would be inherently unstable if we tried to interpret the corresponding R^2 gravity as a complete theory rather than as an effective low energy theory.

On the other hand, eqs. (G.20) and (G.21) reduce, for R^2 gravity, to

$$\frac{1}{2}\left[\bar{\nabla}_\mu \bar{\nabla}_\nu h - \frac{1}{4}\bar{g}_{\mu\nu}\bar{\Box} h\right][1 - 36\lambda_1] - \frac{1 - 24\lambda_1}{2}\left[\bar{\Box} + \frac{2}{\tilde{L}^2}\right]\hat{h}_{\mu\nu} = 0\,, \tag{G.20}$$

and

$$t_{\mu\nu} \equiv \hat{h}_{\mu\nu} - \left[\frac{6\lambda_1}{1 - 24\lambda_1}\right]\left[\bar{\nabla}_\mu \bar{\nabla}_\nu h - \frac{1}{4}\bar{g}_{\mu\nu}\bar{\Box} h\right]\,. \tag{G.21}$$

in agreement with the results obtained in [306, 393].

H
Free field calculation of σ

In [119], the computation of the first fourteen coefficients corresponding to the Taylor expansion of $q(\Omega)$ around $\Omega = \pi$ was made using the techniques of quantum field theory for a free scalar and a free Dirac field.

As we saw in the discussion section, the ratio σ/C_T corresponding to the coefficient σ of the first term in such expansion ($q(\Omega) = \sigma \cdot (\Omega - \pi)^2 + \mathcal{O}(\Omega - \pi)^4$) and the charge in front of the stress tensor two-point function, C_T — see eq. (9.100), appears to satisfy a universal relation of the form

$$\frac{\sigma}{C_T} = \frac{\pi^2}{24} \,. \tag{H.1}$$

This result has been obtained using holographic techniques in the previous sections, and turns out to hold for all the higher-order theories we have considered in this paper. In addition, the numbers obtained using the numerical results in [119] for σ_{scalar} and σ_{fermion} and those for $C_{T\text{scalar}}$ and $C_{T\text{fermion}}$ given in [356] satisfy eq. (H.1) both for the scalar and the fermion with an accuracy better than $\sim 0.0003\%$. While the results obtained in [356] — see eq. (9.118), are exact, the values of σ_{scalar} and σ_{fermion} can only be computed in field theory numerically. In particular, they can be obtained by evaluation of the following monstruous integrals[1]

$$\begin{aligned}
\sigma_{\text{scalar}} &= \frac{1}{2}\int_{1/2}^{+\infty} dm \int_0^{+\infty} db\, \left[-4\pi(1-a)a\, H\, m\, \mu\, \text{sech}^2(\pi b)\right] , && \text{(H.2)}\\
\sigma_{\text{fermion}} &= -4\int_{1/2}^{+\infty} dm \int_0^{+\infty} db\, \left[m\, \mu\, H\, a(1-a)\pi\, \text{cosech}^2(b\pi)\right] && \text{(H.3)}\\
&+ \int_0^{+\infty} db \int_{1/2}^{+\infty} dm\, \left[F\, m\, \text{cosech}^2(b\pi)\right] ,
\end{aligned}$$

where

$$\begin{aligned}
h &\equiv \frac{2\left((a-1)a+m^2\right)\sin^2(\pi a)}{m^2\left(\cos(2\pi a)+\cos\left(\pi\sqrt{1-4m^2}\right)\right)} , && \text{(H.4)}\\
c &\equiv \frac{\pi 2^{2a-1}(1-a)a\sec\left(\frac{1}{2}\pi\left(2a+\sqrt{1-4m^2}\right)\right)\Gamma\left(\frac{1}{2}\left(-2a+\sqrt{1-4m^2}+3\right)\right)}{m\Gamma(2-a)^2\Gamma\left(a+\frac{1}{2}\left(\sqrt{1-4m^2}-1\right)\right)} ,
\end{aligned}$$

[1] We wish to thank Horacio Casini for sending us these integrals, which are the ones originally used in [119].

$$
\begin{aligned}
X_1 &\equiv \frac{2^{-2a}\Gamma(-a)\left(-\frac{1}{2}\pi\sinh\left(\frac{\pi\mu}{2}\right)-\frac{1}{2}i\cosh\left(\frac{\pi\mu}{2}\right)\left(\psi^{(0)}\left(a+\frac{i\mu}{2}+\frac{1}{2}\right)-\psi^{(0)}\left(a-\frac{i\mu}{2}+\frac{1}{2}\right)\right)\right)}{\mu\Gamma(a+1)\Gamma\left(-a-\frac{i\mu}{2}+\frac{1}{2}\right)\Gamma\left(-a+\frac{i\mu}{2}+\frac{1}{2}\right)(\cos(2\pi a)+\cosh(\pi\mu))}\,, \\
X_2 &\equiv X_1 \text{ with } a \text{ replaced by } (1-a), \\
T &\equiv \sqrt{h(a^2-a+(h+1)m^2)}\,, \\
H &\equiv -\frac{8\pi(a-1)ac^2X_1T+8\pi(a-1)ahX_2T-hc}{16hc\pi(a-1)a}\,, \\
F &\equiv -\frac{F_1}{F_2}\,, \\
F_1 &\equiv a^2\left(8\pi c^2\left(m^2+1\right)X_1T+8\pi h\left(m^2+1\right)X_2T-ch\right)-16\pi a^3T\left(c^2X_1+hX_2\right) \\
&+\ a\left(-8\pi c^2m^2X_1T-8\pi hm^2X_2T+ch\right)+8\pi a^4T\left(c^2X_1+hX_2\right)-ch(h+1)m^2\,, \\
F_2 &\equiv \frac{8c\,h\left(a^2-a+m^2\right)^2}{(2a-1)\mu}\,, \\
\mu &\equiv \sqrt{4m^2-1}\,, \\
a &\equiv bi+\frac{1}{2}\ \ (\text{for the scalar})\,, \\
a &\equiv bi\ \ (\text{for the fermion})\,.
\end{aligned}
$$

Notice that (H.3) and (H.2) look very different and without further insights there is a priori no reason to believe they should produce the same result (as they appear to do up to a factor 2).

We can compute integrals (H.2) and (H.3) numerically with arbitrary precision (although the computation time scales notably as we increase the precision). Our results show that both (H.2) and (H.3) exactly produce the results predicted assuming that σ/C_T is a universal constat given by eq. (H.1), *i.e.,*

$$\sigma_{\text{scalar}} = \frac{1}{256} = 0.00390625\,, \qquad \sigma_{\text{fermion}} = \frac{1}{128} = 0.0078125\,. \tag{H.5}$$

We have checked this is the case for a precision of $\sim$ one part in 10^{12}. In particular, we find

$$\sigma_{\text{scalar}} = 0.00390625000000(5)\,, \qquad \sigma_{\text{fermion}} = 0.00781250000000(7)\,, \tag{H.6}$$

where the numbers in brackets are out of the accuracy ranges.

I
Resumen

Esta tesis está dedicada al estudio de varios aspectos relacionados con la supergravedad, los agujeros negros, y la holografía, áreas que juegan un papel crucial en el desarrollo actual de la física teórica de altas energías.

Por un lado, los límites de baja energía de las diversas teorías de cuerdas se corresponden con teorías de supergravedad, lo que teniendo en cuenta las propiedades sorprendentemente buenas (casi todas ellas al menos) de estas como candidatas a unificar la gravedad y las demás interacciones en un marco único, hace a estas últimas dignas de estudio. Por otro lado, si la supersimetría es una simetría fundamental de la naturaleza, lo que podría solucionar en mayor o menor medida varios de los problemas más fundamentales de la física, cierta teoría de supergravedad ha de ser adecuada para describir la física de nuestro universo, en un cierto rango de energías.

Los agujeros negros prometen jugar un papel esencial en la compresión de la naturaleza cuántica de la gravedad. Por un lado, desde el punto de vista semiclásico, los agujeros negros satisfacen las leyes de la termodinámica (y las magnitudes emergentes involucradas están codificadas en diversos objetos relacionados con la geometría de los mismos), lo que pide a gritos una interpretación microscópica de los mismos. Por otro lado, de acuerdo con la imagen clásica, los agujeros negros contienen regiones del espaciotiempo en las que las leyes de la gravedad y la mecánica cuántica son igualmente importantes, por lo que una descripción conjunta de la física en esas regiones se hace necesaria. Es en el contexto de la teoría de cuerdas en el que la descripción microscópica de la termodinámica de ciertas soluciones supersimétricas de tipo agujero negro ha sido llevada a cabo con éxito, lo que sin duda es una prueba altamente no trivial superada por la teoría de cuerdas. Desafortunadamente, este cálculo no ha podido ser extendido a soluciones no extremas, que por otro lado son la mayoría, siendo las extremas, o las supersimetrícas en particular, casos límite de las anteriores. Mucho menos se sabe de las solutionces no extremas, incluso desde el punto de vista de supegravedad.

La palabra holografía hace referencia a la existencia de una descripción físicamente equivalente de un cierto sistema en términos de una teoría en una dimensión menor. La correspondencia AdS/CFT nos proporciona la primera realización del principio holográfico, y un marco perfecto para realizar numerosos cálculos correspondientes a ciertas teorías cuánticas de campos en el regimen de acoplamiento fuerte, que es bastante inaccesible con los métodos usuales. Por otro lado, nos proporciona una puerta fascinante hacia el entendimiento de la naturaleza cuántica de la gravedad y el espaciotiempo.

En esta tesis recopilamos los resultados obtenidos en [87–96]. En [89–91, 95, 96] desarrollamos nuevos métodos para la obtención de soluciones de tipo agujero negro (muchas de las cuales obtenemos explícitamente) en teorías de supergravedad $\mathcal{N}=2$, $d=4$ y teoría de cuerdas, así como otras conteniendo regiones de tipo *hvLf* en [87,88]. Así mismo, desarrollamos un método para la obtención de branas negras en teorías generales, y construimos la *cuerda*-(p, q) no extrema de la teoría de cuerdas tipo-IIB [93]. Además, exploramos la entropía de entrelazamiento holográfica para teorías de alto orden en curvatura para geometrías de tipo *hvLf* [91], así como la contribución del término universal a la entropía de entrelazamiento holográfica para superficies de entrelazamiento que contienen singularidades geométricas [92].

J
Conclusiones

En el capítulo 2 [90] hemos mostrado cómo el formalismo H-FGK puede utilizarse para simplificar la obtención de soluciones de tipo agujero negro en teorías de supergravedad $\mathcal{N} = 2$, $d = 4$. Esto es posible gracias a que cualquier solución queda determinada completamente por las H^M, que son funciones que transforman linealmente bajo el grupo de dualidad de la teoría, y que pueden construirse como una combinación lineal de vectores equivariantes. Además, mostramos cómo el formalismo permite conocer en qué casos las H^M han de contener términos no armónicos a través del cálculo del rango de una matriz. Mostramos explícitamente cómo esta técninca puede utilizarse en ciertos modelos de supergravedad.

En el capítulo 3 [89, 95] definimos el concepto de agujero negro *cuántico* o *cuerdoso* en el contexto de la teoría de cuerdas tipo-IIA compactificada a $d = 4$ en una variedad de tipo Calabi-Yau. En concreto, consideramos una truncación consistente de la teoría que deja de estar bien definida en el límite en el que la corrección perturbativa en α' se anula, y construimos nuevas soluciones de tipo agujero negro genuinamente cuánticos no extremas y con escalares no constantes. Además, consideramos la misma truncación en presencia de la contribución no perturbativa (en α') dominante en el prepotencial de la teoría efectiva, y construimos la primera solución supersimétrica de tipo agujero negro de este tipo. De forma sorprendente, esta solución viene dada en términos de una función bivaluada en los números reales, lo que sugiere la posibilidad de producir una violación de las conjeturas de unicidad usualmente asumidas en la literatura de supergravedad. Mostramos que, no obstante, solo una de las ramas es consistente en el contexto de teoría de cuerdas.

En el capítulo 4 [96] motivados por el último resultado de 3, construimos una teoría de supergravedad altamente no simétrica que admite soluciones de tipo agujero negro regulares que vienen dadas en términos de dicha función bivaluada, y que pueden construirse de forma tal que los valores asintóticos de los escalares así como las cargas de ambas soluciones son iguales. De esta forma, mostramos que la conjetura de no-pelo no funciona completamente en el contexto de supergravedad $\mathcal{N} = 2$, $d = 4$. Argumentamos, no obstante, que las soluciones responsables de la violación podrían tener problemas de estabilidad.

En el capítulo 5 [91] consideramos la teoría de supergravedad $\mathcal{N} = 2$, $d = 4$ acoplada a multipletes SU(2) no abelianos, y construimos las primeras soluciones de tipo agujero negro y monopolos globales con varios centros de la misma.

En el capítulo 6 [93] desarrollamos un formalismo que permite construir automáticamente, dada una solución de tipo brana negra de una cierta teoría altamente genérica, la brana dual electromagnética de la misma. Ilustramos el formalismo construyendo la *cuerda* (p, q) no extrema de la teoría de cuerdas tipo-IIB así como su dual *5-brana* (p, q).

En el capítulo 7 [87,88] demostramos que se pueden construir soluciones asintóticamente *hvLf* (en distintos límites) a partir de soluciones de tipo agujero negro en teorías de supergravedad $\mathcal{N}=2$, $d=4$ no gaugeada. Así mismo, hacemos un estudio pormenorizado de la existencia de soluciones de este tipo en una clase genérica de teorías que contiene a cualquier supergravedad $\mathcal{N}=2$, $d=4$ gaugeada con términos Fayet-Iliopoulos U(1), y construimos soluciones explícitas para un modelo concreto.

En el capítulo 8 [94] estudiamos la entropía de entrelazamiento holográfica para geometrías de tipo *hvLf*. Además de calcular la forma genérica de esta para teorías de curvatura superior generales y obtener los términos universales en algunas de ellas, encontramos que para ciertos valores del parámetro θ se producen nuevas contribuciones logarítmicas a la misma, lo que generaliza el resultado conocido para la gravedad de Einstein.

Por último, en el capítulo 9 [92] estudiamos la contribución a la entropía de entrelazamiento holográfica producida por la presencia de una singularidad geométrica en la superficie de entrelazamiento. Además de extender los resultados conocidos para la gravedad de Einstein y realizar diversas comparaciones con resultados de teoría cuántica de campos, estudiamos los efectos sobre este término de la introducción de términos de curvatura superior en la acción de gravedad. En particular, demostramos que es posible definir dos cargas, κ y σ, que contienen información no trivial y bien definida sobre la teoría en cuestión (para teorías generales). Comparamos estas cargas con otras obtenidas para otros observables físicos. De forma reseñable, encontramos que los cocientes κ/C_T y σ/C_T, donde C_T es la constante análoga que aparece en la expresión de la función a dos puntos del tensor de energía momento holográfico, se mantienen constante para todas las teorías de orden superior consideradas, lo que nos lleva a conjeturar que estas cantidades podrían ser universales para teorías holográficas generales. Comparando este resultado con los correspondientes a un escalar y un fermión libre respectivamente, cuyos valores solo se conocen aproximadamente, encontramos que el cociente σ/C_T se mantiene igual en todos los casos. Esto nos lleva a proponer que $\sigma/C_T = \pi^2/24$ es un resultado universal para teorías de campos conformes generales. Además, utilizamos este resultado para mejorar los resultados de campos libres para σ (apéndice H), que podemos calcular exactamente utilizando el resultado holográfico. Nuestro hallazgo explica también por qué los valores σ_{scalar} y σ_{fermion} parecen diferir (y de hecho difieren) en un factor 2 exactamente.

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