Universidad Autónoma de Madrid
Facultad de Ciencias
Departamento de Física Teórica

Aspects of gauged/massive supergravity and its origin from eleven dimensions

Memoria de Tesis Doctoral realizada por
D. Juan Ignacio Alonso Alberca,
presentada ante el Departamento de Física Teórica
de la Universidad Autónoma de Madrid
para la obtención del Título de Doctor en Ciencias.

Tesis Doctoral dirigida por
D. Tomás Ortín Miguel,
Científico Titular del Consejo Superior de Investigaciones Científicas.

Madrid, Mayo 2003.

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A Javier y Alicia.

A Paloma.

Contents

¡Qué caleidoscopio es el mundo! Y todo con su rotulito a la espalda, por el otro lado, por el que no se ve, todo con su correspondiente explicación. ¡Vaya una ocurrencia que es el mundo!

Miguel de Unamuno, *Amor y pedagogía.*

Introduction

There are four known different ways in which elementary particles can interact in the world we observe. These are the four fundamental interactions: gravity, strong and weak nuclear interactions and the electromagnetic one. It is an extended belief that these interactions are different manifestations of the same phenomenon of Nature, such as electricity and magnetism are different ways in which the electromagnetic interaction manifests. One ambitious challenge in Theoretical Physics is to find a unique theory able to describe all the phenomena we observe: a *theory of everything*.

Symmetry

Our present understanding of the fundamental interactions is based on symmetry principles. Formally, the presence of a symmetry is given by the invariance of a theory under a certain group of transformations. A crucial point is that symmetries in Nature seem to be realized *locally* rather than globally. In Physics, this is considered a principle. There many examples that support it.

The Standard Model of the elementary particles describes the four fundamental interactions. However, the description it provides for the strong, weak and electromagnetic interactions is strickingly different from that for gravity.

On the one hand, the strong, weak and electromagnetic interactions, which are important at short distances, are described by a relativistic quantum field theory with gauge group $SU(3) \times SU(2) \times U(1)$, *i.e.* it is a theory invariant under a *local* $SU(3) \times SU(2) \times U(1)$ *symmetry*. This theory provides a complete description of these three interactions and agrees extremely well with experiments. This is true at least up to the energy scale which it has been experimentally verified (~ 0.2 TeV). But, why does it agree so well with experiments if it does not include gravity? The point is that the gravitational interaction is much weaker than the other three interactions, and therefore its contribution can be completely neglected in current experiments. There are other particular problems inherent to the $SU(3) \times SU(2) \times U(1)$ theory, such as a "huge" number of free parameters (19) which must be imposed by hand in order to agree with experiments.

On the other hand, we have gravity, which is important at very long distances. General Relativity provides the right description of gravity at a classical level. This theory relies on the Principle of General Covariance. This is nothing but considering that the laws of Physics (and therefore the laws of Nature) must be the same independently of the way we describe them.

General Relativity can also be constructed as the gauge theory of the Poincaré group [117], *i.e.* it is a theory invariant under *local Poincaré symmetry.* Contrary to the case of the $SU(3) \times SU(2) \times U(1)$ theory, this construction is not of Yang-Mills type and contains some unusual features[1]. One reason for this, is that the symmetries taken into account for this construction are spacetime symmetries, while in the case of the $SU(3) \times SU(2) \times U(1)$ theory the symmetries are internal.

However, General Relativity is a classical theory. Upon quantization, the theory becomes non-renormalizable. As we have learned from quantum field theory, this may mean that we are neglecting some fundamental structure of the theory. In other words, the theory will be valid only up to a certain energy scale. At higher energy scales, both quantum and gravitational effects become relevant, and a quantum theory of gravity is needed to describe Physics adequately. This takes place at energies at which the structure of spacetime is affected by quantum uncertainties. Such a scale of energy is given by the Planck mass ($\sim 10^{19}$ GeV), which is the energy at which the Schwarzschild radius of a particle becomes comparable with its Compton wavelength. The search for a quantum theory of gravity is one of the major problems in Theoretical Physics.

A crucial property of the $SU(3) \times SU(2) \times U(1)$ theory is that it includes both the carriers of interactions (the gauge bosons) and the particles that interact (fermions). Then, this theory describes both *force* and *matter.* The bosonic or fermionic nature of particles is determined by their spin: integer spin for bosons and half-integer spin for fermions. All known particles are either fermions or bosons. Therefore, in the search for a theory of everything, it is natural to ask if both fermions and bosons can be considered as members of the same entity. From a theoretical point of view, this possibility looks very attractive, as it implies a higher level of unification. If it existed, matter and force would be different manifestations of the same phenomenon!

[1] We will explain this construction in chapter 1.

Examining the S matrix, one finds that it allows for an additional symmetry, which is, precisely, a symmetry rotating bosons into fermions (and vice-versa) [77]. It is known as *supersymmetry.* Unfortunately, there is no direct experimental evidence confirming its existence, but, due to the power and elegance of such an idea, it is strongly believed that an evidence will eventually emerge. In fact, an ambitious challenge of the CERN Large Hadron Collider (LHC) is to look for the supersymmetric partners of the particles in the $SU(3) \times SU(2) \times U(1)$ theory.

If Nature has any form of supersymmetry, it may probably be realized locally. A crucial point is that global supersymmetry transformations generate global spacetime translations, and therefore local supersymmetric field theories are theories of general coordinate transformations, *i.e.* theories of gravity. These are known as *supergravity theories*, and can be understood as supersymmetric extensions of General Relativity.

Analogously to the case of General Relativity, we can construct supergravity theories as gauge theories of an extension of the Poincaré group to include supersymmetry transformations (a Poincaré supergroup)[2]. There are many possible supersymmetric extensions of the Poincaré group, each of them leading to a different supergravity theory.

One way to extend the Poincaré supergroup is to consider higher-dimensional ($d > 4$) groups. Higher-dimensional supergravities can be constructed as their gauge theories. However, if we want to avoid the presence of higher spin fields (spin>2) in interaction or more than one graviton in the supergravity theory, the highest possible spacetime dimension turns out to be eleven [157]. Moreover, only *one* locally supersymmetric field theory can be constructed in eleven dimensions [52, 53]. This theory is known as $N = 1$ $d = 11$ *supergravity* [46].

Another interesting class of supergravity theories are those constructed as gauge theories of anti-de Sitter supergroups. They are known as *gauged supergravities*[3]. These theories typically contain a scalar potential in the Lagrangian which acts effectively as a negative cosmological constant, such that they admit anti-de Sitter spacetime as a vacuum solution (by this we mean a solution for which the anti-de Sitter supergroup is the group of superisometries). Anti-de Sitter supergroups only exist up to seven spacetime dimensions [123], and, therefore, gauged supergravities with anti-de Sitter vacua only be constructed up to $d = 7$. Beyond this limit, gauged theories can be constructed up to ten

[2] This construction presents the same problems as those encountered in the case of General Relativity. We will also see it in chapter 1.

[3] Standard (Poincaré) supergravities are also referred to as *ungauged supergravities.*

spacetime dimensions, but they admit no maximally (super)symmetric solutions. Instead of anti-de Sitter spacetime, they admit *domain wall* vacuum solutions.

An equivalent way to construct gauged supergravities is to make local the R-symmetry group (or a part of it) of an ungauged supergravity. In fact, gauged supergravities have been typically constructed in this way by making use of the Noether method.

Although the introduction of supersymmetry has provided many new possibilities in Physics, it is not enough to cancel the divergences of gravity: supergravity theories are not consistent as quantum theories, since they are non-renormalizable. Then, they are valid up to a certain energy scale, and therefore cannot be theories of everything. Nevertheless, the study of supergravity theories is interesting by itself, but they are specially important because they arise as the low energy limit of *superstring theories.*

Strings

String theory is nowadays the most promising candidate for a unified description of the fundamental interactions. It relies on the idea that the building blocks of Nature are strings instead of pointlike particles. One way to explain why the spatial extension of the strings has not been noticed experimentally is the fact that it may be observable only at energies higher than those reached in current experiments.

The quantization of the bosonic string, only consistent in 26 spacetime dimensions, leads to an spectrum which includes a massless spin-2 field which can be identified with the graviton, the gauge boson of the gravitational interaction.

However, the spectrum also contains a tachyon, a particle which generally signals an instability in the vacuum of the theory. Furthermore, this theory does not contain fermions, whose presence is essential as they are the constituents of matter.

The presence of fermions in the theory can be achieved by introducing supersymmetry, which in addition can be used to remove the tachyon from the spectrum. We end up in this way with *superstring theories.* These theories can be consistently quantized in ten spacetime dimensions, and only five ten-dimensional superstring theories can be constructed: type IIA, type IIB, type I, heterotic $E_8 \times E_8$ and heterotic $SO(32)$.

These five theories differ in their field content and the amount of spacetime supersymmetry. Type I superstring theory is a theory of open and closed unoriented strings, while the others contain only closed strings. A crucial difference between type II superstrings and the other three is that the latter contain a non-Abelian gauge Yang-Mills sector while the former do not.

The problem now is that we were trying to find *one* theory of everything, but we have found five possible candidates. However, there exists a web of transformations, known as *dualities*, which relates the five superstring theories. It is believed that all superstring theories are different perturbative limits of a unique underlying 11-dimensional theory, commonly known as *M-theory*. However, not much is known about this theory.

The study of string dualities led to the discovery of a class of non-perturbative extended solitonic objects which, in addition to strings, appear in string theory. They are dynamical hypersurfaces on which open strings can end, called *D-branes*. There are non-renormalization theorems which imply that these non-perturbative objects must have a faithful description also in the corresponding low-energy limit of the theory. This is in fact the case and, in this limit, they correspond to solutions describing extended objects. Type II superstring theories contain D-branes, and therefore these theories turn out to admit not only closed strings but also open strings, but these must end on a D-brane. In the non-perturbative spectrum, type IIA superstring theory contains Dp-branes with p even, while type IIB theory includes those with p odd.

String/M-theory seems to require ten/eleven spacetime dimensions. In order to make contact with the four-dimensional world we observe, there must be a mechanism ensuring that six of the ten dimensions cannot be noticed at the energy scales we are able to measure.

One possibility is to consider that the extra dimensions are compact and small enough such that they can not be seen in present experiments. Mathematically, we can reduce the number of dimensions via the Kaluza-Klein (or standard) dimensional reduction procedure. Considering that the extra dimensions are periodic (compact), every higher-dimensional field can be expanded in Fourier series. This gives rise to an infinite tower of lower-dimensional fields, some of which are massless. If the extra dimensions are sufficiently small, the contribution of the massive modes cannot be observed at low energies. Therefore, only the zero modes are kept and one arrives to an effectively lower-dimensional theory with no massive Kaluza-Klein modes *at low energy*.

However, there are many inequivalent ways in which a reduction can be performed, each of them leading to a different lower-dimensional theory. Moreover, there is, a priori, no known criterion to choose a preferred compactification. Apparently, string/M-theory is rich in possibilities but does not solve this question. In fact, it still remains as an open problem.

Supergravity theories arise as the low energy limit of superstring theories, *i.e.* they are field theories that give information about superstring theories at low energies, describing only the low energy dynamics of the massless fields of the string spectrum. Solutions to the supergravity equations of motion provide consistent target space backgrounds where strings can propagate. Many non-perturbative properties of superstring theory, such as supersymmetry or dualities, are present at the supergravity level, and provide useful tools to explore the structure of string theory.

The low energy limit of M-theory is supposed to be described by $N = 1, d = 11$ supergravity. Since we believe that any string theory is to be derivable from M-theory in a certain limit, we expect that any supergravity theory describing the low energy classical limit of a string theory should somehow be related to $N = 1$ $d = 11$ supergravity. From the supergravity point of view, this connection is found when we are able to specify a reduction procedure which connects 11-dimensional supergravity with a given theory, up to a duality transformation. An standard example is type IIA supergravity, which can be obtained from $N = 1, d = 11$ supergravity via a Kaluza-Klein reduction on a circle.

Gauged/massive supergravities

Type IIA supergravity presents a problem: it does not allow for an 8-brane solution representing the long range field emitted by a D8-brane, while we expect that *all* the D-branes of type II superstrings have a representation in the corresponding type II supergravity limit. Then, type IIA supergravity should be extended in order to admit 8-brane solutions which could be associated to the D8-branes.

Polchinski noticed [134] that there is a supergravity theory which could describe the low energy limit of type IIA superstring theory including D8-branes. It is known as *Romans' theory* or *massive type IIA supergravity* [140], and was constructed via a deformation of type IIA supergravity by the introduction of a parameter with dimensions of mass. Romans' theory presents many differences with respect to standard type IIA, such as a scalar potential for the dilaton or a mass term for the Kalb-Ramond field. The theory also presents a spontaneous breaking of (super)symmetry, such that there are no maximally (super)symmetric vacuum solutions. In fact, the vacuum solution of this theory is related to the D8-branes of type IIA superstring theory.

While the 11-dimensional origin of standard type IIA supergravity is well-known, Romans' theory is not derivable from $N = 1, d = 11$ supergravity via a Kaluza-Klein reduction. This is also the case of a different kind of supergravity theories, *massive supergravities*, of which Romans' theory is the prime example. The label 'massive' is due to the presence of mass terms for some fields of the theory[4]. Although they are not gauged supergravities, both kind of theories present many common properties, and we consider them as members of the same class. Indeed, there are many gauged/massive supergravities whose 11-dimensional origin is unknown.

Kaluza-Klein reductions on a torus do not lead to any gauged/massive supergravity if the higher dimensional theory is ungauged/massless. In other words, this kind of reduction does not introduce gauge coupling constants or mass parameters in the reduced theory. Since 11-dimensional supergravity does not contain this kind of parameters, toroidal Kaluza-Klein reductions of this theory cannot end up with gauged/massive supergravities.

One possibility is to consider reduction procedures which introduce gauge coupling constants or mass parameters. This can be achieved through a modification of the reduction Ansatz. One example is the Kaluza-Klein reduction on a sphere instead of a torus. This has served to obtain many gauged supergravities from higher-dimensional ungauged supergravities. However, it only works in some cases and, moreover, there is no general understanding of why they work. Notable examples are the S^5 reduction of type IIB supergravity [49], and the S^4 [49, 124] and S^7 [49, 173] reductions of $N = 1, d = 11$ supergravity. They lead, respectively, to gauged maximal supergravities in five, seven and four dimensions.

Scherk and Schwarz developed a reduction procedure, known as *generalized dimensional reduction*, in which some fields of the higher-dimensional theory are allowed to acquire a certain dependence on the internal coordinates in the reduction Ansatz [150, 151]. This dependence is introduced by gauging a global symmetry of the higher-dimensional theory, and is such that, although the higher dimensional fields depend on the internal coordinates, the reduced theory is completely independent of them. In these reductions, parameters with dimensions of mass are introduced in the reduced theory.

[4]Standard supergravities are sometimes called 'massless'.

Gauged supergravity theories play an important role in the context of the *gauge/gravity correspondence*. In its original form [72, 118, 176] (AdS/CFT correspondence), it establishes that $\mathcal{N} = 4$ $d = 4$ super Yang-Mills theory with gauge group $U(N)$ is the dual of type IIB superstring theory on $AdS_5 \times S^5$ in the presence of N units of Ramond-Ramond five-form flux. The superconformal field theory describes the dynamics in the worldvolume of the N coincident D3-branes in the limit of decoupling gravity (low energy), which corresponds to the near-horizon limit of the D3-branes. The AdS/CFT correspondence conjectures that there is an exact map which relates a string phenomenon taking place in the $AdS_5 \times S^5$ background with a phenomenon occurring in the gauge theory.

There are various generalizations of the AdS/CFT correspondence. One of these extensions relates domain wall geometries with supersymmetric quantum field theories (DW/QFT) [28, 94]. There, it is proposed that string/M-theory in a certain domain wall background (which usually breaks one half of the supersymmetry) is dual to a quantum field theory which describes the worldvolume dynamics of N coincident branes. The "near horizon" limit[5] of the supergravity solution of the corresponding brane yields a compactification to the domain wall spacetime proposed by the equivalence. Furthermore, the R-symmetry of the supersymmetric quantum field theory living on the worldvolume of the domain wall must match the gauge group of the dual gauged supergravity.

One interesting example is the case of the D6-brane. The "near-horizon" limit of the D6-brane solution led to deduce the existence of an S^2 compactification of type IIA supergravity which yields an $SU(2)$ $d = 8$ gauged supergravity [28], which does not admit a maximally (super)symmetric vacuum solution but a one half supersymmetric domain wall. This theory was shown to be the maximal $d = 8$ gauged supergravity obtained in [144] from $N = 1, d = 11$ supergravity via a Scherk-Schwarz generalized dimensional reduction on an $SU(2)$ manifold[6].

There are many gauged/massive supergravity theories which cannot be obtained by any sort of (known) dimensional reduction from $d = 11$ supergravity. The most notorious example is precisely Romans' theory.

[5]We use quotes because the near-horizon limit which is taken for the DW/QFT correspondence is not the one of the solution in the string frame. This is because it is not clear if the near-horizon limit of the string frame solution yields a limiting supergravity solution as in the case of the D3-brane [68].

[6]This is completely equivalent to the reduction of type IIA supergravity on a 2-sphere with the RR 2-form field strength proportional to the volume of the 2-sphere.

We are left with the possibility of modifying $N = 1, d = 11$ supergravity such that a standard dimensional reduction of the new theory leads to Romans' theory. However, there is a no-go theorem that asserts that this theory is unique [11, 52, 53]. A way to evade this theorem is to introduce a Killing vector in the Lagrangian. The presence of this vector breaks the 11-dimensional Lorentz symmetry to the 10-dimensional one even if the theory is formally 11-dimensional covariant. Since a Killing vector is dimensionful, a mass parameter must also be introduced in the Lagrangian. This theory is known as "massive 11-dimensional supergravity" [19].

The standard dimensional reduction of "massive 11-dimensional supergravity" in the direction of the Killing vector leads precisely to Romans' theory. In a sense, "massive 11-dimensional supergravity" is nothing but a way of rewriting Romans' theory in an 11-dimensional fashion. The interesting point is that it admits a generalization to include more than one Killing vector [122], and this new theory can be understood as a way of rewriting gauged/massive supergravities in an 11-dimensional manner.

Summary of contents

Chapter 1 is an introductory chapter to supergravity theories. We will try to explain how they can be constructed from symmetry principles and what their relation to superstring/M-theory is.

In chapters 2 and 3 we will try to gain some insight on the 11-dimensional origin of gauged/massive supergravity theories.

In **chapter 2** we will try to find solutions to this problem by making use of generalized dimensional reductions. We will perform a Scherk-Schwarz geometrical reduction of $d = 11$ supergravity on a 3-dimensional manifold, such that we end up with five $d = 8$ gauged maximal supergravities whose gauge groups are the three-dimensional (non-)compact subgroups of $SL(3,\mathbb{R})$. This way we generalize the standard maximal $SO(3)$ gauged supergravity in eight dimensions. We will also construct the most general half-supersymmetric domain wall solutions to these five gauged supergravities and study their upliftings to 11 dimensions.

The original work presented in this chapter is based on Ref. [4].

In **chapter 3** we study various standard dimensional reductions (on an n-torus) of an extension of "massive 11-dimensional supergravity", and show how many massive/gauged maximal supergravities can be systematically obtained this way. We will focus mainly on the reductions to eight and five dimensions, and compare the eight-dimensional theories with those found in chapter 2.

The material contained in this chapter is based on Refs. [2, 3].

Finally, in **chapter 4** we will look for new solutions of gauged $N = 2, d = 4$ supergravity. We will extend the topological Kerr-Newman-AdS solutions by including NUT charge and find generalizations of the Robinson-Bertotti solution to the negative cosmological constant case with different topologies. The supersymmetry properties of the new solutions will also be studied.

The material presented here is based on Ref. [1].

Chapter 1

Supergravity

The main focus of this chapter is to show how supergravity theories can be constructed from symmetry principles and explain how they are related to string/M-theory.

1.1 Supersymmetry

Supersymmetry is a symmetry that rotates bosons into fermions and (vice-versa) by changing the spin by 1/2. Dimensional arguments [125] show that global supersymmetry transformations for a (globally) supersymmetric field theory including bosons B and fermions F (spin 1/2) are of the form

$$\begin{cases} \delta_\epsilon B & \sim \quad \bar{\epsilon} F \,, \\ \\ \delta_\epsilon F & \sim \quad (\partial B)\epsilon \,, \end{cases} \tag{1.1}$$

where ϵ is the spinorial parameter of the transformation. Consider now the action of two consecutive infinitesimal global supersymmetry transformations (1.1) of a bosonic field B (the argument works similarly for fermions F). The first one transforms B into F and the second one rotates F into ∂B, *i.e.*

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] = \tfrac{1}{2}(\bar{\epsilon}_2 \gamma^\mu \epsilon_1)\partial_\mu B \,. \tag{1.2}$$

This means that two supersymmetry transformations lead to a spacetime translation, and therefore supersymmetry is a spacetime symmetry. In fact, it takes its simplest form as a symmetry of *superspace*.

Superspace is an extension of ordinary spacetime to include extra anticommuting coordinates, related to the fermionic degrees of freedom. The functions we define over this space are known as *superfields*. There is a *supergroup*, whose associated *superalgebra* is represented by translations and rotations involving both the spacetime and the anticommuting coordinates. Superfields transform covariantly under these transformations. If superfields are expanded in Taylor series in the fermionic coordinates, we obtain a finite number of terms (due to the fact that these coordinates are anticommuting, *i.e.* their squares vanish), and the coefficients of the expansion are ordinary component fields. The transformations of the component fields follow from a Taylor expansion of the translated and rotated superfields. Though the superspace formulation has been shown to be very useful to study supersymmetric field theories, we will be mainly interested in the component field formulation.

Nature seems to exhibit symmetries locally rather than globally. If supersymmetry is somehow a symmetry of Nature, it is then natural to ask if it could also be realized as local symmetry. The interesting point is that global supersymmetry transformations generate global spacetime translations, and therefore we may expect a locally supersymmetric field theories to be a theories of general coordinate transformations, *i.e.* theories of gravity! Let us think on all this in terms of the algebras of the transformations.

Supersymmetry transformations are generated by spinorial, anticommuting *supercharges* Q^α, which carry an spinorial index α because they are arranged in sets that transform as spinors under the Lorentz group. The infinitesimal parameters of the field transformations will therefore be anticommuting spinors ϵ^α. Now, the anticommutator of two supersymmetry generators is of the form

$$\left\{Q^\alpha, Q^\beta\right\} = i\left(\gamma^a \mathcal{C}^{-1}\right)^{\alpha\beta} P_a\,, \tag{1.3}$$

where $\mathcal{C}$ is the charge conjugation matrix and P_a are the generators of translations. Here we see again that two supersymmetry transformations lead to a spacetime translation. The complete algebra in which (1.3) is included also contains the Poincaré algebra, which is given by

$$\begin{aligned} [M_{ab}, M_{cd}] &= \delta_{ac}M_{bd} + \delta_{bd}M_{ac} - \delta_{ad}M_{bc} - \delta_{bc}M_{ad}\,, \\ [P_a, M_{bc}] &= -2\delta_{a[b}\,P_{c]}\,, \end{aligned} \tag{1.4}$$

and also the additional commutator

$$[Q^\alpha, M_{ab}] = \tfrac{1}{2}\left(\gamma_{ab}\right)^\alpha{}_\beta Q^\beta\,, \tag{1.5}$$

where M_{ab} are the generators of the Lorentz rotations. The complete supersymmetry algebra is called the *Poincaré superalgebra.* Generically, any algebra including supersymmetry generators is referred to as a superalgebra.

The Poincaré algebra contains the generators of translations P_a and of Lorentz rotations M_{ab}. If we gauge translations and Lorentz rotations, we end up with General Relativity [117]. Then, we can also make local all the symmetries represented in a superalgebra and therefore expect to end up with a locally supersymmetric extension of General Relativity which also includes matter (fermions). These theories are known as *supergravity theories*, and will be essential along all our work. But, first of all...

☐ how do we gauge a given set of global symmetries?

We have two possibilities at our dispossal: we can apply the *the Noether method* or even *gauge the (super)algebra* of the transformations. Both methods lead to the same results, but they are quite different. Let us describe them briefly.

The Noether method

The Noether method is a systematic prescription to derive an action with a local symmetry from an action with a global symmetry. From a mathematical point of view, it is not complicated (though it can be can be rather lengthy), and has been shown to be very powerful. It works as follows.

Start with an action which is invariant under certain global transformations. Then, the parameters of the transformation, say σ^I, are constants. If we now use local parameters (functions) instead of global ones, then the variation of the action is proportional to the derivatives of the parameters

$$\delta S \propto \int d^d x \partial_\mu \sigma^I \, j_I^\mu \,, \tag{1.6}$$

which, as expected, vanishes when the parameters are constant. Integration by parts shows that the variation above is (up to a total derivative)

$$\delta S \propto \int d^d x \sigma^I \, \partial_\mu j_I^\mu \,, \tag{1.7}$$

which vanishes iff $\partial_\mu j_I^\mu = 0$. The j_I^μ's are the Noether currents associated to the global symmetry. We have to add new terms to the original action so as to make it invariant under the new *local* symmetry. This leads one to introduce gauge fields which couple to the Noether currents. At the level of the action, new terms must be added until invariance under local transformations is achieved, and we end up with a locally symmetric theory.

The Noether method has been used to construct many field theories, including General Relativity (making local spacetime translations and Lorentz transformations) and many supergravity theories.

Gauging the algebras

A second method to construct a locally symmetric theories is to introduce a gauge field associated to each symmetry, define a curvature for it (*i.e.* a field strength) and construct an action quadratic in derivatives. This kind of action ensures that the gauge fields will propagate. Further, we require that the action is invariant under the gauge symmetries. In Yang-Mills gauge theories, this action is taken to be quadratic in field strengths, but this is not the only possibility. This method is commonly known as *gauging of algebras*[1]. Let us explain the method in a generic case.

Let us explain this method in a generic case. Consider a d-dimensional Lie (super)group $\mathcal{G}$, and let $\{T_A\}$ be a basis for the Lie (super)algebra of $\mathcal{G}$. These generators satisfy the commutation relations[2]

$$[T_A, T_B] = f_{AB}{}^C \, T_C \,, \tag{1.8}$$

where $f_{AB}{}^C$ are the structure constants. Corresponding to every generator we introduce a gauge field, entering the algebra-valued vector field

$$\mathcal{A}_\mu \equiv \mathcal{A}_\mu{}^A \, T_A \,. \tag{1.9}$$

where the vectors $\mathcal{A}_\mu{}^A$ transform in the adjoint representation of $\mathcal{G}$. We define covariant derivatives as

---

[1]Note that this is an abuse of language, since algebras cannot be gauged. What one really gauges is a symmetry (or a set of them). We will construct theories whose vacuum solutions enjoy all the symmetries one gauges. To these vacuum solutions we associate a global algebra corresponding to the set of symmetries (those that we gauge) and therefore we commonly say that we "gauge the algebra".

[2]If we are dealing with a superalgebra there will also appear anticommutators for the fermionic generators.

$$\mathcal{D}_\mu \equiv \partial_\mu + \mathcal{A}_\mu \,. \tag{1.10}$$

Then, the commutator of two covariant derivatives leads to the curvature (field strength)

$$[\mathcal{D}_\mu, \mathcal{D}_\nu] = -R_{\mu\nu} \,, \tag{1.11}$$

such that

$$R_{\mu\nu} \equiv 2\,\partial_{[\mu}\mathcal{A}_{\nu]} + [\mathcal{A}_\mu, \mathcal{A}_\nu] = R_{\mu\nu}{}^A\, T_A \,, \tag{1.12}$$

whose components are given by

$$R_{\mu\nu}{}^A = 2\,\partial_{[\mu}\mathcal{A}_{\nu]}{}^A + f_{BC}{}^A\, \mathcal{A}_\mu{}^B\, \mathcal{A}_\nu{}^C \,. \tag{1.13}$$

The field strengh satisfies the Bianchi identity

$$\mathcal{D}_{[\mu} R_{\nu\rho]} = 0 \,. \tag{1.14}$$

Let us see now the gauge transformations. The local gauge parameters Λ_A associated to the gauge fields enter an algebra-valued scalar field

$$\Lambda \equiv \Lambda^A\, T_A \,. \tag{1.15}$$

By construction, the action of an infinitesimal transformation of the gauge field is the gauge covariant derivative of Λ

$$\delta_\Lambda \mathcal{A}_\mu = \partial_\mu \Lambda\; + [\Lambda, \mathcal{A}_\mu] \;, \tag{1.16}$$

which, in components, reads

$$\delta_\Lambda \mathcal{A}_\mu{}^A = \partial_\mu \Lambda^A + f_{BC}{}^A\, \mathcal{A}_\mu{}^B\, \Lambda^C \,. \tag{1.17}$$

Under gauge transformations the curvature transforms homogeneously

$$\delta_\Lambda R_{\mu\nu}{}^A = f_{BC}{}^A \Lambda^B R_{\mu\nu}{}^C \,. \tag{1.18}$$

Finally, we arrive at the construction of the action. The idea is that this action is quadratic in derivatives of the gauge fields, such that they are propagating fields. A possibility is to try a Yang-Mills type action

$$S = \int d^d x \, R_{\mu\nu}{}^A \, R_{\rho\sigma}{}^B \, Q^{\mu\nu\rho\sigma}_{AB} \,, \tag{1.19}$$

where Q are some constants which are determined by requiring invariance under the gauge transformations. The action one finds is *locally* invariant under the infinitesimal transformations in the algebra of $\mathcal{G}$.

The method of gauging of algebras can been used generically to construct gauge theories. We have pointed in the previous section that the symmetries one makes local in order to obtain General Relativity are those entering the Poincaré algebra. Then, we could try to construct this theory as a *gauge* theory of the Poincaré group. If true, this would provide a strong connection between gauge theories and gravity. This possibility is explored in the next section.

1.2 Gravity as a gauge theory

The starting point is the Poincaré algebra given in (1.4). We are going to gauge the spacetime translations and Lorentz rotations generated by P_a and M_{ab}, respectively. To this end, we introduce a gauge field for each generator: $e_\mu{}^a$ for the translations subalgebra and $\omega_\mu{}^{ab}$ for the Lorentz one. Then, we have

$$\mathcal{A}_\mu \equiv \tfrac{1}{2}\omega_\mu{}^{ab} M_{ab} + e_\mu{}^a P_a \,. \tag{1.20}$$

Now, we introduce the gauge parameters

$$\Lambda \equiv \tfrac{1}{2}\sigma^{ab} M_{ab} + \xi^a P_a \,. \tag{1.21}$$

The effect of the gauge transformations can be shown to be (see *e.g.* [63, 125])

$$\begin{cases} \delta\omega_\mu{}^{ab} &= -\mathcal{D}_\mu \sigma^{ab} \,, \\ \delta e_\mu{}^a &= -\mathcal{D}_\mu \xi^a + \sigma^a{}_b e_\mu{}^b \,, \end{cases} \tag{1.22}$$

where $\mathcal{D}$ stands for the gauge covariant derivative. Observe that, since P_a does not act on objects with Lorentz indices, $\mathcal{D}$ only contains in practice $\omega_\mu{}^{ab}$ and not $e_\mu{}^a$. Moreover, notice that $\mathcal{D}$ has no relation with a geometrical covariant derivative as it contains no Levi-Cività connection. This is due to the fact that, by the moment, we have no metric but only gauge fields.

We now construct the gauge field strength

$$R_{\mu\nu} \equiv 2\,\partial_{[\mu}\mathcal{A}_{\nu]} + [\mathcal{A}_\mu, \mathcal{A}_\nu] = \tfrac{1}{2}R_{\mu\nu}{}^{ab}M_{ab} + R_{\mu\nu}{}^a P_a\,, \tag{1.23}$$

where

$$\begin{cases} R_{\mu\nu}{}^{ab} &= 2\partial_{[\mu}\omega_{\nu]}{}^{ab} - \omega_{[\mu}{}^a{}_c\omega_{\nu]}{}^{cb}\,, \\ R_{\mu\nu}{}^a &= 2\mathcal{D}_{[\mu}e^a{}_{\nu]}\,. \end{cases} \tag{1.24}$$

Finally, we arrive at the construction of the action. If we construct it quadratic in curvatures as in (1.19), we arrive to an action which does *not* reproduce the equations of motion of General Relativity. Then, we have found that this theory cannot be constructed as a pure Yang-Mills gauge theory. But, let us relax the assumption of an action quadratic in curvatures and try to construct *ad hoc* an action leading to General Relativity. It turns out that, in order to arrive to the right result, we must *impose* two conditions:

1. $R_{\mu\nu}{}^a$ must vanish. It must be possible to derive this condition from the action we construct.

2. The gauge fields $e_\mu{}^a$ associated to spacetime translations must be *invertible*, with the inverse fields $e_a{}^\mu$ defined as

$$e_\mu{}^a e_b{}^\mu = \delta^a_b \implies e_\mu{}^a e_a{}^\nu = \delta^\nu_\mu\,, \tag{1.25}$$

The constraint $R_{\mu\nu}{}^a = 0$ can only be implemented if the action is linear in the curvature components $R_{\mu\nu}{}^{ab}$. The right action one must choose is the first order action

$$S \sim \int d^4x\, R_{\mu\nu}{}^{ab} e_\rho{}^c e_\sigma{}^d\, \epsilon^{\mu\nu\rho\sigma}\, \epsilon_{abcd}\,. \tag{1.26}$$

Let us see the equations of motion derived from (1.26). For the field $\omega_\mu{}^{ab}$ we find

$$R_{\mu\nu}{}^{a} = 0\,, \tag{1.27}$$

which turns out to be a relation between the $\omega_\mu{}^{ab}$'s and the $e_\mu{}^a$'s, such that the former can be solved generically in terms of the latter. Explicitely

$$\omega_{abc}(e) = -\Omega_{abc} + \Omega_{bca} - \Omega_{cab}\,, \qquad \Omega_{ab}{}^{c} = e_a{}^{\mu} e_b{}^{\nu} \partial_{[\mu} e^{c}{}_{\nu]}\,. \tag{1.28}$$

Now, the equation of motion for $e_\mu{}^a$ is

$$G_{\mu\nu} = 0\,, \tag{1.29}$$

which is exactly the Einstein equation. Then, (1.28) reveals that the $\omega_\mu{}^{ab}$'s are composite fields, *i.e.* they do not propagate, and therefore all the degrees of freedom of the theory are contained in the $e_\mu{}^a$'s, as expected. In fact, we can rewrite action 1.26 in the form

$$S \sim \int d^4x\, e\, R(e,\omega)\,, \tag{1.30}$$

which we identify with the Einstein-Hilbert action in Vierbein notation. It is then tentative to identify the $e_\mu{}^a$'s with the *Vierbein* and the ω_μ^{ab}'s with the *spin connection*. A metric tensor can be constructed, which, as chosen by Einstein, is $g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab}$, where η_{ab} is the Minkowski tensor.

Now that we have identified the $\omega_\mu{}^{ab}$'s and the $e_\mu{}^a$'s with the spin connection and the Vierbein, it is worth making some comments on the construction we have presented. First, the gauge covariant derivative $\mathcal{D}$ can now be identified with the geometric covariant derivative ∇. Secondly, $R_{\mu\nu}{}^a = 2\mathcal{D}_{[\mu} e^a{}_{\nu]}$ is the torsion of the spacetime, and the constraint $R_{\mu\nu}{}^a = 0$ implies $\nabla_{[\mu} e^a{}_{\nu]} = 0$, which is precisely the Vierbein postulate. Finally, it is interesting to note that, provided $\mathcal{D}_{[\mu} e^a{}_{\nu]} = 0$, Poincaré gauge invariance and reparametrization invariance are related as follows: the effect of an infinitesimal reparametrization generated by the world vector ξ^μ ($\delta x^\mu = \xi^\mu$) is identical to the effect of a P_a gauge transformation with parameter $\xi^a = e_\mu{}^a \xi^\mu$ plus a local Lorentz transformation with parameter $\sigma^{ab} = \xi^\mu \omega_\mu{}^{ab}$.

Then, we have learned that General Relativity can be constructed as a gauge theory of the Poincaré group. Although it is *not* a pure Yang-Mills gauge theory, it contains some of the elements of these kind of theories. The construction is rather *ad hoc*, but it gives the Vierbein a gauge field interpretation, and also reveals which constraints are necessary to relate Poincaré gauge invariance with reparametrization invariance. Nevertheless, there are many questions for which we have no answer, such as the need to impose invertibility of the Vierbein. In this sense we understand that, strictly speaking, General Relativity is not a pure gauge theory of the Poincaré group, but it can be constructed following the same steps, up to a certain point.

One of the facts that differences the gauge theory of the Poincaré group (GR) from a Yang-Mills theory is that the former has an action which is not quadratic in field strengths. A slight improvement of this situation was achieved by MacDowell & Mansouri in [117]. They considered the anti-de Sitter group $SO(2,3)$ and constructed General Relativity with a cosmological constant following the construction we have presented. Upon a Inönü-Wigner contraction (essentially, the zero cosmological constant limit), the group $SO(2,3)$ becomes the Poincaré group $ISO(1,3)$. This algebras can be understood as as a deformation of the Poincaré algebra in which the generators of translations P_a do not commute. After some redefinitions of the $SO(2,3)$ generators, one finds that the algebra can be written as

$$\begin{aligned}
[M_{ab}, M_{cd}] &= \delta_{ac} M_{bd} + \delta_{bd} M_{ac} - \delta_{ad} M_{bc} - \delta_{bc} M_{ad}\,, \\
[P_a, M_{bc}] &= -2\delta_{a[b}\, P_{c]}\,, \\
[P_a, P_b] &= -g^2 M_{ab}\,,
\end{aligned} \tag{1.31}$$

where g is a constant. It is easy to see that, in the limit $g = 0$, one recovers the Poincaré algebra and the results given above. The main point found by MacDowell & Mansouri is that a quadratic action leading to the right results (General Relativity with a cosmological constant $\Lambda \sim g^2$) can be constructed. However,this approach still leaves some questions unanswered, like, *e.g.* the need to impose invertibility of the Vierbein.

If General Relativity can be constructed as a gauge theory of the Poincaré group, then, presumably, extensions of this group will lead (upon gauging) to theories which include General Relativity. Supersymmetric extensions of the Poincaré group should therefore lead to supersymmetric extensions of General Relativity. They are known as *supergravity theories.*

1.3 From supersymmetry to supergravity

Let us start with the construction of the simplest supersymmetric extension of General Relativity. This supergravity theory is obtained by gauging the simplest supersymmetric extension of the Poincaré algebra: the Poincaré superalgebra (see section 1.1). Since it has one more generator, the gauge potential has one more component ${\psi_\mu}^\alpha$, *i.e.*

$$\{M_{ab}\,, P_a\,, Q^\alpha\} \underset{\text{gauging}}{\longrightarrow} \mathcal{A}_\mu \equiv \tfrac{1}{2}{\omega_\mu}^{ab}M_{ab} + {e_\mu}^a P_a + \bar{\psi}_{\mu\alpha}Q^\alpha\,. \tag{1.32}$$

The new field compensates local supersymmetry transformations. It is a Rarita-Schwinger field, which describes a massless spin-3/2 particle: the *gravitino*. It has two possible helicity states ($\pm 3/2$) which are the superpartners of the two helicity states of the graviton (± 2), *i.e.* the graviton and the gravitino constitute a supermultiplet. The quantum field theory will have the same number of bosonic and fermionic states at each mass level, a property of linearly realized supersymmetry.

The curvature is now

$$R_{\mu\nu} = \tfrac{1}{2}{R_{\mu\nu}}^{ab}M_{ab} + {R_{\mu\nu}}^a P_a + \bar{R}_{\mu\nu\alpha}Q^\alpha\,, \tag{1.33}$$

and the action for the theory, known as $N=1, d=4$ *supergravity*, is just the Cartan-Sciama-Kibble (CSK) theory (see, *e.g.* [82,143]) for a Rarita-Schwinger field coupled to gravity

$$S \sim \int d^4x\, e\, \left\{R(e,\omega) + \epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_5\gamma_\nu\mathcal{D}_\rho(\omega)\psi_\sigma\right\}\,. \tag{1.34}$$

This is precisely a CSK theory with a Rarita-Schwinger field which is invariant under local supersymmetry transformations

$$\delta_\epsilon {e^a}_\mu = -i\bar{\epsilon}\gamma^a\psi_\mu\,, \qquad \delta_\epsilon\psi_\mu = \mathcal{D}_\mu\epsilon\,. \tag{1.35}$$

We can now have bilinears of fermions which give rise to non-vanishing torsion, given by

$${T_{\mu\nu}}^a \sim \bar{\psi}_\mu\gamma^a\psi_\nu\,. \tag{1.36}$$

Setting all the fermions to zero is always a consistent truncation and any purely bosonic General Relativity solution will also be a solution of $N = 1, d = 4$ supergravity. In fact, as noted previously, this theory is the simplest supersymmetric extension of General Relativity.

In general, supergravity theories have been constructed by applying the Noether method. However, the gauging of superalgebras provides an interesting connection with gauge theories and we will adopt this point of view to present many possible extensions of the supergravity theory constructed above.

Generalizing the superalgebra we can generalize the supergravity theory. Then, we must think on generalizations of the four-dimensional Poincaré superalgebra. There are various possibilities:

- **Add more supercharges**.
- Consider a **different number of spacetime dimensions**.
- **Relax Poincaré invariance**. This can be achieved in two ways. First, we can add *quasi-central charges*. These are operators that commute with the supercharges but transform as p-forms under Lorentz transformations. The second possibility is to consider a different bosonic subalgebra, such as (anti-)de Sitter or a conformal algebra.

In the following we are going to explain separately these possibilities, but they can be combined, and, in general, the supergravity theories which we are interested in combine more than one of these extensions[3]. We will also give some examples of supergravity theories which will be of interest for us.

1.3.1 Extended supergravity

A possible extension of the Poincaré superalgebra is to consider the addition of more supercharges $Q^{i\alpha}$, $i = 1 \ldots N$, to the algebra. Then, we are considering that there are N different supersymmetry transformations instead of only one. However, the supercharges arrange in sets that transform as spinors under the Lorentz group. In $d = 4$, the irreducible spinors are Majorana (and therefore real) and have four components. The superalgebras one is then left with, usually called N-extended $d = 4$ Poincaré superalgebras, admit *central charges*

[3] There is another possible extension: we could add supersymmetric matter, *i.e.* add fermions which are not contained in the supermultiplet of the graviton. However, we are not interested in this possibility.

$$\mathcal{Z}^{ij} = -\mathcal{Z}^{ji}\,, \tag{1.37}$$

that appear in the anticommutator of two supercharges

$$\left\{Q^{i\alpha}\,, Q^{j\beta}\right\} = i\delta^{ij}(\gamma^a \mathcal{C}^{-1})^{\alpha\beta} P_a - i(\mathcal{C}^{-1})^{\alpha\beta}\mathcal{Z}^{ij}\,, \tag{1.38}$$

and commute with *all* generators, *i.e.* they are central elements of the algebra (hence the label 'central').

In extended supersymmetry, the spinor charges transform reducibly under the Lorentz group and comprise N irreducible spinors. There exists a group H_R of rotations of the spinors which commutes with the Lorentz group and leaves the superalgebra invariant. It is commonly known as the *R-symmetry group*[4]. In four dimensions we have $H_R = U(N)$.

Gauging N-extended Poincaré superalgebras we obtain N-extended Poincaré supergravities. The gauge potential contains N gravitini $\psi_\mu{}^{i\,\alpha}$ and also $N(N-1)/2$ Abelian vector fields $A^{ij}{}_\mu$ associated to the central charges

$$\mathcal{A}_\mu \equiv \tfrac{1}{2}\omega_\mu{}^{ab} M_{ab} + e_\mu{}^a P_a + \tfrac{1}{2}A^{ij}{}_\mu \mathcal{Z}^{ij} + \bar{\psi}^i_{\mu\alpha} Q^{i\alpha}\,. \tag{1.39}$$

The action now contains the kinetic terms of N gravitini and of $N(N-1)/2$ Abelian vector fields $A^{ij}{}_\mu$ with field strengths $F^{ij}{}_{\mu\nu} = 2\,\partial_{[\mu}A^{ij}{}_{\nu]}$, but this is not the whole story, as the counting of bosonic and fermionic states immediately shows: there are additional scalar fields and fermionic fields in the theory that cannot be accounted for with our heuristic formulation[5]. The scalars appear always in a non-linear σ-model, couple in a non-trivial fashion to the vector fields and usually have no potential.

If we do not want to deal with the problem of higher spin fields in interaction or more than one graviton (and we will always try to avoid them), the maximum number of supercharges is 32 [157]. Since four-dimensional irreducible spinors are real and have four components, the 32 supercharges are arranged in 8 sets of irreducible spinors. Therefore, $N = 8$ is the maximum number of allowed supersymmetries in four spacetime dimensions.

[4]To be precise, H_R is defined as the largest subgroup of the automorphism group of the superalgebra that commutes with the Lorentz group.

[5]See [36] for a more rigorous formulation.

1.3.2 Higher dimensional supergravity

If we want to construct higher-dimensional supergravity theories, we simply have to gauge superalgebras whose spacetime bosonic subalgebras are higher-dimensional. In the Poincaré case this is straightforward but only up to $d = 11$, for which we have $N = 1$.

The kind of spinor allowed in a theory depends on the number of spacetime dimensions. As we have seen, in $d = 4$ the irreducible spinors are 4-component Majorana and then we have $N = 8$. In $d = 11$ (with Poincaré invariance), the irreducible spinor is also Majorana and has 32 components, so all the supercharges form a 32-component spinor such that we have $N = 1$. Beyond $d = 11$ we need more that 32 supercharges and therefore we run into the same problems we found in going beyond $N = 8$ in $d = 4$. Similar arguments lead to the maximum number of allowed supersymmetries for other dimensions.

As we will see in section 1.5, higher-dimensional supergravities appear naturally in the context of string/M-theory. A very complete guide to the literature on supergravities in diverse dimensions can be found in [145].

1.3.3 Extended objects

If the condition of Poincaré invariance is relaxed, the superalgebra admits *quasi-central charges* $\mathcal{Z}_{[a_1 \ldots a_p]}$, where a_i are (flat) Lorentz indices. These are operators that commute with the supercharges but transform as p-forms under Lorentz transformations. They appear in the superalgebra as

$$\left\{Q^\alpha\,, Q^\beta\right\} = i(\gamma^a \mathcal{C}^{-1})^{\alpha\beta} P_a + \tfrac{1}{p!}\left(\gamma^{a_1\cdots a_p}\mathcal{C}^{-1}\right)^{\alpha\beta} \mathcal{Z}_{a_1\ldots a_p}\,. \tag{1.40}$$

If we now gauge the superalgebra, the gauge potential must include a new field $C_\mu{}^{a_1\ldots a_p}$

$$\mathcal{A}_\mu \equiv \tfrac{1}{2}\omega_\mu{}^{ab} M_{ab} + e_\mu{}^a P_a + \tfrac{1}{p!} C_\mu{}^{a_1\cdots a_p} \mathcal{Z}_{a_1\ldots a_p} + \bar{\psi}_{\mu\,\alpha} Q^\alpha\,, \tag{1.41}$$

which actually appears in the supergravity action as a $(p+1)$-form potential $C_{\mu_1\cdots\mu_{(p+1)}}$ with field strength $G_{(p+2)} = (p+2)\partial C_{(p+1)}$. The gauge transformations of the new gauge fields are generated by the $\mathcal{Z}_{a_1\ldots a_p}$'s.

This kind of charges were shown in [8] to appear in theories which extended objects, and are of capital importance in the case of higher dimensional theories. This can be seen as follows.

These $(p+1)$-form potentials naturally couple to the worldvolume of extended objects of p spatial dimensions, usually referred to as *p-branes.* Indeed, in the corresponding supergravity theory, one finds classical solutions that include the $(p+1)$-dimensional Poincaré group in their isometry group and represent the long-range fields sourced by a flat p-brane[6]. This explains why Poincaré invariance is broken if quasi-central charges are included, as the presence of the p-branes break the Lorentz symmetry of the vacuum. Moreover, as the $\mathcal{Z}_{a_1 \ldots a_p}$'s generate gauge transformations, their associated conserved charges will be the gauge charge of the corresponding p-brane.

1.3.4 Gauged supergravity

The last extension of the Poincaré superalgebra we are going to consider is the consideration of a different spacetime bosonic subalgebra. Interesting candidates are de Sitter (dS) and anti-de Sitter (AdS). There are more possibilities, like, *e.g.* Heisenberg algebras, but they have never been used to gauge supergravities. dS superalgebras lead to inconsistent field theories (ghosts, non-unitarity...). Then, we will be mainly interested in N-extended AdS superalgebras[7].

Let us restrict, by the moment, to the four-dimensional case: N-extended $d=4, AdS$ superalgebras (we will comment later on the higher-dimensional cases). The supercharges $Q^{i\alpha}$ transform as spinors under the bosonic AdS subalgebra generated by the M_{ab}'s and the P_a's. On top of these, we are forced to introduce bosonic generators T^{ij} that rotate the supercharges and also appear in their anticommutator. These generators are "extensions" of the central charges $\mathcal{Z}^{ij}$'s which appear in extended superalgebras (see Eq. (1.38)). However, due to the modification of the bosonic subalgebra, they are no longer central (*e.g.* they rotate the supercharges). In fact, in the limit in which the Poincaré bosonic subalgebra is recovered (via a Inönü-Wigner contraction in the $(A)dS$ case), the T^{ij} become central and we identify them with the $\mathcal{Z}^{ij}$'s in (1.38).

Now the T^{ij}'s generate the R-symmetry of the theory, or, at least, a part of it. Therefore, we will find a theory with *local* R-symmetry. These theories are commonly known as *gauged supergravities*[8], and contain a set of vectors with gauge group H_R (or $\mathcal{G} \in H_R$). Any of these subgroups can be used in the

[6]There are many interesting and quite complete reviews on p-branes. See, *e.g* [156].

[7]Another interesting possibility is to consider pp-wave algebras, but, to the best of our knowledge, they have not been used yet to construct supergravity theories.

[8]Supergravity theories in which the R-symmetry is not local are therefore commonly referred to as *ungauged.*

construction, but let us focus on the simplest example: $\mathcal{G} = SO(N)$.

The superalgebra for the $SO(N)$ case can be written as in (1.31) plus the additional non-vanishing (anti)commutators

$$\begin{aligned} \left\{Q^{i\alpha}, Q^{j\beta}\right\} &= i\delta^{ij}(\gamma^a \mathcal{C}^{-1})^{\alpha\beta} P_a + g\left(\gamma^{ab}\right) M_{ab} - i(\mathcal{C}^{-1})^{\alpha\beta} T^{ij}\,, \\ \left[T^{ij}, T^{kl}\right] &= g\left(\delta^{ik}T^{jl} + \delta^{jl}T^{ik} - \delta^{il}T^{jk} - \delta^{jk}T^{il}\right)\,, \\ \left[Q^{i\alpha}, T^{jk}\right] &= 2\,g\,\delta^{i[j}Q^{k]\alpha}\,, \end{aligned} \tag{1.42}$$

Written in this form, it is easy to see that, in the $g = 0$ limit, we are left with an extended superalgebra. Now, as the T^{ij}'s generate an $SO(N)$ symmetry, we have to introduce an $SO(N)$ vector field with $N(N-1)/2$ components $A^{ij}{}_{\mu}$, the gauge parameter being g. Moreover, their non-trivial action on the supersymmetry generators imply that the N gravitini $\psi_{\mu}{}^{i\alpha}$ will be charged under the $SO(N)$. Then, we have

$$\mathcal{A}_{\mu} \equiv \tfrac{1}{2}\omega_{\mu}{}^{ab} M_{ab} + e_{\mu}{}^{a} P_a + \tfrac{1}{2} A^{ij}{}_{\mu} T^{ij} + \bar{\psi}^{i}_{\mu\alpha} Q^{i\alpha}\,. \tag{1.43}$$

Gauged supergravities contain in their action terms of order $\mathcal{O}(g)$ and $\mathcal{O}(g^2)$. The terms linear in the gauge coupling are gauge field interactions and fermionic masslike terms. Note that we say 'masslike terms' and not 'mass terms'. This is because any of these terms has the form of a mass term for a fermion in Minkowski spacetime. However, the vacuum solution is now anti-de Sitter, and the mass must be computed with respect to this asymptotically spacetime and not with respect to Minkowski. In anti-de Sitter space, fermions are "massless" precisely iff these terms are present.

One of the $\mathcal{O}(g^2)$ terms in the Lagrangian is interpreted as a cosmological term. In the cases with $N = 1, 2$ this term is a negative cosmological constant, while for $N > 2$, there are also scalars present in the theory and there is no longer a cosmological constant but a scalar potential which has an extremum at a negative value and acts as an effective negative cosmological constant.

So far, we have shown how to construct gauged supergravities in $d = 4$. If $d > 4$, $SO(N)$ is not necessarily a subgroup of H_R (see table 1.1 for the list of the R-symmetry groups for maximal extended supersymmetry in d spacetime dimensions). To understand this we simply have to take into account the kind of spinors one can have in d dimensions. In the case we have studied ($d = 4$), the spinors are Majorana and therefore the supercharges are real, and can be rotated with $SO(N)$.

d	11	10A	10B	9	8	7	6	5	4
H_R	1	1	$SO(2)$	$SO(2)$	$U(2)$	$Usp(4)$	$Usp(4) \times Usp(4)$	$Usp(8)$	$U(8)$

Table 1.1: The R-symmetry groups of the maximal extended supersymmetry in d spacetime dimensions. The labels 'A' and 'B' for the two types of maximal extended Poincaré superalgebras that can be constructed in ten spacetime dimensions. The supergravity theories associated to them are usually known as type IIA and type IIB supergravity.

The study of the representations of supersymmetry in anti-de Sitter [123] reveals that AdS superalgebras exist only up to $d = 7$. This does not mean that gauged supergravities cannot be constructed for $d > 7$. In fact, such constructions are possible up to $d = 10$, but they have been tipically constructed by the gauging of a subgroup of H_R of an ungauged supergravity via the Noether method. In fact, the associated superalgebras are generally not well-known.

We have previously explained that the superalgebra one gauges to construct a supergravity theory corresponds to the superisometries of the asymptotic solutions, *i.e.* vacua. Then, the gauging of an AdS superalgebra leads to theories for which AdS spacetime is a vacuum solution. This is only possible up to $d = 7$. For $d > 7$, the theories present a spontaneous breaking of symmetry and supersymmetry, *i.e.* no maximally symmetric and supersymmetric vacuum solution. The true vacuum solutions turn out to be domain wall spacetimes. We will study theories with this property in section 1.6.

1.3.5 $N = 1$ $d = 11$ supergravity

An important example of higher-dimensional superalgebras is the $N = 1, d = 11$ Poincaré superalgebra, *a.k.a. M-superalgebra*[9]. It admits two quasi-central charges for $p = 2, 5$[10]

$$\left\{\hat{Q}^{\alpha}, \hat{Q}^{\beta}\right\} = \left(\hat{\Gamma}^{\hat{a}}\hat{\mathcal{C}}^{-1}\right)^{\alpha\beta} \hat{P}_{\hat{a}} + \tfrac{1}{2}\left(\hat{\Gamma}^{\hat{a}\hat{b}}\hat{\mathcal{C}}^{-1}\right)^{\alpha\beta} \hat{\mathcal{Z}}_{\hat{a}\hat{b}} + \tfrac{1}{5!}\left(\hat{\Gamma}^{\hat{a}_1\cdots\hat{a}_5}\hat{\mathcal{C}}^{-1}\right)^{\alpha\beta} \hat{\mathcal{Z}}_{\hat{a}_1\cdots\hat{a}_5} \,. \tag{1.44}$$

[9]The label 'M' is due to the fact that it is related to the conjectured M-theory. See [164] for an interesting discussion on the great amount of information this superalgebra contains.

[10]All along our work hats will be used to denote 11-dimensional objects.

Then, the gauge potential includes a potential $\hat{C}_{\hat{\mu}}{}^{\hat{a}\hat{b}}$

$$\hat{\mathcal{A}}_{\hat{\mu}} \equiv \tfrac{1}{2}\hat{\omega}_{\hat{\mu}}{}^{\hat{a}\hat{b}}\hat{M}_{\hat{a}\hat{b}} + \hat{e}_{\hat{\mu}}{}^{\hat{a}}\hat{P}_{\hat{a}} + \tfrac{1}{2}\hat{C}_{\hat{\mu}}{}^{\hat{a}\hat{b}}\hat{\mathcal{Z}}_{\hat{a}\hat{b}} + \bar{\hat{\psi}}_{\hat{\mu}\hat{\alpha}}\hat{Q}^{\hat{\alpha}}\,, \tag{1.45}$$

which appears in the $N=1$ $d=11$ supergravity action as a 3-form potential. One may wonder why there is no gauge field associated to the 5-form central charge. This is because it appears as the dual of the 3-form potential.

The theory was constructed in [46] via the Noether method. We give its action and symmetries in appendix (B). $N=1$ $d=11$ supergravity has the very particular feature that, unlike the $d<11$ cases, is the *only* supersymmetric field theory that can be constructed in eleven dimensions [52, 53].

Note that we have not included a gauge field for the 5-form quasi-central charge. This is because it appears as the *dual* field of the 3-form potential. Therefore, $N=1$ $d=11$ supergravity can couple to a 2-brane and, through the dual 6-form potential, to a 5-brane. The quasi-central charges that appear in the superalgebra correspond to these objects and there are classical solutions associated to them. One of the main properties of these solutions is that they are half supersymmetric.

1.4 Kaluza-Klein dimensional reduction

The idea of extra dimensions was born when Kaluza [98] proposed a five dimensional theory of General Relativity as a candidate to unify gravity and electromagnetism. Later on, Klein proposed that, in this framework, the electric charge could be quantized if the extra dimension was a circle [102]. This is what we usually call *compactification*.

The world we observe seems to be four-dimensional. This is true at least up to the energy scale we are able to measure (∼TeV's). Then, if Nature is really five- (or more) dimensional, there must be some "mechanism" ensuring that the fifth dimension cannot be noticed at least up to the scale of TeV's. This can be achieved by considering that the extra dimension is compact and very small.

Considering that the extra dimensions are compact, the higher-dimensional fields can be expanded in Fourier series. This gives rise to an infinite tower of lower-dimensional fields with different masses, of which only one is massless. The contribution of the massive modes *at low energies* can be neglected if the extra dimensions are sufficiently small. Then, only the zero modes are kept and one arrives to an effectively lower-dimensional theory. This is the notion of *dimensional reduction*[11]. Let us show the idea in detail by the use of a simple example: a free 5-dimensional massless, complex scalar field $\hat{\phi}$[12].

The action that describes the system is

$$\hat{S} = \frac{1}{\hat{\kappa}} \int d^5\hat{x} \left(\partial\hat{\phi}\right)^2 , \tag{1.46}$$

where $\hat{\kappa}$ is a coupling constant. This action leads to the sourceless Klein-Gordon equation

$$\hat{\nabla}^2 \hat{\phi} = 0 . \tag{1.47}$$

Let us *compactify* our theory on a circle down to four dimensions. The coordinates split as $\hat{x}^{\hat{\mu}} = \{x^\mu, z\}$, where z parametrizes a periodic space, for example a circle of radius ℓ. As the space is periodic, any field $\hat{\phi}$ defined over it satisfies

$$\hat{\phi}(x,z) = \hat{\phi}(x, z + 2\pi\ell) , \tag{1.48}$$

and therefore it can be expanded in Fourier series as

$$\hat{\phi}(x,z) = \sum_{n\in\mathbb{Z}} e^{2\pi i n z/\ell} \, \phi^{(n)}(x) . \tag{1.49}$$

Therefore, we have infinite d-dimensional $\phi^{(n)}$ fields (modes) with masses proportional to n/ℓ. If we substitute the expansion (1.49) in (1.47), we find that each mode satisfies a 4-dimensional Klein-Gordon equation for massive fields

$$\left[\nabla^2 - \left(\tfrac{2\pi n}{\ell}\right)^2\right] \phi^{(n)}(x) = 0 . \tag{1.50}$$

Let us now *reduce* the theory. We take the radius of the circle to be sufficiently small, *i.e.* we look at the system from a distance much larger than the radius.

---

[11]Note that compactification and dimensional reduction are different concepts. Compactification means that some dimensions are taken to be compact, while reduction means that, due to certain considerations (small size...), the higher dimensional theory is *effectively* lower dimensional. When we compactify a theory, we usually consider that the compact space is small enough to reduce the theory, but, in general, a compactification does *not* imply a reduction.

[12]In this section, hatted objects are 5-dimensional while no hats are used for 4-dimensional objects.

Let us consider that its size is of the order of the Planck length. Then, the masses of all modes with $n \neq 0$ will be of order of the Planck mass, and therefore *at low energies* we can neglect these modes and consider only the massless one ($n = 0$), which is *independent* of the internal direction. Retaining only this mode is equivalent to take the limit $\ell \to 0$. Hence, the reduced theory will not depend on the internal direction and will be effectively d-dimensional.

Finally, after integration on z, the 5-dimensional action (1.46) is rewritten as a 4-dimensional action

$$S = \frac{1}{\kappa} \int d^4\hat{x} \left(\partial\hat{\phi}^{(0)}\right)^2 , \tag{1.51}$$

where $\kappa = \hat{\kappa}/2\pi\ell$ is interpreted as the effective 4-dimensional coupling constant.

In the case of gravity this analysis cannot be straightforwardly applied, as the Fourier modes cannot be interpreted as 4-dimensional metrics, but the underlying idea is exactly the same. The Fourier expansion leads to an infinite tower of massive modes which combine via a Higgs mechanism and represent massive spin-2 particles, *i.e.* massive gravitons. If the size of the circle is of the order of the Planck length, their massess are of the order of the Planck mass, and therefore can be neglected.

Then, we are left with the zero mode, a 5×5 symmetric matrix. It can be decomposed into a $4{\times}4$ symmetric matrix which we interpret as the 4-dimensional graviton, 4-dimensional vector A_μ and a scalar k. This decomposition can be seen in the 5-dimensional Vielbein as

$$\left(\hat{e}_{\hat{\mu}}{}^{\hat{a}}\right) = \begin{pmatrix} e_\mu{}^a & A_\mu \\ 0 & k \end{pmatrix} , \qquad \left(\hat{e}_{\hat{a}}{}^{\hat{\mu}}\right) = \begin{pmatrix} e_a{}^\mu & -A_a \\ 0 & k^{-1} \end{pmatrix} , \tag{1.52}$$

where $A_a = e_a{}^\mu A_\mu$.

One of the problems of the original Kaluza-Klein idea to unify gravity and electromagnetism, is that they were not able to explain the presence of the scalar field k.

In the context of string/M-theory, the spacetime we have to deal with is 10/11-dimensional. The KK reduction provides a mechanism to reduce the theories down to four dimensions. One generically chooses the compact space to be a direct product of circles, *i.e.* a torus. If the size of the torus is small enough, the massive modes can be neglected and the theory is effectively lower-dimensional. In this context, the scalar field is no longer a problem but plays the role of the *dilaton.*

In appendix B we give some standard dimensional reductions of 11-dimensional supergravity which are of our interest, such as the reduction on a circle, which leads to type IIA supergravity.

1.5 String/M theory and supergravity

We have shown how to construct supergravity theories and we have explained many of their properties. These theories are interesting by themselves and have been extensively studied in the literature. However, they turn out to be specially important because they describe the *low energy limit* of superstring theories.

String theory arises as a theory based on the consideration that the building blocks of Nature are strings instead of pointlike particles. Different particles can be seen as different oscillation modes of the strings. A possible way to explain why we have not been able to notice the spatial extension of the strings is to consider that it can only be observed at energies higher than those that can be reached in present-day experiments.

A fundamental property of string theory is that it can be consistently quantized. The quantization of the simplest model, the bosonic string, leads to an spectrum that includes a massless two-index field, which can be decomposed in its symmetric part (a massless spin-2 field), antisymmetric part (known as the Kalb-Ramond field) and the trace, a scalar field, known as the dilaton. A crucial point is that the a massless spin-2 field can be identified with the gauge boson of the gravitational interaction, the graviton. Then, the quantized version of the bosonic string leads to a quantum theory of gravity.

Requiring Poincaré invariance, the quantum theory turns out to be consistent only in 26 spacetime dimensions, *i.e.* the bosonic string propagates in a 26-dimensional target space. In principle, this does not represent a problem, since we can consider, *e.g.* that the 22 extra dimensions are so small that cannot be observed in current experiments.

One of the problems of the bosonic string is that it only contains bosons, and therefore cannot be a theory of everything. Furthermore, the spectrum also includes a tachyon, a particle which signals an instability in the vacuum of the theory.

Fermions can be introduced in the bosonic string by introducing supersymmetry. Moreover, it turns out that the tachyon can be consistently removed from the spectrum via the GSO projection. Supersymmetric string theories are known as *superstring theories*. If Poincaré invariance is required, the quantum theories are only consistent in ten spacetime dimensions. Apart from non-critical theories, only five ten-dimensional tachyon-free superstring theories can be constructed: type IIA, type IIB, type I, heterotic $E_8 \times E_8$ and heterotic $SO(32)$, which differ in their field contents and the amount of spacetime supersymmetry.

The low energy behaviour of superstring theories is described by supergravity theories, *i.e.* These field theories describe the low energy dynamics of the massless fields of the string spectrum, and a lot of information about superstring theories can be extracted from the study of their corresponding supergravity theories[13]. This is the reason why supergravity theories are so important and why their study is essential to understand string theory.

1.5.1 String dualities and M-theory

A *duality* is a transformation that relates different regimes of a theory or even different theories. In the last decade it was found that the five different superstring theories are connected through duality transformations. In addition, these properties are also reflected in the supergravity limit, and provide useful tools to study the structure of string theory. There are two main types of duality in string theory: T-duality and S-duality.

T-duality is a transformation which relates different compactifications of string theory. Let us consider the simplest example: the compactification on a circle. A closed string theory, say A, is T-dual to another string theory, say B, if theory A compactified on a circle of radius R_A is equivalent to theory B compactified on a circle of radius R_B given by

$$R_B = \frac{\alpha'}{R_A}, \tag{1.53}$$

[13]For instance, solutions to the supergravity equations of motion provide consistent target space backgrounds where strings can propagate.

upon exchange of Kaluza-Klein and winding modes[14]. Higher-dimensional internal spaces can also be considered, and the T-duality group is enlarged by taking into account the symmetries of the internal space. At the level of the superstring theories, type IIA and type IIB are T-dual to each other, as well as heterotic $SO(32)$ and heterotic $E_8 \times E_8$.

S-duality is a non-perturbative duality transformation that relates strong and weak coupling regimes. Basically, it states an equivalence between a string theory at coupling g_s and (perhaps another string theory) at coupling $1/g_s$. Therefore, it provides a powerful tool to extract information of the non-perturbative regime of a theory through the study of a perturbative regime (the S-dual picture). It is thought that type I and heterotic $SO(32)$ superstring theories are S-dual to each other, while type IIB is S-self-dual.

The strong coupling pictures of type IIA and heterotic $E_8 \times E_8$ are quite different because they are believed to be described not by a superstring theory but by an eleven-dimensional theory.

From the supergravity point of view, we know that type IIA supergravity can be derived from $N = 1$ $d = 11$ supergravity upon a Kaluza-Klein reduction on a circle. The radius of the eleventh dimension is proportional to the string coupling constant (the value of the dilaton). At weak coupling, the size of the eleventh dimension is very small and the theory is effectively ten-dimensional. However, at infinite coupling, the eleventh dimension opens up, such that the strong coupling limit of type IIA supergravity is $N = 1$ $d = 11$ supergravity [175].

Horăva and Witten [83] found that something similar occurs in the case of the heterotic $E_8 \times E_8$ theory, whose strong coupling limit is believed to be described by $N = 1$ $d = 11$ supergravity on $S^1/\mathbb{Z}_2$.

All this seems to indicate that the five superstring theories are connected to each other and, indeed, it is believed that they are different perturbative limits of a unique underlying theory, known as *M-theory*. However, microscopic degrees of freedom of this theory are still unknown, One of the few things that we know about it, is that its low energy limit is described by $N = 1$ $d = 11$ supergravity.

An outcome of the study of string dualities was the discovery of *D-branes* [134]. These objects are the carriers of the Ramond-Ramond charges, and form

[14] $l_s = \sqrt{\alpha'}$ is the string length.

a new class of non-perturbative extended solitonic objects which also appear in string theory. These objects are dynamical hypersurfaces on which open strings can end. Type II superstring theories (which are theories of closed strings) contain D-branes, so they can also contain open strings, but these must always end on a D-brane. In the non-perturbative spectrum, type IIA/B superstring theory contain Dp-branes with p even/odd.

There are non-renormalization theorems which imply that D-branes must have a faithful description in the corresponding low-energy limit of the theory, *i.e.* at the supergravity level. In this limit, they correspond to solutions which describe extended objects. Precisely, type IIA supergravity allows for p-brane solutions with p even, while type IIB includes those with p odd.

Many non-perturbative properties of superstring theory, such as supersymmetry or dualities, are present at the supergravity level, and provide very useful tools to explore the structure of string theory. The study of supergravity theories may provide new insights in string/M-theory and may help to understand it better. In this thesis, we study supergravity theories with this purpose.

1.6 Massive supergravity

Gauged supergravities contain a scalar potential which follows from the gauging of the R-symmetry group. Generically, the vacuum of these theories is no longer Minkowski but anti-de Sitter spacetime or a domain wall solution. This does not only occur in gauged supergravities: there is another kind of supergravity theories that also contain a scalar potential, but where no symmetry has been gauged. They are constructed via a deformation of standard supergravity theories through the introduction of a parameter to be interpreted as a *mass* for certain fields[15], and so they are commonly known as *massive supergravities*. The label *massless* will be used to refer to the standard ones.

As the deformations which lead to massive supergravity theories are not a consequence of the gauging of a symmetry group of the theory, the motivation for their construction is, in principle, conceptually different from that of gauged supergravities. However, they share a lot of properties, and therefore we may consider both kind of theories as members of the same class.

The most notorious example of a massive supergravity theory was found

[15]It is, however, strange to have a massive supergravity gauge field. We will comment on this later and see that it is perfectly reasonable in this kind of theories.

by Romans in [140], where, up to quartic fermion terms, a new supergravity theory was constructed via a deformation of type IIA supergravity[16]. This theory introduced no new degrees of freedom and respected gauge and supersymmetry invariance iff these transformations were slightly modified in order to accommodate the deformation, represented by a parameter m with the dimensions of a mass. The new theory, known as *Romans' theory* or *massive type IIA supergravity*, shows many differences with respect to standard type IIA, such as scalar potential for the dilaton, a mass term for the Kalb-Ramond field or a spontaneous breaking of (super)symmetry such that no maximally (super)symmetric vacuum solutions can be found.

Many questions might arise at this point. What is the effect of the scalar potential? Is it reasonable to have a massive Kalb-Ramond if it is one of the fields of a supergravity multiplet in which all the fields zero mass? What is the vacuum of the theory? Let us leave our questions unanswered by the moment.

Romans' theory did not receive much attention until Polchinski [134] argued that this theory is closely related to the existence of the 9-form potential of type IIA superstring theory. Further support to this suggestion was given in [15], where type IIA supergravity is deformed to accommodate a 9-form potential instead of a parameter. We refer to this construction as *BRGPT theory*, and we will see that it is a strong candidate to describe the low energy limit of type IIA superstrings. It was also suggested in [134, 136] that the expectation value of the dual of the 10-form field strength associated to the 9-form potential was basically the mass parameter m used by Romans to deform type IIA supergravity.

In the following subsection we are going to show how to construct Romans' theory at the bosonic level and its relation with superstring theory, such that the questions above will find an answer. This analysis will help us to understand the basic properties of massive supergravities and to establish their possible relation to string theory.

1.6.1 Romans' theory

If supergravity theories are the effective field theories describing the low energy behaviour of the different string theories, then *all* the D-branes must have a representation in the corresponding supergravity theory. However, type IIA/B supergravities only include those representations for the D0,2,4,6- and D(-1),1,3,5,7-branes, respectively, *i.e.* there is a supergravity p-brane solution for each of these branes. The problem is that there is no representation for

[16]It was completely constructed in [35] using superspace methods.

D8/D9-branes in type IIA/IIB supergravities. Both D8/D9-branes introduce no new degrees of freedom as they do not correspond to propagating states, but their presence in the corresponding theory has non-trivial effects. In the following we will analyze the case of the D8-branes[17].

□ Including the D8-brane in type IIA supergravity

A D8-brane is a 9-dimensional object which couples to a 9-form gauge potential $C_{(9)}$, whose field-strength, $G_{(10)}$, is 10-dimensional. It lives in a 10-dimensional spacetime, so, via Hodge duality, the 10-form field strength can be dualized into a 0-form field strength (*i.e.* a scalar field), say $M(x)$, which must satisfy a Bianchi identity, given by

$$dM(x) = 0\,, \tag{1.54}$$

implying that $M(x)$ must be a constant, say m. Therefore, the kinetic term for a D8-brane in a supergravity action will be proportional to m^2, and, as D-branes are represented in supergravity through Ramond-Ramond fields, it is reasonable to expect no dilaton factor for it in the string frame.

□ So... what are we looking for?

If we want to find a type IIA supergravity theory that includes *all* the even D-branes, we need to find a deformation of type IIA supergravity which allows for the presence of a constant parameter in its action, in which must appear, up to a dilaton factor, a constant kinetic term. This parameter represents the D8-brane.

A theory satisfying these requirements is precisely Romans' theory [140], Its bosonic action in the string-frame reads

[17] D9-branes are spacetime filling branes, and therefore open strings are allowed to end anywhere. In this sense, a D9-brane is basically a Neumann boundary condition. Cancellation of tadpoles and anomalies in type IIB with D9-branes requires their number to be 32 and orientifolding the theory, such that the theory presents two orientifold 9-planes which cancel the total Ramond-Ramond charge of the D9-branes. This way one ends up with type I string theory.

$$
\begin{aligned}
S_{Romans} &= \frac{1}{16\pi G_N{}^{(10)}}\int d^{10}x\,\sqrt{|g|}\,\{e^{-2\phi}\left[R-4\left(\partial\phi\right)^2+\tfrac{1}{2\cdot 3!}H^2\right] \\
&\quad -\left[\tfrac{1}{2}m^2+\tfrac{1}{2\cdot 2!}G^2_{(2)}+\tfrac{1}{2\cdot 4!}G^2_{(4)}\right] \\
&\quad -\tfrac{1}{144}\tfrac{1}{\sqrt{|g|}}\,\epsilon\left[\partial C_{(3)}\partial C_{(3)}B+\tfrac{1}{2}m\,\partial C_{(3)}B^3+\tfrac{9}{80}m^2B^5\right]\Big\}\,,
\end{aligned}
\tag{1.55}
$$

where $G^{(2)}$ and $G^{(4)}$ are the RR 2- and 4-form field strengths, defined by

$$
\begin{aligned}
G_{(2)} &= 2\partial C_{(1)}+mB\,, \\
G_{(4)} &= 4\partial C_{(3)}+4\,C_{(1)}\,H+3mB^2\,,
\end{aligned}
\tag{1.56}
$$

and

$$
H=3\partial B\,. \tag{1.57}
$$

is the NSNS 3-form field strength. Note that this action contains the kinetic term we were looking for: it is constant and has no dilaton factor in the string frame. These conditions do not conclude but only support the idea that Romans' theory includes D8-branes. This would be confirmed if we found an 8-brane solution breaking one half of the supersymmetries with mass inversely proportional to the string coupling constant. We will come back to this issue later.

Since the theory has been deformed from type IIA supergravity, it seems reasonable that both field strengths and Lagrangian are now invariant under some new bosonic gauge transformations which are deformations of the massless ones. The new gauge transformations are

$$
\begin{aligned}
\delta B &= 2\partial\Lambda_{(1)}\,, \\
\delta C_{(1)} &= \partial\Lambda_{(0)}-m\Lambda_{(1)}\,, \\
\delta C_{(3)} &= 3\partial\Lambda_{(2)}-3mB\Lambda_{(1)}-H\Lambda_{(0)}\,.
\end{aligned}
\tag{1.58}
$$

The supersymmetry transformation rules that leave invariant the action for Romans' theory are also a deformation of the usual type IIA supersymmetry transformations. The new rules for the fermions of the theory are given by

$$\begin{aligned} \delta_\epsilon \psi_\mu &= \left\{\partial_\mu - \tfrac{1}{4}\left(\not{\omega}_\mu + \tfrac{1}{2}\Gamma_{11}\not{H}_\mu\right)\right\}\epsilon + \tfrac{i}{8} e^{\phi} \Sigma_{n=0}^{n=2} \tfrac{1}{(2n)!}\not{G}^{(2n)}\Gamma_\mu \left(-\Gamma_{11}\right)^n \epsilon\,, \\ \delta_\epsilon \lambda &= \left[\not{\partial}\phi + \tfrac{1}{2\cdot 3!}\Gamma_{11}\not{H}\right]\epsilon + \tfrac{i}{4} e^{\phi} \sum_{n=0}^{n=2} \tfrac{5-2n}{(2n)!}\not{G}^{(2n)}\left(-\Gamma_{11}\right)^n \epsilon\,, \end{aligned} \tag{1.59}$$

where we have identified

$$G^{(0)} = m\,. \tag{1.60}$$

If $m = 0$ is set everywhere, standard type IIA supergravity is recovered (see appendix B).

We have seen how the theory is and its *possible* relation to superstring theory via the inclusion of D8-branes in type IIA supergravity. Let us try to go further in our understanding.

□ What's the effect of 'mass-deforming' type IIA supergravity?

Let us now come back to the massive gauge transformations given in (1.58). The variations containing $\Lambda_{(0)}$ and $\Lambda_{(2)}$ are nothing but the massless gauge transformations, *i.e.* the massive theory is also invariant under $\Lambda_{(0)}$ and $\Lambda_{(2)}$ transformations. However, it is no longer invariant under $\Lambda_{(1)}$ transformations. These are invariances iff $\delta C_{(1)}$ and $\delta C_{(3)}$ are deformed. The required deformation (1.58) implies that the vector field $C_{(1)}$ can be completely gauged away by absorbing it into the Kalb-Ramond field[18], with the consequence that the field strength for $C_{(1)}$ becomes a *mass term* for B, and the action reads

$$\begin{aligned} S_{Romans} = \tfrac{1}{16\pi G_N{}^{(10)}} \int d^{10}x \sqrt{|g|}\,\Big\{ & e^{-2\phi}\left[R - 4\left(\partial\phi\right)^2 + \tfrac{1}{2\cdot 3!}H^2\right] \\ & - \left[\tfrac{1}{2}m^2 + \tfrac{1}{4}m^2 B^2 + \tfrac{1}{2\cdot 4!}G_{(4)}^2\right] \\ & - \tfrac{1}{144}\tfrac{1}{\sqrt{|g|}}\,\epsilon\left[\partial C_{(3)}\partial C_{(3)} B + \tfrac{1}{2} m\,\partial C_{(3)} B^3 + \tfrac{9}{80} m^2 B^5\right]\Big\}\,, \end{aligned} \tag{1.61}$$

Those fields that behave as $C_{(1)}$ are usually called *Stückelberg fields*, and are auxiliary fields whose gauge transformations are as that of $C_{(1)}$ in (1.58).

[18]We usually say that the B field "eats" the vector.

Had we started with action (1.61), which is *not* gauge invariant, and had we wanted to make the theory formally invariant under this kind of transformations, we would have had to introduce the $U(1)$ Stückelberg field $C_{(1)}$. In this way, the action can be written in a formally gauge invariant manner. We would have arrived then to action (1.55). This is the simplest example of a mechanism known as *Stückelberg mechanism* that will be at work in all the cases we are going to study.

The physical interpretation of the gauging away of the vector is that in the physical spectrum of the theory there are no there are no quantum excitations associated to this vector field and the quantum excitations associated to the 2-form are massive.

We can wonder how this can happen in supergravity theories since all the fields in the supergravity multiplet should have the same mass (zero). The reason why, is that supersymmetry is partially and spontaneously broken: massive supergravity theories as formally invariant under a certain modification of the full supersymmetry transformations of the massless theory. However, the vacuum of these theories breaks part of the symmetry and in that vacuum the supergravity multiplet becomes reducible into a massless supergravity multiplet and massive matter multiplets. We have written the theory in a form in which all the gauge symmetries of the massless theory are formally present, but in the vacuum those symmetries responsible for the masslessness of the 2-form are spontaneously broken. This is the reason why we call this theory a massive supergravity and why we say that the supersymmetry is spontaneously broken.

Although we have not written the complete theory, as the fermionic contributions have been omitted, we believe that a Stückelberg mechanism takes place also at the fermionic level. The theory is formally invariant under $N = 2$ supersymmetry transformations, and so there are two gravitinos and two dilatinos. We think that one of the gravitinos is massive, whose mass term is written as a kinetic term for one of the dilatinos. This dilatino is a fermionic Stückelberg field. Both bosonic and fermionic Stückelberg mechanisms together give rise to a *super-Stückelberg mechanism* such the $N = 2$ $d = 10$ supergravity is reduced, via the spontaneous breaking of supersymmetry, into a $N = 1$ $d = 9$ supergravity multiplet plus massive matter multiplets which include both Kalb-Ramond and gravitino massive fields.

□ What's happening in the vacuum?

It is easy to see that Minkowski spacetime is *not* a solution of Romans' theory, so... what is then the new vacuum solution?

We have seen that the D8-brane is represented in the action for Romans' theory (in the string frame) through a kinetic-like term of the form

$$\int d^{10}x\sqrt{|g|}\left[-\frac{1}{2}m^2\right]. \tag{1.62}$$

This kinetic term is different from the usual ones (those for the other D($2n$)-branes) in the sense that it is *fixed*, *i.e.* it is *not* a function of the coordinates but a constant, so it does not lead to any additional degrees of freedom, and therefore is not ruled out automatically by supersymmetry considerations[19]. In the Einstein frame, the kinetic term (1.62) reads

$$\int d^{10}x\sqrt{|g|}\left[-\frac{1}{2}m^2 e^{5\phi/2}\right], \tag{1.63}$$

i.e. it is a scalar potential for the dilaton field ϕ. It is not a proper cosmological constant in the sense that it is not *constant* in the Einstein frame, and, in fact, there is no 10-dimensional (anti-)de Sitter solution. However, it is commonly referred to as the cosmological constant term of Romans' theory and we will also adopt this terminology.

The key point is the following: m represents the contribution of the D8-brane to the theory, so, as it is a parameter of the theory and therefore its value is fixed, it is not possible to switch off the D8-brane. Hence, *all* the solutions of the theory must feel the presence of the D8-brane[20], and in this sense one can interpret the D8-brane as a background for the massless type IIA supergravity, *i.e.*

> *Romans' theory is nothing but massless type IIA supergravity in a background of a D8-brane.*

[19]A similar argument was used many years ago in [7, 54] to introduce a cosmological constant in four-dimensional field theories by making use of a four-form field strength. In [7] a 4-dimensional massive theory was obtained performing a Kaluza-Klein dimensional reduction of a 5-dimensional theory with a 4-form field strength, whose dimensional reduction was dualized to give a cosmological-type term. Hawking used this argument in the search for a mechanism for the variation of the cosmological constant [81].

[20]This is what happens, for example, in General Relativity with non-zero cosmological constant. All solutions of this theory feel the presence of the cosmological constant, the reason being that the cosmological constant is a free parameter of the theory.

With this interpretation[21], we also expect the vacuum solution to feel the presence of the D8, and in fact one could argue naively that the D8-brane itself is a vacuum solution. We are going to think this way, but, as we will see later, it will turn to be a non-trivial problem. Let us first see what our 'vacuum solution' is like.

The D8-brane solution is a domain-wall spacetime (a nine-dimensional hypersurface in a ten-dimensional spacetime) with non-trivial metric $g_{\mu\nu}$ and dilaton ϕ, whose supergravity solution in the string frame is

$$\begin{aligned} ds^2 &= H^{-1/2} d\vec{x}^2_{(9)} - H^{1/2} dy^2 \,, \\ e^{\phi} &= H^{-5/4} \,, \end{aligned} \tag{1.64}$$

where $d\vec{x}^2_{(9)}$ is Minkowski spacetime in nine dimensions representing the world-volume of the D8, y is the transverse direction and $H(y)$ is given by[22]

$$H(y) = \pm m(y - y_0) \,, \tag{1.65}$$

with y_0 an arbitrary integration constant. The Killing spinor equations (1.59) for the 8-brane solution given above are solved for

$$\epsilon = H^{-1/8}\, \epsilon_0 \,, \qquad \Gamma_y\, \epsilon_0 = \pm \epsilon_0 \,. \tag{1.66}$$

The sign in (1.65) depends on the "chirality" choice for the spinor ϵ in (1.66). Reality of the metric implies that H must be positive, and therefore ϵ must change "chirality" at $y = y_0$. We understand that this is possible because the spinor blows up at this point, which, in principle, seems acceptable because also the metric is singular at $y = y_0$. In this sense, the theory admits a solution for which

$$H(y) = m|y - y_0| \,, \tag{1.67}$$

which is a continuous function of y with a delta function singularity at $y = y_0$, where also the curvature tensor finds a delta singularity. This suggests the

[21] It is convenient to anticipate that this interpretation is not the right one, but it helps in our understanding and, as we will see, it is very close to the correct interpretation.

[22] As for every D-brane, H is a harmonic function of the transverse space. In the case at hand, H must also satisfy $\partial_y H = m$.

interpretation of y_0 as the position of the 8-brane along the y-direction. This is in fact the solution one finds considering an action consisting on that for a 8-brane placed at $y = y_0$ and that for the bulk (Romans' Theory).

The spinor solution in (1.66) shows that the 8-brane solution breaks one half of the supersymmetries, as expected from the fact that it represents (at the classical level) one of the string theory D-branes. This result also agrees with the idea that in Romans' theory (and, in general, in all massive supergravities) the supersymmetry is spontaneously broken, *i.e.* the theory has only $N = 1$ supersymmetry (the supersymmetry of the D8-brane vacuum) but, formally, the theory is invariant under $N = 2$ local supersymmetry transformations. Some gauged supergravities also show this massive behaviour when they do not have a maximally supersymmetric background. This will be the case of the gauged $d = 8$ supergravities we will study in chapter 2.

The 8-brane solution also presents $ISO(1,8)$ invariance, as the translational invariance is broken in the y-direction. As the theory is formally invariant under $ISO(1,9)$ transformations, *i.e.* also the g.c.t.'s are spontaneously broken.

□ A problematic vacuum solution

When we first wrote the action for Romans' theory in (1.55) we supposed that the theory included a representation for D8-branes, and it was a supposition because we did not know if the theory admitted an 8-brane solution breaking one half of the supersymmetries. We have shown that this is indeed the case. We also supposed that the tension of this object was inversely proportional to the string coupling constant. Is it really so?

The fact that geometrical conserved charges associated to a source can only be defined and computed asymptotically leads to the notion of supergravity vacuum as the spacetime solution to which a certain class of spacetimes asymptote. Every conserved geometrical charge, including the mass, can only be defined and computed with respect to this vacuum, which, in this sense, asymptotes to itself.

The 8-brane solution (1.64) asymptotes to itself, and so it can be considered as a vacuum solution. This implies that, applying the Abbott & Deser approach, the mass one computes for this spacetime is, by construction, zero, and so it is not useful to calculate the mass of the 8-brane. There is an alternative way to determine the mass for this spacetime: T-duality. If this 8-brane solution is as a field theory realization of the D8-brane of type IIA superstring theory, then T-duality must relate it to the 7- and 9- brane solutions of type IIB supergravity, and, via a chain of T-dualities, to any other brane. This was used in [15] to show that, as expected, the mass of the 8-brane solution is inversely proportional to the string coupling constant, and so it was proposed the identification of the 8-brane solution with the supergravity solution associated to the D8-brane. However, this identification does not refer exactly to (1.64), but to the single 8-brane solution of a theory which is a subtle but crucial reformulation of Romans' theory. Let us see it.

□ One step further

An important advance in the understanding of the role of the 9-form potential in supergravity was performed when Romans' theory was reformulated in terms of a Ramond-Ramond 9-form potential instead of a constant parameter [15], a theory that we refer to as *BRGPT theory*. Moreover, as string theory is supposed to have no free parameters, it is not satisfactory to have one in the action of a possible candidate to describe the low energy limit of type IIA superstrings.

Such a formulation implies a non-trivial generalization of Romans' theory. The solutions of the new theory are conceptually the same as those of Romans', but in the BRGPT theory there is no longer a constant parameter m but a *piecewise constant* function, say M, which has an important implication: one can have solutions in which the value of the cosmological constant is different in different regions of spacetime. The discontinuities are 9-dimensional topological defects, *i.e.* domain walls, which act as sources of the 9-form potential and are interpreted as D8-branes. A special case of this multi 8-brane solution was previously found in [135].

The single 8-brane solution of BRGPT theory is the one which, via T-duality, was shown in [15] to have mass inversely proportional to the string coupling constant. Moreover, while in Romans' theory the mass parameter is a parameter of the action and therefore it cannot be switched off (such that Minkowski is not a solution), in the BRGPT theory the 9-form potential can be set to zero, and Minkowski is therefore a vacuum solution (in which there are no D8-branes).

□ But... what is really m?

It was argued in [136] that the constant parameter m is the gauge charge of the $D8$-brane. This is not exactly so, but m is directly related to the charge. Let us consider the single D8-brane solution in the formulation of massive

Type IIA Supergravity in terms of the 9-form potential. This solution has an associated 10-form field strength given by

$$G_{(10)} = \frac{M(y)}{\sqrt{|g|}}\,\varepsilon_{(10)}\,, \tag{1.68}$$

where $M(y) = \partial_y H$ is a piecewise constant function with the jump at $y = y_0$ (the position of the brane) and $\varepsilon_{(10)}$ is the 10-dimensional Levi-Civita pseudo-tensor.

Gauge field charges are generically computed by integrating the dual of the gauge field strength over a sphere containing the source in the transverse space. In the case at hand the transverse space is 1-dimensional, and so the sphere surrounding the source is the "0-dimensional sphere" S^0, an space formed by two points $y = \pm\ell$, each on one side of the D8. The gauge charge is then given by

$$Q_{D8} = \int_{S^0} {}^*G_{(10)} = M(+\ell) - M(-\ell)\,, \tag{1.69}$$

where $\ell > y_0$. Reality of the metric requires M to be of the form

$$M = \begin{cases} m_1, & y < y_0, \\ m_2, & y > y_0, \end{cases} \tag{1.70}$$

implying $Q_{D8} = m_2 - m_1 \neq 0$. This shows the importance of the BRGPT formulation of massive type IIA supergravity in [15], as in Romans' Theory the value of m is *the same* in all the spacetime, implying that the charge of the D8 is zero, *i.e.* "there is no brane"[23]. Therefore,

> *Romans' theory dictates the dynamics of the different regions of spacetime separated by a D8-brane, and the low energy limit of type IIA superstring theory is described by the BRGPT theory.*

If we consider a multi 8-brane solution by allowing delta singularities of H at $n+1$ ordered points $y = y_0 < y_1 < \ldots < y_n$, the picture is a spacetime with $(n+2)$ regions with different values of m separated by the $(n+1)$ 8-branes, with Romans' theory (with different m's) "living" in each of these regions.

[23]Romans' theory can also be interpreted as type IIA supergravity in the presence of a D8-brane placed *at infinity.*

It is worth noticing that D8-branes do not exactly interpolate between these regions, but simply separate them.

□ Dirac quantization condition

An 8-brane couples to a 9-form potential $C_{(9)}$ with 10-dimensional field strength $G_{(10)}$, whose dual is a 0-dimensional field strength associated to a "(-1)-form potential". The source of this potential is a "(-2)-brane", an object still not provided of physical meaning. Therefore, D8-branes are objects electrically charged with respect to the gauge field $C_{(9)}$, but there are no magnetically charged solutions. Hence, these objects do not satisfy a Dirac quantization condition, a condition satisfied by *all* D-branes but the D8 and the D9, which, at least, is a curious property.

1.7 Unification in eleven dimensions?

It is thought, perhaps ambitiously, that M-theory is powerful enough to unify the superstring theories. Such a unification should be reflected somehow in the low energy limit, *i.e.* it should be possible to relate the supergravity theories associated to the different string theories with the supergravity description of M-theory. Standard dimensional reduction on a torus together with dualities indicate that this seems to be the case. However, these 'tools' are not enough to give all gauged/massive supergravities an 11-dimensional origin. Among these, Romans' theory is the most notorious example, as it seems to have a deep connection with string theory. So

what is the 11-dimensional origin of gauged/massive supergravities?

Two essential ingredients must be taken into account to understand this problem: the 11-dimensional theory and the reduction scheme. We need to introduce somehow gauge couplings and/or mass parameters in order that the reduced theory is gauged/massive. Then, one could think of introducing them in any (or both) of the two ingredients. In chapter 2 we will deal with a possible generalization of the Kaluza-Klein reduction. We will deal with this possibility in chapter 2. The other possibility implies a modification of 11-dimensional supergravity. However, this theory seems to be unique, so... how can it be deformed? We will focus on this possibility in chapter 3.

Chapter 2

Getting masses from extra dimensions

Many ungauged/massless supergravities arise from 11-dimensional supergravity by applying a standard dimensional reduction on a torus. However, this reduction procedure does not lead to any gauged/massive supergravity if the higher dimensional theory is ungauged/massless. This is the case of 11-dimensional supergravity. We need to introduce gauge coupling constants and/or mass parameters in the reduced theory. Then, it is natural to ask if these parameters can be introduced through the reduction.

One way to do this is to perform a Kaluza-Klein reduction on a sphere instead of a torus. This has served to obtain many gauged supergravities from higher-dimensional ungauged supergravities. Some remarkable examples are the S^5 reduction of type IIB supergravity [49], and the S^4 [49, 124] and S^7 [49, 173] reductions of $N = 1, d = 11$ supergravity. These reductions lead to gauged maximal supergravities in five, seven and four dimensions, respectively.

However, Kaluza-Klein sphere reductions are only consistent works in some cases. Then, it is natural to ask if

is it possible to generalize the Kaluza-Klein reduction procedure?

A possible answer to this question was provided by Scherk & Schwarz. Their relevant work in this respect [150, 151] was initially motivated by the study of procedures that could generate theories with spontaneously broken supersymmetry from higher dimensional theories with unbroken supersymmetry. Their method, known as *generalized dimensional reduction*, extends the standard dimensional reduction procedure by the introduction of a certain dependence of some fields on the internal coordinates. A crucial point in the Scherk-Schwarz technique is that, although the higher dimensional fields depend on the internal coordinates, the reduced theory is completely independent of them.

In these reductions, the dependence on the internal coordinates is introduced by gauging a global symmetry of the theory. Although a compact symmetry was used in the original work, other global symmetries of a theory are also valid for this purpose. The result is usually that parameters with dimensions of mass are introduced in the reduced theory. We will distinguish between two versions of generalized dimensional reduction, taking into account the kind of global symmetry one uses to introduce the dependence on the internal coordinates. If no geometrical origin is known for the global symmetry, the procedure will be referred to as *non-geometrical* or *SS1 reduction*. On the other hand, if the global symmetry is one of the symmetries of the compactification manifold (and therefore has a well-known geometrical origin) the reduction scheme will be referred to as *geometrical* or *SS2 reduction*.

The organization of the chapter is as follows. Section 2.1 is devoted to explain the basics of generalized dimensional reductions, both geometrical and non-geometrical. We comment first on the SS1 reduction and on some of its applications. After that, we use a toy model to explain the general features of the SS2 reduction. In section 2.2 we perform an SS2 reduction of 11-dimensional supergravity on a 3-dimensional manifold. This reduction leads to five $d = 8$ gauged maximal supergravities whose gauge groups are the non-compact subgroups of $SL(3, \mathbb{R})$, among which the standard gauged maximal supergravity in $d = 8$ [144] is included. We will classify the different theories according to the Bianchi classification of 3-dimensional Lie groups. In section 2.3 we construct the most general half-supersymmetric domain wall solution to these five gauged supergravities. Section 2.4 contains the uplifting to 11 dimensions of the domain wall solution and its relation to already known solutions. Our conclusions are presented in section 2.5. Appendix C contains some basic material on the classification of 3-dimensional Lie algebras.

2.1 Generalized dimensional reductions

2.1.1 Non-geometrical generalized dimensional reduction

Generalized dimensional reduction associated to global symmetries with no geometrical origin can be seen as the answer to the question 'how do we dimensionally reduce multivalued fields?' Let us consider a toy model which exhibits the general features of this kind of reduction[1]: a real scalar field $\hat{\varphi}$ taking values on a circle of radius m. In practice, one takes a field living on the real line and identifies

$$\hat{\varphi} \sim \hat{\varphi} + 2\pi m\,. \tag{2.1}$$

Now we consider that one of the coordinates (say z) is compact (*e.g.* a circle). A single-valued field has to be a periodic function of the compact coordinate. On the other hand, a multi-valued field is allowed to take a different value as long as it is a multiple of $2\pi m$ because both values of the field are assumed to represent the same Physics

$$\hat{\varphi}(x, z + 2\pi\ell) = \hat{\varphi}(x, z) + 2\pi N m \sim \hat{\varphi}(x, z)\,. \tag{2.2}$$

In standard Kaluza-Klein reduction one only considers single-valued fields, so that the needed Fourier decomposition of any field $\hat{\phi}$ reads

$$\hat{\phi}(x, z) = \sum_{n\in\mathbb{Z}} e^{2\pi i n z/\ell}\, \phi^{(n)}(x)\ . \tag{2.3}$$

Dimensional reduction then means keeping the massless modes, *i.e.* $\phi^{(0)}$, only. In the case of our multivalued field $\hat{\varphi}$, the above Fourier expansion is enhanced to

$$\hat{\varphi}(x, z) = \frac{mNz}{\ell} + \sum_{n\in\mathbb{Z}} e^{2\pi i n z/\ell}\, \varphi^{(n)}(x)\ , \tag{2.4}$$

The term linear in z is the responsible for the multivaluedness. This term is non-dynamical, and therefore it introduces no new degrees of freedom, and therefore the value of N cannot change (at least, classically). The above field configuration is topologically non-trivial: the field is wound N around the compact direction. N labels the different *topological sectors*, and is given by

$$N \;=\; \lim_{x\to\infty} \frac{1}{2\pi\ell m} \oint d\hat{\varphi}\ , \tag{2.5}$$

[1]In this section we use hats for d-dimensional objects and no hats for $(d-1)$-dimensional objects.

which is nothing but the *winding number*.

The dimensional reduction is performed keeping only the zero modes, *i.e.* the massless ones. Then, we consider the Ansatz

$$\hat{\varphi}(x,z) = \frac{mNz}{\ell} + \varphi^{(0)}(x)\ . \tag{2.6}$$

Now, the action for a field living on an S^1 is always invariant under arbitrary shifts of the field, even if the field is to be identified under discrete shifts. This then ensures that the lower dimensional theory does not depend on z. The term linear in z gives rise to a term which plays the role of a mass term.

In the original work of Scherk and Schwarz [150], an Abelian $U(1)$ phase symmetry acting on the spinors was used, such that a spontaneous breaking of supersymmetry was induced in the reduced theory. In its original form, the SS1 was also applied in, *e.g.* the reduction on a 6-torus of the effective action of the heterotic string on a 6-torus [137, 159] to obtain gauged $N = 4$ $d = 4$ supergravity with a positive semidefinite potential.

Other compact global symmetries that have been used to perform SS1 reductions are axion shifts, such that the axion is allowed to have a linear dependence on the compactification coordinate [15]. In addition to the spontaneous supersymmetry breaking, gauge symmetries of some gauge fields are also spontaneously broken in the reduction, with the corresponding appearance of mass terms. Moreover, it is possible to set the masses to zero such that the ungauged/massless theories are recovered. In all these cases, the reduction Ansatz only depends on one of the internal coordinates (apart from the dependence on the 'external' coordinates). Of course, it is possible to have higher dimensional internal spaces, but only one of the internal coordinates appears in the Ansatz.

The axion shift symmetry of type IIB supergravity was exploited in [15] to obtain a massive $N = 2, d = 9$ supergravity. This symmetry is part of a global $SL(2,\mathbb{R})$ of the $N = 2B, d = 10$ theory. If the full $SL(2,\mathbb{R})$ symmetry is exploited, a 3-parameter family of massive $N = 2$ $d = 9$ supergravity theories is obtained [122] (see also [66]), some of which are gauged supergravities [45].

It was shown in [122] that SS1 reductions could be performed by a gauging procedure, *i.e.* by the introduction of a gauge field associated to the global symmetry. However, as the global symmetry is local only in the internal coordinates, one must impose that the gauge field is non-vanishing and constant only in the internal directions. The fields which transform under the global symmetry are given this way a linear dependence on the internal coordinates in the reduction Ansatz.

Global *scaling* symmetries can also been employed in SS1 reductions. An interesting example of this case can be found in [106], where a 10-dimensional gauged supergravity theory was derived from $d = 11$ supergravity via an SS1 reduction which exploits a global scaling symmetry of the 11-dimensional equations of motion. The reduced theory is a massive deformation of type IIA supergravity such that the 1- and 3-form potentials become massive via a Stückelberg mechanism in which these fields "eat" the dilaton and NS-NS 2-form, respectively. However, the vector field of the theory is tachyonic and the theory might be unstable. Unlike Romans' theory, this massive supergravity has no domain wall 8-brane solution but admits a de Sitter one, which uplifted to 11 dimensions leads to flat space. As the global symmetry exploited in the reduction is *only* a symmetry of the equations of motion, one has to resort to the reduction of the equations of motion. Furthermore, there is no action from which one can derive the equations of motion of the reduced theory.

So far, we have seen many cases in which global symmetries (axion shift symmetries or scaling invariances) have been used to perform Scherk-Schwarz reductions. As noted previously, these symmetries have no known geometrical origin. We could now try to apply a similar procedure but considering symmetries with a well-known geometrical origin, *i.e.* symmetries of the compactification manifold.

2.1.2 Geometrical generalized dimensional reduction

The geometrical generalized dimensional reduction [151] corresponds to a reduction in which the global symmetries to be gauged are *symmetries of the compactification manifold.* This gauging is consistent only if every field in the theory carrying an internal index acquires a certain dependence on the internal coordinates in the reduction Ansatz. The main consequences of such a generalization of the KK Ansatz are that the KK vectors become non-Abelian and that the internal space is no longer a torus, but a manifold with the symmetry of the the KK vectors.

As a toy model, we are going to consider $\hat{d}$-dimensional gravity, which exhibits the general features of the reduction we are interested in. The action for our toy model is

$$\hat{S} = \int d^{\hat{d}}\hat{x}\sqrt{|\hat{g}|}\,\hat{R}\,, \tag{2.7}$$

This action is, by construction, invariant under general coordinate transformations (g.c.t.'s). These transformations are generated by the infinitesimal parameters $\hat{\xi}^{\hat{\mu}}$, which satisfy the algebra

$$\left[\delta_{\hat{\xi}_1}, \delta_{\hat{\xi}_2}\right] = \delta_{\hat{\xi}_3}\,, \tag{2.8}$$

where

$$\hat{\xi}_3^{\hat{\mu}} = 2\hat{\xi}^{\hat{\nu}}_{[2|}\,\partial_{\hat{\nu}}\,\hat{\xi}^{\hat{\mu}}_{|1]}\,. \tag{2.9}$$

Under a g.c.t., the $\hat{d}$-dimensional metric transforms as

$$\delta_{\hat{\xi}}\,\hat{g}_{\hat{\mu}\hat{\nu}} = -\hat{\xi}^{\hat{\rho}}\partial_{\hat{\rho}}\hat{g}_{\hat{\mu}\hat{\nu}} - 2\hat{g}_{\hat{\rho}(\hat{\mu}}\partial_{\hat{\nu})}\hat{\xi}^{\hat{\rho}}\,. \tag{2.10}$$

Let us now reduce our toy model down to d dimensions on an n-dimensional manifold ($\hat{d} = d + n$), the splitting of the coordinates being $\hat{x}^{\hat{\mu}} = \{x^{\mu}, z^m\}$. In standard dimensional reductions we truncate the spectrum and consider only the zero modes of a Fourier expansion of the higher-dimensional fields. These zero modes do not depend on the internal coordinates. Then, compatibility of the KK Ansatz with (2.10) implies an splitting of the infinitesimal parameters $\hat{\xi}^{\hat{\mu}}$ of the form

$$\left\{ \begin{array}{rcl} \hat{\xi}^{\mu}(\hat{x}) & = & \xi^{\mu}(x)\,, \\ \\ \hat{\xi}^{m}(\hat{x}) & = & -\alpha_n{}^m\lambda^n(x) + R_n{}^m z^n + a^m\,, \end{array} \right. \tag{2.11}$$

where $\alpha_m{}^n$ are some constants[2]. In the reduced theory, these g.c.t.'s correspond to

- G.c.t's in the reduced theory, with parameters ξ^{μ}.
- $GL(n,\mathbb{R})$ transformations with parameters $R^m{}_n$. These can be decomposed into $SL(n,\mathbb{R})$ rotations, that act in the obvious way on all the fields that have m, n indices, and $SO(1,1)$ rescalings.

[2]They can be set, without loss of generality, to $\delta_m{}^n$.

- Gauge transformations with parameters λ^m. These parameters commute, and so they act in an Abelian manner on the KK vectors $A^m{}_\mu$, *i.e.*

$$\delta_\lambda A^m{}_\mu = \partial_\mu \lambda^m \,. \tag{2.12}$$

- Global shifts of the internal coordinates a^m, that do not act on the lower dimensional fields.

The choice of z^m-dependence in (2.11) is the most general choice one can make in order to make it compatible with the KK Ansatz. However, this dependence can be more general: the constants $\alpha_m{}^n$ can be promoted to some functions of the internal coordinates, such that now we have

$$\left\{ \begin{array}{rcl} \hat{\xi}^\mu(\hat{x}) & = & \xi^\mu(x)\,, \\ \hat{\xi}^m(\hat{x}) & = & -(U^{-1})_n{}^m(z)\lambda^n(x) + R_n{}^m z^n + a^m\,, \end{array} \right. \tag{2.13}$$

where $U(z)$ is a $n \times n$ non-singular matrix. As (2.10) must hold for the choice (2.13), the reduction Ansatz for the Vielbein must now be dependent on the internal coordinates. Then, the Ansatz is of the form

$$\begin{array}{rcl} \left(\hat{e}_{\hat{\mu}}{}^{\hat{a}}\right) & = & \left(\begin{array}{cc} e_\mu{}^a & e_m{}^i A^m{}_\mu \\ 0 & U_m{}^n\, e_n{}^i \end{array} \right), \\ \left(\hat{e}_{\hat{a}}{}^{\hat{\mu}}\right) & = & \left(\begin{array}{cc} e_a{}^\mu & -(U^{-1})_n{}^m\, A^n{}_a \\ 0 & (U^{-1})_n{}^m\, e_i{}^n \end{array} \right), \end{array} \tag{2.14}$$

where the d-dimensional Vielbein $e_\mu{}^a$, vectors $A_\mu{}^m$ and scalars $e_m{}^i$ only depend on the x^μ coordinates.

□ Which symmetries do arise from the new higher dimensional g.c.t.'s?

As before, ξ^μ, λ^m and $R^m{}_n$ are the infinitesimal parameters of the g.c.t's, gauge and $GL(n, \mathbb{R})$ transformations, respectively. Again, the internal global shifts a^m do not act on the lower dimensional fields. The difference arises in the commutator of the gauge parameters, which, with the new Ansatz, leads to

$$\lambda^p(x) = f_{mn}{}^p\, \lambda^m(x)\lambda^n(x)\,, \tag{2.15}$$

where

$$f_{mn}{}^p \equiv -2(U^{-1})_m{}^r(U^{-1})_n{}^s\, \partial_{[r}U_{s]}{}^p\,. \tag{2.16}$$

The functions $f_{mn}{}^p$ can only depend on the internal coordinates, and so, if equation (2.15) is to be consistent, then the $U(z)$'s must be such that the $f_{mn}{}^p$'s are constant[3].

With the new setup, the gauge transformations with parameters λ^m or $\lambda_m{}^n \equiv -f_{mp}{}^n\lambda^p$ act now covariantly on *all* the fields that have m, n indices, *e.g.*

$$\delta_\lambda e_m{}^i = -e_n{}^i\lambda_m{}^n = f_{mp}{}^n\lambda^p e_n{}^i\,, \tag{2.17}$$

except for the vectors $A^m{}_\mu$ that transform as gauge vectors with group $\mathcal{G}$ with structure constants $f_{mn}{}^p$, *i.e.*

$$\delta_\lambda A^m{}_\mu = \partial_\mu\lambda^m - \lambda^p f_{np}{}^m A^n{}_\mu \equiv \mathcal{D}_\mu\lambda^m\,, \tag{2.18}$$

where $\mathcal{D}_\mu$ is the d-dimensional gauge covariant derivative. The structure constants $f_{mn}{}^p$ are dimensionful parameters with dimensions of mass, and determine the gauge couplings and masses in the reduced theory.

In order to ensure invariance of the $\hat{d}$-dimensional action under the new internal g.c.t.'s, the $U(z)$'s must satisfy a certain condition, which turns out to be [151]

$$\partial_m\left[(U^{-1})_n{}^m(z)\,|U|\right] = 0\,, \tag{2.19}$$

where $|U| = \det(U(z))$. This condition is easily shown to be equivalent to

$$f_{mn}{}^m = 0\,. \tag{2.20}$$

Therefore, the reduction of the action is *only* consistent if this condition is satisfied. We will refer to these cases as *class A gauged supergravities.* However, the $\hat{d}$-dimensional equations of motion are invariant even if the structure

[3]This is in fact one of the properties of a Lie group with generators $K_{(m)} = (U^{-1})_m{}^n\partial_n$. See *e.g.* [71].

constants have non-vanishing trace, and so for $f_{mn}{}^m \neq 0$ one has to resort to a reduction of the field equations. These cases lead to the *class B gauged supergravities*. Note that the embedding of the gauge group $\mathcal{G} \subset GL(n,\mathbb{R})$ is described by

$$g_n{}^m = e^{\lambda^k f_{kn}{}^m}\,, \tag{2.21}$$

where λ^k are the parameters of the gauge transformations. Therefore, in the case of a non-vanishing trace, the gauge group $\mathcal{G}$ is a subgroup of $GL(n,\mathbb{R}) = SL(n,\mathbb{R}) \otimes SO(1,1)$ and not just $SL(n,\mathbb{R})$. In other words, $SO(1,1)$ is a global rescaling under which the action is not invariant, such that the symmetry group of the action is only $SL(n,\mathbb{R})$. On the other hand, the equations of motion are unaffected by this rescaling and they are therefore invariant under the full $GL(n,\mathbb{R})$, such that one can employ $GL(n,\mathbb{R})$ in an SS2 reduction of the equations of motion, while only $SL(n,\mathbb{R})$ is valid if we wish to reduce the action. Furthermore, it turns out that the equations of motion of class B supergravities cannot be integrated into an action. This is similar to what happens in the 10-dimensional massive theory derived in [106], where the generalized dimensional reduction employs a global rescaling symmetry of the equations of motion which is not an invariance of the action. Also in this case no action can be found for the reduced theory.

We know now that it is possible to obtain a Lie group from higher dimensional g.c.t.'s through a certain dependence on the internal coordinates, and, therefore, we expect the reduced theory to be a non-Abelian extension of the theory one would obtain via a standard dimensional reduction.

□ What are the $U(z)$'s?

Let us consider the z^m's as a system of coordinates on the manifold of a Lie group $\mathcal{G}$ with n generators. Then, the infinitesimal generators of $\mathcal{G}$ can be defined as

$$K_m = (U^{-1})_m{}^n \partial_n\,, \tag{2.22}$$

whose commutator is

$$[K_m, K_n] = f_{mn}{}^p\, K_p\,, \tag{2.23}$$

where the $f_{mn}{}^p$, defined in (2.16), are the structure constants of $\mathcal{G}$. The 1-forms σ^m dual to the vectors K_m satisfy

$$\langle \sigma^m, K_n \rangle = \delta^m{}_n \,, \tag{2.24}$$

are hence are given by

$$\sigma^m = U_n{}^m \, dz^n \,. \tag{2.25}$$

They satisfy the Maurer-Cartan structure equations

$$d\sigma^m = -\tfrac{1}{2} f_{np}{}^m \sigma^n \wedge \sigma^p \,, \tag{2.26}$$

and so are the Maurer-Cartan 1-forms, such that the functions $U(z)$ are nothing but the components of the Maurer-Cartan 1-forms of the Lie group. This implies that, unless $f_{mn}{}^p = 0$, the internal space is no longer a torus. Note that this does not mean that the group manifold of the Lie group $\mathcal{G}$ is the internal manifold.

Let us now proceed with the reduction of the action. The spin connections derived from (2.14) are[4]

$$\begin{aligned}
\hat{\omega}_{abc} &= \omega_{abc} \,, & \hat{\omega}_{abi} &= -\tfrac{1}{2} e_{m\,i} F^m{}_{ab} \,, \\
\hat{\omega}_{iab} &= \tfrac{1}{2} e_{m\,i} F^m{}_{ab} \,, & \hat{\omega}_{aij} &= -e_{[i|}{}^m \mathcal{D}_a e_{m|j]} \,, \\
\hat{\omega}_{ija} &= -\tfrac{1}{2} e_i{}^m e_j{}^n \mathcal{D}_a G_{mn} \,, & \hat{\omega}_{ijk} &= -\hat{\Omega}_{ijk} - 2\hat{\Omega}_{k(ij)} \,,
\end{aligned} \tag{2.27}$$

where

$$\hat{\Omega}_{ijk} = -\frac{1}{2} f_{mn}{}^p e_i{}^m e_j{}^n e_{p\,k} \,, \tag{2.28}$$

and

[4]It is worth noticing that the spin connection $\hat{\omega}_{ijk} \neq 0$, while in standard dimensional reductions it is zero. This new term makes the internal space be no longer a torus. As we will see, it is also the responsible of the appearance of a scalar potential in the reduced theory. Moreover, it gives rise to mass terms for some gravitinos in the reduced theory [151].

$$F^m = 2\partial A^m - f_{np}{}^m A^n A^p \,. \tag{2.29}$$

We decompose now the internal metric as

$$G_{mn} = -K\,\mathcal{M}_{mn} \tag{2.30}$$

with $K = |\det(G)|^{1/n}$ and $\mathcal{M} \in SL(n,\mathbb{R})$, and perform a conformal rescaling (so as to get the lower dimensional action in the Einstein frame)

$$g_{\mu\nu} = K^{\frac{n}{(2-d)}}\, g_{E\,\mu\nu} \,. \tag{2.31}$$

Rewriting the scalar field K as

$$K = e^{-\frac{1}{3}\sqrt{\frac{2(d-2)}{n}}\varphi} \tag{2.32}$$

and using the standard techniques, the reduced action is found to be

$$\begin{aligned} S \;\; = \;\; C_U \int d^d x \sqrt{|g_E|}\, \Big\{ & R_E + \tfrac{1}{2}\left(\mathcal{D}\varphi\right)^2 + \tfrac{1}{4}\mathrm{Tr}\left(\mathcal{M}^{-1}\mathcal{D}\mathcal{M}\right)^2 \\ & -\tfrac{1}{4}e^{\aleph_1\varphi} F^m \mathcal{M}_{mn} F^n - e^{\aleph_2\varphi}\mathcal{V}(\mathcal{M}) \Big\} \,, \end{aligned} \tag{2.33}$$

where the scalar potential $\mathcal{V}$ is given by

$$\mathcal{V}(\mathcal{M}) = \frac{1}{4}\Big[2\mathcal{M}^{nq} f_{mn}{}^p f_{pq}{}^m + \mathcal{M}^{mq}\mathcal{M}^{nr}\mathcal{M}_{ps} f_{mn}{}^p f_{qr}{}^s \Big] \,, \tag{2.34}$$

and therefore depends on the choice of the gauge group $\mathcal{G}$. The numerical factors $\aleph_{1,2}$ are given by

$$\aleph_1 = -3\sqrt{\frac{2}{n(d-2)}} \,, \qquad \aleph_2 = \left(\frac{n+4-2d}{3}\right)\sqrt{\frac{2}{n(d-2)}} \,. \tag{2.35}$$

The global factor C_U in the action is defined by

$$C_U = \int d^n z\, |U| \,, \tag{2.36}$$

and will converge or not depending on the manifold in which one is reducing. If it converges, then the reduced theory may be seen as a compactification. If, on the other hand, it diverges, the resulting action is a "non-compactification" of the higher dimensional theory. This point should be clarified. In a compactified theory the internal space is, at a certain energy scale, small enough to consider that the theory is effectively lower dimensional. In a "non-compactification", although an Ansatz is chosen, we simply "freeze" the dynamics in the z^m's (for that Ansatz). Nevertheless, the procedure leads to a well-defined lower dimensional gauged theory and, furthermore, can be used as a solution generating transformation of the higher dimensional theory.

□ So... what have we done?

We have extended the infinitesimal generators of the internal g.c.t.'s by allowing an extra dependence on the internal coordinates, *i.e.* we have gauged the isometries of the internal manifold. This produces massive deformations parametrized by the structure constants ${f_{mn}}^p$, which come from the reduction over the group manifold. The choice ${f_{mn}}^p = 0$ is the ungauged case and corresponds to reduction over T^n leading to the trivial gauge group $U(1)^n$, which is so due to the fact that the n-torus is a product of n circles, each of them carrying an $U(1)$ gauge group, *i.e.* compactifying on a n-torus leads to the same result than performing n dimensional reductions, each on a circle. This is not the case if the structure constants are different from zero, since the internal space is not a product of circles.

The full massless supergravity theory has, at least, a global $SL(n,\mathbb{R})$ symmetry group acting in the obvious way on the indices m, n. This symmetry is generically broken if the structure constants are non-zero (such that theory is no longer massless), which, under a $SL(n,\mathbb{R})$ transformation, change as

$$ {f_{mn}}^p \to {f'_{mn}}^p = {R_m}^q {R_n}^r {(R^{-1})_s}^p {f_{qr}}^s \, . \tag{2.37} $$

Only transformations that leave the structure constants invariant (${f_{mn}}^p = {f'_{mn}}^p$) are unbroken by the massive deformations. These include the infinitesimal gauge transformations (2.17). As mentioned before, class A supergravity theories can be obtained by the gauging of a certain symmetry group of the ungauged theory. This group is precisely the unbroken group $\mathcal{G} \subset SL(n,\mathbb{R})$ and is also a subgroup of the R-symmetry group of the ungauged theory.

So far, we have considered the reduction of pure gravity. In the reduction of a theory which also includes gauge fields, one is forced to introduce a dependence on the z^m's for these gauge fields in order that the reduced theory is invariant under the new transformations. This dependence turns out to be a factor of $U(z)$ per lower internal index and $U^{-1}(z)$ per upper internal index.

□ What about the fermions?

In the original work [151], Scherk-Schwarz applied an SS2 reduction to $d = 11$ supergravity to obtain a deformation of $N = 8$ $d = 4$ supergravity with three mass parameters and spontaneously broken supersymmetry. The 11-dimensional gravitino splits down to four dimensions into 8 Majorana spin 3/2 fields and 56 Majorana spin 1/2 fields, 8 of which are eaten by the 8 gravitinos, which become massive and, therefore, break spontaneously supersymmetry. The masses (or the gauge group) can be chosen such that no gravitino is massless and the supersymmetry is completely broken, or, on the other hand, the choice can be made in such a way that only some of the gravitinos are massive and some supersymmetry is preserved. However, there is no general pattern for this behaviour. Some cases will present spontaneously broken supersymmetries and some others will not, depending on the higher dimensional theory, the dimension of the internal space and the choice of gauge group, but there will always be a massive gravitino per spontaneously broken supersymmetry.

2.2 Reduction of $d = 11$ supergravity on a 3-manifold

The standard $d = 8$ gauged maximal supergravity is the $SO(3)$-gauged theory of Salam and Sezgin [144], which was constructed by applying an SS2 reduction procedure to $d = 11$ supergravity. The analysis of [144] can be generalized to a 3-dimensional manifold corresponding to other Lie algebras, not only $SO(3)$.

The SS2 reduction Ansatz for the 11-dimensional Vielbein is

$$\hat{e}_{\hat{\mu}}{}^{\hat{a}} = \begin{pmatrix} e^{-\frac{1}{6}\varphi} e_\mu{}^a & e^{\frac{1}{3}\varphi} L_m{}^i A^{1\,m}{}_\mu \\ 0 & e^{\frac{1}{3}\varphi} L_n{}^i U^n{}_m \end{pmatrix} , \tag{2.38}$$

where the label '1' has been added to the KK vectors for later convenience. The Ansatz for the 3-form potential in flat indices is

$$\hat{C}_{abc} = e^{\frac{1}{2}\varphi}\, C_{abc}\,, \qquad \hat{C}_{abi} = L_i{}^m B_{m\,ab}\,,$$
$$\hat{C}_{aij} = e^{-\frac{1}{2}\varphi}\, \epsilon_{mnp} L_i{}^m L_j{}^n\, A_a^{2\,p}\,, \qquad \hat{C}_{ijk} = e^{-\varphi} \epsilon_{ijk}\ell\,, \tag{2.39}$$

which in curved components reads

$$\begin{aligned}
\hat{C} &= C + 3A^{1\,m} B_m + 3\epsilon_{mnp} A^{1\,m} A^{1\,n} A^{2p} + \epsilon_{mnp}\, \ell\, A^{1\,m} A^{1\,n} A^{1\,p}\,,\\
\hat{C}_m &= U_m{}^q \left[B_q + 2\epsilon_{qnp} A^{1\,n} A^{2\,p} + \epsilon_{qnp}\, \ell\, A^{1\,n} A^{1\,p}\right]\,,\\
\hat{C}_{mn} &= U_m{}^q U_n{}^r \left[\epsilon_{qrp} A^{2\,p} + \epsilon_{qrp} \ell\, A^{1\,p}\right]\,,\\
\hat{C}_{mnp} &= U_m{}^q U_n{}^r U_p{}^s\, \epsilon_{qrs}\, \ell\,.
\end{aligned} \tag{2.40}$$

The eleven-dimensional gauge transformations of the 3-form that preserve the above Ansatz are generated by the 2-form $\hat{\chi}_{\hat{\mu}\hat{\nu}}$ with components

$$\begin{aligned}
\hat{\chi}_{\mu\nu} &= \chi_{\mu\nu} - 2A^{1\,m}_{[\mu}\, \Sigma_{\nu]\,m} + \epsilon_{mnp}\, A^{1\,m}_{[\mu} A^{1\,n}_{\nu]} \lambda^{2\,p}\,,\\
\hat{\chi}_{\mu m} &= U_m{}^q\, \left[\Sigma_{\mu\,q} - \epsilon_{qnp}\, A^{1\,n}{}_{\mu} \lambda^{2\,p}\right]\,,\\
\hat{\chi}_{mn} &= U_m{}^q U_n{}^r\, \epsilon_{qrs}\, \lambda^{2\,s}\,,
\end{aligned} \tag{2.41}$$

which correspond to the following gauge transformations

- Gauge transformations of the 3-form

$$\delta_\chi C = 3\partial\chi\,. \tag{2.42}$$

- Massive gauge transformations

$$\delta_\Sigma C = -3F^{1\,m}\Sigma_m\,, \quad \delta_\Sigma B_m = 2\mathcal{D}\Sigma_m\,, \quad \delta_\Sigma A^{2\,m} = \tfrac{1}{2}\epsilon^{mnp} f_{np}{}^q \Sigma_q\,, \tag{2.43}$$

  where

$$\mathcal{D}\Sigma_m = \partial\Sigma_m - f_{mn}{}^p A^{1\,n}\Sigma_p\,. \tag{2.44}$$

- Gauge transformations of the vector fields $A^{2\,m}$

$$\delta_\Lambda A^{2\,m} = \mathcal{D}\lambda^{2\,m}\,, \quad \delta_\Lambda B_m = -2\epsilon_{mnp}\,\partial A^{1\,n}\lambda^{2p}\,, \tag{2.45}$$

where

$$\mathcal{D}\lambda^{2\,m} = 3\partial\lambda^{2\,m} - \tfrac{1}{2}\epsilon^{mnp} f_{np}{}^{q}\epsilon_{qrs}A^{1\,r}\lambda^{2\,s}\,. \tag{2.46}$$

The field strengths associated to the 8-dimensional potentials which are invariant under the above transformations are

$$\begin{aligned} G &= 4\partial C + 6F^{1\,m}B_m\,, & F^{1\,m} &= 2\partial A^{1\,m} - f_{np}{}^{m}A^{1\,n}A^{1\,p}\,, \\ H_m &= 3\mathcal{D}B_m + 3\epsilon_{mnp}F^{1\,n}A^{2\,p}\,, & F^{2\,m} &= 2\mathcal{D}A^{2\,m} - \tfrac{1}{2}\epsilon^{mnp}f_{np}{}^{q}B_q\,. \end{aligned} \tag{2.47}$$

The two vector field strengths form a doublet $F^{I\,m}$, $I = 1, 2$. The scalar field strength for the scalar fields is now given by

$$\mathcal{D}\mathcal{M}_{mn} = \partial\mathcal{M}_{mn} + 2f_{q(m|}{}^{p}A^{1\,q}\mathcal{M}_{|n)p}\,, \tag{2.48}$$

where we have defined the local $SO(3)$ invariant scalar matrix

$$\mathcal{M}_{mn} = L_m{}^{i}L_n{}^{j}\delta_{ij}\,, \tag{2.49}$$

with δ_{ij} the internal flat metric. As in the massless case (see appendix B), the two-dimensional $SL(2,\mathbb{R})/SO(2)$ scalar coset is parametrized by the dilaton φ and the axion ℓ can be parametrized via the local $SO(2)$ invariant scalar matrix

$$\mathcal{W} = \frac{1}{\Im\mathrm{m}(\tau)}\begin{pmatrix} |\tau|^2 & \Re\mathrm{e}(\tau) \\ \Re\mathrm{e}(\tau) & 1 \end{pmatrix}\,, \tag{2.50}$$

where $\tau = \ell + ie^{\varphi}$.

Finally, the Ansatz for the fermionic fields is the same as for the massless case, namely

$$\hat{\psi}_{\hat{a}} = e^{\varphi/12}\left(\psi_a - \tfrac{1}{6}\Gamma_a\Gamma^i\lambda_i\right), \qquad \hat{\psi}_i = e^{\varphi/12}\lambda_i\,, \qquad \hat{\epsilon} = e^{-\varphi/12}\epsilon\,. \tag{2.51}$$

Hence, the full 8-dimensional field content consists of the following 128 + 128 field components:

$$\{e_\mu{}^a, L_m{}^i, \varphi, \ell, A^{I\,m}, B_m, C, \psi_\mu, \lambda_i\}\,. \tag{2.52}$$

We are now ready to give the complete bosonic action and supersymmetry transformations for the $d = 8$ class A gauged supergravity theories obtained from SS2 reduction of $d = 11$ supergravity. The action is given by

$$\begin{aligned} S = \frac{1}{16\pi G_N^{(11)}} C_U \int d^8x\sqrt{|g_E|}\, \Big\{ & R_E + \frac{1}{4}\mathrm{Tr}\left(\mathcal{D}\mathcal{M}\mathcal{M}^{-1}\right)^2 + \frac{1}{4}\mathrm{Tr}\left(\partial\mathcal{W}\mathcal{W}^{-1}\right)^2 \\ & - \frac{1}{4}F^{I\,m}\mathcal{M}_{mn}\mathcal{W}_{IJ}F^{J\,n} + \frac{1}{2\cdot 3!}H_m\mathcal{M}^{mn}H_n - \frac{1}{2\cdot 4!}e^{\varphi}G^2 - \mathcal{V} \\ & - \frac{1}{6^3\cdot 2^4}\frac{1}{\sqrt{|g_E|}}\,\epsilon\,\big[GG\ell - 8GH_mA^{2\,m} + 12G(F^{2\,m} + \ell F^{1\,m})B_m \\ & -8\epsilon^{mnp}H_mH_nB_p - 8G\partial\ell C - 16H_m(F^{2\,m} + \ell F^{1\,m})C\big]\Big\}\,, \end{aligned} \tag{2.53}$$

where the potential $\mathcal{V}$ reads

$$\mathcal{V} = \tfrac{1}{4}e^{-\varphi}\left[2\mathcal{M}^{nq}f_{mn}{}^pf_{pq}{}^m + \mathcal{M}^{mq}\mathcal{M}^{nr}\mathcal{M}_{ps}f_{mn}{}^pf_{qr}{}^s\right]. \tag{2.54}$$

The $d = 8$ supersymmetry transformations for the fermions are

$$
\begin{aligned}
\delta\psi_\mu &= 2\nabla_\mu\epsilon + \tfrac{1}{2}L_{[i|}{}^m\mathcal{D}_\mu L_{m|j]}\Gamma^{ij}\epsilon + \tfrac{1}{24}e^{-\varphi/2}f_{ijk}\Gamma^{ijk}\Gamma_\mu\epsilon \\
&\quad +\tfrac{1}{24}e^{\varphi/2}\Gamma^i L_i{}^m(\Gamma_\mu{}^{\nu\rho} - 10\delta_\mu{}^\nu\Gamma^\rho)F^1_{m\nu\rho}\epsilon + \tfrac{1}{2}e^{-\varphi}\partial_\mu\ell\epsilon \\
&\quad +\tfrac{i}{96}e^{\varphi/2}(\Gamma_\mu{}^{\nu\rho\delta\epsilon} - 4\delta_\mu{}^\nu\Gamma^{\rho\delta\epsilon})G_{\nu\rho\delta\epsilon}\epsilon \\
&\quad +\tfrac{i}{36}\Gamma^i L_i{}^m(\Gamma_\mu{}^{\nu\rho\delta} - 6\delta_\mu{}^\nu\Gamma^{\rho\delta})H_{m\nu\rho\delta}\epsilon \\
&\quad +\tfrac{i}{48}e^{-\varphi/2}\epsilon_{ijk}\Gamma^i\Gamma^j L_m^k(\Gamma_\mu{}^{\nu\rho} - 10\delta_\mu{}^\nu\Gamma^\rho)(F^{2\,m}_{\nu\rho} + \ell F^{1\,m}_{\nu\rho})\epsilon\,, \\
\delta\lambda_i &= \tfrac{1}{2}L_i{}^m L^{jn}\not{\mathcal{D}}\mathcal{M}_{mn}\Gamma_j\epsilon - \tfrac{1}{3}\not{\partial}\varphi\Gamma_i\epsilon - \tfrac{1}{4}e^{-\varphi/2}(2f_{ijk} - f_{jki})\Gamma^{jk}\epsilon \\
&\quad +\tfrac{1}{4}e^{\varphi/2}L_i{}^m\mathcal{M}_{mn}\not{F}^{1\,n}\epsilon + \tfrac{i}{144}e^{\varphi/2}\Gamma_i\not{G}\epsilon + \tfrac{i}{36}(2\delta_i{}^j - \Gamma_i{}^j)L_j{}^m\not{H}_m\epsilon \\
&\quad +\tfrac{i}{24}e^{-\varphi/2}\epsilon_{ijk}\Gamma^j L_m^k(3\delta_i{}^k - \Gamma_i{}^k)(\not{F}^{2\,m} + \ell\not{F}^{1\,m})\epsilon \\
&\quad +\tfrac{1}{3}e^{-\varphi}\Gamma_i\not{\partial}\ell\epsilon\,,
\end{aligned}
\tag{2.55}
$$

where we have used the abbreviations $f_{ijk} \equiv L_i{}^m L_j{}^n L_{pk} f_{mn}{}^p$. Of course, the ungauged/massless limit ($f_{mn}{}^p = 0$) provides the $d = 8$ ungauged supergravity theory obtained via a standard dimensional reduction on a 3-torus (see appendix B).

2.2.1 The Bianchi classification

The structure constants of all 3-dimensional Lie algebras can be parametrized by a symmetric matrix that we denote by Q^{mn} and which will play the role of mass matrix, and by a vector a_m satisfying $\mathsf{Q}^{mn}a_n = 0$ (see, e.g., [167]):

$$
f_{mn}{}^p = \epsilon_{mnq}\mathsf{Q}^{qp} + 2\delta_{[m}{}^p a_{n]}\,. \tag{2.56}
$$

The trace of the structure constants vanishes if and only if the vector vanishes, *i.e.* $a_m = 0$. Restricting to the class A gauged supergravities we can therefore take $f_{mn}{}^p = \epsilon_{mnq}\mathsf{Q}^{qp}$ and all the different cases that we are going to consider will be characterized by a choice of mass matrix Q.

Note that in terms of the mass matrix the potential (2.54) reads

$$\mathcal{V} = -\tfrac{1}{2}\, e^{-\varphi} \left\{ [\mathrm{Tr}(\mathcal{M}\mathsf{Q})]^2 - 2\mathrm{Tr}(\mathcal{M}\mathsf{Q}\mathcal{M}\mathsf{Q}) \right\}. \qquad (2.57)$$

The symmetric mass matrix Q has six different mass parameters. However, by applying symmetries of the massless 8D theory one can relate different choices of Q^{mn} by field redefinitions, via transformations as (2.37). We would like to use the $GL(3,\mathbb{R})$ symmetry of the massless 8-dimensional theory.

Employing these symmetries we can transform $\mathsf{Q}^{mn} \to \pm(R^T\mathsf{Q}R)^{mn}$ with $R \in GL(3,\mathbb{R})$. Now consider an arbitrary symmetric matrix Q^{mn} with eigenvalues λ_m and orthogonal eigenvectors $\vec{u}_m$. Taking $R = (c_1\vec{u}_1, c_2\vec{u}_2, c_3\vec{u}_3) \in GL(3,\mathbb{R})$ with $c_i \neq 0$ we find that

$$\mathsf{Q}^{mn} \to \pm(R^T\mathsf{Q}R)^{mn} = \pm\mathrm{diag}(\mathrm{c}_1{}^2\lambda_1, \mathrm{c}_2{}^2\lambda_2, \mathrm{c}_3{}^2\lambda_3). \qquad (2.58)$$

Thus, all cases with the same signature are related by field redefinitions. Without loss of generality we will use the freedom of field redefinitions to take

$$\mathsf{Q}^{mn} = \tfrac{1}{2}\mathrm{diag}(q_1, q_2, q_3). \qquad (2.59)$$

The different $d = 8$ gauged supergravities will arise from choosing all possible ranks and signatures for the mass matrix Q^{mn}. Actually, this diagonalization plus the choice $a_m = (a, 0, 0)$ is the basis of the Bianchi classification of all real 3-dimensional Lie algebras from Bianchi type I to Bianchi type IX. Thus, each choice of mass matrix corresponds to a choice of Lie algebra and therefore of gauge group. Restricting to the class A theories we only consider the algebras with $a = 0$[5]:

$$\text{Bianchi types I, II, VI}_0\text{, VII}_0\text{, VIII, IX}. \qquad (2.60)$$

All algebras with $a = 0$ are subalgebras of the Lie algebra of $SL(3,\mathbb{R})$. For useful details about the Bianchi classification, see appendix C. The five non-trivial cases with $a = 0$ are given in table 2.1 while Bianchi type I corresponds to the massless case $\mathsf{Q} = 0$ and thus is an ungauged supergravity. This case corresponds to the Abelian Lie algebra $U(1)^3$.

[5]The sub-index 0 in Bianchi type VI_0 and Bianchi type VII_0 indicate that these class A Lie algebras can be obtained as the limit $a \to 0$ of the class B Bianchi type VI_a and Bianchi type VII_a Lie algebras (see appendix C).

Bianchi	$\mathsf{Q} = \frac{1}{2}$diag	Group
II	$(0, 0, q)$	Heisenberg
VI_0	$(0, -q, q)$	$ISO(1,1)$
VII_0	$(0, q, q)$	$ISO(2)$
VIII	$(q, -q, q)$	$SO(2,1)$
IX	(q, q, q)	$SO(3)$

Table 2.1: The different mass matrices and corresponding Bianchi classifications and gauge groups. The $SO(3)$ result was previously obtained in [144].

The structure constants for class A supergravities in the form we have explained can be generated using a particular frame in the internal directions. The explicit coordinate dependence of the components of the Maurer-Cartan 1-forms in this frame turns out to be

$$U^m{}_n = \begin{pmatrix} 1 & 0 & s_{1,3,2} \\ 0 & c_{2,3,1} & -c_{1,3,2}\, s_{2,3,1} \\ 0 & s_{3,2,1} & c_{1,3,2}\, c_{2,3,1} \end{pmatrix} , \qquad \det U \neq 1 , \tag{2.61}$$

where we have used the following abbreviations $(a, b, c = 1, 2, 3)$:

$$c_{a,b,c} \equiv \cos(\sqrt{\tfrac{1}{4} q_a q_b}\, z^c) , \qquad s_{a,b,c} \equiv \sqrt{q_a/q_b}\, \sin(\sqrt{\tfrac{1}{4} q_a q_b}\, z^c) . \tag{2.62}$$

Note that the U-matrix is independent of z^3. It is always possible to choose a frame where z^3 is a manifest isometry. We distinguish the following three different cases:

1. **The matrix Q is non-singular**. In this case z^3 is the only manifest isometry. In the compact case we are dealing with the Salam-Sezgin case in which the group manifold is equal to S^3. The presence of the manifest z^3-isometry direction is related to the fact that S^3 can be viewed as a Hopf fibration over S^2. One consequence of this fact is that the $d = 8$ class A supergravities can also be obtained by reduction of the massless IIA theory. For instance, the Salam-Sezgin theory can alternatively be obtained by reduction of the massless IIA theory over S^2. The latter

reduction naturally occurs in the context of the DW/QFT correspondence [28]. In the non-compact case the $SO(3)$ gauging gets replaced by an $SO(2,1)$ gauging. This case can be understood as an analytic continuation of the Salam-Sezgin theory or as a "non-compactification" of $d = 11$ supergravity.

2. **The matrix Q is singular**, e.g. $\mathsf{Q} = \frac{1}{2}\text{diag}(0, q_2, q_3)$. In this case there is an additional isometry in the z^2-direction:

$$U^m{}_n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sqrt{q_2/q_3}\sin\alpha \\ 0 & \sqrt{q_3/q_2}\sin\alpha & \cos\alpha \end{pmatrix}, \qquad \det U = 1\,, \quad (2.63)$$

with $\alpha = \sqrt{\frac{1}{4}q_2 q_3}\, z^1$. This means that the resulting $d = 8$ class A gauged supergravities can also be obtained by a reduction of the massless 9-dimensional theory.

3. **The matrix Q is doubly-degenerate**, e.g. $\mathsf{Q} = \frac{1}{2}\text{diag}(0, 0, q_3)$. In this case the U-matrix is given by

$$U^m{}_n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2}q_3 z^1 & 1 \end{pmatrix}, \qquad \det U = 1\,, \qquad (2.64)$$

and again the resulting $d = 8$ class A gauged supergravity has its origin in the massless 9-dimensional theory.

2.3 The domain wall solutions

The aim of this section is to look for domain wall (DW) solutions of the $d = 8$ maximal gauged supergravities of class A found in the previous section. We will require that our solutions preserve one half of the supersymmetry. We consider the following Ansatz:

$$\begin{aligned} ds^2 &= g(y)^2 dx_7{}^2 - f(y)^2 dy^2\,, \qquad \mathcal{M} = \mathcal{M}(y)\,, \\ \varphi &= \varphi(y)\,, \qquad \epsilon = \epsilon(y)\,. \end{aligned} \qquad (2.65)$$

It only includes the metric and the scalars, while all other fields are vanishing except the $SL(2,\mathbb{R})/SO(2)$ scalar ℓ which we have set to some constant. It turns out that there are no half-supersymmetric domain walls for non-constant ℓ.

The Killing spinor equations that our solutions need to satisfy are

$$\begin{aligned}
\delta\psi_\mu &= 2\partial_\mu\epsilon - \tfrac{1}{2}\not{\psi}_\mu\epsilon + \tfrac{1}{2}\not{Q}_\mu\epsilon + \tfrac{1}{24}e^{-\varphi/2}f_{ijk}\Gamma^{ijk}\Gamma_\mu\epsilon = 0\,,\\
\delta\lambda_i &= -\not{P}_{ij}\Gamma^j\epsilon - \tfrac{1}{3}\not{\partial}\varphi\Gamma_i\epsilon - \tfrac{1}{4}e^{-\varphi/2}(2f_{ijk} - f_{jki})\Gamma^{jk}\epsilon = 0\,,
\end{aligned}$$

where we have used

$$P_{\mu ij} + Q_{\mu ij} \equiv L_i{}^m\mathcal{D}_\mu L_{mj}\,, \qquad \not{P}_{ij} \equiv P_{\mu ij}\Gamma^\mu\,, \qquad \not{Q}_\mu \equiv Q_{\mu ij}\Gamma^{ij}\,, \tag{2.66}$$

with P symmetric and traceless and Q antisymmetric. The Killing spinor of our solutions must also satisfy the condition

$$(1 + \Gamma_{y123})\epsilon = 0\,, \tag{2.67}$$

which implies a breaking of one half of the supersymmetries. Indices $1, 2, 3$ refer to the internal manifold directions.

To make life easier in the search for solutions we take the following explicit representative of the $L_m{}^i$'s[6]:

$$L_m{}^i = \begin{pmatrix} e^{-\sigma/\sqrt{3}} & e^{-\phi/2+\sigma/2\sqrt{3}}\chi_1 & e^{\phi/2+\sigma/2\sqrt{3}}\chi_2 \\ 0 & e^{-\phi/2+\sigma/2\sqrt{3}} & e^{\phi/2+\sigma/2\sqrt{3}}\chi_3 \\ 0 & 0 & e^{\phi/2+\sigma/2\sqrt{3}} \end{pmatrix}\,, \tag{2.68}$$

which contains two dilatons, ϕ and σ, and three axions[7], χ_1, χ_2 and χ_3.

The domain wall solutions we present below are valid both for non-singular and singular mass matrices Q. We find the following most general class A solution:

[6] The matrix $L_m{}^i$ describes the five-dimensional $SL(3,\mathbb{R})/SO(3)$ scalar coset of the internal space. It transforms under a global $SL(3,\mathbb{R})$ acting from the left and a local $SO(3)$ symmetry acting from the right. We can fix the $SO(3)$ local gauge symmetry such as to consider an explicit representative.

[7] We call the scalars ℓ, χ_1, χ_2 and χ_3 axions and the scalars φ, ϕ and σ dilatons since (in the ungauged case) the axions only occur with a $d = 8$ spacetime derivative whereas the dilatons also occur without such a derivative.

$$\begin{aligned}
ds^2 &= H^{\frac{1}{12}} dx_7^2 - H^{-\frac{5}{12}} dy^2 \,,\\
e^{\varphi} &= H^{\frac{1}{4}}, \quad e^{\sigma} = H^{-\frac{1}{2\sqrt{3}}} h_1^{\frac{\sqrt{3}}{2}}, \quad e^{\phi} = H^{-\frac{1}{2}} h_1^{-\frac{1}{2}} (h_1 h_2 - C_1^2)\,,\\
\chi_1 &= C_1 h_1^{-1}\,,\\
\chi_2 &= \chi_1 \chi_3 + C_2 h_1^{-1}\,,\\
\chi_3 &= (C_1 C_2 + C_3 h_1)\left(h_1 h_2 - C_1^2\right)^{-1}\,,
\end{aligned} \tag{2.69}$$

where the dependence on the transverse coordinate y is governed by

$$\begin{aligned}
H(y) &= h_1 h_2 h_3 - C_3^2 h_1 - C_2^2 h_2 - C_1^2 h_3 - 2 C_1 C_2 C_3\,,\\
h_1 &\equiv q_1 y + C_4, \quad h_2 \equiv q_2 y + C_5, \quad h_3 \equiv q_3 y + C_6\,.
\end{aligned} \tag{2.70}$$

The corresponding Killing spinor is quite intricate so we will not give it here. Note that the solution is given by three harmonic functions h_1, h_2 and h_3. We will call the general solution a *triple domain wall.*

The general solution has six integration constants $C_1, \ldots, C_6$. The constants C_4, C_5 and C_6 are related to the positioning of the domain walls in the transverse space. These form a threshold bound state of n parallel domain walls, where n equals the rank of the mass matrix. It turns out that, provided that one of the charges q_1, q_2 or q_3 is non-zero, one can eliminate one of the constants C_4, C_5 or C_6 by a redefinition of the variable y. Therefore we effectively always end up with at most two constants.

The first three constants C_1, C_2 and C_3 can be understood to come from the following symmetry. The mass deformations do not break the full global $SL(3,\mathbb{R})$ symmetry; indeed, they gauge the 3-dimensional subgroup of $SL(3,\mathbb{R})$ that leave the mass matrix Q invariant. Thus, one can use the unbroken global subgroup to transform any solution[8], introducing three constants. In our solution these correspond to C_1, C_2 and C_3 and thus these can be set to zero by fixing the $SL(3,\mathbb{R})$ frame. From now on we will always assume the frame choice $C_1 = C_2 = C_3 = 0$ unless explicitly stated otherwise. This results in

$$\begin{aligned}
\mathcal{M} &= H^{-2/3} \mathrm{diag}(h_2 h_3, h_1 h_3, h_1 h_2)\,, \qquad H = h_1 h_2 h_3\,,\\
\chi_1 &= \chi_2 = \chi_3 = 0\,.
\end{aligned} \tag{2.71}$$

[8] Note that one can not use the unbroken local subgroup of $SL(3,\mathbb{R})$ (the gauge transformations) since this would induce non-vanishing gauge vectors and thus would be inconsistent with our Ansatz (2.65).

Very similar structures for domain walls were found in [10]. It is interesting to note that, in this $SL(3,\mathbb{R})$ frame, the expression for the Killing spinor simplifies considerably and reads $\epsilon = H^{1/48}\epsilon_0$.

The triple domain wall can be truncated to double or single domain walls when restricting the constants C_4, C_5 and C_6. The single domain walls correspond to the situation where the positions of the parallel domain walls coincide. In table 2.2 we give the three possible truncations leading to single domain walls. The Bianchi II case was given in [44] and the Bianchi IX case in [28] (up to coordinate transformations). It is interesting to note that the Bianchi VII_0 case has vanishing potential. The domain wall is carried by the non-vanishing massive contributions to the BPS equations. The same mechanism occurs in $SO(2)$ gauged $d = 9$ supergravity [24].

Bianchi	$\mathsf{Q} = \frac{1}{2}\text{diag}$	h_1	h_2	h_3	Uplift
II	$(0,0,q)$	C_4	C_5	$C+qy$	(2.81)
VII_0	$(0,q,q)$	C_4	$C+qy$	$C+qy$	(2.79)
IX	(q,q,q)	$C+qy$	$C+qy$	$C+qy$	(2.76)

Table 2.2: The single domain walls as truncations of the triple domain wall solution. We give the three possible truncations and indicate the equation where the uplifted solution to $d = 11$ is given (see the next section).

The triple domain wall solution we found in this section can be interpreted as follows. One can view the $(0,0,q)$ solution, having one harmonic function, as the basic solution. The other solutions can then be obtained as threshold bound states of this solution with the $SL(3,\mathbb{R})$-rotated solutions $(0,q,0)$ and $(q,0,0)$. This is clear at the level of the charges. We now see how, similarly, a composition rule at the level of the solutions can be established. One can thus view the solutions with a rank-1 mass matrix as building blocks for the general solution.

Reality of the solutions requires that the three functions h_i must be positive. This implies that the solutions are valid only in certain regions (along the y-direction), which depend on the choice of gauge group. Furthermore, we are going to see that one can only take an asymptotic limit in some cases. In the following we analyze the different cases in terms of the possible signatures of the mass matrix. Let us define $\tilde{C}_i \equiv -C_{i+3}$ and $y_i \equiv \tilde{C}_i/q_i$ the values of y where the functions h_i are zero. If the mass matrix is $\mathsf{Q} = \frac{1}{2}\text{diag}(q_1, q_2, q_3)$, then

- **The SO(3) case:** $\mathsf{Q} = \frac{1}{2}\text{diag}(q,q,q)$, with $q > 0$. Then, the h_i's are positive for $y > y_i$. Therefore, the solution is only valid for $y > y_0$, with $y_0 = max(y_1, y_2, y_3)$. The asymptotic region is reached for $y \to +\infty$. In this limit, the triple domain wall solution reads

$$
\begin{aligned}
ds^2 &= H^{\frac{1}{12}} dx_7^2 - H^{-\frac{5}{12}} dy^2 \,, \\
e^{\varphi} &= H^{\frac{1}{4}} \,, \\
\mathcal{M} &= |\mathsf{Q}|^{1/3}\, \mathsf{Q}^{-1} \,,
\end{aligned}
\tag{2.72}
$$

where $H = |\mathsf{Q}|\, y^3$. It is interesting to notice that the matrix of scalars $\mathcal{M}$ in this solution extremizes the scalar potential (2.57) (see appendix D).

- **The SO(2,1) case:** $\mathsf{Q} = \frac{1}{2}\text{diag}(q,q,-q)$, with $q > 0$. Then positivity of h_i requires $y > y_{1,2}$ and $y < y_3$. Therefore, there will exist a domain of validity if and only if $y_3 > y_{1,2}$. The solutions are valid in (y_0, y_3), where now $y_0 = max(y_1, y_2)$. This domain is compact and no asymptotic region can be defined. In a sense, this is because two of the three single domain walls find an asymptotic region for $y \to \infty$, while the third one reaches its asymptotics for $y \to -\infty$.

- **The ISO(2) case:** $\mathsf{Q} = \frac{1}{2}\text{diag}(q,q,0)$, with $q > 0$. The solutions are valid for $y > y_0$, with $y_0 = max(y_1, y_2)$. An asymptotic region can be defined and is reached for $y \to +\infty$.

- **The ISO(1,2) case:** $\mathsf{Q} = \frac{1}{2}\text{diag}(q,-q,0)$, with $q > 0$. The region of validity is (y_1, y_2) and exists if and only if $y_2 > y_1$. No asymptotic region can be defined.

- **The Heisenberg case:** $\mathsf{Q} = \frac{1}{2}\text{diag}(q,0,0)$, with $q > 0$. The solutions are valid for $y > y_1$. The asymptotic region is reached for $y \to +\infty$.

2.4 Uplifting to 11 dimensions

In this section we consider the uplifting of the triple domain wall solutions (2.69) to eleven dimensions. We find that upon uplifting, using the frame of (2.71), the triple domain wall solutions becomes a purely gravitational solutions with a metric of the form

$$\hat{ds}^2 = d{x_7}^2 - d{s_4}^2 \,, \tag{2.73}$$

where

$$d{s_4}^2 = H^{-\frac{1}{2}} dy^2 + H^{\frac{1}{2}} \left(\frac{\sigma_1^2}{h_1} + \frac{\sigma_2^2}{h_2} + \frac{\sigma_3^2}{h_3} \right) . \tag{2.74}$$

Here σ_1, σ_2 and σ_3 are the Maurer-Cartan 1-forms defined in (2.25), $H = h_1 h_2 h_3$ and the three harmonics h_1, h_2 and h_3 are defined in (2.70). The uplifted solutions are all 1/2 BPS.

The solutions (2.74) are cohomogeneity one solutions of different Bianchi types. The $SO(3)$ expression of this 4-dimensional metric was obtained previously in the context of gravitational instanton solutions as self-dual metrics of Bianchi IX type with all directions unequal [12]. We find that the metric of [12] is related to a triple domain wall solution of 8-dimensional $SO(3)$-gauged supergravity. More recently, the Heisenberg, $ISO(1,1)$ and $ISO(2)$ cases and their relations to domain wall solutions were considered in [69, 106], whose results are related to ours via coordinate transformations.

It is remarkable that in all cases the uplifted solutions have metrics that contain a 7d Minkowski metric as a factor. This does not happen for the uplift of domain walls in 4d and 7d gauged maximal supergravities [10]. In the following discussion we will focus on the 4-dimensional part of the eleven-dimensional metric since it characterizes the uplifted domain walls.

In section 2.3 we argued that for non-vanishing q_1, q_2 or q_3, one of the three constants C_4, C_5 or C_6 in the harmonic functions can be eliminated by a redefinition of the variable y. Without loss of generality we can take $q_3 > 0$. In that case, the constant C_6 can be eliminated by the change of variables $y = \frac{1}{4} r^4 - C_6/q_3$ in the metric (2.74). If we also rescale the three charges by $q_i = 4\tilde{q}_i$ we obtain

$$d{s_4}^2 = (k_1 k_2 k_3)^{-1/2} \left[dr^2 + r^2 (k_2 k_3 \sigma_1^2 + k_1 k_3 \sigma_2^2 + k_1 k_2 \sigma_3^2) \right] , \tag{2.75}$$

where $k_i = \tilde{q}_i + s_i r^{-4}$ for $i = 1, 2, 3$, $s_i = C_{i+3} - q_i C_6/q_3$. Note that $s_3 = 0$ and then $k_3 = \tilde{q}_3$. As anticipated, the metric (2.75) depends only on two constant parameters s_1 and s_2, which are restricted by the gauge group dependent condition $-s_i < \tilde{q}_i r^4$, with $i = 1, 2$, in order to satisfy the requirement $k_i(r) > 0$.

In general the metrics (2.75) have curvatures that both go to zero as r^{-6} for large r and diverge at $r = 0$, $r = (-s_1/\tilde{q}_1)^{1/4}$ and $r = (-s_2/\tilde{q}_2)^{1/4}$, producing incomplete metrics [12, 56]. There are two exceptions to this behavior. The first one corresponds to the case in which $s_1 = s_2 = 0$. The constants can take these values because $\tilde{q}_1$ and $\tilde{q}_2$ are non-zero and therefore this solution can be reached only for the non-degenerate cases. It is easy to see that for the $SO(3)$ gauging ($\tilde{q}_1 = \tilde{q}_2 = \tilde{q}_3 = 1$) the metric is locally flat space-time

$$ds_4{}^2 = dr^2 + r^2((\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2)\,, \tag{2.76}$$

where r is the radius of the 3-dimensional spheres. Notice that this is precisely the uplifting of the Bianchi IX single domain wall located at $y = -C/q$.

The second exception corresponds to the $SO(3)$ gauging with $s_1 = s_2 \equiv s < 0$, and is known as the Eguchi-Hanson (EH), or Eguchi-Hanson II, metric [56] ($\tilde{q}_1 = \tilde{q}_2 = \tilde{q}_3 = 1$),

$$ds_4{}^2 = \left(1 + \frac{s}{r^4}\right)^{-1} dr^2 + r^2(\sigma_1^2 + \sigma_2^2) + \left(1 + \frac{s}{r^4}\right)\sigma_3^2\,. \tag{2.77}$$

In fact, the EH metric is the only complete and non-singular hyper-Kähler 4-metric admitting a tri-holomorphic $SO(3)$ action. Its generic orbits are RP^3 [12, 67, 70].

Another case that we want to emphasize, although it is singular, is obtained in the $SO(3)$ gauging by choosing $s_1 \equiv s \neq 0$ and $s_2 = 0$. This metric is called the Eguchi-Hanson I (EH-I) metric [56] ($\tilde{q}_1 = \tilde{q}_2 = \tilde{q}_3 = 1$)

$$ds_4{}^2 = \left(1 + \frac{s}{r^4}\right)^{-1} (dr^2 + r^2\sigma_1^2) + \left(1 + \frac{s}{r^4}\right)(\sigma_2^2 + \sigma_3^2)\,. \tag{2.78}$$

It is possible to give similar expressions for the $SO(2, 1)$ gauging.

The uplifted metrics for the singular mass matrices can be obtained directly from (2.75). As an example of a contraction we take $\tilde{q}_1 = 0$ in (2.75) and consider the special cases of the EH metrics (2.77) and (2.78), *i.e.* we take

$s_1 = s_2 \equiv s < 0$ (EH-II) or $s_1 \equiv s \neq 0, s_2 = 0$ (EH-I). We thus obtain the contracted EH metrics with $ISO(2)$ isometry in which the $SO(3)$ orbits are flattened to $ISO(2)$ orbits. We find that the expression for the contracted EH-I metric is given by $(\tilde{q}_2 = \tilde{q}_3 = 1)$

$$ds_4{}^2 = \left(\frac{s}{r^4}\right)^{-1/2} (dr^2 + r^2\sigma_1^2) + \left(\frac{s}{r^4}\right)^{1/2} (\sigma_2^2 + \sigma_3^2)\,, \tag{2.79}$$

while the expression for the contracted EH-II metric reads $(\tilde{q}_2 = \tilde{q}_3 = 1)$

$$\begin{aligned} ds_4{}^2 = &\left(\frac{s}{r^4}(1+\frac{s}{r^4})\right)^{-1/2} dr^2 + \left(\frac{s}{r^4}(1+\frac{s}{r^4})\right)^{1/2} \sigma_3^2 \; + \\ &+ r^2\Big((1+\frac{r^4}{s})^{1/2}\sigma_1^2 + (1+\frac{r^4}{s})^{-1/2}\sigma_2^2\Big)\,. \end{aligned} \tag{2.80}$$

Notice that the contracted EH-I metric with $ISO(2)$ isometry is precisely the 4-dimensional part of the uplifted metric for the Bianchi VII_0 single domain wall.

The metrics with Heisenberg isometry are obtained by a further contraction $\tilde{q}_1 = \tilde{q}_2 = 0$ in the metric (2.75). Again, among these metrics there is one special case that can also be obtained by a contraction of the contracted EH metric with isometry $ISO(2)$. Notice that it is not possible to have a contracted EH-I metric with Heisenberg isometry since we must satisfy the condition $-s_i < \tilde{q}_i r^4$. The expression for the contracted EH metric with Heisenberg isometry is $(\tilde{q}_3 = 1)$

$$\begin{aligned} ds_4{}^2 &= \left(\frac{s}{r^4}\right)^{-1} dr^2 + r^2(\sigma_1^2 + \sigma_2^2) + \left(\frac{s}{r^4}\right)\sigma_3^2\,, \\ &= \left(\frac{s}{r^4}\right)^{-1} dr^2 + r^2(dz_1{}^2 + dz_2{}^2) + \left(\frac{s}{r^4}\right)(dz_3 + 2z^1 dz_2)^2\,, \end{aligned} \tag{2.81}$$

where $s_2 \equiv s$. This is the 4-dimensional part of the uplifted metric for the Bianchi II single domain wall. This contraction was considered in [70].

2.5 Conclusions

We have reduced $d = 11$ supergravity on a 3-dimensional manifold via the SS2 generalized dimensional reduction. The analysis led us to consider two classes of $d = 8$ gauged maximal supergravity theories which we refer to as class A or B. Class A contains supergravities obtained by reducing the 11-dimensional action and an action for them is available, while the supergravities of class B were derived from the $d = 11$ equations of motion and have no action. Specifically, we have derived and studied only class A, which contains 5 supergravities corresponding to the five different subgroups of $SL(3, \mathbb{R})$: $SO(3), SO(2,1), ISO(2), ISO(1,1)$ and the Heisenberg subgroup.

The results we have found are similar to the $d = 9$ case [24]: in both cases a $GL(11 - d, \mathbb{R})$ group ($d = 8, 9$) and its subgroups are the main characters. The group $GL(11 - d, \mathbb{R})$ appears naturally in ungauged maximal supergravities in d dimensions as part of its duality group since they can be obtained by toroidal compactification of 11-dimensional supergravity. It is natural to expect the existence of gauged supergravities associated to the subgroups of $GL(11 - d, \mathbb{R})$. Some cases are already well known, for instance the $d = 5$ maximal supergravities with gauge groups $SO(6 - l, l)$ (all of them subgroups of $SL(6, \mathbb{R})$) constructed in [73, 131].

An interesting outcome of our analysis is the existence in 8 dimensions of a generic triple domain wall solution (2.69). It can be interpreted as n parallel single domain walls where n is the rank of the mass matrix. For the gauging of $SO(3)$, this result is similar to that of [10]. There, the scalar content was the coset $SL(n, \mathbb{R})/SO(n)$ while we have the product of $n = 3$ with an additional $n = 2$ coset. Note that, in the gauged cases, our coset cannot be reduced to the $SL(3, \mathbb{R})/SO(3)$ by truncation of the $SL(2, \mathbb{R})/SO(2)$ scalars. It is interesting that the structure of [10] extends to more general scalar contents and to the other Bianchi classes. It leads one to expect a similar n-tuple domain wall result in other dimensions. In fact, we verified that in $d = 9$ with scalar content $SO(1,1) \times SL(2, \mathbb{R})/SO(2)$ the earlier results on domain wall solutions in gauged $d = 9$ supergravity [24] can be written as a generic double domain wall solution via coordinate transformations.

The domain wall solutions we have found are not valid everywhere along the y-direction. Reality of the solutions imposes restrictions on the domains of validity, which are dependent on the choice of gauge group. For the gauge groups $SO(2,1)$ and $ISO(1,2)$ the regions of validity are compact and no asymptotic limit can be taken, while for the other three cases the regions are semi-infinite and one can take an asymptotic limit. A nice property of the

$SO(3)$ case is that its asymptotic solution extremizes the scalar potential. This does not occur for the other cases where an asymptotic limit can be taken.

The relation between $d = 8$ domain-wall solutions and gauged supergravities that we have discussed fits naturally in the domain wall/QFT correspondence scheme [28, 94]. As discussed in [28], taking the near-horizon limit of the D6-brane leads to the $d = 8$ $SO(3)$ gauged supergravity. Taking the near-horizon limit of the direct reduction of the D6-brane to $d = 9$ dimensions leads to the $d = 8$ $ISO(2)$-gauged supergravity. A further direct reduction to a 6-brane in $d = 8$ dimensions leads to the $d = 8$ Heisenberg gauged supergravity.

We believe that a crucial difference between class A and B theories is that the Maurer-Cartan 1-forms for traceful structure constants probably have no additional isometry. Therefore, in contrast to the class A case, these reductions cannot be reproduced by any known reduction of the massless IIA theory. Cohomogeneity one solutions of class B Bianchi type have been considered in the literature [69]. It would be interesting to see whether these solutions can be reduced to 1/2 BPS domain wall solutions of the corresponding class B $d = 8$ gauged supergravity.

The uplifting of the tripe domain wall solution to $d = 11$ dimensions leads to a purely gravitational solution whose metric is the direct product of a 7-dimensional Minkowski metric and a non-trivial 4-dimensional Euclidean Ricci-flat metric. The 4-metrics are solutions of 4-dimensional Euclidean gravity. Among them we find generalizations of the Eguchi-Hanson solution to different (class A) Bianchi types. It is interesting to note that the uplifting does not lead to the most general 4-metric with $SO(3)$ isometry. The complete non-singular $SO(3)$-invariant hyper-Kähler metrics in four dimensions are the Eguchi-Hanson, Taub-NUT and Atiyah-Hitchin metrics. The absence of the Taub-NUT and Atiyah-Hitchin metrics in our analysis is related to the fact that only the (generalized) Eguchi-Hanson metric allows a covariantly constant spinor that is independent of the $SO(3)$ isometry directions [65]. In performing the SS2 reduction we have assumed that our spinors are independent of the group manifold coordinates and this assumption is thus not compatible with the Taub-NUT and the Atiyah-Hitchin metrics. It would be interesting to see whether we can relax the SS2 procedure such that the Taub-NUT and Atiyah-Hitchin metrics also obtain a half-supersymmetric domain wall interpretation in $d = 8$ dimensions or whether we should view them as $d = 8$ domain walls with fully broken supersymmetry.

In the same spirit one can hope to extend the SS1 reduction, for example as applied in [24]. In that paper the spinors generally were transforming under the $SL(2, \mathbb{R})$ duality symmetry and, consequently, the spinors were given dependence on the internal direction. However, for contracted manifolds, our Ansatz with dependence on z_1 only, see (2.63), can be interpreted as a reduction from the massless $d = 9$ theory. In this case we have taken the spinors to be independent of the internal direction. We therefore have two reduction Ansatz that only differ in the fermionic sector. Therefore, both the SS1 and SS2 reduction procedures might be amenable to extension and it would be desirable to understand the differences between the resulting gauged supergravities.

The Scherk-Schwarz mechanism is a very powerful and elegant procedure which modifies the standard dimensional reduction scheme in such a way that parameters with dimensions of mass appear in the reduced theories. Therefore, it provides an 11-dimensional origin to many gauged/massive supergravity theories. However, not all of them can be obtained this way, and, in fact, the 11-dimensional origin of many gauged/massive supergravity theories, the most notable example being Romans' theory, remains obscure. So, perhaps, we should adopt a different point of view. If it is not enough to modify the reduction procedure, *why don't we modify the higher dimensional theory?*

Chapter 3

Eleven-dimensional massive supergravity

Generalized dimensional reductions introduce gauge coupling constants and/or mass parameters in the reduction Ansatz and therefore lead to lower dimensional gauged/massive theories even if the higher-dimensional theory is ungauged/massless. In this way, we find an 11-dimensional origin of many gauged/massive supergravities. However, many of these theories cannot be obtained by any sort of (known) dimensional reduction from $d = 11$ supergravity. Romans' massive type IIA supergravity is the prime example.

Another example can be read from the ungauged $N = 2, d = 8$ supergravity. This theory contains two $SU(2)$ triplets of vector fields. The two triplets are related by $Sl(2, \mathbb{R})$ S-duality transformations. It should be possible to gauge $SU(2)$ using as $SU(2)$ gauge fields any of the two triplets. If we gauge the triplet of Kaluza-Klein vectors, we get the theory that Salam & Sezgin obtained by Scherk-Schwarz reduction of 11-dimensional supergravity (see chapter 2). It is not known how to derive from standard 11-dimensional supergravity the "S-dual" theory that one would get gauging the other triplet, that comes from the 11-dimensional 3-form.

$N = 1$ $d = 11$ supergravity is believed to be the 11-dimensional theory from which all supergravity theories describing the classical limit of a string theory should be derived. However, no known reduction procedure leads to the gauged/massive supergravity theories listed above. Then, we could ask if

is it possible to deform 11-dimensional supergravity?

Let us focus on Romans' theory as a relevant example of a gauged/massive supergravity with unknown 11-dimensional origin. This theory is basically the extension of type IIA supergravity which includes D8-branes (see section 1.6). The 11-dimensional objects from which D8-branes might arise (if any) are "M8"- or "M9"-branes, depending on whether the reduction is direct or double, respectively. While there is no indication for the existence of M8-branes, M9-branes have been conjectured to exist from the study of the M-superalgebra [91, 164], which is given by

$$\left\{\hat{Q}^{\alpha},\hat{Q}^{\beta}\right\}=\left(\hat{\Gamma}^{\hat{a}}\hat{\mathcal{C}}^{-1}\right)^{\alpha\beta}\hat{P}_{\hat{a}}+\tfrac{1}{2}\left(\hat{\Gamma}^{\hat{a}\hat{b}}\hat{\mathcal{C}}^{-1}\right)^{\alpha\beta}\hat{\mathcal{Z}}_{\hat{a}\hat{b}}+\tfrac{1}{5!}\left(\hat{\Gamma}^{\hat{a}_1\cdots\hat{a}_5}\hat{\mathcal{C}}^{-1}\right)^{\alpha\beta}\hat{\mathcal{Z}}_{\hat{a}_1\cdots\hat{a}_5}\,.$$

The dual of the 2-form central charge $\hat{\mathcal{Z}}_{\hat{a}\hat{b}}$ is a 9-form, and therefore suggests the existence of an 11-dimensional half-supersymmetric 9-brane. Hence, it seems reasonable to think that in the search for an 11-dimensional origin of Romans' theory one is basically looking for an 11-dimensional supergravity including M9-branes. Standard $d = 11$ supergravity does not (seem to) allow for such solutions, so we can think on possible deformations of $d = 11$ supergravity in order that a standard dimensional reduction of the new theory leads to Romans' theory. Furthermore, since 9-branes in $d = 11$ are domain walls, we are basically looking for extensions of 11-dimensional supergravity which include a cosmological constant term. As we will see, making such an extension turns out to be a non-trivial problem. There are various different proposals.

A 'standard' cosmological constant term in $d = 11$

The simplest possibility is the addition of a cosmological constant term to $d = 11$ supergravity. However, there are no-go theorems [11, 52, 53] which assert that this extension is not possible if 11-dimensional covariance is to be preserved. Then, any modification of this theory must be somehow "exotic". Furthermore, although the extension was possible, the cosmological term in $d = 11$ would not lead to the correct dilaton potential of Romans' theory, and therefore a standard dimensional reduction of such a 'cosmological $d = 11$ supergravity' would not lead to Romans' theory.

MM-theory

It was noticed in [86] that the equations of motion of $d = 11$ supergravity can be slightly modified by solving the superspace constraints with a conformal spin connection $\tilde{\omega} \sim \omega + k$ rather than the usual one ω. This means that ω takes values in the tangent space group Spin(1,10), while $\tilde{\omega}$ takes values in

CSpin(1,10). The only condition to be satisfied is that the conformal part of the curvature vanishes, *i.e.* $dk = 0$. If the spacetime is simply connected (*e.g.* 11-dimensional Minkowski spacetime M_{11}), then k is exact and the modification is nothing but a field redefinition. However, in non-simply connected spacetimes the modification is non-trivial and leads to an extension of the 11-dimensional equations of motion [87]. This theory has been referred to as *MM-theory* [38].

The simplest example of a non-simply connected spacetime, $M_{10} \times S^1$, was considered in [87]. One can take $k = m\,dz$, with m a constant with *dimensions of mass* and dz the tangent vector to the circle. A standard reduction of MM-theory (on $M_{10} \times S^1$) on the circle leads to a massive deformation of type IIA supergravity in which the 1- and 3-form RR fields become massive by "eating" the dilaton and NS-NS 2-form, respectively, the vector field being tachyonic. The reduced theory admits no domain wall 8-brane solution, but a de Sitter solution can be found.

The extension of $d = 11$ supergravity which leads to MM-theory is only possible in the equations of motion, and therefore dimensional reductions can only be applied to this set of equations. Moreover, it has been observed that it is not possible to relate Romans' theory and the massive type IIA supergravity derived from MM-theory by means of any field redefinition [87, 106] and therefore they are considered as different theories. Hence, MM-theory does not provide Romans' theory with an eleven-dimensional origin and, still, our question remains unanswered.

The 10-dimensional theory obtained in [87] is exactly the massive supergravity theory derived in [106] via an SS1 reduction of 11-dimensional supergravity exploiting a global scaling symmetry of the 11-dimensional equations of motion (see chapter 2.). So we have two conceptually different constructions which lead to the same results. Although we will not study these constructions, it would be interesting to understand their relation.

Eleven-dimensional massive supergravity

A somewhat different but complementary way of studying the problem of the eleven dimensional origin of Romans' theory is to study the effective world-volume (WV) actions of the objects present in the theory. Such a point of view was adopted in [111, 127] to study the M-theory origin of the massive D2-brane[1] of Romans' theory.

[1]Of course, all branes are massive in the sense that their gravitational mass is not zero. We refer here to *massless/massive* branes as those branes which are solutions of a mass-

It was shown in [153, 163] that the massless D2-brane worldvolume action can be rewritten as a massless M2-brane action. The main point of it is that it requires that the Born-Infeld vector is Poincaré dualized (in the worldvolume) into an embedding coordinate. The M2-brane WV action leads to the D2-brane WV action upon standard dimensional reduction on a circle. These branes are solutions of two different supergravity theories, which are also related via standard dimensional reduction: $d = 11$ supergravity leads to type IIA supergravity upon dimensional reduction on a circle.

One could follow the same reasoning for the massive case and try to find the WV action of the *massive M2-brane*, an object which, upon dimensional reduction on a circle, would lead to the massive D2-brane. The theory of which the massive M2-brane would be a solution would then be, in principle, an 11-dimensional supergravity theory related to Romans' theory via dimensional reduction.

It was noticed in [111, 127] that the WV action of the massive M2-brane is a gauged sigma model obtainable from the massless M2-brane WV action by the gauging of an isometry. This implies the appearance of a Killing vector in the WV action. A similar property was observed previously for KK-monopoles [18], and their WV action was proposed to be a gauged sigma model. A crucial difference between massive M2-branes and KK-monopoles is that the latter arise as solutions of $d = 11$ supergravity while the former do not.

The study of the massive M2-brane was extended in [19] to a generic massive M-brane, and it was shown that the general WV theory of any of these branes is given by a gauged sigma model. Preserving gauge invariance in the gauging requires the introduction of certain modifications to the 11-dimensional supergravity theory. These modifications were shown to be the same for *all* massive M-branes, and consequently a new supergravity theory for these branes was constructed [19]. We usually refer to this theory as *eleven-dimensional massive supergravity* or, for short, *BLO theory*. It requires the existence of a Killing isometry, such that an explicit Killing vector must be included in the Lagrangian. In coordinates adapted to the isometry, the 11-dimensional fields do not depend on that coordinate. The explicit presence of the Killing vector in the action breaks 11-dimensional covariance, and the action is only 10-dimensional covariant. In this sense, we may say that we have simply rewritten a 10-dimensional theory in an 11-dimensional fashion.

A Killing vector is dimensionful, so it must be accompanied by a parameter with dimensions of mass. When this parameter is set to zero, the Killing

less/massive supergravity theory.

vector disappears from the action and $d = 11$ (massless) supergravity is recovered. Hence, BLO theory can be understood as a deformation of standard 11-dimensional supergravity that breaks the 11-dimensional Lorentz symmetry to the 10-dimensional one. One of the new features that BLO theory introduces is the presence of a cosmological constant-type term. This term is proportional to the (squared) mass parameter and to the (fourth power of the) modulus of the Killing vector. However, the no-go theorems of [11, 52, 53] can now be evaded as the theory is not 11-dimensional covariant.

The standard dimensional reduction of BLO theory on a circle in the direction of the Killing vector leads to Romans' theory, as expected. Indeed, one can understand BLO theory as a way of rewriting Romans' theory in an 11-dimensional fashion.

One could think on generalizing BLO theory so as to include more than one Killing vector. This extension was constructed in [122], and we will refer to it as *BLO_n theory*, where the subindex n indicates the number of isometric directions. As in the original case, the Killing vectors are accompanied by mass parameters, now entering an $n \times n$ symmetric matrix. This theory can also be understood as a deformation of $d = 11$ supergravity which, due to the presence of the n Killing vectors, breaks the 11-dimensional Lorentz symmetry to the $(10 - n)$-dimensional one even if the theory is formally 11-dimensional covariant. Analogously to the case of BLO theory, we can understand BLO_n theory as a way of rewriting an $(11 - n)$-dimensional gauged/massive supergravity in a 11-dimensional manner. Then, we can use it as a systematic prescription to obtain gauged/massive supergravity theories by choosing the number of Killing vectors and reducing the theory on an n-torus [2, 3, 122].

The complete BLO theory including fermions has not been written yet, but the supersymmetry transformation rules were derived in [21], and also reproduce those of Romans' theory after the dimensional reduction. The supersymmetry transformations when n Killing vectors are present were obtained in [66].

The aim of this chapter is to explore some of the different gauged/massive supergravity theories that BLO_n theory leads to upon dimensional reduction on an n-torus.

The organization of the chapter is as follows. In section 3.1 we construct BLO theory and show how it leads to Romans' theory upon dimensional reduction in the direction of the Killing vector. We also comment on a possible interpretation of the theory. The generalization of BLO theory to include a arbitrary number of Killing vectors is presented in section 3.2, where we also comment on the dimensional reduction of the theory on an n-torus. Section 3.4 contains the reduction to nine dimensions in the direction of two Killing vectors, leading to a family of $N=2$ $d=9$ gauged/massive supergravity theories which can also be obtained from a non-geometrical Scherk-Schwarz reduction of type IIB supergravity. Section 3.5 is devoted to the reduction of BLO_3 theory on a 3-torus to obtain $N=2$ $d=8$ gauged/massive supergravity theories, which we compare with the 8-dimensional supergravities obtained by geometrical Scherk-Schwarz reduction of 11-dimensional supergravity in chapter 2. We also outline in this section the reduction in the case of an arbitrary number of Killing vectors. Section 3.6 is devoted to the reduction to five dimensions ($n=6$), where we propose an equivalence with the 5-dimensional gauged supergravity theories constructed in [73, 131]. Finally, our conclusions and discussion are presented in section 3.7.

3.1 BLO theory

The field content of 11-dimensional massive supergravity or BLO theory is the same as that of $d=11$ supergravity, *i.e.* the Elfbein and the 3-form potential, and it presents no new degrees of freedom. We need to introduce a Killing vector in the action such that

$$\pounds_{\hat{k}}\hat{g}_{\hat{\mu}\hat{\nu}} = \pounds_{\hat{k}}\hat{C}_{\hat{\mu}\hat{\nu}\hat{\rho}} = 0\,. \tag{3.1}$$

Let us construct first the 1-form *massive gauge parameter* $\hat{\lambda}_{\hat{\mu}}$

$$\hat{\lambda}^{(m)} \equiv m\,(i_{\hat{k}}\hat{\chi})\,, \tag{3.2}$$

where $\hat{\chi}$ is the 11-dimensional gauge parameter, and $i_v T$ denotes the contraction of the last index of the covariant tensor T with the vector v, *e.g.* $(i_k C)_{\mu\nu} = k^\rho C_{\mu\nu\rho}$. For any 11-dimensional tensor $\hat{T}$, we define the *massive gauge transformations* as

$$\delta_{\hat{\chi}}\hat{T}_{\hat{\mu}_1\cdots\hat{\mu}_r} \;=\; -\hat{\lambda}_{\hat{\mu}_1}\hat{k}^{\hat{\nu}}\hat{T}_{\hat{\nu}\hat{\mu}_2\cdots\hat{\mu}_r} - \cdots - \hat{\lambda}_{\hat{\mu}_r}\hat{k}^{\hat{\nu}}\hat{T}_{\hat{\mu}_1\cdots\hat{\mu}_{r-1}\hat{\nu}}\,. \tag{3.3}$$

According to this general rule, the massive gauge transformation of the 11-dimensional metric $\hat{g}_{\hat{\mu}\hat{\nu}}$ and of any 11-dimensional form of rank r $\hat{A}_{\hat{\mu}_1\cdots\hat{\mu}_r}$ are given by

$$\left\{\begin{array}{rcl}\delta_{\hat{\chi}}\hat{g}_{\hat{\mu}\hat{\nu}} & = & -2\hat{k}_{(\hat{\mu}}\,\hat{\lambda}_{\hat{\nu})}\,,\\ \delta_{\hat{\chi}}\hat{A}_{\hat{\mu}_1\cdots\hat{\mu}_r} & = & (-)^r r\hat{\lambda}_{[\hat{\mu}_1}\left(i_{\hat{k}}\hat{A}\right)_{\hat{\mu}_2\cdots\hat{\mu}_r]}\,,\end{array}\right. \tag{3.4}$$

which, together, imply

$$\left\{\begin{array}{rcl}\delta_{\hat{\chi}}\sqrt{|\hat{g}|} & = & 0\,,\\ \delta_{\hat{\chi}}\hat{A}^2 & = & 0\,.\end{array}\right. \tag{3.5}$$

However, the 3-form of massive 11-dimensional supergravity does not transform homogeneously under massive gauge transformations, but

$$\delta_{\hat{\chi}}\hat{C} = 3\partial\hat{\chi} - 3\hat{\lambda}\left(i_{\hat{k}}\hat{C}\right)\,, \tag{3.6}$$

which allows us to see it as a sort of *connection.* The massive 4-form field strength is given by

$$\hat{G} = 4\partial\hat{C} + m3\left(i_{\hat{k}}\hat{C}\right)^2\,, \tag{3.7}$$

and transforms covariantly, according to the above general rule, so

$$\delta_{\hat{\chi}}\hat{G}^2 = 0\,. \tag{3.8}$$

The action for the proposed *massive 11-dimensional supergravity* then reads

$$\begin{array}{rcl}\hat{S} & = & \frac{1}{16\pi G_N{}^{(11)}}\int d^{11}\hat{x}\sqrt{|\hat{g}|}\,\Big\{\hat{R}(\hat{g}) - \frac{1}{2\cdot 4!}\hat{G}^2 - \hat{K}_{\hat{\mu}\hat{\nu}\hat{\rho}}\hat{K}^{\hat{\nu}\hat{\rho}\hat{\mu}} - \frac{1}{2}m(d\hat{k})_{\hat{\mu}\hat{\nu}}(i_{\hat{k}}\hat{C})^{\hat{\mu}\hat{\nu}}\\ & & -\frac{1}{2}m^2|\hat{k}|^4 - \frac{1}{6^4}\frac{\hat{\epsilon}}{\sqrt{|\hat{g}|}}\left[\partial\hat{C}\partial\hat{C}\hat{C} - \frac{9}{8}m\partial\hat{C}\hat{C}\,(i_{\hat{k}}\,\hat{C})^2 + \frac{27}{80}m^2\hat{C}\,(i_{\hat{k}}\,\hat{C})^4\right]\Big\}\,,\end{array} \tag{3.9}$$

where

$$\hat{K}_{\hat{a}\hat{b}\hat{c}} = \frac{1}{2}\left(\hat{T}_{\hat{a}\hat{c}\hat{b}} + \hat{T}_{\hat{b}\hat{c}\hat{a}} - \hat{T}_{\hat{a}\hat{b}\hat{c}}\right)\,, \tag{3.10}$$

is the *contorsion* tensor, and the *torsion* tensor is defined by

$$\hat{T}_{\hat{\mu}\hat{\nu}}{}^{\hat{\rho}} = m(i_{\hat{k}}\hat{C})_{\hat{\mu}\hat{\nu}}\hat{k}^{\hat{\rho}}\,. \tag{3.11}$$

The action is invariant under massive gauge transformations up to total derivatives.

Finally, the supersymmetry transformations under which BLO theory is invariant (up to total derivatives) are given by [21]

$$\begin{aligned} \tfrac{1}{2}\delta_{\hat{\epsilon}}\hat{\psi}_{\hat{\mu}} \;=\; & \Big\{\hat{\nabla}_{\hat{\mu}}(\hat{\omega}+\hat{K}) + \tfrac{i}{288}\left[\hat{\Gamma}^{\hat{a}\hat{b}\hat{c}\hat{d}}{}_{\hat{\mu}} - 8\hat{\Gamma}^{\hat{b}\hat{c}\hat{d}}\hat{e}_{\hat{\mu}}^{\hat{a}}\right]\hat{G}_{\hat{a}\hat{b}\hat{c}\hat{d}} \\ & -\tfrac{i}{12}m|\hat{k}|^2\hat{\Gamma}_{\hat{\mu}} + \tfrac{i}{2}m\hat{k}_{\hat{\mu}}\hat{k}_{\hat{\nu}}\hat{\Gamma}^{\hat{\nu}}\Big\}\,\hat{\epsilon}\,. \end{aligned} \tag{3.12}$$

As we have seen, the construction of massive eleven-dimensional supergravity theory requires the existence of an isometry. This is reflected in the action with the explicit appearance of the Killing vector associated to the isometry. In adapted coordinates, the fields do not depend on that coordinate, and therefore it is not properly an 11-dimensional theory. Therefore, BLO theory can be considered as way of *rewriting* a 10-dimensional theory in an 11-dimensional fashion. In fact, we expect this 10-dimensional theory to be Romans' theory.

3.1.1 From BLO theory to Romans' theory

Let us now reduce BLO theory in the direction of the Killing vector, say z, and see that we end up with Romans' theory. The reduction Ansatz for the 11-dimensional fields is the Kaluza-Klein one, namely

$$\begin{aligned} \left(\hat{e}_{\hat{\mu}}{}^{\hat{a}}\right) &= \begin{pmatrix} e^{-\frac{1}{3}\phi}e_{\mu}{}^{a} & e^{\frac{2}{3}\phi}C_{(1)\mu} \\ 0 & e^{\frac{2}{3}\phi} \end{pmatrix}, \\ \left(\hat{e}_{\hat{a}}{}^{\hat{\mu}}\right) &= \begin{pmatrix} e^{\frac{1}{3}\phi}e_{a}{}^{\mu} & -e^{\frac{1}{3}\phi}C_{(1)a} \\ 0 & e^{-\frac{2}{3}\phi} \end{pmatrix}, \end{aligned} \tag{3.13}$$

$$\hat{C}_{\mu\nu\rho} = C_{(3)\mu\nu\rho}\,, \qquad \hat{C}_{\mu\nu\underline{z}} = B_{\mu\nu}\,, \tag{3.14}$$

such that the field content (in stringy notation) is

$$\{g_{\mu\nu}, \phi, B_{\mu\nu}, C_{(3)\mu\nu\rho}, C_{(1)\mu}, \psi_\mu, \lambda\} \tag{3.15}$$

and with a mass parameter[2] m.

The standard dimensional reduction techniques lead to the following string-frame bosonic action

$$\begin{aligned} S \;=\; & \frac{1}{16\pi G_N^{(10)}} \int d^{10}x\, \sqrt{|g|}\, \Big\{ e^{-2\phi} \left[R - 4\,(\partial\phi)^2 + \tfrac{1}{2\cdot 3!} H^2 \right] \\ & - \left[\tfrac{1}{2} m^2 + \tfrac{1}{2\cdot 2!} m^2 \left(G_{(2)}\right)^2 + \tfrac{1}{2\cdot 4!} \left(G_{(4)}\right)^2 \right] \\ & - \tfrac{1}{144} \frac{1}{\sqrt{|g|}}\, \epsilon \left[\partial C_{(3)} \partial C_{(3)} B + \tfrac{1}{2} m \partial C_{(3)} B^3 + \tfrac{9}{80} m^2 B^5 \right] \Big\}\,, \end{aligned} \tag{3.16}$$

where ${G_N}^{(10)} = {G_N}^{(11)}/2\pi R$, with R the radius of the circle. In the action above, $G^{(2)}$ and $G^{(4)}$ are the RR 2- and 4-form field strengths

$$G_{(2)} = 2\partial C_{(1)} + mB\,, \qquad G_{(4)} = 4\partial C_{(3)} - 12\partial B C_{(1)} + 3mB^2\,, \tag{3.17}$$

and H is the NSNS 3-form field strength

$$H = 3\partial B\,. \tag{3.18}$$

The bosonic action and field strengths above are exactly those of Romans' theory. Let us see now the gauge invariances and supersymmetry transformations.

The field strengths and the Lagrangian are invariant under the bosonic gauge transformations

$$\begin{aligned} \delta B &= 2\partial\Lambda_{(1)}\,, \\ \delta C_{(1)} &= \partial\Lambda_{(0)} - m\Lambda_{(1)}\,, \\ \delta C^{(3)} &= 3\partial\Lambda_{(2)} - 3mB\Lambda_{(1)} - H\Lambda_{(0)}\,, \end{aligned} \tag{3.19}$$

[2] This is the parameter m_R of refs. [24, 25].

where the gauge parameters $\Lambda_{(2)}, \Lambda_{(1)}, \Lambda_{(0)}$ are related to the 11-dimensional ones $\hat{\chi}_{\hat{\mu}\hat{\nu}}$ and $\hat{\xi}^{\hat{\mu}}$ (the generator of infinitesimal g.c.t.'s) by

$$\hat{\chi}_{\mu\nu} = \Lambda_{(2)\,\mu\nu}\,, \qquad \hat{\chi}_{\mu\underline{z}} = \tfrac{1}{m}\hat{\lambda}_{\mu} = \Lambda_{(1)\,\mu}\,, \qquad \hat{\xi}^{\underline{z}} = \Lambda_{(0)}\,. \tag{3.20}$$

The massive gauge invariance of this theory does not lead to a gauged supergravity just because the dimensional reduction of the massive gauge parameter only gives a 1-form.

The Ansatz for the fermionic fields and supersymmetry parameter is

$$\hat{\epsilon} = e^{-\frac{1}{6}\phi}\,\epsilon\,, \qquad \hat{\psi}_a = e^{\frac{1}{6}\phi}\left(2\psi_a - \tfrac{1}{3}\Gamma_a\lambda\right)\,, \qquad \hat{\psi}_z = \tfrac{2i}{3}e^{\frac{1}{6}\phi}\Gamma_{11}\lambda\,, \tag{3.21}$$

and leads to the supersymmetry transformation rules

$$\left\{\begin{array}{rcl} \delta_\epsilon\psi_\mu & = & \left\{\partial_\mu - \tfrac{1}{4}\left(\not{\omega}_\mu + \tfrac{1}{2}\Gamma_{11}\,\not{H}_\mu\right)\right\}\epsilon + \tfrac{i}{8}e^{\phi}\Sigma_{n=0}^{n=2}\tfrac{1}{(2n)!}\,\not{G}_{(2n)}\Gamma_\mu\left(-\Gamma_{11}\right)^n\epsilon\,, \\ \delta_\epsilon\lambda & = & \left[\not{\partial}\phi + \tfrac{1}{2\cdot 3!}\Gamma_{11}\,\not{H}\right]\epsilon + \tfrac{i}{4}e^{\phi}\sum_{n=0}^{n=2}\tfrac{5-2n}{(2n)!}\,\not{G}_{(2n)}\left(-\Gamma_{11}\right)^n\epsilon\,. \end{array}\right. \tag{3.22}$$

where

$$G^{(0)} = m\,. \tag{3.23}$$

As expected, we have obtained the transformations of Romans' theory. Therefore, BLO theory leads to Romans' upon standard dimensional reduction on a circle, at least at the level of the bosonic action and the supersymmetry transformations. But... is BLO theory just a way of rewriting Romans' theory in an 11-dimensional form?

3.1.2 An interpretation of BLO theory

The mass parameter of BLO theory appears always together with the Killing vector, which implies the existence of an isometric direction. If we set the mass parameter to zero, then the isometry "disappears" and we are left with standard $d = 11$ supergravity. This fact allows for an interesting interpretation of the theory [21]. Let us come back for a moment to Romans' theory to start our discussion.

The inclusion of D8-branes in type IIA supergravity leads to the BRGPT theory, a 'generalization' of Romans' theory. The BRGPT theory admits domain wall solutions which separate regions of spacetime with different values of the cosmological constant, and the dynamics in these regions is described by Romans' theory. One possible value of the cosmological constant is zero, and, in that region, low energy Physics is adequately described by standard (massless) type IIA supergravity.

BLO theory admits a domain-wall solution which breaks one half of the supersymmetries [21]. This solution is nothing but the uplift to $d = 11$ of the 8-brane solution of Romans' theory. However, the 8-brane solution which is supposed to represent the long-range field emitted by the D8-brane is a solution of the BRGPT theory and not of Romans'. Then, we would expect BRGPT theory to have an "11-dimensional" formulation, whose 9-brane solution would be the uplift of the 8-brane solution of BRGPT. One could interpret this solution as the supergravity solution associated to an M9-brane (sometimes called KK9-brane).

To this end, BLO theory was reformulated in terms of a 10-form gauge potential instead of the mass parameter [146][3], where also a target space solution of a KK9-brane was obtained. As expected, it was shown to lead to the D8-brane solution of BRGPT theory upon direct standard dimensional reduction on a circle.

The KK9-brane solution found in [146] is a domain wall which separates regions of spacetime with different values of the mass parameter m. In a region in which the mass parameter is zero, no isometric direction is present and the dynamics is described by standard $d = 11$ supergravity, while the dynamics in regions with $m \neq 0$ is governed by BLO theory. Note that this is the straightforward uplift of the picture of a D8-brane in BRGPT theory.

Therefore, although BLO theory can be understood as just a way of rewriting Romans' theory in an 11-dimensional manner, we could also interpret it (or, better, its generalization to include a 10-form potential) as 11-dimensional supergravity in a background of M9-branes, which implies the existence of an isometric direction [21].

[3] The WV action and Wess-Zumino term for the KK9-brane were constructed as those of gauged sigma models in [57] and [146], respectively.

3.2 Generalizations of BLO theory

In this section we are going to construct BLO$_n$ theory following the lines outlined in section 3.1 for the construction of BLO theory. The latter is nothing but BLO$_n$ theory with only one Killing vector ($n = 1$) and with the mass matrix set to $\mathsf{Q}^{mn} = -m\delta^{mn}$.

The standard assumption in toroidal dimensional reductions is that all the fields of this theory are independent of n internal coordinates z^m. It means that the metric admits n mutually commuting Killing vectors $\hat{k}_{(n)}$ associated to the internal coordinates by

$$\hat{k}_{(m)}{}^{\hat{\mu}}\partial_{\hat{\mu}} = \frac{\partial}{\partial z^m} \equiv \partial_m\,. \tag{3.24}$$

Let us introduce now an arbitrary symmetric *mass* matrix Q^{mn}. With these elements (the Killing vectors and the mass matrix) we are going to deform the massless theory.

The *massive gauge parameter* 1-form is now defined as

$$\hat{\lambda}^{(m)} \equiv -\mathsf{Q}^{mn} i_{\hat{k}_{(n)}}\hat{\chi}\,, \tag{3.25}$$

and the massive gauge transformations for any 11-dimensional tensor $\hat{T}$ are given by

$$\delta_{\hat{\chi}}\hat{T}_{\hat{\mu}_1\cdots\hat{\mu}_r} \;=\; -\hat{\lambda}^{(n)}{}_{\hat{\mu}_1}\hat{k}_{(n)}{}^{\hat{\nu}}\hat{T}_{\hat{\nu}\hat{\mu}_2\cdots\hat{\mu}_r} - \cdots - \hat{\lambda}^{(n)}{}_{\hat{\mu}_r}\hat{k}_{(n)}{}^{\hat{\nu}}\hat{T}_{\hat{\mu}_1\cdots\hat{\mu}_{r-1}\hat{\nu}}\,, \tag{3.26}$$

which for the 11-dimensional metric and for any 11-dimensional form $\hat{A}$ of rank r read

$$\left\{\begin{array}{rcl}\delta_{\hat{\chi}}\hat{g}_{\hat{\mu}\hat{\nu}} & = & -2\hat{k}_{(m)\,(\hat{\mu}}\,\hat{\lambda}^{(m)}{}_{\hat{\nu})}\,,\\[2ex] \delta_{\hat{\chi}}\hat{A}_{\hat{\mu}_1\cdots\hat{\mu}_r} & = & (-)^r r\hat{\lambda}^{(n)}{}_{[\hat{\mu}_1}\left(i_{\hat{k}_{(n)}}\hat{A}\right)_{\hat{\mu}_2\cdots\hat{\mu}_r]}\,.\end{array}\right. \tag{3.27}$$

As in the $n = 1$ case, the transformations above imply

$$\left\{\begin{array}{rcl}\delta_{\hat{\chi}}\sqrt{|\hat{g}|} & = & 0\,,\\[2ex] \delta_{\hat{\chi}}\hat{A}^2 & = & 0\,.\end{array}\right. \tag{3.28}$$

The massive gauge transformations of 11-dimensional 3-form are

$$\delta_{\hat{\chi}}\hat{C} = 3\partial\hat{\chi} - 3\hat{\lambda}^{(n)} i_{\hat{k}_{(n)}}\hat{C}\,. \tag{3.29}$$

The massive 4-form field strength is given by

$$\hat{G} = 4\partial\hat{C} - 3\mathsf{Q}^{mn}\, i_{\hat{k}_{(m)}}\hat{C}\, i_{\hat{k}_{(n)}}\hat{C}\,, \tag{3.30}$$

and transforms covariantly, so

$$\delta_{\hat{\chi}}\hat{G}^2 = 0\,. \tag{3.31}$$

The action for BLO_n theory then reads

$$\begin{aligned}\hat{S} \;=\; \tfrac{1}{16\pi G_N{}^{(11)}}\int d^{11}\hat{x}\sqrt{|\hat{g}|}\,\Big\{&\hat{R}(\hat{g}) - \tfrac{1}{2\cdot 4!}\hat{G}^2 - \hat{K}_{\hat{\mu}\hat{\nu}\hat{\rho}}\hat{K}^{\hat{\nu}\hat{\rho}\hat{\mu}} + \tfrac{1}{2}\mathsf{Q}^{mn} d\hat{k}_{(m)} i_{\hat{k}_{(n)}}\hat{C}\\ &+\tfrac{1}{2}(\mathsf{Q}^{mn}\hat{k}_{(m)}{}^{\hat{\mu}}\hat{k}_{(n)\,\hat{\mu}})^2 - (\mathsf{Q}^{mn}\hat{k}_{(m)\,\hat{\mu}}\hat{k}_{(m)\,\hat{\nu}})^2\\ &-\tfrac{1}{6^4}\tfrac{\hat{\epsilon}}{\sqrt{|\hat{g}|}}\Big[\partial\hat{C}\partial\hat{C}\hat{C} + \tfrac{9}{8}\mathsf{Q}^{mn}\partial\hat{C}\hat{C}\, i_{\hat{k}_{(m)}}\hat{C}\, i_{\hat{k}_{(n)}}\hat{C}\\ &+\tfrac{27}{80}\mathsf{Q}^{mn}\mathsf{Q}^{pq}\hat{C}\, i_{\hat{k}_{(m)}}\hat{C}\, i_{\hat{k}_{(n)}}\hat{C}\, i_{\hat{k}_{(p)}}\hat{C}\, i_{\hat{k}_{(q)}}\hat{C}\Big]\Big\}\,,\end{aligned} \tag{3.32}$$

where the contorsion and torsion tensors are defined, respectively, as

$$\hat{K}_{\hat{a}\hat{b}\hat{c}} = \tfrac{1}{2}\left(\hat{T}_{\hat{a}\hat{c}\hat{b}} + \hat{T}_{\hat{b}\hat{c}\hat{a}} - \hat{T}_{\hat{a}\hat{b}\hat{c}}\right)\,, \tag{3.33}$$

and

$$\hat{T}_{\hat{\mu}\hat{\nu}}{}^{\hat{\rho}} = -\mathsf{Q}^{mn}(i_{\hat{k}_{(m)}}\hat{C})_{\hat{\mu}\hat{\nu}}\hat{k}_{(n)}{}^{\hat{\rho}}\,. \tag{3.34}$$

Let us now comment briefly on the fermions of the theory. The above theory is a straightforward generalization to arbitrary n of the $n=2$ case obtained by uplifting of the gauged/massive $N=2, d=9$ supergravities constructed in Ref. [122] by non-geometrical Scherk-Schwarz reduction of the $N=2B, d=10$ theory. This construction was made for the bosonic sector only, but can be made for the full supergravity Lagrangian, as shown in Ref. [66]. Once the full gauged/massive $N = 2, d = 9$ supergravity is constructed it can be uplifted to $d = 11$ and then generalized to arbitrary n. This was done for the fermionic supersymmetry transformation rules in [66] and they were found to have the form

$$\begin{aligned}\tfrac{1}{2}\delta_{\hat{\epsilon}}\hat{\psi}_{\hat{\mu}} \;&=\; \Big\{\hat{\nabla}_{\hat{\mu}}(\hat{\omega}+\hat{K}) + \tfrac{i}{288}\left[\hat{\Gamma}^{\hat{a}\hat{b}\hat{c}\hat{d}}{}_{\hat{\mu}} - 8\hat{\Gamma}^{\hat{b}\hat{c}\hat{d}}\hat{e}^{\hat{a}}_{\hat{\mu}}\right]\hat{G}_{\hat{a}\hat{b}\hat{c}\hat{d}} \\ &\quad -\tfrac{i}{12}\hat{k}_{(n)\hat{\nu}}\mathsf{Q}^{nm}\hat{k}^{\hat{\nu}}_{(m)}\hat{\Gamma}_{\hat{\mu}} + \tfrac{i}{2}\hat{k}_{(n)\hat{\mu}}\mathsf{Q}^{nm}\hat{k}_{(m)\hat{\nu}}\hat{\Gamma}^{\hat{\nu}}\Big\}\,\hat{\epsilon}\,.\end{aligned} \tag{3.35}$$

3.3 Dimensional reduction of BLO$_n$ theory

By construction, BLO$_n$ theory is meant to be compactified in the n-dimensional torus parametrizes by the coordinates z^m. After that dimensional reduction, the explicit Killing vectors in the action disappear and one gets a genuine $(11-n)$-dimensional field theory. In the next few sections we will consider it as a systematic prescription to get gauged/massive supergravity theories in $(11-n)$ dimensions, which we will study for several values of n. We have previously shown how BLO theory ($n = 1$) leads to Romans' theory upon dimensional reduction on a circle. Let us now study the $n > 1$ cases. We will mainly concentrate in the case with $n = 2, 3, 6$, though a reduction for arbitrary n is outlined in section 3.5. In all cases, we will use the Kaluza-Klein Ansatz, *i.e.* the same Ansatz we employ for the reduction of the standard 11-dimensional supergravity.

Also by construction, there is a natural action of the group $Gl(n,\mathbb{R})$ in these theories, all the objects carrying m, n indices (including the mass matrix) transforming in the tensor representations. The subgroup of $Gl(n,\mathbb{R})$ that preserves the mass matrix will be a symmetry group of the theory.

The gauge invariances of the gauged/massive supergravities that we will obtain are encoded in the 11-dimensional *massive gauge transformations* parametrized by the 1-forms $\hat{\lambda}^{(m)}$. Their dimensional reduction will give rise to further massive gauge transformations parametrized by 1-forms and associated to massive 2-forms $\lambda^{(m)}_{\mu}$ and will also give rise to (Yang-Mills) gauge transformations parametrized by the scalars $\lambda^{(m)}_{n}$ where the subindex n corresponds to an internal direction. These scalars exist when there is more than one Killing vector and are antisymmetric in the indices m, n and correspond to orthogonal gauge groups. This is consistent with the fact that the gauge vectors come from the components $\hat{C}_{\mu mn}$ and naturally carry a pair of antisymmetric indices corresponding to the adjoint representation of an orthogonal group.

The field content of the reduced theories and the structure of the theories themselves depend on the number of isometric directions one is considering. However, there are common properties:

- there will always be n 2-form potentials coming from the 11-dimensional 3-form which become massive by "eating" n Stückelberg vectors, which are the vectors coming from the metric;
- the gauge vector fields, which are only present for $n \geq 2$, always come from the 11-dimensional 3-form;
- the scalar sector coming from the metric are a dilaton and $n(n+1)/2$ scalar fields entering a matrix $L_m{}^i$ which parametrizes an $Sl(n,\mathbb{R})/SO(n)$ coset.
- there will be a scalar potential for the scalar fields coming from the metric. We comment on it in the next subsection. For $n \geq 3$, also the scalar fields coming from the 11-dimensional 3-form enter the potential.

It is important to recall that a fully supersymmetric theory is obtained for each value of n. In the next sections we are going to see how known gauged/massive supergravities arise in the dimensional reduction of the action for BLO$_n$ theory in the direction of the n Killing vectors $\hat{k}_{(m)}$.

The scalar potential

Before turning to some explicit dimensional reductions, let us comment on the general reduction "cosmological constant" term in BLO$_n$ theory. This term is of the form

$$\int d^{11}\hat{x}\sqrt{|\hat{g}|}\,\left\{\tfrac{1}{2}(\mathsf{Q}^{mn}\hat{k}_{(m)}{}^{\hat{\mu}}\hat{k}_{(n)\,\hat{\mu}})^2 - (\mathsf{Q}^{mn}\hat{k}_{(m)\,\hat{\mu}}\hat{k}_{(m)\,\hat{\nu}})^2\right\}\,, \qquad (3.36)$$

and reproduces, upon dimensional reduction, a scalar potential term. In the $n = 1$ case, the scalar potential is nothing but the dilaton potential of Romans' theory. For $n \geq 2$ it gives rise to a term which in the Einstein frame reads[4]

[4]This result is obtained writing the Ansatz for the internal metric as $G_{mn} = e^{2\varphi/n}\mathcal{M}_{mn}$.

$$\int d^{11-n}x \sqrt{|g_E|}\,\{-\mathcal{V}\}\ , \tag{3.37}$$

where the potential $\mathcal{V}$ reads

$$\mathcal{V} = -\frac{1}{2}\, e^{\aleph\varphi}\ \left\{[\mathrm{Tr}(\mathcal{M}\mathsf{Q})]^2 - 2\mathrm{Tr}\,(\mathcal{M}\mathsf{Q}\mathcal{M}\mathsf{Q})\right\}\ , \tag{3.38}$$

with $\aleph$ a constant given by

$$\aleph = -1 - \frac{4}{n} + \frac{d}{3}\sqrt{\frac{n}{2(d-2)}}\,, \tag{3.39}$$

and $\mathcal{M}$ is an $Sl(n,\mathbb{R})$ matrix given by

$$\mathcal{M}_{mn} = L_m{}^i L_n{}^j \delta_{ij}\ . \tag{3.40}$$

The extrema of the potential are found to be (see appendix D)

$$\mathcal{M}_{0\,mn} = |\mathsf{Q}|^{1/n}\,(\mathsf{Q}^{-1})_{mn}\,, \tag{3.41}$$

where the potential reaches the value

$$\mathcal{V}_0 = -\frac{1}{2}n(n-2)\,|\det\mathsf{Q}|^{2/n}\,e^{\aleph\varphi}\,. \tag{3.42}$$

Note that the signature of $\mathcal{M}$ is $(+\cdots+)$ and therefore, in virtue of (3.41), the potential can only be minimized when the signature of the mass matrix is $(+\cdots+)$. In the cases with such a mass matrix, the potential is extremized with respect to $\mathcal{M}$ but not with respect to the dilaton φ. This is basically telling us that the effective potential in the 'vacuum' is a dilaton potential, such that the 'vacuum' solutions are domain walls. However, there is an exception: $n = 6$ ($d = 5$). In that case one finds $\aleph = 0$, such that the potential only depends on $\mathcal{M}$. The effective potential is then a constant, and the vacuum solution is not a domain wall but anti-de Sitter spacetime. This looks reasonable as we expect BLO_6 compactified on a 6-torus to be the five dimensional gauged supergravity constructed in [73, 75, 131]. We will come back to this in section 3.6.

3.4 Gauged/massive $N = 2$ $d = 9$ supergravities from BLO_2

The reduction of the $n = 2$ case in the direction of the two Killing vectors gives gauged/massive $N = 2, d = 9$ supergravities characterized by the mass matrices Q^{mn} [24,25,45,92,122,126]. The theories obtained this way were also found via a non-geometrical Scherk-Schwarz reduction from $N = 2B, d = 10$ supergravity [122].

The field content of these theories is

$$\{g_{\mu\nu}, \varphi, L_m{}^i, C_{\mu\nu\rho}, B_{m\,\mu\nu}, V_\mu, A^m{}_\mu, \psi^i_\mu, \lambda^i\}\,. \tag{3.43}$$

The $L_m{}^i$ parametrize an $Sl(2,\mathbb{R})/SO(2)$ coset. The field V_μ comes from the 11-dimensional 3-form components $\hat{C}_{\mu mn}$ and will be a gauge field. Its presence is the main new feature with respect to the $n = 1$ case. The gauge group will depend on the choice of mass matrix, as we are going to see. As in all cases, there will always be the same number of 2-forms $B_{\mu\nu\,m}$ and Kaluza-Klein vectors $A^m{}_\mu$ that play the role of Stückelberg fields for the 2-forms.

Explicitly, the Kaluza-Klein Ansatz for the bosonic fields is

$$\begin{aligned}
\left(\hat{e}_{\hat{\mu}}{}^{\hat{a}}\right) &= \begin{pmatrix} e^{-\frac{1}{3\sqrt{7}}\varphi} e_\mu{}^a & e^{\frac{\sqrt{7}}{6}\varphi} L_m{}^i A^m{}_\mu \\ 0 & e^{\frac{\sqrt{7}}{6}\varphi} L_m{}^i \end{pmatrix}, \\
\left(\hat{e}_{\hat{a}}{}^{\hat{\mu}}\right) &= \begin{pmatrix} e^{\frac{1}{3\sqrt{7}}\varphi} e_a{}^\mu & -e^{\frac{1}{3\sqrt{7}}\varphi} A^m{}_a \\ 0 & e^{\frac{-\sqrt{7}}{6}\varphi} L_i{}^m \end{pmatrix},
\end{aligned} \tag{3.44}$$

and[5]

$$\left\{\begin{aligned}
\hat{C}_{\mu\nu\rho} &= C_{\mu\nu\rho} - \tfrac{3}{2} A^m{}_{[\mu} B_{m|\,\nu\rho]} + 3\eta_{mn} V_{[\mu} A^m{}_\nu A^n{}_{\rho]}\,, \\
\hat{C}_{\mu\nu m} &= B_{m\,\mu\nu} - 2\eta_{mn} V_{[\mu} A^n{}_{\nu]}\,, \\
\hat{C}_{\mu mn} &= \eta_{mn} V_\mu\,.
\end{aligned}\right. \tag{3.45}$$

[5]The definition of $C_{\mu\nu\rho}$ is not the most naive $\hat{C}_{abc} \sim C_{abc}$ because in this case one is interested in recovering exactly the theories obtained by non-geometrical Scherk-Schwarz reduction from $N = 2B, d = 10$ supergravity [122].

The gauge parameter $\hat{\chi}_{\hat{\mu}\hat{\nu}}$ gives rise to a scalar parameter σ, two vector parameters $\lambda_{m\,\mu}$ and a 2-form parameter $\chi_{\mu\nu}$:

$$\hat{\chi}_{\mu\nu} = \chi_{\mu\nu}\,, \qquad \hat{\chi}_{\mu m} = \lambda_{m\,\mu}\,, \qquad \hat{\chi}_{mn} = \eta_{mn}\sigma\,, \tag{3.46}$$

The gauge vector V_μ transforms under the group generated by the single[6] local parameter $\sigma(x)$

$$\delta_\sigma V_\mu = \partial_\mu \sigma\,. \tag{3.47}$$

To find which is the one-parameter gauge group we have to look at the δ_σ transformations of the fields that carry $Sl(2,\mathbb{R})$ indices m,n:

$$\begin{aligned} \delta_\sigma L_m{}^i &= -\sigma L_n{}^i \mathsf{m}^n{}_m\,, \\ \delta_\sigma A^m{}_\mu &= \sigma \mathsf{m}^m{}_n A^n{}_\mu\,, \\ \delta_\sigma B_{\mu\nu\,m} &= -\sigma B_{m\,\mu\nu}\mathsf{m}^n{}_m + 2\eta_{mn}\partial_{[\mu}\sigma A^n{}_{\nu]}\,, \end{aligned} \tag{3.48}$$

that leave invariant all the field strengths except for that of $B_{m\,\mu\nu}$ that transforms covariantly. This tells us that the gauge group of the 9-dimensional theory is the group generated by the 2×2 traceless matrix $\mathsf{m}^m{}_n = -\mathsf{Q}^{mp}\eta_{pn}$, which is a generator of a subgroup of $Sl(2,\mathbb{R})$. By construction, it is the subgroup that preserves the mass matrix Q^{mn}: it transforms according to

$$\mathsf{Q}' = \Lambda \mathsf{Q} \Lambda^T\,, \qquad \Lambda = e^{\sigma \mathsf{m}}\,, \tag{3.49}$$

such that the condition that it is preserved $\Lambda^{-1}\mathsf{Q} = \mathsf{Q}\Lambda^T$ translates into

$$\mathsf{mQ} = -\mathsf{Qm}^T\,, \tag{3.50}$$

which is trivially satisfied for $\mathsf{m} = -\mathsf{Q}\eta$ on account of the property $\mathsf{m}^T = -\eta\mathsf{m}\eta^{-1}$.

It is clear that the theories obtained can be classified first by the sign of the determinant of the mass matrix $\alpha^2 = -4\det \mathsf{Q}$, which is an $SL(2,\mathbb{R})$ invariant:

[6] Some of the gauged/massive $N=2, d=9$ theories presented in [25] have a 2-parameter non-Abelian gauge group and, therefore, cannot be described in this framework even if we allowed for more general, non-symmetric mass matrices.

class I $\alpha^2 = 0$, class II $\alpha^2 > 0$ and class III $\alpha^2 < 0$ [24,92]. These classes should be subdivided further into $Sl(2,\mathbb{Z})$ equivalence classes since the theories are equivalent only when they are related by $Sl(2,\mathbb{Z})$ transformations. However, it should be clear that theories within the same α^2 class have the same gauge group, the difference being a change of basis which is an $Sl(2,\mathbb{R})$ but not an $Sl(2,\mathbb{Z})$ transformation.

Thus, all theories in class III ($\alpha^2 < 0$) have gauge group $SO(2)$ and all theories in class II ($\alpha^2 > 0$) have gauge group $SO(1,1)$. The theories in class I ($\alpha^2 = 0$) are all equivalent to one with

$$\mathsf{Q} = \begin{pmatrix} -m & 0 \\ 0 & 0 \end{pmatrix}, \tag{3.51}$$

which is just the reduction of the $n = 1$ case (Romans' theory) considered in the previous section. The group is now $SO(1,1)$, with

$$\Lambda = \begin{pmatrix} 1 & \sigma m \\ 0 & 1 \end{pmatrix}. \tag{3.52}$$

The transformation laws of the fields of this theory are rather unconventional but the theory is still a gauged supergravity.

From a combination of different terms we get the scalar potential

$$\mathcal{V}(\varphi, \mathcal{M}) = \tfrac{1}{2} e^{\frac{4}{\sqrt{7}}\varphi} \mathrm{Tr}\left(\mathsf{m}^2 + \mathsf{m}\mathcal{M}\mathsf{m}^T\mathcal{M}^{-1}\right). \tag{3.53}$$

Its presence suggests the existence of domain-wall (7-brane) solutions which will be the vacua of the different theories obtained from different mass matrices. In fact, these domain-wall solutions correspond to different 7-brane solutions of the $N = 2B, d = 10$ theory: each kind of 10-dimensional 7-brane is characterized by its $Sl(2,\mathbb{Z})$ monodromy Λ and it is possible to reduce the $N = 2B, d = 10$ theory in the Scherk-Schwarz generalized fashion admitting this monodromy for the different fields. The result is a gauged/massive $N = 2, d = 9$ supergravity with a mass matrix Q related to $\Lambda = e^{2\pi\ell\mathsf{m}}$ as explained above. The domain-wall solutions and their 10-dimensional origin and monodromies have been studied in detail in Ref. [24]. A study of the non-conformal 8-dimensional field theories living in the "boundaries" of these solutions and their relations is still lacking.

The 2-forms $B_{m\,\mu\nu}$ are also invariant under the standard 2-form gauge transformations

$$\delta_\lambda B_{m\,\mu\nu} = 2\partial_{[\mu}\lambda_{|m|\nu]}\,. \tag{3.54}$$

This is possible because these transformations are supplemented by the massive gauge transformations of the KK vectors

$$\delta_\lambda A^m{}_\mu = \mathsf{Q}^{mn}\lambda_{n\,\mu}\,, \tag{3.55}$$

that leave invariant the field strength

$$F^m{}_{\mu\nu} = 2\partial_{[\mu}A^m{}_{\nu]} - \mathsf{Q}^{mn}B_{n\,\mu\nu}\,, \tag{3.56}$$

which would allow us to gauge them away giving explicit mass terms to the 2-forms. It is in this way (Stückelberg mechanism) that there is no clash between the gauge invariance under δ_σ and the 2-form gauge transformations δ_λ.

The full bosonic action for this theory can be found in [122], and the fermionic supersymmetry transformation rules were derived in [66].

3.5 Gauged/massive $N = 2$ $d = 8$ supergravities from BLO_3

The reduction of next case $n = 3$ in the direction of the three Killing vectors gives 8-dimensional gauged theories [2, 3]. In this section we are going to use the general formalism and field definitions that will be valid in any dimension to show that in the general case n one can get $SO(n-l,l)$-gauged $(11-n)$-dimensional supergravities.

The field content of these theories is

$$\{g_{\mu\nu}, \varphi, \ell, L_m{}^i, C_{\mu\nu\rho}, B_{m\,\mu\nu}, V_{mn\,\mu}, A^m{}_\mu, \psi^i_\mu, \lambda^i\}\,, \tag{3.57}$$

where the indices $m, n = 1, 2, 3$ are $Sl(3,\mathbb{R})$ indices and also, simultaneously, gauge indices. The $L_m{}^i$ parametrize now an $Sl(3,\mathbb{R})/SO(3)$ coset and the three vector fields $V_{mn\,\mu}$ gauge a 3-dimensional group which should be a subgroup of $Sl(3,\mathbb{R})$[7].

[7]In the general case we will have $Sl(n,\mathbb{R})$ indices, the $L_m{}^i$ will parametrize an $Sl(n,\mathbb{R})/SO(n)$ coset and instead of one scalar ℓ one gets ℓ_{mnp} (here $\ell_{mnp} = \epsilon_{mnp}\ell$). Further, we will have $n(n-1)/2$ gauge vectors $V_{mn\,\mu}$.

The Ansatz for the bosonic fields is the same as the one used in appendix B for the standard dimensional reduction of massless 11-dimensional supergravity on a 3-torus. The 11-dimensional field strength is decomposed with the same structure, but with differences in the definition of the field strengths, due to the presence of massive terms.

The new 8-dimensional field strengths are defined as

$$\begin{aligned}
G_{\mu\nu\rho\sigma} &= 4\partial_{[\mu}C_{\nu\rho\sigma]} + 6B_{m\,[\mu\nu}F^{1\,m}{}_{\rho\sigma]} - 3B_{m\,[\mu\nu|}\mathsf{Q}^{mn}B_{n\,|\rho\sigma]}\,,\\
H_{m\,\mu\nu\rho} &= 3\partial_{[\mu|}B_{m|\,\nu\rho]} + 3V_{mn\,[\mu}F^{1\,n}{}_{\nu\rho]}\,,\\
F_{mn\,\mu\nu} &= 2\partial_{[\mu|}V_{mn|\,\nu]} + 2V_{mp\,[\mu|}\mathsf{Q}^{pq}V_{nq|\,\nu]}\,,\\
F^{1\,m}{}_{\mu\nu} &= 2\partial_{[\mu}A^{m}{}_{\nu]} - \mathsf{Q}^{mn}B_{m\,\mu\nu}\,.
\end{aligned} \tag{3.58}$$

to which we have to add the covariant derivative of the $Sl(3,\mathbb{R})/SO(3)$ scalars[8]

$$\mathcal{D}_\mu L_m{}^i = \partial_\mu L_m{}^i - V_{mp\,\mu}\mathsf{Q}^{pq}L_q{}^i\,. \tag{3.60}$$

The complete $d=8$ gauged/massive action can be written as

[8]In the general case we also have to add the covariant derivative of the ℓ_{mnp} scalars

$$\mathcal{D}_\mu \ell_{mnp} = \partial_\mu \ell_{mnp} - 3V_{[m|q\,\mu}\mathsf{Q}^{qr}\ell_{|np]r}\,, \tag{3.59}$$

that reduces to a partial derivative in $d=8$ when $\ell_{mnp} = \epsilon_{mnp}\ell$.

$$
\begin{aligned}
S \;=\; & \frac{1}{16\pi G_N{}^{(8)}}\int d^8x\sqrt{|g_E|}\;\Big\{R_E+\tfrac{1}{4}\mathrm{Tr}\left(\mathcal{D}\mathcal{M}\mathcal{M}^{-1}\right)^2+\tfrac{1}{4}\mathrm{Tr}\left(\partial\mathcal{W}\mathcal{W}^{-1}\right)^2 \\
& -\tfrac{1}{4}F^{i\,m}\mathcal{M}_{mn}\mathcal{W}_{ij}F^{j\,n}+\tfrac{1}{2\cdot 3!}H_m\mathcal{M}^{mn}H_n-\tfrac{1}{2\cdot 4!}e^{\varphi}G^2-\mathcal{V} \\
& -\tfrac{1}{6^3\cdot 2^4}\frac{1}{\sqrt{|g_E|}}\,\epsilon\,\Big[GG\ell-8GH_mA^{2\,m}+12GG^m_{(2)}B_m-16H_mG^m_{(2)}C \\
& -8G\partial\ell C-8\epsilon^{mnp}H_mH_nB_p+2H_m\mathsf{Q}^{mn}B_n\left(C\ell+6B_pA^{2\;p}\right) \\
& -3B_m\mathsf{Q}^{mn}B_n\left(G\ell+2H_mA^{2\;m}+3B_mG^m_{(2)}+2C\partial\ell\right) \\
& +4Cf_{pq}{}^{m}A^{2\;p}A^{2\;q}H_m-12Cf_{pq}{}^{m}A^{2\;p}G^q_{(2)}B_m \\
& +\tfrac{3^2\cdot 11}{2}\left(B_n\mathsf{Q}^{nm}B_m\right)^2\ell-\tfrac{2^2\cdot 3^3\cdot 11}{5}B_n\mathsf{Q}^{nm}B_mf_{pq}{}^{r}A^{2\;p}A^{2\;q}B_r\Big]\Big\}\,,
\end{aligned}
\tag{3.61}
$$

where $G_N{}^{(8)}=G_N{}^{(11)}/\mathrm{Vol}(T^3)$. In the action above we have written $F_{mn}=\epsilon_{mnp}F^{2p}$ and, as for the massless case (see appendix B), we have introduced the abbreviation

$$
G^m_{(2)} \;=\; F^{2\;m}\;+\;\ell F^{1\;m}\,. \tag{3.62}
$$

$\mathcal{W}$ is a scalar symmetric $Sl(2,\mathbb{R})/SO(2)$ matrix

$$
\mathcal{W}=\frac{1}{\Im\mathrm{m}(\tau)}\left(\begin{array}{cc}|\tau|^2 & \Re\mathrm{e}(\tau)\\ \Re\mathrm{e}(\tau) & 1\end{array}\right)\,, \tag{3.63}
$$

τ being the complex scalar field

$$
\tau=\ell+ie^{\varphi}\,. \tag{3.64}
$$

The scalar potential reads

$$
\mathcal{V}=-\tfrac{1}{2}e^{-\varphi}\left(\ell^2+e^{2\varphi}\right)\,\left\{\left[\mathrm{Tr}\left(\mathsf{Q}\mathcal{M}\right)\right]^2-2\mathrm{Tr}\left(\mathsf{Q}\mathcal{M}\right)^2\right\}\,. \tag{3.65}
$$

Let us now analyze the different gauge symmetries of the theory. The 2-form $\hat{\chi}_{\hat{\mu}\hat{\nu}}$ decomposes now into a 2-form $\chi_{\mu\nu}$, 3 vector parameters $\lambda_{m\,\mu}$ which will be associated to the massive gauge invariances of the 3 2-forms $B_{m\,\mu\nu}$, and 3 scalars $\sigma_{mn} = -\sigma_{nm}$[9]

$$\hat{\chi}_{\mu\nu} = \chi_{\mu\nu}\,, \qquad \hat{\chi}_{\mu m} = \lambda_{m\,\mu}\,, \qquad \hat{\chi}_{mn} = \sigma_{mn}\,. \tag{3.66}$$

It is also convenient to define

$$\sigma^{m}{}_{n} = \mathsf{Q}^{mp}\sigma_{pn}\,. \tag{3.67}$$

These are going to be the infinitesimal generators of the gauge transformations. Observe that, depending on the choice of Q^{mp}, $\sigma^{m}{}_{n}$ can contain an equal or smaller number of independent components than σ_{pn} and, thus, the gauge group can have dimension 3 or smaller.

Under the δ_σ transformations[10]

$$\begin{aligned}
\delta_\sigma L_m{}^i &= -L_n{}^i\sigma^n{}_m\,,\\
\delta_\sigma A^m{}_\mu &= \sigma^m{}_n A^n{}_\mu\,,\\
\delta_\sigma V_{mn\,\mu} &= \mathcal{D}_\mu\sigma_{mn}\,,\\
\delta_\sigma B_{m\,\mu\nu} &= -B_{n\,\mu\nu}\sigma^n{}_m + 2\partial_{[\mu|}\sigma_{mn}A^n{}_{|\nu]}\,,\\
\delta_\sigma C_{\mu\nu\rho} &= 3\partial_{[\mu|}\sigma_{mn}A^m{}_{|\nu}A^n{}_{\rho]}\,,
\end{aligned} \tag{3.69}$$

the field strengths and covariant derivatives transform covariantly, *i.e.*

$$\begin{gathered}
\delta_\sigma G = 0\,, \quad \delta_\sigma H_m = -H_n\sigma^n{}_m\,, \quad \delta_\sigma F_{mn} = -2F_{p[n}\sigma^p{}_{m]}\,, \quad \delta_\sigma F^m = \sigma^m{}_n F^n\,,\\
\delta_\sigma \mathcal{D}L_m{}^i = -(\mathcal{D}L_n{}^i)\sigma^n{}_m\,, \quad \delta_\sigma\mathcal{D}a_{mnp} = -3(\mathcal{D}a_{q[np})\sigma^q{}_{m]}\,.
\end{gathered} \tag{3.70}$$

[9]In the general case we will get n vector parameters associated to the massive gauge invariances of the n 2-forms $B_{m\,\mu\nu}$ and $n(n-1)/2$ scalars $\sigma_{mn} = -\sigma_{nm}$.

[10]In the general case the gauge transformations of these fields take the same form but the scalars ℓ_{mnp} transform covariantly

$$\delta_\sigma \ell_{mnp} = -3\ell_{q[np}\sigma^q{}_{m]}\,. \tag{3.68}$$

This transformation vanishes in $d = 8$ when $\ell_{mnp} = \epsilon_{mnp}\ell$.

Finally, the parameters $\lambda_{m\,\mu}$ generate massive gauge transformations under which

$$
\begin{aligned}
\delta_\lambda A^m{}_\mu &= \mathsf{Q}^{mn}\lambda_{n\,\mu}\,, \\
\delta_\lambda B_{m\,\mu\nu} &= 2\partial_{[\mu|}\lambda_{m\,|\nu]}\,, \\
\delta_\lambda C_{\mu\nu\rho} &= -6A^m{}_{[\mu}\partial_{\nu|}\lambda_{m|\,\rho]}\,,
\end{aligned}
\tag{3.71}
$$

leaving invariant all the field strengths. In this and all cases this invariance can be used to eliminate the 3 (n) KK vector fields $A^m{}_\mu$ giving masses to the 3 (n) 2-forms $B_{m\,\mu\nu}$.

So far, we have obtained a set of gauged supergravity theories in which the gauge group depends on the choice of mass matrix Q^{mn}. The gauge group is the subgroup of $Sl(3,\mathbb{R})$ ($Sl(n,\mathbb{R})$ in the general case) obtained by exponentiation of $\sigma^m{}_n$ that preserves the mass matrix Q^{mn}. For $n = 3$ it is not difficult to classify all the gauge groups that appear by comparing with the Bianchi classification of all real 3-dimensional Lie algebras[11]. We have already done this classification in chapter 2, where we found a similar set of theories via geometrical generalized dimensional reduction from $d = 11$ supergravity on a 3-dimensional manifold. However, the set of theories we have obtained now is different from that of chapter 2. Let us comment on their relation.

Both set of theories contain gauged supergravities. The field contents are identical, only the couplings are different: in the SS case (that of chapter 2) the gauge vector fields are the KK ones $A^{1\,m}$ and the Stückelberg vector fields are the ones coming from the 3-form $A^{2\,m}$, while in our case these roles are interchanged (the 2-forms are always massive). Some of the couplings to the scalars φ and ℓ are also different.

Actually it is convenient to describe the differences between both 8-dimensional theories through the action of the global $Sl(2,\mathbb{R})$ duality symmetry that the (equations of motion of the) massless theory enjoys (see appendix B). The scalars φ and ℓ can be combined in the complex scalar $\tau = a + ie^{\varphi}$ that parametrizes the coset $Sl(2,\mathbb{R})/SO(2)$ and undergoes fractional-linear transformations under $Sl(2,\mathbb{R})$. Under this group, the vector fields form 3 doublets $(A^{1\,m}, A^{2\,m})$ while the 2-forms are singlets[12]. The 4-form field strength G undergoes electric-magnetic duality rotations.

[11] This study is more complicated for $n > 3$ and, further, the real Lie algebras are not classified in general, but only for $n = 3$ (the Bianchi classification) and $n = 4$. See e.g. Ref. [105] and references therein.

[12] Actually, the 2-forms are singlets after a field redefinition.

The differences between the two 8-dimensional gauged theories are associated, precisely, to the $Sl(2,\mathbb{R})$ transformation

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.72}$$

that interchanges the vector fields $A^{1\,m}$ and $A^{2\,m}$ and transforms τ into $-1/\tau$. It also interchanges G^1 and G^2, where

$$G^1 \equiv G\,, \qquad G^2 \equiv -e^{\varphi\star}G - \ell G\,, \tag{3.73}$$

i.e. G^i (in the massless case) transforms as a doublet under $Sl(2,\mathbb{R})$ (just like the doublet $F^{i\,m}$). This duality is reflected in the scalar potential, which in our case is given by

$$\mathcal{V} = -\tfrac{1}{2}\frac{|\tau|^2}{\Im\mathrm{m}(\tau)}\left\{[\mathrm{Tr}\,(\mathrm{Q}\mathcal{M})]^2 - 2\mathrm{Tr}\,(\mathrm{Q}\mathcal{M})^2\right\}\,, \tag{3.74}$$

while in SS's case is (see eq. (2.57))

$$\mathcal{V}_{\mathrm{SS}} = -\tfrac{1}{2}\,g^2\frac{1}{\Im\mathrm{m}(\tau)}\left\{(\mathrm{Tr}\,\mathcal{M})^2 - 2\mathrm{Tr}\,(\mathcal{M}^2)\right\}\,. \tag{3.75}$$

Thus, in a sense, we can view our theory as the S-dual of SS's although, in practice, one cannot perform such a transformation directly on the gauged theories and, rather, one has to do it in the ungauged one. The difference between the theories lies in which triplet of vectors do we gauge.

3.6 Gauged/massive $N = 8$ $d = 5$ supergravities from BLO$_6$

From the discussions and examples in the previous sections it should be clear that in the $n = 4$ case we will obtain $SO(4-l,l)$-gauged 7-dimensional supergravities etc. A particularly interesting case is the $n = 6$ one, in which we expect to obtain $SO(6-l,l)$-gauged $N = 8, d = 5$ supergravities which were constructed in Refs. [73, 131]. This offers us the possibility to check our construction and show that, as we have claimed, it systematically gives gauged/massive supergravities.

The derivation of the 5-dimensional theory from "massive 11-dimensional supergravity" offers no new technical problems and the action, field strengths etc. can be found applying the general recipes explained in the previous section and are given explicitly in appendix E. One of the highlights of this derivation is the field content, which is of the general form

$$\{g_{\mu\nu}, \varphi, a, \ell_{mnp}, L_m{}^i, B_{m\,\mu\nu}, V_{mn\,\mu}, A^m{}_\mu, \psi^i_\mu, \lambda^i\}\,, \tag{3.76}$$

where now the m, n, p indices are $Sl(6,\mathbb{R})$ indices and where we have dualized the 3-form $C_{\mu\nu\rho}$ into the scalar a. The scalars φ and a can be combined again into the complex τ that parametrizes an $Sl(2,\mathbb{R})/SO(2)$ coset. In the ungauged/massless theory this $Sl(2,\mathbb{R})$ global symmetry and the more evident $Sl(6,\mathbb{R})$ are part of the E_6 duality group of the theory that only becomes manifest after the 6 2-forms are dualized into 6 additional vector fields[13].

As usual, this is also the field content of the ungauged theory. This is already a surprise since in Refs. [73, 131] it was argued that the theory could only be consistently gauged if the 6 KK vector fields $A^m{}_\mu$ were dualized into 6 2-forms $\tilde{B}_{m\,\mu\nu}$ which, together with the already existing 6 2-forms $B_{m\,\mu\nu}$ and via a self-dual construction, could describe 6 massive 2-forms. Once there are no massless higher-rank fields with $Sl(6,\mathbb{R})$ indices left, the theory can be consistently gauged. In the theory that we get, the same goal is achieved by the Stückelberg mechanism: the 6 KK vector fields $A^m{}_\mu$ are not dualized but are gauged away leaving mass terms for the already existing 6 2-forms $B_{m\,\mu\nu}$.

Another interesting point is the form of the scalar potential $\mathcal{V}(\varphi, \ell_{mnp})$, given in (E.6). The first term, which is universal for all the gauged/massive theories we are studying and is the only one that survives the consistent truncation $a_{mnp} = 0$, is independent of the scalar φ that measures the volume of the internal manifold. As shown in appendix D, this universal term is always minimized for $\mathcal{M} = \mathbb{I}_{n\times n}$ when $\mathsf{Q} = g\,\mathbb{I}_{n\times n}$ and the value of the potential for $n = 6$ is constant, such that the vacuum solution is AdS_5 as in Refs. [73, 131]. Not only the vacuum is the same: in Ref. [75] it was shown that there is a consistent truncation of the scalars that leaves the same potential (the first term of Eq. (E.6)) for the remaining scalars and thus, in spite of the apparent differences it is plausible that the two untruncated potentials are completely equivalent.

If the field content is equivalent, the symmetries of the theory are the same, the vacuum is the same and, presumably, the potentials are equivalent, we can

[13] The bosonic action of the massless theory with $C_{\mu\nu\rho}$ dualized into a and the $B_{m\,\mu\nu}$ dualized into vector fields $N^m{}_\mu$ is given in section B.4.

expect to have obtained a completely equivalent form of the $SO(6-l,l)$-gauged $N=8, d=5$ theories constructed in Refs. [73, 131]. To make more plausible this equivalence we would like to show that these theories have identical equations of motion, but this is extremely complicated for the full theories and we will content ourselves with showing the equivalence of the self-dual and Stückelberg Lagrangians for charged 2-forms ignoring all the scalars for the sake of simplicity.

3.6.1 Self-duality versus Stückelberg

The gauging of $N=4, d=7$ [130] and $N=8, d=5$ [73, 131] supergravity theories presents many peculiar features and problems absent in other cases. All these problems were resolved using the *self-duality mechanism* [161, 162]. Before comparing it with the Stückelberg mechanism, we will review the above mentioned problems and the reasoning that lead to the use of the self-duality mechanism to solve them in the 5-dimensional case.

In the usual gauging procedure one gauges the symmetry group of all the vector fields present in the ungauged theory. In one version of $N=8, d=5$ ungauged supergravity in which all 2-forms have been dualized into vectors, there are 27 vector fields, but there is no 27-dimensional simple Lie group, and therefore the standard recipe does not work. The origin of the gauged theory from IIB supergravity compactified on S^5 [48, 74, 101] suggested the gauging of the isometry group of the internal space, the 15-dimensional $SO(6)$. $E_{6(6)}$ being the global symmetry group of the ungauged theory and $Usp(8)$ the local composite one, the idea was to gauge an $SO(6)$ subgroup of the $Usp(8)$ embedded in $E_{6(6)}$[14]. All bosonic fields are in irreducible representations of $E_{6(6)}$ and in general transform as reducible representations under $SO(6)$. In particular, the **27** of vector fields breaks, under $SO(6)$, as $\mathbf{27} = \mathbf{15} + \mathbf{6} + \mathbf{6}$. The **15** is precisely the adjoint of $SO(6)$. This raises a second problem: how to couple the two sextets of Abelian vector fields to the 15 $SO(6)$ Yang-Mills fields.

On the other hand, the superalgebra of the gauged theory was expected to be $SU(2,2|4)$. The irreducible representation of this superalgebra in which the graviton is contained also contains two sextets of 2-index antisymmetric tensor fields (2-forms). This and other reasons [74, 101] suggested the replacement of the two sextets of Abelian vector fields by two sextets of 2-form fields, but there is also a problem in coupling these fields to the Yang-Mills ones: it is not pos-

[14] $Usp(8)$ contains $Sl(2,\mathbb{R}) \times Sl(6,\mathbb{R})$ as a subgroup, and the $SO(6)$ to be gauged is in $Sl(6,\mathbb{R})$. One may also gauge a non-compact group $SO(6-l,l)$ instead of $SO(6)$.

sible to reconcile both gauge invariances simultaneously. Replacing ordinary derivatives by Yang-Mills covariant ones breaks the local gauge invariance of the antisymmetric fields, which means that there are more modes propagating than in the ungauged theory. But there is a way out: the antisymmetric fields must satisfy *self-dual equations of motion*. We will describe them below.

Once the twelve vectors have been replaced by the self-dual 2-form fields one finds that the latter do not satisfy Bianchi identities, and for consistency the model must be gauged [73,131]. This, in turn, implies that, naively, the gauged theory does not have a good $g \to 0$ limit, although the limit can be taken after elimination of one of the 2-form sextets [162]. In the (Stückelberg) formulation we have derived, the $g \to 0$ limit can always be taken.

In the next two subsections we are going to construct Stückelberg formulations for a massive, uncharged 2-form field and for a sextet of massive 2-form fields charged under $SO(6)$ Yang-Mills fields and we will show that they lead to equations of motion fully equivalent to those obtained from self-dual formulations. The Stückelberg formulations we present below are just simplifications of the gauged/massive $N = 8, d = 5$ supergravity theory we have obtained.

Uncharged case

We start from the standard action for a massive 2-form field

$$S[B] = \int d^5x \left\{ \tfrac{1}{2\cdot 3!} H^2 - \tfrac{1}{4} m^2 B^2 \right\} , \tag{3.77}$$

where $H = 3\partial B$. The equation of motion for B derived from (3.77) is the Proca equation

$$(\Box + m^2) B_{\mu\nu} = 0 \,. \tag{3.78}$$

The action given in (3.77) is not gauge invariant. To recover formally gauge invariance we introduce in the action a Stückelberg field A_μ, such that the action is now

$$S[B, A] = \int d^5x \left\{ \tfrac{1}{2\cdot 3!} H^2 - \tfrac{1}{4} F^2 \right\} , \tag{3.79}$$

where

$$H = 3\partial B\,,$$
$$F = 2\partial A - mB\,. \tag{3.80}$$

The equations of motion for these fields are

$$\partial_\mu H^{\mu\nu\rho} - mF^{\nu\rho} = 0\,,$$
$$\partial_\mu F^{\mu\nu} = 0\,, \tag{3.81}$$

and now we have invariance under the following "massive gauge transformations":

$$\delta A = m\Lambda\,,$$
$$\delta B = 2\,\partial\Lambda\,. \tag{3.82}$$

The vector A_μ does not propagate any degrees of freedom, since it can be completely gauged away. In fact, setting $A_\mu = 0$ we recover the Proca equation. So, as we know, the introduction of the Stückelberg field is just a way of re-writing the theory described by (3.77) in a formally gauge invariant way.

Now, to connect with the self-dual formulation, we dualize the vector A_μ into a two-form $\tilde{B}_{\mu\nu}$, for which we add a Lagrange multiplier term in the action. Then

$$S[B,\tilde{B},F] = \int d^5x \left\{ \tfrac{1}{2\cdot 3!}H^2 - \tfrac{1}{4}F^2 + \tfrac{1}{4}\epsilon\partial\tilde{B}(F + mB) \right\}\,. \tag{3.83}$$

The equation of motion for $F = dA$ is

$$F = {}^*\tilde{H} = \tfrac{1}{3!}\epsilon\tilde{H}, \tag{3.84}$$

where $\tilde{H} = 3\partial\tilde{B}$. Inserting Eq. (3.84) into (3.83) one gets

$$S[B,\tilde{B}] = \int d^5x \left\{ \tfrac{1}{2\cdot 3!}H^2 + \tfrac{1}{2\cdot 3!}\tilde{H}^2 + \tfrac{m}{12}\epsilon\tilde{H}B \right\}\,. \tag{3.85}$$

The action above contains two 2-forms, but it describes the degrees of freedom of only one massive 2-form. Observe that this action is invariant under the gauge transformations

$$\delta B = 2\partial\Sigma\,, \qquad \delta\tilde{B} = 2\partial\tilde{\Sigma}\,. \tag{3.86}$$

Using this gauge invariance, the equations of motion derived from (3.85) can always be integrated to yield[15]

$$\begin{aligned} {}^*H &= +m\tilde{B}\,, \\ {}^*\tilde{H} &= -mB\,, \end{aligned} \tag{3.87}$$

which are precisely the equations of motion that one can derive directly from the *self-dual* action:

$$S_{SD}[B,\tilde{B}] = \int d^5x \left\{ -\tfrac{1}{4}m^2B^2 - \tfrac{1}{4}m^2\tilde{B}^2 - \tfrac{m}{12}\epsilon\tilde{H}B \right\}\,. \tag{3.88}$$

Therefore, the self-dual action (3.88) and the action (3.85) (and, therefore, the Stückelberg action Eq. (3.79)) are classically equivalent, since they lead to the same equations of motion.

Our next step is to establish a relation between the Stückelberg and self-dual actions for a sextet of $SO(6)$-charged, massive 2-forms.

The $SO(6)$ charged case

Let us consider now six massive two forms coupled to the 15 $SO(6)$-vector fields V_{mn}. The Stückleberg action for them can be read off from the action describing the 5-dimensional gauged/massive supergravity, given explicitly in appendix E setting $\mathsf{Q}^{mn} = m\delta^{mn}$. To simplify matters we truncate all the fields that are not relevant for our problem (in particular, all the scalars) and we work in flat spacetime. We are left with

$$S[B_m, A_m, V_{mn}] = \int d^5x \left\{ \tfrac{1}{2\cdot 3!}\mathbb{H}_m\mathbb{H}_m - \tfrac{1}{4}F_mF_m - \tfrac{1}{4}\mathcal{F}_{mn}\mathcal{F}_{mn} \right\}\,, \tag{3.89}$$

[15]These two equations can be combined to get the Proca equation.

where

$$\mathbb{H}_m = 3\partial B_m + 3V_{mn}F_n \equiv H_m + 3V_{mn}F_n\,,$$
$$F_m = 2\partial A_m - mB_m\,, \tag{3.90}$$
$$\mathcal{F}_{mn} = 2\partial V_{mn} + 2mV_{mp}V_{np}\,.$$

where $\mathcal{D}$ is the $SO(6)$ covariant derivative. This action is invariant under

$$\delta A_m = \sigma_{mn}A_n + m\lambda_m\,,$$
$$\delta V_{mn} = \mathcal{D}\sigma_{mn}\,, \tag{3.91}$$
$$\delta B_m = 2\partial\lambda_m + 2\partial\sigma_{mn}A_n + m\sigma_{mn}B_n\,,$$

In order to dualize the vectors A_m into two-forms $\tilde{B}_m$ we follow exactly the same steps as in the uncharged case, and the (much more complicated) equation we find for F_m is

$$F_m{}^{\mu\nu} = \mathcal{P}^{-1}(V)_{mn}{}^{\mu\nu}{}_{\rho\sigma}\left[({}^*\tilde{H})_n{}^{\rho\sigma} + H_p{}^{\rho\sigma\lambda}V_{pn\,\lambda}\right]\,, \tag{3.92}$$

where

$$\mathcal{P}_{mn}{}^{\rho\sigma}{}_{\mu\nu}(V) = \delta_{mn}\,\eta^{[\rho\sigma]}{}_{\mu\nu} - 3\eta^{[\rho\sigma}{}_{\mu\nu}V_{np}{}^{\lambda]}V_{mp\,\lambda}\,, \tag{3.93}$$

Then, the action in terms of the dual fields $\tilde{B}_m$ reads

$$\begin{aligned} S[B_m,\tilde{B}_m,V_{mn}] \;=\; & \int d^5x\,\Big\{\tfrac{1}{2\cdot 3!}H_mH_m + \tfrac{1}{4}({}^*\tilde{H}_m + H_pV_{pm})\mathcal{P}^{-1}_{mn}({}^*\tilde{H}_n + H_qV_{qn}) \\ & -\tfrac{1}{4}\mathcal{F}_{mn}\mathcal{F}_{mn} + \tfrac{1}{12}\epsilon\tilde{H}_mB_m\Big\}\,. \end{aligned} \tag{3.94}$$

The action given in (3.94) describes only the degrees of freedom of the six massive 2-forms B_m coupled to the vector fields V_{mn}. This action is invariant under the following gauge transformations

$$
\begin{aligned}
&\delta V_{mn} = \mathcal{D}\sigma_{mn}\,, \\
&\delta B_m = \mathcal{P}^{-1}_{mn}\left\{\left(d\lambda_n - \tfrac{1}{2}\epsilon\, d\tilde{\lambda}_p V_{np}\right) - \tfrac{1}{2}\epsilon\left[\left(\tilde{B}_p - \tfrac{1}{2}\epsilon B_q V_{pq}\right)\mathcal{D}\sigma_{np}\right]\right\}\,, \\
&\delta \tilde{B}_m = \mathcal{P}^{-1}_{mn}\left\{\left(d\tilde{\lambda}_n - \tfrac{1}{2}\epsilon\, d\lambda_p V_{np}\right) - \tfrac{1}{2}\epsilon\left[\left(B_p - \tfrac{1}{2}\epsilon \tilde{B}_q V_{pq}\right)\mathcal{D}\sigma_{np}\right]\right\}\,.
\end{aligned}
\tag{3.95}
$$

The equations of motion derived from (3.94) are

$$
\begin{aligned}
&\mathcal{D}_\mu \mathcal{F}_{mn}{}^{\mu\nu} = \tfrac{1}{4}m^2\epsilon^{\nu\rho\sigma\delta\lambda}B_{[m|\,\rho\sigma}\tilde{B}_{|n]\,\delta\lambda}\,, \\
&{}^*H_m = +m[\tilde{B}_m + \tfrac{1}{2}\epsilon V_{mn}B_n]\,, \\
&{}^*\tilde{H}_m = -m[B_m + \tfrac{1}{2}\epsilon V_{mn}\tilde{B}_n]\,,
\end{aligned}
\tag{3.96}
$$

which can also be derived from the *self-dual* action:

$$
S_{SD}[\vec{B}_m, V_{mn}] = \int d^5x \left\{-\tfrac{1}{4}m^2\vec{B}^T_m\vec{B}_m - \tfrac{1}{4}\mathcal{F}_{mn}\mathcal{F}_{mn} - \tfrac{m}{8}\epsilon\vec{B}^T_m\eta\mathcal{D}\vec{B}_m\right\}\,, \tag{3.97}
$$

where

$$
\vec{B}_m = \begin{pmatrix} B_m \\ \tilde{B}_m \end{pmatrix}\,, \qquad \eta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,, \tag{3.98}
$$

and $\mathcal{D}$ is the $SO(6)$ covariant derivative acting on $\vec{B}_m$:

$$
\mathcal{D}\vec{B}_m = \begin{pmatrix} \partial B_m - mV_{mn}B_n \\ \\ \partial\tilde{B}_m + mV_{mn}\tilde{B}_n \end{pmatrix}\,. \tag{3.99}
$$

Observe that the $SO(6)$ charges of B_m and $\tilde{B}_m$ are opposite.

Action (3.97) is precisely the kind of action that appears in the standard form of $N = 8, d = 5$ gauged supergravity.

3.7 Conclusions

In this chapter we have reviewed the "massive 11-dimensional supergravity", *a.k.a.* BLO theory [19] and its generalization to include n Killing vectors, known as BLO$_n$ theory [122]. This generalization can understood as a way of rewriting $(11-n)$-dimensional supergravity theories in an 11-dimensional fashion. In fact, we have considered BLO$_n$ theory as a systematic prescription to get gauged/massive supergravity theories in $(11-n)$ dimensions, which we have studied for several values of n.

The reduction of the $n=2$ theory in the direction of the two Killing vectors turns out to give all the $SO(2-l,l)$-gauged $N=2, d=9$ supergravities obtained by non-geometrical Scherk-Schwarz reduction from $N=2B, d=10$ supergravity [66,122]: each of these theories is determined by a traceless 2×2 matrix $m^m{}_n$ of the $sl(2,\mathbb{R})$ Lie algebra which is related to the mass matrix.

The reduction of the $n=3$ theory gives rise to the "S-dual" theories of the gauged supergravities we obtained in chapter 2 from $d=11$ supergravity via a geometrical generalized dimensional reduction. The "S-duality" group is broken in the gauging and therefore is only valid in the ungauged/massless limit, though it is reflected in the scalar potential.

Generically, the theories obtained in this way are $(11-n)$-dimensional supergravity theories with 32 supercharges determined by a mass matrix Q^{mn}. They are *covariant* under global $Sl(n,\mathbb{R})$ duality transformations that in general transform Q^{mn} into the mass matrix of another theory[16] of the same family.

The subgroup of $Sl(n,\mathbb{R})$ that preserves the mass matrix is a symmetry of the theory and at the end it will be the gauge group. If we use $Sl(n,\mathbb{R})$ transformations and rescalings to diagonalize the mass matrix so it has only $+1,-1,0$ in the diagonal, it is clear that $SO(n,n-l)$ will be amongst the possible gauge groups and corresponds to a non-singular mass matrix. These theories with non-singular mass matrices have $n(n-1)/2$ vector fields coming from the $\hat{C}_{\mu mn}$ components of the 11-dimensional 3-form and transforming as $SO(n-l,l)$ $l=0,\ldots,n$ gauge vector fields plus n 2-forms with the same origin and n Kaluza-Klein vectors coming from the 11-dimensional metric that transform as $SO(n-l,l)$ n-plets . The n vectors act as Stückelberg fields for the 2-forms which become massive. In this way the theory is consistent with

[16] In all cases we expect the entries of the mass matrix Q^{mn} to be quantized and take integer values in appropriate units, since they are related to tensions and charges of branes which are quantized in string theory. The duality group is then broken to $Sl(n,\mathbb{Z})$ [90]. Theories related by $Sl(n,\mathbb{Z})$ transformations should be considered equivalent from the string theory point of view.

the $SO(n-l,l)$ gauge symmetry.

Finally, all these theories have a scalar potential that contains a universal term of the form

$$\mathcal{V} = -\tfrac{1}{2} e^{\aleph\varphi} \left\{ [\mathrm{Tr}(\mathsf{Q}\mathcal{M})]^2 - 2\mathrm{Tr}(\mathsf{Q}\mathcal{M})^2 \right\} \,, \tag{3.100}$$

where $\mathcal{M}$ is a (symmetric) $Sl(n,\mathbb{R})/SO(n)$ scalar matrix, plus, possibly, other terms form the scalars that come from the 3-form. That scalar potentials of this form appears in several gauged supergravities was already noticed in Refs. [75, 144]. The $d=5$ case is special because $\aleph = 0$. This is related to the invariance of the Lagrangian under the $N=2B, d=10$ $Sl(2,\mathbb{R})$ symmetry.

Some of these theories are known, albeit in a very different form. The case $n=6$ is particularly interesting: we get $SO(6-l,l)$-gauged $N=8, d=5$ supergravities which were constructed by explicit gauging in Refs. [73, 131], with 15 gauge vectors that originate in the 3-form, 6 Kaluza-Klein vector fields that originate in the metric and give mass by the Stückelberg mechanism to 6 2-forms that come from the 3-form. That is: the field content (but not the couplings nor the spectrum) is the same as that of the ungauged theory that one would obtain by straightforward toroidal dimensional reduction. In fact, the ungauged theory can be recovered by taking the limit $\mathsf{Q} \to 0$ which is non-singular. In Refs. [73, 131] the gauged theories were constructed by dualizing first the 6 vectors into 2-forms that, together with the other 6 2-forms, satisfy self-duality equations [161] and describe also the degrees of freedom of 6 massive 2-forms. In this theory the massless limit is singular and can only be taken after the elimination of the 6 unphysical 2-forms [162].

Thus, we have, presumably, two different versions of the same theory in which the 6 massive 2-forms are described using the Stückelberg formalism or the self-duality formalism. However, we have not shown the whole equivalence between both formulations (at the classical level), but only worked out some simple cases. It is worth noticing that something similar happens in $d=7$, although we get $SO(4-l,l)$-gauged theories and in the literature only $SO(5-l,l)$-gauged theories have been constructed [129, 130].

The cosmological term in BLO theory allows for the existence of domain wall solutions which separate regions of spacetime with different values of the cosmological constant. However, this is seen when the mass parameter is dualized into a 10-form potential and source terms for this potential are included. This is exactly what happens in the case of Romans' theory, whose "dual picture" is BRGPT theory. The latter admits domain wall solutions separating regions in which the dynamics is described by Romans' theory.

The rewriting of BRGPT in an 11-dimensional fashion can be seen as the "dual picture" of BLO theory. This theory admits domain wall solutions separating regions with different values of the mass parameter, BLO theory describing the dynamics between the 10-dimensional domain walls. Then, as the mass parameter in BLO theory is related to the existence of an isometric direction, one can have regions with $m = 0$, such that the isometry disappears and $d = 11$ standard supergravity describes adequately the dynamics.

What about BLO_n theory? In this theory there are $\frac{1}{2}n(n+1)$ mass parameters. There is no available formulation with all the mass parameters dualized into gauge fields. If it existed, we would expect the theory to admit domain-wall solutions separating regions with different number n of isometric directions.

Chapter 4

Topological Kerr-Newman-Taub-NUT-AdS spacetimes

The presence of a negative cosmological constant is enough to invalidate the classical theorems [64,80] in which it is proven that at any given time black-hole horizons are always topologically spheres. In fact, asymptotically anti-de Sitter ($aAdS$) black-hole solutions are known such that the constant-time sections of their event horizons are not topologically spheres [6,26,31,88,107–109,119,120,154]. In particular, $aAdS$ Schwarzschild black holes with horizons with the topology of Riemann surfaces of arbitrary genus (henceforth called *topological* black holes) were given in Ref. [166]. The charged generalization in the framework of the Einstein-Maxwell theory with a negative cosmological constant (topological $aAdS$ Reissner-Nordström (RN-AdS) black holes) was studied in Ref. [29]. The generalization to the rotating case (topological $aAdS$ Kerr-Newman (KN-AdS) black holes) was found and studied in Ref. [103] using the general Petrov type D solution of Plebanski and Demianski (PD solution) [133] (which contains in different limits all these topological black-hole solutions) and other methods. $AAdS$ black holes with exotic horizons with different topologies are also known in higher dimensions [26], in theories with dilaton [32] and Lovelock gravity [33].

The supersymmetry properties of $aAdS$ black holes were first studied by Romans in the context of $N = 2, d = 4$ gauged supergravity [141] for RN-AdS black holes with spherical horizons. Later on, Kostelecký and Perry studied the supersymmetry properties of KN-AdS black holes [104]. Caldarelli and Klemm extended Romans' results to the case of topological RN-AdS black holes and extended and corrected Kostelecký and Perry's in the spherical KN-AdS case in Ref. [34].

The supersymmetry properties known are far from being understood. In the recent years we have learned how to interpret many supersymmetric solutions as intersections of "elementary" supersymmetric solutions preserving half of the supersymmetries. Each additional object in the intersection breaks an additional half of the remaining supersymmetry [1].

In $N = 2, d = 4$ gauged supergravity, however, Romans discovered solutions that preserve just 1/4 of the supersymmetry, characterized by a magnetic charge inversely proportional to the coupling constant. The simplest of those solutions only has magnetic charge (zero mass and electric charge) equal to the minimal amount of magnetic charge allowed by Dirac's quantization condition. It is really difficult to understand this fact using the paradigm of intersection of elementary objects.

Our goal in this chapter is to try to gain some insight into this problem by examining more general cases an calculating, if possible, the amount of supersymmetries preserved by the solutions [1]. We first present topological Kerr-Newman-Taub-NUT-AdS solutions and cosmological generalizations of the Robinson-Bertotti solution and then study their supersymmetry properties together with those of the general Plebanski-Demianski solution from which all of them can be obtained through different contractions. We will see that, generically, these solutions preserve only 1/4 of the available supersymmetries in presence of angular momentum. Our second main result will be the identification of a sort of electric-magnetic duality symmetry of the *supersymmetric* Plebanski-Demianski solutions that involves the mass and NUT charge.

The chapter is organized as follows: in section 4.1 we describe $N = 2, d = 4$ gauged supergravity. In section 4.2 we describe the solutions whose supersymmetry properties we are going to study. Section 4.3 is devoted to the study of the integrability conditions of the Killing spinor equation for the topological KN-TN-AdS solutions. In section 4.3.3 and section 4.3.4 we perform the same analysis for RB-AdS and the general PD solutions respectively. Finally, we present our conclusions in section 4.4.

[1]Except in Hanany-Witten-like cases in which one can add one more object to an intersection without breaking any further supersymmetry. Needless to say that here we use "object" in a loose and general way that may include gravitational instantons, certain kinds of singularities, etc.

4.1 $N = 2, d = 4$ gauged supergravity

The $N = 2, d = 4$ supergravity multiplet consists of the Vierbein, a couple of real gravitini and a vector field

$$\{e_\mu{}^a, \psi_\mu = \begin{pmatrix} \psi^1_\mu \\ \psi^2_\mu \end{pmatrix}, A_\mu\}, \tag{4.1}$$

respectively. With this multiplet one can construct two different supergravity theories: standard (ungauged) $N = 2, d = 4$ supergravity and gauged $N = 2, d = 4$ supergravity. The former can be understood as the zero-coupling limit of the latter and the second as the theory one obtains by gauging the $SO(2)$ symmetry that rotates the gravitini. The gauged $N = 2, d = 4$ supergravity action for these fields in the 1.5 formalism is

$$\begin{aligned} S &= \int d^4x\, e \left\{ R(e,\omega) + 6g^2 + 2e^{-1}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_5\gamma_\nu \left(\hat{\mathcal{D}}_\rho + igA_\rho\sigma^2\right)\psi_\sigma - \mathcal{F}^2 \right. \\ &\quad \left. + \mathcal{J}_{(m)}{}^{\mu\nu}(\mathcal{J}_{(e)\mu\nu} + \mathcal{J}_{(m)\mu\nu}) \right\}, \end{aligned} \tag{4.2}$$

where $\hat{\mathcal{D}}$ is the $SO(2,3)$ gauge covariant derivative

$$\hat{\mathcal{D}}_\mu = \nabla_\mu - \tfrac{i}{2} g\gamma_\mu, \tag{4.3}$$

F is the standard vector field strength, $\tilde{F}$ is the supercovariant field strength and we also define for convenience $\mathcal{F}$ by

$$\left\{ \begin{aligned} F_{\mu\nu} &= 2\partial_{[\mu}A_{\nu]}, \\ \tilde{F}_{\mu\nu} &= F_{\mu\nu} + \mathcal{J}_{(e)\mu\nu}, \\ \mathcal{F}_{\mu\nu} &= \tilde{F}_{\mu\nu} + \mathcal{J}_{(m)\mu\nu}, \end{aligned} \right. \tag{4.4}$$

where we have also defined

$$\left\{ \begin{aligned} \mathcal{J}_{(e)\mu\nu} &= i\bar{\psi}_\mu\sigma^2\psi_\nu, \\ \mathcal{J}_{(m)\mu\nu} &= -\tfrac{1}{2e}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\rho\gamma_5\sigma^2\psi_\sigma. \end{aligned} \right. \tag{4.5}$$

We see that the gauge coupling constant g is related to the cosmological constant by

$$\Lambda = -3g^2 \,. \tag{4.6}$$

The equation of motion for ${\omega_\mu}^{ab}$ implies that it is given by

$$\left\{\begin{array}{rcl} \omega_{abc} & = & -\Omega_{abc} + \Omega_{bca} - \Omega_{cab} \,, \\ {\Omega_{\mu\nu}}^a & = & {\Omega_{\mu\nu}}^a(e) + \frac{1}{2}{T_{\mu\nu}}^a \,, \\ \Omega_{abc}(e) & = & {e^\mu}_a {e^\nu}_b \partial_{[\mu} e_{\nu]c} \,, \\ {T_{\mu\nu}}^a & = & i\bar{\psi}_\mu \gamma^a \psi_\nu \,. \end{array}\right. \tag{4.7}$$

It is assumed that this equation has been used everywhere (1.5 formalism).

The Maxwell equation and Bianchi identity are

$$\left\{\begin{array}{rcl} \partial_\mu (e\,\mathcal{F}^{\mu\nu}) & = & \frac{ig}{2}\epsilon^{\nu\lambda\rho\sigma}\bar{\psi}_\lambda \gamma_5 \gamma_\rho \sigma^2 \psi_\sigma \,, \\ \partial_\mu (e\,{}^\star F^{\mu\nu}) & = & 0 \,. \end{array}\right. \tag{4.8}$$

Observe that the divergences of $\mathcal{J}_e$ and $\mathcal{J}_m$ are two topologically conserved currents that appear as electric-like and magnetic-like sources for the vector field in the Maxwell equation

$$\partial_\mu (eF^{\mu\nu}) = +\partial_\mu (e\mathcal{J}_e^{\nu\mu}) + \partial_\mu (e\mathcal{J}_m^{\nu\mu}) + \tfrac{ig}{2}\epsilon^{\nu\lambda\rho\sigma}\bar{\psi}_\lambda \gamma_5 \gamma_\rho \sigma^2 \psi_\sigma \,. \tag{4.9}$$

They are naturally associated to the electric and magnetic central charges of the $N = 2, d = 4$ supersymmetry algebra. The third term in the r.h.s. of the above equation is associated to the gravitino electric charge and it is, therefore, proportional to the gauge coupling constant. Finally, the Einstein and Rarita-Schwinger equations are

$$\left\{\begin{array}{rcl} 0 & = & {G_a}^\mu - 3g^2 {e_a}^\mu - 2T(\psi)_a{}^\mu - 2\tilde{T}(A)_a{}^\mu \,, \\ 0 & = & e^{-1}\epsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_\nu \left(\hat{\mathcal{D}}_\rho + igA_\rho\sigma^2\right)\psi_\sigma - i\left(\tilde{F}^{\mu\nu} + i^\star\tilde{F}^{\mu\nu}\gamma_5\right)\sigma^2\psi_\nu \,, \end{array}\right. \tag{4.10}$$

where the equation of motion for $\omega_\mu{}^{ab}$ has been used and where

$$\left\{\begin{array}{rcl} T(\psi)_a{}^\mu & = & -\frac{1}{2e}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\nu\gamma_5\gamma_a\left(\hat{\mathcal{D}}_\rho + igA_\rho\sigma^2\right)\psi_\sigma \\ & & -\frac{ig}{4e}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\nu\gamma_5\gamma_{\rho a}\psi_\sigma\,, \\ \tilde{T}(A)_a{}^\mu & = & \tilde{F}_a{}^\rho\tilde{F}^\mu{}_\rho - \frac{1}{4}e_a{}^\mu\tilde{F}^2\,. \end{array}\right. \tag{4.11}$$

Apart from invariance under general coordinate and local Lorentz transformations the theory is invariant under $U(1)$ gauge transformations

$$\left\{\begin{array}{rcl} A'_\mu & = & A_\mu + \partial_\mu\chi\,, \\ \psi'_\mu & = & e^{-ig\chi\sigma^2}\psi_\mu\,, \end{array}\right. \tag{4.12}$$

and local $N=2$ supersymmetry transformations

$$\left\{\begin{array}{rcl} \delta_\epsilon e_\mu{}^a & = & -i\bar{\epsilon}\gamma^a\psi_\mu\,, \\ \delta_\epsilon A_\mu & = & -i\bar{\epsilon}\sigma^2\psi_\mu\,, \\ \delta_\epsilon\psi_\mu & = & \tilde{\hat{\mathcal{D}}}_\mu\epsilon\,, \end{array}\right. \tag{4.13}$$

where the $\tilde{\hat{\mathcal{D}}}_\mu$ is the supercovariant derivative defined by

$$\tilde{\hat{\mathcal{D}}}_\mu = \hat{\mathcal{D}}_\mu + igA_\mu\sigma^2 + \frac{1}{4}\,\not{\tilde{F}}\gamma_\mu\sigma^2\,. \tag{4.14}$$

In the ungauged case, the theory enjoys *chiral-dual* invariance which interchanges the Maxwell and Bianchi equations and the topologically conserved electric and magnetic charges (and, therefore, the associated central charges). In the gauged theory, the gauge coupling breaks this invariance.

We are going to work with purely bosonic solutions of this theory. They obey the bosonic equations of motion

$$\left\{\begin{array}{rcl} \nabla_\mu F^{\mu\nu} & = & 0\,, \\ \nabla_\mu{}^\star F^{\mu\nu} & = & 0\,, \\ R_{\mu\nu} & = & 2T_{\mu\nu}(A) - 3g^2 g_{\mu\nu}\,, \end{array}\right. \tag{4.15}$$

where $T_{\mu\nu}(A)$ is just the standard energy-momentum tensor for an Abelian gauge field:

$$T_{\mu\nu}(A) = F_{\mu}{}^{\rho}F_{\rho\nu} - \tfrac{1}{4}g_{\mu\nu}F^2\,. \tag{4.16}$$

These equations of motion are duality-invariant. However, the gravitino supersymmetry rule (even with fermionic fields set to zero) is not duality-invariant and the supersymmetry properties of duality-related bosonic solutions are not, in general, the same.

4.2 The solutions

In this section we display and describe the solutions whose supersymmetry properties will later be studied. For simplicity we start with the unrotating RN-TN-AdS although they are included in the general KN-TN-AdS case.

4.2.1 Topological RN-TN-AdS Solutions

These solutions generalize, by including NUT charge N, the topological RN-AdS black hole solutions found in Ref. [29]. There are three cases labeled by the parameter $\aleph$ whose value is essentially the sign of one minus the genus of the horizon and therefore takes the values $1, 0, -1$ for the sphere (genus zero), the torus (genus 1) and higher genus Riemann surfaces, respectively. In the three cases the metric can be written in the form

$$\left\{ \begin{array}{rcl} ds^2 & = & \dfrac{\lambda}{R^2}\left(dt + \omega_{\aleph} d\varphi\right)^2 - \dfrac{R^2}{\lambda}dr^2 - R^2 d\Omega^2_{\aleph}\,, \\[2ex] \lambda & = & \left[g^2R^4 + (\aleph + 4g^2N^2)(r^2 - N^2) - 2Mr + |Z|^2\right], \\[2ex] R^2 & = & r^2 + N^2\,, \end{array} \right. \tag{4.17}$$

where $d\Omega^2_{\aleph}$ is the metric of the unit sphere, the plane and the upper half plane respectively

$$d\Omega^2_{\aleph} = \left\{ \begin{array}{lrcl} d\theta^2 + \sin^2\theta d\varphi^2\,, & \aleph & = & +1\,, \\[2ex] d\theta^2 + d\varphi^2\,, & \aleph & = & 0\,, \\[2ex] d\theta^2 + \sinh^2\theta d\varphi^2\,, & \aleph & = & -1\,, \end{array} \right. \tag{4.18}$$

$\omega_\aleph$ is the function

$$\omega_\aleph = \begin{cases} 2N\cos\theta\,, & \aleph = +1\,, \\ -2N\theta\,, & \aleph = 0\,, \\ -2N\cosh\theta\,, & \aleph = -1\,, \end{cases} \tag{4.19}$$

and the vector potential is given by

$$\begin{aligned} A_t &= (Qr - NP)/R^2\,, \\ A_\varphi &= \begin{cases} \cos\theta\big[P(r^2-N^2)+2NQr\big]/R^2\,, & \text{for } \aleph = +1\,, \\ -\theta\big[P(r^2-N^2)+2NQr\big]/R^2\,, & \text{for } \aleph = 0\,, \\ -\cosh\theta\big[P(r^2-N^2)+2NQr\big]/R^2\,, & \text{for } \aleph = -1\,. \end{cases} \end{aligned} \tag{4.20}$$

It is understood that one has to take the equation of the last two spacetimes by a discrete group in order to get a torus or a Riemann surface of arbitrary genus.

These solutions are valid in the $g = 0$ case. In that limit (with $N = 0$), we can speak of black holes only if $\aleph = +1$, which are solutions which can have a regular event horizon, in agreement with [64, 80]. With $g \neq 0$ (still with $N = 0$) and we recover the solutions of Ref. [29] in which M is the mass, Q the electric charge, P the magnetic charge and $Z = Q + iP$. Some of these cases are black holes with regular horizons of different topologies.

For $g = 0, N \neq 0$ we recover the standard RN-TN solutions in which those parameters are still the physical parameters[2] and N is the NUT charge. When the product $gN \neq 0$ it is no longer clear that M, Q, P are the true mass, electric and magnetic charges that appear in the superalgebra. This is similar to what happens in the rotating case [104] in which the true charges are combinations of the parameters M, P, Q appearing in the solution with the product ga.

[2] A definition of the mass of Taub-NUT spaces cannot be given in the standard form because these solutions do not go asymptotically to any other vacuum solution. The same happens in the 5-dimensional KK monopole solution, studied in Refs. [27, 51]. However, as different from the KK monopole, the TN solution is not ultrastatic and the tricks used in those references to define and calculate the mass of the KK monopole do not seem to apply to this case. A definition inspired in the AdS/CFT correspondence has been given in Refs. [37, 121].

It is useful to have a general form of the solutions valid for the three cases $\aleph = 1, 0, -1$. To have such a general expression we define the coordinate u

$$u \equiv \begin{cases} -\cos\theta\,, & \aleph = +1\,, \\ \theta\,, & \aleph = 0\,, \\ \cosh\theta\,, & \aleph = -1\,, \end{cases} \tag{4.21}$$

and then

$$\begin{cases} ds^2 = \dfrac{\lambda}{R^2}\left(dt - 2Nud\varphi\right)^2 - \dfrac{R^2}{\lambda}dr^2 - \dfrac{R^2}{S(u)}du^2 - R^2\, S(u)d\varphi^2\,, \\ A_t = \left(Qr - NP\right)/R^2\,, \\ A_\varphi = -u\left[P\left(r^2 - N^2\right) + 2NrQ\right]/R^2\,, \\ S(u) = \aleph(1-u^2) + 1 - \aleph^2\,, \end{cases} \tag{4.22}$$

where λ and R are as above.

4.2.2 Topological KN-TN-AdS Solutions

These solutions generalize the topological KN-AdS solutions given in Ref. [34, 103] to the non-zero NUT charge case. In the t, r, u, φ coordinate system (which is Boyer-Lindquist-type) they can be written as follows:

$$\left\{\begin{array}{rcl}
ds^2 &=& \frac{\lambda}{R^2(r,u)}\left\{dt-\left[2Nu-a\left(\aleph^2-u^2\right)\right]d\varphi\right\}^2-\frac{R^2(r,u)}{\lambda}dr^2 \\
&& -\frac{R^2(r,u)}{\mathcal{S}(u)}du^2-\frac{\mathcal{S}(u)}{R^2(r,u)}\left[\left(r^2+N^2+\aleph^2a^2\right)d\varphi+adt\right]^2 . \\
A_t &=& [Qr-P(N+au)]/R^2(r,u)\,, \\
A_\varphi &=& \frac{1}{a}\sqrt{r^2+N^2+\aleph^2a^2}\,[Qr-P(N+au)]/R^2(r,u)\,, \\
\lambda &=& g^2r^4+(\aleph+\aleph^2a^2g^2+6g^2N^2)\,r^2-2Mr+|Z|^2 \\
&& -N^2\,(\aleph-3\aleph^2a^2g^2+3g^2N^2)+a^2\,(1+\aleph-\aleph^2)\ , \\
\mathcal{S}(u) &=& S(u)+(a^2g^2\,u^2\,+\,4ag^2N\,u)\,(u^2-\aleph^2)\ , \\
R^2(r,u) &=& r^2\,+\,(N\,+\,au)^2\ ,
\end{array}\right. \tag{4.23}$$

with $S(u)$ as above.

The above form of the potentials is valid only for $a \neq 0$. However, the field strength components read

$$\begin{array}{rcl}
F_{01} &=& R(r,u)^{-4}\,[Q(r^2-(N+au)^2)\,-\,2Pr(N+au)]\ , \\
F_{23} &=& -R(r,u)^{-4}\,\{P(r^2-(N+au)^2)\,+\,2Qr(N+au)\}\ ,
\end{array} \tag{4.24}$$

such that the $a \to 0$ limit of the field strength is perfectly well defined.

4.2.3 Topological RB Solutions

In ungauged $N=2, d=4$ supergravity, the extremal RN black hole can be seen as a soliton interpolating between two supersymmetric vacua: Minkowski spacetime at infinity and RB in the near-horizon limit. The RB spacetime is the product $AdS_2 \times S^2$ where both factors are maximally symmetric spaces with opposite curvatures that cancel each other. The same thing occurs with other p-branes in higher dimensions [28,68] where the role of the RB spacetime is played by $AdS_{p+2} \times S^{8-p}$. Here we present a generalization of the RB spacetime to the case of gauged $N = 2, d = 4$ supergravity (cosmological Einstein-Maxwell theory) whose supersymmetry properties we will study later. They are the product of AdS_2 with a sphere S^2, a torus T^2 or a higher-genus Riemann surface Σ_g in which now the curvature of the AdS_2 spacetime is not completely canceled by the other factor space but they add up to the 4-dimensional cosmological constant

$$\left\{\begin{array}{lcl} ds^2 & = & K^2 r^2 dt^2 - \frac{1}{K^2 r^2} dr^2 - L^{-2} S(u)^{-1} du^2 - L^{-2} S(u) d\varphi^2 \,, \\ F_{01} & = & \alpha \\ F_{23} & = & -\beta \,, \end{array}\right. \tag{4.25}$$

where the constants K, L, α, β satisfy

$$\begin{array}{rcl} g^2 & = & \frac{1}{6}\{K^2 - \aleph L^2\} \,, \\ \alpha^2 + \beta^2 & = & \frac{1}{2}(K^2 + \aleph L^2) \,. \end{array} \tag{4.26}$$

The field strength is covariantly constant and in this coordinate system has constant components which correspond to the vector potential components

$$\left\{\begin{array}{lcl} A_t & = & -\alpha r \,, \\ A_\varphi & = & -\beta / L^2 u \,. \end{array}\right. \tag{4.27}$$

The $\aleph = -1, K^2 = 2L^2$ solution, which has special supersymmetry properties, has been given in [30].

4.2.4 PD Solutions

Plebanski and Demianski found *most general Petrov type D* solution of the cosmological Einstein-Maxwell theory [133]. This general solution contains as limiting cases all the known solutions, and, in particular the topological KN-TN-AdS solutions presented above (which in their turn, also contain the RN-TN-AdS solutions presented at the beginning). We will show this explicitly at the end of the section.

The PD solution depends on the constants[3] $\mathsf{M}, \mathsf{N}, \mathsf{Q}, \mathsf{P}, \mathsf{E}, \alpha$ and, of course, g, and, in Boyer coordinates τ, σ, p, q, reads [133]

[3]These constants are different from the constants M, N, Q, P that appear in the previous solutions.

$$\left\{\begin{array}{rcl} ds^2 & = & \frac{\mathcal{Q}(q)}{p^2+q^2}\left(d\tau - p^2 d\sigma\right)^2 - \frac{p^2+q^2}{\mathcal{Q}(q)}dq^2 - \frac{p^2+q^2}{\mathcal{P}(p)}dp^2 - \frac{\mathcal{P}(p)}{p^2+q^2}\left(d\tau + q^2 d\sigma\right)^2 , \\ F_{01} & = & (q^2+p^2)^{-2}\left[\mathsf{Q}(q^2-p^2) - 2\mathsf{P}pq\right] , \\ F_{23} & = & -(q^2+p^2)^{-2}\left[\mathsf{P}(q^2-p^2) + 2\mathsf{Q}pq\right] , \\ \mathcal{Q}(q) & = & g^2q^4 + \mathsf{E}q^2 - 2\mathsf{M}q + \mathsf{Q}^2 + \alpha , \\ \mathcal{P}(p) & = & g^2p^4 - \mathsf{E}p^2 + 2\mathsf{N}p - \mathsf{P}^2 + \alpha . \end{array}\right. \tag{4.28}$$

This class of solutions has a scaling invariance given by

$$\begin{array}{lllllllll} q & \rightarrow & \kappa q , & \mathsf{M} & \rightarrow & \kappa^3\mathsf{M} , & \mathsf{N} & \rightarrow & \kappa^3\mathsf{N} , \\ p & \rightarrow & \kappa p , & \mathsf{Q} & \rightarrow & \kappa^2\mathsf{Q} , & \mathsf{P} & \rightarrow & \kappa^2\mathsf{P} , \\ \tau & \rightarrow & \kappa^{-1}\tau , & \mathsf{E} & \rightarrow & \kappa^2\mathsf{E} , & \alpha & \rightarrow & \kappa^4\alpha , \\ \sigma & \rightarrow & \kappa^{-3}\sigma , & & & & & & \end{array} \tag{4.29}$$

which can be used to bring one of the charges to a given value. This scaling freedom remains if one of the charges happens to be nil.

The curvature is determined by M, N, Q and P and one can see that when they are zero, the Weyl tensor vanishes. This then means that, in that case, the solution is locally AdS_4.

Obtaining the Topological KN-TN-AdS Metric from PD's

Performing in Eq. (4.28) the coordinate change (see the analogous discussion in [103])

$$\begin{array}{lllllll} q & = & r , & \tau & = & t + \left[a^{-1}N^2 + a\aleph^2\right]\varphi , \\ p & = & N + au , & \sigma & = & a^{-1}\phi , \end{array} \tag{4.30}$$

and the following redefinitions of the parameters $\mathsf{M} = M$, $\mathsf{Q} = Q$, $\mathsf{P} = P$,

$$
\begin{aligned}
\mathsf{E} &= \aleph + \aleph^2 a^2 g^2 + 6g^2 N^2 \,, \\
\mathsf{N} &= N\left(\aleph - \aleph^2 a^2 g^2 + 4g^2 N^2\right) \,, \\
\alpha &= a^2\left(1+\aleph-\aleph^2\right) - N^2\left(\aleph - 3\aleph^2 a^2 g^2 + 3g^2 N^2\right) + P^2 \,,
\end{aligned} \tag{4.31}
$$

we go from the PD metric to the KN-TN-AdS metric as written down in Eqs. (4.23).

Note that the choice of the redefinitions is largely dictated by the factorizability of $\mathcal{P}$.

4.3 Supersymmetry and integrability conditions

The bosonic part of the supercovariant derivative for gauged $N = 2$ supergravity is

$$
\tilde{\hat{\mathcal{D}}}_\mu = \hat{\nabla}_\mu + gA_\mu i\sigma^2 + \tfrac{1}{4}\, \not{F}\gamma_\mu\sigma^2 \,, \tag{4.32}
$$

where $\hat{\nabla}_\mu$ is the $SO(2,3)$ gauge-covariant derivative. The Killing spinor equation is

$$
\tilde{\hat{D}}_\mu \epsilon = 0 \,, \tag{4.33}
$$

and a necessary condition for it to have solutions is the integrability condition

$$
\left[\tilde{\hat{\mathcal{D}}}_\mu, \tilde{\hat{\mathcal{D}}}_\nu\right]\epsilon = 0 \,. \tag{4.34}
$$

One finds [141]

$$
\begin{aligned}
\left[\tilde{\hat{\mathcal{D}}}_\mu, \tilde{\hat{\mathcal{D}}}_\nu\right]\epsilon &= -\tfrac{1}{4}\left\{C_{\mu\nu}{}^{ab}\gamma_{ab} + 2i\, \not{\nabla}\left(F_{\mu\nu} + i\,{}^\star F_{\mu\nu}\gamma_5\right) i\sigma^2 \right. \\
&\quad \left. + \tfrac{g}{2} F_{ab}\left(3\gamma^{ab}\gamma_{\mu\nu} + \gamma_{\mu\nu}\gamma^{ab}\right) i\sigma^2 \right\}\epsilon = 0 \,.
\end{aligned} \tag{4.35}
$$

We study first the non-rotating case RN-TN-AdS case.

4.3.1 Supersymmetry of Topological RN-TN-AdS Solutions

Introducing the Vierbein 1-forms

$$
\begin{aligned}
e^0 &= \lambda^{1/2}/R\,(dt+\omega_{\aleph}d\varphi)\,, & e^1 &= \lambda^{-1/2}R dr\,,\\
e^2 &= R\,d\theta\,, & e^3 &= R\,\Omega_{\aleph}\,d\varphi\,,
\end{aligned}
\tag{4.36}
$$

we find

$$
\begin{aligned}
F_{01} &= \left(Q(r^2-N^2)-2NPr\right)/R^4\,,\\
F_{23} &= -\left(P(r^2-N^2)+2NQr\right)/R^4\,,
\end{aligned}
\tag{4.37}
$$

$$
\begin{aligned}
\nabla_1 F_{01} &= -2\lambda^{1/2}/R^7\left[Q(r^3-3rN^2)-P(3r^2N-N^3)\right]\,,\\
\nabla_1{}^\star F_{01} &= -2\lambda^{1/2}/R^7\left[P(r^3-3rN^2)+Q(3r^2N-N^3)\right]\,,
\end{aligned}
\tag{4.38}
$$

(the remaining components of $\nabla_a F_{bc}$ can be found using the Bianchi identities or the Maxwell equations, which are satisfied) and

$$
\begin{aligned}
-\tfrac{1}{2}C_{01}{}^{01} &= C_{02}{}^{02}=C_{03}{}^{03}=C_{12}{}^{12}=C_{13}{}^{13}=-\tfrac{1}{2}C_{23}{}^{23}=C_1\,,\\
C_{02}{}^{13} &= -C_{03}{}^{12}=C_{12}{}^{03}=-C_{13}{}^{02}=-\tfrac{1}{2}C_{23}{}^{01}=C_2\,,\\
C_1 &= \left[Mr^3-\left(3N^2(\aleph-4g^2N^2)+|Z|^2\right)r^2\right.\\
&\quad \left.-3N^2Mr+N^2\left(N^2(\aleph-4g^2N^2)+|Z|^2\right)\right]/R^6\,,\\
C_2 &= -N\left[(\aleph-4g^2N^2)r^3+3Mr^2\right.\\
&\quad \left.-\left(3N^2(\aleph-4g^2N^2)+2|Z|^2\right)r-MN^2\right]/R^6\,,
\end{aligned}
\tag{4.39}
$$

Plugging all this into the integrability conditions we get the following conditions on the parameters:

$$
0 = g\left[MP\,+\,QN(\aleph+4g^2N^2)\right]\,, \tag{4.40}
$$

$$
0 = \mathcal{B}_+\mathcal{B}_-\,, \tag{4.41}
$$

(4.42)

where we have defined

$$
\begin{array}{rcl}
\mathcal{B}_{\pm} & \equiv & (M \mp gNQ)^2 + N^2(\aleph \pm gP + 4g^2N^2)^2 \\
& - & (\aleph \pm 2gP + 5g^2N^2)|Z|^2 \,.
\end{array}
\tag{4.43}
$$

The first condition plays the role of a constraint which is automatically satisfied in the well-known $g = 0$ case, while the second implies $\mathcal{B}_{\pm} = 0$ which should be the (saturated) Bogomol'nyi bound of gauged $N = 2, d = 4$ supergravity and actually reduces to the well-known Bogomol'nyi bound of ungauged $N = 2, d = 4$ supergravity in asymptotically flat spaces ($\aleph = +1$), generalized so as to include NUT charge (see Refs. [5, 14, 96]), *i.e.* $M^2 + N^2 = Q^2 + P^2$. For $g = 0$ and arbitrary $\aleph$ we get

$$
M^2 + \aleph^2 N^2 = \aleph(Q^2 + P^2) \,. \tag{4.44}
$$

A detailed analysis of the different cases in which the constraints is satisfied and the Bogomol'nyi bound is saturated gives as a result the four cases represented in table 4.1.

The first case corresponds to AdS_4 itself in standard spherical coordinates, which is maximally symmetric and preserves all supersymmetries. The second case can be shown to describe, at least locally, AdS_4 as well (the Weyl tensor vanishes and the space is maximally symmetric). There are, thus, two different values of the parameter N that correspond to the same spacetime.

In the third and fourth cases we have taken for the sake of convenience Q and N as independent parameters. The third case is a generalization to $gN \neq 0$ of the standard $M = |Q|$ case of ungauged $N = 2, d = 4$ supergravity where Q is arbitrary which preserves 1/2 of the supersymmetries. Here, a non-vanishing magnetic charge proportional to N is induced. As a matter of fact, it admits the limits $g \to 0$ and/or $N \to 0$ with the same amount of supersymmetry preserved.

There are two particularly interesting limits: the often neglected $g = 0, \aleph = 0$ case which (setting $N = 0$ for simplicity and rescaling the coordinates θ, φ which do not represent angles anymore) corresponds to the solution

$$
\left\{
\begin{array}{rcl}
ds^2 & = & \frac{Q^2}{r^2} dt^2 - \frac{r^2}{Q^2}\left(dr^2 + d\theta^2 + d\varphi^2\right) \,, \\
\\
A_t & = & \frac{Q}{r} \,.
\end{array}
\right.
\tag{4.45}
$$

This solution belongs to the Papapetrou-Majumdar class

$$\left\{\begin{array}{rcl} ds^2 & = & H^{-2}dt^2 - H^2 d\vec{x}^{\,2}\,, \\ A_t & = & \pm H^{-1}\,, \\ \partial_{\underline{i}}\partial_{\underline{i}}H & = & 0\,, \end{array}\right. \tag{4.46}$$

where the harmonic function H has been chosen to depend on only one coordinate $H = |Q|x$ and not on y, z.

The second interesting limit $Q \to 0$ also gives a supersymmetric configuration that preserves 1/2 of the supersymmetries with only magnetic and NUT charge and zero mass.

The fourth case in table 4.1 preserves 1/4 of the supersymmetries and only exists for $g \neq 0$. It is a generalization to $N \neq 0$ of Romans' global monopole solution [141]. We see that the presence of both NUT and electric charge implies that the mass parameter has to be finite. On the other hand, it admits the limits $Q \to 0$ and/or $N \to 0$ with the same amount of supersymmetry preserved.

4.3.2 Supersymmetry of KN-TN-AdS Solutions

We choose the Vierbein 1-forms

$$\begin{array}{rcl} e^0 & = & \dfrac{\lambda^{1/2}}{R(r,u)}\left[dt - \big(2Nu - a(\aleph^2 - u^2)\big)\,d\varphi\right], \\ e^1 & = & \dfrac{R(r,u)}{\lambda^{1/2}}dr\,, \\ e^2 & = & \dfrac{R(r,u)}{\mathcal{S}^{1/2}(u)}du\,, \\ e^3 & = & \dfrac{\mathcal{S}^{1/2}(u)}{R(r,u)}\left[(r^2 + N^2 + \aleph^2 a^2)d\varphi\, +\, adt\right], \end{array} \tag{4.47}$$

on which the field strength components read

$$\begin{array}{rcl} F_{01} & = & R(r,u)^{-4}\left[Q(r^2 - (N+au)^2)\, -\, 2Pr(N+au)\right]\,, \\ F_{23} & = & -R(r,u)^{-4}\left\{P(r^2 - (N+au)^2)\, +\, 2Qr(N+au)\right\}\,, \end{array} \tag{4.48}$$

M	N	Q	P	$\aleph$	SUSY
0	0	0	0	$+1$	1
0	$\pm\frac{1}{2g}$	0	0	-1	1
$\lvert Q\sqrt{\aleph+4g^2N^2}\rvert$	any	any	$\pm N\sqrt{\aleph+4g^2N^2}$	any	$\frac{1}{2}$
$\lvert 2gNQ\rvert$	any	any	$\pm\frac{\aleph+4g^2N^2}{2g}$	any	$\frac{1}{4}$

Table 4.1: In this table we represent the different combinations of values for the parameters $M, N, Q, P, \aleph$ of the general RN-TN-AdS solution Eq. (4.17) for which there are Killing spinors and the fraction of supersymmetry preserved. The first two cases correspond locally to AdS. The last two cases are the two general solutions of the constraint and Bogomol'nyi bound equations and admit different limits with the same amount of supersymmetries preserved. In particular, the third case preserves the same amount of supersymmetry in the particular cases $Q = 0$, $N = 0, \aleph = +1$ (for any Q) and $N = \pm 1/2g, \aleph = -1$. The fourth case preserves the same amount of supersymmetry in the cases $Q = 0$, $N = 0$ (for any Q) and $g = 0$. In this last case, electric-magnetic invariance is preserved and Q can be substituted by $\sqrt{Q^2 + P^2}$.

We only need to calculate

$$
\begin{aligned}
C_{0101} &= -2R^{-6}\left[M(r^3-3rX^2)+N(\aleph-\aleph^2a^2g^2+4g^2N^2)(3r^2X-X^3)-Z^2(r^2-X^2)\right]\,, \\
C_{0123} &= 2R^{-6}\left[M(3r^2X-X^3)+N(\aleph-\aleph^2a^2g^2+4g^2N^2)(3rX^2-r^3)-2Z^2rX\right]\,, \\
\nabla_1 F_{01} &= -2R^{-7}\lambda^{1/2}\left[Q(r^3-3rX^2)-P(3r^2X-X^3)\right]\,, \\
\nabla_2 F_{01} &= -2aR^{-7}\mathcal{S}^{1/2}\left\{P(r^3-3rX^2)+Q(3r^2X-X^3)\right\}\,,
\end{aligned}
\tag{4.49}
$$

where we used the abbreviation $X = N + au$. As in the RN-TN-AdS case the other components of the integrability condition turn out to be proportional to the 01 component. From this, one obtains the constraint and generalization of the Bogomol'nyi bound

$$
\begin{aligned}
0 &= g\left[MP+NQ(\aleph-\aleph^2a^2g^2+4g^2N^2)\right]\,, \\
0 &= \mathcal{B}_+\mathcal{B}_-\,,
\end{aligned}
\tag{4.50}
$$

where now

$$
\begin{aligned}
\mathcal{B}_\pm \equiv\; & M^2+N^2(\aleph-\aleph^2a^2g^2+4g^2N^2)-\Big[(\aleph+\aleph^2a^2g^2+6g^2N^2) \\
& \pm 2g\sqrt{a^2(1+\aleph-\aleph^2)-N^2\left(\aleph-3\aleph^2a^2g^2+3g^2N^2\right)}\Big]\,Z^2\,,
\end{aligned}
\tag{4.51}
$$

The fact that the bound factorizes into the product $\mathcal{B}_+\mathcal{B}_-$ is difficult to see directly from the calculation but easy to deduce from the results we will find in the general PD case. It can be checked that the (saturated) bound obtained is exactly the same, when $N = 0$, as the one given by Caldarelli and Klemm in Ref. [34].

We can now try to analyze different solutions to these two equations. This is a very complex problem and it would only make sense to explain in detail a classification of the solutions if the different classes had different amounts of unbroken supersymmetry. However, in all the cases that we have been able to analyze we have not found any single supersymmetric solution with $a \neq 0$ preserving $1/2$ of the supersymmetries. In fact, adding angular momentum to the RN-TN-AdS solutions that do preserve $1/2$ of the supersymmetries always seems to break an another half leaving only $1/4$ unbroken.

For instance, the solution with $M = Q = 0, P = \pm(2g)^{-1}(\aleph - \aleph^2 a^2 g^2 + 4g^2N^2), \aleph = \pm 1$ preserves 1/2 with $a = 0$ and only 1/4 with $a \neq 0$. The same effect takes place in all the instances studied.

4.3.3 Supersymmetry of Topological RB Solutions

To check supersymmetry of the topological RB solutions we only need

$$C_{0101} = \tfrac{1}{3}\left\{K^2 - \aleph L^2\right\} = 2g^2\,, \tag{4.52}$$

since the vector field strengths is covariantly constant. The integrability condition then reads

$$g\left[g\mathbb{I} \;-\; \alpha\gamma^{01}i\sigma^2 \;+\; \beta\gamma^{23}i\sigma^2\right]\epsilon \;=\; 0\,. \tag{4.53}$$

Obviously, for $g = 0$ one finds Robinson-Bertotti which does not break any supersymmetry. When $g \neq 0$ however, one finds, just by taking the determinant of the above equation, that one has to satisfy

$$(g \pm \beta)^2 \;+\; \alpha^2 \;=\; 0 \quad \rightarrow \quad \left\{\begin{array}{lcl}\alpha & = & 0\\ \beta & = & \pm g\end{array}\right. \tag{4.54}$$

which then break half of the available supersymmetry. Plugging the above equations into Eq. (4.26), one finds that

$$\aleph \;=\; -1 \quad , \quad K^2 \;=\; 2L^2\,, \tag{4.55}$$

which means that $K^2 = 4g^2$ and $L^2 = 2g^2$. This is the solution found in Ref. [30].

We could have found this solution also as the near-horizon limit of the $\aleph$ generalization of Romans' global monopole [141]. In that case we have $P = \frac{\pm\aleph}{2g}$ and with $\aleph = -1$ and all other charges vanishing we find that there is a horizon at $2g^2r^2 = 1$. At this radius the solution can be approximated by

$$\begin{aligned} ds^2 \;&=\; 4g^2r^2dt^2 - \tfrac{1}{4g^2r^2}dr^2 - \tfrac{1}{2g^2}\left(d\theta^2 \;+\; \sinh^2(\theta)d\phi^2\right)\,,\\ F_{23} \;&=\; -\tfrac{\pm 1}{2g}\cdot\left(\tfrac{1}{2g^2}\right)^{-1} \;=\; \mp g\,. \end{aligned} \tag{4.56}$$

which is just the supersymmetric RB-like solution discussed above. We then see that we have supersymmetry enhancement at the horizon from 1/4 to 1/2. Observe that the presence of electric charge would have meant the complete annihilation of supersymmetry at the horizon.

4.3.4 Supersymmetry of the PD General Solution

As in the foregoing cases, one finds that all the components of the integrability condition are equivalent, so we will only write down the components of the Weyl tensor and the covariant derivative of the vector field strength to calculate the integrability condition in the 01 direction.

$$
\begin{aligned}
C_{1010} &= \frac{-2(p+q)^3}{(1+p^2q^2)^3}\left[-\mathsf{M}(1-3p^2q^2)+\mathsf{N}(3pq-p^3q^3)+\mathsf{Z}^2(p-q)(1-p^2q^2)\right] ,\\
C_{1023} &= \frac{-2(p+q)^3}{(1+p^2q^2)^3}\left[-\mathsf{M}(3pq-3p^3q^3)-\mathsf{N}(1-3p^2q^2)+\mathsf{Z}^2 2pq(p-q)\right] ,\\
\nabla_1 F_{01} &= \frac{2(p+q)^2\mathcal{Q}^{1/2}}{(1+p^2q^2)^{7/2}}\left[\mathsf{Q}(1-3p^3q+p^5q^3-3p^2q^2)+\mathsf{P}(3pq-p^3q^3+p^2-3p^4q^2)\right] ,\\
\nabla_2 F_{01} &= \frac{2(p+q)^2\mathcal{P}^{1/2}}{(1+p^2q^2)^{7/2}}\left[\mathsf{P}(1-3p^3q+p^5q^3-3p^2q^2)-\mathsf{Q}(3pq-p^3q^3+p^2-3p^4q^2)\right] ,
\end{aligned}
\tag{4.57}
$$

Plugging these expressions into the integrability condition and calculating the determinant, one finds that the following conditions need to be satisfied in order for the solution to be supersymmetric

$$
\begin{aligned}
0 &= g\left[\mathsf{MP}\ +\ \mathsf{NQ}\right] ,\\
0 &= \mathcal{B}_+\mathcal{B}_- ,
\end{aligned}
\tag{4.58}
$$

where, now

$$
\mathcal{B}_\pm \equiv \mathsf{W}^2 - (\mathsf{E} \pm 2g\alpha^{1/2})\mathsf{Z}^2 , \tag{4.59}
$$

and we have defined $\mathsf{W}^2 = \mathsf{M}^2 + \mathsf{N}^2$ and $\mathsf{Z}^2 = \mathsf{Q}^2 + \mathsf{P}^2$. One can check that these conditions are invariant under the scalings in Eq. (4.29) and they give the integrability equations of the RN-TN-AdS and KN-TN-AdS cases after the redefinitions (4.31).

Again we find a constraint on the charges and a generalization of the (saturated) Bogomol'nyi bound $\mathcal{B}_\pm = 0$. The advantage of the parametrization of the PD solution is, first of all, that the second integrability condition factorizes completely and that $\mathcal{B}_\pm$ is extremely simple and is almost identical to the standard bound for asymptotically flat, ungauged, $N = 2, d = 4$ supergravity solutions, being electric-magnetic duality-invariant and invariant under gravito-electric-magnetic duality that rotates M into N and vice-versa. These duality invariances are broken by the constraint $g\,[\mathsf{MP} + \mathsf{NQ}] = 0$ which is, nevertheless invariant under *simultaneous* rotations with the same angle

$$\left\{ \begin{array}{rcl} \mathsf{M}' & = & \cos\theta \mathsf{M} - \sin\theta \mathsf{N}\,, \\ \\ \mathsf{N}' & = & \sin\theta \mathsf{M} + \cos\theta \mathsf{N}\,, \end{array} \right. \qquad \left\{ \begin{array}{rcl} \mathsf{Q}' & = & \cos\theta \mathsf{Q} + \sin\theta \mathsf{P}\,, \\ \\ \mathsf{P}' & = & -\sin\theta \mathsf{Q} + \cos\theta \mathsf{P}\,. \end{array} \right. \tag{4.60}$$

Actually, assuming that $g \neq 0$ one can eliminate completely the constraint, getting a pair of equations

$$\left\{ \begin{array}{rcl} \mathsf{M}^2 & = & (\mathsf{E} \pm 2g\alpha^{1/2})\mathsf{Q}^2\,, \\ \\ \mathsf{N}^2 & = & (\mathsf{E} \pm 2g\alpha^{1/2})\mathsf{P}^2\,, \end{array} \right. \tag{4.61}$$

which hold even if some of these charges (but not g) vanish. These equations rotate into each other under the above duality transformations.

The rotation parameter is always bounded above:

$$\alpha^{1/2} \leq \pm \mathsf{E}/2g\,. \tag{4.62}$$

When this bound is saturated, then both $\mathsf{M} = 0$ and $\mathsf{N} = 0$, while Q and P remain arbitrary. This is always the case when $\mathsf{E} = 0$. Finally, the only supersymmetric solution with $\mathsf{Z} = 0$ is AdS_4.

A calculation of the rank of the integrability condition shows that all these configurations will generically break three-fourths of the available supersymmetries. This was to be expected from our results in the KN-TN-AdS case. On the other hand, we have not been able to find any combination preserving up to 1/2 of the available supersymmetry which is not the RN-TN-AdS solution.

4.4 Conclusions

In this chapter we have presented new solutions which generalize the already known topological black holes and the standard Robinson-Bertotti solution. We have considered the presence of a non-vanishing (negative) cosmological constant, and we have looked for solutions with non-zero NUT charge. We have also explored their supersymmetry properties finding that generically they preserve only 1/4 of the supersymmetry. The only solutions that preserve 1/2 are non-rotating ones and the addition of angular momentum seems to break a further half of the remaining supersymmetries.

A somewhat surprising result that deserves further study is the fact that the most general family of supersymmetric solutions of this theory (*i.e.* the supersymmetric Plebanski-Demianski solutions) is invariant under a continuous $SO(2)$ group of electric-magnetic duality transformations. Had we not included in our study NUT charge, the existence of that symmetry would have passed completely unnoticed. Its meaning is, however, obscure. After all, the charges that undergo the duality rotation in its simplest, linear form, are not the physical charges. In terms of the physical charges, the duality transformations are very nonlinear.

Summary and conclusions

Let us now summarize the main results we have presented.

Chapter 2

In this chapter we have reduced $d = 11$ supergravity on a 3-dimensional manifold via the SS2 generalized dimensional reduction. The reduced theories are $d = 8$ gauged maximal supergravity theories which we classify in two classes, A and B.

Class A contains supergravities obtained by reducing the 11-dimensional action, and an action for the reduced theories is therefore available. The structure constants of the gauge groups in class A are traceless and are completely specified by a 3×3 mass matrix. On the other hand, supergravities in class B can only be obtained from the reduction of the $d = 11$ equations of motion. The gauge groups can now have traceful constants, and there is no action from which the equations of motions of these theories can be derivable.

We have derived and studied only class A, which contains five gauged supergravities whose gauge groups correspond to the five different subgroups of $SL(3,\mathbb{R})$. For class B supergravities, the gauge groups are subgroups of $GL(3,\mathbb{R})$.

Also, we have found a generic half-supersymmetric domain wall solution to all class A supergravities. This solution can be interpreted as n parallel single domain walls where n is the rank of the mass matrix. We have compared our solution with previously known domain wall solutions in 8- and 9-dimensional gauged supergravities [10, 24]. Another point that we have discussed is how the relation between $d = 8$ domain-wall solutions and gauged supergravities fits naturally in the domain wall/QFT correspondence scheme [28, 94].

The Maurer-Cartan 1-forms of class A gauged supergravities can always be written in a frame with a manifest isometry. Then, all theories in class A can be derived from standard type IIA supergravity. We believe that the

Maurer-Cartan 1-forms for class B theories probably have no additional isometry. Therefore, in contrast to the class A case, these reductions cannot be reproduced by any known reduction of the massless IIA theory.

We have uplifted the triple domain wall solution to 11 dimensions and found that it leads to a purely gravitational solution whose metric is the direct product of a 7-dimensional Minkowski metric and a non-trivial 4-dimensional Euclidean Ricci-flat metric. The 4-metrics are solutions of 4-dimensional Euclidean gravity. Some of these solutions are generalizations of the Eguchi-Hanson solution to different (class A) Bianchi types.

However, all possible non-singular $SO(3)$-invariant hyper-Kähler metrics in four dimensions are the Eguchi-Hanson, Taub-NUT and Atiyah-Hitchin metrics, while only the (generalized) Eguchi-Hanson metric is found in the uplifting. This is due to the fact we have considered an Ansatz in which the spinors are independent of the internal coordinates, which is not compatible with the Taub-NUT and the Atiyah-Hitchin metrics. Only the (generalized) Eguchi-Hanson metric allows a covariantly constant spinor that is independent of the $SO(3)$ isometry directions [65]. We could try to perform a Scherk-Schwarz reduction in which the spinors were allowed to depend on the internal coordinates, and see if the Taub-NUT and Atiyah-Hitchin metrics admit a half-supersymmetric domain wall interpretation in eight dimensions.

One can hope to extend in this way the SS1 reduction (as in [24]), so both the SS1 and SS2 reduction procedures might admit an extension, and, if so, it would be interesting to understand the differences between the resulting gauged supergravities.

Chapter 3

"Massive 11-dimensional supergravity" (BLO theory) was originally little more than the straightforward uplift of Romans' theory. This is nothing but a way of rewriting the latter in an 11-dimensional fashion. For consistence, the theory must include in the action an explicit Killing vector, which is accompanied by a mass parameter (the mass parameter of Romans' theory). By construction, the reduction of the theory in the direction of the Killing vector leads to Romans' theory.

A similar construction with n Killing vectors (BLO_n theory) contains an $n\times n$ mass matrix. This theory can also be understood as a way of rewriting $(11-n)$-dimensional supergravity theories in an 11-dimensional fashion. In this chapter we have used BLO_n theory as a systematic prescription to get gauged/massive supergravity theories in $(11-n)$ dimensions, focusing on the reductions to eight and five dimensions.

The reduction of the $n=3$ case gives rise to a set of 8-dimensional gauged supergravities, which can be considered as the "S-dual" theories of the gauged supergravities obtained in chapter 2 from $d=11$ supergravity via an Scherk-Schwarz generalized dimensional reduction. The "S-duality" group is broken in the gauging and therefore is only valid in the ungauged/massless limit, but it is reflected in the scalar potential.

For the $n=6$ case we find a family of $SO(6-l,l)$-gauged $N=8, d=5$ supergravities which we identify with those which were constructed by explicit gauging in Refs. [73, 131]. Presumably, we have found an alternative way of writing these theories with a Stückelberg formalism, instead of self-duality formalism of Refs. [73,131]. However, we have not shown the whole equivalence between both formulations (at the classical level), but only worked out some simple cases. A nice feature of the Stückelberg formulation is that there is no problem in taking the massless limit, while in the self-duality formalism this limit is singular and can only be taken after the elimination of some unphysical fields needed to gauge the theory [162].

Chapter 4

In this chapter we have presented first new solutions of gauged $N=2$ $d=4$ supergravity generalizing the already known topological black holes and standard Robinson-Bertotti solution. We have also identified the limits in which some of the new solutions can be obtained from the most general Petrov type D solution (Plebanski-Demianski solution).

We have studied the supersymmetry properties of the new solutions, and we have found that generically only 1/4 of the supersymmetry is preserved. The only solutions that preserve 1/2 are non-rotating ones and the addition of angular momentum seems to break a further half of the remaining supersymmetries.

An outcome of our analysis is the fact that the most general family of supersymmetric solutions of this theory is invariant under a continuous $SO(2)$ group of electric-magnetic duality transformations. However, its meaning is not clear to us.

Appendix A

Conventions

In this appendix we give the conventions we have used.

A.1 Geometry

Greek indices $\mu, \nu, \rho, \ldots$ for world (coordinate basis) tensor indices, and Latin $a, b, c \ldots$ indices $0, 1, \ldots$ are tangent space (Lorentz, Vielbein basis) indices. We use hats for 11-dimensional objects and no hats for 8-dimensional objects. We symmetrize and antisymmetrize with weight one.

We use mostly minus signature $(+ - \cdots -)$. η is the Minkowski spacetime metric and the spacetime metric is g. Lorentz and world indices are related by the Vielbeins ${e_a}^\mu$ and inverse Vielbeins ${e_\mu}^a$, that satisfy

$$ {e_a}^\mu {e_b}^\nu g_{\mu\nu} = \eta_{ab} \,, \qquad {e_\mu}^a {e_\nu}^b \eta_{ab} = g_{\mu\nu} \,. \tag{A.1} $$

We denote by ∇ the total covariant derivative (with respect to g.c.t.'s and local Lorentz transformations) and by $\mathcal{D}$ the Lorentz covariant derivative. Acting on tensors and spinors (ψ) they are defined by

$$ \begin{aligned} \nabla_\mu \xi^\nu &= \partial_\mu \xi^\nu + {\Gamma_{\mu\rho}}^\nu \xi^\rho \,, \\ \mathcal{D}_\mu \xi^a &= \partial_\mu \xi^a + {\omega_{\mu b}}^a \xi^b \,, \\ \nabla_\mu \psi &= \partial_\mu \psi - \tfrac{1}{4} {\omega_\mu}^{ab} \Gamma_{ab} \psi \,, \end{aligned} \tag{A.2} $$

where Γ_{ab} stands for the antisymmetric product of two gamma matrices to be defined shortly.

We define the corresponding curvature tensors (and torsion) by the Ricci identities

$$
\begin{aligned}
[\nabla_\mu, \nabla_\nu]\ \xi^\rho &= R_{\mu\nu\sigma}{}^\rho(\Gamma)\,\xi^\sigma + T_{\mu\nu}{}^\sigma \nabla_\sigma \xi^\rho\,, \\
[\mathcal{D}_\mu, \mathcal{D}_\nu]\ \xi^a &= R_{\mu\nu b}{}^a(\omega)\xi^b\,,
\end{aligned}
\tag{A.3}
$$

and the curvatures are given in terms of the connections by

$$
\begin{aligned}
R_{\mu\nu\rho}{}^\sigma(\Gamma) &= 2\partial_{[\mu}\Gamma_{\nu]\rho}{}^\sigma + 2\Gamma_{[\mu|\lambda}{}^\sigma \Gamma_{\nu]\rho}{}^\lambda\,, \\
R_{\mu\nu a}{}^b(\omega) &= 2\partial_{[\mu}\,\omega_{\nu]a}{}^b - 2\omega_{[\mu|a}{}^c\,\omega_{|\nu]c}{}^b\,.
\end{aligned}
\tag{A.4}
$$

Imposing the Vielbein postulate

$$\nabla_\mu e_a{}^\mu = 0\,, \tag{A.5}$$

the two connections are related by

$$\omega_{\mu a}{}^b = \Gamma_{\mu a}{}^b + e_a{}^\nu \partial_\mu e_\nu{}^b\,, \tag{A.6}$$

and the curvatures of both connections are now related by

$$R_{\mu\nu\rho}{}^\sigma(\Gamma) = e_\rho{}^a e^\sigma{}_b R_{\mu\nu a}{}^b(\omega)\,. \tag{A.7}$$

If we impose the metric postulate

$$\nabla_\mu g_{\rho\sigma} = 0\,, \tag{A.8}$$

then the connection can always be written in the form

$$\Gamma_{\mu\nu}{}^\rho = \left\{ {\rho \atop \mu\,\nu} \right\} + K_{\mu\nu}{}^\rho = \Gamma_{\mu\nu}{}^\rho(g) + K_{\mu\nu}{}^\rho\,, \tag{A.9}$$

where

$$\left\{ {\rho \atop \mu\,\nu} \right\} = \tfrac{1}{2} g^{\rho\sigma} \left\{ \partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right\}\,. \tag{A.10}$$

are the *Christoffel symbols* and K is the contorsion tensor given in terms of the torsion tensor by

$$K_{\mu\nu}{}^{\rho} = \tfrac{1}{2} g^{\rho\sigma} \left\{ T_{\mu\sigma\nu} + T_{\nu\sigma\mu} - T_{\mu\nu\sigma} \right\} . \qquad \text{(A.11)}$$

If, on top of the metric postulate, we impose the Vielbein postulate, then the relation between Γ and ω implies

$$\begin{aligned} \omega_{abc} &= \omega_{abc}(e) + K_{abc} \,, \\ \omega_{abc}(e) &= -\Omega_{abc} + \Omega_{bca} - \Omega_{cab} \,, \\ \Omega_{ab}{}^{c} &= e_a{}^{\mu} e_b{}^{\nu} \partial_{[\mu} e^{c}{}_{\nu]} \,. \end{aligned} \qquad \text{(A.12)}$$

$\omega(e)$ is the spin connection related to the Levi-Cività connection $\Gamma(g)$.

A.2 Eleven-dimensional gamma matrices and spinors

Our 11-dimensional gamma matrices satisfy the anticommutation relations

$$\left\{ \hat{\Gamma}^{\hat{a}}, \hat{\Gamma}^{\hat{b}} \right\} = +2\hat{\eta}^{\hat{a}\hat{b}} \,, \qquad \text{(A.13)}$$

It is possible to choose (in a way consistent with all the properties that we are going to enumerate) the eleventh gamma matrix $\hat{\Gamma}^{10}$ to be

$$\hat{\Gamma}^{10} = i\hat{\Gamma}^{0} \ldots \hat{\Gamma}^{9} \,. \qquad \text{(A.14)}$$

They are in a purely imaginary (*i.e.* Majorana) representation, i.e.

$$\hat{\Gamma}^{\hat{a}\,\star} = -\hat{\Gamma}^{\hat{a}} \,. \qquad \text{(A.15)}$$

We have the property

$$\hat{\Gamma}^{0}\,\hat{\Gamma}^{\hat{a}}\,\hat{\Gamma}^{0} = \hat{\Gamma}^{\hat{a}\,\dagger} \,. \qquad \text{(A.16)}$$

The Dirac-conjugation matrix $\hat{\mathcal{D}}$ is the real antisymmetric matrix

$$\hat{\mathcal{D}} = i\hat{\Gamma}^{0}\,, \tag{A.17}$$

and, thus, we have

$$\hat{\mathcal{D}}\,\hat{\Gamma}^{\hat{a}_1\cdots\hat{a}_n}\,\hat{\mathcal{D}}^{-1} = (-1)^{[n/2]}\left(\hat{\Gamma}^{\hat{a}_1\cdots\hat{a}_n}\right)^{\dagger}\,. \tag{A.18}$$

We choose a charge conjugation matrix equal to the Dirac conjugation matrix

$$\hat{\mathcal{C}} = \hat{\mathcal{D}} = i\hat{\Gamma}^{0}\,, \tag{A.19}$$

which satisfies

$$\hat{\mathcal{C}}^{\,T} = \hat{\mathcal{C}}^{\,\dagger} = \hat{\mathcal{C}}^{-1} = -\hat{\mathcal{C}}\,, \tag{A.20}$$

and

$$\hat{\mathcal{C}}\,\hat{\Gamma}^{\hat{a}}\,\hat{\mathcal{C}}^{-1} = -\hat{\Gamma}^{\hat{a}\,T}\,. \tag{A.21}$$

This last property implies

$$\hat{\mathcal{C}}\,\hat{\Gamma}^{\hat{a}_1\cdots\hat{a}_n}\,\hat{\mathcal{C}}^{-1} = (-1)^{n+[n/2]}\left(\hat{\Gamma}^{\hat{a}_1\cdots\hat{a}_n}\right)^{T}\,. \tag{A.22}$$

The standard definition of the Dirac conjugates and Majorana conjugates and our specific choice of Dirac and charge conjugation matrices $\hat{\mathcal{C}} = \hat{\mathcal{D}}$ imply that the Majorana condition

$$\bar{\hat{\lambda}} = \hat{\lambda}^{c}\,, \tag{A.23}$$

is equivalent to requiring that all components of a Majorana spinor are real.

Finally, we have the useful identity

$$\hat{\Gamma}^{\hat{a}_1\cdots\hat{a}_n} = \frac{(-1)^{[n/2]+1}}{(11-n)!}\hat{\epsilon}^{\hat{a}_1\cdots\hat{a}_n\hat{b}_1\cdots\hat{b}_{11-n}}\hat{\Gamma}_{\hat{b}_1\cdots\hat{b}_{11-n}}\,. \tag{A.24}$$

Appendix B

Eleven-dimensional supergravity and standard dimensional reductions

In this appendix we write down some of the supergravity actions (and their symmetries) which may be useful all along our work. We start writing down the action for 11-dimensional supergravity and the gauge and supersymmetry transformations that leave it invariant. After that, we perform a standard dimensional reduction on a circle and arrive to type IIA supergravity. For convenience, further Kaluza-Klein reductions to eight and five dimensions are also included.

B.1 Eleven-dimensional supergravity

The fields of $N = 1$ $d = 11$ supergravity are the metric, a 3-form potential and a 32-component Majorana gravitino

$$\left\{\hat{g}_{\hat{\mu}\hat{\nu}}, \hat{C}_{\hat{\mu}\hat{\nu}\hat{\rho}}, \hat{\psi}_{\hat{\mu}}\right\} . \tag{B.1}$$

The field strength for the 3-form potential is

$$\hat{G} = 4\,\partial\,\hat{C}\,. \tag{B.2}$$

which is obviously invariant under the gauge transformation

$$\delta\,\hat{C} = 3\,\partial\,\hat{\chi}\,, \tag{B.3}$$

where $\hat{\chi}$ is a 2-form.

The action for the bosonic fields of the theory is

$$\hat{S} = \frac{1}{16\pi G_N{}^{(11)}} \int d^{11}\hat{x}\sqrt{|\hat{g}|}\left[\hat{R} - \tfrac{1}{2\cdot 4!}\hat{G}^2 - \tfrac{1}{6^4}\tfrac{1}{\sqrt{|\hat{g}|}}\,\hat{\epsilon}\,\partial\hat{C}\,\partial\hat{C}\,\hat{C}\right], \qquad \text{(B.4)}$$

where $G_N{}^{(11)}$ is the 11-dimensional Newton's constant.
The supersymmetry transformations under which the action (B.4) is invariant are

$$\begin{aligned}
\delta_{\hat{\epsilon}}\hat{e}_{\hat{\mu}}{}^{\hat{a}} &= -\tfrac{i}{2}\bar{\hat{\epsilon}}\,\hat{\Gamma}^{\hat{a}}\,\hat{\psi}_{\hat{\mu}}\,, \\
\delta_{\hat{\epsilon}}\hat{\psi}_{\hat{\mu}} &= 2\hat{\nabla}_{\hat{\mu}}\hat{\epsilon} + \tfrac{i}{144}\left(\hat{\Gamma}^{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}{}_{\hat{\mu}} - 8\hat{\Gamma}^{\hat{\beta}\hat{\gamma}\hat{\delta}}\hat{\eta}_{\hat{\mu}}{}^{\hat{\alpha}}\right)\hat{\epsilon}\,\hat{G}_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}\,, \qquad \text{(B.5)} \\
\delta_{\hat{\epsilon}}\hat{C}_{\hat{\mu}\hat{\nu}\hat{\rho}} &= \tfrac{3}{2}\bar{\hat{\epsilon}}\,\hat{\Gamma}_{[\hat{\mu}\hat{\nu}}\,\hat{\psi}_{\hat{\rho}]}\,,
\end{aligned}$$

up to bilinears in fermions, where $\hat{\epsilon}$, the parameter of the transformations, is a Majorana spinor.

B.2 Reduction on a circle: type IIA supergravity

In this section we give the dimensional decomposition of the 11-dimensional fields which leads to type IIA supergravity. The reduction is performed on a direction, say z, which lives in a circle S^1. Of course, all the fields are independent of this coordinate.

Considering the following Ansatz for the Elfbein

$$\left(\hat{e}_{\hat{\mu}}{}^{\hat{a}}\right) = \begin{pmatrix} e^{-\frac{1}{3}\phi}e_\mu{}^a & e^{\frac{2}{3}\phi}C^{(1)}{}_\mu \\ 0 & e^{\frac{2}{3}\phi} \end{pmatrix}, \qquad \left(\hat{e}_{\hat{a}}{}^{\hat{\mu}}\right) = \begin{pmatrix} e^{\frac{1}{3}\phi}e_a{}^\mu & -e^{\frac{1}{3}\phi}C^{(1)}{}_a \\ 0 & e^{-\frac{2}{3}\phi} \end{pmatrix}, \qquad \text{(B.6)}$$

the 3-form potential $\hat{C}_{\hat{\mu}\hat{\nu}\hat{\rho}}$

$$\hat{C}_{\mu\nu\rho} = C^{(3)}{}_{\mu\nu\rho}\,, \qquad \hat{C}_{\mu\nu\underline{z}} = B_{\mu\nu}\,, \qquad \text{(B.7)}$$

gauge parameter $\hat{\chi}_{\hat{\mu}\hat{\nu}}$

$$\hat{\chi}_{\mu\nu} = \Lambda_{\mu\nu(2)}\,, \qquad \hat{\chi}_{\mu\underline{z}} = \Lambda_{\mu(1)}\,, \tag{B.8}$$

and the generator of 11-dimensional infinitesimal g.c.t.'s

$$\hat{\xi}^{\underline{z}} = \Lambda_{(0)}\,, \tag{B.9}$$

we find the type IIA supergravity bosonic field content

$$\{g_{\mu\nu}, \phi, B_{\mu\nu}, C^{(3)}{}_{\mu\nu\rho}, C^{(1)}{}_{\mu}\}\,. \tag{B.10}$$

The field strengths for the NS-NS 2-form and the Ramond-Ramond potentials are

$$\begin{aligned} H &= 3\partial B\,, \\ G_{(2)} &= 2\partial C_{(1)}\,, \\ G_{(4)} &= 4\partial C_{(3)} + 4\,C_{(1)}\,H\,, \end{aligned} \tag{B.11}$$

which are invariant under the following gauge transformations

$$\begin{aligned} \delta B &= 2\partial\Lambda_{(1)}\,, \\ \delta C_{(1)} &= \partial\Lambda_{(0)}\,, \\ \delta C_{(3)} &= 3\partial\Lambda_{(2)} - H\Lambda_{(0)}\,. \end{aligned} \tag{B.12}$$

The bosonic action (in the string-frame) is

$$\begin{aligned} S_{IIA} = \frac{1}{16\pi G_N{}^{(10)}}\int d^{10}x\,\sqrt{|g|}\,\Big\{ & e^{-2\phi}\left[R - 4\,(\partial\phi)^2 + \tfrac{1}{2\cdot 3!}H^2\right] \\ & -\tfrac{1}{2\cdot 2!}G_{(2)}^2 - \tfrac{1}{2\cdot 4!}G_{(4)}^2 - \tfrac{1}{144}\frac{1}{\sqrt{|g|}}\,\epsilon\,\partial C_{(3)}\partial C_{(3)}B \Big\}\,, \end{aligned} \tag{B.13}$$

where $G_N{}^{(10)} = G_N{}^{(11)}/2\pi R$, with R the radius of compactification[1]. The following Ansatz for the fermionic fields and supersymmetry parameter

[1]Note that we have taken the coupling constant equal to one.

$$\hat{\epsilon} = e^{-\frac{1}{6}\phi}\,\epsilon\,, \qquad \hat{\psi}_a = e^{\frac{1}{6}\phi}\left(2\psi_a - \tfrac{1}{3}\Gamma_a\lambda\right)\,, \qquad \hat{\psi}_z = \tfrac{2i}{3}e^{\frac{1}{6}\phi}\Gamma_{11}\lambda\,, \tag{B.14}$$

and the decomposition for the gamma matrices

$$\begin{aligned} \hat{\Gamma}^a &= \Gamma^a\,, \quad a = 0,1\ldots 9\,, \\ \hat{\Gamma}^{10} &= i\hat{\Gamma}^0\ldots\hat{\Gamma}^9\,, \end{aligned} \tag{B.15}$$

we arrive to the type IIA supersymmetry transformation rules for the fermions

$$\begin{aligned} \delta_\epsilon\psi_\mu &= \nabla_\mu\epsilon - \tfrac{1}{8}\Gamma_{11}\,\not{H}_\mu\epsilon + \tfrac{i}{8}e^\phi\sum_{n=1}^{n=2}\tfrac{1}{(2n)!}\,\not{G}^{(2n)}\Gamma_\mu\left(-\Gamma_{11}\right)^n\epsilon\,, \\ \delta_\epsilon\lambda &= \left[\not{\partial}\varphi + \tfrac{1}{2\cdot 3!}\Gamma_{11}\,\not{H}\right]\epsilon + \tfrac{i}{4}e^\phi\sum_{n=1}^{n=2}\tfrac{5-2n}{(2n)!}\,\not{G}^{(2n)}\left(-\Gamma_{11}\right)^n\epsilon\,, \end{aligned} \tag{B.16}$$

where ϵ is the parameter of the transformations.

B.3 Reduction on a 3-torus: (ungauged) supergravity in eight dimensions

In chapters 2 and 3 we study some massive/gauged supergravity theories that arise, respectively, from generalized dimensional reduction of standard (massless) 11-dimensional supergravity on a 3-manifold and from dimensional reduction reduction of 11-dimensional massive supergravity on a 3-torus. Both ways lead to massive/gauged supergravity theories which are deformations of the massless/ungauged ones, which are recovered when the mass parameters are set to zero. Therefore, it is convenient to have at hand the action, fields and (super)symmetry transformation rules for the massless/ungauged theories. In this section we are going to perform now the standard dimensional reduction of 11-dimensional supergravity on a 3-torus.

The Kaluza-Klein Ansatz for the Elfbein is

$$\left(\hat{e}_{\hat{\mu}}{}^{\hat{a}}\right) = \begin{pmatrix} e_\mu{}^a & e_m{}^i A^m{}_\mu \\ 0 & e_m{}^i \end{pmatrix}\,, \qquad \left(\hat{e}_{\hat{a}}{}^{\hat{\mu}}\right) = \begin{pmatrix} e_a{}^\mu & -A^m{}_a \\ 0 & e_i{}^m \end{pmatrix}\,, \tag{B.17}$$

where $A^m{}_a = e_a{}^\mu A^m{}_\mu$. We define the internal metric on T^3 by

$$G_{mn} = e_m{}^i e_{n\,j} = -e_m{}^i e_n{}^j \delta_{ij}\,. \tag{B.18}$$

Under global transformations in the internal space

$$x^{m\,\prime} = \left(R^{-1\,T}\right)^m{}_n\, x^n + a^m\,, \qquad R \in GL(3,\mathbb{R})\,, \tag{B.19}$$

objects with internal space indices (the internal metric $G = (G_{mn})$ and the KK vectors $\vec{A}_\mu = (A^m{}_\mu)$) transform as follows:

$$G' = RGR^T\,, \qquad \vec{A}'_\mu = (R^{-1})^T \vec{A}_\mu\,. \tag{B.20}$$

We know that $GL(3,\mathbb{R})$ can be decomposed in $Sl(3,\mathbb{R}) \times \mathbb{R}^+ \times \mathbb{Z}_2$ and any matrix R, forgetting its $\mathbb{Z}_2$ part (we will focus on $GL(3,\mathbb{R})/\mathbb{Z}_2 \sim Sl(3,\mathbb{R}) \times \mathbb{R}^+$), can therefore be decomposed into

$$R = c\Lambda\,, \quad \Lambda \in Sl(3,\mathbb{R})\,, \quad c \in \mathbb{R}^+\,. \tag{B.21}$$

We want to separate fields that transform under the different factors. First we define the symmetric $Sl(3,\mathbb{R})$ matrix[2]

$$\mathcal{M} = -G/|\text{det }G|^{1/3}\,, \tag{B.23}$$

and the scalar

$$\sqrt{|\text{det }G|} = e^{\varphi}\,. \tag{B.24}$$

Now, under $Sl(3,\mathbb{R})$ only $\mathcal{M}$ and $\vec{A}_\mu$ transform:

$$\mathcal{M}' = \Lambda\mathcal{M}\Lambda^T\,, \quad \vec{A}'_\mu = (\Lambda^{-1})^T \vec{A}_\mu\,, \tag{B.25}$$

[2]The scalars matrix $\mathcal{M}$ it can be expressed in terms of the *internal Driebein* $L_m{}^i$ as

$$\mathcal{M}_{mn} = L_m{}^i L_n{}^j\,\delta_{ij}\,, \tag{B.22}$$

where δ_{ij} is the internal flat metric. Although in this section we will keep $\mathcal{M}$ explicitely, in general we will introduce the internal Driebein notation in the Ansatz for the Elfbein.

that is, $\vec{A}_\mu$ transforms contravariantly, while under $\mathbb{R}^+$ rescalings only φ and $\vec{A}_\mu$ transform:

$$\varphi' = \varphi - \log c\,, \quad \vec{A}'_\mu = c\vec{A}_\mu\,. \tag{B.26}$$

For future convenience, we label the KK vector with an upper index 1, *i.e.* $A^{1\,m}{}_\mu$.

Using the standard techniques with the above Elfbein Ansatz, and rescaling the resulting 8-dimensional metric to the Einstein frame

$$g_{\mu\nu} = e^{-\varphi/3} g_{E\ \mu\nu}\,, \tag{B.27}$$

one finds

$$\begin{aligned}\int \mathrm{Vol}(T^3)\, d^{11}\hat{x}\, \sqrt{|\hat{g}|}\ \left[\, \hat{R}\,\right] \ &= \ \int d^8x\, \sqrt{|g_E|}\ \Big[R_E + \tfrac{1}{2}\left(\partial\varphi\right)^2 \\ &+ \tfrac{1}{4}\mathrm{Tr}\left(\partial\mathcal{M}\mathcal{M}^{-1}\right)^2 - \tfrac{1}{4}e^{-\varphi}F^{1\,m}\mathcal{M}_{mn}F^{1\,n}\Big]\,,\end{aligned} \tag{B.28}$$

where

$$F^{1\,m} = 2\partial A^{1\,m}\,. \tag{B.29}$$

The kinetic term for $\mathcal{M}$ is just an $Sl(3,\mathbb{R})/SO(3)$ sigma model.

The fields arising from $\hat{C}_{\hat{\mu}\hat{\nu}\hat{\rho}}$ are $\{C_{\mu\nu\rho}, B_{\mu\nu\,m}, A^{2\,m}{}_\mu, a\}$. We decompose the 11-dimensional 3-form by identifying objects with flat 11- and 8-dimensional flat indices (up to factors coming from the rescaling of the metric) as

$$\begin{aligned}\hat{C}_{abc} &= e^{\varphi/2} C_{abc}\,, \\ \hat{C}_{ab\,m} &= e^{\varphi/3} B_{m\,ab}\,, \\ \hat{C}_{a\,mn} &= \epsilon_{mnp} e^{\varphi/6} A^{2\,p}{}_a\,,\end{aligned} \tag{B.30}$$

which implies, for curved components

$$\hat{C}_{\mu\nu\rho} = C_{\mu\nu\rho} + 3A^{1\,m}{}_{[\mu}B_{|m|\,\nu\rho]} + 3\epsilon_{mnp}A^{1\,m}{}_{[\mu}A^{1\,n}{}_{\nu}A^{2\,p}{}_{\rho]} + \epsilon_{mnp}\,\ell\,A^{1\,m}{}_{[\mu}A^{1\,n}{}_{\nu}A^{1\,p}{}_{\rho]}\,,$$

$$\hat{C}_{\mu\nu\,m} = B_{m\,\mu\nu} + 2\epsilon_{mnp}A^{1\,n}{}_{[\mu}A^{2\,p}{}_{\nu]} + \epsilon_{mnp}\,\ell\,A^{1\,n}{}_{[\mu}A^{1\,p}{}_{\nu]}\,,$$

$$\hat{C}_{\mu\,mn} = \epsilon_{mnp}A^{2\,p}{}_{\mu} + \epsilon_{mnp}\ell\,A^{1\,p}{}_{\mu}\,,$$

$$\hat{C}_{mnp} = \epsilon_{mnp}\,\ell\,. \tag{B.31}$$

These fields inherit the following gauge transformations from 11-dimensional gauge and general coordinate transformations of $\hat{\hat{C}}$:

$$\delta C = 3\partial\chi - 6A^{1\,m}\partial\Sigma_m + 3\epsilon_{mnp}A^{1\,m}A^{1\,n}\partial\lambda^{2\,p}\,,$$

$$\delta B_m = 2\partial\Sigma_m - 2\epsilon_{mnp}A^{1\,n}\partial\lambda^{2\,p}\,, \tag{B.32}$$

$$\delta A^{2\,m} = \partial\lambda^{2\,m}\,.$$

In particular we see that this choice implies that these potentials do not transform under reparametrizations of the internal torus $\delta A^{1\,m} = \partial\Lambda^{1\,m}$. The gauge-invariant field strengths of the above fields are

$$G = 4\partial C + 6F^{1\,m}B_m\,,$$

$$H_m = 3\partial B_m + 3\epsilon_{mnp}F^{1\,n}A^{2\,p}\,, \tag{B.33}$$

$$F^{2\,m} = 2\partial A^{2\,m}\,,$$

and lead to the following non-trivial Bianchi identities:

$$\partial G = 2F^{1\,m}H_m\,,$$

$$\partial H_m = \tfrac{3}{2}\epsilon_{mnp}F^{1\,n}F^{2\,p}\,, \tag{B.34}$$

Using now

$$e_i{}^m\,e_j{}^n\,e_k{}^p\,\epsilon_{mnp} = \det(e^{-1})\,\epsilon_{ijk}\,, \tag{B.35}$$

with $\det(e^{-1}) = e^{-\varphi}$, we get the following decomposition of the 11-dimensional 4-form field strength into the above 8-dimensional field strengths:

$$
\begin{aligned}
\hat{G}_{abcd} &= e^{2\varphi/3} G_{abcd}\,, \\
\hat{G}_{abc\,i} &= e^{\varphi/2} e_i{}^m\, H_{m\,abc}\,, \\
\hat{G}_{ab\,ij} &= e^{-2\varphi/3}\,\epsilon_{ijk}\,e_p{}^k\,[F^{2\,p}{}_{ab} + \ell\, F^{1\,p}{}_{ab}]\,, \\
\hat{G}_{a\,ijk} &= e^{-5\varphi/6}\,\epsilon_{ijk}\,\partial_a\,\ell\,.
\end{aligned}
\tag{B.36}
$$

We can now reduce the kinetic term of the 11-dimensional 3-form. The result can be combined with the result of the reduction of the Einstein-Hilbert term, giving

$$
\begin{aligned}
&\int d^{11}\hat{x}\sqrt{|\hat{g}|}\left[\hat{R} - \tfrac{1}{2\cdot 4!}\hat{G}^2\right] \;= \\
&\quad \mathrm{Vol}(T^3)\int d^8x\sqrt{|g_E|}\;\Big[R_E + \tfrac{1}{4}\mathrm{Tr}\left(\partial\mathcal{M}\mathcal{M}^{-1}\right)^2 + \tfrac{1}{4}\mathrm{Tr}\left(\partial\mathcal{W}\mathcal{W}^{-1}\right)^2 \\
&\quad -\tfrac{1}{4}F^{i\,m}\mathcal{M}_{mn}\mathcal{W}_{ij}F^{j\,n} + \tfrac{1}{2\cdot 3!}H_m\mathcal{M}^{mn}H_n - \tfrac{1}{2\cdot 4!}e^{\varphi}G^2\Big]\,,
\end{aligned}
\tag{B.37}
$$

where we have introduced the symmetric $Sl(2,\mathbb{R})/SO(2)$ matrix

$$
\mathcal{W} = \frac{1}{\Im\mathrm{m}(\tau)}\begin{pmatrix} |\tau|^2 & \Re\mathrm{e}(\tau) \\ \Re\mathrm{e}(\tau) & 1 \end{pmatrix}\,,
\tag{B.38}
$$

where τ is the standard complex combination

$$
\tau = \ell + ie^{\varphi}\,.
\tag{B.39}
$$

Let us now reduce the Chern-Simons term also using tangent space indices and taking into account the definition, in any dimension

$$
\epsilon^{\mu_1\cdots\mu_d} = \sqrt{|g|}\,\epsilon^{a_1\cdots a_d}e_{a_1}{}^{\mu_1}\ldots e_{a_d}{}^{\mu_d}\,.
\tag{B.40}
$$

The final result is

$$
\begin{aligned}
S \;=\; & \tfrac{1}{16\pi G_N{}^{(8)}}\int d^8x\sqrt{|g_E|}\,\Big\{R_E+\tfrac{1}{4}\mathrm{Tr}\,(\partial\mathcal{M}\mathcal{M}^{-1})^2+\tfrac{1}{4}\mathrm{Tr}\,(\partial\mathcal{W}\mathcal{W}^{-1})^2 \\
& -\tfrac{1}{4}F^{i\,m}\mathcal{M}_{mn}\mathcal{W}_{ij}F^{j\,n}+\tfrac{1}{2\cdot 3!}H_m\mathcal{M}^{mn}H_n-\tfrac{1}{2\cdot 4!}e^{\varphi}G^2\,, \\
& -\tfrac{1}{6^3\cdot 2^4}\tfrac{1}{\sqrt{|g_E|}}\,\epsilon\,[GG\ell-8GH_mA^{2\,m}+12G(F^{2\,m}+\ell F^{1\,m})B_m \\
& -8\epsilon^{mnp}H_mH_nB_p-8G\partial aC-16H_m(F^{2\,m}+\ell F^{1\,m})C]\Big\}\,,
\end{aligned}
\tag{B.41}
$$

where $G_N{}^{(8)}=G_N{}^{(11)}/\mathrm{Vol}(T^3)$. The kinetic terms (except for that of C) are explicitly invariant under $Sl(2,\mathbb{R})$ transformations

$$
\mathcal{W}'=\Lambda\mathcal{W}\Lambda^T\,,\qquad F^{i\,m\,\prime}=F^{j\,m}\left(\Lambda^{-1}\right)_j{}^i\,,\qquad \Lambda\in Sl(2,\mathbb{R})\,,
\tag{B.42}
$$

and $Sl(3,\mathbb{R})$ transformations

$$
\begin{aligned}
\mathcal{M}' &= K\mathcal{M}K^T\,,\quad F^{i\,m\,\prime}=F^{i\,n}\,(K^{-1})_n{}^m\,,\\
H'_m &= K_m{}^nH_n\,,\quad K\in Sl(3,\mathbb{R})\,.
\end{aligned}
\tag{B.43}
$$

The kinetic term of C and the Chern-Simons term are not invariant as a matter of fact. However, let us look into the equations of motion of C. We can write them, together with the Bianchi identity, in the following form:

$$
\partial G^i=2F^{i\,m}H_m\,,
\tag{B.44}
$$

where

$$
G^1\equiv G\,,\qquad G^2\equiv -e^{\varphi\star}G-\ell G\,.
\tag{B.45}
$$

G^i transforms as a doublet under $Sl(2,\mathbb{R})$ (just like the doublet $F^{i\,m}$) and therefore, the above equation of motion is covariant under $Sl(2,\mathbb{R})$ electric-magnetic duality transformations. The remaining equations of motion are covariant under $Sl(2,\mathbb{R})$ transformations as well. The structures are very similar to those of $N=4, d=4$ supergravity (see *e.g.* Ref. [112]), the obvious difference being that in four dimensions we dualize 2-form field strengths and in eight dimensions we dualize 4-form field strengths. This duality was first described in Ref. [95] and is part of a series of electric-magnetic dualities present in type II theories in any dimension (the 6-dimensional version was studied in Ref. [13] and a general discussion can be found in Ref. [113]).

The representation for the gamma matrices we use is

$$\begin{aligned}\hat{\Gamma}^{a} &= \Gamma^{a}\times\mathbb{I}_{3}\,,\\ \hat{\Gamma}^{i} &= \Gamma_{9}\times\sigma^{i}\,,\quad \Gamma_{9}=i\Gamma^{0}\Gamma^{1}\cdots\Gamma^{7}\,,\quad \Gamma_{9}^{2}=1\,,\end{aligned}\tag{B.46}$$

which, together with the following Ansatz for the 11-dimensional gravitino and supersymmetry parameter

$$\begin{aligned}\hat{\psi}_{\hat{a}} &= e^{\varphi/12}\left(\psi_{a}-\tfrac{1}{6}\Gamma_{a}\Gamma^{i}\lambda_{i}\right)\,,\\ \hat{\psi}_{i} &= e^{\varphi/12}\lambda_{i}\,,\\ \hat{\epsilon} &= e^{-\varphi/12}\epsilon\,,\end{aligned}\tag{B.47}$$

lead the supersymmetry transformation rules for the 8-dimensional fermionic fields

$$\begin{aligned}\delta\psi_{\mu} &= 2\nabla_{\mu}\epsilon+\tfrac{1}{2}L_{[i|}{}^{m}\partial_{\mu}L_{m|j]}\Gamma^{ij}\epsilon+\tfrac{1}{24}e^{\varphi/2}\Gamma^{i}L_{i}{}^{m}(\Gamma_{\mu}{}^{\nu\rho}-10\delta_{\mu}{}^{\nu}\Gamma^{\rho})F^{1}_{m\nu\rho}\epsilon\\ &\quad+\tfrac{1}{2}e^{-\varphi}\partial_{\mu}a\epsilon+\tfrac{i}{96}e^{\varphi/2}(\Gamma_{\mu}{}^{\nu\rho\delta\epsilon}-4\delta_{\mu}{}^{\nu}\Gamma^{\rho\delta\epsilon})G_{\nu\rho\delta\epsilon}\epsilon\\ &\quad+\tfrac{i}{36}\Gamma^{i}L_{i}{}^{m}(\Gamma_{\mu}{}^{\nu\rho\delta}-6\delta_{\mu}{}^{\nu}\Gamma^{\rho\delta})H_{m\nu\rho\delta}\epsilon\\ &\quad+\tfrac{i}{48}e^{-\varphi/2}\epsilon_{ijk}\Gamma^{i}\Gamma^{j}L_{m}^{k}(\Gamma_{\mu}{}^{\nu\rho}-10\delta_{\mu}{}^{\nu}\Gamma^{\rho})(F^{2\,m}_{\nu\rho}+\ell F^{1\,m}_{\nu\rho})\epsilon\,,\\ \delta\lambda_{i} &= \tfrac{1}{2}L_{i}{}^{m}L^{jn}\partial\!\!\!/\mathcal{M}_{mn}\Gamma_{j}\epsilon-\tfrac{1}{3}\partial\!\!\!/\varphi\Gamma_{i}\epsilon+\tfrac{1}{4}e^{\varphi/2}L_{i}{}^{m}\mathcal{M}_{mn}F\!\!\!/^{1\,n}\epsilon\\ &\quad+\tfrac{i}{144}e^{\varphi/2}\Gamma_{i}G\!\!\!/\epsilon+\tfrac{i}{36}(2\delta_{i}{}^{j}-\Gamma_{i}{}^{j})L_{j}{}^{m}H\!\!\!/_{m}\epsilon\\ &\quad+\tfrac{i}{24}e^{-\varphi/2}\epsilon_{ijk}\Gamma^{j}L_{m}^{k}(3\delta_{i}{}^{k}-\Gamma_{i}{}^{k})(F\!\!\!/^{2\,m}+\ell F\!\!\!/^{1\,m})\epsilon\\ &\quad+\tfrac{1}{3}e^{-\varphi}\Gamma_{i}\partial\!\!\!/\ell\epsilon\,,\end{aligned}\tag{B.48}$$

Let us summarize our results. The full 8-dimensional field content consists of the following 128 + 128 field components (omitting spacetime indices on the potentials):

$$\{e_\mu{}^a, \mathcal{M}_{mn}, \varphi, \ell, A^m, V^m, B_m, C, \psi_\mu, \lambda_i\}\,, \tag{B.49}$$

with bosonic field strengths given by Eqs. (B.29, B.33) and bosonic action given by Eq. (B.41). The scalars parametrize $Sl(3,\mathbb{R})/SO(3)$ and $Sl(2,\mathbb{R})/SO(2)$ sigma models. The action has the global invariance group $Sl(3,\mathbb{R})$ but the equations of motion are also invariant under $Sl(2,\mathbb{R})$ electric-magnetic duality transformations.

B.4 Reduction on a 6-torus: (ungauged) supergravity in five dimensions

In chapter 3 we study the massive/gauged supergravity theories arising from dimensional reduction of 11-dimensional massive supergravity on a 6-torus. It is again convenient to have at hand the explicit expressions for the massless/ungauged theories.

The Kaluza-Klein Ansatz for the Elfbein is

$$\begin{aligned}
\left(\hat{e}_{\hat{\mu}}{}^{\hat{a}}\right) &= \begin{pmatrix} e^{-\frac{1}{3}\varphi} e_\mu{}^a & e^{\frac{1}{6}\varphi} L_m{}^i A^m{}_\mu \\ 0 & e^{\frac{1}{6}\varphi} L_m{}^i \end{pmatrix}, \\
\left(\hat{e}_{\hat{a}}{}^{\hat{\mu}}\right) &= \begin{pmatrix} e^{\frac{1}{3}\varphi} e_a{}^\mu & -e^{\frac{1}{3}\varphi} A^m{}_a \\ 0 & e^{-\frac{1}{6}\varphi} L_i{}^m \end{pmatrix},
\end{aligned} \tag{B.50}$$

and for the 3-form potential

$$\begin{aligned}
\hat{C}_{abc} &= e^{\varphi} C_{abc}\,, & \hat{C}_{abi} &= e^{\varphi/2} L_i{}^m B_{m\,ab}\,, \\
\hat{C}_{aij} &= L_i{}^m L_j{}^n V_{mn\,a}\,, & \hat{C}_{ijk} &= e^{-\varphi/2} L_i{}^m L_j{}^n L_k{}^p \ell_{mnp}\,,
\end{aligned} \tag{B.51}$$

which, in curved components, are

$$
\begin{aligned}
\hat{C}_{\mu\nu\rho} &= C_{\mu\nu\rho} + 3A^m{}_{[\mu|}B_{m|\,\nu\rho]} + 3V_{mn\,[\mu}A^m_\nu A^n_{\rho]} + \ell_{mnp}A^m{}_{[\mu}A^n{}_\nu A^p{}_{\rho]}\,,\\
\hat{C}_{\mu\nu m} &= B_{m\,\mu\nu} - 2V_{mn\,[\mu}A^n{}_{\nu]} + \ell_{mnp}A^n{}_\mu A^p{}_\nu\,,\\
\hat{C}_{\mu mn} &= V_{mn\,\mu} + \ell_{mnp}A^p{}_\mu\,,\\
\hat{C}_{mnp} &= \ell_{mnp}\,.
\end{aligned}
\tag{B.52}
$$

The 11-dimensional 4-form field strength decomposes as

$$
\begin{aligned}
\hat{G}_{abcd} &= e^{\frac{4}{3}\varphi}G_{abcd}\,,\\
\hat{G}_{abc\,i} &= e^{\frac{5}{6}\varphi}L_i{}^m\,H_{m\,abc}\,,\\
\hat{G}_{ab\,ij} &= e^{\frac{1}{3}\varphi}\,L_i{}^mL_j{}^n\left[F_{mn\,ab} + \ell_{mnp}\,F^p{}_{ab}\right]\,,\\
\hat{G}_{a\,ijk} &= e^{-\frac{1}{6}\varphi}\,L_i{}^mL_j{}^nL_k{}^p\partial_a\,\ell_{mnp}\,,
\end{aligned}
\tag{B.53}
$$

such that the field strengths are

$$
\begin{aligned}
&G_{\mu\nu\rho\sigma} = 4\partial_{[\mu}C_{\nu\rho\sigma]} + 6B_{m\,[\mu\nu}F^m{}_{\rho\sigma]}\,,\\
&H_{m\,\mu\nu\rho} = 3\partial_{[\mu|}B_{m|\,\nu\rho]} + 3V_{mn\,[\mu}F^n{}_{\nu\rho]}\,,\\
&F_{mn\,\mu\nu} = 2\partial_{[\mu|}V_{mn|\,\nu]}\,,\\
&F^m{}_{\mu\nu} = 2\partial_{[\mu}A^m{}_{\nu]}\,.
\end{aligned}
\tag{B.54}
$$

The 5-dimensional supergravity action is therefore

$$
\begin{aligned}
S \;=\; & \tfrac{1}{16\pi {G_N}^{(5)}}\int d^5x\,\sqrt{|g_E|}\;\Big\{R_E+\tfrac{1}{2}\,(\partial\varphi)^2+\tfrac{1}{4}\mathrm{Tr}\,(\partial\mathcal{M}\mathcal{M}^{-1})^2\\
& -\tfrac{1}{4}e^{\varphi}F^m(A)\mathcal{M}_{mn}F^n(A)-\tfrac{1}{2\cdot 4!}e^{2\varphi}G^2+\tfrac{1}{2\cdot 3!}e^{\varphi}H_m\mathcal{M}^{mn}H_n\\
& -\tfrac{1}{8}\mathcal{M}^{mp}\mathcal{F}_{mn}\mathcal{M}^{nq}\mathcal{F}_{pq}\;+\tfrac{1}{18}e^{-\varphi}\mathcal{M}^{mq}\mathcal{M}^{nr}\mathcal{M}^{ps}\partial\ell_{mnp}\,\partial\ell_{qrs}\\
& -\tfrac{1}{2^6\cdot 3^4}\tfrac{\epsilon}{\sqrt{|g_E|}}\epsilon^{mnpqrs}\,[\,2\,G\,\partial\ell_{mnp}\,\ell_{qrs}+12\,H_m\mathcal{F}_{np}\ell_{qrs}\\
& +24\,H_m\partial\ell_{npq}V_{rs}+27\,\mathcal{F}_{mn}\mathcal{F}_{pq}V_{rs}+36\,\mathcal{F}_{mn}\partial\ell_{pqr}B_s\\
& +4\,\partial\ell_{mnp}\,\partial\ell_{qrs}C\,]\,\Big\}\;.
\end{aligned}
\tag{B.55}
$$

where ${G_N}^{(5)}={G_N}^{(11)}/\mathrm{Vol}(T^6)$, and

$$
\mathcal{F}_{mn}=F_{mn}(V)+\ell_{mnp}F^p(A)\,. \tag{B.56}
$$

Dualizing now the three-form and two-form fields C and B_m into the scalar and vector potentials a and N^m, respectively, as

$$
\begin{aligned}
G &= e^{-2\varphi}\,{}^*\tilde{G}\,,\\
H_m &= e^{-\varphi}\,{}^*\tilde{H}_m\,,
\end{aligned}
\tag{B.57}
$$

where

$$
\begin{aligned}
\tilde{G} &= \partial a-\tfrac{1}{6^3}\,\epsilon^{mnpqrs}\,\partial\ell_{mnp}\,\ell_{qrs}\,,\\
\tilde{H}_m &= \mathcal{M}_{mn}\,\left[2\partial N^n-\tfrac{1}{36}\epsilon^{npqrsu}\mathcal{F}_{pq}\ell_{rsu}+a\,F^n(A)\right]\;.
\end{aligned}
\tag{B.58}
$$

we find that, in terms of the dual fields, action (B.55) reads

$$
\begin{aligned}
S \;=\; & \frac{1}{16\pi G_N{}^{(5)}}\int d^5x\,\sqrt{|g_E|}\;\Big\{R_E + \tfrac{1}{4}\mathrm{Tr}\left(\partial\mathcal{M}\mathcal{M}^{-1}\right)^2 + \tfrac{1}{2}\left(\partial\varphi\right)^2 + \tfrac{1}{2}e^{-2\varphi}\tilde{G}^2 \\
& -\tfrac{1}{4}e^{\varphi}F^m(A)\mathcal{M}_{mn}F^n(A) - \tfrac{1}{4}e^{-\varphi}\tilde{H}_m\mathcal{M}^{mn}\tilde{H}_n \\
& -\tfrac{1}{8}\mathcal{M}^{mp}\mathcal{F}_{mn}\mathcal{M}^{nq}\mathcal{F}_{pq} + \tfrac{1}{18}e^{-\varphi}\mathcal{M}^{mq}\mathcal{M}^{nr}\mathcal{M}^{ps}\partial\ell_{mnp}\,\partial\ell_{qrs} \\
& -\tfrac{1}{2^6\cdot 3^4}\tfrac{\epsilon}{\sqrt{|g_E|}}\left[\epsilon^{mnpqrs}\left(27\mathcal{F}_{mn}\mathcal{F}_{pq}V_{rs} - 12\ell_{mnp}F_{qr}(V)F^u(A)V_{us}\right)\right. \\
& \left. -6^4\,F^m(A)F_{mn}(V)N^n\,\right]\Big\}\;. \qquad \text{(B.59)}
\end{aligned}
$$

Appendix C

Bianchi classification of 3-dimensional Lie algebras

In this appendix we will discuss the Bianchi classification of three-dimensional Lie algebras. We will also show how different algebras are related via analytic continuation or group contraction. We compare our results with the $CSO(p,q,r)$ notation which is often used in the supergravity literature.

We assume that the generators of the three-dimensional Lie group satisfy the commutation relations ($m = 1, 2, 3$)

$$[T_m, T_n] = f_{mn}{}^p T_p \,, \tag{C.1}$$

with constant structure coefficients $f_{mn}{}^p$ subject to the Jacobi identity

$$f_{m[n}{}^p f_{qr]p} = 0 \,. \tag{C.2}$$

For three-dimensional Lie groups the structure constants have nine components and can be conveniently parameterized by

$$f_{mn}{}^p = \epsilon_{mnq} \mathsf{Q}^{pq} + 2\delta_{[m}{}^p a_{n]} \,. \tag{C.3}$$

Here Q^{pq} is a symmetric matrix. The Jacobi identity implies $\mathsf{Q}^{pq} a_q = 0$. Having $a_q = 0$ corresponds to an algebra with traceless structure constants: $f_{mn}{}^n = 0$.

Of course, Lie algebras are only defined up to changes of basis $T_m \to R_m{}^n T_n$. This can always be used [152, 167] to transform Q^{pq} into a diagonal form and a_q to have only one component. We will take $\mathsf{Q}^{pq} = \frac{1}{2}\mathrm{diag}(q_1, q_2, q_3)$ and $a_q = (a, 0, 0)$. The commutation relations then take the form

$$[T_1, T_2] = \tfrac{1}{2}q_3 T_3 - aT_2\,, \quad [T_2, T_3] = \tfrac{1}{2}q_1 T_1\,, \quad [T_3, T_1] = \tfrac{1}{2}q_2 T_2 + aT_3\,. \qquad \text{(C.4)}$$

The different three-dimensional Lie algebras have been classified and are given in table C. There are 11 different algebras, two of which are a one-parameter family. Of these, only $SO(3)$ and $SO(2,1)$ are simple, while the rest are all non-semi-simple [78, 152]. Note that for $a \neq 0$ the rank of Q can not exceed two due to the Jacobi identity.

Class	Bianchi	a	(q_1, q_2, q_3)	Group	$CSO(p,q,r)$
A	I	0	$(0,0,0)$	$U(1)^3$	$(0,0,3)$
A	II	0	$(0,0,1)$	Heisenberg	$(1,0,2)$
A	VI_0	0	$(0,-1,1)$	$ISO(1,1)$	$(1,1,1)$
A	VII_0	0	$(0,1,1)$	$ISO(2)$	$(2,0,1)$
A	VIII	0	$(1,-1,1)$	$SO(2,1)$	$(2,1,0)$
A	XI	0	$(1,1,1)$	$SO(3)$	$(3,0,0)$
B	V	1	$(0,0,0)$		
B	IV	1	$(0,0,1)$		
B	III	1	$(0,-1,1)$		
B	VI_a	a	$(0,-1,1)$		
B	VII_a	a	$(0,1,1)$		

Table C.1: The different three-dimensional Lie algebras in terms of components of their structure constants and the Bianchi and $CSO(p,q,r)$ classification. Note that there are two one-parameter families VI_a and VII_a with special case VI_0, VII_0 and VI_1=III.

We will now show relations between all algebras of Class A, *i.e.* having $a = 0$. Our starting point will be $SO(3)$. Its generators take, in our basis with $\mathsf{Q} = \frac{1}{2}\mathrm{diag}(1,1,1)$, the form

$$T_1 = \tfrac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T_2 = \tfrac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad T_3 = \tfrac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}. \tag{C.5}$$

One can obtain the other algebras with $a = 0$ from these $SO(3)$ generators by the analytic continuation and/or contraction. Define the operations A_1 (analytic continuation) and C_1 (contraction) by

$$T_2 \to \lambda^{-1} T_2\,, \quad T_3 \to \lambda^{-1} T_3\,, \tag{C.6}$$

with $\lambda = i$ for A_1 and $\lambda \to 0$ for C_1. Its effect on the parameters of the algebra reads

$$\mathsf{Q} = \tfrac{1}{2}\mathrm{diag}(q_1, q_2, q_3) \to \mathsf{Q} = \tfrac{1}{2}\mathrm{diag}(\lambda^2 q_1, q_2, q_3)\,. \tag{C.7}$$

Thus from $SO(3)$ one can obtain $SO(2,1)$ by an A operation and $ISO(2)$ by a C operation. Similarly, the other Class A algebras are related by various analytic continuations and contractions, as shown in figure C.1.

It is instructive to compare the discussion of the previous paragraph with the $CSO(p,q,r)$ notation which is often used in the supergravity literature, see e.g. [89, 93]. In our case $p + q + r = 3$ but the $CSO(p,q,r)$ classification of contracted algebras is valid more generally. The $CSO(p,q,r)$ group is a group contraction of $SO(p+r,q)$ and can be obtained as follows. One defines the starting point $CSO(p,q,0) = SO(p,q)$. The effect of analytic continuation in one of the p directions is

$$A_p: \qquad CSO(p,q,r) \to CSO(p-1, q+1, r)\,, \tag{C.8}$$

while the effect of contraction is

$$C_p: \qquad CSO(p,q,r) \to CSO(p-1, q, r+1)\,. \tag{C.9}$$

This defines all Class A algebras in terms of the $CSO(p,q,r)$ classification, as shown in table C. These can all be obtained from the semi-simple algebras $SO(3)$ or $SO(2,1)$ by various contractions. Using the fact that $CSO(p,q,r) \sim CSO(q,p,r)$ one can see that Class A exhausts the possibilities of distributing p, q, r subject to $p + q + r = 3$.

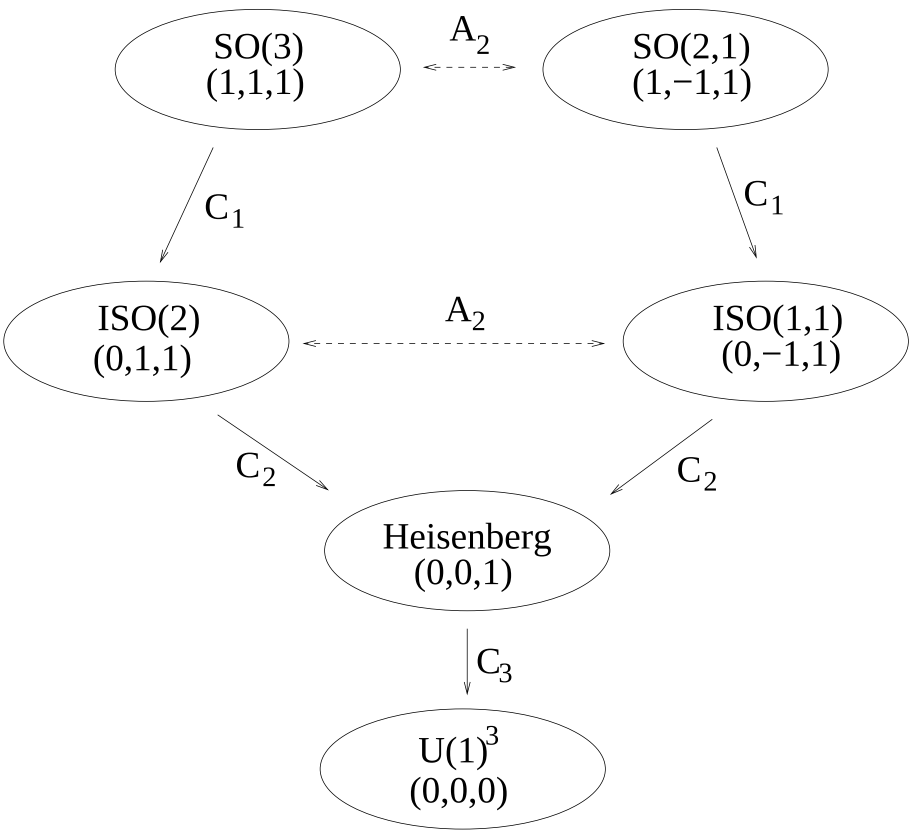

Figure C.1: Relations between groups associated to the 3D Class A Lie algebras. The boxes give the groups and the components $\mathsf{Q}^{mn} = \frac{1}{2}\text{diag}(q_1, q_2, q_3)$ of the structure constants. The arrows give the operations: the dashed arrow corresponds to the reversible analytic continuation, the solid arrow to the irreversible group contraction. These analytic continuations and contractions are defined in (C.8) and (C.9).

Appendix D

Extrema of the scalar potential

In this appendix we extremize the generic potential appearing in the supergravity theories which we study in chapters 2 and 3.

The generic potential reads

$$\mathcal{V} = -\tfrac{1}{2}\, e^{\aleph\varphi} \left\{ [\mathrm{Tr}(\mathcal{M}\mathsf{Q})]^2 - 2\mathrm{Tr}(\mathcal{M}\mathsf{Q}\mathcal{M}\mathsf{Q}) \right\} . \tag{D.1}$$

where $\aleph$ is a constant. $\mathcal{M}$ is an $Sl(n,\mathbb{R})/SO(n)$ scalar matrix and Q is a $n \times n$ mass matrix.

Due to the exponential dependence on the dilaton, there are no extrema with respect to this field (if $\aleph \neq 0$), and therefore it is reasonable to think on the simplest vacua as those that minimize $\mathcal{V}$ with respect to $\mathcal{M}$. From the fact that the potential is not extremized with respect to the dilaton follows that the dynamics of this field will be non-trivial.

In order to extremize the potential we must take into account the constraints on $\mathcal{M}$, namely

$$\begin{aligned} \mathcal{M}\ \ \textit{symmetric} \ \ &: \qquad g_1 = \mathcal{M}_{mn} - \mathcal{M}_{nm} = 0\,, \\ \textit{Unit determinant} \ \ &: \qquad g_2 = \det \mathcal{M} - 1 = 0\,, \end{aligned} \tag{D.2}$$

which, together with the Euler-Lagrange equation

$$\frac{\partial \mathcal{V}}{\partial \mathcal{M}_{mn}} + \lambda_1 \frac{\partial g_1}{\partial \mathcal{M}_{mn}} + \lambda_2 \frac{\partial g_2}{\partial \mathcal{M}_{mn}} = 0\,, \tag{D.3}$$

provide the solution for $\mathcal{M}$ that extremizes the potential. The auxiliary fields $\lambda_{1,2}$ are the Lagrange multipliers associated to the constraints $g_{1,2}$. The first constraint in (D.2) simply implies symmetricity with respect to the internal indices in the equations, so we can solve the equations taking into account only the second constraint and then imposing symmetricity. However, symmetricity appears automatically in these calculations, such that we must only solve

$$\begin{aligned} &\textit{Euler} - \textit{Lagrange} \quad : \qquad \frac{\partial \mathcal{V}(\mathcal{M})}{\partial \mathcal{M}_{mn}} + \lambda \frac{\partial g}{\partial \mathcal{M}_{mn}} = 0\,, \\ &\textit{Unit determinant} \quad : \qquad g = \det \mathcal{M} - 1 = 0\,. \end{aligned} \tag{D.4}$$

where λ is the Lagrange multiplier associated to the constraint g. Then, equation (D.3) reads

$$2\,\mathrm{Tr}\,(\mathcal{M}\mathsf{Q})\,\mathsf{Q}^{mn} - 4\,(\mathsf{Q}\mathcal{M}\mathsf{Q})^{mn} + \lambda\,\mathcal{M}^{mn} = 0\,, \tag{D.5}$$

which is solved for

$$\mathcal{M}_{0\,mn} = |\mathsf{Q}|^{1/n}\,(\mathsf{Q}^{-1})_{mn}\,, \tag{D.6}$$

with

$$\lambda = 2(2-n)|\mathsf{Q}|^{2/n}\,, \tag{D.7}$$

where $|\mathsf{Q}|$ is the determinant of the mass matrix. The value that the potential reaches in these extrema is

$$\mathcal{V}_0 = -\tfrac{1}{2}n(n-2)\,|\mathsf{Q}|^{2/n}\,e^{\aleph\varphi}\,. \tag{D.8}$$

Two cases deserve special mention:

- If $\aleph = 0$ the potential is also extremized with respect to the dilaton, and therefore the potential in the extrema behaves as a pure cosmological constant. The sign of the cosmological constant corresponds to anti-de Sitter space. This is the case of the gauged supergravity theories in five dimensions we derive from BLO_6 in chapter 3, in which a vacuum solution is precisely anti-de Sitter spacetime.

- If $n = 2$ the value that the potential reaches in the extrema is zero. This case corresponds to 9-dimensional gauged/massive supergravities in chapter 3, in which, although the potential is zero, the superpotential is not and hence domain wall solutions can be found.

Appendix E

Reduction of BLO_6 theory to five dimensions

In this appendix we give explicitely the reduction of BLO_6 theory on a 6-torus in the direction of the six Killing vectors. The general procedure can be read from section 3.5 and so we will present the results with no further explanation.

The Ansatz for the Elfbein and 3-form potential are the same as those for the massless case, given, respectively, in (B.50) and (B.51). The 11-dimensional 4-form field strength decomposes as

$$\begin{aligned}
\hat{G}_{abcd} &= e^{\frac{4}{3}\varphi} G_{abcd}\,, \\
\hat{G}_{abc\,i} &= e^{\frac{5}{6}\varphi} L_i{}^m\, H_{m\,abc}\,, \\
\hat{G}_{ab\,ij} &= e^{\frac{1}{3}\varphi}\, L_i{}^m L_j{}^n \left[F_{mn\,ab} + \ell_{mnp}\, F^p{}_{ab}\right]\,, \\
\hat{G}_{a\,ijk} &= e^{-\frac{1}{6}\varphi}\, L_i{}^m L_j{}^n L_k{}^p \mathcal{D}_a\, \ell_{mnp}\,, \\
\hat{G}_{ijkl} &= e^{\frac{1}{6}\varphi}\, L_i{}^m L_j{}^n L_k{}^p L_l{}^q \left[-3\mathsf{Q}^{rs}\ell_{r[mn}\ell_{pq]s}\right]\,,
\end{aligned} \tag{E.1}$$

where $\mathcal{D}$ is the covariant derivative. Note that the term $\hat{G}_{ijkl}$ is 'new' in our analysis in the sense that it is zero for $n \leq 3$. This term gives a new contribution to the scalar potential.

The new 5-dimensional field strengths are defined as

$$
\begin{aligned}
G_{\mu\nu\rho\sigma} &= 4\partial_{[\mu}C_{\nu\rho\sigma]} + 6B_{m\,[\mu\nu}F^{1\,m}{}_{\rho\sigma]} - 3B_{m\,[\mu\nu|}\mathsf{Q}^{mn}B_{n\,|\rho\sigma]}\,, \\
H_{m\,\mu\nu\rho} &= 3\partial_{[\mu|}B_{m|\,\nu\rho]} + 3V_{mn\,[\mu}F^{1\,n}{}_{\nu\rho]}\,, \\
F_{mn\,\mu\nu} &= 2\partial_{[\mu|}V_{mn|\,\nu]} + 2V_{mp\,[\mu|}\mathsf{Q}^{pq}V_{nq|\,\nu]}\,, \\
F^{1\,m}{}_{\mu\nu} &= 2\partial_{[\mu}A^{m}{}_{\nu]} - \mathsf{Q}^{mn}B_{m\,\mu\nu}\,.
\end{aligned}
\tag{E.2}
$$

to which we have to add the covariant derivative of the scalars

$$
\mathcal{D}_{\mu}\ell_{mnp} = \partial_{\mu}\ell_{mnp} - 3V_{[m|q\,\mu}\mathsf{Q}^{qr}\ell_{|np]r}\,. \tag{E.3}
$$

The expressions for the massive gauge transformations are the same as those given in (3.69) and (3.71).

Finally, the $d = 5$ massive action reads

$$
\begin{aligned}
S \;=\; & \frac{1}{16\pi G_N^{(5)}}\int d^5x\sqrt{|g_E|}\;\Big\{R_E+\tfrac{1}{2}\left(\partial\varphi\right)^2+\tfrac{1}{4}\mathrm{Tr}\left(\mathcal{D}\mathcal{M}\mathcal{M}^{-1}\right)^2 \\
& -\tfrac{1}{4}e^{-\varphi}F^m(A)\mathcal{M}_{mn}F^n(A)-\tfrac{1}{2\cdot 4!}e^{-2\varphi}G^2+\tfrac{1}{2\cdot 3!}e^{-\varphi}H_m\mathcal{M}^{mn}H_n \\
& -\tfrac{1}{8}\mathcal{M}^{mp}\mathcal{F}_{mn}\mathcal{M}^{nq}\mathcal{F}_{pq}\;+\tfrac{1}{18}e^{\varphi}\mathcal{M}^{mq}\mathcal{M}^{nr}\mathcal{M}^{ps}\mathcal{D}\ell_{mnp}\,\mathcal{D}\ell_{qrs}\;-\mathcal{V} \\
& -\tfrac{1}{2^6\cdot 3^4}\tfrac{\epsilon}{\sqrt{|g_E|}}\epsilon^{mnpqrs}\;\{\,2\,G\,\mathcal{D}\ell_{mnp}\,\ell_{qrs}+12\,H_m\mathcal{F}_{np}\ell_{qrs}\;+24\,H_m\mathcal{D}\ell_{npq}V_{rs} \\
& +27\,\mathcal{F}_{mn}\mathcal{F}_{pq}V_{rs}+36\,\mathcal{F}_{mn}\mathcal{D}\ell_{pqr}B_s+4\,\mathcal{D}\ell_{mnp}\,\mathcal{D}\ell_{qrs}C \\
& +9\mathsf{Q}^{vw}\left[2\left(GV_{mn}+4H_mB_n+2\mathcal{F}_{mn}C\right)\ell_{pqv}\ell_{rsw}\right. \\
& +2\left(G\ell_{mnp}+12H_mV_{np}+18\mathcal{F}_{mn}B_p+3\mathcal{D}\ell_{mnp}C\right)V_{qv}\ell_{rsw} \\
& +\;\left(4H_m\ell_{npq}+18\mathcal{F}_{mn}V_{pq}+9\mathcal{D}\ell_{mnp}B_q\right)B_v\ell_{rsw} \\
& +4\left(2H_m\ell_{npq}+9\mathcal{F}_{mn}V_{pq}+6\mathcal{D}\ell_{mnp}B_q\right)V_{rv}V_{sw} \\
& \left.+12\left(\mathcal{F}_{mn}\ell_{pqr}+2\mathcal{D}\ell_{mnp}V_{qr}\right)B_vV_{sw}\;+3\mathcal{D}\ell_{mnp}\ell_{qrs}B_vB_w\,\right] \\
& +\tfrac{9}{10}\,\mathsf{Q}^{vw}\,\mathsf{Q}^{xy}\left[9\left(4\ell_{mnp}\ell_{qrv}V_{sw}+3V_{mn}\ell_{pqv}\ell_{rsw}\right)B_xB_y\right. \\
& +8\left(9V_{mn}\ell_{pqv}V_{rw}+16B_m\ell_{npv}\ell_{qrw}\right)V_{sx}B_y \\
& +24\left(3V_{mn}V_{pv}V_{qw}+6B_m\ell_{npv}V_{qw}+12C\ell_{mnv}\ell_{pqw}\right)V_{rx}V_{sy} \\
& \left.+36C\ell_{mnv}\ell_{pqw}\ell_{rsx}B_y\,\right]\}\Big\}\;,
\end{aligned}
\tag{E.4}
$$

where $G_N{}^{(5)}=G_N{}^{(11)}/\mathrm{Vol}(T^6)$, and we have defined

$$
\mathcal{F}_{mn}=F_{mn}(V)+\ell_{mnp}F^p(A)\,. \tag{E.5}
$$

The scalar potential $\mathcal{V}$ is given by

$$
\begin{aligned}
\mathcal{V} \;=\; & -\tfrac{1}{2}\,\Big\{\left[\mathrm{Tr}(\mathcal{M}\mathsf{Q})\right]^2 - 2\,\mathrm{Tr}(\mathcal{M}\mathsf{Q}\mathcal{M}\mathsf{Q}) \\
& -\tfrac{1}{2}e^{\varphi}\left[(\mathsf{Q}\mathcal{M}\mathsf{Q})^{mq}\mathcal{M}^{nr}\mathcal{M}^{ps} - 2\,\mathsf{Q}^{mq}\mathcal{M}^{nr}\mathsf{Q}^{ps}\right]\ell_{mnp}\ell_{qrs} \\
& -\tfrac{1}{24}e^{2\varphi}\mathcal{M}^{mr}\mathcal{M}^{ns}\mathcal{M}^{pt}\mathcal{M}^{qu}\left(3\,\ell_{v[mn}\ell_{p]qw}\mathsf{Q}^{vw}\right)\left(3\,\ell_{x[rs}\ell_{t]uy}\mathsf{Q}^{xy}\right)\Big\}\ .
\end{aligned} \tag{E.6}
$$

A consistent truncation of this theory is to keep only the metric and to set all the fields to zero (the dilaton can be set to a constant). The vacuum solution is anti-de Sitter spacetime. This is a further indication that these theories are in fact the gauged supergravities obtained in [73, 131].

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