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Solutions and Democratic Formulation of Supergravity Theories in Four Dimensions

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Soluciones y Formulación Democrática de Teorías de Supergravedad en Cuatro Dimensiones

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„Es ist mir in den Wissenschaften gegangen wie einem, der früh aufsteht, in der Dämmerung die Morgenröte, sodann aber die Sonne ungeduldig erwartet und doch, wie sie hervortritt, geblendet wird“

Johann Wolfgang von Goethe

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Chapter 1

Introduction

This thesis deals with four-dimensional Supergravity theories and solutions thereto. Supergravities are very interesting theories for many reasons. In this introduction we shall give a short overview of the main motivations for introducing Supersymmetry, Supergravity and, last but not least, Superstring Theory. We will briefly describe how supersymmetry might help to address some "problems" of the Standard Model, then we shortly summarize some basic facts about Superstring Theory and its low energy limit, Supergravity. In the second part of this introduction we will discuss gaugings of Supergravity and its implications, focussing on the so-called tensor hierarchy. In the section 1.3 we will describe schematically how to find supersymmetric solutions to a given Supergravity theory. The outline of this thesis is given in the last section of this introduction.

1.1 Supersymmetry, Supergravity and Superstring Theory

In the last decades of the past century a new theory, Superstring Theory, arose. There are two basic ingredients of Superstring Theory. First, there is the assumption that the fundamental constituents of matter are not pointlike particles, but oscillating one-dimensional objects: strings. The second basic ingredient of Superstring theory is Supersymmetry (SUSY). We start by giving an overview of some open open questions which supersymmetry, especially in the framework of Superstring Theory, might help to answer.

The Standard Model (SM) of elementary particle physics is a spectacularly successful theory of the known particles and their electroweak and strong interactions [1]. Experiments have verified its predictions with incredible precision, and all the particles predicted by this theory have been found apart from the Higgs boson, which is expected to be detected soon at particle accelerators, such as e.g. at LHC at CERN. However, the Standard Model does not explain everything. For example, gravity is not included in the Standard Model of particle physics. Due to its weakness (at a typical energy-scale of particle physics, it is about 10^{-25} times weaker than the weak force and 10^{-38} times than the strong nuclear force[1]) gravity is irrelevant for describing the interactions of the matter studied by particle physicists.

While the electroweak and strong forces are transmitted by spin-1 particles, gravity is supposed to be transmitted by a particle which carries spin 2, and in contrast to the other forces, it acts on *every* particle. On the one hand, Quantum Field Theory is used to explain the fundamental interactions at small distances, while on the other hand the large scale structure of the universe is governed by gravitational interactions described accurately by Einstein's General Relativity. Trying to add gravity to the Standard Model and in particular to combine General Relativity with Quantum Mechanics leads to inconsistencies [2]. From a theoretical and conceptual point of view this is fairly unsatisfactory since we assume that there should be a way to describe the four fundamental forces within the framework of a unique underlying theory. The biggest problems of the Standard Model, as recognized by its practitioners, are:

- The SM is a Yang-Mills gauge theory, in which the gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$ is spontaneously broken to $SU(3)_c \times U(1)_{EM}$ by the non-vanishing vacuum expectation value (VEV) of a fundamental scalar field, the Higgs field. Phenomenologically, the mass of the Higgs boson associated with electroweak symmetry breaking must be in the electroweak range $\langle h \rangle \sim 246$ GeV. However, the contribution of radiative corrections to the Higgs boson mass is *nonzero, divergent and positive*. While the corrections to the electron mass are themselves proportional to the electron mass and quite small, even if we use the Planck scale as cut-off the mass of Higgs particles is very sensitive to the scale.the $(\text{mass})^2$ of the Higgs boson receives radiative corrections from higher-order terms in perturbation theory and a fine tuning of 28 orders of magnitude is necessary in order to obtain a phenomenologically viable Higgs mass. This is possible but very unnatural. This is the so-called *hierarchy problem* and it is the main motivation for introducing supersymmetry at the weak scale.

  The best studied way of achieving this kind of cancellation of quadratic terms (also known as the cancellation of the quadratic divergencies) is supersymmetry (SUSY) [3]. Supersymmetry is a symmetry relating bosons and fermions: it relates particles with integer spin to those of half-integer spin and vice versa, thus assigning every particle a "superpartner" with spin differing by $\frac{1}{2}$. This essentially means that the two basic groups of particles of the Standard Model of Particle Physics, namely matter constituents (those with half-integer spin) and intermediate particles, which carry the forces (those whose spin is an integer), become related to each other. In principle every fermion is accompanied by a

[1] The exact strengths depend on the particles and energies involved.

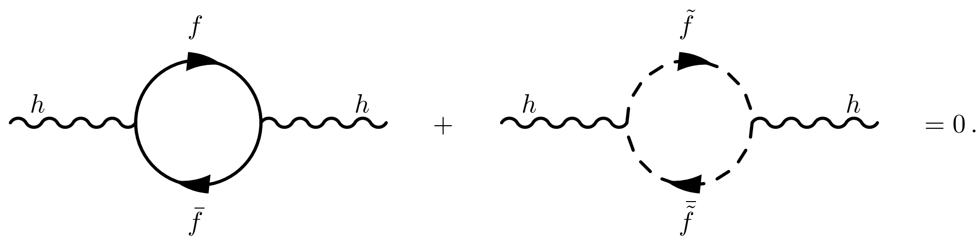

Figure 1.1.1: **Left:** A Higgs boson dissociating into a virtual fermion-antifermion pair in the Standard Model. **Right:** A Higgs boson dissociating into a virtual sfermion-antisfermion pair. This diagram cancels the one on the left.

bosonic superpartner with the same mass[2] and vice versa for the bosons. For example, the quarks, which are fermions, are accompanied by squarks, which are bosons. Similarly, the gluons, being bosons, are accompanied by gluinos, which are fermions [2]. Thus, supersymmetric theories are characterized by equal numbers of bosonic and fermionic degrees of freedom. In the supersymmetric extension of the Standard Model the quadratic corrections to the Higgs boson mass are automatically canceled to all orders in perturbation theory. This is due to the contributions of superpartners of ordinary particles. The contributions from bosonic loops cancel those from the fermionic ones because of an additional factor -1 arising from Fermi statistics, as shown in Fig.1.1.1.

- The Standard Model cannot describe accurately the unification of the gauge couplings in the framework of a The Standard Model fails to deliver gauge coupling unification as envisaged by the paradigm of a Grand Unified Theory (GUT). Supersymmetric extensions of the Standard Model do a far better job.

  The philosophy of Grand Unification is based on a hypothesis: gauge symmetry increases with energy in the sense that at high energies all $(\text{mass})^2$ become negligible. Bearing in mind the unification of all forces of Nature on a common basis and, neglecting gravity for the time being, the idea of GUTs is the following: all known interactions are different branches of a unique interaction associated

[2]Since at today's particle accelerators none of the predicted superpartners has been found yet, if Supersymmetry is a symmetry of Nature, it must be broken (at least at low energy scale) by some appropriate mechanism.

to a simple (in the mathematical sense) gauge group.

Low energy					$\Longrightarrow$	**High energy**
$SU(3)_C$	$\otimes$	$SU(2)_L$	$\otimes$	$U(1)_Y$	$\longrightarrow$	G_{GUT}
g_3		g_2		g_1	$\longrightarrow$	g_{GUT}

Table 1.1.1: Unification of gauge couplings in a Grand Unified Theory.

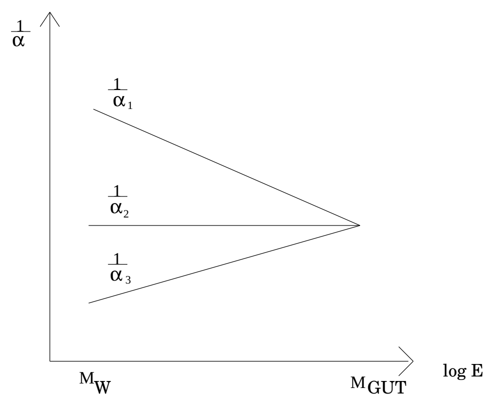

Figure 1.1.2: Coupling constant unification in supersymmetric theories [4]. The constants α_3, α_2 and α_1 correspond to the three factors in $SU(3)\times SU(2)\times U(1)$.

Although there is a big difference in the values of the coupling constants at low energies of strong, weak and electromagnetic interactions, a unification is possible at high energy [3]. The crucial point is the running of the coupling constants. Their values depend on the energy scale at which they are measured as well as on the particle content of the theory. After the precise measurement of the $SU(3) \times SU(2) \times U(1)$ coupling constants, it has become possible to test the unification numerically. Using their values measured at low energies one can extrapolate them to higher energies. It turns out that if one does so in the framework of the Standard Model of particle physics the three coupling constants do not meet in one point, whereas when taking the Minimal Supersymmetric Standard Model (MSSM) they indeed do unify in one point $M_{\rm GUT}$, as schematically shown in Fig. 1.1.2 (supposing that the SUSY masses are of the order of 1 TeV [3]).

- Many attempts have been made to make General Relativity consistent with Quantum Field Theory, especially within the framework of a theory which combines gravity with the strong and electroweak interactions. It is interesting that in some of the most successful attempts Supersymmetry is used, either as a global symmetry or as a local symmetry, therefore containing Supergravity. "Super"-symmetry is a special instance of a Lie superalgebra, which roughly speaking is a Lie algebra containing anticommutators as well as commutators.

The simplest four-dimensional Supersymmetry algebra is the so-called $N = 1$ SUSY algebra. It is the simplest extension of the Poincaré algebra obtained by adding one fermionic chiral generator Q, also called supercharge, with commutation relation. The $N = 1$ SUSY algebra can be written as [5]

$$\{Q, \bar{Q}\} = 2\sigma^\mu P_\mu \,, \tag{1.1.1}$$

$$\{Q, Q\} = \{\bar{Q}, \bar{Q}\} = 0 \,. \tag{1.1.2}$$

The commutator of two infinitesimal SUSY transformations is

$$[\xi Q, \overline{\eta}\overline{Q}] = 2\xi\sigma^\mu\overline{\eta}P_\mu, \tag{1.1.3}$$

with anti-commuting, also called *Grassmann*, parameters ξ and η. In the case of global SUSY this describes a translation along the vector $\xi\sigma^\mu\overline{\eta}$. Choosing the parameters ξ and η to be local, i.e. functions of a space-time point, one finds that the right-hand side of Eq. (1.1.3) becomes $2\xi(x)\sigma^\mu\overline{\eta}(x)P_\mu$ which can be understood as a local coordinate transformation. We see that SUSY is not an internal symmetry, but a spacetime symmetry related through the SUSY algebra to spacetime translations. The theory which is invariant under a general coordinate transformation (GCTs) is General Relativity. Thus, making SUSY local, one obtains General Relativity, or a supersymmetric generalization thereof, Supergravity. In this sense Supergravity is the (non-Abelian) gauge theory of Supersymmetry. After the construction of rigid supersymmetric theories in the early 1970's, $N = 1$ $d = 4$ Supergravity was constructed in 1976 [6,7]. Note that the SM does include Special Relativity, but does not include General Relativity or gravity. Therefore we are led to look for extensions of it and it seems natural to include supersymmetry.

- With the ingredients of the Standard Model of particle physics alone we cannot understand why its particle content is the way it is. The existence of three families, for example, is an experimental fact and is built into the Standard Model. The couplings of the Higgs field to fermions generate masses of quarks and leptons, however their values are free parameters of the SM. There seems to be no reason why the mass spectrum of quarks and leptons should stretch over six orders of magnitude between the masses of the electron and the top quark.

- Other evidence for the existence of Physics Beyond the Standard Model is the cold dark matter (CDM) of the universe, because the Standard Model does not provide a viable candidate for it. Under certain assumptions the lightest supersymmetric particle (LSP) is neutral and stable and hence provides an excellent candidate for CDM.

Thus, despite its spectacular success, the Standard Model of particle physics is not "The End of Science" [2], but may be the low energy limit of some more fundamental underlying theory.

Apart from the arguments given above, there are also more theoretical motivations to study supersymmetry. The first to be mentioned is the Haag-Łopuszanski-Sohnius theorem [8], which states that extended Supersymmetry is the most general extension of the Poincaré and Yang-Mills-type symmetries of the S-matrix. Another reason why Supersymmetry is believed to play an important role in particle physics is that it yields non-renormalization theorems which work to all orders in perturbation theory. This is due to the fact that many divergences in fermionic and bosonic loop diagrams cancel, as is shown in Fig. (1.1.1) for the quadratic divergences for the Higgs mass. Non-renormalization theorems avoid a mixing between low and high energy mass scales, thus solving the hierarchy problem (see above). Furthermore, supersymmetry often makes it possible to extrapolate results from the weak-coupling to the strong-coupling regime, thereby providing information about strongly coupled theories:
Hitherto we restricted ourselves to the $N = 1$ Supersymmetry algebra. Although this seems to be the only phenomenologically viable option, it is very interesting to study *extended Supersymmetry algebras*, i.e. Supersymmetry algebras with more than one supercharge ($N \geq 2$). They play for example an important role in the study of the properties of String Theory. The main implication of including N supercharges Q^A ($A = 1 \ldots N$) is the modification of the anticommutators Eqs. (1.1.1) and (1.1.2), which for extended Supersymmtry take the form [5]

$$\{Q^A, \bar{Q}_B\} = 2\delta^A{}_B \sigma^\mu P_\mu \,, \tag{1.1.4}$$
$$\{Q^A, Q^B\} = Z^{AB} \,, \tag{1.1.5}$$

where Z^{AB} is referred to as a *central charge*, since it commutes with everything. Extended supersymmetry algebras with central charges have special representations, so-called short multiplets. The states in these representations, the BPS states, are annihilated by some of the generators of the supersymmetry algebra. They are characterized by the fact that they saturate the Bogomolny'i bound $M \leq |Z|$, an inequality between its mass and its charge. Even though both mass and charge may undergo renormalization, this definite mass-charge relationship for BPS states is expected to be protected from quantum corrections, since it is a consequence of the supersymmetry algebra assuming that the full theory is supersymmetric.[3] If it were violated, then new states would appear out of nowhere and quantum corrections are not expected

[3] Thus the equality of mass and charge of BPS states is protected against quantum corrections, but mass and charge separately may receive corrections, which depend on the particular theory one is dealing with, especially on the number of supercharges.

to produce these new degrees of freedom. This property of BPS states means that supersymmetry plays a crucial role in the theory of supersymmetric black holes. It turns out that unbroken supersymmetry is an important ingredient in the stringy calculation of the black hole entropy by the counting of microstates of supersymmetric black holes.

String Theory originally arose as an attempt to understand the strong nuclear force between hadrons. It turns out that if one wants String Theory to include also spacetime fermions, one needs to include Supersymmetry, which leads to Superstring Theory. According to String Theory, different kinds of particles (with different charges, masses ...) correspond to the same fundamental object, the string, in different excitation modes. Since the string's length is of the order of the Planck scale ($10^{-35}m$) they are far too small to be identified as extended objects at today's particle colliders. During the *First Superstring Revolution* in the 1980s it was found that there are actually five different spacetime supersymmetric Superstring Theories, each of them living in ten spacetime-dimensions: *type I, type IIA, type IIB, heterotic* $SO(32)$ *and heterotic* $E_8 \times E_8$, which, as it was discovered later, are related to each other by *dualities* (see below). All these five theories live in ten spacetime dimensions and seem to be just special limits of a single underlying eleven-dimensional theory called *M-Theory*. This immediately leads to the idea of compactification, in order to make contact with our four-dimensional world.

One of the problems arising in String Theory is the so-called *vacuum selection problem*: different compactifications of Superstring Theory down to four dimensions may lead to very different physics , because the spectrum (and gauge group) of the four-dimensional theory depends on the choice of six-dimensional internal manifold. Supersymmetric compactifications provide a promising setting for obtaining realistic supersymmetric models of particle physics: by compactifying down to four spacetime dimensions, one might hope to make contact with particle physics phenomenology.

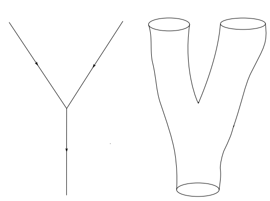

Figure 1.1.3:
Worldline *vs* world-sheet

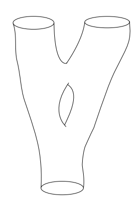

Figure 1.1.4:
$g = 1$ surface

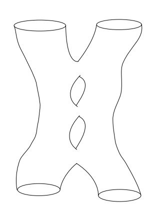

Figure 1.1.5:
$g = 2$ surface

There is only one fundamental (dimensionful) constant in String Theory, which governs the scale of the massive string excitations. This constant can be expressed in terms of the *Regge slope parameter* α' (which has mass dimension -2), the string tension (energy per unit length) $T = \frac{1}{2\pi\alpha'}$ or in terms of the string length scale $l_s^2 = 2\alpha'$.

Massive string excitations have masses of the order $M \sim \frac{1}{\sqrt{\alpha'}}$ which are typically of the order of the Planck mass. By definition, the low-energy limit of string theory only involves processes at an energy scale E far below the Planck scale, i.e.

$$E^2\alpha' \ll 1. \tag{1.1.6}$$

This means that in the low-energy approximation one can restrict the analysis to the massless modes only and describe them by an effective theory. The massive states of String Theory become important only at energy scales that are currently out of reach. The low-energy effective theories of spacetime supersymmetric String Theories always contain in their spectra a massless spin-2 particle (together with its corresponding spin-3/2 superpartner) and consistency requires that these theories are Supergravity (SUGRA) theories. As indicated above this is a good approximation, as long as one considers processes with energies far below the Planck mass. At energy scales much lower than the Planck scale, that is at length scales much larger than the string length $l_s = \sqrt{\alpha'}$, the string behaves like a pointlike particle. Effects due to the extension of the string are hidden in stringy α'-corrections.

Superstring Theory is well suited for constructing a quantum theory that unifies the description of gravity and the other fundamental forces of nature. One of the most important feature of Superstring Theory is that gravity is automatically incorporated in the theory. The theory gets modified at very short distances/high energies but at ordinary distances and energies gravity is present in exactly the form proposed by Einstein. While ordinary Quantum Field Theory does not seem to be compatible with gravity, String Theory *requires* gravity.

Supergravity plays for many reasons a key role in our understanding of String Theory. It is very difficult to study full string theories, but studying its low-energy effective theory, i.e. Supergravity, can give insight in concepts such as string dualities, which for instance can relate strong-coupling and weak-coupling regimes in String Theory.

All five String Theories contain a massless scalar field, the dilaton ϕ, whose vacuum expectation value $\phi_0 = \langle\phi\rangle$ determines the string coupling constant $g_S = e^{\phi_0}$. Just as Feynman diagrams in Quantum Electro Dynamics, in Superstring Theory one can do a power series expansion in the dimension-less string coupling constant. The String Theory Feynman diagram is represented by a 2-dimensional Riemann surface (see Fig.1.1.3), i.e. for oriented closed strinsg an orientable and closed surface of genus g (a surface with g handles), which comes along with an factor g_S^{2g} [4]. As an example the world-sheet in Fig.1.1.4 is of genus 1, the one in Fig.1.1.5 is of genus 2. However, there is a priori no reason why the string coupling constant g_S should be small. For this reason a lot of effort is made to understand non-perturbative aspects of string theory. After the discovery of dualities in the last decade of the past century (*Second Superstring Revolution*) it was shown that Superstring Theory contains, apart from the 1-dimensional strings, also higher-dimensional objects with $p \geq 2$ spacial dimensions, referred to as *p-branes*. Of special interest is a subclass thereof, the so-called D-branes: p-branes on which open strings can end. One of the most important applications of D-brane physics is the counting of black hole microstates. According to the Bekenstein-Hawking formula the entropy of a (classical) black hole is given by $S_{\mathrm{BH}} = \frac{1}{4}A$, where A denotes the area of the black hole event horizon. The Bekenstein-Hawking entropy plays the role of the macroscopic or thermodynamical entropy. Considering, then, the macroscopic Supergravity description of a black hole to be an effective description of an underlying microscopic quantum theory, the macroscopic Bekenstein-Hawking entropy should match the microscopic entropy

$$S_{\mathrm{BH}} = S_{\mathrm{micro}}, \tag{1.1.7}$$

where the microscopic or statistical entropy is given by

$$S_{\mathrm{micro}} = \ln N(M, J, Q), \tag{1.1.8}$$

and where N is the number of different microstates of a black hole characterized by the macroscopic variables M, J and Q.

D-brane techniques can be used to count the black hole microstates and it turns out that the macroscopic and microscpic entropies of supersymmetric or “near supersymmetric” black holes indeed agree. This was done first for a class of 5-dimensional extremal black holes by Strominger and Vafa [9] and later on for other kinds of black holes.

In this thesis we deal with different kinds of Supergravity theories in four spacetime dimensions. Some, but not all d-dimensional Supergravities can be obtained as the low-energy limit of some Superstring Theory compactified on a $(10 - d)$-dimensional manifold (we will discuss this in some more detail in Section 1.2). But there is also another point of view, not taking into account any relation to (higher-dimensional) String Theory, to study Supergravity for its own sake. The basic ingredients of Supergravity are General Relativity (GR) and Supersymmetry. General Relativity is a purely bosonic theory. Making GR supersymmetric then means introducing fermionic, anti-commuting coordinates, thus generalizing the standard bosonic spacetime to superspacetime. Depending on the dimension of the spacetime, one can introduce different kinds of Supergravity theories.

1.2 Gauged Supergravity and the p-form hierarchy

Gauged Supergravities can be considered as deformations of the ungauged theories. While the undeformed theories by definition do no include a potential for the scalar fields nor a cosmological constant, gauging Supergravity introduces a scalar potential and the theory is no longer determined by its kinetic terms only. The gauge coupling constant plays the role of the deformation parameter. However, there are also other types of deformations, which are not due to gaugings. In $N = 1$ Supergravity, for example, one can always introduce a superpotential, independently of making some global symmetry group local or not. Another way to deform supergravities are *massive deformations*, see e..g. Romans' massive $N = 2A$ $d = 10$ Supergravity [10].

There are two ways of obtaining gauged Supergravity from the ungauged theory: on the one hand one can consider the higher dimensional origin of gaugings by compactification of ten or eleven-dimensional Supergravity on manifolds with fluxes; or, on the other hand, one can directly deform the four-dimensional theory. If, for example, we compactify ten-dimensional Supergravity on a six-torus T^6, we obtain maximal $N = 8$ Supergravity in four dimensions (see Chapter 2.2.1). Note that compactification on a torus does not break any supersymmetry, such that the lower-dimensional theories are maximally supersymmetric. If one compactifies on a manifold which allows for some of the higher dimensional p-form fields to acquire background fluxes or a manifold provided with torsion etc., one generically ends up with a gauged Supergravity theory in lower dimensions. In this thesis we will focus our attention on the first approach and shall discuss how to obtain the gauged version of a given four-dimensional theory by promoting some subgroup G of the global symmetry group H to a local symmetry.

The first examples of gauged Supergravity were constructed in the early 1980's, and recent research has shown that gauged Supergravities can be constructed in a systematic way by means of the so-called *embedding tensor formalism* [11]. This formalism is independent of the dimension and the number of supersymmetries of the respective theory. Furthermore, from the higher-dimensional point of view, it allows us to encode some, but not obligatorily all [12], the flux/deformation parameters in a single tensorial object, the embedding tensor [13].

We will denote collectively the electric and magnetic vector fields by the symplectic vector $A^M{}_\mu$, because the global symmetry group G will always act on A^M as a subgroup of $Sp(2n, \mathbb{R})$, where n denotes the number of (electric) vector fields appearing in the theory, even though G can be a larger group than $Sp(2n, \mathbb{R})$ and/or not be contained in it (see Section 3.2.1). The fact that one can always dualize the electric vectors appearing in the standard formulation of four-dimensional Supergravity into magnetic vectors, is a property of the four-dimensional theory. We will see in the following chapters how this works in detail. These Abelian vector fields are invariant under the Abelian gauge transformations

$$\delta_\Lambda A^M{}_\mu = -\partial_\mu \Lambda^M \,, \tag{1.2.1}$$

where $\Lambda^M(x)$ is a symplectic vector of local gauge parameters.

As mentioned before, we are going to construct gauged Supergravity as a deformation of the ungauged theory, thus our starting point will be the ungauged theory with global symmetry group G. The generators T_A of the Lie algebra $\mathfrak{g}$ of the symmetry group G satisfy the commutation relations

$$[T_A, T_B] = -f_{AB}{}^C T_C \,. \tag{1.2.2}$$

where $f_{AB}{}^C$ are the structure constants of $\mathfrak{g}$.

Under this non-Abelian global symmetry the vectors of the theory transform as

$$\delta_\alpha A^M = \alpha^A T_{A\,N}{}^M A^N \,, \tag{1.2.3}$$

where $T_{A\,M}{}^N$ are the components of the matrices T_A, $T_{A\,M}{}^N = (T_A)_M{}^N$, that generate the Lie algebra $\mathfrak{g}$.

In order to gauge the symmetry group G we must promote the global parameters α^A to arbitrary spacetime functions $\alpha^A(x)$ and make the theory invariant under these new transformations. This is achieved by identifying these arbitrary functions with a subset of the (Abelian) gauge parameters of the vector fields, Λ^M and subsequently using the corresponding vectors as gauge fields. This identification is conveniently made through the use of the *embedding tensor* $\theta^A{}_M$ [11, 14–17]

$$\alpha^A(x) \equiv \Lambda^M(x) \vartheta_M{}^A \,. \tag{1.2.4}$$

The embedding tensor approach provides a systematic way to study the most general gaugings of a Supergravity theory and is a powerful technique to construct gauged Supergravity theories for different gauge groups in a unified way. The embedding tensor indicates what vector fields (electric or magnetic) gauge what symmetry., allowing us to treat all vector fields, gauged or not, on the same footing. Symplectic invariance can, thus, be formally preserved after the gauging. This is one of the main virtues of this formalism. The choice of the embedding tensor $\theta^A{}_M$ determines completely a particular gauging of the theory, i.e. it determines G.

The embedding tensor is not completely arbitrary but must satisfy a number of constraints which guarantee the consistency of the theory. In the case discussed in this thesis, namely the four-dimensional one, the embedding tensor has to fullfill three different constraints: two quadratic constraints and a linear one, the so-called *representation constraint*. In the gauged theory, we then have to replace partial derivatives by covariant derivatives, schematically:

$$d \longrightarrow \mathfrak{D} = d + \Gamma(T_A)\theta^A{}_M A^M \,. \tag{1.2.5}$$

Here no gauge coupling constant g appears explicitly, but it is contained in the embedding tensor, taking into account that different choices of the embedding tensor correspond to different gaugings and thus describing in a natural and unified way multiple gauge groups.

When constructing a matter-coupled Supergravity theory one usually concentrates on the lowest rank fields that describe the physical states of the theory in question.

Generically the bosonic states are represented by the graviton, and a set of matter fields that generically are differential forms of low rank $(d-2)/2 \geq p \geq 0$ for d even and $(d-3)/2 \geq p \geq 0$ for d odd, respectively. To describe the coupling of Supergravity to branes one is naturally led to consider the dual $(d-p-2)$-form potentials as well. For $p \neq 0$ and at leading order, the construction of the dual potentials is rather straightforward as the original low-rank differential form fields always occur via their curvatures. However, it might not always be possible to eliminate the original potentials from the action in favour of their (magnetic) duals, since the bosonic gauge transformations of the $(d-p-2)$-forms might become rather complicated and involve the gauge transformations of their dual p-form fields. The first example for this was found in [18], where the 3-form potential of eleven-dimensional Supergravity was dualized into a 6-form potential, which turned out to transform under the gauge transformations of the 3-form. In [19] a democratic formulation of ten-dimensional type II Supergravity was achieved, i.e. a formulation of IIA/B Supergravity where all R-R potentials $C^{(p)}$ $(p = 0 \dots 9)$ are treated in a unified way (p odd in case of IIA and p even for IIB, respectively). By virtue of the Bianchi identities of the curvatures of the electric and magnetic potentials, the second-order equations of motion can be derived as integrability conditions of the duality relations:

$$\text{Bianchi identities} \;\;\&\;\; \text{duality relations} \;\Leftrightarrow\; \text{equations of motion}\,. \tag{1.2.6}$$

For instance, in the case of IIA/IIB Supergravity the supersymmetry algebra can be realized on all p-forms $(0 \leq p \leq 10)$ with p odd (IIA) or p even (IIB). The Bianchi identities and duality relations then give all equations of motion (except for the Einstein equation). This is often referred to as the *democratic formulation* of IIA/IIB Supergravity [19].

The idea of deriving the equations of motion of Supergravity from an underlying set of Bianchi identities and first-order differential equations has been pursued in several contexts in the Supergravity literature. It already occurs in the work of [18] for the case of maximal Supergravity including massive IIA Supergravity [20]. Similar duality relations are natural in the E_{11}-approach to Supergravity [21–24]. Duality relations also play an important role in encoding the integrability of a system, for instance in maximal two-dimensional Supergravity [25].

The most important physical application of introducing all higher degree dual potentials is related to the fact that, just as pointlike particles naturally couple to 1-form potentials), higher degree p-forms couple naturally to objects with $p-1$ spatial dimensions. Part of this thesis is dedicated to the study of string-solutions of four-dimensional $N=2$ Supergravity [26] [27] and their coupling to 2-form potentials, which are obtained when dualizing the scalars of the theory [28]. We will show how, once the supersymmetry transformation law for the 2-form is known, to construct the most general space-time supersymmetric worldsheet-action for the supersymmetric string solutions. In four dimensions, apart from 2-forms, one can construct 3 and 4-form potentials, to which domain-walls and spacetime-filling branes, respectively,

couple.
Before discussing the introduction of all possible p-form potentials in four dimensions, let us consider the bosonic fields which appear in the standard formulation of four-dimensional Supergravity. The basic constituent is the Supergravity multiplet, which contains at least the graviton and a certain number N of gravitini (this is what we will refer to as N*-extended Supergravity*). Further it contains $\frac{N(N-1)}{2}$ vectors (graviphotons) for $N \geq 2$ and a number of scalars for $N \geq 4$. The gravitino has spin $3/2$ and plays the role of the gauge field for Supersymmetry. The maximal number of gravitini depends on the dimension of space-time. In $4 = 3 + 1$ dimensions one can have $N = 1$ upto $N = 8$ gravitinos, for larger values of N one would need particles with spin larger than 2 and no consistent interacting theories exist for these cases [29]. The field content of four-dimensional Supergravity multiplets for different numbers of supersymmetries is given in table 1.2.1.
A Supergravity whose field content is contained exclusively in the gravity multiplet, is referred to as *pure* or *minimal* Supergravity. Further, for $N \leq 4$, one can couple different kinds of matter to the pure Supergravity theories. The kind of matter which can be added depends on the number of supersymmetries. In Table 1.2.2 possible mattermultiplets are summarized for four-dimensional Supergravities. **V** denotes possible vector multiplets, **S** multiplets whose bosonic content is only scalar fields.

Vector multiplets are those containing $s = 1$ fields (vectors) as highest spin fields, and the multiplets for $N \leq 2$ with spin $\leq \frac{1}{2}$ are called *hypermultiplets* for $N = 2$ and *chiral multiplets* for $N = 1$, respectively. Their field contents are given in Table 1.2.3. An arbitrary number of these matter multiplets can be used for rigid supersymmetry or can be added to the gravity multiplet in local supersymmetry (Supergravity).

The vectors in the matter multiplets and those possibly contained in the gravity multiplet can be used to gauge a (possibly non-Abelian) global symmetry group. As can be seen in Table 1.2.1, apart from $N = 1$ there is always at least one vector in the gravity supermultiplet, which means that for $N \geq 2$ one can gauge pure Supergravity, i.e. without coupling it to additional "external" matter.
One of the aims of this thesis is to study the extension of the set of standard bosonic fields of four-dimensional Supergravity. We are going to show that we can consistently add dual magnetic vectors, 2-forms, 3-forms and 4-forms to the standard set of bosonic fields, which we were discussing in the previous paragraph. By "consistently" we mean that we can define supersymmetry transformations for them such that the local supersymmetry algebra closes on-shell. First we are going to consider the ungauged theory. The inclusion of magnetic vector fields and 2-forms B was worked out in detail in [28] and [30] for $N = 2$ and $N = 1$ ungauged Supergravity, respectively. It turns out that gauging the theory leads to an entanglement between higher degree forms, which does not appear in the ungauged case. Although 3- and 4-form fields need not appear in the ungauged theory, since for vanishing coupling constant the hierarchy can be consistently truncated, they appear naturally in the gauged theory.

s	$N=1$	$N=2$	$N=3$	$N=4$	$N=5$	$N=6$	$N=8$
2	1	1	1	1	1	1	1
$\frac{3}{2}$	1	2	3	4	5	6	8
1		1	3	6	10	16	28
$\frac{1}{2}$			1	4	11	26	56
0				2	10	30	70

Table 1.2.1: *Pure Supergravity multiplets in four dimensions according to spin s*

susy	32	24	20	16	12	8	4
M	$N=8$	$N=6$	$N=5$	$N=4$ $\mathbf{V}$	$N=3$ $\mathbf{V}$	$N=2$ $\mathbf{V},\mathbf{S}$	$N=1$ $\mathbf{V},\mathbf{S}$

Table 1.2.2: *Possible types of matter multiplets in four-dimensional Supergravity*

s	$N=1$	$N=2$	$N=3,4$
1	1	1	1
$\frac{1}{2}$	1	2	4
0		2	6

s	$N=1$	$N=2$
$\frac{1}{2}$	1	2
0	2	4

Table 1.2.3: *Field content of matter multiplets in four dimensions*

Our starting point will be the generalization of electromagnetic duality in four dimensions. While the standard electric vector fields appear in the action and carry propagating degrees of freedom, the dual magnetic vectors are defined as their on-shell Hodge duals. Let us consider the magnetic vector fields in the ungauged theory first.

The bosonic action of four-dimensional Supergravity generically takes the form

$$\begin{aligned} S \;=\; \int d^4x\sqrt{|g|}\,\big[R + 2\mathcal{G}_{ij^*}\partial_\mu Z^i\partial^\mu Z^{*\,j^*} \\ +\Im\mathrm{m}\mathcal{M}_{\Lambda\Sigma}F^{\Lambda\,\mu\nu}F^{\Sigma}{}_{\mu\nu} - \Re\mathrm{e}\mathcal{M}_{\Lambda\Sigma}F^{\Lambda\,\mu\nu\star}F^{\Sigma}{}_{\mu\nu}\big]\,, \end{aligned} \tag{1.2.7}$$

where Z^i denote the complex scalars of the theory which parameterize a Kähler manifold[4] and

$$\star F^{\Sigma}{}_{\mu\nu} \equiv \tfrac{1}{2\sqrt{|g|}}\,\epsilon_{\mu\nu\rho\sigma}F^{\Sigma\,\rho\sigma}\,. \tag{1.2.8}$$

The metric on the Kähler manifold is denoted by $\mathcal{G}_{ij^*}$, where the index (j^*) i is a (anti-)holomorphic index. The field strengths of the n_V (Abelian) vectors $A^{\Lambda}{}_{\mu}$ ($\Lambda = 1\ldots n_V$) are $F^{\Lambda} = dA^{\Lambda}$. The scalars couple to the vectors via some scalar-dependent complex matrix $\mathcal{M}_{\Lambda\Sigma}(Z^i, Z^{*i^*})$. Moreover, the matrix $\Im\mathrm{m}\mathcal{M}_{\Lambda\Sigma}$ must be negative-definite to ensure the right sign of the vector kinetic term. Note that for constant $\mathcal{M}_{\Lambda\Sigma}$ the last term in (1.2.7) is just a total derivative, while for $\mathcal{M}_{\Lambda\Sigma}(Z^i, Z^{*\,i^*})$ a function of the scalars, it describes a non-trivial coupling of the scalars to the vector fields.

The field strengths F^{Λ} of the vector potentials A^{Λ} satisfy the Bianchi identity

$$\star\mathcal{B}^{\Lambda} \;\equiv\; -dF^{\Lambda} = 0\,, \tag{1.2.9}$$

and the equations of motion have the form

$$\star\mathcal{E}_{\Lambda} \;\equiv\; -dF_{\Lambda}\,, \tag{1.2.10}$$

where we defined the dual field strength F_{Λ}

$$F_{\Lambda} \;\equiv\; \frac{1}{4\sqrt{|g|}}\frac{\delta S}{\delta\star F^{\Lambda}}\,. \tag{1.2.11}$$

The Maxwell equations can be interpreted as Bianchi identities for the dual field strengths, F_{Λ}, ensuring the local existence of n dual vector potentials A_{Λ} such that

$$F_{\Lambda} = dA_{\Lambda}\,. \tag{1.2.12}$$

[4]For $N = 1$ the scalars parameterize a *Kähler-Hodge* manifold, for $N = 2$ a *special Kähler* manifold. This will be discussed in detail in Chapter 2.

It is convenient to combine the standard, electric, field strengths and potentials and their duals Eq. (1.2.11) into a single $2n_V$-dimensional symplectic vector

$$F^M \equiv \begin{pmatrix} F^\Lambda \\ F_\Lambda \end{pmatrix} = dA^M \equiv d \begin{pmatrix} A^\Lambda \\ A_\Lambda \end{pmatrix}, \tag{1.2.13}$$

which allows us to write the Maxwell equations and Bianchi identities in the compact form

$$dF^M = 0. \tag{1.2.14}$$

This set of extended equations of motion (Maxwell equations plus Bianchi identities) is invariant under general linear transformations

$$\begin{pmatrix} F^\Lambda \\ F_\Lambda \end{pmatrix}' = \begin{pmatrix} A_\Sigma{}^\Lambda & B^{\Sigma\Lambda} \\ C_{\Sigma\Lambda} & D^\Sigma{}_\Lambda \end{pmatrix} \begin{pmatrix} F^\Sigma \\ F_\Sigma \end{pmatrix}. \tag{1.2.15}$$

However, consistency with the definition of F_Λ Eq. (1.2.11) requires that the kinetic matrix $\mathcal{M}$ appearing in the action Eq. (1.2.7) transforms at the same time and then one finds that the Maxwell equations and Bianchi identities are formally invariant under the transformations

$$F'^M \equiv M_N{}^M F^N, \tag{1.2.16}$$

with $M \in Sp(2n_V, \mathbb{R})$ [31].

Note that the fact, that the vectors A^Λ and A_Λ appear in pairs is a special property of four-dimensional Supergravity, since only in four dimensions are vectors dual to vectors. In general in even dimensions $d = 2k$ there is a duality between $(k-1)$-forms and $(k-1)$-forms.

In the gauged theory the story is slightly more complicated. It turns out that for general gaugings, i.e. using electric as well as magnetic vectors as gauge fields, one needs to introduce a set of 2-forms in F^M, in order to have a covariantly transforming field strength for the vector fields

$$F^M = dA^M + \tfrac{1}{2} X_{[NP]}{}^M A^N \wedge A^P + Z^{MA} B_A, \tag{1.2.17}$$

where X_M denote the generators of the gauge group and Z^{MA} is essentially the embedding tensor. It can be shown then that in order to have a covariant field strength for the 2-form fields one needs to introduce 3-forms and so on. This bootstrap procedure ends with the introduction of the top-form potentials. In this way one obtains a complete tensor-hierarchy, i.e. a set of p-form fields, with $1 \leq p \leq 4$, which realizes an off-shell algebra of bosonic gauge transformations. Schematically the covariant field strengths $F_{(p+1)}$ of the p-form field $A_{(p)}$ take the form

$$F_{(p+1)} = \mathfrak{D} A_{(p)} + \cdots + Y_{(p+1)} A_{(p+1)} \tag{1.2.18}$$

where the constants $Y_{(p+1)}$ depend on the embedding tensor, showing clearly that in the ungauged theory the hierarchy decouples.

The only input required for this construction is the number of electric $p \geq 1$-form potentials, the global symmetries of the theory and the representations of this group under which the p-forms transform. Changing these data leads to different theories that can be seen as different *realizations* of the low-rank sector of the same tensor hierarchy.

The magnetic $(d-p-2)$-forms do not introduce any new degrees of freedom. As we just saw in the example of the vector field strength, this is ensured by first-order duality-relations, which generically relate the electric p-forms to the magnetic $(d-p-2)$-forms.

Dual potentials are not only relevant to describe the coupling to branes but play also a crucial role in the construction of a supersymmetric action for certain gauged Supergravity theories.

Although usually supersymmetric actions involve, apart from the metric, only electric potentials, using the embedding tensor approach, we are going to show that the action must also contain a dual 2-form potential[5], if one wants to consider a magnetic gauging in $d = 4$, i.e. a gauging involving a magnetic vector field. In general dimensions, p-form potentials of even higher rank are introduced. For instance, the action corresponding to certain gaugings in $d = 6$ requires magnetic 2-form and 3-form potentials [32]. This leads to the notion of a *tensor hierarchy*, which consists of a system of potentials of all degrees ($p = 1, \ldots, d$) and their respective curvatures, which are related by Bianchi identities [17, 33].

1.3 Supersymmetric configurations and solutions of Supergravity

Supersymmetric classical solutions of Supergravity theories have played, and continue to play, a key role in many of the most important developments in string theory. They are an important tool in the current research on many topics in superstring theory, ranging from the AdS/CFT correspondence to stringy black-hole physics. Not all locally supersymmetric solutions are necessarily interesting or need be useful in the end, but it is clearly important to find and classify them all for every possible Supergravity theory.

This goal has been pursued and reached in several lower-dimensional theories and families of theories. The pioneering work was done in 1983 by Tod [34] in pure, ungauged, $N = 2, d = 4$ Supergravity. It was subsequently extended to the gauged case in Ref. [35], to include the coupling to general (ungauged) vector multiplets and hypermultiplets in Refs. [26] and [27], respectively and some partial results on the

[5] In the context of $N = 2, d = 4$ Supergravity it has been shown how the local supersymmetry algebra can be closed on some of these dual 2-form fields [28].

theory with gauged vector multiplets have been recently obtained [36]. Research on pure $N = 4, d = 4$ Supergravity was started in Ref. [37] and completed in Ref. [38].

In $d = 5$, the minimal $N = 1$ (sometimes referred as $N = 2$) theory was worked out in Ref. [39] and the results were extended to the gauged case in Ref. [40]. The coupling to an arbitrary number of vector multiplets and their Abelian gaugings was considered in Refs. [41, 42][6]. The inclusion of (ungauged) hypermultiplets was considered in [45][7] and the extension to the most general gaugings with vector multiplets and hypermultiplets was worked out in [49].

The minimal $d = 6$ SUGRA was dealt with in Refs. [50, 51], some gaugings were considered in Ref. [52] and the coupling to hypermultiplets was fully solved in Ref. [53].

All these works are essentially based on the method pioneered by Tod and made more accessable by Gauntlett et al. in Ref. [39] using non-4d-specific techniques, which we will use here. An alternative method is that of spinorial geometry, developed in Ref. [54]. Some further works on this subject in 4 or higher dimensions are Refs. [55].

Another motivation to study supersymmetric solutions of Supergravity theories is their importance for black hole thermodynamics: a microscopic interpretation of black hole entropy in String Theory is best understood for supersymmetric black holes, and various kinds of supersymmetric solutions have transformed our understanding of quantum field theory via the AdS/CFT correspondence and its generalizations.

Let us denote symbolically by B and F the bosonic and fermionic fields of the theory, respectively. Then, the Supersymmetry transformations of the fields are schematically of the form

$$\delta_\epsilon B \sim \bar{\epsilon} F \tag{1.3.1}$$

$$\delta_\epsilon F \sim \partial\epsilon + B\epsilon, \tag{1.3.2}$$

where $\epsilon(x)$ denotes a spinorial parameter. A classical bosonic configuration (i.e. a configuration $B = \{$metric $g_{\mu\nu}$, vectors A_μ, scalars ϕ and possibly higher-degree form fields$\}$, depending on the specific Supergravity theory, with vanishing fermionic fields $F = 0$) is invariant under the infinitesimal supersymmetry transformation generated by ϵ if it satisfies

$$\delta_\epsilon F \sim \partial\epsilon + B\epsilon = 0. \tag{1.3.3}$$

These equations are called *Killing Spinor Equations (KSEs)* and an $\epsilon(x)$ satisfying the KSEs is accordingly called a *Killing spinor*. In Supergravities (which may have one or more than one supercharge, $N \geq 1$) a configuration is called *supersymmetric* if there is at least one Killing spinor.

It is essential for the understanding of what follows to distinguish between *supersymmetric configurations* and *supersymmetric solutions* of a theory. A set of bosonic fields which admits a Killing spinor is called a supersymmetric configuration and may

[6] Previous work on these theories can be found in Refs. [43, 44].

[7] Previous partial results on that problem were presented in Refs. [46–48].

not fullfill the equations of motion. By supersymmetric solution we mean a supersymmetric bosonic field configuration, that leaves unbroken at least some amount of supersymmetry and fullfills the bosonic equations of motion. We will see what Supersymmetry can tell about solutions of the field equations and how it restricts the number of independent equations of motion, in the sense that once dealing with a supersymmetric configuration one does not have to impose all of the equations of motion, but only a subset of them, in order to be sure that all the equations of motion are satisfied.

Therefore, to achieve our goal of finding all the supersymmetric solutions of a given Supergravity theory, it is in general much simpler to start with finding supersymmetric configurations, since the equations of motion are second order differential equations, whereas the KSEs are only of first order. Further, the supersymmetric field configurations satisfy the so-called *Killing Spinor Identities (KSIs)*, which can be derived from the integrability conditions of the KSEs. These equations relate the different (bosonic) equations of motion and their content is highly non-trivial, even if each term vanishes separately on-shell. Since in this way they reduce the number of independent equations that need to be imposed, they are of great avail in finding supersymmetric solutions. This is reflected by the fact that supersymmetric solutions are generically given in terms of a very small number of independent functions. The general Killing Spinor Identities, which the bosonic equations of motion have to satisfy in supersymmetric theories if the solutions admit Killing spinors, were found in [56] and applied to the problem of finding the minimal set of equations of motion in [57].

The Killing spinor identities can be derived from the supersymmetry variation of the action in the following way [57]: demanding invariance of a generic action S under supersymmetry transformations means

$$\delta_\epsilon S = \int d^d x (\delta_B S \delta_\epsilon B + \delta_F S \delta_\epsilon F) + \text{surface terms} = 0\,, \tag{1.3.4}$$

where $S,_B = \delta_B S = \frac{\delta S}{\delta B}$ is the equation of motion of the fermion field B and analogously for the fermions. Summation over the indices F, B is understood. Now we vary this equation w.r.t. the fermionic fields

$$\{S,_{BF_2}\,\delta_\epsilon B + S,_B\,(\delta_\epsilon B),_{F_2} + S,_{F_1F_2}\,\delta_\epsilon F_1 + S,_{F_1}\,(\delta_\epsilon F),_{F_2}\,\}|_{F=0} = 0. \tag{1.3.5}$$

Since we are only interested in bosonic backgrounds, we are now going to set the fermionic fields to zero, $F = 0$. The bosonic equations of motion $S,_B$ and the supersymmetry variations of the fermions $\delta_\epsilon F$ are necessarily even in fermions and thus vanish for vanishing fermions, but on the first and the fourth term in Eq. (1.3.5) we have to impose:

$$S,_{BF_2}\,|_{F=0} = 0, \qquad (\delta_\epsilon F),_{F_2} = 0\,. \tag{1.3.6}$$

This leaves us with

$$\{S,_B\,(\delta_\epsilon B),_{F_2} + S,_{F_1F_2}\,\delta_\epsilon F_1\}|_{F=0} = 0. \tag{1.3.7}$$

These equations are valid for arbitrary values of the bosonic fields and the supersymmetry parameter ϵ. We are interested in supersymmetric bosonic configurations, i.e. field configurations which admit (at least) one Killing spinor κ. In our schematic way of writing the KSE, Eq. (1.3.3), is written as

$$\delta_\kappa F|_{F=0} = 0\,, \tag{1.3.8}$$

which implies tha a supersymmetric configuration always satisfies the *Killing spinor identities (KSIs)*

$$S_{,B}\,(\delta_\kappa B)_{,F}\,|_{F=0} = 0. \tag{1.3.9}$$

Written in this form it is easy to see that the KSIs relate the bosonic equations of motion of the theory, as already mentioned in the previous paragraph. In this sense the KSIs help us to remarkably reduce the amount of work one needs to do in order to verify that a supersymmetric configuration is also a solution to the classical equations of motion. Note that while Eq. (1.3.4) relates bosonic equations of motion to fermionic ones, the KSIs relate bosonic equations of motion to bosonic ones.

Observe that the Bianchi identities (involving vector field strengths, in the case treated in this thesis, or $p+1$-form field strengths in the general case) do not appear in the Killing spinor identities because the procedure used to derive them assumes the existence of the potentials and, therefore, the vanishing of the Bianchi identities. Since it is convenient to treat Maxwell equations and Bianchi identities on equal footing to preserve the electric-magnetic dualities of the theory, it is sometimes convenient to have the duality-covariant version of the above KSIs. These can be found by performing duality rotations of the above identities or from the integrability conditions of the KSEs.

How to find supersymmetric solutions?

Since one of the purposes of this thesis is to systematically find all the supersymmetric solutions of $d = 4$ Supergravity, we should say a few words about what we mean by ”finding solutions” and how we are going to proceed in order to find all of them. Finding supersymmetric configurations of the theory means expressing the bosonic fields of the theory in terms of a minimal set of independent variables and/or structures in such a way that they admit Killing spinors, i.e. the Killing spinor equations are sastisfied for at least one Killing spinor whose existence is to be proved. The next step is to check which of these field configurations fullfill the equations of motions, *viz.* to find supersymmetric solutions.

The basic strategy to find supersymmetric solutions of a given Supergravity theory is to assume the existence of at least one Killing spinor, and to derive consistency conditions (necessary conditions) in terms of bilinears constructed out of the Killing spinor(s). In more detail:[8]

[8]We follow the procedure of [58], which we rewrite here for the sake of completeness.

I Translate the Killing spinor equations and KSIs into tensorial equations.

Depending on the theory under consideration out of the Killing spinor ϵ one can construct scalar, vector, and p- form bilinears $M \sim \bar{\epsilon}\epsilon\,, \quad V_\mu \sim \bar{\epsilon}\gamma_\mu\epsilon\,, \cdots$ that are related by Fierz identities. These bilinears satisfy certain equations because they are made out of Killing spinors, for instance, if the KSE is of the general form

$$\delta_\epsilon\psi_\mu = \tilde{\mathcal{D}}_\mu\epsilon = [\nabla_\mu + \Omega_\mu]\epsilon = 0\,, \;\Rightarrow\; \nabla_\mu M + 2\Omega_\mu M = 0\,, \tag{1.3.10}$$

The set of all such equations for the bilinears should be equivalent to the original spinorial equation or at least it should contain most of the information contained in it (but not necessarily all of it).

II One of the vector bilinears (say V_μ) is always a Killing vector which can be timelike or null. These two cases are treated separately and are called *timelike case* and *null case*, respectively.

III One can get an expression of all the gauge field strengths of the theory using the Killing equation for those scalar bilinears: Ω_μ is usually of the form $F_{\mu\nu}V^\nu$ and, then Eq. (1.3.10) tells us that $F_{\mu\nu}V^\nu \sim \nabla_\mu \log M$. When V is timelike this determines F completely and, when it is null, it determines the general form of F. Of course, Eq. (1.3.10) is an oversimplified KSE and in real-life situations there are additional scalar factors, $SU(N)$ indices etc.

IV Up to now we found expressions for the bosonic fields of the theory which fullfill certain conditions, which we derived from the KSEs as necessary conditions for supersymmetry. The next step is to prove their sufficiency, that is we have to show the existence of the Killing spinor(s) we assumed to exist. This may lead to additional conditions on the Killing spinors, which may tell us the minimal amount of unbroken supersymmetry in the most general setup. Once the existence of the Killing spinor(s) is ensured, we have found all supersymmetric configurations of the theory.

V The KSIs relate the Maxwell equations, Bianchi identities and the other bosonic equations of motion and guarantee that these sets of equations are combinations of a reduced number of simple equations involving a reduced number of scalar unknowns. solutions of the theory. The tricky part is, usually, identifying the right variables that satisfy simple equations and finding these equations as combinations of the Maxwell, Einstein etc. equations.

VI The equations of motion have to be imposed in order to find the supersymmetric solutions of the theory. As outlined above, the KSIs are of great help at this.

VII Find interesting examples. Some of them are given in Chapter 5.

1.4 Outline of this thesis

In Chapter 2 we are going to introduce the ungauged Supergravity theories we are going to work with in this thesis. We describe the action, symmetries, bosonic equations of motion and supersymmetry transformations rules for ungauged $N = 1, 2$ Supergravities. Our next step will be to gauge these four-dimensional theories.

In Chapter 3 we are first going to introduce the embedding tensor formalism in order to study the most general gaugings of four-dimensional Supergravity in a unified way. Then we compute the complete 4-dimensional tensor hierarchy, i.e. a set of p-form fields, with $1 \leq p \leq 4$, which realize an off-shell algebra of bosonic gauge transformations. We show how this tensor hierarchy can be put on-shell by introducing a set of duality relations, whereby introducing additional scalars and a metric tensor. This so-called duality hierarchy encodes the equations of motion of the bosonic part of the most general gauged Supergravity theories in four dimensions, including the (projected) scalar equations of motion. We construct the gauge-invariant action that includes all the fields in the tensor hierarchy and elucidate the relation between the gauge transformations of the p-form fields in the action and those of the same fields in the tensor hierarchy. The content of Chapter 3 is based on ref. [33].

After having introduced the gaugings of a generic four-dimensional Supergravity theory, we are going to apply our results to $N = 1, 2$ Supergravity in Chapter 4. We discuss $N = 1$ matter-coupled Supergravity with electric and magnetic gaugings and $N = 2$ Einstein-Yang-Mills Supergravity. There we study the closure, up to duality relations, of the $N = 1$ supersymmetry algebra on all the bosonic p-form fields of the hierarchy, applying the results about the general four-dimensional tensor hierarchy from the previous chapter, which was purely bosonic, including fermions. The content of Chapter 4 is based on ref. [59, 60].

In Chapter 5 we will use the procedure described in section 1.3 in order to find supersymmetric solutions to $N = 2$ Supergravity. In section 5.1 we will consider ungauged $d = 4$, $N = 2$ Supergravity coupled to vector and hypermultiplets and completely classify all its supersymmetric solutions. In section 5.2 we discuss the solutions to $N = 2$ Einstein-Yang-Mills (EYM) Supergravity. This chapter is based on refs. [27, 36, 59, 61].

In the last chapter of this thesis we extend the system of ungauged $N = 2$, $d = 4$ Supergravity coupled to vector multiplets and hypermultiplets with 2-form potentials and show that the local supersymmetry algebra can be closed on them. We will discuss the coupling of the 2-forms to the 1/2 BPS 1-brane solutions (*stringy cosmic strings*) found in Chapter 5. This coupling to the one-dimensional solutions found earlier [26, 27] was the main motivation for introducing 2-forms in $N = 2$ four-dimensional Supergravity [28], which was done before the stucture of the general four-dimensional tensor hierarchy was found. Further we construct the half-supersymmetric bosonic world-sheet actions for these strings and discuss the properties of the corresponding stringy cosmic string solutions. Chapter 6.1 is based on [28].

A complete list of the publications which lead to this thesis can be found in Appendix G.

Chapter 2

Ungauged $N = 1, 2$ Supergravity in four dimensions

In this chapter we are going to describe briefly ungauged four-dimensional Supergravity with four and eight supercharges, respectively $N = 1$ and $N = 2$ theories, in order to introduce the basic concepts needed for the investigations in the following chapters. We will consider possible matter couplings, i.e. coupling to chiral and vector-multiplets for $N = 1$ Supergravity and to vector- and hypermultiplets for the $N = 2$ case (see Section 1.2). In Section 2.2.1 we will address the question of how matter-coupled four-dimensional $N = 2$ Supergravity is obtained when compactifying ten-dimensional type II Sugra on a Calabi-Yau threefold. Gaugings of $N = 1, 2$ $d = 4$ Supergravity theories will be considered in Chapter 4.

2.1 Ungauged matter coupled $N = 1$ Supergravity

The basic[1] field content of any $N = 1, d = 4$ ungauged supergravity theory is

Gravity multiplet

- Graviton ${e_a}^\mu$

[1]In the ungauged classical theory (this work is only concerned with the classical theory) linear multiplets can always be dualized into chiral multiplets and so we do not need to deal with them. After the gauging, this is not possible in general, but the embedding tensor formalism will allow us to introduce the 2-forms in at a later stage in a consistent form.

- Gravitino Ψ_μ

n_C **chiral multiplets,** $i = 1 \ldots n_C$

- Complex scalar Z^i
- Chiralino χ^i

n_V **Vector multiplets,** $\Lambda = 1, \cdots, n_V$

- Vector field $A^\Lambda{}_\mu$
- gauginos λ^Λ

The conventions used here are essentially those of Refs. [30] and [62]. The complex scalars Z^i parametrize an arbitrary Kähler-Hodge manifold with metric $\mathcal{G}_{ij^*}$ and the field strengths of the Abelian vector fields A^Λ are given by $F^\Lambda = dA^\Lambda$.

In the ungauged theory the couplings between the above fields are determined by the Kähler metric[2] $\mathcal{G}_{ij^*}$, an arbitrary holomorphic kinetic matrix $f_{\Lambda\Sigma}(Z)$ with positive-definite imaginary part and an arbitrary holomorphic superpotential $W(Z)$ which appears through the covariantly holomorphic section of Kähler weight $(1, -1)$ $\mathcal{L}(Z, Z^*)$:

$$\mathcal{L}(Z, Z^*) = W(Z) e^{\mathcal{K}/2}\,, \tag{2.1.1}$$

so its Kähler-covariant derivative given in Eq. (B.0.16) for $\bar{q} = -1$ is

$$\mathcal{D}_{i^*}\mathcal{L} = e^{\mathcal{K}/2}\partial_{i^*} W = 0\,. \tag{2.1.2}$$

In absence of scalar fields, it is possible to introduce a constant superpotential $\mathcal{L} = W = w$.

The chirality of the spinors is related to their Kähler weight: ψ_μ, λ^Σ and χ^i have the same chirality and ψ_μ, λ^Σ and χ^{*i^*} have the same Kähler weight $(1/2, -1/2)$ so their covariant derivatives take the form of Eq. (B.0.18) with $q = 1/2$.

The action for the bosonic fields in the ungauged theory is

[2]The elements of Kähler geometry needed in this paper are reviewed in Appendix B.

$$S_{\rm u} = \int \left[\star R - 2\mathcal{G}_{ij^*} dZ^i \wedge \star dZ^{*\,j^*} - 2\Im{\rm m} f_{\Lambda\Sigma} F^\Lambda \wedge \star F^\Sigma + 2\Re{\rm e} f_{\Lambda\Sigma} F^\Lambda \wedge F^\Sigma - \star V_{\rm u}\right] , \tag{2.1.3}$$

where the scalar potential $V_{\rm u}$ is given by

$$V_{\rm u}(Z, Z^*) = -24|\mathcal{L}|^2 + 8\mathcal{G}^{ij^*}\mathcal{D}_i\mathcal{L}\mathcal{D}_{j^*}\mathcal{L}^* . \tag{2.1.4}$$

In absence of scalar fields the constant superpotential $\mathcal{L} = W = w$ leads to an anti-de Sitter-type cosmological constant

$$V_{\rm u} = -24|w|^2 . \tag{2.1.5}$$

The supersymmetry transformation rules for the fermions (to first order in fermions) are

$$\delta_\epsilon \psi_\mu = \mathcal{D}_\mu \epsilon + i\mathcal{L}\gamma_\mu\epsilon^* = \left[\nabla_\mu + \tfrac{i}{2}\mathcal{Q}_\mu\right]\epsilon + i\mathcal{L}\gamma_\mu\epsilon^* , \tag{2.1.6}$$

$$\delta_\epsilon \lambda^\Lambda = \tfrac{1}{2}\not{F}^{\Lambda +}\epsilon , \tag{2.1.7}$$

$$\delta_\epsilon \chi^i = i\not{\partial} Z^i \epsilon^* + 2\mathcal{G}^{ij^*}\mathcal{D}_{j^*}\mathcal{L}^*\epsilon . \tag{2.1.8}$$

The last terms in Eqs. (2.1.6) and (2.1.8) are *fermion shifts* associated to the superpotential which contribute quadratically to the potential $V_{\rm u}$.

In absence of scalar fields and with constant superpotential $\mathcal{L} = W = w$ the fermion shift in Eq. (2.1.6) can be interpreted as part of an anti-de Sitter covariant derivative

$$\delta_\epsilon \psi_\mu = \nabla_\mu \epsilon + iw\gamma_\mu\epsilon^* . \tag{2.1.9}$$

The supersymmetry transformation rules for the bosonic fields (to the same order in fermions) are

$$\delta_\epsilon e^a{}_\mu = -\tfrac{i}{4}\bar{\psi}_\mu\gamma^a\epsilon^* + \text{c.c.} , \tag{2.1.10}$$

$$\delta_\epsilon A^\Lambda{}_\mu = \tfrac{i}{8}\bar{\lambda}^\Lambda\gamma_\mu\epsilon^* + \text{c.c.} , \tag{2.1.11}$$

$$\delta_\epsilon Z^i = \tfrac{1}{4}\bar{\chi}^i\epsilon . \tag{2.1.12}$$

Note that $N = 1$ $d = 4$ Supergravity can be obtained by truncation of the $N = 2$ $d = 4$ theory [30].

2.1.1 Perturbative symmetries of the ungauged theory

The possible matter couplings of $N = 1, d = 4$ supergravities are quite unrestricted. As a result, the global symmetries of these theories can be very different from case to case: depending on the couplings it is possible to have, at the same time, symmetry transformations that only act on certain fields and not on the rest and symmetry transformations that act simultaneously on all of them. Thus, it is not easy to describe all the possible global symmetry groups in a form that is at the same time unified and detailed without introducing a very complicated notation with several different kinds of indices. We are going to try to find an equilibrium between simplicity and usefulness.

Therefore, we are going to denote the group of all the global symmetries of the theory we work with[3] by G and its generators by T_A with $A, B, C = 1, \cdots, \operatorname{rank} G$. They satisfy the Lie algebra

$$[T_A, T_B] = -f_{AB}{}^{C} T_C \,. \tag{2.1.13}$$

We denote by $G_{\rm bos}$ the subgroup of transformations of G that act on the bosonic fields and its generators by $T_{\rm a}$ with $\rm a, b, c = 1, \cdots, \operatorname{rank} G_{\rm bos} \leq \operatorname{rank} G$. They satisfy the Lie subalgebra

$$[T_{\rm a}, T_{\rm b}] = -f_{\rm ab}{}^{\rm c} T_{\rm c} \,. \tag{2.1.14}$$

In $N = 1, d = 4$ supergravity we have $G = G_{\rm bos} \times U(1)_R$ and $\operatorname{rank} G_{\rm bos} = \operatorname{rank} G - 1$. We split the indices accordingly as $A = ({\rm a}, \sharp)$. We may introduce a further splitting of the indices of $G_{\rm bos}$, ${\rm a} = (\mathbf{a}, \underline{\rm a})$ to distinguish between those that act on the scalars (holomorphic isometries, belonging to the group[4] $G_{\rm iso} \subset G_{\rm bos}$) and those that do not. These will be the subgroup $G_{\rm V} \subset G_{\rm bos}$ of those that only act on the vector (super)fields and leave invariant the kinetic matrix $f_{\Lambda\Sigma}$, as we will see. We have, then, $G_{\rm bos} = G_{\rm iso} \times G_{\rm V}$.

Let us describe the $U(1)_R$ transformations first. Under a $U(1)_R$ transformation with constant parameter $\alpha^\sharp$, objects with Kähler weight q are multiplied by the phase $e^{-iq\alpha^\sharp}$. All the fermions $\psi_\mu, \lambda^\Sigma, \chi^{*\,i^*}$, have a non-vanishing Kähler weight 1/2, though. All the bosons have zero Kähler weight and do not transform under $U(1)_R$.

The superpotential $\mathcal{L}$ has a non-vanishing Kähler weight and therefore transforms under $U(1)_R$ in spite of the invariance of the scalar fields. As a general rule, in presence of a non-vanishing superpotential, $U(1)_R$ will only be a symmetry of $N = 1, d = 4$

[3] In this section we will use this notation only for the perturbative symmetries and later on we will use the same notation for all symmetries. It should be easy to recognize from the context which case we are talking about.

[4] Not all the isometries of the metric will be perturbative or even non-perturbative symmetries of the full theory. They have to satisfy further conditions that we are going to study next. It is understood that, in order not to have a complicated notation, we denote by $G_{\rm iso}$ only those isometries which really are symmetries of the full theory and not the full group of isometries of $\mathcal{G}_{ij^*}$ (although they may eventually coincide).

supergravity if the phase factor acquired by $\mathcal{L}$ in a $U(1)_R$ transformation can be compensated by a transformation of the scalars that leaves invariant the rest of the action. These transformations, which are necessarily isometries of the Kähler metric will be described next, but we can already give two examples to clarify the above statement.

1. Let us consider the case with no chiral superfields and, therefore, no scalars and a constant $\mathcal{L} = W = w$ giving rise to the potential Eq. (2.1.5) and the gravitino supersymmetry transformation Eq. (2.1.9). In this case $U(1)_R$ transforms the complex constant w into $e^{-i\alpha^\sharp}w$ and, therefore it is not a symmetry since symmetry transformations act on fields, not on coupling constants. Certainly, we can never gauge these transformations since the local phases would transform a constant into a function which is not a field.

2. Let us consider a theory with just one chiral supermultiplet, with Kähler potential $\mathcal{K} = |Z|^2$ and superpotential $W(Z) = wZ$ where w is some complex constant so $\mathcal{L} = wZe^{|Z|^2/2}$. In this case $U(1)_R$ transforms $\mathcal{L}(Z, Z^*)$ into $\mathcal{L}'(Z, Z^*) = we^{-i\alpha^\sharp}Ze^{|Z|^2/2}$. This transformation can be seen as a transformation of the scalar $Z' = e^{-i\alpha^\sharp}Z$ which happens to leave invariant the Kähler potential, metric etc. In this case $U(1)_R$ is a symmetry when combined with the transformation of the scalar.

The G_{iso} transformations with constant parameters $\alpha^{\mathbf{a}}$ act on the complex scalars Z^i as reparametrizations

$$\delta_\alpha Z^i = \alpha^{\mathbf{a}} k_{\mathbf{a}}{}^i(Z)\,. \tag{2.1.15}$$

If these transformations are symmetries of the full theory they must, first, preserve the metric $\mathcal{G}_{ij^*}$ and its Hermitean structure, which implies that the $k_{\mathbf{a}}{}^i$s are the holomorphic components of a set of Killing vectors $\{K_{\mathbf{a}} = k_{\mathbf{a}}{}^i\partial_i + k^{*i^*}_{\mathbf{a}}\partial_{i^*}\}$ that satisfy the Lie algebra of the group G_{iso}

$$[K_{\mathbf{a}}, K_{\mathbf{b}}] = -f_{\mathbf{ab}}{}^{\mathbf{c}}K_{\mathbf{c}}\,. \tag{2.1.16}$$

The holomorphic and antiholomorphic components satisfy, separately, the same Lie algebra.

We can formally add to this algebra, vanishing "Killing vectors" $K_{\underline{\mathrm{a}}}$ associated to the transformations that do not act on the scalars (but do act on the vectors), so we have the full algebra of G_{bos}

$$[K_{\mathrm{a}}, K_{\mathrm{b}}] = -f_{\mathrm{ab}}{}^{\mathrm{c}}K_{\mathrm{c}}\,. \tag{2.1.17}$$

Further, we can also add another vanishing Killing vector $K_\sharp$, formally associated to $U(1)_R$ and write the full Lie algebra of G

$$[K_A, K_B] = -f_{AB}{}^C K_C\,, \tag{2.1.18}$$

so the reparametrizations of the scalars Z^i can be written

$$\delta_\alpha Z^i = \alpha^A k_A{}^i(Z)\,. \tag{2.1.19}$$

The Killing property of the reparametrizations only ensures the invariance of the kinetic term for the scalars. In order to be symmetries of the full theory they must preserve the entire Kähler-Hodge structure and leave invariant the superpotential and the kinetic terms for the vector fields.

1. Let us start with the Kähler structure. The reparametrizations must leave the Kähler potential invariant up to Kähler transformations, i.e., for each Killing vector K_A

   $$\pounds_A \mathcal{K} \equiv \pounds_{K_A}\mathcal{K} = k_A{}^i\partial_i\mathcal{K} + k^*_A{}^{i^*}\partial_{i^*}\mathcal{K} = \lambda_A(Z) + \lambda^*_A(Z^*)\,. \tag{2.1.20}$$

   This relation is consistent for $A = \underline{\mathrm{a}}, \sharp$, if

   $$\Re\mathrm{e}\,\lambda_{\underline{\mathrm{a}}} = \Re\mathrm{e}\,\lambda_\sharp = 0\,. \tag{2.1.21}$$

   Furthermore, the reparametrizations must preserve the Kähler 2-form $\mathcal{J}$

   $$\pounds_A\mathcal{J} = 0\,. \tag{2.1.22}$$

   The closedness of $\mathcal{J}$ implies that $\pounds_A\mathcal{J} = d(i_{k_A}\mathcal{J})$ and therefore the preservation of the Kähler structure implies the existence of a set of real functions $\mathcal{P}_A$ called *momentum maps* such that

   $$i_{K_A}\mathcal{J} = d\mathcal{P}_A\,, \tag{2.1.23}$$

   which is also consistent for $A = \underline{\mathrm{a}}, \sharp$ if the corresponding

   $$\mathcal{P}_{\underline{\mathrm{a}}} = \mathcal{P}_\sharp = \text{ constant}\,. \tag{2.1.24}$$

   There is a further constraint that the momentum map has to satisfy (equivariance): Eq. (B.1.34)

   It implies that these constant momentum maps can only be different from zero for Abelian factors. These constants will be associated after gauging to the *D*- or *Fayet-Iliopoulos* terms.

   A local solution to Eq. (B.1.29) is provided by

$$i\mathcal{P}_A = k_A{}^i \partial_i \mathcal{K} - \lambda_A \,, \tag{2.1.25}$$

which, on account of Eq. (B.1.26) is equivalent to

$$i\mathcal{P}_A = -(k^*_A{}^{i^*} \partial_{i^*} \mathcal{K} - \lambda^*_A) \,, \tag{2.1.26}$$

which implies, for $A = \underline{\mathrm{a}}, \sharp$

$$\lambda_{\underline{\mathrm{a}}} = -i\mathcal{P}_{\underline{\mathrm{a}}} \,, \qquad \lambda_\sharp = -i\mathcal{P}_\sharp \,. \tag{2.1.27}$$

where $\mathcal{P}_{\underline{\mathrm{a}}}$ and $\mathcal{P}_\sharp$ are real constants (Eq. (2.1.24)).

The momentum map can be used as a prepotential from which the Killing vectors can be derived:

$$k_{A\, i^*} = i\partial_{i^*} \mathcal{P}_A \,. \tag{2.1.28}$$

Observe that this equation is consistent with the triviality of the "Killing vectors" $K_{\underline{\mathrm{a}}}, K_\sharp$ and the constancy of the corresponding momentum maps Eq. (2.1.24).

2. If the Kähler-Hodge structure is preserved, any section Φ of Kähler weight (p, q) must transform as[5]

$$\delta_\alpha \Phi = -\alpha^A (\mathbb{L}_A - K_A)\Phi \,, \tag{2.1.29}$$

where $\mathbb{L}_A$ stands for the symplectic and Kähler-covariant Lie derivative w.r.t. K_A and is given by

$$\mathbb{L}_A \Phi \equiv \{\pounds_A + [T_A + \tfrac{1}{2}(p\lambda_A + q\lambda^*_A)]\}\Phi \,, \tag{2.1.30}$$

where the T_A are the matrices that generate G in the representation in which the section transforms and satisfy the Lie algebra Eq. (B.1.37). This means that the gravitino ψ_μ transforms according to

$$\delta_\alpha \psi_\mu = -\tfrac{i}{2} \alpha^A \Im\mathrm{m}\, \lambda_A \psi_\mu \,. \tag{2.1.31}$$

For $A = \underline{\mathrm{a}}, \sharp$ we have just $U(1)_R$ transformations for each component $\mathcal{P}_{\underline{\mathrm{a}}}, \mathcal{P}_\sharp$ different from zero. For $A = \mathbf{a}$ the transformations are still global but the $\Im\mathrm{m}\, \lambda_A$s are in general functions of Z, Z^*. These cannot be compensated by $U(1)_R$ transformations.

[5]We do not write explicitly any spacetime, target space etc. indices.

The chiralinos χ^i transform according to

$$\delta_\alpha \chi^i = \alpha^A \{\partial_j k_A{}^i \chi^j + \tfrac{i}{2} \Im\mathrm{m}\, \lambda_A \chi^i\}\,, \tag{2.1.32}$$

and the transformations of the gauginos will be discussed after we discuss the transformations of the vector fields.

3. Let us now consider the invariance of the superpotential W. We can require, equivalently, that the section $\mathcal{L}$ be invariant up to Kähler transformations. A Kähler-weight (p, q) section Φ will be invariant if[6]

$$\mathbb{L}_{\mathbf{a}}\Phi = 0\,, \quad \Rightarrow \quad \mathscr{L}_{\mathbf{a}}\Phi = -[T_{\mathbf{a}} + \tfrac{1}{2}(p\lambda_{\mathbf{a}} + q\lambda^*_{\mathbf{a}})]\Phi\,. \tag{2.1.33}$$

Therefore, we must require for all $A = \mathbf{a}$

$$K_{\mathbf{a}}\mathcal{L} = -i\Im\mathrm{m}\, \lambda_{\mathbf{a}}\mathcal{L}\,, \quad \Rightarrow \quad \delta_\alpha \mathcal{L} = -i\alpha^{\mathbf{a}}\Im\mathrm{m}\, \lambda_{\mathbf{a}}\mathcal{L}\,, \tag{2.1.34}$$

but we cannot extend straightforwardly the same expression for all A since, as discussed at the beginning of this section, the corresponding transformations (constant phase multiplications) are only symmetries when $\mathcal{L} = 0$ or when they are associated to transformations of the scalars and this is, by definition, not the case when $A = \underline{\mathrm{a}}, \sharp$.

We, therefore, write

$$\delta_\alpha \mathcal{L} = -i\alpha^A \Im\mathrm{m}\, \lambda_A \mathcal{L}\,, \tag{2.1.35}$$

imposing at the same time the constraint[7]

$$(\alpha^{\underline{\mathrm{a}}}\Im\mathrm{m}\, \lambda_{\underline{\mathrm{a}}} + \alpha^\sharp \Im\mathrm{m}\, \lambda_\sharp)\mathcal{L} = (\alpha^{\underline{\mathrm{a}}}\mathcal{P}_{\underline{\mathrm{a}}} + \alpha^\sharp \mathcal{P}_\sharp)\mathcal{L} = 0\,. \tag{2.1.36}$$

4. The kinetic term for the vector fields A^Λ in the action will be invariant[8] if the effect of a reparametrization on the kinetic matrix $f_{\Lambda\Sigma}$ is equivalent to a rotation on its indices that can be compensated by a rotation of the vectors, or a constant Peccei-Quinn-type shift i.e.

[6] This condition only makes sense for transformations $K_{\mathbf{a}}$ that really act on the scalars.

[7] This constraint should be understood as a way to consider the cases $\mathcal{L} = 0$ and $\mathcal{L} \neq 0$ simultaneously: when $\mathcal{L} \neq 0$ the symmetry transformations must satisfy $(\alpha^{\underline{\mathrm{a}}}\mathcal{P}_{\underline{\mathrm{a}}} + \alpha^\sharp \mathcal{P}_\sharp) = 0$ and they are unrestricted when $\mathcal{L} = 0$.

[8] It is at this point that the restriction to perturbative symmetries (symmetries of the action) is made.

$$
\begin{aligned}
\delta_\alpha f_{\Lambda\Sigma} &\equiv -\alpha^{\mathrm{a}} \pounds_{\mathrm{a}} f_{\Lambda\Sigma} = \alpha^{\mathrm{a}} [T_{\mathrm{a}\,\Lambda\Sigma} - 2T_{\mathrm{a}\,(\Lambda}{}^{\Omega} f_{\Sigma)\Omega}]\,, \qquad (2.1.37)\\
\delta_\alpha A^\Lambda &= \alpha^{\mathrm{a}} T_{\mathrm{a}\,\Sigma}{}^{\Lambda} A^\Sigma\,, \qquad (2.1.38)
\end{aligned}
$$

where the shift generator is symmetric $T_{\mathrm{a}\,\Lambda\Sigma} = T_{\mathrm{a}\,\Sigma\Lambda}$ to preserve the symmetry of the kinetic matrix.

Observe that for $\mathrm{a} = \underline{\mathrm{a}}$, $\pounds_{\underline{\mathrm{a}}} f_{\Lambda\Sigma} = 0$, and, for consistency, we must have $T_{\underline{\mathrm{a}}\,(\Lambda}{}^{\Omega} f_{\Sigma)\Omega} = 0$, i.e. the transformations $T_{\underline{\mathrm{a}}}$ are those that preserve the kinetic matrix. This is why we call the group generated by $T_{\underline{\mathrm{a}}}$ the invariance group G_{V} of the complex vector kinetic matrix.

The iteration of two of these infinitesimal transformations indicates that they can be described by the $2n_V \times 2n_V$ matrices[9]

$$
T_{\mathrm{a}} \equiv \begin{pmatrix} T_{\mathrm{a}\,\Lambda}{}^{\Sigma} & 0 \\ & \\ T_{\mathrm{a}\,\Lambda\Sigma} & T_{\mathrm{a}}{}^{\Lambda}{}_{\Sigma} \end{pmatrix}, \qquad T_{\mathrm{a}}{}^{\Lambda}{}_{\Sigma} \equiv -T_{\mathrm{a}\,\Sigma}{}^{\Lambda}\,, \qquad (2.1.39)
$$

satisfying the Lie algebra

$$
[T_{\mathrm{a}}, T_{\mathrm{b}}] = -f_{\mathrm{ab}}{}^{\mathrm{c}} T_{\mathrm{c}}\,. \qquad (2.1.40)
$$

As we have discussed some of the transformations generated by the $K_{\mathbf{a}}$ may only act on the scalars and not on the vectors, for instance, because the kinetic matrix does not depend on the relevant scalars. We assume that the corresponding subset of $2n_V \times 2n_V$ matrices $T_{\mathbf{a}}$ are identically zero. On the other hand, we can formally add to these matrices another identically vanishing $2n_V \times 2n_V$ matrix $T_\sharp$ so we have a full set of $2n_V \times 2n_V$ matrices T_A satisfying the Lie algebra of G, Eq. (B.1.37).

Combining all these results we conclude that the gauginos transform according to

$$
\delta_\alpha \lambda^\Sigma = -\alpha^A [T_{A\,\Omega}{}^{\Sigma} \lambda^\Omega + \tfrac{i}{2} \Im\mathrm{m}\, \lambda_A \lambda^\Sigma]\,. \qquad (2.1.41)
$$

At this point there is no restriction on the group G nor on the $n_V \times n_V$ matrices $T_{A\,\Lambda}{}^{\Sigma}$, although one can already see that the lower-triangular $2n_V \times 2n_V$ matrices T_A are generators of the symplectic group.

[9] Observe that this group is the semidirect product of the group that rotates the vectors, generated by the matrices $T_{\mathrm{a}\,\Sigma}{}^{\Lambda}$ and the Abelian group of shifts generated by the matrices $T_{\mathrm{a}\,\Lambda\Sigma}$. Evidently, some of these matrices identically vanish. This is the price we have to pay to use the same indices $\mathrm{a}, \mathrm{b}, \mathrm{c}, \ldots$ for the generators of both groups.

2.1.2 Non-perturbative symmetries of the ungauged theory

The non-perturbative symmetries to be considered are symmetries of the "extended" equations of motion of the ungauged theory which are the standard equations of motion plus the Bianchi identities of the vector field strengths:

$$dF^{\Lambda} = 0\,. \tag{2.1.42}$$

The Maxwell equations that one obtains from the action Eq. (2.1.3) can be written as Bianchi identities for the 2-forms G_{Λ}

$$dG_{\Lambda} = 0\,, \qquad G_{\Lambda}{}^{+} \equiv f_{\Lambda\Sigma}(Z) F^{\Sigma\,+}\,, \tag{2.1.43}$$

where $F^{\Sigma +} \equiv \frac{1}{2}\left(F^{\Sigma} + i \star F^{\Sigma}\right)$.

This set of extended equations of motion (Maxwell equations plus Bianchi identities) is invariant under general linear transformations

$$\left(\begin{array}{c} F^{\Lambda} \\ G_{\Lambda} \end{array}\right)' = \left(\begin{array}{cc} A_{\Sigma}{}^{\Lambda} & B^{\Sigma\Lambda} \\ C_{\Sigma\Lambda} & D^{\Sigma}{}_{\Lambda} \end{array}\right) \left(\begin{array}{c} F^{\Sigma} \\ G_{\Sigma} \end{array}\right)\,. \tag{2.1.44}$$

However, consistency with the definition of G_{Λ} Eq. (2.1.43) requires that the kinetic matrix transforms at the same time as

$$f' = (C + Df)(A + Bf)^{-1}\,. \tag{2.1.45}$$

Then f' will be symmetric if

$$A^T C - C^T A = 0\,, \qquad B^T D - D^T B = 0\,, \qquad A^T D - C^T B = \xi \mathbb{I}_{n_V \times n_V}\,, \tag{2.1.46}$$

where ξ is a constant whose value is found to be $\xi = 1$ by the requirement of invariance of the Einstein equations.

These conditions can be reexpressed in a better form after introducing some notation. We define the contravariant tensor of 2-forms G^M, the symplectic metric Ω_{MN} and its inverse Ω^{MN} which we will use to, respectively, lower and raise indices

$$G^M \equiv \left(\begin{array}{c} F^{\Lambda} \\ G_{\Lambda} \end{array}\right)\,, \qquad \Omega_{MN} = \left(\begin{array}{cc} 0 & \mathbb{I}_{n_V \times n_V} \\ -\mathbb{I}_{n_V \times n_V} & 0 \end{array}\right)\,, \qquad \Omega^{MN}\Omega_{NP} = -\delta^{M}{}_{P}\,. \tag{2.1.47}$$

Then, the Maxwell equations and Bianchi identities are formally invariant under the transformations

$$G'^M \equiv M_N{}^M G^N\,, \qquad M = (M_N{}^M) = \left(\begin{array}{cc} A & B \\ C & D \end{array}\right)\,, \tag{2.1.48}$$

satisfying

$$M^T \Omega M = \Omega \,. \tag{2.1.49}$$

i.e. $M \in Sp(2n_V, \mathbb{R})$ [31]. Infinitesimally[10]

$$M_N{}^M \sim \mathbb{I}_{2n_V \times 2n_V} + \alpha^A T_{A\,N}{}^M = \alpha^A \begin{pmatrix} T_{A\,\Sigma}{}^\Lambda & T_A{}^{\Sigma\Lambda} \\ & \\ T_{A\,\Sigma\Lambda} & T_A{}^\Sigma{}_\Lambda \end{pmatrix} , \tag{2.1.50}$$

and the condition $M \in Sp(2n_V, \mathbb{R})$ reads

$$T_{A\,[MN]} \equiv T_{A\,[M}{}^P \Omega_{N]P} = 0 \,. \tag{2.1.51}$$

These transformations change the kinetic matrix and will only be symmetries of all the extended equations of motion if they can be compensated by reparametrizations, i.e. $f_{\Lambda\Sigma}$ has to satisfy

$$\alpha^A k_A{}^i \partial_i f_{\Lambda\Sigma} = \alpha^A \{-T_{A\,\Lambda\Sigma} + 2T_{A\,(\Lambda}{}^\Omega f_{\Sigma)\Omega} - T_A{}^{\Omega\Gamma} f_{\Omega\Lambda} f_{\Gamma\Sigma}\} \,. \tag{2.1.52}$$

The subalgebra of matrices that generate symmetries of the action (perturbative symmetries) are those with $T_A{}^{\Sigma\Lambda} = 0$, i.e. the lower-triangular matrices of Eq. (2.1.39).

Observe that the transformations acting on the vectors are constrained to belong to $Sp(2n_V, \mathbb{R})$. This does not mean that the global symmetry group $G \subset Sp(2n_V, \mathbb{R})$, but that the group that we can gauge must be contained (embedded) in $Sp(2n_V, \mathbb{R})$. The generators T_A corresponding to non-symplectic symmetries (in particular $U(1)_R$), must necessarily vanish.

The transformation rule of the kinetic matrix $f_{\Lambda\Sigma} \equiv R_{\Lambda\Sigma} + iI_{\Lambda\Sigma}$ Eq. (2.1.45) can be alternatively expressed using the $Sp(2n_V, \mathbb{R})$ matrix

$$\left(\mathcal{M}^{MN}\right) \equiv \begin{pmatrix} I^{\Lambda\Sigma} & I^{\Lambda\Omega} R_{\Omega\Sigma} \\ & \\ R_{\Lambda\Omega} I^{\Omega\Sigma} & I_{\Lambda\Sigma} + R_{\Lambda\Omega} I^{\Omega\Gamma} R_{\Gamma\Sigma} \end{pmatrix} , \qquad I^{\Lambda\Omega} I_{\Omega\Sigma} = \delta^\Lambda{}_\Sigma \,, \tag{2.1.53}$$

which transforms linearly

$$\mathcal{M}' = M \mathcal{M} M^T \,. \tag{2.1.54}$$

[10] We include identically vanishing generators associated to $U(1)_R$ etc. On the other hand, it is clear that the index A refers now to more symmetries than in the perturbative case.

2.2 Ungauged matter coupled $N = 2$ Supergravity

In this thesis we are going to study $N = 2$, $d = 4$ Supergravity coupled to n_V vector multiplets and n_H hypermultiplets, thus we are dealing with the following fields:

Gravity multiplet

- Graviton ${e_a}^{\mu}$
- A pair of gravitinos $\Psi_{I\mu}$, $I = 1, 2$
- Vector field A_{μ}

n_V **Vector multiplets,** $i = 1 \ldots n_V$

- Complex scalar Z^i
- A pair of gauginos λ^{Ii}, $I = 1, 2$
- Vector field ${A^i}_{\mu}$

n_H **Hypermultiplets**

- 4 real scalars q^u, $u = 1 \ldots 4n_H$
- 2 hyperinos ζ^{α}, $\alpha = 1 \ldots 2n_H$

In the coupled theory we denote the vector fields collectively by ${A^{\Lambda}}_{\mu}$, $\Lambda = 1 \ldots \bar{n}$ where $\bar{n} = n_V + 1$.

The action of the bosonic fields of the theory is

$$
\begin{aligned}
S &= \int d^4x \sqrt{|g|} \left[R + 2\mathcal{G}_{ij^*} \partial_{\mu} Z^i \partial^{\mu} Z^{*\, j^*} + 2\mathsf{h}_{uv} \partial_{\mu} q^u \partial^{\mu} q^v \right. \\
&\qquad \left. + 2\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma} F^{\Lambda\,\mu\nu} {F^{\Sigma}}_{\mu\nu} - 2\Re\mathrm{e}\mathcal{N}_{\Lambda\Sigma} F^{\Lambda\,\mu\nu\star} {F^{\Sigma}}_{\mu\nu} \right] ,
\end{aligned}
\tag{2.2.1}
$$

The coupling of scalars to scalars is described by a non-linear σ-model with Kähler metric $\mathcal{G}_{ij^*}(Z, Z^*)$ (see Appendix B), and the coupling to the vector fields by a complex scalar-field-valued matrix $\mathcal{N}_{\Lambda\Sigma}(Z, Z^*)$. These two couplings are related by a structure called special Kähler geometry, described in Appendix C. The symmetries of these two sectors will be related and this relation will be discussed shortly. The $4n_H$ hyperscalars parameterize a quaternionic Kähler manifold (defined and studied in Appendix D) with metric $\mathsf{h}_{uv}(q)$ [63]. Observe that the hypermultiplets do not couple to the vector multiplets.

For convenience, we denote the bosonic equations of motion by

$$\mathcal{E}_a{}^\mu \equiv -\frac{1}{2\sqrt{|g|}}\frac{\delta S}{\delta e^a{}_\mu}\,, \qquad \mathcal{E}_i \equiv -\frac{1}{2\sqrt{|g|}}\frac{\delta S}{\delta Z^i}\,, \tag{2.2.2}$$

$$\mathcal{E}_\Lambda{}^\mu \equiv \frac{1}{8\sqrt{|g|}}\frac{\delta S}{\delta A^\Lambda{}_\mu}\,, \qquad \mathcal{E}^u \equiv -\frac{1}{4\sqrt{|g|}}\mathsf{h}^{uv}\frac{\delta S}{\delta q^v}\,. \tag{2.2.3}$$

and the Bianchi identities for the vector field strengths by

$$\mathcal{B}^{\Lambda\,\mu} \equiv \nabla_\nu{}^\star F^{\Lambda\,\nu\mu}\,. \tag{2.2.4}$$

The explicit forms of the equations of motion can be found to be

$$\begin{aligned}
\mathcal{E}_{\mu\nu} &= G_{\mu\nu} + 2\mathcal{G}_{ij^*}[\partial_\mu Z^i \partial_\nu Z^{*\,j^*} - \tfrac{1}{2}g_{\mu\nu}\partial_\rho Z^i \partial^\rho Z^{*\,j^*}] \\
&\quad +8\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,+}{}_\mu{}^\rho F^{\Sigma\,-}{}_{\nu\rho} + \ 2\mathsf{h}_{uv}\ [\partial_\mu q^u \partial_\nu q^v - \tfrac{1}{2}g_{\mu\nu}\partial_\rho q^u \partial_\rho q^v]\,, \qquad (2.2.5)\\
\mathcal{E}_i &= \nabla_\mu(\mathcal{G}_{ij^*}\partial^\mu Z^{*\,i^*}) - \partial_i \mathcal{G}_{jk^*}\partial_\rho Z^j \partial^\rho Z^{*\,k^*} + \partial_i[F_\Lambda{}^{\mu\nu\star}F^\Lambda{}_{\mu\nu}]\,, \qquad (2.2.6)\\
\mathcal{E}_\Lambda{}^\mu &= \nabla_\nu{}^\star F_\Lambda{}^{\nu\mu}\,, \qquad (2.2.7)\\
\mathcal{E}^u &= \mathfrak{D}_\mu \partial^\mu q^u = \nabla_\mu \partial^\mu q^u + \Gamma_{vw}{}^u \partial^\mu q^v \partial_\mu q^w\,, \qquad (2.2.8)
\end{aligned}$$

where we have defined the dual vector field strength F_Λ by

$$F_{\Lambda\,\mu\nu} \equiv -\frac{1}{4\sqrt{|g|}}\frac{\delta S}{\delta^\star F^\Lambda{}_{\mu\nu}} = \Re\mathrm{e}\mathcal{N}_{\Lambda\Sigma}F^\Sigma{}_{\mu\nu} + \Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}{}^\star F^\Sigma{}_{\mu\nu}\,. \tag{2.2.9}$$

Note that the Bianchi identities Eq. (2.2.4) and the Maxwell equations Eq. (2.2.7), respectively, can be written using differential form notation in the following way:

$$\star\mathcal{B}^{\Lambda} = dF^{\Lambda}\,, \tag{2.2.10}$$

$$\star\mathcal{E}_{\Lambda} = dF_{\Lambda}\,. \tag{2.2.11}$$

The equation of motion (2.2.7) can be interpreted as a Bianchi identity for the dual field strength F_{Λ},

$$dF_{\Lambda} = 0\,, \tag{2.2.12}$$

implying the local existence of $\bar{n} = n_V + 1$ dual vector fields A_{Λ}, i.e. locally $F_{\Lambda} = dA_{\Lambda}$. Now we define a vector of $2\bar{n}$ 2-forms

$$\mathcal{F} \equiv \begin{pmatrix} F^{\Lambda} \\ \\ F_{\Lambda} \end{pmatrix} \tag{2.2.13}$$

and then can summarize the equation of motion and Bianchi identity for A^{Λ}, Eqs. (2.2.7) and (2.2.4), respectively, as

$$d\mathcal{F} = 0\,. \tag{2.2.14}$$

The symmetries of this set of equations of motion are the isometries of the Kähler manifold and those of the quaternionic manifold. A prerequisite to understand the following development is a study of the symplectic transformations. These are duality symmetries of four dimensions, which are a generalization of electromagnetic duality [64]. The Maxwell and Bianchi identities can be rotated into each other by $GL(2\bar{n}, \mathbb{R})$ transformations under which they are a $2\bar{n}$-dimensional vector:

$$\mathcal{E}^{\mu} \equiv \begin{pmatrix} \mathcal{B}^{\Lambda\,\mu} \\ \\ \mathcal{E}_{\Lambda}{}^{\mu} \end{pmatrix} \longrightarrow \begin{pmatrix} A & B \\ \\ C & D \end{pmatrix} \begin{pmatrix} \mathcal{B}^{\Lambda\,\mu} \\ \\ \mathcal{E}_{\Lambda}{}^{\mu} \end{pmatrix}, \tag{2.2.15}$$

where A, B, C and D are $\bar{n} \times \bar{n}$ matrices. These transformations act in the same form on the vector $\mathcal{F}$

$$\mathcal{F}' = S\mathcal{F} \;\text{ where }\; S \in GL(2\bar{n}, \mathbb{R})\,. \tag{2.2.16}$$

The $(2n_V + 2)$-dimensional vector of potentials

$$\mathcal{A} \equiv \begin{pmatrix} A^{\Lambda} \\ A_{\Lambda} \end{pmatrix}, \tag{2.2.17}$$

whose local existence is implied by Eqs. (2.2.14), transforms in the same way. However, since the dual potentials, A_{Λ}, depend in a non-local way on the 'fundamental'

ones, A^Λ, these transformations are non-local and are not symmetries of the action, which only depends on the fundamental potentials, but only of the Maxwell equations and Bianchi identities.
Now we are going to see, that consistency of this transformation rule with the definition of $\tilde{F}$ Eq. (2.2.9) requires the matrix

$$S = \begin{pmatrix} D & C \\ & \\ B & A \end{pmatrix} \tag{2.2.18}$$

to belong to the symplectic subgroup of the general linear group:

$$S \in Sp(2\bar{n}, \mathbb{R}) \subset GL(2\bar{n}, \mathbb{R}) \,, \tag{2.2.19}$$

or, which es equivalent,

$$\mathcal{S}^T \Omega \mathcal{S} = \Omega \quad \text{ with } \quad \Omega \equiv \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} . \tag{2.2.20}$$

While the duality rotation Eq. (2.2.13) is performed on the field strengths and their duals, also the scalar fields are transformed (since they belong to the same multiplets) by a diffeomorphism of the scalar manifold and, as a consequence, the matrix $\mathcal{N}_{\Lambda\Sigma}$ changes. By definition it is

$$F'_\Lambda = \Re\mathrm{e}\mathcal{N}'_{\Lambda\Sigma} F'^{\Sigma} + \Im\mathrm{m}\mathcal{N}'_{\Lambda\Sigma} {}^\star F'^{\Sigma} \,, \tag{2.2.21}$$

and for the transformations to be consistently defined, they must act on the period matrix $\mathcal{N}$ according to

$$\mathcal{N}' = (D\mathcal{N} + C)(B\mathcal{N} + A)^{-1} \equiv \mathcal{N}(Z', Z'^*) \,. \tag{2.2.22}$$

Furthermore, the transformations must preserve the symmetry of the period matrix, which requires

$$D^T B = B^T D \,, \quad C^T A = A^T C \quad \text{and} \quad D^T A - B^T C = \mathbb{1} \,, \tag{2.2.23}$$

i.e. the transformations must belong to $Sp(2\bar{n}, \mathbb{R})$ and only this subgroup of elements $\mathcal{S} \in GL(2n_V + 2, \mathbb{R})$ can be a symmetry of all the equations of motion of the theory[11]. The above transformation rules for the vector field strength and period matrix imply

$$\Im\mathrm{m}\mathcal{N}' = (B\mathcal{N}^* + A)^{-1\,T} \Im\mathrm{m}\mathcal{N} (B\mathcal{N} + A)^{-1} \,, \qquad F'^{\Lambda\,+} = (B\mathcal{N}^* + A)_{\Lambda\Sigma} F^{\Sigma\,+} \,, \tag{2.2.24}$$

[11] This, in fact, is the largest possible electro-magnetic duality group of any Lagrangian depending on Abelian field strengths, scalars and derivatives of scalars as well as spinor fields [31].

so the combination $\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,+}{}_{\mu}{}^{\rho}F^{\Lambda\,+}{}_{\nu\rho}$ that appears in the energy-momentum tensor is automatically invariant. These transformations have to be symmetries of the σ-model as well, which implies that only the isometries of the special Kähler manifold which are embedded in $Sp(2\bar{n},\mathbb{R})$ and those of the quaternionic manifold parameterized by the hyperscalars are symmetries of all the equations of motion of the theory (dualities of the theory).
For vanishing fermions, the supersymmetry transformation rules of the fermions are

$$\delta_\epsilon \psi_{I\,\mu} \;=\; \mathfrak{D}_\mu \epsilon_I + \varepsilon_{IJ}\, T^{+}{}_{\mu\nu}\gamma^{\nu}\, \epsilon^{J}\,, \tag{2.2.25}$$

$$\delta_\epsilon \lambda^{iI} \;=\; i\, \not{\partial} Z^i \epsilon^I \;+\; \varepsilon^{IJ}\, \not{G}^{i\,+}\, \epsilon_J\,. \tag{2.2.26}$$

$$\delta_\epsilon \zeta_\alpha \;=\; -i\mathbb{C}_{\alpha\beta}\, \mathsf{U}^{\beta I}{}_{u}\, \varepsilon_{IJ}\;\; \not{\partial} q^u\, \epsilon^J\,, \tag{2.2.27}$$

Here $\mathfrak{D}$ is the Lorentz and Kähler-covariant derivative of Ref. [26] supplemented by (the pullback of) an $SU(2)$ connection $\mathsf{A}_I{}^J$ described in Appendix D, acting on objects with $SU(2)$ indices I,J and, in particular, on ϵ_I as:

$$\mathfrak{D}_\mu \epsilon_I = (\nabla_\mu \;+\; \tfrac{i}{2}\,\mathcal{Q}_\mu)\,\epsilon_I \;+\; \mathsf{A}_{\mu\,I}{}^{J}\,\epsilon_J\,. \tag{2.2.28}$$

$\mathsf{U}^{\beta I}{}_u$ is a Quadbein, *i.e.* a quaternionic Vielbein, and $\mathbb{C}_{\alpha\beta}$ the $Sp(m)$-invariant metric, both of which are described in Appendix D.

From this point on we will refer to the upper case Greek indices as symplectic indices and to vectors X given by

$$X = \begin{pmatrix} X^\Lambda \\ X_\Lambda \end{pmatrix} \tag{2.2.29}$$

as symplectic vectors. Given two symplectic vectors X and Y we define the symplectic-invariant inner product, $\langle X \mid Y\rangle$, by [12]

$$\langle\, X \mid Y\,\rangle = -X^T\Omega Y = X_\Lambda Y^\Lambda - X^\Lambda Y_\Lambda\,. \tag{2.2.31}$$

When writing forms inside a symplectic inner product we will implicitly assume that we are taking the exterior product of both. One should then keep in mind that $\langle X_{(p)} \mid TY_{(q)}\rangle = (-1)^{pq}\langle Y_{(q)} \mid TX_{(p)}\rangle$, where $X_{(p)}$ and $Y_{(q)}$ are p- and q-forms,

[12]Note that when dealing with gauged supergravities, we were using a slightly different notation. Here we suppress the index M in the symplectic vector X, which in the previous sections we referred to as X^M. The symplectic-invariant inner product could equivalently written, using our former notation, as

$$\langle\, X \mid Y\,\rangle = -X^M\Omega_{MN}Y^N = X_M Y^M = -X^M Y_M\,. \tag{2.2.30}$$

respectively. Note that in the variation of the gravitini the hyperscalars only appear via the $SU(2)$ connection $\mathsf{A}_{\mu\, I}{}^{J}$, while in the variation of the gaugini the hyperscalars do not appear at all.

The supersymmetry transformations of the bosons are

$$
\begin{aligned}
\delta_\epsilon e^a{}_\mu &= -\tfrac{i}{4}(\bar{\psi}_{I\,\mu}\gamma^a\epsilon^I + \bar{\psi}^I{}_\mu\gamma^a\epsilon_I)\,, && (2.2.32)\\
\delta_\epsilon A^\Lambda{}_\mu &= \tfrac{1}{4}(\mathcal{L}^{\Lambda\,*}\varepsilon^{IJ}\bar{\psi}_{I\,\mu}\,\epsilon_J \;+\; \mathcal{L}^\Lambda\varepsilon_{IJ}\bar{\psi}^I{}_\mu\,\epsilon^J)\\
&\quad +\tfrac{i}{8}(f^\Lambda{}_i\varepsilon_{IJ}\bar{\lambda}^{iI}\gamma_\mu\epsilon^J + f^{\Lambda *}{}_{i^*}\varepsilon^{IJ}\bar{\lambda}^{i^*}{}_I\gamma_\mu\epsilon_J)\,, && (2.2.33)\\
\delta_\epsilon Z^i &= \tfrac{1}{4}\bar{\lambda}^{iI}\epsilon_I\,, && (2.2.34)\\
\delta_\epsilon q^u &= \mathsf{U}_{\alpha I}{}^u(\bar{\zeta}^\alpha\epsilon^I + \mathbb{C}^{\alpha\beta}\epsilon^{IJ}\bar{\zeta}_\beta\epsilon_J)\,. && (2.2.35)
\end{aligned}
$$

Observe that the fields of the hypermultiplet and the fields of the gravity and vector multiplets do not mix in any of these supersymmetry transformation rules. This means that the KSIs associated to the gravitinos and gauginos will have the same form as in Ref. [26] and in the KSIs associated to the hyperinos only the hyperscalars equations of motion will appear.

For convenience, we denote the bosonic equations of motion by

$$
\mathcal{E}_a{}^\mu \equiv -\frac{1}{2\sqrt{|g|}}\frac{\delta S}{\delta e^a{}_\mu}\,, \quad \mathcal{E}_i \equiv -\frac{1}{2\sqrt{|g|}}\frac{\delta S}{\delta Z^i}\,, \tag{2.2.36}
$$

$$
\mathcal{E}_\Lambda{}^\mu \equiv \frac{1}{8\sqrt{|g|}}\frac{\delta S}{\delta A^\Lambda{}_\mu}\,, \quad \mathcal{E}^u \equiv -\frac{1}{4\sqrt{|g|}}\mathsf{H}^{uv}\frac{\delta S}{\delta q^v}\,. \tag{2.2.37}
$$

and the Bianchi identities for the vector field strengths by

$$
\mathcal{B}^{\Lambda\,\mu} \equiv \nabla_\nu{}^\star F^{\Lambda\,\nu\mu}\,. \tag{2.2.38}
$$

Then, using the action Eq. (2.2.1), we find that all the equations of motion of the bosonic fields of the gravity and vector supermultiplets take the same form as if there were no hypermultiplets, as in Ref. [26], except for the Einstein equation, which obviously is supplemented by the energy-momentum tensor of the hyperscalars

$$
\mathcal{E}_{\mu\nu} \;=\; \mathcal{E}_{\mu\nu}(q=0) \;+\; 2\mathsf{H}_{uv}\,[\partial_\mu q^u\partial_\nu q^v - \tfrac{1}{2}g_{\mu\nu}\partial_\rho q^u\partial_\rho q^v]\,. \tag{2.2.39}
$$

Furthermore, the equation of motion for the hyperscalars reads

$$\mathcal{E}^u = \mathfrak{D}_\mu \partial^\mu q^u = \nabla_\mu \partial^\mu q^u + \Gamma_{vw}{}^u \partial^\mu q^v \partial_\mu q^w \,, \tag{2.2.40}$$

where $\Gamma_{vw}{}^u$ are the Christoffel symbols of the 2^{nd} kind for the metric H_{uv}.

The symmetries of this set of equations of motion are the isometries of the Kähler manifold parametrized by the $\bar{n} - 1$ complex scalars Z^is embedded in $Sp(2\bar{n}, \mathbb{R})$ and those of the quaternionic manifold parametrized by the $4m$ real scalars q^u.

2.2.1 $N = 2$, $d = 4$ Supergravity from String Theory

In this chapter we are going to review the higher-dimensional origin of $N = 2$, $d = 4$ Supergravity, i.e. how it arises from compactification of ten-dimensional Superstring Theory.

Type II Supergravity theories, being the low energy limits of type II superstring theory, live in ten dimensions. To recover the four-dimensional spacetime of everyday experience, we have to compactify the ten-dimensional theory on a six-dimensional internal manifold. The four-dimensional theory obtained upon compactification heavily depends on the topology of the internal manifold (see below). If we compactify ten-dimensional type II String Theory, which has 32 supersymmetries, on a six-torus T^6 for example, we are left with $N = 8$ supersymmetry in four dimensions because, due to its trivial holonomy, a torus does not break any supersymmetry. If, on contrary, one compactifies on a Calabi-Yau manifold[13] CY_n, which by definition has $SU(n)$ holonomy, some fraction of the available amount of supersymmetry is broken. In case of compactification on a Calabi-Yau threefold CY_3 three quarters of the supersymmetries are broken. Schematically this can be explained in the following way: for an orientable six-dimensional manifold parallel transport of a spinor along a closed curve generically gives a rotation by a $SO(6) \sim SU(4)$ matrix, this is the generic holonomy group. The **16** Weyl representation of the ten dimensional Lorentz group $SO(1, 9)$ decomposes with respect to $SO(1, 3) \otimes SO(6)$ as

$$\mathbf{16} \Rightarrow (\mathbf{2_L}, \bar{\mathbf{4}}) + (\mathbf{2_R}, \mathbf{4}) \tag{2.2.41}$$

The largest subgroup of $SU(4)$ for which a spinor of definite chirality can be invariant is $SU(3)$. The reason is that the **4** has an $SU(3)$ decomposition

$$\mathbf{4} \Rightarrow \mathbf{3} \oplus \mathbf{1}, \tag{2.2.42}$$

i.e. it decomposes into a triplet and a singlet, which is invariant under $SU(3)$. Since the condition for $N = 1$ unbroken supersymmetry in four dimensions is the existence of a covariantly constant spinor on the internal six-dimensional manifold, and only the singlet pieces of **4** and $\bar{\mathbf{4}}$ in Eq. (2.2.42) lead to covariantly constant spinors,

[13] A compact Kähler manifold with vanishing first Chern class is called a *Calabi-Yau manifold*. For details about Kähler geometry see Appendix B.

compactification on a manifold with $SU(3)$ holonomy breaks 3/4 of the original supersymmetries. Imposing the Majorana condition in ten dimensions, it follows that type II supergravity on a CY_3 leads to $N = 2$ supergravity in four dimensions. Thus, from the 32 supercharges we have in ten dimensions in case of type II supergravities, we are left with 8 in four dimensions. In this way CY_3 compactification of type II supergravity leads to $N = 2$, $d = 4$ supergravity coupled to n_V vector and n_H hypermultiplets, where the numbers of multiplets is given in terms of topological invariants of the Calabi-Yau manifold one is compactifying on.

The massless Kaluza-Klein modes associated with various fields in ten dimensions, compactified on a Calabi-Yau space are given in Table 2.2.1. Let us see in some more detail how the massless scalars in four dimensions are related to the ten-dimensional theory, taking IIB as example. The bosonic fields of IIB supergravity are:[14]

$$G_{MN},\, B_{MN},\, \phi,\, C,\, C_{MN},\, C_{MNPQ}\,. \tag{2.2.43}$$

Additionally the supergravity multiplet contains 2 gravitini and two dilatini with the same chirality. The metric G_{MN}, the dilaton ϕ and the two-form B_{MN} come from the NS-NS sector, whereas the axion C, the 2-form and 4-form C_{MN} and C_{MNPQ} come from the R-R sector.

The axion, the dilaton and the duals of $B_{\mu\nu}$ and $C_{\mu\nu}$ lead to 4 real scalars, combined in the so-called *universal hypermultiplet*, independently of the specific choice of Calabi-Yau manifold; the topological origin of this fact is that $h^{0,0} = 1$ for any Calabi-Yau threefold, where $h^{p,q}$ are the Hodge numbers of the Calabi-Yau. The Hodge numbers of a generic Calabi-Yau threefold are conveniently displayed in the so-called "Hodge diamond":

$$\begin{array}{ccccccc}
 & & & h^{3,3} & & & \\
 & & h^{3,2} & & h^{2,3} & & \\
 & h^{3,1} & & h^{2,2} & & h^{1,3} & \\
h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\
 & h^{2,0} & & h^{1,1} & & h^{0,2} & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & & & h^{0,0} & & &
\end{array}
\quad = \quad
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{1,2} & & h^{1,2} & & 1 \\
 & 0 & & h^{1,1} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}$$

[14] In the following upper case Latin indices $M, L\dots$ denote ten-dimensional indices, while Greek indices $\mu, \nu\dots$ live in four dimensions and lower case Latin indices $i, j\dots$ in the internal six-dimensional space.

Now let us consider metric deformations of the Calabi-Yau manifold. After fixing the diffeomorphism invariance and taking into account the Ricci-flatness of Calabi-Yau manifolds, the deformations δg_{ij} and $\delta g_{i\bar{j}}$ decouple and thus can be considered separately. The purely holomorphic or anti-holomorphic components g_{ij} and $g_{\bar{i}\bar{j}}$, respectively, are zero. However, one can consider variations to non-zero values, thereby changing the complex structure[15].

Thus metric deformations of the Calabi-Yau manifold give two types of moduli [66], [67]:

- *Kähler moduli*: $h^{1,1}$ real scalars due to deformations of $g_{i\bar{j}}$:

$$\delta g_{i\bar{j}} = \sum_{\alpha=1}^{h^{1,1}} t^\alpha b^\alpha{}_{i\bar{j}}, \tag{2.2.44}$$

  where we expanded $\delta g_{i\bar{j}}$ in a basis of real $(1,1)$-forms, which we denoted by b^α, $\alpha = 1 \dots h^{1,1}$, and t^α are the Kähler moduli, and

- *Complex structure moduli*: $h^{1,2}$ complex scalars due to the deformations of δg_{ij}:

$$\Omega_{ijk}\delta g^k{}_{\bar{l}} = \sum_{a=1}^{h^{2,1}} t^a b^a{}_{ij\bar{l}} \tag{2.2.45}$$

  where a complex $(2,1)$ form is associated to each variation of the complex structure. Here b^a, $a = 1 \dots h^{2,1}$, denote a basis of harmonic $(2,1)$-forms and the complex parameters t^a are called the *complex structure moduli.* Ω denotes the unique holomorphic $(3,0)$-form of Calabi-Yau threefolds. It turns out that the metric on the complex structure moduli space is Kähler with Kähler potential given by [67]

$$\mathcal{K} = -\log(i \int \Omega \wedge \Omega^*). \tag{2.2.46}$$

The 2-forms lead to 2 $h^{1,1}$ scalars $B_{i\bar{j}}$ and $C_{i\bar{j}}$ and taking into account the self-duality of the 5-form field-strength of the 4-form, there are $h^{2,2} = h^{1,1}$ scalars $C_{ij\bar{k}\bar{l}}$ arising from C_{MNPQ}. These $4h^{1,1}$ scalars are part of $h^{1,1}$ additional hypermultiplets. Finally the $h^{1,2}$ complex scalars (complex structure moduli) are associated to $h^{1,2}$ vector multiplets.

Further, the spectrum of the low dimensional theory contains $h^{3,0}$ $(= 1)$ vectors $C_{\mu ijk}$ in the gravity multiplet and $h^{2,1} = h^{1,2}$ vectors $C_{\mu ij\bar{k}}$ associated to the vector multiplets.

[15]Since a Calabi-Yau manifold is a Kähler manifold it admits by definition a complex structure. A complex structure is a $(1,1)$-tensor $\mathbf{J}$ that satisfies $\mathbf{J}^2 = -1$ (for more details see [65]).

A	B	field	spin-2	spin-1	spin-0
1	1	g_{MN}	1	0	$h^{1,1}$ real + $h^{1,2}$ complex
1	2	ϕ	0	0	1
1	0	A_M	0	1	0
1	2	A_{MN}	0	0	$(h^{1,1}+1)$ real
1	0	A_{MNP}	0	$h^{1,1}$	$(h^{1,2}+1)$ complex
0	1	$\left[A_{MNPQ}\right]_{\pm}$	0	$h^{1,2}+1$	$h^{1,1}$ real

Table 2.2.1: Massless Kaluza-Klein modes associated with various fields in ten dimensions, compactified on a Calabi-Yau space. The first two columns specify the number of these fields contained in IIA or IIB supergravity in ten space-time dimensions [68].

In the case of the type IIA theory the massless bosonic fields in ten dimensions are

$$G_{MN},\, B_{MN},\, \phi,\, C_M,\, C_{MNP}\ . \tag{2.2.47}$$

Additionally the supergravity multiplet contains 2 gravitini and two dilatini with opposite chiralities. Note that just as for type IIB G_{MN}, B_{MN}, and ϕ arise from the NS-NS sector, whereas in the case at hand the R-R fields are forms of odd degree.

The NS-NS fields give the same number of massless scalars as in the IIB case, namely one real scalar from the dilaton, $2h^{1,2} + h^{1,1}$ real scalars from the metric and $h^{1,1} + 1$ real scalars from the NS-NS 2-form. Now the R-R 3-form leads to $h^{2,1} = h^{1,2}$ complex scalars $C_{ij\bar{k}}$ and $h^{3,0} = 1$ complex scalar C_{ijk}.

The 1-form leads to one vector field C_μ (which will be contained in the supergravity multiplet) and the 3-form to $h^{1,1}$ vectors $C_{\mu i\bar{j}}$, contained in the vector multiplets. Grouping all these fields again into multiplets, one obtains gravity coupled to $h^{1,1}$ vector multiplets and $h^{1,2}$ hypermultiplets in four dimensions. With these results it is easy to count the number of bosonic massless states that emerge in the compactification of IIA and IIB supergravity on a Calabi-Yau manifold [69]:

$$
\text{Type IIA Sugra}: \quad \begin{array}{ll} 1 \text{ spin-}1 + 1 \text{ spin-}2 & \text{gravity multiplet} \\ \left.\begin{array}{l} h^{1,1} \text{ spin-}1 \\ h^{1,1} \text{ complex spin-}0 \end{array}\right\} & h^{1,1} \text{ vector multiplets} \\ h^{1,2}+1 \text{ quaternionic spin-}0 & h^{1,2}+1 \text{ hypermultiplets} \end{array} \tag{2.2.48}
$$

$$
\text{Type IIB Sugra}: \quad \begin{array}{ll} 1 \text{ spin-}1 + 1 \text{ spin-}2 & \text{gravity multiplet} \\ \left.\begin{array}{l} h^{1,2} \text{ spin-}1 \\ h^{1,2} \text{ complex spin-}0 \end{array}\right\} & h^{1,2} \text{ vector multiplets} \\ h^{1,1}+1 \text{ quaternionic spin-}0 & h^{1,1}+1 \text{ hypermultiplets} \end{array} \tag{2.2.49}
$$

The field content of four-dimensional supergravity associated to the field content of ten-dimensional type IIA/B supergravity is summarized in Table 2.2.1.

The total target manifold parameterized by the various scalars factorizes as a product of vector and hypermultiplet manifolds:

$$
\begin{aligned}
\mathcal{M}_{scalar} &= \mathcal{SM} \otimes \mathcal{HM}, \\
\dim_{\mathbf{C}} \mathcal{SM} &= n_V, \\
\dim_{\mathbf{R}} \mathcal{HM} &= 4\, n_H,
\end{aligned}
$$

where $\mathcal{SM}$, $\mathcal{HM}$ are respectively *special Kähler* and *quaternionic Kähler* and n_V, n_H are respectively the number of *vector multiplets* and *hypermultiplets* contained in the theory. The direct product structure Eq. (2.2.50) imposed by supersymmetry precisely reflects the fact that the quaternionic and special Kähler scalars belong to different supermultiplets [70].

An important implication is the following: since the string coupling constant is given by the vacuum expectation value of the dilaton $g_s \equiv e^{-\phi/2}$ and the the four-dimensional reduction of the dilaton always belongs to a hypermultiplet, the hypermultiplet sector receives both perturbative and non-perturbative g_s corrections [71]. Non-perturbative corrections arise from instantons and/or branes wrapping cycles in the Calabi-Yau. The vector multiplet geometry remains unaffected.

Up to now we were only considering the higher dimensional origin of the massless states in four dimension. However, also the coupling of the vector multiplet scalars to the vectors is encoded in the Calabi Yau geometry, namely in a holomorphic function called the *prepotential* (see also Appendix C.1). To start with we introduce a real symplectic basis $(\alpha_\Lambda, \beta^\Sigma)$ [72] of 3-forms of $H^3(CY) = H^{(3,0)} \oplus H^{(2,1)} \oplus H^{(1,2)} \oplus H^{(0,3)}$, $\alpha_\Lambda \in H^{(3,0)} \oplus H^{(2,1)}$ and $\beta^\Lambda \in H^{(0,3)} \oplus H^{(1,2)}$, chosen such that they satisfy

$$\int_{A^\Lambda} \alpha_\Sigma = \int \alpha_\Sigma \wedge \beta^\Lambda = \delta^\Lambda{}_\Sigma \ , \tag{2.2.50}$$

$$\int_{B_\Lambda} \beta^\Sigma = \int \beta^\Sigma \wedge \alpha_\Lambda = -\delta^\Sigma{}_\Lambda \ , \tag{2.2.51}$$

$$\int \alpha_\Lambda \wedge \alpha_\Sigma = \int \beta^\Lambda \wedge \beta^\Sigma = 0, \tag{2.2.52}$$

where (A^Λ, B_Σ) denotes the dual homology basis of 3-cycles [16] with intersection numbers

$$A^\Lambda \cap B_\Sigma = -B_\Sigma \cap A^\Lambda = \delta^\Lambda{}_\Sigma \ , \qquad \text{and} \qquad A^\Lambda \cap A^\Sigma = B_\Lambda \cap B_\Sigma = 0, \tag{2.2.53}$$

and $\Lambda, \Sigma = 0 \dots h^{2,1}$. Now we can define coordinates on the moduli space[17] by the periods of the holomorphic 3-form Ω

$$\mathcal{X}^\Lambda = \int_{A^\Lambda} \Omega = \int \Omega \wedge \beta^\Lambda. \tag{2.2.54}$$

In this way we define one more coordinate than we have moduli fields, but the additional degree of freedom is killed by fixing the $U(1)$ gauge freedom, as described in Appendix C.1. In order not to have more independent variables, the B periods

$$\mathcal{F}_\Lambda = \int_{B_\Lambda} \Omega = \int \Omega \wedge \alpha_\Lambda \tag{2.2.55}$$

must be functions of $\mathcal{X}$, whence Ω, which is just a 3-form, can be expanded in the basis of 3-forms

$$\Omega = \mathcal{X}^\Lambda \alpha_\Lambda - \mathcal{F}_\Lambda \beta^\Lambda \,. \tag{2.2.56}$$

Using Eq. (2.2.46) the Kähler potential takes the form

$$\mathcal{K} = -\log\left(i(\mathcal{X}^{*\Lambda}\mathcal{F}_\Lambda - \mathcal{X}^\Lambda \mathcal{F}^*{}_\Lambda)\right). \tag{2.2.57}$$

As under a change of the complex structure Eq. (2.2.45) dz becomes a linear combination of dz and $d\bar{z}$, the holomorphic $(3,0)$-form Ω becomes a linear combination of $(3,0)$ and $(2,1)$-forms [66]

$$\partial_\Lambda \Omega \in H^{(3,0)} \oplus H^{(2,1)}, \tag{2.2.58}$$

[16] Observe that the 3-form α_Λ is the Poincaré dual of the 3-cycle B_Λ and β^Σ of A^Σ, respectively.

[17] Loosely speaking, we mean by moduli space the scalars in the lower-dimensional theory which encode the geometric properties, such as shape and size, of the internal manifold.

it follows

$$\Omega \wedge \partial_\Lambda \Omega = 0. \tag{2.2.59}$$

Integrating the last equation over the Calabi-Yau threefold and taking into account the basic properties of the basis of 3-forms, Eqs. (2.2.50)-(2.2.52), this implies

$$\mathcal{F}_\Lambda = \mathcal{X}^\Sigma \partial_\Lambda \mathcal{F}_\Sigma, \tag{2.2.60}$$

where

$$\mathcal{F} = \frac{1}{2}\mathcal{X}^\Lambda \mathcal{F}_\Lambda. \tag{2.2.61}$$

This function is exactly the prepotential of $N = 2$ supergravity in four dimensions (Appendix C.1).

Notice that the results in case IIA/B are the same upon the exchange $h^{p,q} \longleftrightarrow h^{3-p,q}$. This phenomenon for Calabi-Yau threefolds is part of what is called *mirror symmetry*: type IIA theory compactified on a Calabi-Yau threefold M is equivalent to type IIB compactified on the mirror Calabi-Yau threefold W. The mirror map associates to a Calabi-Yau threefold M another one W such that

$$h^{p,q}(M) = h^{3-p,q}(W). \tag{2.2.62}$$

This means that mirror symmetry maps the complex structure moduli space of type IIB compactified on M to the Kähler structure moduli space of type IIA on W. But apart from the fact that the low energy spectrum of type IIA on M and IIB on the mirror manifold W are the same (up to now we were only considering the massless Kaluza-Klein modes), the mirror symmetry proposal implies much more. Actually mirror symmetry claims the two theories to be exactly equivalent to all orders of α', i.e. including stringy effects . The α' corrections are controlled by the Kähler moduli, which for type $IIB(IIA)$ appear in the lower-dimensional theory through the scalars in a hypermultiplet (vector multiplet). This implies that the result obtained for type IIB on M, the vector multiplet moduli space, i.e. the complex structure moduli space, does not suffer from α' corrections, and the result obtained in the supergravity approximation is exact to all orders in α'. Mirror symmetry thus allows us to obtain information about the α'-corrections of the hypermultiplet sector in type IIA on the mirror manifold W, which are highly non-trivial.

Thence mirror symmetry is a very useful concept, e.g. to compute the holomorphic prepotential of the effective action, although it has not been proven yet [73].

Chapter 3

Gauging Supergravity and the four-dimensional tensor hierarchy

In this chapter we are going to study gaugings of four-dimensional Supergravities. Considering the most general (electro-magnetic) gaugings will lead to the construction of the complete $d = 4$ tensor hierarchy. We use as our starting point Ref. [15]. We use the same formalism, impose the same constraints on the embedding tensor and follow the same steps up to the 2-form level reproducing exactly the same results, but we carry out the program to its completion, determining explicitly all the 3- and 4-forms and their gauge transformations. Here we find already a surprise in the sense that in $D = 4$ we find more top-form potentials than follow from the expectations formulated in Refs. [17, 74][1]. Our results and the general results and conjectures of these references[2] cannot be straightforwardly compared, though, since in these works on the general structure of tensor hierarchies only one possible constraint on the embedding tensor (the *standard* quadratic constraint) is considered, while in the 4-dimensional setup of Ref. [15] the embedding tensor is subject to two additional constraints, one quadratic and one linear. They are ultimately responsible for the existence of additional 4-forms, which we find to be in one-to-one correspondence with the constraints[3].

[1]For instance, we find in $D = 4$ not only top-forms that correspond to quadratic constraints of the embedding tensor but also top-forms that are related to certain linear constraints, see subsection 3.2.4.

[2]There are no direct computations of tensor hierarchies up to the 4-form level in the literature. All we know about them, up to now, is based on general arguments.

[3]Note added in proof: it has recently been shown in Ref. [60] that the introduction of these additional 4-forms is consistent with $N = 1, D = 4$ supergravity. Furthermore, it has been shown that the gauging of particular classes of theories (e.g. $N = 1, D = 4$ supergravity with a non-vanishing

Next, we will make precise how a set of dynamical equations can be defined by the introduction of first-order duality relations. Besides the p-form potentials these duality relations also contain the scalars and the metric tensor defining the theory. The set of dynamical equations not only contains the equations of motion putting all electric potentials on-shell but it also involves the (projected) scalar equations of motion. The tensor hierarchy supplemented by this set of duality relations will be called the *duality hierarchy*. This set of duality relations cannot be derived from an action, though the relation to a possible action will be elucidated in a last step.

For the readers' convenience we briefly outline our program, which can be summarized by the following 3-step procedure.

1. The first step consists of the general construction of the tensor hierarchy, which is an *off-shell* system. The structure in generic dimension has been given in [16, 17]. The explicit form of the *complete* $D = 4$ tensor hierarchy, however, is not available in the literature since it was constructed in [15] only up to the 2-form level. (For the construction of the tensor hierarchy of maximal and half-maximal 4-dimensional supergravities, see [75] and references therein.) The complete $D = 3$ tensor hierarchy has been discussed in [16, 76]. To construct the tensor hierarchy one usually starts from the p-form potential fields of all degrees $p = 1, \ldots, D$ and then constructs the gauge-covariant field strengths of all degrees $p = 2, \ldots, D$. These field strengths are related to each other via a set of Bianchi identities of all degrees $p = 3, \ldots, D$. Usually, one starts with the construction of the covariant field strength for 1-form potentials which, for general gaugings, requires the introduction of 2-form potentials. The corresponding 3-form Bianchi identity relates the 2-form field strength to a 3-form field strength for the 2-form potential, whose construction requires the introduction of a 3-form potential, etc. This bootstrap procedure ends with the introduction of the top-form potentials. The only input required for this construction is the number of electric $p \geq 1$-form potentials, the global symmetries of the theory and the representations of this group under which the p-forms transform. Changing these data leads to different theories that can be seen as different *realizations* of the low-rank sector of the same tensor hierarchy.

   A trick that simplifies the construction outlined above and which makes the construction of the complete $D = 4$ tensor hierarchy feasible is to first construct the set of all Bianchi identities relating the $(p + 1)$-form field strengths to the $(p + 2)$-field strengths. This systematic construction of the Bianchi identities can be carried out *even if we do not know explicitly the transformation rules of the potentials*. These can be found afterwards by using the covariance of the different field strengths. The resulting gauge transformations form an algebra that closes off-shell: at no stage of the construction equations of motions are

superpotential) may require additional constraints on the embedding tensor, which lead to extensions of the tensor hierarchy and, in particular, to additional 4-forms related to the new constraints.

used.

2. The second step is to complement the tensor hierarchy with a set of duality relations and as such to promote it to what we have called duality hierarchy. The duality relations contain more 'external' information about the particular theory we are dealing with. The duality hierarchy will introduce the scalars and the metric tensor field that were not involved in the construction of the tensor hierarchy[4]. More precisely, some of the duality relations contain the scalar fields via functions that define all scalar couplings, i.e. the Noether currents, the (scalar derivative of the) scalar potential and functions that define the scalar-vector couplings. In this way the duality hierarchy contains all the information about the particular realization of the tensor hierarchy as a field theory.

   The duality hierarchy leads to a set of dynamical equations that not only contains the equations of motion for the electric potentials but it also involves the (projected) scalar equations of motion according to the rule:

$$\text{Tensor hierarchy} \ \& \ \text{duality relations} \quad \Leftrightarrow \quad \text{dynamical equations}\,. \qquad (3.0.1)$$

   The gauge algebra of the tensor hierarchy closes off-shell even in the presence of the duality relations. However, in the context of the duality hierarchy this is a basis-dependent statement. We are free to modify the gauge transformations by adding terms that are proportional to the duality relations. Of course, in this new basis the gauge algebra will close on-shell, i.e. up to terms that are proportional to the duality relations. We will call the original basis with off-shell closed algebra the off-shell basis.

3. The third and last step is the construction of a gauge-invariant action for all p-form potentials, scalars and metric[5]. In this last step we encounter a few subtleties that need and will be clarified. In particular, we will answer the following questions:

   (a) How are the equations of motion that follow from the gauge-invariant action related to the set of dynamical equations defined by the duality hierarchy?

   (b) How are the gauge transformations of the p-form potentials occurring in the action related to the gauge transformations that follow from the tensor hierarchy?

   It turns out that the construction of a gauge-invariant action requires that the gauge transformations of the duality hierarchy are given in a particular basis that can be obtained from the off-shell basis by a change of basis that will be

[4]The dual scalars, i.e. the $(D-2)$-form potentials, are included in the tensor hierarchy.
[5]Strictly speaking, in $D = 4$ not all 2-forms enter the action, see sec. 3.4.

described in this paper. To be specific, the two sets of transformation rules (those corresponding to the off-shell tensor hierarchy and those that leave the action invariant) differ by terms that are proportional to the duality relations. It is important to note that once a gauge-invariant action is specified the gauge transformations that leave this action invariant are not anymore related to the off-shell basis by a legitimate basis transformation *from the action point of view*. This is due to the fact that from the point of view of the action one is not allowed to remove terms that are *not* proportional to one of the equations of motion following from this action[6]. However, although some projected duality relations follow by extremizing the action, this is not the case for all duality relations of the duality hierarchy. Therefore, from the point of view of the action, the gauge transformations that leave the action invariant are not equivalent to the gauge transformations of the duality hierarchy in the off-shell basis. Indeed, the gauge transformations in the off-shell basis do not leave the action invariant.

3.1 The embedding tensor formalism

We start by giving a brief review of the the embedding tensor formalism [11,14,16,17]. Readers familiar with this technique may skip this part.

The embedding tensor formalism is a convenient tool to study gaugings of supergravity theories in a universal and general way, that does not require a case-by-case analysis. This technique formally maintains covariance with respect to the global invariance group G of the ungauged theory, even though in general G will ultimately be broken by the gauging to the subgroup that is gauged. It turns out that all couplings that deform an ungauged supergravity into a gauged one, as Yukawa couplings, scalar potentials, etc., can be given in terms of a special tensor, called the embedding tensor. Thus, gauged supergravities are classified by the embedding tensor, subject to a number of algebraic or group-theoretical constraints, some of which we will discuss below.

To be more precise, the embedding tensor $\Theta_M{}^\alpha$ pairs the generators t_α of the group G with the vector fields $A_\mu{}^M$ used for the gauging. The indices $\alpha, \beta, \ldots$ label the adjoint representation of G and the indices $M, N, \ldots$ label the representation $\mathcal{R}_V$ of G, in which the vector fields that will be used for the gauging transform. Thus, the choice of $\Theta_M{}^\alpha$, which generally will not have maximal rank, determines which combinations of vectors

$$A_\mu{}^M \Theta_M{}^\alpha \,, \tag{3.1.1}$$

can be seen as the gauge fields associated to (a subset of) the generators t_α of the group G, and, simultaneously, or alternatively, which combinations of group generators

[6]One may only change the gauge transformations by adding so-called "equations of motion symmetries".

$$X_M \;=\; \Theta_M{}^\alpha\, t_\alpha \tag{3.1.2}$$

can be seen as the generators of the gauge group. Consequently, the embedding tensor can be used to define covariant derivatives

$$D_\mu \;=\; \partial_\mu - A_\mu{}^M\, \Theta_M{}^\alpha\, t_\alpha \;=\; \partial_\mu - A_\mu{}^M\, X_M\ , \tag{3.1.3}$$

which shows that the embedding tensor can also be interpreted as a set of gauge coupling constants[7] of the theory. Even though $\Theta_M{}^\alpha$ has been introduced as a tensor of the duality group G, it is not taken to transform according to its index structure, i.e. in the tensor product $\mathcal{R}_V \otimes \mathrm{Adj}^*$, but must be inert under G for consistency. This requirement leads to the so-called quadratic constraints, which state that the embedding tensor is invariant under the gauge group. If we denote the generators of G (with structure constants $f_{\alpha\beta}{}^\gamma$) in the representation $\mathcal{R}_V$ by $(t_\alpha)_M{}^N$, this amounts to the condition

$$\delta_P \Theta_M{}^\alpha \;=\; \Theta_P{}^\beta t_{\beta M}{}^N \Theta_N{}^\alpha + \Theta_P{}^\beta f_{\beta\gamma}{}^\alpha \Theta_M{}^\gamma \;=\; 0\ . \tag{3.1.4}$$

Therefore, seemingly G-covariant expressions actually break the duality group to the subgroup which is gauged.

In the next sections we will frequently make use of the objects

$$X_{MN}{}^P \;\equiv\; \Theta_M{}^\alpha t_{\alpha N}{}^P \;=\; X_{[MN]}{}^P + Z^P{}_{MN}\ , \tag{3.1.5}$$

with $Z^P{}_{MN}$ denoting the symmetric part of $X_{MN}{}^P$, in terms of which the quadratic constraints read

$$\Theta_P{}^\alpha Z^P{}_{MN} \;=\; 0\ . \tag{3.1.6}$$

Thus, the antisymmetry of the 'structure constants' of the gauge group holds only upon contraction with the embedding tensor. Similar relations, that are familiar from ordinary gauge theories but hold in the present context only upon contraction with Θ, will be encountered at several places in the next sections. Note that standard closure of the gauge group follows from (3.1.4) in that

$$[X_M, X_N] \;=\; -X_{MN}{}^P X_P \;=\; -X_{[MN]}{}^P X_P \tag{3.1.7}$$

by virtue of (3.1.6).

[7] G may have a product structure and each factor may have a different coupling constant, which is contained in the embedding tensor. We, therefore, do not write any other explicit coupling constants apart from $\Theta_M{}^\alpha$.

So far, the discussion has been quite general. In the remaining part of this paper we are going to discuss the $D=4$ and $D=3$ tensor hierarchies in full detail. For these cases the embedding tensor can be specialized according to the known representation of the vector fields. Also, our notation for the indices will slightly differ from the general case to accord with the literature. In the $D=4$ case we will work with electric vectors $A^{\Lambda}{}_{\mu}$, with $\Lambda = 1, \ldots, \bar{n}$, and magnetic vectors $A_{\Lambda\mu}$. Together, these vectors will be combined into a symplectic contravariant vector $A^{M}{}_{\mu}$ with M labeling the fundamental representation of $Sp(2\bar{n}, \mathbb{R})$. Also the adjoint index of the global symmetry group will be denoted by A instead of α. This leads to the following notation for the $D=4$ embedding tensor:

$$D=4: \qquad \Theta_{M}{}^{\alpha} \quad \rightarrow \quad \Theta_{M}{}^{A}. \tag{3.1.8}$$

We now discuss the $D=4$ tensor hierarchy in sections 3.2, 3.3 and 3.4.

3.2 The $D=4$ tensor hierarchy

3.2.1 The setup

The (bosonic) electric fields of any 4-dimensional field theory are the metric, scalars and (electric) vectors. Only the latter are needed in the construction of the tensor hierarchy. We denote them by $A^{\Lambda}{}_{\mu}$ where $\Lambda, \Sigma, \ldots = 1, \cdots, \bar{n}$. In 4-dimensional ungauged theories one can always introduce their magnetic duals which we denote by a similar index in lower position $A_{\Lambda\,\mu}$.

The symmetries of the equations of motion of 4-dimensional theories that act on the electric and magnetic vectors are always subgroups of $Sp(2\bar{n}, \mathbb{R})$ [31] . Thus, it is convenient to define the symplectic contravariant vector

$$A^{M}{}_{\mu} = \begin{pmatrix} A^{\Lambda}{}_{\mu} \\ A_{\Lambda\,\mu} \end{pmatrix}. \tag{3.2.1}$$

It is also convenient to define the symplectic metric Ω_{MN} by

$$\Omega_{MN} = \begin{pmatrix} 0 & \mathbb{I}_{\bar{n}\times\bar{n}} \\ -\mathbb{I}_{\bar{n}\times\bar{n}} & 0 \end{pmatrix}, \tag{3.2.2}$$

and its inverse Ω^{MN} by

$$\Omega^{MN}\Omega_{NP} = -\delta^{M}{}_{P}. \tag{3.2.3}$$

They will be used, respectively, to lower and raise symplectic indices, *e.g.*[8]

$$A_{M} \equiv \Omega_{MN}A^{N} = (A_{\Lambda}\,, -A^{\Lambda})\,, \qquad A^{M} = A_{N}\Omega^{NM}. \tag{3.2.4}$$

[8] In what follows we will mostly use differential-form language and suppress the spacetime indices.

The contraction of contravariant and covariant symplectic indices is, evidently, equivalent to the symplectic product: $A^M B_M = A^M \Omega_{MN} B^N = -A_M B^M$.

We denote the global symmetry group of the theory by G and its generators by T_A, $A, B, C, \ldots = 1, \cdots, \mathrm{rank}\, G$. These satisfy the commutation relations

$$[T_A, T_B] = -f_{AB}{}^C T_C \,. \tag{3.2.5}$$

G can actually be larger than $Sp(2\bar{n}, \mathbb{R})$ and/or not be contained in it[9], but, according to the above discussion, it will always act on A^M as a subgroup of it, i.e. infinitesimally

$$\delta_\alpha A^M = \alpha^A T_{A\,N}{}^M A^N \,, \qquad \delta_\alpha A_M = -\alpha^A T_{A\,M}{}^N A_N \,, \tag{3.2.6}$$

where

$$T_{A\,[MN]} \equiv T_{A\,[M}{}^P \Omega_{N]P} = 0 \,. \tag{3.2.7}$$

This is an important general property of the 4-dimensional case. It is implicit in this formalism that some of the matrices $T_{A\,M}{}^N$ may act trivially on the vectors, i.e. they may vanish. Otherwise we could only deal with $G \subset Sp(2\bar{n}, \mathbb{R})$.

Apart from its global symmetries, an ungauged theory containing $\bar{n}$ Abelian vector fields will always be invariant under the $2\bar{n}$ Abelian gauge transformations

$$\delta_\Lambda A^M{}_\mu = -\partial_\mu \Lambda^M \,, \tag{3.2.8}$$

where $\Lambda^M(x)$ is a symplectic vector of local gauge parameters.

To gauge a subgroup of the global symmetry group G we must promote the global parameters α^A to arbitrary spacetime functions $\alpha^A(x)$ and make the theory invariant under these new transformations. This is achieved by identifying these arbitrary functions with a subset of the (Abelian) gauge parameters Λ^M of the vector fields and subsequently using the corresponding vectors as gauge fields. This identification is made through the embedding tensor $\Theta_M{}^A \equiv (\Theta_\Lambda{}^A, \Theta^{\Lambda\,A})$:

$$\alpha^A(x) \equiv \Lambda^M(x) \Theta_M{}^A \,. \tag{3.2.9}$$

The embedding tensor allows us to keep treating all vector fields, used for gaugings or not, on the same footing. It hence allows us to formally preserve the symplectic invariance even after gauging.

As discussed in section 3.1 the embedding tensor must satisfy a number of constraints which guarantee the consistency of the theory. Some of these constraints have already been discussed in section 3.1. In total we have three constraints which we list below. First of all, in the $D = 4$ case we must impose the following quadratic constraint

$$Q^{AB} \equiv \tfrac{1}{4} \Theta^{M\,[A} \Theta_M{}^{B]} = 0 \,, \tag{3.2.10}$$

[9] The symmetries of a set of scalars decoupled from the vectors are clearly unconstrained.

which guarantees that the electric and magnetic gaugings are mutually local [15]. Observe that the antisymmetry of Ω^{MN} and the above constraint imply $\Theta^{M\,A}\Theta_M{}^B = 0$. This constraint is a particular feature of the 4-dimensional case.

As mentioned in section 3.1 there is a second quadratic constraint which encodes the fact that the embedding tensor has to be itself invariant under gauge transformations. If the gauge transformations of objects with contravariant and covariant symplectic indices are

$$\delta_\Lambda \xi^M = \Lambda^N \Theta_N{}^A T_{A\,P}{}^M \xi^P\,, \qquad \delta_\Lambda \eta_M = -\Lambda^N \Theta_N{}^A T_{A\,M}{}^P \xi_P\,, \tag{3.2.11}$$

and the gauge transformations of objects with contravariant and covariant adjoint indices are written in the form

$$\delta_\Lambda \pi^A = \Lambda^M \Theta_M{}^B f_{BC}{}^A \pi^C\,. \qquad \delta_\Lambda \zeta_A = -\Lambda^M \Theta_M{}^B f_{BA}{}^C \zeta_C\,, \tag{3.2.12}$$

then

$$\delta_\Lambda \Theta_M{}^A = -\Lambda^N Q_{NM}{}^A\,, \qquad Q_{NM}{}^A \equiv \Theta_N{}^A T_{A\,M}{}^P \Theta_P{}^A - \Theta_N{}^A \Theta_M{}^B f_{AB}{}^A\,, \tag{3.2.13}$$

and the second quadratic constraint reads

$$Q_{NM}{}^A = 0\,. \tag{3.2.14}$$

The third constraint applies to all 4-dimensional supergravity theories that are free of gauge anomalies [77] and can be expressed using the X generators introduced in section 3.1, see Eq. (3.1.5):

$$X_M \equiv \Theta_M{}^A T_A\,, \qquad X_{MN}{}^P \equiv \Theta_M{}^A T_{A\,N}{}^P\,. \tag{3.2.15}$$

This constraint (the so-called *representation constraint*) is linear in $\Theta_M{}^A$ and reads as follows [15]:

$$L_{MNP} \equiv X_{(MNP)} = X_{(MN}{}^Q \Omega_{P)Q} = 0\,. \tag{3.2.16}$$

The three constraints that the embedding tensor has to satisfy are not independent, but are related by

$$Q_{(MN)}{}^A - 3L_{MNP} Z^{PA} - 2Q^{AB} T_{BMN} = 0\,. \tag{3.2.17}$$

This relation can be used to show that the constraint $Q^{AB} = 0$ follows from the constraint $Q_{(MN)}{}^A = 0$ when the linear constraint $L_{MNP} = 0$ is explicitly solved, whenever the action of the global symmetry group on the vectors is faithful. We will neither solve explicitly the linear constraint by choosing to work only with representations allowed by it, nor we will assume the action of the global group on the vectors to be faithful, since there are many interesting situations in which this is not the case and we aim to be as general as possible. In (half-) maximal supergravities, though, the global symmetry group always acts faithfully on the vector fields.

These two choices, which differ from those made in the explicit examples found in the literature (see e.g. Ref. [75]) will have important consequences in the field content of the tensor hierarchy and are the reason why our results also differ from those obtained in them.
Before we go on we wish to collect a few properties of the X generators $X_{MN}{}^{P}$ in a separate subsection.

The X generators and their properties

We first discuss the symmetry properties of the X generators. By their definition, and due to the symplectic property of the $T_{A\,N}{}^{P}$ generators, see Eq. (2.1.51), we have

$$X_{MNP} = X_{MPN}\,. \tag{3.2.18}$$

From the definition of the quadratic constraint Eq. (3.2.14) it follows that

$$X_{(MN)}{}^{P}\Theta_{P}{}^{C} = Q_{(MN)}{}^{C}\,, \tag{3.2.19}$$

and so it will vanish[10], although, in general, we will have

$$X_{(MN)}{}^{P} \neq 0\,. \tag{3.2.20}$$

This implies, in particular

$$X_{(MN)P} = -\tfrac{1}{2}X_{PMN} + \tfrac{3}{2}L_{MNP} \;\; \Rightarrow X_{(MN)}{}^{P} = Z^{PA}T_{AMN} + \tfrac{3}{2}L_{MN}{}^{P}\,, \tag{3.2.21}$$

where we have defined

$$Z^{PA} \equiv -\tfrac{1}{2}\Omega^{NP}\Theta_{N}{}^{A} = \begin{cases} +\tfrac{1}{2}\Theta^{\Lambda A}\,, \\ \\ -\tfrac{1}{2}\Theta_{\Lambda}{}^{A}\,, \end{cases} . \tag{3.2.22}$$

Z^{PA} will be used to project in directions orthogonal to the embedding tensor since, due to the first quadratic constraint Eq. (3.2.10), we find that

$$Z^{MA}\Theta_{M}{}^{B} = -\tfrac{1}{2}Q^{AB}\,. \tag{3.2.23}$$

[10]Here we will keep the terms proportional to constraints for later use, including the linear constraints in (3.2.21).

We next discuss some properties of the products of two X generators. From the commutator of the T_A generators and the definition of the generators X_M and the matrices $X_{MN}{}^P$ we find the commutator of the X_M generators to be

$$[X_M, X_N] = Q_{MN}{}^C T_C - X_{MN}{}^P X_P\,. \tag{3.2.24}$$

This reduces to (cf. to Eq. (3.1.7))

$$[X_M, X_N] = -X_{[MN]}{}^P X_P\,, \tag{3.2.25}$$

upon use of the above constraint and $Q_{MN}{}^C = 0$. From the commutator Eq. (3.2.24) one can derive the analogue of the Jacobi identities

$$\begin{aligned} X_{[MN]}{}^Q X_{[PQ]}{}^R + X_{[NP]}{}^Q X_{[MQ]}{}^R + X_{[PM]}{}^Q X_{[NQ]}{}^R &= \\ = -\tfrac{1}{3}\{X_{[MN]}{}^Q X_{(PQ)}{}^R + X_{[NP]}{}^Q X_{(MQ)}{}^R + X_{[PM]}{}^Q X_{(NQ)}{}^R\} \\ -Q_{[MN|}{}^C T_{C\,|P]}{}^R\,. \end{aligned} \tag{3.2.26}$$

We finally present two more useful identities that can be derived from the commutators:

$$X_{(MN)}{}^Q X_{PQ}{}^R - X_{PN}{}^Q X_{(MQ)}{}^R - X_{PM}{}^Q X_{(NQ)}{}^R = -Q_{P(M|}{}^C T_{C\,|N)}{}^R\,, \tag{3.2.27}$$

$$X_{[MN]}{}^Q X_{PQ}{}^R - X_{PN}{}^Q X_{[MQ]}{}^R + X_{PM}{}^Q X_{[NQ]}{}^R = Q_{P[M|}{}^C T_{C\,|N]}{}^R\,. \tag{3.2.28}$$

3.2.2 The vector field strengths F^M

We now return to the construction of the field strengths of the different p-form potentials. In what follows we will set all the constraints explicitly to zero in order to simplify the expressions. In this section we consider the vector field strengths.

To construct the vector field strength it is convenient to start from the covariant derivative. This derivative acting on objects transforming according to $\delta\phi = \Lambda^M \delta_M \phi$ is defined by

$$\mathfrak{D}\phi = d\phi + A^M \delta_M \phi\,. \tag{3.2.29}$$

For instance, the covariant derivative of a contravariant symplectic vector

$$\mathfrak{D}\xi^M = d\xi^M + X_{NP}{}^M A^N \xi^P\,, \tag{3.2.30}$$

transforms covariantly provided that

$$\delta A^M = -\mathfrak{D}\Lambda^M + \Delta A^M\,, \qquad \Theta_M{}^A \Delta A^M = 0\,. \tag{3.2.31}$$

The Ricci identity of the covariant derivative on Λ^N can be written in the form

$$\mathfrak{D}\mathfrak{D}\Lambda^M \quad = \quad X_{NP}{}^M F^N \Lambda^P\,, \tag{3.2.32}$$

for some 2-form F^M. Since this expression is gauge-covariant, F^M, contracted with the embedding tensor, will automatically be gauge-covariant, whatever it is and it is natural to identify it with the gauge-covariant vector field strength. The above expression defines it up to a piece ΔF^M which is projected out by the embedding tensor, just like ΔA^M in δA^M. An explicit calculation gives

$$F^M = dA^M + \tfrac{1}{2} X_{[NP]}{}^M A^N \wedge A^P + \Delta F^M\,, \qquad \Theta_M{}^A \Delta F^M = 0\,. \tag{3.2.33}$$

The possible presence of ΔF^M is a novel feature of the embedding tensor formalism. Its gauge transformation rule can be found by using the gauge covariance of F^M. Under Eq. (3.2.31), using $\Theta_M{}^A \Delta F^M = 0$, we find that

$$\delta F^M = \Lambda^P X_{PN}{}^M F^N + \mathfrak{D}\Delta A^M - 2X_{(NP)}{}^M (\Lambda^N F^P + \tfrac{1}{2} A^N \wedge \delta A^P) + \delta \Delta F^M\,, \tag{3.2.34}$$

so that F^M transforms covariantly provided that we take

$$\delta \Delta F^M = -\mathfrak{D}\Delta A^M + 2Z^{MA} T_{A\,NP} (\Lambda^N F^P + \tfrac{1}{2} A^N \wedge \delta A^P)\,, \tag{3.2.35}$$

where we have used Eq. (3.2.21). Since both ΔA^M and ΔF^M are annihilated by the embedding tensor, we conclude that in the generic situation we are considering here[11] $\Delta F^M = Z^{MA} B_A$ where B_A is some 2-form field in the adjoint of G and $\Delta A^M = -Z^{MA}\Lambda_A$ where Λ_A is a 1-form gauge parameter in the same representation. Then

$$F^M \quad = \quad dA^M + \tfrac{1}{2} X_{[NP]}{}^M A^N \wedge A^P + Z^{MA} B_A\,, \tag{3.2.36}$$

$$\delta A^M \quad = \quad -\mathfrak{D}\Lambda^M - Z^{MA}\Lambda_A\,, \tag{3.2.37}$$

$$\delta B_A \quad = \quad \mathfrak{D}\Lambda_A + 2T_{A\,NP}[\Lambda^N F^P + \tfrac{1}{2} A^N \wedge \delta A^P] + \Delta B_A\,, \tag{3.2.38}$$

[11] The only information we have about the embedding tensor in a generic situation is provided by the three constraints $Q_{NP}{}^E = 0\,,\ Q^{AB} = 0\,,\ L_{MNP} = 0$. There is only one which we can write in the form $\Theta_M{}^A \times \text{Something}^M = 0$, which is the constraint $Q^{AB} = 0$ and that uniquely identifies $\text{Something}^M = Z^{MB}$ up to a proportionality constant.

where ΔB_A is a possible additional term which is projected out by Z^{MA}, i.e.

$$Z^{MA}\Delta B_A = 0\,, \tag{3.2.39}$$

and can be determined by studying the construction of a gauge-covariant field strength H_A for the 2-form B_A.

3.2.3 The 3-form field strengths H_A

We continue to determine the form of H_A using the Bianchi identity for F^M just as we used the Ricci identity to find an expression for F^M. An explicit computation using Eq. (3.2.36) gives

$$\mathfrak{D}F^M = Z^{MA}\{\mathfrak{D}B_A + T_{A\,RS}A^R \wedge [dA^S + \tfrac{1}{3}X_{NP}{}^S A^N \wedge A^P]\}\,. \tag{3.2.40}$$

It is clear that the expression in brackets must be covariant and it defines a 3-form field strength H_A up to terms ΔH_A that are projected out by Z^{MA}, i.e.

$$\mathfrak{D}F^M = Z^{MA}H_A\,, \tag{3.2.41}$$

$$H_A = \mathfrak{D}B_A + T_{A\,RS}A^R \wedge [dA^S + \tfrac{1}{3}X_{NP}{}^S A^N \wedge A^P] + \Delta H_A \tag{3.2.42}$$

with $Z^{MA}\Delta H_A = 0$. Both ΔB_A and ΔH_A are determined by requiring gauge covariance of H_A. An explicit calculation gives

$$\begin{aligned}\delta H_A = {}& -\Lambda^M\Theta_M{}^B f_{BA}{}^C H_C \\ & -Y_{AM}{}^C[\Lambda^M H_C - \delta A^M \wedge B_C - F^M \wedge \Lambda_C - \tfrac{1}{3}T_{C\,NP}A^M \wedge A^N \wedge \delta A^P] \\ & +\mathfrak{D}\Delta B_A + \delta\Delta H_A\,.\end{aligned} \tag{3.2.43}$$

We have defined the Y-tensor as

$$Y_{AM}{}^C \equiv \Theta_M{}^B f_{AB}{}^C - T_{A\,M}{}^N\Theta_N{}^C\,. \tag{3.2.44}$$

and it satisfies the condition

$$Z^{MA}Y_{AN}{}^C = \tfrac{1}{2}\Omega^{PM}Q_{PN}{}^C = 0\,. \tag{3.2.45}$$

The 3-form field strengths H_A transform covariantly provided that the last two lines in Eq. (3.2.43) vanish. A natural solution is to take

$$\Delta B_A \equiv -Y_{AM}{}^{C}\Lambda_C{}^{M}\,, \qquad \Delta H_A \equiv Y_{AM}{}^{C} C_C{}^{M}\,, \tag{3.2.46}$$

where $\Lambda_C{}^M$ is a 2-form gauge parameter and $C_C{}^M$ is a 3-form field about which we will not make any assumptions for the moment. In particular, we will not assume it to satisfy any constraints in spite of the fact that we expect it to be "dual" to the embedding tensor, which is a constrained object. We are going to see that, actually, we are not going to need any such explicit constraints to construct a fully consistent tensor hierarchy. On the other hand, we are going to find Stückelberg shift symmetries acting on $C_C{}^M$ whose role is, precisely, to compensate for the lack of explicit constraints and, potentially, allow us to remove the same components of $C_C{}^M$ which would be eliminated by imposing those constraints. We anticipate that those Stückelberg shift symmetries require the existence of 4-forms in order to construct gauge-covariant 4-form field strengths $G_C{}^M$. It should come as no surprise after this discussion, that the 4-forms are in one-to-one correspondence with the constraints of the embedding tensor. Working with unconstrained fields is simpler and it is one of the advantages of our approach.

We then, find

$$H_A = \mathfrak{D}B_A + T_{A\,RS}A^R \wedge [dA^S + \tfrac{1}{3}X_{NP}{}^{S}A^N \wedge A^P] + Y_{AM}{}^{C}C_C{}^{M} \tag{3.2.47}$$

$$\delta B_A = \mathfrak{D}\Lambda_A + 2T_{A\,NP}[\Lambda^N F^P + \tfrac{1}{2}A^N \wedge \delta A^P] - Y_{AM}{}^{C}\Lambda_C{}^{M}\,, \tag{3.2.48}$$

$$\begin{aligned}\delta C_C{}^M = {}& \mathfrak{D}\Lambda_C{}^M + \Lambda^M H_C - \delta A^M \wedge B_C - F^M \wedge \Lambda_C \\ & -\tfrac{1}{3}T_{C\,NP}A^M \wedge A^N \wedge \delta A^P + \Delta C_C{}^M\,,\end{aligned} \tag{3.2.49}$$

where we have introduced a possible additional term $\Delta C_C{}^M$ analogous to ΔA^M and ΔB_A which now is projected out by $Y_{AM}{}^C$

$$Y_{AM}{}^{C}\Delta C_C{}^{M} = 0\,, \tag{3.2.50}$$

and which will be determined by requiring gauge covariance of the 4-form field strength $G_C{}^M$.

3.2.4 The 4-form field strengths $G_C{}^M$

To determine the 4-form field strengths $G_C{}^M$ we use the Bianchi identity of H_A. We can start by taking the covariant derivative of both sides of the Bianchi identity of F^M Eq. (3.2.41) and then using the Ricci identity. We thus get

$$Z^{MA}\mathfrak{D}H_A = X_{NP}{}^M F^N \wedge F^P = Z^{MA} T_{A\,NP} F^N \wedge F^P\,. \tag{3.2.51}$$

This implies that

$$\mathfrak{D}H_A = T_{A\,MN} F^M \wedge F^N + \Delta\mathfrak{D}H_A\,, \tag{3.2.52}$$

where

$$Z^{MA}\Delta\mathfrak{D}H_A = 0\,, \tag{3.2.53}$$

suggesting that

$$\Delta\mathfrak{D}H_A \sim Y_{AM}{}^C G_C{}^M\,. \tag{3.2.54}$$

A direct calculation yields the result

$$\begin{aligned} G_C{}^M \quad = \quad & \mathfrak{D}C_C{}^M + F^M \wedge B_C - \tfrac{1}{2} Z^{MA} B_A \wedge B_C \\ & + \tfrac{1}{3} T_{C\,SQ} A^M \wedge A^S \wedge (F^Q - Z^{QA} B_A) \\ & - \tfrac{1}{12} T_{C\,SQ} X_{NT}{}^Q A^M \wedge A^S \wedge A^N \wedge A^T \\ & + \Delta G_C{}^M\,, \end{aligned} \tag{3.2.55}$$

where

$$Y_{AM}{}^C \Delta G_C{}^M = 0\,. \tag{3.2.56}$$

The Bianchi identity then takes the form

$$\mathfrak{D}H_A = Y_{AM}{}^C G_C{}^M + T_{A\,MN} F^M \wedge F^N\,. \tag{3.2.57}$$

$\Delta C_C{}^M$ and $\Delta G_C{}^M$ must now be determined by using the gauge covariance of the full field strength $G_C{}^M$. It is tempting to repeat what we did in the previous cases. However, the calculation is, now, much more complicated and it would be convenient to have some information about the new tensor(s) orthogonal to $Y_{AM}{}^C$ that we may expect.

Given that the projectors arise naturally in the computation of the Bianchi identities, we are going to "compute" the Bianchi identity of $G_C{}^M$ obviating the fact that it is already a 4-form, and in $D = 4$ its Bianchi identity is trivial. We have not used the dimensionality of the problem so far (except in the existence of magnetic vector fields that gives rise to the symplectic structure and in the assignment of adjoint indices to the 2-forms) and, in any case, our only goal in performing this computation is to find the relevant invariant tensor(s).

Thus, we apply $\mathfrak{D}$ to both sides of Eq. (3.2.57) using the Bianchi identity of F^M Eq. (3.2.41) and the Ricci identity. This leads to the following identity

$$Y_{AM}{}^C\{\mathfrak{D}G_C{}^M - F^M \wedge H_C\} = 0\,, \tag{3.2.58}$$

from which it follows that

$$\mathfrak{D}G_C{}^M = F^M \wedge H_C + \Delta\mathfrak{D}G_C{}^M\,, \qquad Y_{AM}{}^C \Delta\mathfrak{D}G_C{}^M = 0\,. \tag{3.2.59}$$

Acting again with $\mathfrak{D}$ on both sides of this last equation and using the Ricci and Bianchi identities, we get in an straightforward manner

$$\begin{aligned}\mathfrak{D}\Delta\mathfrak{D}G_C{}^M &= W_C{}^{MAB} H_A \wedge H_B \\ &\quad + W_{CNPQ}{}^M F^N \wedge F^P \wedge F^Q \\ &\quad + W_{CNP}{}^{EM} F^N \wedge G_E{}^P\,,\end{aligned} \tag{3.2.60}$$

where

$$W_C{}^{MAB} \equiv -Z^{M[A}\delta_C{}^{B]}\,, \tag{3.2.61}$$

$$W_{CNPQ}{}^M \equiv T_{C\,(NP}\delta_{Q)}{}^M\,, \tag{3.2.62}$$

$$W_{CNP}{}^{EM} \equiv \Theta_N{}^D f_{CD}{}^E \delta_P{}^M + X_{NP}{}^M \delta_C{}^E - Y_{CP}{}^E \delta_N{}^M\,. \tag{3.2.63}$$

We thus found the desired new tensors. The Y-tensor annihilates the three new W tensors in virtue of the 3 constraints satisfied by the embedding tensor

$$Y_{AM}{}^C W_C{}^{MAB} = Y_{AM}{}^C W_{CNPQ}{}^M = Y_{AM}{}^C W_{CNP}{}^{EM} = 0\,, \tag{3.2.64}$$

as expected. Note that the first and third W-tensors are linear in Θ but that the second W-tensor is independent of Θ. Other important sets of identities satisfied by these W-tensors can be found in Appendix E.2.

Coming back to our original problem of determining the form of $\Delta G_C{}^M$ and $\Delta C_C{}^M$, we conclude from the previous analysis that

$$\Delta C_C{}^M = -W_C{}^{MAB}\Lambda_{AB} - W_{CNPQ}{}^M \Lambda^{NPQ} - W_{CNP}{}^{EM}\Lambda_E{}^{NP}\,, \tag{3.2.65}$$

$$\Delta G_C{}^M = W_C{}^{MAB} D_{AB} + W_{CNPQ}{}^M D^{NPQ} + W_{CNP}{}^{EM} D_E{}^{NP}\,, \tag{3.2.66}$$

where $\Lambda_{AB}, \Lambda^{NPQ}, \Lambda_E{}^{NP}$ are 3-form gauge parameters and $D_{AB}, D^{NPQ}, D_E{}^{NP}$ are possible 4-forms whose presence will be justified in $G_C{}^M$ if their gauge transformations are non-trivial in order to make the 4-form field strengths gauge covariant. Taking into account the symmetries of the W-tensors, it is easy to see that $D_{AB} = D_{[AB]}$, $D^{NPQ} = D^{(NPQ)}$ and analogously for the gauge parameters $\Lambda_{AB}, \Lambda^{NPQ}$. $D_E{}^{NP}$ and $\Lambda_E{}^{NP}$ have no symmetries.

We observe that the three 4-form D-potentials seem to be associated to the three constraints Q^{AB}, L_{NPQ}, $Q_{NP}{}^{E}$ given in Eqs. (3.2.10), (3.2.14) and (3.2.16) in the sense that they carry the same representations. Only the last one was expected according to the general formalism developed in Ref. [16] and the specific study of the top forms performed in Ref. [17, 74]. We find that in 4 dimensions there are more top-form potentials due to the additional structures (e.g. the symplectic one) and properties of 4-dimensional theories.

Knowing the different W tensors it is now a relatively straightforward task to obtain by a direct calculation the expression for $\delta G_C{}^M$, collect the terms proportional to the three W-structures and determine the gauge transformations of the three different 4-form D-potentials by requiring gauge-covariance of $G_C{}^M$. An explicit calculation gives

$$\begin{aligned}
\delta D_{AB} &= \mathfrak{D}\Lambda_{AB} + \alpha B_{[A} \wedge Y_{B]P}{}^{E}\Lambda_E{}^P + \mathfrak{D}\Lambda_{[A} \wedge B_{B]} - 2\Lambda_{[A} \wedge H_{B]} \\
&\quad +2T_{[A|NP}[\Lambda^N F^P - \tfrac{1}{2}A^N \wedge \delta A^P] \wedge B_{|B]}\,, \qquad (3.2.67)\\
\delta D_E{}^{NP} &= \mathfrak{D}\Lambda_E{}^{NP} - [F^N - \tfrac{1}{2}(1-\alpha)Z^{NA}B_A] \wedge \Lambda_E{}^P + C_E{}^P \wedge \delta A^N \\
&\quad +\tfrac{1}{12}T_{EQR}A^N \wedge A^P \wedge A^Q \wedge \delta A^R + \Lambda^N G_E{}^P\,, \qquad (3.2.68)\\
\delta D^{NPQ} &= \mathfrak{D}\Lambda^{NPQ} - 2A^{(N} \wedge (F^P - Z^{PA}B_A) \wedge \delta A^{Q)} \\
&\quad +\tfrac{1}{4}X_{RS}{}^{(N}A^{P|} \wedge A^R \wedge A^S \wedge \delta A^{|Q)} - 3\Lambda^{(N}F^P \wedge F^{Q)}\,, \qquad (3.2.69)
\end{aligned}$$

where α is an arbitrary real constant. We hence find that there is a 1-parameter family of solutions to the problem of finding a gauge-covariant field strength for the 3-form. The origin of this freedom resides in the presence of a Stückelberg-type symmetry which we discuss in the next subsection.

Stückelberg symmetries

Differentiating (3.2.17) with respect to $\Theta_Q{}^C$ using Eqs. (E.2.7)-(E.2.9) gives the following identity among the W tensors:

$$W_{C(MN)}{}^{AQ} - 3W_{CMNP}{}^{Q}Z^{PA} - 2W_C{}^{QAB}T_{B\,MN} = \tfrac{3}{2}L_{MN}{}^{Q}\delta_C{}^{A}\,. \qquad (3.2.70)$$

The relation (3.2.70) gives rise to symmetries under Stückelberg shifts of the 4-forms in the 4-form field strength $G_C{}^M$

$$
\begin{aligned}
\delta D_E{}^{NP} &= \Xi_E{}^{(NP)}\,, \\
\delta D_{AB} &= -2\Xi_{[A}{}^{MN} T_{B]MN}\,, \\
\delta D^{NPQ} &= -3 Z^{(N|A} \Xi_A{}^{|PQ)}\,.
\end{aligned} \tag{3.2.71}
$$

This shift symmetry, which allows us to remove the part symmetric in NP of $D_E{}^{NP}$, also leaves the 4-form field strengths $G_C{}^M$ invariant.

If we multiply (3.2.17) by Z^{NE} we find another relation between constraints

$$
Q^{AB} Y_{BP}{}^E - \tfrac{1}{2} Z^{NA} Q_{NP}{}^E = 0\,. \tag{3.2.72}
$$

Differentiating it again with respect to the embedding tensor we find the following relation between W-tensors[12]:

$$
W_C{}^{MAB} Y_{BP}{}^E - \tfrac{1}{2} Z^{NA} W_{CNP}{}^{EM} = \tfrac{1}{4} Q^M{}_P{}^E \delta_C{}^A - Q^{AB}[\delta_P{}^M f_{BC}{}^E - T_{B\,P}{}^M \delta_C{}^E]\,, \tag{3.2.73}
$$

which implies that the Stückelberg shift

$$
\begin{aligned}
\delta D_E{}^{NP} &= \tfrac{1}{2} Z^{NB} \Xi_{BE}{}^P\,, \\
\delta D_{AB} &= Y_{[A|P}{}^E \Xi_{B]E}{}^P\,,
\end{aligned} \tag{3.2.74}
$$

leaves invariant the 4-form field strength $G_C{}^M$ up to terms proportional to the quadratic constraints, which are taken to vanish identically in the tensor hierarchy. This shift symmetry is associated to the arbitrary parameter α in the gauge transformations of D_{AB} and $D_E{}^{NP}$. Observe that, even though it is based on the identity Eq. (3.2.73) which we can get from Eq. (3.2.70), this symmetry is genuinely independent from that in Eq. (3.2.71).

This finishes the construction of the 4-dimensional tensor hierarchy. The field strengths, Bianchi identities and gauge transformations of the hierarchy's p-form fields are collected in Appendix E.3. By construction the algebra of all bosonic gauge transformations closes off-shell on all p-form potentials. No equations of motion are needed at this stage.

3.3 The $D = 4$ duality hierarchy

In this section we are going to introduce dynamical equations for the tensor hierarchy via the introduction of first-order duality relations, see Eq. (3.0.1). This promotes the tensor hierarchy to a *duality hierarchy*. We will see that the dynamical equations

[12]This identity can also be obtained multiplying Eq. (3.2.70) by Z^{NE}.

will not only contain the equations of motions of the p-form potentials but also the (projected) scalar equations of motion. These scalars, together with the metric, will be introduced via the duality relations. In particular, the scalar couplings enter into the duality relations via functions that can be identified with the Noether currents, the (scalar derivative of the) scalar potential and the kinetic matrix describing the coupling of the scalars to the vectors. In this way the duality hierarchy puts the tensor hierarchy on-shell and establishes a link with a Yang-Mills-type gauge field theory containing a metric, scalars and p-form potentials. This field theory can be viewed as the bosonic part of a gauged supergravity theory. We stress that at this point we only compare equations of motion. It is only in the last and third step that we consider an action for the fields of the hierarchy. We will assume that the Yang-Mills-type gauge field theory has an action but we will only consider its equations of motion in order to properly identify in the duality relations the Noether current, scalar potential and the scalar-vector kinetic function.

In the next subsection we will first consider a Yang-Mills-type gauge field theory with purely electric gaugings, i.e. only electric 1-forms are involved in the gauging. In particular we will compare the equations of motion of this field theory with the dynamical equations of the duality hierarchy. This example shows us how to introduce the metric and scalars in the duality hierarchy. In the next subsection we will first consider a formally symplectic-covariant generalization of the equations of motion with purely electric gaugings. This generalization necessarily involves electric and magnetic gaugings. We will see that this generalization does not lead to gauge-invariant answers unless we also include the equations of motion corresponding to the magnetic 2-form potentials. In this way we recover the observation of [15–17, 78, 79] that magnetic gaugings require the introduction of magnetic 2-form potentials in the action of the field theory.

3.3.1 Purely electric gaugings

Having $N = 1, D = 4$ supergravity in mind, we consider complex scalars Z^i ($i = 1, \cdots, n$) with Kähler metric $\mathcal{G}_{ij^*}$ admitting holomorphic Killing vectors $K_A = k_A{}^i\partial_i+$ c.c.. The index A of the Killing vectors must be associated to those of the generators of the global symmetry group G. In general, not all the global symmetries will act on the scalars. Therefore, we assume that some of the K_A may be identically zero just as some of the matrices $T_{A\,M}{}^N$ can be zero for other values of A. The action for the electrically gauged theory is

$$S_{\text{elec}}[g, Z^i, A^\Lambda] = \int \left\{\star R - 2\mathcal{G}_{ij^*}\mathfrak{D}Z^i \wedge \star\mathfrak{D}Z^{*\,j^*} + 2F^\Sigma \wedge G_\Sigma - \star V\right\} , \tag{3.3.1}$$

where $\mathfrak{D}Z^i$ is given by

$$\mathfrak{D}Z^i = dZ^i + A^\Lambda\Theta_\Lambda{}^A k_A{}^i\,, \tag{3.3.2}$$

and where G_Λ denotes the combination of scalars and electric vector field strengths defined by

$$G_\Lambda{}^+ = f_{\Lambda\Sigma}(Z) F^{\Sigma\,+} \,, \tag{3.3.3}$$

where $F^{\Sigma\,+} = \frac{1}{2}(F^\Sigma + i \star F^\Sigma)$. It is assumed that the scalar-dependent kinetic matrix $f_{\Lambda\Sigma}(Z)$ is invariant under the global symmetry group, i.e.[13]

$$\pounds_A f_{\Lambda\Sigma} = 2 T_{A\,(\Lambda}{}^\Omega f_{\Sigma)\Omega} \,, \tag{3.3.4}$$

where $\pounds_A$ stands for the Lie derivative with respect to K_A, since this is a pre-condition to gauge the theory. However, the potential needs only be invariant under the gauge transformations, because the gauging usually adds to the globally-invariant potential of the ungauged theory another piece. Thus, we must have

$$\pounds_A V = Y_{A\Lambda}{}^C \frac{\partial V}{\partial \Theta_\Lambda{}^C} \,, \tag{3.3.5}$$

where $Y_{A\Lambda}{}^C$ is the electric component of the tensor defined in Eq. (3.2.44). Indeed, using this property, one can show that under the gauge transformations

$$\begin{aligned} \delta Z^i &= \Lambda^\Lambda \Theta_\Lambda{}^A k_A{}^i \,, \\ \delta A^\Lambda &= -\mathfrak{D}\Lambda^\Lambda \,, \end{aligned} \tag{3.3.6}$$

the scalar potential V is gauge invariant:

$$\delta V = \Lambda^\Sigma \Theta_\Sigma{}^A \pounds_A V = \Lambda^\Sigma Q_\Sigma{}^{\Lambda C} \frac{\partial V}{\partial \Theta_\Lambda{}^A} = 0 \,, \tag{3.3.7}$$

on account of the quadratic constraint.

The equations of motion (plus the Bianchi identity for F^Λ) corresponding to the action (3.3.1) are given by

[13] Here we are only considering a restricted type of perturbative symmetries of the theory, excluding Peccei-Quinn-type shifts of the kinetic matrix for simplicity. We will consider these shifts together with the possible non-perturbative symmetries in the general gaugings' section.

$$\begin{aligned}
\mathcal{E}_{\mu\nu} &\equiv -\star\frac{\delta S}{\delta g^{\mu\nu}} = G_{\mu\nu} + 2\mathcal{G}_{ij^*}[\mathfrak{D}_\mu Z^i \mathfrak{D}_\nu Z^{*\,j^*} - \tfrac{1}{2}g_{\mu\nu}\mathfrak{D}_\rho Z^i \mathfrak{D}^\rho Z^{*\,j^*}] \\
&\quad -4\Im\mathrm{m} f_{\Lambda\Sigma} F^{\Lambda\,+}{}_\mu{}^\rho F^{\Sigma\,-}{}_{\nu\rho} + \tfrac{1}{2}g_{\mu\nu}V\,, \qquad (3.3.8)
\end{aligned}$$

$$\mathcal{E}_i \equiv \tfrac{1}{2}\frac{\delta S}{\delta Z^i} = \mathcal{G}_{ij^*}\mathfrak{D}\star\mathfrak{D}Z^{*\,j^*} - \partial_i G_\Sigma{}^+ \wedge F^{\Sigma +} - \star\tfrac{1}{2}\partial_i V\,, \qquad (3.3.9)$$

$$\begin{aligned}
\mathcal{E}_\Lambda &\equiv -\tfrac{1}{4}\star\frac{\delta S}{\delta A^\Lambda} = \mathfrak{D}G_\Lambda - \tfrac{1}{4}\Theta_\Lambda{}^A \star j_A\,, \\
\mathcal{E}^\Lambda &\equiv \mathfrak{D}F^\Lambda\,, \qquad (3.3.10)
\end{aligned}$$

where

$$j_A \equiv 2k^*_{Ai}\mathfrak{D}Z^i + \text{c.c.}\,, \qquad (3.3.11)$$

is the covariant Noether current.

According to the second Noether theorem there is an off-shell relation between equations of motion of a theory associated to each gauge invariance. For instance, associated to general covariance we find the well-known identity

$$\nabla^\mu \mathcal{E}_{\mu\nu} - (\mathfrak{D}_\nu Z^i \mathcal{E}^*_i + \text{c.c.}) + 2F^\Lambda{}_{\nu\rho}(\star\mathcal{E}_\Lambda)^\rho = 0\,, \qquad (3.3.12)$$

which implies the on-shell covariant conservation of the energy-momentum tensor. Similarly, the identity associated to the Yang-Mills-type gauge invariance of the theory is given by

$$\mathfrak{D}\mathcal{E}_\Lambda + \tfrac{1}{2}\Theta_\Lambda{}^A(k_A{}^i\mathcal{E}_i + \text{c.c.}) = 0\,. \qquad (3.3.13)$$

Using the Ricci identity for the covariant derivative and Eqs. (3.3.4) and (3.3.5) we find that this equation is indeed satisfied because the Noether current satisfies the identity

$$\mathfrak{D}\star j_A = -2(k_A{}^i\mathcal{E}_i + \text{c.c.}) + 4T_{A\,\Sigma}{}^\Gamma F^\Sigma \wedge G_\Gamma + \star Y_{A\Lambda}{}^C\frac{\partial V}{\partial \Theta_\Lambda{}^C}\,. \qquad (3.3.14)$$

We are now going to establish a relation between the tensor hierarchy and the equations of motion for the vector fields, their Bianchi identities and the following projected scalar equations of motion:

$$\begin{aligned}
\mathfrak{D}G_\Lambda - \tfrac{1}{4}\Theta_\Lambda{}^A \star j_A &= 0\,, &(3.3.15)\\
\mathfrak{D}F^\Lambda &= 0\,, &(3.3.16)\\
k_A{}^i\Big[\mathcal{G}_{ij^*}\mathfrak{D}\star\mathfrak{D}Z^{*\,j^*} - \partial_i G_\Sigma{}^+ \wedge F^{\Sigma+} - \star\tfrac{1}{2}\partial_i V\Big] + \text{c.c.} &= 0\,. &(3.3.17)
\end{aligned}$$

Note that, unlike the tensor hierarchy, these equations contain not only p-form potentials but also the metric and scalars.

In order to derive the above equations of motion from the tensor hierarchy we must complement the tensor hierarchy with a set of duality relations that reproduces the scalar and metric dependence of these equations. Besides the usual $\mathfrak{D}^2 Z$ term in the last equation the scalar dependence of (3.3.15)-(3.3.17) resides in the magnetic 2-forms G_Λ, the Noether currents j_A and the derivatives $\partial_i V$ of the scalar potential V. The latter derivative is equivalently represented, via the invariance property (3.3.5), by the derivative $\dfrac{\partial V}{\partial \Theta_\Lambda{}^A}$ of the scalar potential with respect to the embedding tensor. These are precisely the objects that occur in the following set of duality relations that we introduce:

$$\begin{aligned}
G_\Lambda &= F_\Lambda\,,\\
j_A &= -2\star H_A\,,\\
\frac{\partial V}{\partial \Theta_\Lambda{}^A} &= -2\star G_A{}^\Lambda\,,
\end{aligned} \qquad (3.3.18)$$

where the magnetic 2-form field strengths F_Λ, the 3-form field strengths H_A and the 4-form field strengths $G_A{}^\Lambda$ are those of the tensor hierarchy. The tensor hierarchy, together with the above duality relations, forms the duality hierarchy. Upon hitting the duality relations (3.3.18) with a covariant derivative and next applying one of the Bianchi identities of the tensor hierarchy we precisely obtain the equations of motion (3.3.15)-(3.3.17). In the case of the scalar equations of motion we first obtain the identity

$$\mathfrak{D}\star j_A - 4T_{A\,\Sigma}{}^\Gamma F^\Sigma \wedge G_\Gamma - \star Y_{A\Lambda}{}^C \frac{\partial V}{\partial \Theta_\Lambda{}^A} = 0\,. \qquad (3.3.19)$$

Next, by comparing this equation with the Noether identity (3.3.14) we derive the projected scalar equations of motion (3.3.17), i.e.

$$k_A{}^i \mathcal{E}_i + \text{c.c.} = 0. \qquad (3.3.20)$$

It also works the other way around. By substituting the duality relations into the equations of motion the scalar and metric dependence of these equations can be eliminated and one recovers the hierarchy's Bianchi identities for a purely electric embedding tensor $\Theta^{\Sigma A} = 0$. To be precise, Eqs. (3.3.15) and (3.3.16) are mapped into the 3-form Bianchi identities (3.2.41). Furthermore, Eq. (3.3.19), which is equivalent to (3.3.17) upon use of the Noether identity (3.3.14), is mapped into the 4-form Bianchi identities (3.2.57).

We conclude that, at least in this case, the duality hierarchy encodes precisely the vector equations of motion and the projected scalar equations of motion via the duality rules (3.3.18).

3.3.2 General gaugings

In this subsection we wish to consider the more general case of electric and magnetic gaugings. Our starting point is the formally symplectic-covariant generalization of the equations of motion (3.3.15)-(3.3.17)[14]

$$\begin{aligned}
\mathcal{E}_{\mu\nu} &= G_{\mu\nu} + 2\mathcal{G}_{ij^*}[\mathfrak{D}_\mu Z^i \mathfrak{D}_\nu Z^{*\,j^*} - \tfrac{1}{2} g_{\mu\nu} \mathfrak{D}_\rho Z^i \mathfrak{D}^\rho Z^{*\,j^*}] - G^M{}_{(\mu|}{}^\rho \star G_{M|\nu)\rho} + \tfrac{1}{2} g_{\mu\nu} V\,, \\
\mathcal{E}_i &= \mathcal{G}_{ij^*} \mathfrak{D} \star \mathfrak{D} Z^{*\,j^*} - \partial_i G_M{}^+ \wedge G^{M+} - \star \tfrac{1}{2} \partial_i V\,, \qquad (3.3.21) \\
\mathcal{E}_M &\equiv \mathfrak{D} G_M - \tfrac{1}{4} \Theta_M{}^A \star j_A\,,
\end{aligned}$$

where we have defined

$$(G^M) \equiv \begin{pmatrix} F^\Sigma \\ G_\Sigma \end{pmatrix}, \qquad G_\Sigma{}^+ = f_{\Sigma\Gamma}(Z) F^{\Gamma +}\,, \qquad (3.3.22)$$

and where the electric and magnetic field strengths F^M are defined as in the tensor hierarchy, i.e. including the 2-form B_A for which we do not want to have an independent equation of motion to preserve the original number of degrees of freedom.

The requirement that the kinetic matrix is invariant under the global symmetry group G and that the potential is gauge-invariant leads to the conditions

$$\pounds_A f_{\Lambda\Sigma} = -T_{A\,\Lambda\Sigma} + 2T_{A\,(\Lambda}{}^\Omega f_{\Sigma)\Omega} - T_A{}^{\Omega\Gamma} f_{\Omega\Lambda} f_{\Gamma\Sigma}\,, \qquad (3.3.23)$$

$$\pounds_A V = Y_{AM}{}^C \frac{\partial V}{\partial \Theta_M{}^C}\,, \qquad (3.3.24)$$

[14]The Einstein and scalar equations of motion are just a rewriting of the original ones, which are already symplectic-invariant.

from which it follows that

$$k_A{}^i \partial_i G_M{}^+ \wedge G^{M+} = k_A{}^i \partial_i f_{\Lambda\Sigma} F^{\Lambda+} \wedge F^{\Sigma+} = -T_{A\,MN} G^M \wedge G^N \,. \tag{3.3.25}$$

A direct computation using the above properties leads to the following identity for the covariant Noether current:

$$\mathfrak{D} \star j_A = -2(k_A{}^i \mathcal{E}_i + \text{c.c.}) - 2T_{A\,MN} G^M \wedge G^N + \star Y_A{}^{\Lambda C} \frac{\partial V}{\partial \Theta_\Lambda{}^C} \,. \tag{3.3.26}$$

On the other hand, the Ricci identity gives

$$\mathfrak{D}\mathfrak{D} G_M = -X_{NM}{}^P F^N \wedge G_P = X_{NPM} F^N \wedge G^P \,. \tag{3.3.27}$$

Taking the covariant derivative of the full $\mathcal{E}_M$ and using Eqs. (3.3.26) and (3.3.27) we find

$$\mathfrak{D}\mathcal{E}_M + \tfrac{1}{2}\Theta_M{}^A (k_A{}^i \mathcal{E}_i + \text{c.c.}) = X_{NPM}(F^N - G^N) \wedge G^P = \Theta^{\Sigma A}(F_\Sigma - G_\Sigma) \wedge T_{A\,PM} G^P \,. \tag{3.3.28}$$

This is the gauge identity associated to the standard electric and magnetic gauge transformations of the vectors and scalars

$$\begin{aligned} \delta Z^i &= \Lambda^M \Theta_M{}^A k_A{}^i \,, \\ \delta A^M &= -\mathfrak{D}\Lambda^M \,, \end{aligned} \tag{3.3.29}$$

provided that the right-hand side of the equation vanishes. Since this is not the case we conclude that the equations of motion are not gauge-invariant. Hence, a naive symplectic covariantization of the electric gauging case is not enough to obtain a gauge-invariant answer involving magnetic gaugings.

In order to re-obtain gauge invariance we extend the set of equations of motion, adding, arbitrarily, as equation of motion of the 2-forms B_A

$$\mathcal{E}^A \equiv \Theta^{MA}(F_M - G_M) = -\Theta^{\Sigma A}(F_\Sigma - G_\Sigma) \,, \tag{3.3.30}$$

so that the above identity becomes again a relation between equations of motion

$$\mathfrak{D}\mathcal{E}_M + \tfrac{1}{2}\Theta_M{}^A (k_A{}^i \mathcal{E}_i + \text{c.c.}) + T_{A\,MP} \mathcal{E}^A \wedge G^P = 0 \,, \tag{3.3.31}$$

that we can interpret as the gauge identity associated to an off-shell gauge invariance of the extended set of equations of motion.

The price we may have to pay for doing this is the possible modification of the equations of motion of the vector fields: the above gauge identities are associated to the gauge transformations of B_A

$$\delta B_A = 2T_{A\,MP}\Lambda^M G^P + 2R_{A\,M} \wedge \delta A^M\,, \tag{3.3.32}$$

where $R_{A\,M}$ is a 1-form that is cancelled in the above gauge identity by an extra term in the equation of motion of the vector fields:

$$\mathcal{E}'_M = \mathcal{E}_M + R_{A\,M}\mathcal{E}^A \wedge A^M\,. \tag{3.3.33}$$

The 1-forms $R_{A\,M}$ must be such that the infinitesimal gauge transformations form a closed algebra. The gauge identity takes now the form

$$\mathfrak{D}\mathcal{E}'_M + \tfrac{1}{2}\Theta_M{}^A(k_A{}^i\mathcal{E}_i + \text{c.c.}) + T_{A\,MP}\mathcal{E}^A \wedge G^P - \mathfrak{D}(R_{A\,M}\mathcal{E}^A \wedge A^M) = 0\,. \tag{3.3.34}$$

In order to make contact with the tensor hierarchy we take

$$R_{A\,M} = \tfrac{1}{2}X^P{}_{MN}A^N \wedge (F_P - G_P)\,. \tag{3.3.35}$$

We observe that the equations of motion also satisfy the relation

$$\mathfrak{D}\mathcal{E}^A - \tfrac{1}{2}T_{B\,MN}\Theta^{PA}A^N \wedge \mathcal{E}^B + \Theta^{MA}\mathcal{E}_M = 0\,, \tag{3.3.36}$$

which can be interpreted as the gauge identity associated to the symmetry

$$\begin{aligned} \delta A^M &= Z^{MA}\Lambda_A\,, \\ \delta B_A &= \mathfrak{D}\Lambda_A - \tfrac{1}{2}T_{AMN}\Theta^{NB}A^M \wedge \Lambda_B\,. \end{aligned} \tag{3.3.37}$$

As we did in the electric gauging case, we are now going to establish a relation between the tensor hierarchy and the following equations of motion:

$$\mathcal{E}'_M = \mathfrak{D}G_M - \tfrac{1}{4}\Theta_M{}^A \star j_A + \tfrac{1}{2}T_{A\,MN}A^N \wedge \Theta^{PA}(F_P - G_P) = 0\,, \tag{3.3.38}$$

$$\mathcal{E}^A = \Theta^{MA}(F_M - G_M) = 0\,, \tag{3.3.39}$$

$$k_A{}^i\mathcal{E}_i = k_A{}^i\left[\mathcal{G}_{ij^*}\mathfrak{D}\star\mathfrak{D}Z^{*\,j^*} - \partial_i G_M{}^+ \wedge G^{M+} - \star\tfrac{1}{2}\partial_i V\right] = 0\,. \tag{3.3.40}$$

These equations are invariant under the gauge transformations

$$\delta_a Z^i = \delta_h Z^i\,, \tag{3.3.41}$$

$$\delta_a A^M = \delta_h A^M\,, \tag{3.3.42}$$

$$\delta_a B_A = \delta_h B_A - 2T_{A\,NP}\Lambda^N(F^P - G^P)\,, \tag{3.3.43}$$

where we have denoted by δ_a the gauge transformations that leave this system of equations invariant and by δ_h those derived in the construction of the 4-dimensional tensor hierarchy (summarized in Appendix E.3). $\delta_a B_A$ is, therefore, just $\delta_h B_A$ with F^P replaced by G^P.

Following the electric gauging case, in order to derive the above equations of motion from the tensor hierarchy, we introduce the following set of duality relations:

$$\begin{aligned} G^M &= F^M\,, \\ j_A &= -2\star H_A\,, \\ \frac{\partial V}{\partial \Theta_M{}^A} &= -2\star G_A{}^M\,. \end{aligned} \tag{3.3.44}$$

We note that the gauge-covariance of the first duality relation is more subtle in that G^M transforms not only covariantly, but also into $G^M - F^M$, see [77]. Note that the equation of motion of the magnetic 2-form potentials, $\mathcal{E}^A = 0$, is identified as a projected duality relation. To recover the other equations of motion we have to again hit the duality relations (3.3.44) with a covariant derivative and next apply one of the Bianchi identities of the tensor hierarchy. To derive the projected scalar equations of motion we first obtain the identity

$$\mathfrak{D}\star j_A + 2T_{A\,MN}G^M\wedge G^N - \star Y_A{}^{\Lambda C}\frac{\partial V}{\partial \Theta_\Lambda{}^A} = 0 \tag{3.3.45}$$

from the duality hierarchy and, next, apply the Noether identity (3.3.26).

The gauge identities guarantee the existence of a gauge-invariant action from which the equations of motion $\mathcal{E}'_M$ and $\mathcal{E}^A$ can be derived. This action has actually been constructed in Ref. [15]. In our conventions, it is given by

$$\begin{aligned} S[g_{\mu\nu}, Z^i, A^M, B_A] = \int & \big\{\star R - 2\mathcal{G}_{ij^*}\mathfrak{D}Z^i\wedge\star\mathfrak{D}Z^{*\,j^*} + 2F^\Sigma\wedge G_\Sigma - \star V \\ & -4Z^{\Sigma A}B_A\wedge\big(F_\Sigma - \tfrac{1}{2}Z_\Sigma{}^B B_B\big) \\ & -\tfrac{4}{3}X_{[MN]\Sigma}A^M\wedge A^N\wedge\big(F^\Sigma - Z^{\Sigma B}B_B\big) \\ & -\tfrac{2}{3}X_{[MN]}{}^\Sigma A^M\wedge A^N\wedge\big(dA_\Sigma - \tfrac{1}{4}X_{[PQ]\Sigma}A^P\wedge A^Q\big)\big\}\,. \end{aligned} \tag{3.3.46}$$

A general variation of the above action gives

$$\delta S = \int\left\{\delta g^{\mu\nu}\frac{\delta S}{\delta g^{\mu\nu}} + \left(\delta Z^i\frac{\delta S}{\delta Z^i} + \text{c.c.}\right) - \delta A^M\wedge\star\frac{\delta S}{\delta A^M} + 2\delta B_A\wedge\star\frac{\delta S}{\delta B_A}\right\}, \tag{3.3.47}$$

where

$$\frac{\delta S}{\delta g^{\mu\nu}} = \star\mathbb{I}\mathcal{E}_{\mu\nu}\,, \tag{3.3.48}$$

$$-\tfrac{1}{2}\frac{\delta S}{\delta Z^{i}} = \mathcal{E}_{i}\,, \tag{3.3.49}$$

$$-\tfrac{1}{4}\star\frac{\delta S}{\delta A^{M}} = \mathcal{E}'_{M}\,, \tag{3.3.50}$$

$$\star\frac{\delta S}{\delta B_{A}} = \mathcal{E}^{A}\,. \tag{3.3.51}$$

3.3.3 The unconstrained case

In this subsection we briefly comment on the meaning of the top-form and next to top-form potentials. Experience shows that these higher-rank potentials can be related to constraints: the constancy of $\Theta_M{}^A$, $\mathfrak{D}\Theta_M{}^A = 0$, can be associated to the 3-form potential, and the quadratic and linear constraints $Q_{NP}{}^E = 0$, $Q^{AB} = 0$, $L_{NPQ} = 0$ can be associated to the 4-form potentials $D_E{}^{NP}, D_{AB}, D^{NPQ}$ that we have providentially found. We would like to stress, however, that prior to relaxing the constraints one is forced to introduce these potentials if one requires that the field equations are derivable as compatibility conditions from the duality relations, as we showed in the previous section.

In view of the discussion of an action principle with Lagrange multipliers in the next section, we reconsider the gauge identities of the equations $\mathcal{E}'_M, \mathcal{E}^A$ defined in the previous subsections assuming that those constraints are not satisfied. We then denote the embedding tensor by $\vartheta_M{}^A = \vartheta_M{}^A(x)$ in order to indicate that it is now space-time dependent. Evidently, we are going to get extra terms proportional to the constraints which we will reinterpret as equations of motion of the 3- and 4-form potentials, obtaining new gauge identities that involve the equations of motion of all fields. Thus, off-shell gauge invariance will have been preserved by the same mechanism used in the previous case. The price that we will have to pay is the same: modifying the gauge transformations and the equations of motion.

This procedure is too complicated in this case, though. As an example, let us take the covariant derivative of $\mathcal{E}^A$:

$$\mathfrak{D}\mathcal{E}^A = -\mathfrak{D}\vartheta_M{}^A \wedge (F^M - G^M) + \vartheta^{MA}(\mathfrak{D}F_M - \mathfrak{D}G_M)\,. \tag{3.3.52}$$

The unconstrained Bianchi identity for F^M is

$$\begin{aligned}\mathfrak{D}F^{M} &= Z^{MB}[H_B - Y_{BN}{}^{C}C_C{}^{N}] + L^{M}{}_{RS}[\tfrac{3}{2}A^{R}\wedge dA^{S} + \tfrac{1}{2}X_{NP}{}^{S}A^{R}\wedge A^{N}\wedge A^{P}] \\ &\quad +\mathfrak{D}\vartheta_N{}^{A}\wedge[\tfrac{1}{2}\Omega^{NM}B_A + \tfrac{1}{2}T_{A\,P}{}^{M}A^{N}\wedge A^{P}] + \tfrac{1}{3}Q_{NP}{}^{E}T_{E\,R}{}^{M}A^{N}\wedge A^{P}\wedge A^{R}\,,\end{aligned} \tag{3.3.53}$$

and, using the equation of motion $\mathcal{E}'_M$ we can write the following gauge identity

$$\begin{aligned}\mathfrak{D}\mathcal{E}^{A} - \tfrac{1}{2}T_{B\,MN}\vartheta^{MA}A^{N}\wedge\mathcal{E}^{B} + \vartheta^{MA}\mathcal{E}'_M + Q^{AB}[2(H_B + \tfrac{1}{2}\star j_B) - 2Y_{BN}{}^{C}C_C{}^{N}] \\ +\mathfrak{D}\vartheta_M{}^{B}\wedge[\tfrac{1}{2}\vartheta^{MA}B_B + \tfrac{1}{2}T_{B\,P}{}^{Q}\vartheta_Q{}^{A}A^{M}\wedge A^{P} + \delta_B{}^{A}(F^{M} - G^{M})] \\ +L_{MRS}\vartheta^{MA}[-\tfrac{3}{2}A^{R}\wedge dA^{S} - \tfrac{1}{2}X_{NP}{}^{S}A^{R}\wedge A^{N}\wedge A^{P}] \\ -\tfrac{1}{3}Q_{NP}{}^{E}T_{E\,R}{}^{M}\vartheta_M{}^{A}A^{N}\wedge A^{P}\wedge A^{R} &= 0\,.\end{aligned} \tag{3.3.54}$$

It is very difficult to infer directly from this and similar identities all the gauge transformations of the fields and the modifications of the equations of motion. Thus, we are going to adopt a different strategy in the next section: we are going to construct directly a gauge-invariant action.

3.4 The $D = 4$ action

In this section we perform the third and last step of our procedure: the construction of an action for the fields of the tensor hierarchy[15]. Our starting point is the action Eq. (3.3.46), which we will denote by S_0 in what follows and which includes, besides the metric, only scalars, 1-forms and 2-forms and which is invariant under the gauge transformations Eqs. (3.3.41)-(3.3.43). We now want to add to it 3- and 4-forms as Lagrange multipliers enforcing the covariant constancy of the embedding tensor (which we promote to an unconstrained scalar field $\Theta_M{}^{A}(x)$) and the three algebraic constraints Q^{AB}, L_{NPQ}, $Q_{NP}{}^{E}$ that we have imposed on the embedding tensor. The new terms must be metric-independent ("topological") and scalar-independent in order to leave unmodified the scalar and Einstein equations of motion (3.3.21) which are derived from the action S_0 given in Eq. (3.3.46).

Thus, we add to S_0 the following piece ΔS given by[16]

$$\Delta S = \int\left\{\mathfrak{D}\vartheta_M{}^{A}\wedge\tilde{C}_A{}^{M} + Q_{NP}{}^{E}\tilde{D}_E{}^{NP} + Q^{AB}\tilde{D}_{AB} + L_{NPQ}\tilde{D}^{NPQ}\right\}\,. \tag{3.4.1}$$

[15] Actually, not all the 2-forms B_A will appear in the action but only $\Theta^{\Lambda A}B_A$.

[16] Observe that $\mathfrak{D}\Theta_M{}^{A} = d\Theta_M{}^{A} - Q_{NM}{}^{A}A^{N}$ and, therefore, the covariant constancy of the embedding tensor plus the quadratic constraint $Q_{NP}{}^{E} = 0$ imply $d\Theta_M{}^{A} = 0$.

The tildes in $\tilde{C}_C{}^M$, $\tilde{D}_{AB}$, $\tilde{D}^{NPQ}$ and $\tilde{D}_E{}^{NP}$ indicate that these 3- and 4-form fields need not be identical to those found in the hierarchy, although we expect them to be related by field redefinitions.

The action S_0 is no longer gauge invariant under the gauge transformations involving 0- and 1-form gauge parameters Λ^M, Λ_A, without imposing any constraints on the embedding tensor, but the non-vanishing terms in the transformation can only be proportional to the l.h.s.'s of the constraints $\mathfrak{D}\vartheta_M{}^C = 0$, $Q_{NP}{}^E = 0$, $Q^{AB} = 0$ and $L_{NPQ} = 0$ and, by choosing appropriately the gauge transformations of $\tilde{C}_C{}^M$, $\tilde{D}_{AB}$, $\tilde{D}^{NPQ}$ and $\tilde{D}_E{}^{NP}$ we can always make the variation of the action $S \equiv S_0 + \Delta S$ vanish. Having done that we would like to relate the tilded fields with the untilded ones in the hierarchy.

Let us start by computing the general variation of the action. Taking into account the fact that the fields $g_{\mu\nu}$, Z^i and $B_{A\,\mu\nu}$ only occur in S_0, that the field $A^M{}_\mu$ occurs in S_0 and in the term $\mathfrak{D}\vartheta_M{}^A\tilde{C}_A{}^M$ in ΔS and that the new fields $\tilde{C}_C{}^M$, $\tilde{D}_{AB}$, $\tilde{D}^{NPQ}$ and $\tilde{D}_E{}^{NP}$ only occur in ΔS, we find

$$\begin{aligned}\delta S \;=\; & \int \Bigg\{ \delta g^{\mu\nu}\frac{\delta S_0}{\delta g^{\mu\nu}} + \left(\delta Z^i \frac{\delta S_0}{\delta Z^i} + \text{c.c.}\right) - \delta A^M \wedge \star\frac{\delta S_0}{\delta A^M} + 2\delta B_A \wedge \star\frac{\delta S_0}{\delta B_A} \\ & + \mathfrak{D}\vartheta_M{}^A \wedge \delta\tilde{C}_A{}^M + Q_{NP}{}^E(\delta\tilde{D}_E{}^{NP} - \delta A^N \wedge \tilde{C}_E{}^P) + Q^{AB}\delta\tilde{D}_{AB} \\ & + L_{NPQ}\delta\tilde{D}^{NPQ} + \delta\vartheta_M{}^A \frac{\delta S}{\delta\vartheta_M{}^A} \Bigg\}\,. \end{aligned} \tag{3.4.2}$$

The scalar and Einstein equations of motion are as in Eqs. (3.3.21) and (3.3.48),(3.3.49). The variations of the old action S_0 with respect to A^M and B_A are modified by terms proportional to the constraints. We can write them in the form

$$\begin{aligned} -\tfrac{1}{4}\star\frac{\delta S_0}{\delta A^M} \;=\; & \mathfrak{D}F_M - \tfrac{1}{4}\vartheta_M{}^A \star j_A - \tfrac{1}{3}dX_{[PQ]M} \wedge A^P \wedge A^Q - \tfrac{1}{2}Q_{(NM)}{}^E A^N \wedge B_E \\ & - L_{MNP}A^N \wedge \left(dA^P + \tfrac{3}{8}X_{[RS]}{}^P A^R \wedge A^S\right) + \tfrac{1}{8}Q_{NP}{}^A T_{A\,QM} A^N \wedge A^P \wedge A^Q \\ & - d(F_M - G_M) - X_{[MN]}{}^P A^N \wedge (F_P - G_P)\,, \end{aligned} \tag{3.4.3}$$

$$\star\frac{\delta S_0}{\delta B_A} \;=\; \vartheta^{PA}(F_P - G_P) + Q^{AB}B_B\,. \tag{3.4.4}$$

In deriving these equations we have used the unconstrained Bianchi identity for F^Λ, given by the upper component of Eq. (3.3.53), to replace H_A in the equation of motion of A_Λ. This has allowed us to write a symplectic-covariant expression for the equation of motion of A^M.

The only non-trivial variation that remains to be computed in Eq. (3.4.2) is the equation of motion of the embedding tensor. We get

$$\begin{aligned}\frac{\delta S}{\delta \vartheta_M{}^A} &= -\mathfrak{D}\tilde{C}_A{}^M + Z^{MB}B_B \wedge B_A - 2(F^M - G^M) \wedge B_A - \star\frac{\partial V}{\partial \vartheta_M{}^A} \\ &+W_{ANP}{}^{EM}\tilde{D}_E{}^{NP} + W_A{}^{BCM}\tilde{D}_{BC} + W_{ANPQ}{}^M\tilde{D}^{NPQ} \\ &+A^M \wedge \Big\{-\star j_A + Y_{AN}{}^C\tilde{C}_C{}^N - T_{AN}{}^P A^N \wedge (F_P - G_P) \\ &-\tfrac{4}{3}T_{ANR}A^N \wedge \big[dA^R + \tfrac{3}{8}X_{[PQ]}{}^R A^P \wedge A^Q + \tfrac{3}{2}Z^{RB}B_B\big]\Big\}\ .\end{aligned} \tag{3.4.5}$$

We are going to use this equation to find the relation between the tilded fields and the hierarchy fields. Using Eqs. (3.3.44) and the definitions of the tensor hierarchy's field strengths H_A and $G_A{}^M$, we are left with

$$\begin{aligned}\tfrac{1}{2}\frac{\delta S}{\delta \vartheta_M{}^A} &= \mathfrak{D}(-\tfrac{1}{2}\tilde{C}_A{}^M - C_A{}^M - A^M \wedge B_A) \\ &+Y_{AP}{}^C A^M \wedge (\tfrac{1}{2}\tilde{C}_C{}^P + C_C{}^P + A^P \wedge B_C) + W_A{}^{BCM}(\tfrac{1}{2}\tilde{D}_{BC} - D_{BC}) \\ &+W_{ANP}{}^{EM}(\tfrac{1}{2}\tilde{D}_E{}^{NP} - D_E{}^{NP} + \tfrac{1}{2}A^N \wedge A^P \wedge B_E) \\ &+W_{ANPQ}{}^M(\tfrac{1}{2}\tilde{D}^{NPQ} - D^{NPQ})\,,\end{aligned} \tag{3.4.6}$$

which is satisfied if we identify

$$\begin{aligned}\tilde{C}_A{}^M &= -2(C_A{}^M + A^M \wedge B_A)\,, & \tilde{D}_E{}^{NP} &= 2D_E{}^{NP} - A^N \wedge A^P \wedge B_E\,, \\ \tilde{D}_{BC} &= 2D_{BC}\,, & \tilde{D}^{NPQ} &= 2D^{NPQ}\,.\end{aligned} \tag{3.4.7}$$

Using these identifications ΔS reads

$$\begin{aligned}\Delta S &= \int \big\{-2\mathfrak{D}\vartheta_M{}^A \wedge (C_A{}^M + A^M \wedge B_A) + 2Q_{NP}{}^E(D_E{}^{NP} - \tfrac{1}{2}A^N \wedge A^P \wedge B_E) \\ &+2Q^{AB}D_{AB} + 2L_{NPQ}D^{NPQ}\big\}\ ,\end{aligned} \tag{3.4.8}$$

and a general variation of the total action $S = S_0 + \Delta S$ is given by

$$\begin{aligned}\delta S &= \int \Big\{ \delta g^{\mu\nu} \frac{\delta S_0}{\delta g^{\mu\nu}} + \Big(\delta Z^i \frac{\delta S_0}{\delta Z^i} + \text{c.c.}\Big) - \delta A^M \wedge \star \frac{\delta S_0}{\delta A^M} + 2\delta B_A \wedge \star \frac{\delta S_0}{\delta B_A} \\ &\quad + \mathfrak{D}\vartheta_M{}^A \wedge [-2\delta C_A{}^M - 2\delta A^M \wedge B_A - 2A^M \wedge \delta B_A] + Q^{AB}[2\delta D_{AB}] \\ &\quad + Q_{NP}{}^E[2\delta D_E{}^{NP} + 2\delta A^N \wedge C_E{}^P + 2\delta A^{(N} \wedge A^{P)} \wedge B_E - A^N \wedge A^P \wedge \delta B_E] \\ &\quad + L_{NPQ}[2\delta D^{NPQ}] + \delta\vartheta_M{}^A \frac{\delta S}{\delta \vartheta_M{}^A} \Big\} \,. \end{aligned} \tag{3.4.9}$$

The first variation of the total action S with respect to $\vartheta_M{}^A$ can be written in the form

$$\begin{aligned}\tfrac{1}{2}\frac{\delta S}{\delta\vartheta_M{}^A} &= (G_A{}^M - \tfrac{1}{2}\star\partial V/\partial\vartheta_M{}^A) - A^M \wedge (H_A + \tfrac{1}{2}\star j_A) \\ &\quad -\tfrac{1}{2}T_{AN}{}^P A^M \wedge A^N \wedge (F_P - G_P) - (F^M - G^M) \wedge B_A \,. \end{aligned} \tag{3.4.10}$$

We can now check the gauge invariance of the total action S. We are going to use for the gauge transformations of all the fields (except for the scalars and vectors) the Ansatz $\delta_a = \delta_h + \Delta$ where Δ is a piece to be determined. If we assume that the embedding tensor is exactly invariant[17], i.e. $\delta\vartheta_M{}^A = 0$, we find

$$\Delta B_A = -2T_{A\,NP}\Lambda^N (F^P - G^P)\,, \tag{3.4.11}$$

$$\Delta C_A{}^M = \Lambda_A \wedge (F^M - G^M) - \Lambda^M (H_A + \tfrac{1}{2}\star j_A)\,, \tag{3.4.12}$$

$$\Delta D_{AB} = 2\Lambda_{[A} \wedge (H_{B]} + \tfrac{1}{2}\star j_{B]}) - 2T_{[A|\,NP}\Lambda^N (F^P - G^P) \wedge B_{|B]}\,, \tag{3.4.13}$$

$$\Delta D_E{}^{NP} = -\Lambda^N (G_E{}^P - \tfrac{1}{2}\star\partial V/\partial\vartheta_P{}^E) + (F^N - G^N) \wedge \Lambda_E{}^P\,, \tag{3.4.14}$$

$$\begin{aligned}\Delta D^{NPQ} &= -3\delta A^{(N} \wedge A^P \wedge (F^{Q)} - G^{Q)}) + 6\Lambda^{(N} F^P \wedge (F^{Q)} - G^{Q)}) \\ &\quad -3\Lambda^{(N}(F^P - G^P) \wedge (F^{Q)} - G^{Q)})\,, \end{aligned} \tag{3.4.15}$$

[17] One could also allow ϑ_M^A to transform according to its indices as $\delta\vartheta_M^A = -Q_{NM}{}^A\Lambda^N$. This is like adding a term proportional to an equation of motion, that of $D_A{}^{NM}$, to the zero variation.

where we have used in this calculation the non-trivial Ricci identities[18]

$$\begin{aligned}
\vartheta_M{}^C \mathfrak{D}\mathfrak{D}\Lambda_C{}^M &= \mathfrak{D}\vartheta_M{}^A \wedge (-Y_{AP}{}^E A^M \wedge \Lambda_E{}^P) + Q_{NP}{}^E [(F^N - Z^{NA} B_A) \wedge \Lambda_E{}^P \\
&\quad -\tfrac{1}{2} Y_{EQ}{}^C A^N \wedge A^P \wedge \Lambda_C{}^Q] \,, \qquad (3.4.16)\\
\mathfrak{D}\mathfrak{D} F_M &= X_{NPM} F^N \wedge F^P - 2Q^{AB} T_{A\,PM} F^P \wedge B_B + dX_{NPM} \wedge A^N \wedge F^P \\
&\quad -\tfrac{1}{2} Q_{NP}{}^E T_{E\,MQ} A^N \wedge A^P \wedge F^Q \,, \qquad (3.4.17)
\end{aligned}$$

and the variations of the kinetic matrix and the potential Eqs. (3.3.23) and (3.3.24).

We observe that all terms in the extra variations Δ vanish when we use the duality relations (3.3.44). Actually, all of them, except for just one term in ΔD^{NPQ}, are such that the variations δ_a are obtained from the tensor hierarchy variations δ_h simply by replacing the scalar-independent field strengths $F^M, H_A, G_A{}^M$ by the corresponding scalar-dependent objects $G^M, j_A, \dfrac{\partial V}{\partial \vartheta_\Lambda{}^A}$ via the duality relations (3.3.44).

Finally, we note that the variations δ_a and δ_h are equivalent from the point of view of the duality hierarchy. The two sets of transformation rules differ by terms that are proportional to the duality relations. The only difference is that the commutator algebra corresponding to δ_h closes off-shell whereas the algebra corresponding to δ_a closes up to terms that are proportional to the duality relations. The two sets of transformation rules are not equivalent from the action point of view in the sense that only one of them, the one with transformation rules δ_a, leaves the action invariant, whereas the other, with transformations δ_h, does not.

[18] If the constraints are satisfied, $\vartheta_M{}^C \mathfrak{D}\mathfrak{D}\Lambda_C{}^M = \mathfrak{D}\mathfrak{D}(\vartheta_M{}^C \Lambda_C{}^M) = dd(\vartheta_M{}^C \Lambda_C{}^M) = 0$. Therefore, when they are not satisfied, $\vartheta_M{}^C \mathfrak{D}\mathfrak{D}\Lambda_C{}^M$ must be proportional to them.

Chapter 4

Applications: Gauging $N = 1, 2$ Supergravity

In this Chapter we are going to apply the general results of Chapter 3 to specific Supergravity theories, i.e. $N = 1$ (section 4.1) and $N = 2$ (section 4.2) Supergravity. We start with electric gaugings of the perturbative symmetries of matter coupled $N = 1$ Sugra in section 4.1.1 Our next step will be to consider the most general gauging of $N = 1, d = 4$ supergravity, using perturbative and non-perturbative global symmetries and using electric and magnetic vectors (4.1.2). To do so, we introduce magnetic vector fields and magnetic gauginos, in order to have well-defined covariant derivatives acting on the bosonic fields. As was discussed in Chapter 3, general gaugings of four-dimensional Supergravities imply the existence of a complete hierarchy of p-form fields with degrees $p \geq 1$. We are going to find the hierarchy fields predicted by the general $4d$ tensor hierarchy for $N = 1$ Supergravity and their supersymmetry transformations in section 4.1.3. However, we will find some more fields not predicted by the hierarchy and discuss their origin. We will show that the local supersymmetry algebra closes on all these "extensions" of $N = 1$ Supergravity. In section 4.2 we are going to study $N = 2$ $d = 4$ Einstein-Yang-Mills (EYM) Supergravity, i.e. the gaugings of $N = 2$ $d = 4$ Supergravity coupled to non-Abelian vector supermultiplets.

4.1 Gauged $N = 1$ Supergravity

4.1.1 Electric gaugings of perturbative symmetries

We are now going to gauge the symmetries described in the section 2.1.1 using as gauge fields the electric 1-form potentials A^{Λ}. This requires the introduction of the (electric) *embedding tensor* $\vartheta_{\Lambda}{}^{A}$ to indicate which global symmetry T_A is gauged by which gauge field A^{Λ} and, equivalently, to identify the parameters of global symmetries α^A that are going to be promoted to local parameters with the gauge parameters $\Lambda^{\Sigma}(x)$ of the 1-forms:

$$\alpha^{A}(x) \equiv \Lambda^{\Sigma}(x)\vartheta_{\Sigma}{}^{A}\,. \qquad (4.1.1)$$

We will write now the constraint Eq. (2.1.36) in the form[1]

$$(\vartheta_{\Sigma}{}^{\underline{a}}\mathcal{P}_{\underline{a}} + \vartheta_{\Sigma}{}^{\sharp}\mathcal{P}_{\sharp})\mathcal{L} = 0\,. \qquad (4.1.2)$$

Taking into account Eq. (2.1.19) and this definition, the gauge transformations of the complex scalars will be

$$\delta Z^{i} = \Lambda^{\Sigma}\vartheta_{\Sigma}{}^{A}k_{A}{}^{i}\,. \qquad (4.1.3)$$

The embedding tensor cannot be completely arbitrary. To start with, it is clear that it has to be invariant under gauge transformations, which we denote by δ:

$$\delta\vartheta_{\Lambda}{}^{A} = -\Lambda^{\Sigma}Q_{\Sigma\Lambda}{}^{A}\,, \qquad Q_{\Sigma\Lambda}{}^{A} \equiv \vartheta_{\Sigma}{}^{B}T_{B\,\Lambda}{}^{\Omega}\vartheta_{\Omega}{}^{A} - \vartheta_{\Sigma}{}^{B}\vartheta_{\Lambda}{}^{C}f_{BC}{}^{A}\,. \qquad (4.1.4)$$

Then, the embedding tensor has to satisfy the quadratic constraint

$$Q_{\Sigma\Lambda}{}^{A} = 0\,. \qquad (4.1.5)$$

The gauge fields A^{Λ} effectively couple to the generators

$$X_{\Sigma\Omega}{}^{\Gamma} \equiv \vartheta_{\Sigma}{}^{A}T_{A\,\Omega}{}^{\Gamma}\,, \qquad X_{\Sigma\Omega\Gamma} \equiv \vartheta_{\Sigma}{}^{A}T_{A\,\Omega\Gamma}\,, \qquad X_{\Sigma} \equiv \vartheta_{\Sigma}{}^{A}T_{A}\,. \qquad (4.1.6)$$

From the definition of the quadratic constraint Eq. (4.1.5)

$$X_{(\Lambda\Sigma)}{}^{\Omega}\vartheta_{\Omega}{}^{A} = 0\,, \qquad (4.1.7)$$

and so it will vanish, although, in general, we will have

$$X_{(\Lambda\Sigma)}{}^{\Omega} \neq 0\,. \qquad (4.1.8)$$

From the commutator of the matrices T_A and using the quadratic constraint we find the commutator of X generators

$$[X_{\Lambda}, X_{\Sigma}] = -X_{\Lambda\Sigma}{}^{\Omega}X_{\Omega}\,, \qquad (4.1.9)$$

from which we can derive the analogue of the Jacobi identities.

[1] Again, this constraint and other constraints of the same kind that will follow, should be understood as a way to consider the cases $\mathcal{L} = 0$ and $\mathcal{L} \neq 0$ simultaneously: when $\mathcal{L} \neq 0$ the embedding tensor must satisfy $(\vartheta_{\Sigma}{}^{\underline{a}}\mathcal{P}_{\underline{a}} + \vartheta_{\Sigma}{}^{\sharp}\mathcal{P}_{\sharp}) = 0$ and it is unrestricted when $\mathcal{L} = 0$.

We are now ready to gauge the theory. We will not attempt to give the full supersymmetric Lagrangian and supersymmetry transformation rules, but only those elements that allow its construction to lowest order.

First, we have to replace the partial derivatives of the scalars in their kinetic term by the covariant derivatives

$$\mathfrak{D}Z^i \equiv dZ^i + A^\Lambda \vartheta_\Lambda{}^A k_A{}^i \,, \tag{4.1.10}$$

where the gauge potentials transform according to

$$\delta A^\Sigma = -\mathfrak{D}\Lambda^\Sigma \equiv -(d\Lambda^\Sigma + X_{\Lambda\Omega}{}^\Sigma A^\Lambda \Lambda^\Omega) \,. \tag{4.1.11}$$

We also replace in the action the vector field strengths by the gauge-covariant field strengths

$$F^\Sigma = dA^\Sigma + \tfrac{1}{2} X_{\Lambda\Omega}{}^\Sigma A^\Lambda \wedge A^\Omega \,. \tag{4.1.12}$$

Observe that we have not introduced a coupling constant g as it is standard in the literature since the embedding tensor already plays the role of coupling constant and even of different coupling constants if we deal with products of groups. Observe also that $\vartheta_\sharp{}^A$ does not appear in any of these expressions because $K_\sharp = T_\sharp = 0$.

We have to replace the (Kähler- and Lorentz-) covariant derivatives of the spinors in their kinetic terms by gauge-covariant derivatives:

$$\mathfrak{D}_\mu \psi_\nu = \{\mathcal{D}_\mu - \tfrac{i}{2} A^\Lambda{}_\mu \vartheta_\Lambda{}^A \mathcal{P}_A\}\psi_\nu \,, \tag{4.1.13}$$

$$\mathfrak{D}\chi^i = \mathcal{D}\chi^i + \Gamma_{jk}{}^i \mathfrak{D}Z^j \chi^k - A^\Lambda \vartheta_\Lambda{}^A \partial_j k_A{}^i \chi^j + \tfrac{i}{2} A^\Lambda \vartheta_\Lambda{}^A \mathcal{P}_A \chi^i \,, \tag{4.1.14}$$

$$\mathfrak{D}\lambda^\Sigma = \{\mathcal{D} - \tfrac{i}{2} A^\Lambda \vartheta_\Lambda{}^A \mathcal{P}_A\}\lambda^\Sigma - X_{\Lambda\Omega}{}^\Sigma A^\Lambda \lambda^\Omega \,. \tag{4.1.15}$$

The components $\vartheta_\Lambda{}^\sharp$ occur in all these covariant derivatives. The components $\vartheta_\Lambda{}^{\underline{\mathrm{a}}}$ only occur in the last term of $\mathfrak{D}\lambda^\Sigma$.

The supersymmetry transformations of the bosonic fields do not change with the gauging, but those of the fermions do by the addition of a new fermion shift term in the gauginos supersymmetry transformation rule. To first order in fermions, we have

$$\delta_\epsilon \psi_\mu = \mathfrak{D}_\mu \epsilon + i\mathcal{L}\gamma_\mu \epsilon^* \,, \tag{4.1.16}$$

$$\delta_\epsilon \lambda^\Sigma = \tfrac{1}{2}\left[\not{F}^{\Sigma +} + i\mathcal{D}^\Sigma\right]\epsilon \,, \tag{4.1.17}$$

$$\delta_\epsilon \chi^i = i\not{\mathfrak{D}} Z^i \epsilon^* + 2\mathcal{G}^{ij^*}\mathcal{D}_{j^*}\mathcal{L}^*\epsilon \,, \tag{4.1.18}$$

where

$$\mathcal{D}^{\Lambda} \equiv -\Im\mathrm{m}\, f^{\Lambda\Sigma} \vartheta_{\Sigma}{}^{A} \mathcal{P}_{A}\,. \tag{4.1.19}$$

where we use the notation

$$\Im\mathrm{m}\, f^{\Lambda\Sigma} \equiv (\Im\mathrm{m}\, f)^{-1|\Lambda\Sigma}\,. \tag{4.1.20}$$

The new term leads to corrections of the scalar potential of the ungauged theory V_{u}, given in Eq. (2.1.4), which now takes the form

$$V_{\mathrm{eg}} = V_{\mathrm{u}} - \mathcal{D}^{\Lambda} \vartheta_{\Lambda}{}^{A} \mathcal{P}_{A} = V_{\mathrm{u}} + \tfrac{1}{2} \Im\mathrm{m}\, f^{\Lambda\Sigma} \vartheta_{\Lambda}{}^{A} \vartheta_{\Sigma}{}^{B} \mathcal{P}_{A} \mathcal{P}_{B}\,. \tag{4.1.21}$$

The action for the bosonic fields of the $N = 1, d = 4$ gauged supergravity of the kind considered here is obtained by replacing the partial derivatives and field strengths by gauge-covariant derivatives and field strengths, replacing the potential V_{u} by V_{eg} above and by adding a Chern–Simons term [80, 81] which is necessary to make the action gauge invariant

$$\begin{aligned} S_{\mathrm{eg}} \;\; = \;\; & \int \big\{ \star R - 2\mathcal{G}_{ij^*} \mathfrak{D} Z^i \wedge \star \mathfrak{D} Z^{*\,j^*} - 2\Im\mathrm{m} f_{\Lambda\Sigma} F^{\Lambda} \wedge \star F^{\Sigma} + 2\Re\mathrm{e} f_{\Lambda\Sigma} F^{\Lambda} \wedge F^{\Sigma} \\ & - \star\, V_{\mathrm{eg}} - \tfrac{4}{3} X_{\Lambda\Sigma\Omega} A^{\Lambda} \wedge A^{\Sigma} \wedge [dA^{\Omega} + \tfrac{3}{8} X_{\Gamma\Delta}{}^{\Omega} A^{\Gamma} \wedge A^{\Delta}] \big\}\,. \end{aligned} \tag{4.1.22}$$

Gauge-invariance can be achieved only if

$$X_{(\Lambda\Sigma\Omega)} = 0\,, \tag{4.1.23}$$

which is a constraint that also follows from supersymmetry.

4.1.2 General gaugings of $N = 1, d = 4$ supergravity

In this section we will discuss the most general gaugings of $N = 1, d = 4$ supergravity by using as gauge group any subgroup of $G = G_{\mathrm{iso}} \times G_{\mathrm{V}} \times U(1)_R$ that can be embedded into $Sp(2n_{\mathrm{V}}, \mathbb{R})$.

From the purely bosonic point of view it would suffice to use the results of Refs. [15, 33] taking into account the particular structure of the global symmetry group of $N = 1, d = 4$ supergravity. This involves the introduction of new p-form fields $p = 2, 3, 4$ which, together with the electric and magnetic (to be defined) 1-forms of the theory, combined into A^M, constitute the standard 4-dimensional tensor hierarchy, reviewed in Appendices E.1 and E.3. Its field content is

$$\{A^M, B_A, C_A{}^M, D_{AB}, D_E{}^{NP}, D^{NPQ}\}\,.$$

At the level of the action, is is not necessary to introduce all these fields, though. It is enough to introduce the magnetic 1-forms A_Λ and 2-forms B_A.

This procedure, however, must be compatible with $N = 1, d = 4$ supersymmetry. A supersymmetrization of the tensor hierarchy and the action is necessary. The supersymmetrization of the tensor hierarchy is a first step towards the construction of a fully supersymmetry action with electric and magnetic gaugings and this is going to be our goal in this section.

Thus, we are going to repeat the construction of the 4-dimensional tensor hierarchy checking at each step its consistency with $N = 1, d = 4$ supersymmetry: for each new p-form field we will construct a supersymmetry transformation and we will check the closure of the local $N = 1, d = 4$ supersymmetry algebra on it. The commutator of two $N = 1, d = 4$ local supersymmetry transformations acting on bosonic p-form fields is expected to have the general form

$$[\delta_\eta\,, \delta_\epsilon] = \delta_{\text{g.c.t.}} + \delta_{\text{gauge}} + \text{duality relations}\,, \tag{4.1.24}$$

where $\delta_{\text{g.c.t.}}$ is a general coordinate transformation and δ_{gauge} is a gauge transformation that should coincide with the one predicted by the bosonic tensor hierarchy purely on gauge-invariance arguments. We also expect in general additional terms proportional to duality relations between the new fields and the original fields of the ungauged $N = 1, d = 4$ supergravity. These duality relations project the tensor hierarchy onto the physical theory reducing the number of independent fields.

Contrary to that expectation, we are going to see that, at least for some fields, it is possible to construct supersymmetry transformations such that the local $N = 1, d = 4$ supersymmetry algebra closes without the use of any duality relation, i.e.

$$[\delta_\eta\,, \delta_\epsilon] = \delta_{\text{g.c.t.}} + \delta_{\text{gauge}}\,. \tag{4.1.25}$$

To make this possible we will have to introduce the additional p-form fields of the tensor hierarchy in supermultiplets constructing, as a matter of fact, a supersymmetric tensor hierarchy. Now, to project the supersymmetric tensor hierarchy onto the physical theory we will use duality relations both for the bosons and fermions.

We have succeeded in supersymmetrizing in this way the hierarchy up to 2-forms (which requires the introduction of linear multiplets) but these results strongly indicate that the same should be possible for all p-forms in the tensor hierarchy.

Studying the closure of the local $N = 1, d = 4$ supersymmetry algebra we are going to see that it is necessary to add more bosonic p-form fields to the standard tensor hierarchy. The main reason for this is the existence of the constraint Eq. (4.1.2) which will be conveniently generalized to the electric-magnetic case in Eq. (4.1.40). This constraint restricts simultaneously the terms $\mathcal{P}_{\underline{\text{a}}}, \mathcal{P}_\sharp$ and the symmetries that can be gauged and reflects the breaking of the $U(1)_R$ symmetry by the presence of a non-vanishing superpotential $\mathcal{L}$.

The breaking of this symmetry will manifest itself in the existence of a new Stückelberg shift of the 2-forms $B_{\underline{\text{a}}}, B_\sharp$

$$\delta B_{\underline{\mathrm{a}}} \sim \mathcal{P}_{\underline{\mathrm{a}}}\Lambda\,, \qquad \delta B_{\sharp} \sim \mathcal{P}_{\sharp}\Lambda\,, \tag{4.1.26}$$

where Λ is a 2-form that appears whenever $\mathcal{L} \neq 0$. We can only find this shift by studying the closure of the local supersymmetry algebra. Therefore, it is necessary to simultaneously construct the tensor hierarchy and study its supersymmetrization.

To construct the respective gauge-covariant 3-form field strengths $H_{\underline{\mathrm{a}}}, H_{\sharp}$ the existence of one new 3-form C is required. We will find consistent supersymmetry transformations for the needed 3-form C and also for another 3-form C' and for a set of 4-forms D^M. The extended hierarchy of $N = 1, d = 4$ supergravity will, thus, have the total bosonic field content

$$\{A^M, B_A, C_A{}^M, C, C', D_{AB}, D_E{}^{NP}, D^{NPQ}, D^M\}\,.$$

We now want to consider the most general gauging of $N = 1, d = 4$ supergravity, using perturbative (see section 2.1.1) and non-perturbative (see section 2.1.2) global symmetries and using electric and magnetic vectors, to be introduced next. In the ungauged theory we can introduce n_V 1-form potentials A_Λ and their field strengths $F_\Lambda = dA_\Lambda$. The Maxwell equations can be replaced by the first-order duality relation

$$G_\Lambda = F_\Lambda\,, \tag{4.1.27}$$

since now the Bianchi identity $dF_\Lambda = 0$ implies the standard Maxwell equation $dG_\Lambda = 0$. The magnetic vectors A_Λ will be introduced in the theory as auxiliary fields and we will study them from the supersymmetry point of view later on. The electric A^Λ and magnetic A_Λ vectors will be combined into a symplectic vector A^M

$$A^M \equiv \begin{pmatrix} A^\Lambda \\ A_\Lambda \end{pmatrix}, \qquad A_M \equiv \Omega_{MN}A^N = (A_\Lambda\,, -A^\Lambda)\,, \qquad A^M = A_N\Omega^{NM}\,, \tag{4.1.28}$$

and used as the gauge fields of the symmetries described in the previous subsection.

In order to use all the 1-forms A^M as gauge fields we need to add a magnetic component to the embedding tensor, which becomes a covariant symplectic vector

$$\vartheta_M{}^A \equiv (\vartheta^{\Lambda A}\,, \vartheta_\Lambda{}^A\,)\,, \tag{4.1.29}$$

where the index A ranges over all the generators of $G = G_{\text{bos}} \times U(1)_R$, so we have now

$$\alpha^A(x) \equiv \Lambda^M(x)\vartheta_M{}^A\,, \tag{4.1.30}$$

and the gauge transformations of the complex scalars, for instance, take the form

$$\delta Z^i = \Lambda^M \vartheta_M{}^A k_A{}^i\,. \tag{4.1.31}$$

The embedding tensor, then, provides an embedding of the gauge group into the group $Sp(2n_V, \mathbb{R})$ which acts on the vectors. If the global symmetry group is bigger than $Sp(2n_V, \mathbb{R})$ we will not be able to gauge it completely. Further constraints will decrease the rank of the group that we can actually gauge.

For instance, we must impose the constraint

$$Q^{AB} \equiv \tfrac{1}{4}\vartheta^{[A|M}\vartheta^{B]}{}_M = 0\,, \quad \Rightarrow \quad \vartheta^{AM}\vartheta_M{}^B = 0\,, \tag{4.1.32}$$

which guarantees that the electric and magnetic gaugings are mutually local [15] and we can go to a theory with only purely electric gaugings by a symplectic transformation.

The embedding tensor must satisfy further conditions. We define the matrices

$$X_{MN}{}^P \equiv \vartheta_M{}^A T_{A\,N}{}^P\,, \tag{4.1.33}$$

which satisfy

$$X_{MNP} = X_{MPN}\,, \tag{4.1.34}$$

on account of Eq. (2.1.51). Observe that the components $\vartheta_M{}^\sharp$ are no present in the X_{MNP} tensors. Further, we impose the quadratic constraint[2]

$$Q_{NM}{}^A \equiv \vartheta_N{}^A T_{A\,M}{}^P \vartheta_P{}^A - \vartheta_N{}^A \vartheta_M{}^B f_{AB}{}^A = 0\,, \tag{4.1.35}$$

to ensure invariance of $\vartheta_M{}^A$ and the *representation constraint* [15]

$$L_{MNP} \equiv X_{(MNP)} = X_{(MN}{}^Q \Omega_{P)Q} = 0\,. \tag{4.1.36}$$

This constraint is required by gauge invariance and supersymmetry[3]. It implies Eq. (4.1.23) and also

$$X_{(MN)P} = -\tfrac{1}{2}X_{PMN} \;\; \Rightarrow X_{(MN)}{}^P = Z^{PA}T_{AMN}\,, \tag{4.1.37}$$

where we have defined

$$Z^{PA} \equiv -\tfrac{1}{2}\Omega^{NP}\vartheta_N{}^A\,. \tag{4.1.38}$$

This definition and that of the other projectors that appear in the 4-dimensional hierarchy are collected in Appendix E.1. The tensor Z^{PA} will be used to project in directions orthogonal to the embedding tensor since, due to the first quadratic constraint Eq. (3.2.10)

$$Z^{MA}\vartheta_M{}^B = 0\,. \tag{4.1.39}$$

[2] Observe that $\vartheta_M{}^\sharp$ does not occur in $Q_{NM}{}^A$ either.

[3] In Ref. [77] it has been shown how this constraint gets modified in the presence of anomalies and the modifications can cancel exactly the lack of gauge invariance of the classical action.

Finally, it should be clear that the constraint Eq. (4.1.2) on the triple product of embedding tensor, momentum maps and superpotential should be generalized to

$$(\vartheta_M{}^{\underline{a}}\mathcal{P}_{\underline{a}} + \vartheta_M{}^{\sharp}\mathcal{P}_{\sharp})\mathcal{L} = 0\,. \tag{4.1.40}$$

With these properties we can define gauge-covariant derivatives of objects transforming according to $\delta\phi = \Lambda^M \delta_M \phi$ by

$$\mathfrak{D}\phi = d\phi + A^M \delta_M \phi\,. \tag{4.1.41}$$

if the gauge fields transform according to

$$\delta A^M = -\mathfrak{D}\Lambda^M + \Delta A^M = -(d\Lambda^M + X_{NP}{}^M A^N \Lambda^P) + \Delta A^M\,, \tag{4.1.42}$$

where ΔA^M is a piece that we can add to this gauge transformation if it satisfies

$$\vartheta_M{}^A \Delta A^M = 0\,. \tag{4.1.43}$$

The covariant derivatives of the scalars, gravitino and chiralinos read

$$\mathfrak{D}Z^i = dZ^i + A^M \vartheta_M{}^A k_A{}^i\,, \tag{4.1.44}$$

$$\mathfrak{D}_\mu \psi_\nu = \{\mathcal{D}_\mu - \tfrac{i}{2} A^M{}_\mu \vartheta_M{}^A \mathcal{P}_A\}\psi_\nu\,, \tag{4.1.45}$$

$$\mathfrak{D}\chi^i = \mathcal{D}\chi^i + \Gamma_{jk}{}^i \mathfrak{D}Z^j \chi^k - A^M \vartheta_M{}^A \partial_j k_a{}^i \chi^j + \tfrac{i}{2} A^M \vartheta_M{}^A \mathcal{P}_A \chi^i\,. \tag{4.1.46}$$

Observe that ΔA^M drops automatically from the gauge transformations of these expressions because A^M always comes projected by $\vartheta_M{}^A$.

It is clear that we need to introduce auxiliary "magnetic gauginos" λ_Λ in order to construct a symplectic vector of gauginos λ^M whose covariant derivative is

$$\mathfrak{D}\lambda^M = \{\mathcal{D} - \tfrac{i}{2} A^N \vartheta_N{}^A \mathcal{P}_A\}\lambda^M - X_{NP}{}^M A^N \lambda^P\,. \tag{4.1.47}$$

The magnetic gauginos are the supersymmetric partners of the magnetic 1-forms. We will discuss their supersymmetry transformation rules later.

So far, to introduce the general 4-dimensional embedding-tensor formalism we have introduced magnetic 1-forms A_Λ and gauginos λ_Λ. As discussed at the beginning of this section, we have to find supersymmetry transformations for them and check the closure of the local $N = 1, d = 4$ supersymmetry algebra.

4.1.3 Supersymmetric tensor hierarchy of $N = 1$, $d = 4$ supergravity

Before we deal with the supersymmetry transformations of the magnetic 1-forms that we have introduced, we take one step back and study the closure of the local $N = 1, d = 4$ supersymmetry algebra on the 0-forms.

The scalars Z^i

Their supersymmetry transformations are given by Eq. (2.1.12), which we rewrite here for convenience:

$$\delta_\epsilon Z^i = \tfrac{1}{4}\bar{\chi}^i\epsilon\,. \tag{4.1.48}$$

At leading order in fermions,

$$\delta_\eta\delta_\epsilon Z^i = \tfrac{1}{4}\overline{(\delta_\eta\chi^i)}\epsilon\,, \tag{4.1.49}$$

and all we need is the supersymmetry transformation for χ^i. This is given in Eq. (4.1.18), which we also rewrite here

$$\delta_\eta\chi^i = i\,\not{\mathfrak{D}}Z^i\eta^* + 2\mathcal{G}^{ij^*}\mathcal{D}_{j^*}\mathcal{L}^*\eta\,, \tag{4.1.50}$$

where we have to take into account that the covariant derivative $\mathfrak{D}Z^i$ is now given by Eq. (4.1.44). We get

$$[\delta_\eta\,,\,\delta_\epsilon]Z^i = \delta_{\text{g.c.t.}}Z^i + \delta_h Z^i\,, \tag{4.1.51}$$

where $\delta_{\text{g.c.t.}}Z^i$ is a g.c.t. with infinitesimal parameter ξ^μ

$$\delta_{\text{g.c.t.}}Z^i \quad = \quad \pounds_\xi Z^i = +\xi^\mu\partial_\mu Z^i\,, \tag{4.1.52}$$

$$\xi^\mu \quad \equiv \quad \tfrac{i}{4}(\bar{\epsilon}\gamma^\mu\eta^* - \bar{\eta}\gamma^\mu\epsilon^*)\,, \tag{4.1.53}$$

and where $\delta_h Z^i$ is the gauge transformation Eq. (4.1.31) with gauge parameter Λ^M

$$\delta Z^i \quad = \quad \Lambda^M\vartheta_M{}^A k_A{}^i\,, \tag{4.1.54}$$

$$\Lambda^M \quad \equiv \quad \xi^\mu A^M{}_\mu\,. \tag{4.1.55}$$

This is just a small generalization of the standard result in which electric and magnetic gauge parameters appear. As expected, no duality relations are required to close the local supersymmetry algebra on the Z^i.

The 1-form fields A^M

As we have mentioned before, to define supersymmetry transformations for the magnetic vectors A_Λ it is convenient to introduce simultaneously magnetic gauginos[4] λ_Λ. This is equivalent to introducing n_V auxiliary vector supermultiplets. Symplectic covariance suggests that we can write the following supersymmetry transformation rules for the electric and magnetic 1-forms and gauginos:

$$\delta_\epsilon A^M{}_\mu = -\tfrac{i}{8}\bar{\epsilon}^*\gamma_\mu\lambda^M + \text{c.c.}\,, \tag{4.1.56}$$

$$\delta_\epsilon \lambda^M = \tfrac{1}{2}\left[\not{F}^{M+} + i\mathcal{D}^M\right]\epsilon\,, \tag{4.1.57}$$

where F^M is the gauge-covariant 2-form field strength of A^M, to be defined shortly. and where we have defined the symplectic vector

$$\mathcal{D}^M \equiv \begin{pmatrix} \mathcal{D}^\Lambda \\ \mathcal{D}_\Lambda \end{pmatrix} \equiv \begin{pmatrix} \mathcal{D}^\Lambda \\ f_{\Lambda\Sigma}\mathcal{D}^\Sigma \end{pmatrix}, \tag{4.1.58}$$

where now, the electric $\mathcal{D}^\Lambda$ has been redefined, with respect to the purely electric gauging case, to include a term with the magnetic component of the embedding tensor $\vartheta^{\Lambda A}$:

$$\mathcal{D}^\Lambda = -\Im\text{m} f^{\Lambda\Sigma}\,(\vartheta_\Sigma{}^A + f^*_{\Sigma\Omega}\vartheta^{\Omega A})\mathcal{P}_A\,. \tag{4.1.59}$$

Although at this point we do not need it, it is important to observe that there is a duality relation between the magnetic gauginos and the electric ones

$$\lambda_\Lambda = f_{\Lambda\Sigma}\lambda^\Sigma\,. \tag{4.1.60}$$

The gaugino duality relation is local and takes the same form as the duality relation between the magnetic and the electric vector field strengths:

$$F_\Lambda{}^+ = f_{\Lambda\Sigma}F^{\Sigma\,+}\,, \tag{4.1.61}$$

which is obtained from the duality between electric and magnetic vectors $F_\Lambda = G_\Lambda$, combined with Eq. (2.1.43). These duality relations relate the supersymmetry transformation $\delta_\epsilon\lambda^\Lambda$ to $\delta_\epsilon\lambda_\Lambda$.

Now we can check the closure of the local supersymmetry algebra on A^M. It is, however, convenient to know which kind of gauge transformations with should expect in the right hand side. The gauge transformations of A^M are given in Eq. (3.2.31) up to a term ΔA^M which is determined in the construction of its gauge-covariant field strength F^M. This term is also needed to have well-defined supersymmetry transformations for all the gauginos.

[4] Magnetic gauginos have also been introduced in Ref. [79].

As shown in Ref. [15], this requires the introduction of a set of 2-forms B_A in F^M, which takes the form

$$F^M = dA^M + \tfrac{1}{2}X_{[NP]}{}^M A^N \wedge A^P + Z^{MA}B_A\,, \tag{4.1.62}$$

and is gauge-covariant under the transformations[5]

$$\delta_h A^M \quad = \quad -\mathfrak{D}\Lambda^M - Z^{MA}\Lambda_A\,, \tag{4.1.63}$$

$$\delta_h B_A \quad = \quad \mathfrak{D}\Lambda_A + 2T_{A\,NP}[\Lambda^N F^P + \tfrac{1}{2}A^N \wedge \delta_h A^P] + \Delta B_A\,, \tag{4.1.64}$$

where

$$Z^{MA}\Delta B_A = 0\,. \tag{4.1.65}$$

Let us now compute the commutator of two supersymmetry transformations on A^M. To leading order in fermions, Eq. (4.1.56) gives

$$\delta_\eta\delta_\epsilon A^M = -\tfrac{i}{8}\bar{\epsilon}^*\gamma_\mu\delta_\eta\lambda^M + \text{c.c.} \tag{4.1.66}$$

Using Eq. (4.1.57) with the parameter η, we find

$$[\delta_\eta\,,\,\delta_\epsilon]A^M = \xi^\nu F^M{}_{\nu\mu} + Z^{MA}\mathcal{P}_A\xi_\mu\,, \tag{4.1.67}$$

where ξ^μ is given by Eq. (4.1.53) and we have used

$$\Im\mathrm{m}\mathcal{D}^M = 2Z^{MA}\mathcal{P}_A\,, \tag{4.1.68}$$

which follows from the definitions Eqs. (4.1.58), (4.1.59) and (E.1.1). We always expect a general coordinate transformation on the right hand side of the form

$$\delta_{\text{g.c.t.}}A^M{}_\mu = \pounds_\xi A^M{}_\mu = \xi^\mu\partial_\mu A^M{}_\mu + \partial_\mu\xi^\mu A^M{}_\mu\,. \tag{4.1.69}$$

Using the explicit form of the field strength F^M Eq. (3.2.36) we can rewrite it as

$$\delta_{\text{g.c.t.}}A^M{}_\mu = \xi^\mu F^M{}_{\mu\nu} + \mathfrak{D}_\mu(A^M{}_\nu\xi^\nu) + Z^{MA}[B_{A\mu\nu}\xi^\nu - T_{A\,NP}A^N{}_\mu A^P{}_\nu\xi^\nu]\,. \tag{4.1.70}$$

Using this expression in the commutator and the definition Eq. (4.1.55) of the gauge parameter Λ^M, we arrive at

$$[\delta_\eta\,,\,\delta_\epsilon]A^M = \delta_{\text{g.c.t.}}A^M + \delta_h A^M\,, \tag{4.1.71}$$

where, in complete agreement with the tensor hierarchy, $\delta_h A^M$ is the gauge transformation in Eq. (4.1.63) with the 1-form gauge parameter Λ_A given by

[5] The label h in the gauge transformations will be explained soon.

$$\Lambda_A \equiv -T_{A\,MN} A^N \Lambda^M + b_A - \mathcal{P}_A \xi \,, \tag{4.1.72}$$

$$b_{A\,\mu} \equiv B_{A\,\mu\nu} \xi^\nu \,. \tag{4.1.73}$$

Observe that no duality relation has been needed to close the local supersymmetry algebra on the magnetic vector fields. This result is a consequence of using fully independent magnetic gauginos as supersymmetric partners of the magnetic vector fields, i.e. transforming as $\delta_\epsilon \lambda_\Sigma \sim \not{F}_\Sigma{}^+$ instead of $\delta_\epsilon \lambda_\Sigma \sim \not{G}_\Sigma{}^+$. In the later case we would have gotten additional $G_\Sigma - F_\Sigma$ terms to be cancelled by using the duality relation.

The 2-form fields B_A

In order to have a gauge-covariant field strength F^M for the 1-forms we have been forced to introduce a set of 2-forms B_A and now we want to study the consistency of this addition to the theory from the point of view of supersymmetry and gauge invariance. We will first study the closure of the supersymmetry algebra on the 2-forms B_A without introducing its supersymmetric partners and, later on, we will introduce the 2-forms as components of linear supermultiplets. In the first case, the local $N = 1, d = 4$ supersymmetry algebra will close up to the use of duality relations while in the second case it will close exactly.

It is useful to know beforehand what to expect in the right hand side of the commutator of supersymmetry transformations acting on the 2-forms B_A. The gauge transformations of the 2-forms are given in Eq. (4.1.64) up to a term ΔB_A which is constraint to satisfy $Z^{MA} \Delta B_A = 0$. In Ref. ([33]) it was found that, in general,

$$\Delta B_A = -Y_{AM}{}^C \Lambda_C{}^M \,, \tag{4.1.74}$$

for some 2-form parameters $\Lambda_C{}^M$. $Y_{AM}{}^C$ is the projector given in Eq. (3.2.44) and is annihilated by Z^{NA} in virtue of the quadratic constraint Eq. (4.1.5) (see Eq. (E.1.6)), as required by the gauge-covariance of F^M. $Y_{AM}{}^C$ is the only tensor with this property in generic 4-dimensional theories in which we can only use the constraint $Q_{NP}{}^E = 0$. At this point we have to remind ourselves that in $N = 1, d = 4$ supergravity there is another constraint that may be used, given in Eq. (4.1.40). To confirm it we need to compute the commutator of supersymmetry transformations on B_A.

In any case, the generic tensor hierarchy prediction is that, with the gauge transformations Eq. (E.3.2), which we rewrite here

$$\delta_h B_A = \mathfrak{D} \Lambda_A + 2 T_{A\,NP} [\Lambda^N F^P + \tfrac{1}{2} A^N \wedge \delta_h A^P] - Y_{AM}{}^C \Lambda_C{}^M \,, \tag{4.1.75}$$

the gauge-covariant field strength of B_A is as given in Eq. (E.3.7)

$$H_A = \mathfrak{D}B_A + T_{A\,RS}A^R \wedge [dA^S + \tfrac{1}{3}X_{NP}{}^S A^N \wedge A^P] + Y_{AM}{}^C C_C{}^M\,, \qquad (4.1.76)$$

where $C_C{}^M$ is a 3-form whose gauge transformations are determined to be

$$\delta_h C_C{}^M = \mathfrak{D}\Lambda_C{}^M - F^M \wedge \Lambda_C - \delta_h A^M \wedge B_C - \tfrac{1}{3}T_{C\,NP}A^M \wedge A^N \wedge \delta_h A^P + \Lambda^M H_C + \Delta C_C{}^M\,, \qquad (4.1.77)$$

where

$$Y_{AM}{}^C \Delta C_C{}^M = 0\,. \qquad (4.1.78)$$

Another constraint would mean that one more 2-form shift can be added to $\delta_h B_A$ and, correspondingly, another 3-form C must appear in H_A. We are going to see that this is indeed what supersymmetry implies.

Inspired by the results of Ref. [28], we found that, for the 2-forms B_A, the supersymmetry transformation is given by

$$\delta_\epsilon B_{A\mu\nu} = \tfrac{1}{4}[\partial_i \mathcal{P}_A \bar{\epsilon}\gamma_{\mu\nu}\chi^i + \text{c.c.}] + \tfrac{i}{2}[\mathcal{P}_A \bar{\epsilon}^* \gamma_{[\mu}\psi_{\nu]} - \text{c.c.}] + 2T_{AMN}A^M{}_{[\mu}\delta_\epsilon A^N{}_{\nu]}\,. \qquad (4.1.79)$$

The commutator of two of these supersymmetry transformations closes up to a duality relation to be described later on a general coordinate transformation plus a gauge transformation of the form

$$\delta'_h B_A = \delta_h B_A - (\delta_A{}^{\underline{a}}\mathcal{P}_{\underline{a}} + \delta_A{}^\sharp \mathcal{P}_\sharp)\Lambda\,, \qquad (4.1.80)$$

where $\delta_h B_A$ is the standard hierarchy's gauge transformation Eq. (E.3.2) with the 2-form parameters Λ and $\Lambda_C{}^M$ given by

$$\Lambda_C{}^M \equiv -\Lambda^M B_C - c_C{}^M - \tfrac{1}{3}T_{CQP}\Lambda^P A^M \wedge A^Q\,, \qquad (4.1.81)$$

$$\Lambda \equiv -c + 2\Re\text{e}(\phi\mathcal{L})\,, \qquad (4.1.82)$$

$$\phi_{\mu\nu} \equiv \bar{\epsilon}^*\gamma_{\mu\nu}\eta^* = -\bar{\eta}^*\gamma_{\mu\nu}\epsilon^*\,, \qquad (4.1.83)$$

$$c_C{}^M{}_{\mu\nu} \equiv C_C{}^M{}_{\mu\nu\rho}\xi^\rho\,, \qquad (4.1.84)$$

$$c_{\mu\nu} \equiv C_{\mu\nu\rho}\xi^\rho\,, \qquad (4.1.85)$$

and where the parameters Λ^M and Λ_A are, again, given by Eqs. (4.1.55) and (4.1.72) respectively. We have introduced the anticipated 3-form C with the gauge transformation

$$\delta'_h C = -d\Lambda\,, \tag{4.1.86}$$

to take care of the Stückelberg shift parameter Λ. Strictly speaking we only need to introduce C when $\mathcal{L} \neq 0$, so, according to the constraint Eq. (4.1.40) $(\vartheta_M{}^{\underline{a}}\mathcal{P}_{\underline{a}} + \vartheta_M{}^{\sharp}\mathcal{P}_{\sharp}) = 0$. We can express this as a "constraint"

$$(\vartheta_M{}^{\underline{a}}\mathcal{P}_{\underline{a}} + \vartheta_M{}^{\sharp}\mathcal{P}_{\sharp})C = 0\,. \tag{4.1.87}$$

so

$$(\vartheta_M{}^{\underline{a}}\mathcal{P}_{\underline{a}} + \vartheta_M{}^{\sharp}\mathcal{P}_{\sharp})\Lambda = 0\,, \tag{4.1.88}$$

This constraint and Eq. (4.1.40) ensure that $Z^{MA}\Delta B_A = 0$ and F^M remains gauge-covariant under $\delta'_h B_A$.

The hierarchy's gauge-covariant field strength H_A given in Eq. (E.3.7) has to be modified:

$$H'_A \equiv H_A - (\delta_A{}^{\underline{a}}\mathcal{P}_{\underline{a}} + \delta_A{}^{\sharp}\mathcal{P}_{\sharp})C\,, \tag{4.1.89}$$

and the duality constraint that has to be imposed in order to close the local supersymmetry algebra reads

$$H_A = -\tfrac{1}{2} \star j_A\,, \tag{4.1.90}$$

where

$$j_A \equiv 2k^*_{A\,i}\mathfrak{D}Z^i + \text{c.c.}\,, \tag{4.1.91}$$

is the covariant Noether current 1-form. Observe that it vanishes for $A = \underline{a}, \sharp$. For these case we expect to have currents bilinear in fermions which cannot appear at the order in fermions we are working at.

Technically, the difference between the cases $A = \mathbf{a}$ and $A = \underline{a}, \sharp$ lies in the fact that the identity

$$\partial^{i^*}\mathcal{P}_{\mathbf{a}}\mathcal{D}_{i^*}\mathcal{L}^* - \mathcal{P}_{\mathbf{a}}\mathcal{L}^* = 0\,, \tag{4.1.92}$$

which is crucial to cancel terms coming from the supersymmetry variation of the first and second terms of Eq. (4.1.79) cannot be extended to the cases $A = \underline{a}, \sharp$ in which we have introduced *fake* (vanishing) Killing vectors.

The supermultiplet of B_A

We are now going to show that if we add to the tensor hierarchy full linear multiplets[6] $\{B_{A\,\mu\nu}, \varphi_A, \zeta_A\}$ where φ_A is a real scalar and ζ_A is a Weyl spinor, instead of just the 2-forms B_A, as in the preceding section, we can close the local $N = 1, d = 4$ supersymmetry algebra on the 2-forms exactly without the use of the duality relation Eq. (4.1.90).

We will construct the supersymmetry rules of the linear supermultiplet first for the case $A = \mathbf{a}$ after which this result will be generalized to include also the cases $A = \underline{a}, \sharp$. The above supersymmetry transformation rule Eq. (4.1.79) suggests the fermionic duality rule

$$\zeta_{\mathbf{a}} = \partial_i \mathcal{P}_{\mathbf{a}} \chi^i = i k^*_{\mathbf{a}\,i} \chi^i \,, \tag{4.1.93}$$

so we would have

$$\delta_\epsilon B_{\mathbf{a}\mu\nu} = \tfrac{1}{4}[\bar{\epsilon}\gamma_{\mu\nu}\zeta_{\mathbf{a}} + \text{c.c.}] + \tfrac{i}{2}[\mathcal{P}_{\mathbf{a}}\bar{\epsilon}^*\gamma_{[\mu}\psi_{\nu]} - \text{c.c.}] + 2T_{\mathbf{a}\,MN}A^M{}_{[\mu}\delta_\epsilon A^N{}_{\nu]} \,. \tag{4.1.94}$$

The supersymmetry transformation rule of $\zeta_{\mathbf{a}}$ follows from the above duality rule:

$$\delta_\epsilon \zeta_{\mathbf{a}} = i k^*_{\mathbf{a}\,i}\delta_\epsilon\chi^i = -k^*_{\mathbf{a}\,i}\,\not{\mathfrak{D}} Z^i \epsilon^* + 2\partial_i\mathcal{P}_{\mathbf{a}}\mathcal{G}^{ij^*}\mathcal{D}_{j^*}\mathcal{L}^*\epsilon \,. \tag{4.1.95}$$

Using next the duality rule Eq. (4.1.90) $j_{\mathbf{a}} = 4\Re\mathrm{e}(k^*_{\mathbf{a}\,i}\mathfrak{D}Z^i) = -2\star H_{\mathbf{a}}$ we find

$$\delta_\epsilon\zeta_{\mathbf{a}} = -i[\tfrac{i}{12}\not{H}_{\mathbf{a}} + \Im\mathrm{m}(k^*_{\mathbf{a}\,i}\mathfrak{D}_\mu Z^i)\gamma^\mu]\epsilon^* + 2\mathcal{P}_{\mathbf{a}}\mathcal{L}^*\epsilon \,. \tag{4.1.96}$$

To make contact with the standard linear multiplet supersymmetry transformations we should be able to identify consistently

$$\Im\mathrm{m}(k^*_{\mathbf{a}\,i}\mathfrak{D}Z^i) \equiv \mathfrak{D}\varphi_{\mathbf{a}} \,, \tag{4.1.97}$$

for some real scalar $\varphi_{\mathbf{a}}$. The integrability condition of this equation can be obtained by acting with $\mathfrak{D}$ on both sides. Using on the l.h.s. the property

$$\mathfrak{D}k^*_{\mathbf{a}\,i} = \mathfrak{D}Z^{*j^*}\nabla_{j^*}k^*_{\mathbf{a}\,i} \,, \tag{4.1.98}$$

and the Killing property, the integrability condition takes the form

$$-iF^M\vartheta_M{}^{\mathbf{b}}k^*_{[\mathbf{a}|i}k_{|\mathbf{b}]}{}^i = f_{\mathbf{ab}}{}^{\mathbf{c}}F^M\vartheta_M{}^{\mathbf{b}}\varphi_{\mathbf{c}} \,, \tag{4.1.99}$$

which is solved by

$$-ik^*_{[\mathbf{a}|i}k_{|\mathbf{b}]}{}^i = f_{\mathbf{ab}}{}^{\mathbf{c}}\varphi_{\mathbf{c}} \,. \tag{4.1.100}$$

[6] Similar supermultiplets have been introduced in electro-magnetically gauged globally supersymmetric $N = 2, d = 4$ field theory [79].

Given that the Killing vectors can be derived from the Killing prepotential $\mathcal{P}_{\mathbf{a}}$ which is equivariant, it follows that

$$k^*_{[\mathbf{a}|i}k_{|\mathbf{b}]}{}^i = \tfrac{i}{2}\pounds_{\mathbf{a}}\mathcal{P}_{\mathbf{b}} = -\tfrac{i}{2}f_{\mathbf{ab}}{}^{\mathbf{c}}\mathcal{P}_{\mathbf{c}}\,, \tag{4.1.101}$$

and we can finally identify

$$\Im\mathrm{m}(k^*_{\mathbf{a}\,i}\mathfrak{D}Z^i) = -\tfrac{1}{2}\mathfrak{D}\mathcal{P}_{\mathbf{a}}\,. \tag{4.1.102}$$

The supersymmetry transformations of the linear multiplet $\{B_{\mathbf{a}\,\mu\nu}, \varphi_{\mathbf{a}}, \zeta_{\mathbf{a}}\}$ are given by

$$\begin{aligned}
\delta_\epsilon\zeta_{\mathbf{a}} &= -i[\tfrac{1}{12}\not{H}_{\mathbf{a}} + \not{\mathfrak{D}}\varphi_{\mathbf{a}}]\epsilon^* - 4\varphi_{\mathbf{a}}\mathcal{L}^*\epsilon\,, && (4.1.103)\\
\delta_\epsilon B_{\mathbf{a}\mu\nu} &= \tfrac{1}{4}[\bar\epsilon\gamma_{\mu\nu}\zeta_{\mathbf{a}} + \text{c.c.}] - i[\varphi_{\mathbf{a}}\bar\epsilon^*\gamma_{[\mu}\psi_{\nu]} - \text{c.c.}] + 2T_{\mathbf{a}\,MN}A^M{}_{[\mu}\delta_\epsilon A^N{}_{\nu]}\,, && (4.1.104)\\
\delta_\epsilon\varphi_{\mathbf{a}} &= -\tfrac{1}{8}\bar\zeta_{\mathbf{a}}\epsilon + \text{c.c.}\,. && (4.1.105)
\end{aligned}$$

The duality relations needed to relate these fields to the fundamental fields of the $N=1, d=4$ gauged supergravity are

$$\begin{aligned}
\zeta_{\mathbf{a}} &= \partial_i\mathcal{P}_{\mathbf{a}}\chi^i\,, && (4.1.106)\\
H_{\mathbf{a}} &= -\tfrac{1}{2}\star j_{\mathbf{a}}\,, && (4.1.107)\\
\varphi_{\mathbf{a}} &= -\tfrac{1}{2}\mathcal{P}_{\mathbf{a}}\,. && (4.1.108)
\end{aligned}$$

The supersymmetry algebra closes on all the fields of the linear multiplet without the use of any duality relation.

Now that we know the supersymmetry transformation rules for $A = \mathbf{a}$ we will generalize them to all values of A. The supersymmetry transformations of the linear multiplet $\{B_{A\,\mu\nu}, \varphi_A, \zeta_A\}$ are given by

$$\begin{aligned}
\delta_\epsilon\zeta_A &= -i[\tfrac{1}{12}\not{H}'_A + \not{\mathfrak{D}}\varphi_A]\epsilon^* - 4\delta_A{}^{\mathbf{a}}\varphi_{\mathbf{a}}\mathcal{L}^*\epsilon\,, && (4.1.109)\\
\delta_\epsilon B_{A\mu\nu} &= \tfrac{1}{4}[\bar\epsilon\gamma_{\mu\nu}\zeta_A + \text{c.c.}] - i[\varphi_A\bar\epsilon^*\gamma_{[\mu}\psi_{\nu]} - \text{c.c.}] + 2T_{A\,MN}A^M{}_{[\mu}\delta_\epsilon A^N{}_{\nu]}\,, && (4.1.110)\\
\delta_\epsilon\varphi_A &= -\tfrac{1}{8}\bar\zeta_A\epsilon + \text{c.c.}\,. && (4.1.111)
\end{aligned}$$

The duality relations that project these fields onto those of the physical one are

$$\zeta_A = \partial_i \mathcal{P}_{\mathbf{a}} \chi^i \, , \, , \tag{4.1.112}$$

$$H'_A = -\tfrac{1}{2} \star j_A \, , \tag{4.1.113}$$

$$\varphi_A = -\tfrac{1}{2} \mathcal{P}_A \, . \tag{4.1.114}$$

Observe that some terms on the right hand side are zero for $A = \underline{a}, \sharp$, at least to leading order in fermions.

Now the gauge parameters that appear on the right hand side of the commutator of two supersymmetry transformations are different from those we found in the previous section and, therefore, do not match with those we found in the case of the 1-forms. To relate the parameters of the supersymmetry algebra in the case with and without the linear supermultiplets we also need to use the above duality relations. For instance, Λ_A is given by Eq. (4.1.72) with $\mathcal{P}_A$ replaced by $-2\varphi_A$. This means that, in order to supersymmetrize consistently the tensor hierarchy we also must replace $\mathcal{P}_A$ by $-2\varphi_A$ in the supersymmetry transformation rules of the gauginos Eq. (4.1.57) (i.e. in the definition of $\mathcal{D}^M$ Eqs. (4.1.58) and (4.1.59)). There are furthermore also 3-forms contained in the transformations rule for ζ_A. Thus, if we continue this program we need to find a way to close the algebra on all the 3-forms without using any duality relations.

However, we will not pursue here any further the supersymmetrization of the tensor hierarchy for the higher-rank p-forms but we think that the above results strongly suggest that an extension with additional fermionic and bosonic fields of the tensor hierarchy on which the local supersymmetry algebra closes without the use of duality relations must exist. The duality relations must project the supersymmetric tensor hierarchy on to the $N = 1$ supersymmetric generalization of the action which will be given later in Eq. (3.3.46).

As we have seen in the vector and 2-form cases, the duality relations among the additional fields (fermionic λ_Σ, ζ^A and bosonic φ_A) are local as opposed to those involving the original bosonic ones, which are non-local and related via Hodge-duality.

The 3-form fields $C_A{}^M$

We will be brief here because the construction of the field strength and the determination of the gauge transformations of the 3-forms $C_A{}^M$ are similar to those of the other fields.

We first remark that, in order to make the standard hierarchy's field strength $G_C{}^M$ gauge-invariant under the new gauge transformations, we must modify it as follows:

$$G'_C{}^M \equiv G_C{}^M + (\delta_A{}^{\underline{a}}\mathcal{P}_{\underline{a}} + \delta_A{}^{\sharp}\mathcal{P}_{\sharp})D^M\,, \tag{4.1.115}$$

where $G_C{}^M$ is given in Eq. (E.3.8) and D^M is a 4-form transforming as

$$\delta'_h D^M = \mathfrak{D}\Sigma^M + (F^M - \tfrac{1}{2}Z^{MA}B_A)\wedge\Lambda\,, \tag{4.1.116}$$

and where we must also modify the gauge transformation rules of the 3-forms $C_C{}^M$ to be

$$\delta'_h C_A{}^M = \delta_h C_A{}^M - (\delta_A{}^{\underline{a}}\mathcal{P}_{\underline{a}} + \delta_A{}^{\sharp}\mathcal{P}_{\sharp})\mathfrak{D}\Sigma^M\,. \tag{4.1.117}$$

In order to prove this result we have made use of the constraint Eq. (4.1.40) and also of the fact, mentioned in Section 2.1.1, that the directions $A = \underline{a}$ for which $\mathcal{P}_{\underline{a}} \neq 0$ must necessarily be Abelian, so

$$Y_{CM}{}^A(\delta_A{}^{\underline{a}}\mathcal{P}_{\underline{a}} + \delta_A{}^{\sharp}\mathcal{P}_{\sharp})\mathcal{L} = 0\,, \tag{4.1.118}$$

etc.

Then, the supersymmetry transformations of the 3-forms $C_A{}^M$ are given by

$$\delta_\epsilon C_A{}^M{}_{\mu\nu\rho} = -\tfrac{i}{8}[\mathcal{P}_A\bar{\epsilon}^*\gamma_{\mu\nu\rho}\lambda^M - \text{c.c.}] - 3B_{A\,[\mu\nu|}\delta_\epsilon A^M{}_{|\rho]} - 2T_{A\,PQ}A^M{}_{[\mu}A^P{}_{\nu|}\delta_\epsilon A^Q{}_{|\rho]}\,. \tag{4.1.119}$$

The local $N = 1, d = 4$ supersymmetry algebra closes on $C_A{}^M$ upon the use of a duality relation to be discussed later. The gauge transformations of $C_A{}^M$ that appear on the right hand side are the ones described above with

$$\Lambda_{BC} = d_{BC} + B_{[B}\wedge b_{C]} + 2T_{[B|\,NP}\Lambda^P A^N \wedge B_{C]}\,, \tag{4.1.120}$$

$$\Lambda^{NPQ} = d^{NPQ} + 2\Lambda^{(P}A^N\wedge(F^{Q)} - Z^{Q)C}B_C) - \tfrac{1}{4}X_{RS}{}^{(Q}\Lambda^P A^{N)}\wedge A^R\wedge A^S\,, \tag{4.1.121}$$

$$\Lambda_E{}^{NP} = d_E{}^{NP} - \Lambda^N C_E{}^P + \tfrac{1}{2}T_{E\,QR}\Lambda^Q A^N\wedge A^R\wedge A^P\,, \tag{4.1.122}$$

where $d_{BC\mu\nu\rho} = D_{BC\mu\nu\rho\sigma}\xi^\sigma$, and similarly for d^{NPQ} and $d_E{}^{NP}$. The gauge transformation parameters Λ^M, $\Lambda_{\mathbf{a}}$ and $\Lambda_{\mathbf{a}}{}^M$ are, again, given by Eqs. (4.1.55), (4.1.72) and (4.1.81), respectively.

In the closure of the local supersymmetry algebra we have made use of the duality relation

$$G'_A{}^M = -\tfrac{1}{2}\star\Re\mathrm{e}(\mathcal{P}_{\mathrm{A}}\mathcal{D}^{\mathrm{M}})\,. \tag{4.1.123}$$

According to the results of Ref. [33], the duality relation has the general form

$$G'_A{}^M = \tfrac{1}{2} \star \frac{\partial V}{\partial \vartheta_M{}^A} \,. \tag{4.1.124}$$

Comparing these two expressions and using the relation between the potential of the supergravity theory and the fermion shifts, we conclude that, after the general electric-magnetic gauging the potential of $N=1, d=4$ supergravity is given by

$$V_{\mathrm{e-mg}} = V_{\mathrm{u}} - \tfrac{1}{2}\Re\mathrm{e}\, \mathcal{D}^M \vartheta_M{}^A \mathcal{P}_A = V_{\mathrm{u}} + \tfrac{1}{2}\mathcal{M}^{MN}\vartheta_M{}^A\vartheta_N{}^A\mathcal{P}_A\mathcal{P}_B \,, \tag{4.1.125}$$

where $\mathcal{M}$ is the symplectic matrix defined in Eq. (2.1.53). It satisfies

$$\partial V_{\mathrm{e-mg}}/\partial\vartheta_M{}^A = -\Re\mathrm{e}(\mathcal{D}^M\mathcal{P}_A)\,. \tag{4.1.126}$$

There may exist a supermultiplet containing the 3-forms $C_A{}^M$ such that the supersymmetry algebra closes without the need to use a duality relation. We leave it to future work to study its possible (non-)existence.

The 3-form C and the dual of the superpotential

We have seen that the consistency of the closure of the local supersymmetry algebra on the 2-forms $B_{\underline{\mathrm{a}}}$ and $B_\sharp$ requires the existence of a 3-form field that we have denoted by C, whose gauge transformation cancels the Stückelberg shift of those 2-forms.

An Ansatz for the supersymmetry transformation of C can be made by writing down 3-form spinor bilinears that have zero Kähler weight and that are consistent with the chirality of the fermionic fields. Further, from Eq. (4.1.82) it follows that there will be no gauge potential terms needed in the Ansatz. We thus make the following Ansatz

$$\delta_\epsilon C_{\mu\nu\rho} = -3i\eta\mathcal{L}\,\bar{\epsilon}^*\gamma_{[\mu\nu}\psi^*_{\rho]} - \tfrac{1}{2}\eta\mathcal{D}_i\mathcal{L}\bar{\epsilon}^*\gamma_{\mu\nu\rho}\chi^i + \mathrm{c.c.}\,, \tag{4.1.127}$$

where η is a constant to be found. It turns out that the local supersymmetry algebra closes for two different reality conditions for η, which leads to the existence of two different 3-forms that we will call C and C'.

1. For $\eta = -i$ the algebra closes into the gauge transformations required by the 2-forms $B_{\underline{\mathrm{a}}}$ and $B_\sharp$ provided that the field strength $G = dC$ vanishes. As discussed earlier there may be non-vanishing contributions if we were to construct the supersymmetry algebra at the quartic fermion order.

2. For $\eta \in \mathbb{R}$ the algebra closes into the following gauge transformation

$$\delta_{\mathrm{gauge}} C' = -d\Lambda'\,, \tag{4.1.128}$$

where the 2-form Λ' is given by

$$\Lambda' = c' - 2\eta\Im\mathrm{m}(\mathcal{L}\phi)\,, \qquad c'_{\mu\nu} \equiv C'_{\mu\nu\rho}\xi^{\rho}\,, \tag{4.1.129}$$

provided the field strength $G' = dC'$ satisfies the duality relation

$$G' = \star\eta(-24|\mathcal{L}|^2 + 8\mathcal{G}^{ij^*}\mathcal{D}_i\mathcal{L}\mathcal{D}_{j^*}\mathcal{L}^*)\,. \tag{4.1.130}$$

Observe that the right hand side is nothing but the part of the scalar potential Eq. (4.1.125) that depends on the superpotential. Actually, if we rescale the superpotential by $\mathcal{L} \to \eta\mathcal{L}$, then we can rewrite the above duality relation in the standard fashion

$$G' = \tfrac{1}{2}\star\frac{\partial V_{\mathrm{e-mg}}}{\partial\eta}\,, \tag{4.1.131}$$

and, therefore, we can see the 3-form C' as the dual of the deformation parameter associated to the superpotential, just as we can see the 3-forms $C_A{}^M$ as the duals of the deformation parameters $\vartheta_M{}^A$.

Observe that, had we chosen to work with a vanishing superpotential we would have found the duality rule $G' = 0$. This suggests a possible interpretation of the 3-form C to be explored: that it may be related to another, as yet unknown, deformation of $N = 1, d = 4$ supergravity which has not been used. The full supersymmetric action is needed to confirm this possibility or to find, perhaps, a term bilinear in fermions which is dual to C.

Finally, observe that neither of the 3-forms C, C' was predicted by the standard tensor hierarchy. C, though, is predicted by the extension associated to the constraints Eqs. (4.1.40) and (4.1.118).

The 4-form fields $D_E{}^{NP}, D_{AB}, D^{NPQ}, D^M$

In the previous sections we have introduced four 4-forms $D_E{}^{NP}, D_{AB}, D^{NPQ}, D^M$ in order to close the local supersymmetry algebra and have fully gauge-covariant field strengths. We thus expect that we can also find consistent supersymmetry transformations for all these 4-forms.

For the three 4-forms $D_E{}^{NP}, D_{AB}, D^{NPQ}$ there is a slight complication that has to do with the existence of extra Stückelberg shift symmetries. There are two such shift symmetries and in Appendix E.3 they correspond to the parameters $\tilde{\Lambda}_E{}^{(NP)}$ and $\Lambda_{BE}{}^P$. The origin of these symmetries lies in the fact that the W tensors that appear in the field strengths of the 3-forms are not all independent. The symmetries result from the identities E.1.10 and E.1.11 together with the constraints $L_{NPQ} = Q^{AB} = Q_{NM}{}^A = 0$. This means that if we want to realize $N = 1$ supersymmetry on the 4-forms $D_E{}^{NP}, D_{AB}, D^{NPQ}$ the parameters $\tilde{\Lambda}_E{}^{(NP)}$ and $\Lambda_{BE}{}^P$ will appear on the right hand side of commutators as part of the local algebra.

Most of these features are already visible in the simpler case of the ungauged theory[7], i.e. for $\vartheta_M{}^A = 0$ and even when the ungauged case has no symmetries that act on the vectors, i.e. when all the matrices $T_A = 0$. We will restrict ourselves to realizing the supersymmetry algebra on the 4-forms for the ungauged theory with $T_A = 0$ for all A for simplicity. The 4-form supersymmetry transformations in this simple setting are given by

$$\delta_\epsilon D_{AB} = -\tfrac{i}{2} \star \mathcal{P}_{[A}\partial_i \mathcal{P}_{B]}\bar{\epsilon}\chi^i + \text{c.c.} - B_{[A} \wedge \delta_\epsilon B_{B]}\,, \tag{4.1.132}$$

$$\delta_\epsilon D^{NPQ} = 10 A^{(N} \wedge F^P \wedge \delta_\epsilon A^{Q)}\,, \tag{4.1.133}$$

$$\delta_\epsilon D_E{}^{NP} = C_E{}^P \wedge \delta_\epsilon A^N\,. \tag{4.1.134}$$

$$\delta_\epsilon D^M = -\tfrac{i}{2} \star \mathcal{L}^* \bar{\epsilon}\lambda^M + \text{c.c.} + C \wedge \delta_\epsilon A^M\,. \tag{4.1.135}$$

When $\vartheta_M{}^A = 0$ and $T_A = 0$ the only place where there still appears a Stückelberg shift parameter is in the gauge transformation of $D_E{}^{NP}$. From the commutators we find that

$$\tilde{\Lambda}_E{}^{(NP)} = -2\Lambda^{(N} F^{P)} \wedge B_E\,. \tag{4.1.136}$$

4.1.4 The gauge-invariant bosonic action

It turns out that in order to write an action for the bosonic fields of the theory with electric and magnetic gaugings of perturbative and non-perturbative symmetries it is enough to add to the fundamental (electric) fields just the magnetic 1-forms A_Λ and the 2-forms B_A. The gauge-invariant action takes the form

$$\begin{aligned} S_{\text{e-mg}} = \int \Big\{ & \star R - 2\mathcal{G}_{ij^*}\mathfrak{D}Z^i \wedge \star\mathfrak{D}Z^{*\,j^*} - 2\Im\text{m} f_{\Lambda\Sigma}F^\Lambda \wedge \star F^\Sigma + 2\Re\text{e} f_{\Lambda\Sigma}F^\Lambda \wedge F^\Sigma \\ & - \star V_{\text{e-mg}} - 4Z^{\Sigma A}B_A \wedge \left(F_\Sigma - \tfrac{1}{2}Z_\Sigma{}^B B_B\right) \\ & -\tfrac{4}{3}X_{[MN]\Sigma}A^M \wedge A^N \wedge \left(F^\Sigma - Z^{\Sigma B}B_B\right) \\ & -\tfrac{2}{3}X_{[MN]}{}^\Sigma A^M \wedge A^N \wedge \left(dA_\Sigma - \tfrac{1}{4}X_{[PQ]\Sigma}A^P \wedge A^Q\right) \Big\}\,. \end{aligned} \tag{4.1.137}$$

[7]Note that the hierarchy remains non-trivial for $\vartheta_M{}^A = 0$.

The scalar potential $V_{\text{e-mg}}$ is given by Eq. (4.1.125). Furthermore, the gauge transformations that leave invariant the above action (δ_a) are those of the extended hierarchy (δ'_h) except for the 2-forms[8]:

$$\delta_a B_A = \delta'_h B_A - 2T_{A\,NP}\Lambda^N (F^P - G^P)\,. \tag{4.1.138}$$

The action contains the 2-forms B_A always contracted with Z^{MA} so that we do not need to worry about the different behavior of $B_{\mathbf{a}}$ and $B_{\underline{\mathrm{a}}}, B_\sharp$ under gauge transformation due to the extra constraint Eq. (4.1.88).

A general variation of the above action gives

$$\delta S = \int \left\{ \delta g^{\mu\nu}\frac{\delta S}{\delta g^{\mu\nu}} + \left(\delta Z^i \frac{\delta S}{\delta Z^i} + \text{c.c.}\right) - \delta A^M \wedge \star\frac{\delta S}{\delta A^M} + 2\delta B_A \wedge \star\frac{\delta S}{\delta B_A}\right\}\,, \tag{4.1.139}$$

where the first variations with respect to the different fields are given by

$$\begin{aligned} -\star\,\frac{\delta S}{\delta g^{\mu\nu}} &= G_{\mu\nu} + 2\mathcal{G}_{ij^*}[\mathfrak{D}_\mu Z^i \mathfrak{D}_\nu Z^{*\,j^*} - \tfrac{1}{2}g_{\mu\nu}\mathfrak{D}_\rho Z^i \mathfrak{D}^\rho Z^{*\,j^*}] \\ &\quad - G^M{}_{(\mu|}{}^\rho \star G_{M|\nu)\rho} + \tfrac{1}{2}g_{\mu\nu}V_{\text{e-mg}}\,, \end{aligned} \tag{4.1.140}$$

$$-\tfrac{1}{2}\frac{\delta S}{\delta Z^i} = \mathcal{G}_{ij^*}\mathfrak{D}\star\mathfrak{D}Z^{*\,j^*} - \partial_i G_M{}^+ \wedge G^{M+} - \star\tfrac{1}{2}\partial_i V_{\text{e-mg}}\,, \tag{4.1.141}$$

$$-\tfrac{1}{4}\star\frac{\delta S}{\delta A^M} = \mathfrak{D}G_M - \tfrac{1}{4}\vartheta_M{}^A \star j_A + \tfrac{1}{2}T_{A\,MN}A^N \wedge \vartheta^{PA}(F_P - G_P)\,, \tag{4.1.142}$$

$$\star\frac{\delta S}{\delta B_A} = \vartheta^{PA}(F_P - G_P)\,. \tag{4.1.143}$$

The above equations are formally symplectic-covariant and, therefore, electric-magnetic duality symmetric. Both the Maxwell equations and the "Bianchi identities" have now sources to which they couple with a strength determined by the embedding tensor's electric and magnetic components.

It is expected to be possible to find a gauge-invariant action in which all the hierarchy's fields appear (as was done in [33]) if one assumes that none of the constraints on the embedding tensor is satisfied. Then, the 3-forms $C_A{}^M$ and the 4-forms $D_E{}^{NP}, D_{AB}, D^{NPQ}$, D^M are introduced as Lagrange multipliers enforcing the constancy of the embedding tensor and the algebraic constraints $Q_{NP}{}^E = 0, Q^{AB} = 0$, $L_{NPQ} = 0$ and $(\vartheta_M{}^{\underline{\mathrm{a}}}\mathcal{P}_{\underline{\mathrm{a}}} + \vartheta_M{}^\sharp\mathcal{P}_\sharp)\mathcal{L} = 0$, respectively, but we will not study this possibility here.

[8] The piece ΔB_A in the gauge transformation of the B_As does not play any role here because the B_as always appear projected with Z^{MA}.

It should be stressed that, even though the action Eq. (3.3.46) contains $2n_V$ vectors and some number n_B of 2-forms B_a it does not carry all those degrees of freedom. To make manifest the actual number of degrees of freedom we briefly repeat here the arguments of [15] regarding the gauge fixing of the action (3.3.46). First, we choose a basis of magnetic vectors and generators such that the non-zero entries of $\vartheta^{\Lambda a}$ arrange themselves into a square invertible submatrix ϑ^{Ii}. We split accordingly $A_{\Lambda\mu} = (A_{I\mu}, A_{U\mu})$. It can be shown by looking at the vector equations of motion that the Lagrangian does not depend on the $A_{U\mu}$, i.e. $\delta\mathcal{L}/\delta A_{U\mu} = 0$. Further, the electric vectors $A^I{}_\mu$ that are dual to the magnetic vectors $A_{I\mu}$, which are used in some gauging, have massive gauge transformations, $\delta A^I{}_\mu = -\mathfrak{D}_\mu\Lambda^I - \vartheta^{Ii}\Lambda_{i\mu}$ and can be gauged away. The n_B 2-forms B_i can by eliminated from the Lagrangian by using their equations of motion Eq. (4.1.143). The 2-forms appear without derivatives in Eq. (4.1.143) so that it is possible to solve for them and to substitute the on-shell expression back into the action. This is allowed as the 2-forms appear everywhere (up to partial integrations) without derivatives. One then ends up with an action depending on n_B magnetic vectors $A_{I\mu}$ and $n_V - n_B$ electric vectors $A^U{}_\mu$.

4.1.5 Summary

In sections 4.1.1-4.1.4 we have discussed the possible symmetries of $N = 1, d = 4$ supergravity and their gauging using as gauge fields both electric and magnetic vectors.

When using both electric and magnetic 1-forms as gauge fields at the same time one is compelled to introduce 2-forms B_A, associated to all the possible symmetries of the theory. For each electric vector A^Λ whose magnetic dual A_Λ is gauged, corresponding to non-vanishing magnetic components of the embedding tensor $\vartheta^{\Lambda A}$, one introduces a 2-form $\vartheta^{\Lambda A}B_A$ in its field strength. A^Λ has a massive gauge transformation and it forms a Stückelberg pair with the 2-form $\vartheta^{\Lambda A}B_A$. By electro-magnetic duality we end up with Stückelberg pairs $A^M, \vartheta_M{}^A B_A$.

The embedding tensor-projected 2-forms $\vartheta_M{}^{\mathbf{a}}B_{\mathbf{a}}$ are dual to embedding tensor-projected Noether currents associated to gauged isometry directions $\vartheta_M{}^{\mathbf{a}}j_{\mathbf{a}}$. The rest are pure gauge at zero order in fermions, but it is to be expected that they are actually dual to the Noether currents associated to the respective symmetries, which are bilinear in fermions. To properly test this idea one would have to construct the supersymmetry algebra at quartic order in fermions, which is left for future work.

We have seen that the presence of a non-vanishing superpotential breaks the global symmetries of the ungauged theory that involve Kähler transformations with constant parameters. Thus, if $\mathcal{L} \neq 0$, we must set $(\vartheta_M{}^{\underline{a}}\mathcal{P}_{\underline{a}} + \vartheta_M{}^{\sharp}\mathcal{P}_\sharp) = 0$, which is a new constraint that the embedding tensor must satisfy. We have written it in the form Eq. (4.1.40) to handle the cases $\mathcal{L} = 0$ and $\mathcal{L} \neq 0$ simultaneously. When $\mathcal{L} \neq 0$, then, $N = 1, d = 4$ supersymmetry implies that the 2-forms $B_{\underline{a}}, B_\sharp$ transform under new Stückelberg shifts parametrized by a 2-form Λ. Still, since $\Lambda \neq 0$ only when $\mathcal{L} \neq 0$

and in this case we have to impose the new constraint (something we have expressed through Eq. (4.1.88)), the gauge transformations of the projected 2-forms $Z^{MA}B_A$ are the same. The field strengths F^M and the action keep their standard form.

In the standard tensor hierarchy it is necessary to introduce 3-forms $C_A{}^M$ to construct gauge-covariant field strengths H_A for the 2-forms B_A. These 3-forms are the dual of the embedding tensor $\vartheta_M{}^A$. However, when $\mathcal{L} \neq 0$, the standard tensor hierarchy field strengths H_A need to be modified by the addition of a 3-form C, into H'_A Eq. (4.1.89). C must absorb the new Stückelberg shifts of the 2-forms $B_{\underline{a}}, B_\sharp$, but one has to show that $N = 1, d = 4$ supergravity allows for such a 3-form.

We have found consistent supersymmetry transformation rules for two 3-forms C and C' the first of which has precisely the required gauge transformations. C' is unexpected from the hierarchy point of view but turns out to be the dual of the superpotential, seen as a deformation of the ungauged theory. The fact that it is not predicted by the hierarchy (even in its extended form which includes the constraint Eq. (4.1.40)) is due to the fact that the superpotential is not associated to any gauge symmetry, which is the keystone of the tensor hierarchy. On the other hand, the existence of the 3-form C suggests the possible existence of another deformation of $N = 1, d = 4$ supergravity unrelated to gauge symmetry and to the superpotential.

Again, in the $\mathcal{L} \neq 0$ case the field strengths $G_C{}^M$ need to be modified by the addition of new 4-forms D^M not predicted by the standard hierarchy, which must absorb gauge transformations related to Λ. In the standard hierarchy the 4-forms $D_E{}^{NP}, D_{AB}, D^{NPQ}$ are associated to the constraints $Q_{NP}{}^E, Q^{AB}, L_{NPQ}$. The fourth 4-form that appears when $\mathcal{L} \neq 0$ in $N = 1, d = 4$ supergravity could well be related to the constraint $(\vartheta_M{}^{\underline{a}}\mathcal{P}_{\underline{a}} + \vartheta_M{}^\sharp\mathcal{P}_\sharp) = 0$ that the embedding tensor must satisfy. This can only be fully confirmed by the construction of a supersymmetric action containing all the p-forms as in [33]. Nevertheless, it is clear that, when we vary the action without any constraints imposed on the embedding tensor, we expect it to be necessary to introduce a 4-form D^M multiplying that constraint. The gauge transformations of the 4-forms D^M should compensate for this lack of gauge invariance.

4.2 $N = 2$ Einstein-Yang-Mills Supergravity

In this section we will describe the theory of $N = 2$ $d = 4$ Supergravity coupled to non-Abelian vector supermultiplets to which we will refer to as $N = 2$ Einstein-Yang-Mills (EYM) Supergravity. These theories can be obtained from the ungauged theory with vector supermultiplets by gauging the isometries of the special-Kähler manifold parametrized by the scalars in the vector supermultiplets[9]. Some definitions and formulae related to the gauging of holomorphic isometries of special Kähler manifolds are contained in Appendix C.2.

[9] For a more detailed description see Refs. [62] or [82], the review Ref. [83], and the original works Refs. [80, 84]. Our conventions are contained in Refs. [26, 27].

The action restricted to the bosonic fields of these theories is

$$\begin{aligned} S &= \int d^4x\sqrt{|g|}\,\big[R+2\mathcal{G}_{ij^*}\mathfrak{D}_\mu Z^i\mathfrak{D}^\mu Z^{*\,j^*}+2\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,\mu\nu}F^{\Sigma}{}_{\mu\nu} \\ &\qquad -2\Re\mathrm{e}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,\mu\nu\,\star}F^{\Sigma}{}_{\mu\nu}-V(Z,Z^*)\big]\,, \end{aligned} \tag{4.2.1}$$

where the potential $V(Z,Z^*)$, is given by

$$V(Z,Z^*)=2\mathcal{G}_{ij^*}W^iW^{*j^*}\,, \tag{4.2.2}$$

where

$$W^i \;\equiv\; \tfrac{1}{2}g\mathcal{L}^{*\,\Lambda}k_\Lambda{}^i\,. \tag{4.2.3}$$

In these expressions g is the gauge coupling constant, the $k_\Lambda{}^i(Z)$ are holomorphic Killing vectors of $\mathcal{G}_{ij^*}$ and $\mathfrak{D}$ the gauge covariant derivative (also Kähler-covariant when acting on fields of non-trivial Kähler weight) and is defined in Appendix C.2.

This is not the most general gauged $N=2, d=4$ supergravity: if the $\mathfrak{sp}(2\bar{n})$ matrices $\mathcal{S}_\Lambda$ that provide a representation of the Lie algebra of the gauge group G_V, see Eq. (B.1.37), are written in the form

$$\mathcal{S}_\Lambda=\begin{pmatrix} a_\Lambda{}^\Omega{}_\Sigma & b_\Lambda{}^{\Omega\Sigma} \\ & \\ c_{\Lambda\Omega\Sigma} & d_{\Lambda\Omega}{}^\Sigma \end{pmatrix}\,, \tag{4.2.4}$$

we are then considering only the cases in which $b=0$, so that only symmetries of the action are gauged, and $c=0$. This last restriction is only made for the sake of simplicity as theories in which symmetries with $c\neq 0$ are gauged have complicated Chern-Simons terms.

Within this restricted class of theories, then, we can use Eqs. (C.2.16) and (C.2.18) to rewrite the potential as

$$V(Z,Z^*)=\tfrac{1}{2}g^2f^{*\Lambda\,i}f^{\Sigma}{}_i\mathcal{P}_\Lambda\mathcal{P}_\Sigma=-\tfrac{1}{4}g^2(\Im\mathrm{m}\mathcal{N})^{-1|\Lambda\Sigma}\mathcal{P}_\Lambda\mathcal{P}_\Sigma\,. \tag{4.2.5}$$

Then, since $\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}$ is negative definite and the momentum map is real, the potential is positive semi-definite $V(Z,Z^*)\geq 0$. For constant values of the scalars $V(Z,Z^*)$ behaves as a non-negative cosmological constant $\Lambda=V(Z,Z^*)/2$ which leads to Minkowski ($\Lambda=0$) or dS ($\Lambda>0$) vacua. The latter cannot be maximally supersymmetric, however.

For convenience, we denote the bosonic equations of motion by

$$\mathcal{E}_a{}^\mu\equiv-\frac{1}{2\sqrt{|g|}}\frac{\delta S}{\delta e^a{}_\mu}\,,\qquad \mathcal{E}^i\equiv-\frac{\mathcal{G}^{ij^*}}{2\sqrt{|g|}}\frac{\delta S}{\delta Z^{*j^*}}\,,\qquad \mathcal{E}_\Lambda{}^\mu\equiv\frac{1}{8\sqrt{|g|}}\frac{\delta S}{\delta A^\Lambda{}_\mu}\,. \tag{4.2.6}$$

and the Bianchi identities for the vector field strengths by

$$\mathcal{B}^{\Lambda\,\mu} \equiv \mathfrak{D}_\nu \star F^{\Lambda\,\nu\mu}\,, \quad \star\mathcal{B}^{\Lambda} \equiv -\mathfrak{D}F^{\Lambda}\,. \tag{4.2.7}$$

Then, using the action Eq. (4.2.1), we find

$$\begin{aligned}
\mathcal{E}_{\mu\nu} &= G_{\mu\nu} + 2\mathcal{G}_{ij^*}[\mathfrak{D}_\mu Z^i \mathfrak{D}_\nu Z^{*\,j^*} - \tfrac{1}{2}g_{\mu\nu}\mathfrak{D}_\rho Z^i \mathfrak{D}^\rho Z^{*\,j^*}] \\
&\quad +8\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,+}{}_{\mu}{}^{\rho}F^{\Sigma\,-}{}_{\nu\rho} + \tfrac{1}{2}g_{\mu\nu}V(Z,Z^*)\,, &(4.2.8)\\
\mathcal{E}_{\Lambda}{}^{\mu} &= \mathfrak{D}_\nu \star F_{\Lambda}{}^{\nu\mu} + \tfrac{1}{2}g\Re\mathrm{e}(k_{\Lambda\,i^*}\mathfrak{D}^\mu Z^{*i^*})\,, &(4.2.9)\\
\mathcal{E}^{i} &= \mathfrak{D}^2 Z^i + \partial^i \tilde{F}_{\Lambda}{}^{\mu\nu}\star F^{\Lambda}{}_{\mu\nu} + \tfrac{1}{2}\partial^i V(Z,Z^*)\,. &(4.2.10)
\end{aligned}$$

In differential-form notation, the Maxwell equation takes the form

$$-\star\hat{\mathcal{E}}_{\Lambda} = \mathfrak{D}F_{\Lambda} - \tfrac{1}{2}g\star\Re\mathrm{e}\,(k^*_{\Lambda\,i}\mathfrak{D}Z^i)\,. \tag{4.2.11}$$

For vanishing fermions, the supersymmetry transformation rules of the fermions are

$$\begin{aligned}
\delta_\epsilon \psi_{I\,\mu} &= \mathfrak{D}_\mu \epsilon_I + \epsilon_{IJ}T^{+}{}_{\mu\nu}\gamma^\nu\epsilon^J\,, &(4.2.12)\\
\delta_\epsilon \lambda^{Ii} &= i\,\not{\mathfrak{D}}Z^i\epsilon^I + \epsilon^{IJ}[\not{\mathcal{G}}^{i\,+} + W^i]\epsilon_J\,. &(4.2.13)
\end{aligned}$$

$\mathfrak{D}_\mu\epsilon_I$ is given in Eq. (C.2.11).

The supersymmetry transformations of the bosons are the same as in the ungauged case

$$\begin{aligned}
\delta_\epsilon e^a{}_\mu &= -\tfrac{i}{4}(\bar{\psi}_{I\,\mu}\gamma^a\epsilon^I + \bar{\psi}^I{}_\mu\gamma^a\epsilon_I)\,, &(4.2.14)\\
\delta_\epsilon A^{\Lambda}{}_\mu &= \tfrac{1}{4}(\mathcal{L}^{\Lambda\,*}\epsilon^{IJ}\bar{\psi}_{I\,\mu}\epsilon_J + \mathcal{L}^{\Lambda}\epsilon_{IJ}\bar{\psi}^I{}_\mu\epsilon^J) \\
&\quad +\tfrac{i}{8}(f^{\Lambda}{}_i\epsilon_{IJ}\bar{\lambda}^{Ii}\gamma_\mu\epsilon^J + f^{\Lambda *}{}_{i^*}\epsilon^{IJ}\bar{\lambda}_I{}^{i^*}\gamma_\mu\epsilon_J)\,, &(4.2.15)\\
\delta_\epsilon Z^i &= \tfrac{1}{4}\bar{\lambda}^{Ii}\epsilon_I\,. &(4.2.16)
\end{aligned}$$

Chapter 5

Supersymmetric solutions

Chapter we are going to study the supersymmetric solutions of $N = 2$ Supergravity in four dimensions[1]. We confine ourselves to the study of the ungauged theory with the most general matter couplings, section 5.1, and $N = 2$ Einstein-Yang-Mills theory, which is done in section 5.2.

5.1 Ungauged $N = 2$ Supergravity coupled to vector and hyper-multiplets

In what follows we are going to study the supersymmetric solutions of ungauged $N = 2$ SUGRA coupled to vector and hypermultiplets. The solutions to $N = 2$ ungauged Supergravity coupled to only vector- multiplets were studied in ref. [26]. Among these solutions we will find supersymmetric 1-brane solutions, which wew refer to as *stringy cosmic strings* in analogy with the terminology in ref. [85].

5.1.1 Supersymmetric configurations: generalities

As we mentioned in Section 2.2 the supersymmetry transformation rules of the bosonic fields indicate that the KSIs associated to the gravitinos and gauginos are going to have the same form as in absence of hypermultiplets. This is indeed the case, and the integrability conditions of the KSEs $\delta_\epsilon \psi_{I\mu} = 0$ and $\delta_\epsilon \lambda^{iI} = 0$ confirm the results. Of course, now the Einstein equation includes an additional term: the hyperscalars energy-momentum tensor. In the KSI approach the origin of this term is clear. In

[1]Using the same formalism as we are going to use in what follows, the solutions of $N = 1$ $d = 4$ Supergravity were found in [30] and the supersymmetric configurations for the $N = 4$ $d = 4$ case were classified in [38]

the integrability conditions it appears through the curvature of the $SU(2)$ connection and Eq. (D.0.27). The results coincide for $\lambda = -1$.

There is one more set of KSIs associated to the hyperinos which take the form

$$\mathcal{E}^u \ \mathsf{U}^{\alpha I}{}_u \ \epsilon_I \ = \ 0 \,, \tag{5.1.1}$$

and which can be obtained from the integrability condition $\not{\mathfrak{D}}\delta_\epsilon\zeta_\alpha \ = \ 0$ using the covariant constancy of the Quadbein, Eq. (D.0.21).

The KSIs involving the equations of motion of the bosonic fields of the gravity and vector multiplets take, of course, the same form as in absence of hypermultiplets. Acting with $\bar{\epsilon}^J$ from the left on the new KSI Eq. (5.1.1) we get

$$X\mathcal{E}^u \ \mathsf{U}^{\alpha I}{}_u \ = \ 0 \,, \tag{5.1.2}$$

which implies, in the timelike $X \neq 0$ case, that all the supersymmetric configurations satisfy the hyperscalars equations of motion automatically:

$$\mathcal{E}^u = 0 \,. \tag{5.1.3}$$

In the null case, parametrizing the Killing spinors by $\epsilon_I = \phi_I \epsilon$, we get just

$$\mathcal{E}^u \ \mathsf{U}^{\alpha I}{}_u \ \phi_I \ \epsilon \ = \ 0 \,. \tag{5.1.4}$$

As usual, there are two separate cases to be considered: the one in which the vector bilinear $V^\mu \equiv i\bar{\epsilon}^I\gamma^\mu\epsilon_I$, which is always going to be Killing, is timelike (Section 5.1.2) and the one in which it is null (Section 5.1.3). The procedure we are going to follow is almost identical to the one we followed in Ref. [26].

5.1.2 The timelike case

As mentioned before, the presence of hypermultiplets only introduces an $SU(2)$ connection in the covariant derivative $\mathfrak{D}_\mu\epsilon_I$ in $\delta_\epsilon\psi_{I\mu} = 0$ and has no effect on the KSE $\delta_\epsilon\lambda^{iI} = 0$. Following the same steps as in Ref. [26], by way of the gravitino supersymmetry transformation rule Eq. (2.2.25), we arrive at

$$\mathfrak{D}_\mu X \ = \ -iT^{+}{}_{\mu\nu}V^\nu \,, \tag{5.1.5}$$

$$\mathfrak{D}_\mu V_J{}^I{}_\nu \ = \ i\delta^I{}_J(XT^{*-}{}_{\mu\nu} - X^*T^{+}{}_{\mu\nu}) - i(\epsilon^{IK}T^{*-}{}_{\mu\rho}\Phi_{KJ}{}^\rho{}_\nu - \epsilon_{JK}T^{+}{}_{\mu\rho}\Phi^{IK\rho}{}_\nu \tag{5.1.6}$$

The $SU(2)$ connection does not occur in the first equation, simply because $X = \frac{1}{2}\epsilon^{IJ}M_{IJ}$ is an $SU(2)$ scalar, but it does occur in the second, although not in its trace. This means that V^μ is, once again, a Killing vector and the 1-form $\hat{V} = V_\mu dx^\mu$ satisfies the equation

$$d\hat{V} = 4i(XT^{*\,-} - X^{*}T^{+})\,. \tag{5.1.7}$$

The remaining 3 independent 1-forms[2]

$$\hat{V}^{x} \;\equiv\; \tfrac{1}{\sqrt{2}}(\sigma_x)_I{}^J\; V_J{}^I{}_\mu\; dx^\mu\,, \tag{5.1.8}$$

however, are only $SU(2)$-covariantly exact

$$d\hat{V}^{x} \;+\; \varepsilon^{xyz}\;\mathsf{A}^{y}\wedge\hat{V}^{z} \;=\; 0\,. \tag{5.1.9}$$

From $\delta_\epsilon\lambda^{iI} = 0$ we get exactly the same equations as in absence of hypermultiplets. In particular

$$V^\mu\partial_\mu Z^i \;\;=\;\; 0\,, \tag{5.1.10}$$

$$2iX^*\partial_\mu Z^i + 4iG^{i\,+}{}_{\mu\nu}V^\nu \;\;=\;\; 0\,. \tag{5.1.11}$$

Combine Eqs. (5.1.5) and (5.1.11), we get

$$V^\nu F^{\Lambda\,+}{}_{\nu\mu} = \mathcal{L}^{*\,\Lambda}\mathfrak{D}_\mu X + X^* f^\Lambda{}_i\partial_\mu Z^i = \mathcal{L}^{*\,\Lambda}\mathfrak{D}_\mu X + X^*\mathfrak{D}_\mu\mathcal{L}^\Lambda\,, \tag{5.1.12}$$

which, in the timelike case at hand, is enough to completely determine through the identity

$$C^{\Lambda\,+}{}_\mu \equiv V^\nu F^{\Lambda\,+}{}_{\nu\mu} \;\;\Rightarrow\;\; F^{\Lambda\,+} = V^{-2}[\hat{V}\wedge\hat{C}^{\Lambda\,+} + i\,{}^\star(\hat{V}\wedge\hat{C}^{\Lambda\,+})]\,. \tag{5.1.13}$$

Observe that this equation does not involve the hyperscalars in any explicit way, as was to be expected due to the absence of couplings between the vector fields and the hyperscalars.

Let us now consider the new equation $\delta_\epsilon\zeta_\alpha = 0$. Acting on it from the left with $\bar{\epsilon}^K$ and $\bar{\epsilon}^K\gamma_\mu$ we get, respectively

$$\mathsf{U}^{\alpha I}{}_u\;\varepsilon_{IJ}\;V^J{}_K{}^\mu\;\partial_\mu q^u \;\;=\;\; 0\,, \tag{5.1.14}$$

$$X^*\mathsf{U}^{\alpha K}{}_u\;\partial_\mu q^u \;+\; \mathsf{U}^{\alpha I}{}_u\;\varepsilon_{IJ}\;\Phi^{KJ}{}_\mu{}^\rho\;\partial_\rho q^u \;\;=\;\; 0\,. \tag{5.1.15}$$

Using $\varepsilon_{IJ}\;V^J{}_K \;=\; \varepsilon_{KJ}\;V^J{}_I \;+\; \varepsilon_{IK}\;V$ in the first equation we get

[2] $\sigma_x{}_J{}^I$, $(x = 1,2,3)$ are the Pauli matrices satisfying Eq. (D.0.12).

$$\mathsf{U}^{\alpha I}{}_{u}\ V^{J}{}_{I}{}^{\mu}\partial_{\mu}q^{u}\ -\ \mathsf{U}^{\alpha J}{}_{u}\ V^{\mu}\partial_{\mu}q^{u}\ =\ 0\,. \tag{5.1.16}$$

It is not difficult to see that the second equation can be derived from this one using the Fierz identities that the bilinears satisfy in the timelike case (see Ref. [38]), whence the only equations to be solved are (5.1.16).

The metric

If we define the time coordinate t by

$$V^{\mu}\partial_{\mu}\equiv\sqrt{2}\partial_{t}\,, \tag{5.1.17}$$

then $V^2=4|X|^2$ implies that $\hat{V}$ must take the form

$$\hat{V}=2\sqrt{2}|X|^{2}(dt+\omega)\,, \tag{5.1.18}$$

where ω is a 1-form to be determined later.

Since the $\hat{V}^x$s are not exact, we cannot simply define coordinates by putting $\hat{V}^x\equiv dx^x$. We can, however, still use them to construct the metric: using

$$g_{\mu\nu}=2V^{-2}[V_{\mu}V_{\nu}-V_{J}{}^{I}{}_{\mu}V_{I}{}^{J}{}_{\nu}]\,, \tag{5.1.19}$$

and the decomposition

$$V_{J}{}^{I}{}_{\mu}\ =\ \tfrac{1}{2}V_{\mu}\ \delta_{J}{}^{I}\ +\ \tfrac{1}{\sqrt{2}}\ (\sigma_{x})_{J}{}^{I}\ V^{x}{}_{\mu}\,, \tag{5.1.20}$$

we find that the metric can be written in the form

$$ds^{2}=\frac{1}{4|X|^{2}}\hat{V}\otimes\hat{V}-\frac{1}{2|X|^{2}}\delta_{xy}\hat{V}^{x}\otimes\hat{V}^{y}\,. \tag{5.1.21}$$

The $\hat{V}^x$ are mutually orthogonal and also orthogonal to $\hat{V}$, which means that they can be used as a Dreibein for a 3-dimensional Euclidean metric

$$\delta_{xy}\hat{V}^{x}\otimes\hat{V}^{y}\ \equiv\ \gamma_{\underline{mn}}dx^{m}dx^{n}\,, \tag{5.1.22}$$

and the 4-dimensional metric takes the form

$$ds^{2}\ =\ 2|X|^{2}(dt+\omega)^{2}-\frac{1}{2|X|^{2}}\gamma_{\underline{mn}}dx^{m}dx^{n}\,. \tag{5.1.23}$$

The presence of a non-trivial Dreibein and the corresponding 3D metric $\gamma_{\underline{mn}}$ is the main (and only) novelty brought about by the hyperscalars!

In what follows we will use the Vierbein basis

$$e^{0}=\frac{1}{2|X|}\hat{V}\,,\qquad e^{x}=\frac{1}{\sqrt{2}|X|}\hat{V}^{x}\,, \tag{5.1.24}$$

that is

$$(e^a{}_\mu) = \begin{pmatrix} \sqrt{2}|X| & \sqrt{2}|X|\omega_{\underline{m}} \\ 0 & \frac{1}{\sqrt{2}|X|}V^x{}_{\underline{m}} \end{pmatrix}, \qquad (e^\mu{}_a) = \begin{pmatrix} \frac{1}{\sqrt{2}|X|} & -\sqrt{2}|X|\omega_x \\ 0 & \sqrt{2}|X|V_x{}^{\underline{m}} \end{pmatrix}. \tag{5.1.25}$$

where $V_x{}^{\underline{m}}$ is the inverse Dreibein $V_x{}^{\underline{m}}V^y{}_{\underline{m}} = \delta^y{}_x$ and $\omega_x = V_x{}^{\underline{m}}\omega_{\underline{m}}$. We shall also adopt the convention that all objects with flat or curved 3-dimensional indices refer to the above Dreibein and the corresponding metric.

Our choice of time coordinate Eq. (5.1.10) means that the scalars Z^i are time-independent, whence $\imath_V \mathcal{Q} = 0$. Contracting Eq. (5.1.5) with V^μ we get

$$V^\mu \mathfrak{D}_\mu X = 0\,, \;\Rightarrow\; V^\mu\partial_\mu X = 0\,, \tag{5.1.26}$$

so that also X is time-independent.

We know the $\hat{V}^x$s to have no time components. If we choose the gauge for the pullback of the $SU(2)$ connection $\mathsf{A}^x{}_t = 0$, then the $SU(2)$-covariant constancy of the $\hat{V}^x$ (Eq. (5.1.9)) states that the pullback of A^x, the $\hat{V}^x$s and, therefore, the 3-dimensional metric $\gamma_{\underline{mn}}$ are also time-independent. Eq. (5.1.9) can then be interpreted as Cartan's first structure equation for a torsionless connection ϖ in 3-dimensional space

$$d\hat{V}^x - \varpi^{xy} \wedge \hat{V}^y = 0\,, \tag{5.1.27}$$

which means that the 3-dimensional spin connection 1-form $\varpi_x{}^y$ is related to the pullback of the $SU(2)$ connection A^x by

$$\varpi_{\underline{m}}{}^{xy} = \varepsilon^{xyz}\mathsf{A}^z{}_u\,\partial_{\underline{m}}q^u\,, \tag{5.1.28}$$

implying the embedding of the internal group $SU(2)$ into the Lorentz group of the 3-dimensional space as discussed in the introduction.

The $\mathfrak{su}(2)$ curvature will also be time-independent and Eq. (D.0.27) implies that the pullback of the Quadbein is also time-independent and its time component vanishes:

$$\mathsf{U}^{\alpha I}{}_u\, V^\mu\partial_\mu q^u = 0\,. \tag{5.1.29}$$

Let us then consider the 1-form ω: following the same steps as in Ref. [26], we arrive at

$$(d\omega)_{xy} = -\frac{i}{2|X|^4}\varepsilon_{xyz}(X^*\mathfrak{D}^z X - X\mathfrak{D}^z X^*)\,. \tag{5.1.30}$$

This equation has the same form as in the case without hypermultiplets, but now the Dreibein is non-trivial and, in curved indices, it takes the form

$$(d\omega)_{\underline{mn}} = -\frac{i}{2|X|^4\sqrt{|\gamma|}}\varepsilon_{\underline{mnp}}(X^*\mathfrak{D}^{\underline{p}}X - X\mathfrak{D}^{\underline{p}}X^*)\,. \tag{5.1.31}$$

Introducing the real symplectic sections $\mathcal{I}$ and $\mathcal{R}$

$$\mathcal{R} \equiv \Re\mathrm{e}(\mathcal{V}/X)\,, \qquad\qquad \mathcal{I} \equiv \Im\mathrm{m}(\mathcal{V}/X)\,, \tag{5.1.32}$$

where $\mathcal{V}$ is the symplectic section

$$\mathcal{V} = \begin{pmatrix} \mathcal{L}^\Lambda \\ \mathcal{M}_\Sigma \end{pmatrix}, \qquad \langle \mathcal{V} \mid \mathcal{V}^* \rangle \equiv \mathcal{L}^{*\,\Lambda}\mathcal{M}_\Lambda - \mathcal{L}^\Lambda \mathcal{M}^*_\Lambda = -i\,, \tag{5.1.33}$$

we can rewrite the equation for ω to the alternative form

$$(d\omega)_{xy} = 2\epsilon_{xyz}\langle \mathcal{I} \mid \partial^z \mathcal{I}\,\rangle\,, \tag{5.1.34}$$

whose integrability condition is

$$\langle \mathcal{I} \mid \nabla_{\underline{m}}\partial^{\underline{m}}\mathcal{I}\,\rangle = 0\,, \tag{5.1.35}$$

and will be satisfied by harmonic functions on the 3-dimensional space, *i.e.* by those real symplectic sections satisfying $\nabla_{\underline{m}}\partial^{\underline{m}}\mathcal{I} = 0$. In general the harmonic functions will have singularities leading to non-trivial constraints like those studied in Refs. [86,87].

Solving the Killing spinor equations

We are now going to see that it is always possible to solve the KSEs for field configurations with metric of the form (5.1.23) where the 1-form ω satisfies Eq. (5.1.30) and the 3-dimensional metric has spin connection related to the $SU(2)$ connection by Eq. (5.1.28), vector fields of the form (5.1.12) and (5.1.13), time-independent scalars Z^i and, most importantly, hyperscalars satisfying

$$\mathsf{U}^{\alpha J}{}_x\ (\sigma_x)_J{}^I \;=\; 0\,, \qquad \mathsf{U}^{\alpha J}{}_x \;\equiv\; V_x{}^{\underline{m}}\partial_{\underline{m}}q^u\ \mathsf{U}^{\alpha J}{}_u\,, \tag{5.1.36}$$

which results from Eqs. (5.1.16), (5.1.29) and (5.1.20).

Let us consider first the $\delta_\epsilon\zeta_\alpha = 0$ equation. Using the Vierbein Eq. (5.1.25) and multiplying by γ^0 it can be rewritten in the form

$$\mathsf{U}_{\alpha I\,x}\ \gamma^{0x}\ \epsilon^I \;=\; 0\,, \tag{5.1.37}$$

which can be solved using Eq. (5.1.36) if the spinors satisfy a constraint

$$\Pi^x{}_I{}^J\ \epsilon_J \;=\; 0 \quad , \quad \Pi^x{}_I{}^J \;\equiv\; \tfrac{1}{2}[\ \delta_I{}^J \;-\; \gamma^{0(x)}\ (\sigma_{(x)})_I{}^J\ ] \qquad \text{(no sum over } x\text{)}, \tag{5.1.38}$$

for each non-vanishing $\mathsf{U}_{\alpha I\,x}$. These three operators are projectors, *i.e.* they satisfy $(\Pi^x)^2 = \Pi^x$, and commute with each other. From $(\sigma_{(x)})_I{}^K\ \Pi^{(x)}{}_K{}^J\ \epsilon_J\ =\ 0$ we find

$$(\sigma_{(x)})_I{}^J\epsilon_J\ =\ \gamma^{0(x)}\epsilon_I\,, \tag{5.1.39}$$

which solves $\delta_\epsilon\zeta_\alpha = 0$ together with Eq. (5.1.36) and tells us that the embedding of the $SU(2)$ connection in the Lorentz group requires the action of the generators of $\mathfrak{su}(2)$ to be identical to the action of the three Lorentz generators $\frac{1}{2}\gamma^{0x}$ on the spinors. When we impose these constraints on the spinors, each of the first two reduces by a factor of 1/2 the number of independent spinors, but the third condition is implied by the first two and does not reduce any further the number of independent spinors.

Observe that

$$\Pi^{x\,I}{}_J\ \equiv\ (\Pi^x{}_I{}^J)^*\ =\ -\varepsilon^{IK}\ \Pi^x{}_K{}^L\ \varepsilon_{LJ}\,. \tag{5.1.40}$$

Let us now consider the equation $\delta_\epsilon\lambda^{iI} = 0$. It takes little to no time to realize that it reduces to the same form as in absence of hypermultiplets

$$\delta_\epsilon\lambda^{iI}\ =\ i\ \partial\!\!\!/ Z^i\ (\epsilon^I + i\gamma_0 e^{-i\alpha}\varepsilon^{IJ}\epsilon_J\ )\ = 0\ \ , \tag{5.1.41}$$

the only difference being in the implicit presence of the non-trivial Dreibein in $\partial\!\!\!/ Z^i$. Therefore, as before, this equation is solved by imposing the constraint

$$\epsilon^I\ +\ i\gamma_0\ e^{-i\alpha}\varepsilon^{IJ}\ \epsilon_J\ =\ 0\,, \tag{5.1.42}$$

which can be seen to commute with the projections Π^x since, by virtue of Eq. (5.1.40),

$$\Pi^{x\,K}{}_I\ (\epsilon^I + i\gamma_0 e^{-i\alpha}\varepsilon^{IJ}\epsilon_J)\ =\ (\Pi^{x\,K}{}_I\epsilon^I) + i\gamma_0 e^{-i\alpha}\varepsilon^{KJ}(\Pi^x{}_J{}^L\epsilon_L)\,. \tag{5.1.43}$$

Let us finally consider the equation $\delta_\epsilon\lambda^{iI} = 0$: in the $SU(2)$ gauge $\mathsf{A}^x{}_t = 0$ the 0th component of the equation is automatically solved by time-independent Killing spinors using the above constraint. Again, the equation takes the same form as without hypermultiplets but with a non-trivial Dreibein. In the same gauge, the spatial (flat) components of the $\delta_\epsilon\lambda^{iI} = 0$ equation can be written, upon use of the above constraint and the relation Eq. (5.1.28) between the $SU(2)$ and spatial spin connection, in the form

$$X^{1/2}\partial_y(X^{-1/2}\epsilon_I)\ +\ \tfrac{i}{2}\mathsf{A}^x{}_y\ [(\sigma_x)_I{}^J\epsilon_J - \gamma^{0x}\epsilon_I]\ =\ 0 \quad , \quad \mathsf{A}^x{}_y\ =\ \mathsf{A}^x{}_u\partial_{\underline{m}}q^u\ V_y{}^{\underline{m}}\,, \tag{5.1.44}$$

which is solved by

$$\epsilon_I = X^{1/2}\epsilon_{I\,0}\,, \qquad \partial_\mu\epsilon_{I\,0} = 0\,, \qquad \epsilon_{I\,0} + i\gamma_0\varepsilon_{IJ}\epsilon^J{}_0 = 0\,, \qquad \Pi^x{}_I{}^J\ \epsilon_{J\,0} = 0\,, \tag{5.1.45}$$

where the constraints Eq. (5.1.38) are imposed for each non-vanishing component of the $SU(2)$ connection.

Equations of motion

According to the KSIs, all the equations of motion of the supersymmetric solutions will be satisfied if the Maxwell equations and Bianchi identities of the vector fields are satisfied. Before studying these equations it is important to notice that supersymmetry requires Eqs. (5.1.36) to be satisfied. We will assume here that this has been done and we will study in the next section possible solutions to these equations.

Using Eqs. (5.1.12) and (5.1.13) we can write the symplectic vector of 2-forms in the form

$$F = \frac{1}{2|X|^2}\{\hat{V}\wedge d[|X|^2\mathcal{R}] - {}^\star[\hat{V}\wedge \Im\mathrm{m}(\mathcal{V}^*\mathfrak{D}X + X^*\mathfrak{D}\mathcal{V})]\}\,, \tag{5.1.46}$$

which can be rewritten in the form

$$F = -\tfrac{1}{2}\{d[\mathcal{R}\hat{V}] + {}^\star[\hat{V}\wedge d\mathcal{I}]\}\,. \tag{5.1.47}$$

The Maxwell equations and Bianchi identities $dF = 0$ are, therefore, satisfied if

$$d^\star[\hat{V}\wedge d\mathcal{I}] = 0\,,\ \Rightarrow \nabla_{\underline{m}}\partial^{\underline{m}}\mathcal{I} = 0\,, \tag{5.1.48}$$

i.e. if the $2\bar{n}$ components of $\mathcal{I}$ are as many real harmonic functions in the 3-dimensional space with metric $\gamma_{\underline{mn}}$.

Summarizing, the timelike supersymmetric solutions are determined by a choice of Dreibein and hyperscalars such that Eq. (5.1.36) is satisfied and a choice of $2\bar{n}$ real harmonic functions in the 3-dimensional metric space determined by our choice of Dreibein $\mathcal{I}$. This choice determines the 1-form ω. The full $\mathcal{V}/X$ is determined in terms of $\mathcal{I}$ by solving the stabilization equations and with $\mathcal{V}/X$ one constructs the remaining elements of the solution as explained in Ref. [26].

The cosmic string scrutinized

It is always convenient to have an example that shows that we are not dealing with an empty set of solutions. As mentioned in the introduction we can find relatively simple non-trivial examples using the c-map on known supersymmetric solutions with only fields in the vector multiplets excited. A convenient solution is the cosmic string for the case $n = 1$ with scalar manifold $Sl(2,\mathbb{R})/U(1)$ and prepotential $\mathcal{F} = -\frac{i}{4}\mathcal{X}^0\mathcal{X}^1$. Parametrizing the scalars as $\mathcal{X}^0 = 1$ and $\mathcal{X}^1 = -i\tau$, we find from the formulae in appendix (D.2.1) that the only non-trivial fields of the c-dual solution are the spacetime metric

$$ds^2 \ = \ 2du\ dv \ - \ 2\ \mathrm{Im}(\tau)\ dzdz^*\ , \tag{5.1.49}$$

with $\tau = \tau(z)$, and the pull-back of the Quadbein is given by

$$\not{U}^{\alpha I} \;=\; [2\mathrm{Im}(\tau)]^{-3/2} \left(\begin{array}{ll} 0 & 0 \\ 0 & 0 \\ \partial_z \tau \;\gamma^z & 0 \\ 0 & \partial_{z^*}\tau^* \;\gamma^{z^*} \end{array} \right) . \tag{5.1.50}$$

From this form, then, it should be clear that the hyperscalar equation (2.2.27) is satisfied by

$$\gamma^z \epsilon^2 \;=\; \gamma^{z^*}\epsilon^1 \;=\; 0 \quad\longrightarrow\quad \gamma^z \epsilon_1 \;=\; \gamma^{z^*}\epsilon_2 \;=\; 0\;, \tag{5.1.51}$$

so that we have to face the fact that this solution can be at most 1/2-BPS.

Since we are dealing with a situation without vector multiplets and with a vanishing graviphoton, the gravitino variation (2.2.25) reduces to

$$0 \;=\; \nabla\, \epsilon_I \;+\; \mathsf{A}_I{}^J \epsilon_J\;. \tag{5.1.52}$$

For the c-mapped cosmic string, we have from Eqs. (D.0.10) and (D.2.20), that $\mathsf{A}_I{}^J \;=\; \frac{i}{2}\;\mathcal{Q}\;\sigma_3\,{}_I{}^J$. Also, for the metric at hand, the 4-d spin connection is readily calculated to be $\frac{1}{2}\omega_{ab}\gamma^{ab} = i\mathcal{Q}\;\gamma^{zz^*}$ (See *e.g.* [38]).

Due to the constraint (5.1.51), however, one can see that $\gamma^{zz^*}\epsilon_I \;=\; \sigma_3\,{}_I{}^J\;\epsilon_J$, which, when mixed with the rest of the ingredients, leads to, dropping the I-indices,

$$\text{Eq. (5.1.52)} \;=\; d\epsilon \;-\; \tfrac{1}{4}\omega_{ab}\gamma^{ab}\epsilon \;+\; \tfrac{i}{2}\mathcal{Q}\sigma_3\epsilon \;=\; d\epsilon\;, \tag{5.1.53}$$

so that the c-mapped cosmic string is a 1/2-BPS solution with, as was to be expected, a constant Killing spinor.

5.1.3 The null case

In the null case[3] the two spinors ϵ_I are proportional: $\epsilon_I = \phi_I\epsilon$. The complex functions ϕ_I, normalized such that $\phi^I\phi_I = 1$ and satisfying $\phi_I^* = \phi^I$, carry a -1 $U(1)$ charge *w.r.t.* the imaginary connection

$$\zeta \;\equiv\; \phi^I\;\mathfrak{D}\;\phi_I \quad\to\quad \zeta^* \;=\; -\zeta\,, \tag{5.1.54}$$

opposite to that of the spinor ϵ, whence ϵ_I is neutral. On the other hand, the ϕ_Is are neutral with respect to the Kähler connection, and the Kähler weight of the spinor ϵ is the same as that of the spinor ϵ_I, i.e. 1/2. The $SU(2)$-action is the one implied by the I-index structure.

The substitution of the null-case spinor condition into the KSEs (2.2.25–2.2.27) immediately yields

[3] The details concerning the normalization of the spinors and the construction of the bilinears in this case are explained in the Appendix of Ref. [38], which you are strongly urged to consult at this point.

$$\mathfrak{D}_\mu\phi_I\epsilon + \phi_I\mathfrak{D}_\mu\epsilon + \varepsilon_{IJ}\phi^J T^+{}_{\mu\nu}\gamma^\nu\epsilon^* \;=\; 0\,, \tag{5.1.55}$$

$$\phi^I \,\not{\partial} Z^i\epsilon^* + \varepsilon^{IJ}\phi_J \,\not{G}^{i\,+}\epsilon \;=\; 0\,, \tag{5.1.56}$$

$$\mathbb{C}_{\alpha\beta}\mathsf{U}^{\beta I}{}_u\varepsilon_{IJ}\,\not{\partial} q^u\phi^J\epsilon^* \;=\; 0. \tag{5.1.57}$$

Contracting Eq. (5.1.55) with ϕ^I results in

$$\mathfrak{D}_\mu\epsilon \;=\; -\phi^I\,\mathfrak{D}_\mu\phi_I\,\epsilon \;\longleftarrow\; \tilde{\mathfrak{D}}_\mu\epsilon \;\equiv\; (\mathfrak{D}_\mu \;+\; \zeta_\mu)\epsilon \;=\; 0\,, \tag{5.1.58}$$

which is the only differential equation for ϵ. Substituting Eq. (5.1.58) into Eq. (5.1.55) as to eliminate the $\mathfrak{D}_\mu\epsilon$ term, we obtain

$$\left(\tilde{\mathfrak{D}}_\mu\phi_I\right)\,\epsilon \;+\; \varepsilon_{IJ}\phi^J\;T^+{}_{\mu\nu}\gamma^\nu\;\epsilon^* \;=\; 0\,, \qquad \tilde{\mathfrak{D}}_\mu\phi_I \equiv (\mathfrak{D}_\mu - \zeta_\mu)\phi_I\,, \tag{5.1.59}$$

which is a differential equation for ϕ_I and, at the same time, an algebraic constraint for ϵ. Two further algebraic constraints can be found by acting with ϕ^I on Eq. (5.1.56):

$$\not{\partial} Z^i\,\epsilon^* \;=\; \not{G}^{i\,+}\;\epsilon \;=\; 0\,. \tag{5.1.60}$$

Finally, we add to the set-up an auxiliary spinor η, with the same chirality as ϵ but with all $U(1)$ charges reversed, and impose the normalization condition

$$\bar{\epsilon}\eta = \tfrac{1}{2}\,. \tag{5.1.61}$$

This normalization condition will be preserved if and only if η satisfies the differential equation

$$\tilde{\mathfrak{D}}_\mu\eta \;+\; \mathfrak{a}_\mu\;\epsilon \;=\; 0\,, \tag{5.1.62}$$

for some $\mathfrak{a}$ with $U(1)$ charges -2 times those of ϵ, i.e.

$$\tilde{\mathfrak{D}}_\mu\;\mathfrak{a}_\nu \;=\; (\nabla_\mu \;-\; 2\zeta_\mu \;-\; i\mathcal{Q}_\mu)\;\mathfrak{a}_\nu\,. \tag{5.1.63}$$

$\mathfrak{a}$ is to be determined by the requirement that the integrability conditions of the above differential equation be compatible with those for ϵ.

Killing equations for the vector bilinears and first consequences

We are now ready to derive equations involving the bilinears, in particular the vector bilinears which we construct with ϵ and the auxiliary spinor η introduced above. First we deal with the equations that do not involve derivative of the spinors. Acting with $\bar{\epsilon}$ on Eq. (5.1.59) and with $\bar{\epsilon}\gamma^\mu$ on Eq. (5.1.60) we find

$$T^+{}_{\mu\nu}\, l^\nu \;=\; G^{i\,+}{}_{\mu\nu}\, l^\nu \;=\; 0 \quad \longrightarrow \quad F^{\Lambda\,+}{}_{\mu\nu}\, l^\nu \;=\; 0\,, \tag{5.1.64}$$

which implies

$$F^{\Lambda\,+} \;=\; \tfrac{1}{2}\,\varphi^\Lambda\; \hat{l}\wedge\hat{m}^*\,, \tag{5.1.65}$$

for some complex functions φ^Λ. Acting with $\bar{\eta}$ on Eq. (5.1.59) we get

$$\tilde{\mathfrak{D}}_\mu\phi_I + i\sqrt{2}\varepsilon_{IJ}\phi^J T^+{}_{\mu\nu}m^\nu = 0\,, \tag{5.1.66}$$

and substituting Eq. (5.1.65) into it, we arrive at

$$\tilde{\mathfrak{D}}_\mu\, \phi_I \;-\; \tfrac{i}{\sqrt{2}}\,\varepsilon_{IJ}\phi^J\; \mathcal{T}_\Lambda\varphi^\Lambda\; l_\mu \;=\; 0\,. \tag{5.1.67}$$

Finally, acting with $\bar{\epsilon}$ and $\bar{\eta}$ on Eq. (5.1.60) we get

$$l^\mu\; \partial_\mu Z^i \;=\; m^\mu\; \partial_\mu Z^i \;=\; 0 \quad \longrightarrow \quad dZ^i \;=\; A^i\; \hat{l} \;+\; B^i\; \hat{m}\,, \tag{5.1.68}$$

for some functions A^i and B^i.

The relevant differential equations specifying the possible spacetime dependencies for the tetrad follow from Eqs. (5.1.58) and (5.1.62). *I.e.*

$$\nabla_\mu\, l_\nu \;\;=\;\; 0\,, \tag{5.1.69}$$

$$\tilde{\mathfrak{D}}_\mu\, n_\nu \;\;\equiv\;\; \nabla_\mu\, n_\nu \;=\; -\mathfrak{a}^*_\mu\, m_\nu \;-\; \mathfrak{a}_\mu\, m^*_\nu\,, \tag{5.1.70}$$

$$\tilde{\mathfrak{D}}_\mu m_\nu \;\;\equiv\;\; (\nabla_\mu \;-\; 2\zeta_\mu \;-\; i\mathcal{Q}_\mu)\; m_\nu \;=\; -\mathfrak{a}_\mu\; l_\nu\,. \tag{5.1.71}$$

Equations of motion and integrability constraints

As was discussed in Sec. (3.2.1), the KSIs in the case at hand don't vary a great deal, with respect to the ones derived in [26], and so we can be brief: the only equations of motion that are automatically satisfied are the ones for the graviphoton and the ones for the scalars from the vector multiplets. As one can see from Eq. (5.1.4), the same thing cannot be said about the equation of motion for the hyperscalar, but as we shall see in a few pages, it is anyhow identically satisfied. The, at the moment, relevant KSI is

$$\left(\mathcal{E}_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu}\,\mathcal{E}_{\sigma}{}^{\sigma}\right)\, l^{\nu} \;=\; \left(\mathcal{E}_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu}\,\mathcal{E}_{\sigma}{}^{\sigma}\right)\, m^{\nu} \;=\; 0\,, \tag{5.1.72}$$

where the relation of the equation of motion with and without hypermultiplets is given in Eq. (2.2.39).

Substituting the expressions (5.1.68) and (5.1.65) into the above KSIs we find the two conditions

$$0 \;=\; [\, R_{\mu\nu} \;+\; 2\mathsf{H}_{uv}\,\partial_\mu q^u\,\partial_\nu q^v]\; l^{\nu}\,, \tag{5.1.73}$$

$$0 \;=\; [\, R_{\mu\nu} \;+\; 2\mathsf{H}_{uv}\,\partial_\mu q^u\,\partial_\nu q^v]\; m^{\nu} \;-\; \mathcal{G}_{ij^*}\,\left(A^i l_\mu \;+\; B^i m_\mu\right)\, B^{*\,j^*}\,. \tag{5.1.74}$$

Comparable equations can be found from the integrability conditions of Eq. (5.1.58), *i.e.*

$$0 \;=\; [\, R_{\mu\nu} \;+\; 2(d\zeta)_{\mu\nu}]\; l^{\nu}\,, \tag{5.1.75}$$

$$0 \;=\; [\, R_{\mu\nu} \;+\; 2(d\zeta)_{\mu\nu}]\; m^{*\,\nu} \;-\; \mathcal{G}_{ij^*} B^i\,(A^{*\,j^*}\; l_\mu \;+\; B^{*\,j^*}\; m^*_\mu)\,, \tag{5.1.76}$$

and those of Eq. (5.1.62)

$$0 \;=\; [\, R_{\mu\nu} \;-\; 2(d\zeta)_{\mu\nu}]\; m^{\nu} \;-\; \mathcal{G}_{ij^*}\,(A^i l_\mu + B^i m_\mu)\, B^{*\,j^*} \;+\; 2(\tilde{\mathfrak{D}}\mathfrak{a})_{\mu\nu} l^{\nu}, \tag{5.1.77}$$

$$0 \;=\; [\, R_{\mu\nu} - 2(d\zeta)_{\mu\nu}]\; n^{\nu} \;+\; 2(\tilde{\mathfrak{D}}\mathfrak{a})_{\mu\nu}\; m^{*\,\nu}\,. \tag{5.1.78}$$

In the derivation of these last identities use has been made of the formulae

$$(d\mathcal{Q})_{\mu\nu}\; m^{*\nu} \;=\; i\mathcal{G}_{ij^*}\; B^i\; B^{*\,j^*}\; m^*_\mu\,, \qquad (d\mathcal{Q})_{\mu\nu}\; l^{\nu} \;=\; (d\mathcal{Q})_{\mu\nu}\; n^{\nu} \;=\; 0\,, \tag{5.1.79}$$

which follow from the definition of the Kähler connection and from Eq. (5.1.68).

Comparing these three sets of equations, we find that they are compatible if

$$(d\zeta)_{\mu\nu}\; l^{\nu} \;=\; \mathsf{H}_{uv}\;\partial_\mu q^u\; l^{\nu}\partial_\nu q^v\;, \tag{5.1.80}$$

$$(d\zeta)_{\mu\nu}\; m^{*\nu} \;=\; \mathsf{H}_{uv}\;\partial_\mu q^u\; m^{*\nu}\partial_\nu q^v\;, \tag{5.1.81}$$

and

$$(\tilde{\mathfrak{D}}\;\mathfrak{a})_{\mu\nu}\; l^{\nu} \;=\; 0\,. \tag{5.1.82}$$

Please observe that, due to the positive definiteness of H, Eq. (5.1.80) implies $l^{\nu}\partial_\nu q^v = 0$, but that Eq. (5.1.81) need not imply $m^{*\nu}\partial_\nu q^v = 0$.

A coordinate system, some more consistency and an anti-climax

In order to advance in our quest, it is useful to introduce a coordinate representation for the tetrad and hence also for the metric. Since $\hat{l}$ is a covariantly constant vector, we can introduce coordinates u and v through $l^{\mu}\partial_{\mu} = \partial_v$ and $l_{\mu}dx^{\mu} = du$. We can also define a complex coordinates z and z^* by

$$\hat{m} \;=\; e^{U}\; dz \quad , \quad \hat{m}^* \;=\; e^{U}\; dz^*\,, \tag{5.1.83}$$

where U may depend on z, z^* and u, but not v. Eq. (5.1.68) then implies that the scalars Z^i are just functions of z and u:

$$Z^i = Z^i(z,u)\,, \tag{5.1.84}$$

wherefore the functions A^i and B^i defined in Eq. (5.1.68) are

$$A^i \;=\; \partial_{\underline{u}} Z^i\,, \qquad e^{U}B^i \;=\; \partial_{\underline{z}} Z^i\,,\; \Rightarrow \partial_{\underline{z}^*}(e^{U}B^i) \;=\; 0\,. \tag{5.1.85}$$

Finally, the most general form that $\hat{n}$ can take in this case is

$$\hat{n} = dv + Hdu + \hat{\omega}\,, \qquad \hat{\omega} = \omega_{\underline{z}}dz + \omega_{\underline{z}^*}dz^*\,, \tag{5.1.86}$$

where all the functions in the metric are independent of v. The above form of the null tetrad components leads to a Brinkmann pp-wave metric [88][4]

$$ds^2 \;=\; 2du\,(dv \;+\; Hdu \;+\; \hat{\omega}) \;-\; 2e^{2U}\; dzdz^*\,. \tag{5.1.87}$$

As we now have a coordinate representation at our disposal, we can start checking out the consistency conditions in this representation: Let us expand the connection ζ as

$$\zeta \;=\; i\zeta_n\;\hat{n} \;+\; i\zeta_l\;\hat{l} \;+\; \zeta_m\hat{m} \;-\; \zeta_{m^*}\hat{m}^*\,, \tag{5.1.88}$$

where ζ_l and ζ_n are real functions, whereas ζ_m is complex. Likewise expand

$$\hat{\mathfrak{a}} \;=\; \mathfrak{a}_l\;\hat{l} \;+\; \mathfrak{a}_m\;\hat{m} \;+\; \mathfrak{a}_{m^*}\;\hat{m}^* \;+\; \mathfrak{a}_n\;\hat{n}\,, \tag{5.1.89}$$

and

$$\mathcal{Q} \;=\; \mathcal{Q}_l\;\hat{l} \;+\; \mathcal{Q}_m\;\hat{m} \;+\; \mathcal{Q}_{m^*}\;\hat{m}^* \;+\; \mathcal{Q}_n\;\hat{n}, \tag{5.1.90}$$

where, due to the reality of $\mathcal{Q}$, $(\mathcal{Q}_m)^* = \mathcal{Q}_{m^*}$. Let us now consider the tetrad integrability equations (5.1.69)-(5.1.71): Eq. (5.1.69) is by construction identically satisfied. Eq. (5.1.71), with our choice of coordinate z Eq. (5.1.83), implies

$$\begin{aligned} 0 \;&=\; e^{-U}\partial_{\underline{z}^*}U \;+\; 2\zeta_{m^*} \;-\; i\mathcal{Q}_{m^*}\,, && (5.1.91)\\ 0 \;&=\; -2i\zeta_n \;-\; i\mathcal{Q}_n\,, && (5.1.92)\end{aligned}$$

[4] The components of the connection and the Ricci tensor of this metric can be found in the Appendix of Ref. [38].

and

$$\hat{\mathfrak{a}} \;=\; \left[\dot{U} \;-\; 2i\zeta_l \;-\; i\mathcal{Q}_l\right] \;\hat{m} \;+\; \mathfrak{a}_l \;\hat{l}\,, \tag{5.1.93}$$

where $\mathfrak{a}_l = \mathfrak{a}_l(z, z^*, u)$ is a functions to be determined and dots indicate partial derivation *w.r.t.* the coordinate u. Eq. (5.1.84) implies that $\zeta_n = \mathcal{Q}_n = 0$ and from Eq. (5.1.91) we obtain

$$\partial_{\underline{z}^*}(U + \tfrac{1}{2}\mathcal{K}) = -2\zeta_{\underline{z}^*}\,. \tag{5.1.94}$$

This last equation states that ζ^*_m, whence also ζ_m, can be eliminated by a gauge transformation, after which we are left with

$$\hat{\zeta} \;=\; i\zeta_l \;\hat{l}\,. \tag{5.1.95}$$

At this point it is wise to return to Eq. (5.1.81) and to deduce

$$\begin{aligned} \mathsf{H}_{uv}\;\partial_\mu q^u\; m^{*\nu}\partial_\nu q^v \;=\; (d\zeta)_{\mu\nu}\; m^{*\;\nu} \;&=\; 2e^{-U}(\partial_{\underline{z}}\zeta_l\; m_{[\mu}\; l_{\nu]} \;+\; \partial_{\underline{z}^*}\zeta_l\; m^*_{[\mu}l_{\nu]})m^{*\nu} \\ &=\; e^{-U}\partial_{\underline{z}^*}\zeta_l\; l_\mu\,. \end{aligned} \tag{5.1.96}$$

This equation implies that $dq^u \sim \hat{l}$, and we are therefore obliged to accept the fact that in the null case, the hyperscalars can only depend on the spacetime coordinate u!

Had we been hoping for the hyperscalars to exhibit some interesting spacetime dependency, then this result would have been a bit of an anti-climax. But then, the fact that the hyperscalars can only depend on u, means that we can eliminate the connection A from the initial set-up, which means that as far as solutions to the Killing Spinor equations is concerned, the problem splits into two disjoint parts: one is the solution to the KSEs in the null case of $N = 2$ $d = 4$ supergravity, which are to be found in [26, 34], and the solutions to Eq. (2.2.27).

In the case at hand Eq. (2.2.27) reduces to

$$0 \;=\; \mathsf{U}_v^{\alpha I}\varepsilon_{IJ}\;\partial_u q^v\;\gamma^u\epsilon^J\;, \tag{5.1.97}$$

so that either we take the hyperscalars to be constant or impose the condition $\gamma^u\epsilon^I = 0$. This last condition is however always satisfied by any non-maximally supersymmetric solution of the null case, to wit Minkowski space and the 4D Kowalski-Glikman wave. It is however obvious that these solutions are incompatible with u-dependent hyperscalars, and its reason takes us to the last point in this exposition: the equations of motion.

As far as the equations of motion are concerned, it is clear that, since we are dealing with a pp-wave metric, the hyperscalar equation of motion is identically satisfied. As the only coupling between vector multiplets and hypermultiplets is through the gravitational interaction, see Eq. (2.2.39), the only equation of motion that changes is the one in the uu-direction. More to the point, its sole effect is to change the differential equation [26, (5.91)] determining the wave profile H in (5.1.87).

A fitting example of a solution demonstrating just this, consider the deformation of the cosmic string solution found in Ref. [26][5]:

$$\begin{aligned} ds^2 &= 2\,du\,\left(dv + \mathsf{H}(\dot{q},\dot{q})\,|z|^2\right) - 2e^{-\mathcal{K}}\,dz\,dz^*\,, & Z^i &= Z^i(z)\,, \\ F^\Lambda &= 0\,, & q^w &= q^w(u)\,, \end{aligned} \tag{5.1.99}$$

which is a 1/2-BPS solution.

5.1.4 Summary

Let us summarize our results:

1. In the timelike case supersymmetric the configurations are completely determined by

   (a) A 3-dimensional space metric

   $$\gamma_{\underline{mn}}dx^m dx^n\,, \quad m,n = 1,2,3\,, \tag{5.1.100}$$

   and a mapping $q^u(x)$ from it to the quaternionic hyperscalar manifold such that the 3-dimensional spin connection[6] $\varpi_x{}^y$ is related to the pullback of the quaternionic $SU(2)$ connection A^x by

   $$\varpi_{\underline{m}}{}^{xy} = \varepsilon^{xyz}\mathsf{A}^z{}_u\,\partial_{\underline{m}}q^u\,, \tag{5.1.101}$$

   and such that

   $$\mathsf{U}^{\alpha J}{}_x\,(\sigma_x)_J{}^I = 0\,, \qquad \mathsf{U}^{\alpha J}{}_x \equiv V_x{}^{\underline{m}}\partial_{\underline{m}}q^u\,\mathsf{U}^{\alpha J}{}_u\,, \tag{5.1.102}$$

   where $\mathsf{U}^{\alpha I}{}_u$ is the *Quadbein* defined in Appendix D.

   (b) A choice of a symplectic vector $\mathcal{I} \equiv \Im\mathrm{m}(\mathcal{V}/X)$ whose components are real harmonic functions with respect to the above 3-dimensional metric:

   $$\nabla_{\underline{m}}\partial^{\underline{m}}\mathcal{I} = 0\,. \tag{5.1.103}$$

[5] This solutions reads

$$\begin{aligned} ds^2 &= 2\,du\,dv - 2e^{-\mathcal{K}}\,dz\,dz^*\,, & Z^i &= Z^i(z)\,, \\ F^\Lambda &= 0\,, & q^u &= \text{const.}\,, \end{aligned} \tag{5.1.98}$$

[6] In this paper we use $x, y, z = 1, 2, 3$ as tangent-space indices or as $SU(2)$ indices.

Given $\mathcal{I}$, $\mathcal{R} \equiv \Re\mathrm{e}(\mathcal{V}/X)$ can in principle be found by solving the generalized stabilization equations and then the metric is given by

$$ds^2 = |M|^2(dt + \omega)^2 - |M|^{-2}\gamma_{\underline{mn}}dx^m dx^n\,, \tag{5.1.104}$$

where

$$|M|^{-2} \;=\; \langle \mathcal{R} \mid \mathcal{I}\rangle\,, \tag{5.1.105}$$

$$(d\omega)_{xy} \;=\; 2\epsilon_{xyz}\langle \mathcal{I} \mid \partial^z \mathcal{I}\,\rangle\,. \tag{5.1.106}$$

The second equation implicitly contains the Dreibein of the 3-dimensional metric γ and its integrability condition is

$$\langle \mathcal{I} \mid \nabla_{\underline{m}}\partial^{\underline{m}}\mathcal{I}\rangle = 0\,. \tag{5.1.107}$$

As is discussed in *e.g.* Refs. [86, 87], this condition will lead to non-trivial constraints.

The vector field strengths are given by

$$F = -\tfrac{1}{\sqrt{2}}\{d[|M|^2\mathcal{R}(dt+\omega)] - {}^{\star}[|M|^2 d\mathcal{I}\wedge(dt+\omega)]\}\,, \tag{5.1.108}$$

and the scalar fields Z^i can be computed by taking the quotients

$$Z^i = (\mathcal{V}/X)^i/(\mathcal{V}/X)^0\,. \tag{5.1.109}$$

The hyperscalars $q^u(x)$ are just the mapping whose existence we assumed from the onset.

These solutions can therefore be seen as deformations of those devoid of hypers, originally found in Ref. [89].

As for the number of unbroken supersymmetries, the presence of non-trivial hyperscalars breaks 1/2 or 1/4 of the supersymmetries of the related solution without hypers, which may have all or 1/2 of the original supersymmetries. Therefore, we will have solutions with $1/2, 1/4$ and 1/8 of the original supersymmetries. The Killing spinors take the form

$$\epsilon_I = X^{1/2}\epsilon_{I\,0}\,, \qquad \partial_\mu\epsilon_{I\,0} = 0\,, \qquad \epsilon_{I\,0} + i\gamma_0\varepsilon_{IJ}\epsilon^J{}_0 = 0\,, \qquad \Pi^x{}_I{}^J\,\epsilon_{J\,0} = 0\,, \tag{5.1.110}$$

where the first constraint is imposed only if there are non-trivial vector multiplets and each of the other three constraints is imposed for each non-vanishing component of the $SU(2)$ connection. Each constraint breaks 1/2 of the supersymmetries independently, but the third constraint $\Pi^x{}_I{}^J\ \epsilon_{J\,0} = 0$ is implied by the first two. Finally, the meaning of these last three constraints is that they enforce the embedding of the gauge connection into the gauge connection since they are in different representations.

2. In the null case the hyperscalars can only depend on the null coordinate u and the solutions take essentially the same form as in the case without hypermultiplets (See Ref. [26]).

5.2 $N = 2$ Einstein-Yang-Mills Supergravity

In this section we are going to find the supersymmetric configurations and solutions of $N = 2$ Einstein-Yang-Mills Supergravity[7] in four dimensions. Further we are going to study how the *attractor mechanism* works in case of non-Abelian black holes. The existence of the *attractor mechanism* for the values of the scalars is one of the most interesting aspects of the supersymmetric black holes of ungauged $N = 2, d = 4$ Supergravity [90, 91]: independently of their asymptotic values, the values of the scalars on the event horizon are fully determined by the conserved charges. As a result, the Bekenstein-Hawking entropy only depends on conserved charges which is, by itself, a strong indication that it admits a microscopic interpretation. It is of utmost interest, then, to study if and how the attractor mechanism works for the supersymmetric non-Abelian black holes in these theories.

5.2.1 Supersymmetric configurations: general setup

Our first goal is to find all the bosonic field configurations $\{g_{\mu\nu}, F^{\Lambda}{}_{\mu\nu}, Z^i\}$ for which the Killing spinor equations (KSEs):

$$\delta_\epsilon \psi_{I\,\mu} \quad = \quad \mathfrak{D}_\mu \epsilon_I + \epsilon_{IJ} T^{+}{}_{\mu\nu}\gamma^\nu \epsilon^J = 0\,, \tag{5.2.1}$$

$$\delta_\epsilon \lambda^{Ii} \quad = \quad i\not{\mathfrak{D}} Z^i \epsilon^I + \epsilon^{IJ}[\not{G}^{i\,+} + W^i]\epsilon_J = 0\,, \tag{5.2.2}$$

admit at least one solution.

Our second goal will be to identify among all the supersymmetric field configurations those that satisfy all the equations of motion (including the Bianchi identities).

Let us initiate the analysis of the KSEs by studying their integrability conditions.

[7] Remember that $N = 2$ $d = 4$ Supergravity coupled to non-Abelian vector supermultiplets we refer to as $N = 2$ Einstein-Yang-Mills (EYM) Supergravity.

Killing Spinor Identities (KSIs)

The off-shell equations of motion of the bosonic fields of bosonic supersymmetric configurations satisfy certain relations known as (*Killing spinor identities*, KSIs) [56,57]. If we assume that the Bianchi identities are always identically satisfied everywhere, the KSIs only depend on the supersymmetry transformation rules of the bosonic fields. These are identical for the gauged and ungauged theories, implying that their KSIs are also identical. If we do not assume that the Bianchi identities are identically satisfied everywhere, then they also occur in the KSIs, which now have to be found via the integrability conditions of the KSEs. In the ungauged case they occur in symplectic-invariant combinations, as one would expect, and take the form [26]

$$\mathcal{E}_a{}^{\mu}\gamma^a\epsilon^I - 4i\epsilon^{IJ}\langle\, \mathcal{E}^{\mu} \mid \mathcal{V}\,\rangle\epsilon_J \quad = \quad 0\,, \tag{5.2.3}$$

$$\mathcal{E}^i\epsilon^I - 2i\epsilon^{IJ}\langle\, \not{\mathcal{E}} \mid \mathcal{U}^{*i}\,\rangle\epsilon_J \quad = \quad 0\,, \tag{5.2.4}$$

where

$$\mathcal{E}^a \equiv \begin{pmatrix} \mathcal{B}^{\Lambda\, a} \\ \mathcal{E}_{\Lambda}{}^a \end{pmatrix}. \tag{5.2.5}$$

We have checked through explicit computation that these relations remain valid in the non-Abelian gauged case at hand.

Taking products of these expressions with Killing spinors and gamma matrices, one can derive KSIs involving the bosonic equations and tensors constructed as bilinears of the commuting Killing spinors.[8] In the case in which the bilinear $V^{\mu} \equiv i\bar{\epsilon}^I\gamma^{\mu}\epsilon_I$ is a timelike vector (referred to as the *timelike case*), one obtains [87] the following identities (*w.r.t.* an orthonormal frame with $e_0{}^{\mu} \equiv V^{\mu}/|V|$)

$$\mathcal{E}^{ab} \quad = \quad \eta^a{}_0\eta^b{}_0\mathcal{E}^{00}\,, \tag{5.2.6}$$

$$\langle\, \mathcal{V}/X \mid \mathcal{E}^a\,\rangle \quad = \quad \tfrac{1}{4}|X|^{-1}\mathcal{E}^{00}\delta^a{}_0\,, \tag{5.2.7}$$

$$\langle\, \mathcal{U}^*_{i^*} \mid \mathcal{E}^a\,\rangle \quad = \quad \tfrac{1}{2}e^{-i\alpha}\mathcal{E}_{i^*}\delta^a{}_0\,, \tag{5.2.8}$$

where $X \equiv \frac{1}{2}\varepsilon_{IJ}\bar{\epsilon}^I\epsilon^J$ and is non-zero in the timelike case.

As discussed in Ref. [87], these identities contain a great deal of physical information. In this paper we shall exploit only one fact, namely the fact that if the Maxwell equation and the Bianchi identity are satisfied for a supersymmetric configuration,

[8] See the appendix in Ref. [38] for the definitions and properties of these bilinears.

then so are the rest of the equations of motion. The strategy to be followed is, therefore, to first identify the supersymmetric configurations and impose the Maxwell equations and the Bianchi identities. This will lead to some differential equations that need be solved in order to construct a supersymmetric solution.

In the case in which V^μ is a null vector (the *null case*), renaming it as l^μ for reasons of clarity, one gets

$$(\mathcal{E}_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}\mathcal{E}^\rho{}_\rho)l^\nu = (\mathcal{E}_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}\mathcal{E}^\rho{}_\rho)m^\nu = 0\,, \tag{5.2.9}$$

$$\mathcal{E}_{\mu\nu}l^\nu = \mathcal{E}_{\mu\nu}m^\nu = 0\,, \tag{5.2.10}$$

$$\langle \mathcal{V} \mid \mathcal{E}^\mu \rangle = 0\,, \tag{5.2.11}$$

$$\langle \mathcal{U}^*_{i^*} \mid \mathcal{E}^\mu \rangle\, l_\mu = \langle \mathcal{U}^*_{i^*} \mid \mathcal{E}^\mu \rangle\, m^*_\mu = 0\,, \tag{5.2.12}$$

$$\mathcal{E}^i = 0\,, \tag{5.2.13}$$

where l, n, m, m^* is a null tetrad constructed with the Killing spinor ϵ^I and an auxiliary spinor η as explained in Ref. [26].

These identities imply that the only independent equations of motion that one has to check on supersymmetric configurations are $\mathcal{E}_{\mu\nu}n^\mu n^\nu$ and $\langle \mathcal{U}^*_{i^*} \mid \mathcal{E}_\mu \rangle\, n^\mu$. As before, these are the equations that need to be imposed in order for a supersymmetric configuration to be a supersymmetric solution.

Killing equations for the bilinears

In order to find the most general background admitting a solution to the KSEs, Eqs. (5.2.1) and (5.2.2), we shall assume that the background admits one Killing spinor. Using this assumption we will derive consistency conditions that the background must satisfy, after which we will prove that these necessary conditions are also sufficient.

It is convenient to work with spinor bilinears, and consequently we start by deriving equations for these bilinears by contracting the KSEs with gamma matrices and Killing spinors.

From the gravitino supersymmetry transformation rule Eq. (4.2.12) we get the independent equations

$$\mathfrak{D}_\mu X = -iT^+{}_{\mu\nu}V^\nu \,, \tag{5.2.14}$$

$$\mathfrak{D}_\mu V^I{}_{J\,\nu} = i\delta^I{}_J[XT^{*-}{}_{\mu\nu} - X^*T^+{}_{\mu\nu}] \tag{5.2.15}$$

$$-i[\epsilon^{IK}T^{*-}{}_{\mu\rho}\Phi_{KJ}{}^\rho{}_\nu - \epsilon_{JK}T^+{}_{\mu\rho}\Phi^{KI\,\rho}{}_\nu]\,, \tag{5.2.16}$$

which have the same functional form as their equivalents in the ungauged case. Hence, as in the ungauged case, V^μ is a Killing vector and the 1-form $\hat{V} \equiv V_\mu dx^\mu$ satisfies the equation

$$d\hat{V} = 4i[XT^{*-} - X^*T^+]\,. \tag{5.2.17}$$

The remaining 3 independent 1-forms $\hat{V}^x \equiv \frac{1}{\sqrt{2}}V^I{}_{J\,\mu}\sigma^{x\,J}{}_I dx^\mu$ ($x = 1,2,3$ and the σ^x are the Pauli matrices) are exact, *i.e.*

$$d\hat{V}^x = 0\,. \tag{5.2.18}$$

From the gauginos' supersymmetry transformation rules, Eqs. (4.2.13), we obtain

$$V^I{}_K{}^\mu\mathfrak{D}_\mu Z^i + \epsilon^{IJ}\Phi_{KJ}{}^{\mu\nu}G^{i\,+}{}_{\mu\nu} + W^i\epsilon^{IJ}M_{KJ} = 0\,, \tag{5.2.19}$$

$$iM^{KI}\mathfrak{D}_\mu Z^i + i\Phi^{KI}{}_\mu{}^\nu\mathfrak{D}_\nu Z^i - 4i\epsilon^{IJ}V^K{}_J{}^\nu G^{i\,+}{}_{\mu\nu} - iW^i\epsilon^{IJ}V^K{}_{J\,\mu} = 0\,. \tag{5.2.20}$$

The trace of the first equation gives

$$V^\mu\mathfrak{D}_\mu Z^i + 2XW^i = 0\,, \tag{5.2.21}$$

while the antisymmetric part of the second equation gives

$$2X^*\mathfrak{D}_\mu Z^i + 4G^{i\,+}{}_{\mu\nu}V^\nu + W^iV_\mu = 0\,. \tag{5.2.22}$$

The well-known special geometry completeness relation implies that

$$F^{\Lambda\,+} = i\mathcal{L}^{*\,\Lambda}T^+ + 2f^\Lambda{}_iG^{i\,+}\,, \tag{5.2.23}$$

which allows us to combine Eqs. (5.2.14) and (5.2.22), as to obtain

$$\begin{aligned} V^\nu F^{\Lambda\,+}{}_{\nu\mu} &= i\mathcal{L}^{*\,\Lambda}V^\nu T^+{}_{\nu\mu} + 2f^\Lambda{}_iV^\nu G^{i\,+}{}_{\nu\mu} \\ &= \mathcal{L}^{*\,\Lambda}\mathfrak{D}_\mu X + X^*\mathfrak{D}_\mu\mathcal{L}^\Lambda + \tfrac{1}{2}W^iV_\mu\,. \end{aligned} \tag{5.2.24}$$

Multiplying this equation by V^μ and using Eq. (5.2.21), we find

$$V^\mu \mathfrak{D}_\mu X = 0\,. \tag{5.2.25}$$

At this point in the investigation, it is convenient to take into account the norm of the Killing vector V^μ: we shall investigate the timelike case in Section 5.2.2 and the null case in Section 5.2.4.

5.2.2 The timelike case

The vector field strengths

As is well-known, the contraction of the (anti-) self-dual part of a 2-form with a non-null vector, such as V^μ in the current timelike case, completely determines the 2-form, *i.e.*

$$C^{\Lambda\,+}{}_\mu \equiv V^\nu F^{\Lambda\,+}{}_{\nu\mu} \;\;\Rightarrow\;\; F^{\Lambda\,+} = V^{-2}[\hat{V} \wedge \hat{C}^{\Lambda\,+} + i \star(\hat{V} \wedge \hat{C}^{\Lambda\,+})]\,. \tag{5.2.26}$$

As $C^{\Lambda\,+}{}_\mu$ is given by Eq. (5.2.24), the vector field strengths are written in terms of the scalars Z^i, X and the vector V. Observe that the component of $C^{\Lambda\,+}{}_\mu$ proportional to V^μ is projected out in this formula: this implies that the field strengths have the same functional form as in the ungauged case. The covariant derivatives that appear in the r.h.s., however, contain explicitly the vector potentials.

The next item on the list is the determination of the spacetime metric:

The metric

As in the ungauged case we define a time coordinate t by

$$V^\mu \partial_\mu \equiv \sqrt{2}\partial_t\,. \tag{5.2.27}$$

Unlike the ungauged case, however, the scalars in a supersymmetric configuration need not automatically be time-independent: with respect to the chosen t-coordinate Eq. (5.2.21) takes the form

$$\partial_t Z^i + gA^\Lambda{}_t k_\Lambda{}^i + \sqrt{2}XW^i = \partial_t Z^i + g(A^\Lambda{}_t + \tfrac{1}{\sqrt{2}}X\mathcal{L}^{*\Lambda})k_\Lambda{}^i = 0\,. \tag{5.2.28}$$

It is convenient to choose a G_V gauge in which the complex fields Z^i are time-independent, and one accomplishing just that is

$$A^\Lambda{}_t \;=\; -\sqrt{2}\Re\mathrm{e}\,(X\mathcal{L}^{*\Lambda}) \;=\; -\sqrt{2}|X|^2\Re\mathrm{e}\,(\mathcal{L}^{*\Lambda}/X^*)\,. \tag{5.2.29}$$

This gauge choice reduces Eq. (5.2.28) to

$$\partial_t Z^i - \tfrac{1}{\sqrt{2}}gX^*\mathcal{L}^\Lambda k_\Lambda{}^i \;=\; \partial_t Z^i \;=\; 0\,, \tag{5.2.30}$$

on account of Eq. (C.2.17). It should be pointed out that this gauge choice is identical to the expression for A_t obtained in ungauged case in Refs. [26, 27]. Further, using the above t-independence and gauge choice in Eq. (5.2.25), we can derive

$$\begin{aligned} \partial_t X + i\mathcal{Q}_t X + igA^\Lambda{}_t\mathcal{P}_\Lambda &= \partial_t X + \tfrac{1}{2}(\partial_t Z^i \partial_i \mathcal{K} - \text{c.c})X + igA^\Lambda{}_t\mathcal{P}_\Lambda X \\ &= \partial_t X - \sqrt{2}ig|X|^2\Re\text{e}\,(\mathcal{L}^{*\Lambda}/X^*)\mathcal{P}_\Lambda X \\ &= \partial_t X = 0\,, \end{aligned} \tag{5.2.31}$$

where we made use of Eq. (C.2.16) and the reality of $\mathcal{P}_\Lambda$. Thus, with the standard coordinate choice and the gauge choice (5.2.29) the scalars Z^i and X are time-independent.

Using the exactness of the 1-forms $\hat{V}^x$ to define spacelike coordinates x^x by

$$\hat{V}^x \equiv dx^x\,, \tag{5.2.32}$$

the metric takes on the form

$$ds^2 = 2|X|^2(dt+\hat{\omega})^2 - \frac{1}{2|X|^2}dx^x dx^x \qquad (x, y = 1, 2, 3)\,, \tag{5.2.33}$$

where $\hat{\omega} = \omega_{\underline{i}} dx^i$ is a time-independent 1-form. This 1-form is determined by the following condition

$$d\hat{\omega} = \tfrac{i}{2\sqrt{2}} \star \left[\hat{V} \wedge \frac{X\mathfrak{D}X^* - X^*\mathfrak{D}X}{|X|^4}\right] \tag{5.2.34}$$

Observe that this equation has, apart from a different definition of the covariant derivative, the same functional form as in the ungauged case; before we start rewriting the above result in order to get to the desired result, however, we would like to point out that due to the stationary character of the metric, the resulting covariant derivatives on the transverse $\mathbb{R}^3$ contain a piece proportional to $\omega_{\underline{x}}$. The end-effect of this pull-back is that we introduce a new connection on $\mathbb{R}^3$, denoted by $\tilde{\mathfrak{D}}_{\underline{x}}$, which is formally the same as $\mathfrak{D}_{\underline{x}}$ but for a redefinition of the gauge field, *i.e.*

$$\tilde{A}^\Lambda{}_{\underline{x}} = A^\Lambda{}_{\underline{x}} - \omega_{\underline{x}}\, A^\Lambda{}_t\,. \tag{5.2.35}$$

In order to compare the results in this article with the ones found in [26], we introduce the real symplectic sections $\mathcal{I}$ and $\mathcal{R}$ defined by

$$\mathcal{R} \equiv \Re\text{e}(\mathcal{V}/X)\,, \qquad \mathcal{I} \equiv \Im\text{m}(\mathcal{V}/X)\,. \tag{5.2.36}$$

$\mathcal{V}$ is the symplectic section defining special geometry and thence satisfies

$$\mathcal{V} = \begin{pmatrix} \mathcal{L}^\Lambda \\ \mathcal{M}_\Sigma \end{pmatrix}, \qquad \langle \mathcal{V} \mid \mathcal{V}^* \rangle \equiv \mathcal{L}^{*\,\Lambda}\mathcal{M}_\Lambda - \mathcal{L}^\Lambda \mathcal{M}^*_\Lambda = -i\,. \tag{5.2.37}$$

This then implies that our gauge choice can be expressed in the form

$$A^\Lambda{}_t = -\sqrt{2}|X|^2\mathcal{R}^\Lambda\,, \tag{5.2.38}$$

and that the metric function $|X|$ can be written as

$$\frac{1}{2|X|^2} = \langle\, \mathcal{R} \mid \mathcal{I}\,\rangle\,, \tag{5.2.39}$$

Similar to the ungauged case, we can then rewrite Eq. (5.2.34) as

$$(d\hat{\omega})_{xy} = 2\epsilon_{xyz}\langle \mathcal{I} \mid \tilde{\mathfrak{D}}_z\mathcal{I}\,\rangle\,, \tag{5.2.40}$$

whose integrability condition reads

$$\langle\, \mathcal{I} \mid \tilde{\mathfrak{D}}_x\tilde{\mathfrak{D}}_x\mathcal{I}\,\rangle = 0\,, \tag{5.2.41}$$

and we shall see that, apart from possible singularities [86, 87], the integrability condition is identically satisfied for supersymmetric solutions.

Solving the Killing spinor equations

In the previous sections we have found that timelike supersymmetric configurations have a metric and vector field strengths given by Eqs. (5.2.33,5.2.24) and (A.1.16) in terms of the scalars X, Z^i. It is easy to see that all configurations of this form admit spinors ϵ_I that satisfy the Killing spinor equations (5.2.1,5.2.2). The Killing spinors have exactly the same form as in the ungauged case [26]

$$\epsilon_I = X^{1/2}\epsilon_{I\,0}\,, \qquad \partial_\mu\epsilon_{I\,0} = 0\,, \qquad \epsilon_{I\,0} + i\gamma_0\epsilon_{IJ}\epsilon^J{}_0 = 0\,. \tag{5.2.42}$$

We conclude that we have identified all the supersymmetric configurations of the theory.

Equations of motion

The results of Section 5.2.1 imply that in order to have a classical solution, we only need to impose the Maxwell equations and Bianchi identities on the supersymmetric configurations. In this section, then, we will discuss the differential equations arising from the applying the Maxwell and Bianchi equations on the supersymmetric configurations obtained thus far.

As we mentioned in Section 5.2.2 the field strengths of supersymmetric configurations take the same form as in the ungauged case [26] with the Kähler-covariant derivatives replaced by Kähler- and G_V-covariant derivatives. Therefore, the symplectic vector of field strengths and dual field strengths takes the form

$$F = \frac{1}{2|X|^2}\left\{\hat{V}\wedge\mathfrak{D}(|X|^2\mathcal{R}) - \star[\hat{V}\wedge\Im\mathrm{m}(\mathcal{V}^*\mathfrak{D}X + X^*\mathfrak{D}\mathcal{V})]\right\}\,. \tag{5.2.43}$$

Operating in the first term we can rewrite it in the form

$$F = -\tfrac{1}{2}\left\{\mathfrak{D}(\mathcal{R}\hat{V}) - 2\sqrt{2}|X|^2\mathcal{R}d\hat{\omega} + \star\left[\hat{V}\wedge\frac{\Im\mathrm{m}(\mathcal{V}^*\mathfrak{D}X + X^*\mathfrak{D}\mathcal{V})}{|X|^2}\right]\right\}\,, \tag{5.2.44}$$

and using the equation of 1-form $\hat{\omega}$, Eq. (5.2.34), which is also identical to that of the ungauged case with the same substitution of covariant derivatives, we arrive at

$$F = -\tfrac{1}{2}\left\{\mathfrak{D}(\mathcal{R}\hat{V}) + \star(\hat{V}\wedge\mathfrak{D}\mathcal{I})\right\}\,. \tag{5.2.45}$$

In what follows we shall use the following Vierbein (e^0, e^x) and the corresponding directional derivatives (θ_0, θ_a), normalized as $e^a(\theta_b) = \delta^a{}_b$, that are given by

$$\begin{array}{rclcrcl} e^0 & = & \sqrt{2}|X|\,(dt\,+\,\omega)\,, & & \theta_0 & = & \frac{1}{\sqrt{2}}|X|^{-1}\,\partial_t\,, \\ e^x & = & \frac{1}{\sqrt{2}}|X|^{-1}\,dx^x\,, & & \theta_x & = & \sqrt{2}|X|\,(\partial_{\underline{x}}\,-\,\omega_{\underline{x}}\partial_t)\,. \end{array} \tag{5.2.46}$$

With respect to this basis we

$$V^\mu\partial_\mu \;=\; 2|X|\,\theta_0\,, \qquad \hat{V} \;=\; 2|X|\,e^0\,, \tag{5.2.47}$$

and the gauge fixing (5.2.29) and the constraint (5.2.28) read

$$A^\Lambda{}_0 \;=\; -|X|\,\mathcal{R}^\Lambda\,, \qquad X^*\,\mathfrak{D}_0 Z^i \;=\; -|X|\,W^i\,. \tag{5.2.48}$$

The equation that the spacelike components of the field strengths $F^\Lambda{}_{\underline{xy}}$ satisfy can be rewritten in the form

$$\tilde{F}^\Lambda{}_{\underline{xy}} \;=\; -\tfrac{1}{\sqrt{2}}\epsilon_{xyz}\tilde{\mathfrak{D}}_{\underline{z}}\mathcal{I}^\Lambda\,, \tag{5.2.49}$$

where the tilde indicates that the gauge field that appears in this equation is the combination $\tilde{A}^\Lambda{}_{\underline{x}}$ defined in Eq. (5.2.35).

This equation is easily recognized as the well-known Bogomol'nyi equation [92] for the connection $\tilde{A}^\Lambda{}_{\underline{x}}$ and the real "Higgs" field $\mathcal{I}^\Lambda$ on $\mathbb{R}^3$. Its integrability condition uses the Bianchi identity for the 3-dimensional gauge connection $\tilde{A}^\Lambda{}_{\underline{x}}$ and, as it turns out, is equivalent to the complete Bianchi identity for the 4-dimensional gauge connection $A^\Lambda{}_\mu$. It takes the form

$$\tilde{\mathfrak{D}}_{\underline{x}}\tilde{\mathfrak{D}}_{\underline{x}}\,\mathcal{I}^{\Lambda} \;=\; 0\;. \tag{5.2.50}$$

Taking the Maxwell equation in form notation Eq. (4.2.11) and using heavily the formulae in Appendix C.2 we find that all the components are satisfied (as implied by the KSIs) except for one which leads to the equation

$$\tilde{\mathfrak{D}}_{\underline{x}}\tilde{\mathfrak{D}}_{\underline{x}}\mathcal{I}_{\Lambda} \;=\; \tfrac{1}{2}g^{2}\;\left[f_{\Lambda(\Sigma}{}^{\Gamma}f_{\Delta)\Gamma}{}^{\Omega}\;\mathcal{I}^{\Sigma}\mathcal{I}^{\Delta}\right]\,\mathcal{I}_{\Omega}\;. \tag{5.2.51}$$

Plugging the above equation and the Bianchi identity (5.2.50) into the integrability condition for ω, Eq. (5.2.41), leads to

$$\langle\,\mathcal{I}\,|\,\tilde{\mathfrak{D}}_{\underline{x}}\tilde{\mathfrak{D}}_{\underline{x}}\mathcal{I}\,\rangle \;=\; -\mathcal{I}^{\Lambda}\tilde{\mathfrak{D}}_{\underline{x}}\tilde{\mathfrak{D}}_{\underline{x}}\mathcal{I}_{\Lambda} \;=\; -\tfrac{1}{2}g^{2}f_{\Lambda(\Sigma}{}^{\Gamma}f_{\Delta)\Gamma}{}^{\Omega}\;\mathcal{I}^{\Lambda}\mathcal{I}^{\Sigma}\mathcal{I}^{\Delta}\;\mathcal{I}_{\Omega} \;=\; 0\;, \tag{5.2.52}$$

which is, ignoring possible singularities, therefore identically satisfied.

Construction of supersymmetric solutions of $N = 2, d = 4$ SEYM

According to the KSIs, the supersymmetric configurations that satisfy the pair of Eqs. (5.2.50) and (5.2.51), or, equivalently, the pair of Eqs. (5.2.49) and (5.2.51) solve all the equations of motion of the theory. This implies that one can give a step-by-step prescription to construct supersymmetric solutions of any $N = 2, d = 4$ SEYM starting from any solution of the YM-Higgs Bogomol'nyi equations on $\mathbb{R}^3$:

1. Take a solution $\tilde{A}^{\Lambda}{}_{\underline{x}}$, $\mathcal{I}^{\Lambda}$ to the equations

$$\tilde{F}^{\Lambda}{}_{\underline{xy}} \;=\; -\tfrac{1}{\sqrt{2}}\epsilon_{xyz}\tilde{\mathfrak{D}}_{\underline{z}}\mathcal{I}^{\Lambda}\;.$$

   As we have stressed repeatedly, these equations are nothing but YM-Higgs Bogomol'nyi equations on $\mathbb{R}^3$ and there are plenty of solutions available in the literature. However, since in most cases the authors' goal is to obtain regular monopole solutions on $\mathbb{R}^3$, there are many solutions to the same equations that have been discarded because they present singularities. We know, however, that in the Abelian case, the singularities might be hidden by an event horizon[9]. Therefore, we will not require the solutions to the Bogomol'nyi equations to be globally regular on $\mathbb{R}^3$.

2. Given the solution $\tilde{A}^{\Lambda}{}_{\underline{x}}$, $\mathcal{I}^{\Lambda}$, Eq. (5.2.51), which we write here again for the sake of clarity (as we will do with other relevant equations):

$$\tilde{\mathfrak{D}}_{\underline{x}}\tilde{\mathfrak{D}}_{\underline{x}}\,\mathcal{I}_{\Lambda} \;=\; \tfrac{1}{2}g^{2}\;\left[f_{\Lambda(\Sigma}{}^{\Gamma}f_{\Delta)\Gamma}{}^{\Omega}\;\mathcal{I}^{\Sigma}\mathcal{I}^{\Delta}\right]\,\mathcal{I}_{\Omega}\;.$$

[9]More precisely they turn out to be coordinate singularities in the full spacetime and correspond, not to a singular point, but to an event horizon.

becomes a linear equation for the $\mathcal{I}_\Lambda$s alone which has to be solved. For compact gauge groups a possible solution is

$$\mathcal{I}_\Lambda = \mathcal{J}\mathcal{I}^\Lambda \,, \tag{5.2.53}$$

for an arbitrary real constant $\mathcal{J}$ (the r.h.s. of Eq. (5.2.51) vanishes for this Ansatz).

3. The first two steps provide $\mathcal{I} = (\mathcal{I}^\Lambda, \mathcal{I}_\Lambda) = \Im\mathrm{m}\,(\mathcal{V}/X)$. The next step, then, is to obtain $\mathcal{R} = (\mathcal{R}^\Lambda, \mathcal{R}_\Lambda) = \Re\mathrm{e}\,(\mathcal{V}/X)$ as functions of $\mathcal{I}$ by solving the model-dependent *stabilization equations*. The stabilization equations depend only on the specific model one is considering and does not depend on whether the model is gauged or not.

4. Given $\mathcal{R}$ and $\mathcal{I}$, one can compute the metric function $|X|$ using Eq. (5.2.39)

$$\frac{1}{2|X|^2} = \langle\, \mathcal{R} \mid \mathcal{I} \,\rangle\,;$$

the n physical complex scalars Z^i by

$$Z^i \equiv \frac{\mathcal{L}^i}{\mathcal{L}^0} = \frac{\mathcal{L}^i/X}{\mathcal{L}^0/X} = \frac{\mathcal{R}^i + i\mathcal{I}^i}{\mathcal{R}^0 + i\mathcal{I}^0}\,, \tag{5.2.54}$$

and the metric 1-form $\hat{\omega}$ using Eq. (5.2.40)

$$(d\hat{\omega})_{\underline{xy}} = 2\epsilon_{xyz}\langle\, \mathcal{I} \mid \tilde{\mathfrak{D}}_{\underline{z}}\mathcal{I} \,\rangle\,.$$

This last equation can always be solved locally, as according to Eq. (5.2.52) its integrability equation is solved automatically, at least locally: Since the solutions to the covariant Laplace equations are usually local (they generically have singularities), the integrability condition may fail to be satisfied everywhere, as discussed for example in Refs. [86, 87, 93], leading to singularities in the metric. The solution Eq. (5.2.53), however, always leads to exactly vanishing $\hat{\omega}$, whence to static solutions.

$|X|$ and $\hat{\omega}$ completely determine the metric of the supersymmetric solutions, given in Eq. (5.2.33)

$$ds^2 = 2|X|^2(dt + \hat{\omega})^2 - \frac{1}{2|X|^2}dx^x dx^x \qquad (x, y = 1, 2, 3)\,.$$

5. Once $\mathcal{I}, \mathcal{R}, |X|$ and $\hat{\omega}$ have been determined, the 4-dimensional gauge potential can be found from Eq. (5.2.38)

$$A^{\Lambda}{}_{t} = -\sqrt{2}|X|^2\mathcal{R}^{\Lambda}\,,$$

and from the definition of $\tilde{A}^{\Lambda}{}_{\underline{x}}$ Eq. (5.2.35)

$$A^{\Lambda}{}_{\underline{x}} = \tilde{A}^{\Lambda}{}_{\underline{x}} + \omega_{\underline{x}}\; A^{\Lambda}{}_{t}\;.$$

The procedure we have followed ensures that this is the gauge potential whose field strength is given in Eq. (5.2.45).

In the next section we are going to construct, following this procedure, several solutions.

5.2.3 Monopoles and hairy black holes

As we have seen, the starting point in the construction of $N = 2, d = 4$ SEYM supersymmetric solutions is the Bogomol'nyi equation on $\mathbb{R}^3$. Of course, the most interesting solutions to the Bogomol'nyi equations are the monopoles that can be characterised by saying that they are finite energy solutions that are everywhere regular. The fact that the gauge fields are regular does, however, not imply that the full supergravity solution is regular. Indeed, the metric and the physical scalar fields are built out of the "Higgs field", *i.e.* $\mathcal{I}$, and the precise relations are model dependent and requires knowing the solutions to the stabilization equation.

As the Higgs field in a monopole asymptotes to a non-trivial constant configuration, it asymptotically breaks the gauge group through the Higgs effect. In fact, as we are dealing with supergravity and supersymmetry preserving solutions, monopoles in our setting would have to implement the super-Higgs effect as for example discussed in Refs. [94]. If we were to insist on an asymptotic supersymmetric effective action, we would be forced to introduce hypermultiplets in order to fill out massive supermultiplets, but this point will not be pursued in this article.

The Bogomol'nyi equations admit more than just regular solutions, and we shall give families of solutions, labelled by a continuous parameter $s > 0$, having the same asymptotic behaviour as the monopole solutions. As they are singular on $\mathbb{R}^3$, however, we will use them to construct metrics describing the regions outside regular black holes: as will be shown, the members of a given family lead to black holes that are not distinguished by their asymptotic data, such as the moduli or the asymptotic mass, nor by their entropy and as such illustrate the non-applicability of the no-hair theorem to supersymmetric EYM theories. Furthermore, in all examples considered, the attractor mechanisms is at work, meaning that the physical scalars at the horizon and the entropy depend only on the asymptotic charges and not on the moduli nor on the parameter s.

The plan of this section is as follows: in section (5.2.3) we shall repeat briefly the embedding of the spherically symmetric solutions to the $SO(3)$ Bogomol'nyi equations in the $\overline{\mathbb{CP}}^3$ models. In all but one of these solutions, the asymptotic gauge symmetry breaking is maximal, *i.e.* the $SO(3)$ gauge symmetry is broken down to $U(1)$. In section (5.2.3), we will investigate the embedding of solutions that manifest a non-maximal asymptotic symmetry breaking: for this we take E. Weinberg's spherically symmetric $SO(5)$-monopole [95] embedded into $\overline{\mathbb{CP}}^{10}$. This monopole breaks the $SO(5)$ down to $U(2)$ and has the added characteristic that, unlike the 't Hooft-Polyakov monopole, the Higgs field does not vanish at the origin.

An interesting question is whether one can embed monopoles also into more complicated models. This question will be investigated in Section 5.2.3, where we consider gauged "Magic" supergravities.

Spherically symmetric solutions in $SO(3)$ gauged $\overline{\mathbb{CP}}^3$

Before discussing the solutions we need to make some comments on the model: the model we shall consider in this and the next section is the so-called $\overline{\mathbb{CP}}^n$ model.[10] In this model the metric on the scalar manifold is that of the symmetric space $SU(1,n)/U(n)$ and the prepotential is given by

$$\mathcal{F} \;=\; \tfrac{1}{4i}\,\eta_{\Lambda\Sigma}\,\mathcal{X}^\Lambda\,\mathcal{X}^\Sigma\,, \qquad \eta \;=\; \mathrm{diag}\,(\,+\;,\;[-]^n\;)\;, \tag{5.2.55}$$

which is manifestly $SO(1,n)$ invariant.

The Kähler potential is straightforwardly derived by fixing $\mathcal{X}^0 = 1$ and introducing the notation $\mathcal{X}^i \;=\; Z^i$; this results in

$$e^{-\mathcal{K}} \;=\; |\mathcal{X}^0|^2 \;-\; \sum_{i=1}^n\;|\mathcal{X}^i|^2 \;=\; 1 \;-\; \sum_{i=1}^n\;|Z^i|^2 \equiv 1 \;-\; |Z|^2\;. \tag{5.2.56}$$

Observe that this expression for the Kähler potential implies that the Z's are constrained by $0 \leq |Z|^2 < 1$.

As the model is quadratic, the stabilization equations are easily solved and leads to

$$\mathcal{R}_\Lambda \;=\; \tfrac{1}{2}\eta_{\Lambda\Sigma}\,\mathcal{I}^\Sigma \quad , \quad \mathcal{R}^\Lambda \;=\; -2\eta^{\Lambda\Sigma}\,\mathcal{I}_\Sigma\;. \tag{5.2.57}$$

With this solution to the stabilization equation, we can express the metrical factor, Eq. (5.2.39), in terms of the $\mathcal{I}$ as

$$\frac{1}{2|X|^2} \;=\; \tfrac{1}{2}\,\eta_{\Lambda\Sigma}\,\mathcal{I}^\Lambda\mathcal{I}^\Sigma + 2\eta^{\Lambda\Sigma}\,\mathcal{I}_\Lambda\mathcal{I}_\Sigma \;=\; \tfrac{1}{2}\,\eta_{\Lambda\Sigma}\,\mathcal{I}^\Lambda\mathcal{I}^\Sigma\;, \tag{5.2.58}$$

[10] The solutions in this and the next section can also be embedded into the $\mathcal{ST}$-models, with similar conclusions. Contrary to Ref. [36], however, we have chosen not to deal with this model explicitly, and refer the reader to Appendix C.4 for more details.

where in that last step we used the fact that in this article we shall consider only purely magnetic solutions, so that $\mathcal{I}_\Lambda = 0$. The fact that we choose to consider magnetic embeddings only, implies be means of Eq. (5.2.40) that we will be dealing with static solutions.

In order to finish the discussion of the model, we must discuss the possible gauge groups that can occur in the $\overline{\mathbb{CP}}^n$-models: as we saw at the beginning of this section, these models have a manifest $SO(1,n)$ symmetry, under which the $\mathcal{X}$'s transform as a vector. Furthermore, as we are mostly interested in monopole-like solutions, we shall restrict our attention to compact simple groups, which, as implied by Eq. (C.2.22), must be subgroups of $SO(n)$. In fact, Eq. (C.2.22) and Eq. (C.2.13) make the stronger statement that given a gauge algebra $\mathfrak{g}$, the action of $\mathfrak{g}$ on the $\mathcal{X}$'s must be such that only singlets and the adjoint representation appear. For the $\overline{\mathbb{CP}}^n$-models there is no problem whatsoever as we can choose n to be large enough as to accomodate any Lie algebra. Indeed, as is well-known any compact simple Lie algebra $\mathfrak{g}$ is a subalgebra of $\mathfrak{so}(\dim(\mathfrak{g}))$ and the branching of the latter's vector representation is exactly the adjoint representation of $\mathfrak{g}$.

The simplest possibility, namely the $SO(3)$-gauged model on $\overline{\mathbb{CP}}^3$, will be used in the remainder of this section, and the $SO(5)$-gauged $\overline{\mathbb{CP}}^{10}$ model will be used in section (5.2.3). The $SO(4)$- and the $SU(3)$-gauged models will not be treated, but solutions to these models can be created with great ease using the information in this section and Appendix F.

As we are restricting ourselves to purely magnetic solutions, which are automatically static, the construction of explicit supergravity solutions goes through the explicit solutions to the $SO(3)$ Bogomol'nyi equation (5.2.49). Having applications to the attractor mechanism in mind, and being fully aware of the fact that this class consists of only the tip of the iceberg of solutions, we shall restrict ourselves to spherically symmetric solutions to the Bogomol'nyi equations.

Working in gauge theories opens up the possibility of compensating the spacetime rotations with gauge transformations, and in the case of an $SO(3)$ gauge group this means that the gauge connection and the Higgs field, $\mathcal{I}$, after a suitable gauge fixing, takes on the form (See *e.g.* [96])

$$A^i{}_m \;=\; -\varepsilon_{mn}{}^i\; x^n\; P(r) \quad , \quad \mathcal{I}^i \;=\; -\sqrt{2}\, x^i\; H(r)\; . \tag{5.2.59}$$

Substituting this Ansatz into the Bogomol'nyi equation we find that H and P must satisfy

$$r\partial_r\left(H+P\right) \;=\; gr^2\; P\;\left(H+P\right)\; , \tag{5.2.60}$$

$$r\partial_r P \;+\; 2P \;=\; H\;\left(1+gr^2P\right)\; . \tag{5.2.61}$$

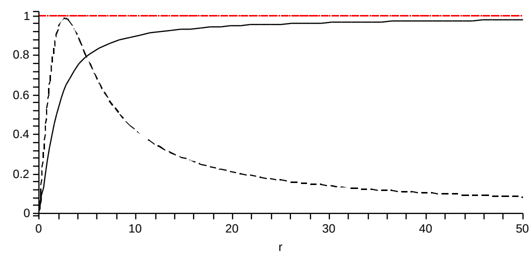

Figure 5.2.1: The profiles of the functions $\overline{P}$ and $\overline{H}$.

All the solutions to the above equations were found in Ref. [97] and all but one of them contain singularities. Furthermore, not all of them have the correct asymptotics to lead to asymptotic flat spaces and only part of the ones that do can be used to construct regular supergravity solutions [36, 98]. Here, by regular supergravity solutions we mean that the solutions is either free of singularities, which is what is meant by a globally regular solution, or has a singularity but, like the black hole solutions in the Abelian theories, has the interpretation of describing the physics outside the event horizon of a regular black hole. The criterion for this last to occur is that the geometry near the singularity is that of a Robinson-Bertotti/$aDS_2 \times S^2$ spacetime, implying that the black hole has a non-vanishing horizon area, whence also entropy.

The suitable solutions, then, break up into 3 classes:

(I) 't Hooft-Polyakov monopole

This is the most famous solution and reads

$$
\begin{aligned}
H &= -\frac{\mu}{gr}\left[\coth(\mu r) - \frac{1}{\mu r}\right] \equiv -\frac{\mu}{gr}\,\overline{H}(r) \;, \\
P &= -\frac{1}{gr^2}\left[1 \;-\; \mu r \sinh^{-1}(\mu r)\right] \equiv -\frac{\mu}{gr}\,\overline{P}(r) \;,
\end{aligned}
\tag{5.2.62}
$$

where μ is a positive constant. The profile of the functions $\overline{P}$ and $\overline{H}$ are given Fig. (1). These functions are regular and bound between 0 and 1 and . Thus, we see that $\mathcal{I}$ (whence also $\mathcal{I}^a$ and $\mathcal{I}_a$) are regular at $r = 0$. The YM fields of this solution are those of the 't Hooft-Polyakov monopole [99].

The renowned regularity of the 't Hooft-Polyakov monopole opens up the possibility of creating a globally regular solution to the supergravity equations which is in fact trivial to achieve: for the moment we have been ignoring $\mathcal{I}^0$, which, since it is uncharged under the gauge group, is just a real, spherically symmetric harmonic function we can parametrize as

$$\mathcal{I}^0 = \sqrt{2}(h + p/r)\,. \tag{5.2.63}$$

It is clear, however, that if we want to avoid singularities, we must take $p = 0$, so that the only free parameter is h.

Let us then discuss the regularity conditions imposed by the metric: as was said before, the solutions are automatically static, so that if singularities in the metric are to appear, they arise from the metrical factor $|X|^2$. Plugging the solution for the Higgs field into the expression (5.2.58), we find

$$\frac{1}{2|X|^2} \;=\; h^2 \;-\; \frac{\mu^2}{g^2}\,\overline{H}^2(r)\;. \tag{5.2.64}$$

As one can infer from its definition in Eq. (5.2.62), the function $\overline{H}$ is a monotonic, positive semi-definite function on $\mathbb{R}^+$ and vanishes only at $r = 0$, where it behaves as $\overline{H} \sim \mu r/3 + \mathcal{O}(r^2)$; its behaviour for large r is given by $\overline{H} = 1 - 1/(\mu r)$, which means that we should choose h large enough in order to ensure the positivity of the metrical factor. A convenient choice for h is given by imposing that asymptotically we recover the standard Minkowskian metric in spherical coordinates: this condition gives $h^2 = 1 + \mu^2 g^{-2}$ from which we find the final metrical factor and can then also calculate the asymptotic mass, *i.e.*

$$\frac{1}{2|X|^2} \;=\; 1 \;+\; \frac{\mu^2}{g^2}\left[1 \;-\; \overline{H}^2\right] \;\;\rightarrow\;\; M \;=\; \frac{\mu}{g^2}\,. \tag{5.2.65}$$

Written in this form, it is paramount that the metric is globally regular and interpolates between two Minkowksi spaces, one at $r = 0$ and one at $r = \infty$.

In order to show that the solution is a globally regular supergravity solution, we should show that the physical scalars are regular. In the $\overline{\mathbb{CP}}^n$-models the scalars are given by (introducing the outward-pointing unit vector $\vec{n} = \vec{x}/r$)

$$Z^i \;\equiv\; \frac{\mathcal{R}^i + i\mathcal{I}^i}{\mathcal{R}^0 + i\mathcal{I}^0} \;=\; \frac{\mathcal{I}^i}{\mathcal{I}^0} \;=\; \frac{\mu}{gh}\,\overline{H}\,n^i\;, \tag{5.2.66}$$

so that the regularity is obvious. The scalars also respect the bound $0 \leq |Z|^2 < 1$ as can be seen from the fact that the bound corresponds to the positivity of the metrical factor. This regularity of the scalars and that of the spacetime metric are related [87].

(II) Hairy black holes

A generic class of singular solutions is indexed by a free parameter $s > 0$, called the *Protogenov hair*, and can be seen as a deformation of the 't Hooft-Polyakov monopole, *i.e.*

$$H = -\frac{\mu}{gr}\left[\coth(\mu r + s) - \frac{1}{\mu r}\right] \equiv -\frac{\mu}{gr}\,\overline{H}_s(r) , \tag{5.2.67}$$

$$P = -\frac{1}{gr^2}\left[1 - \mu r \sinh^{-1}(\mu r + s)\right] . \tag{5.2.68}$$

The effect of introducing the parameter s is to shift the singularity of the cotangent from $r = 0$ to $\mu r = -s$, *i.e.* outside the domain of r, but leaving unchanged its asymptotic behaviour.[11] This not only means that the function $\overline{H}_s$ vanishes at some $r_s > 0$, but also that it becomes singular at $r = 0$, so that in order to build a regular solution we must have $p \neq 0$. Using then the general Ansatz for $\mathcal{I}^0$, Eq. (5.2.63), in order to calculate the metrical factor, we find in stead of Eq. (5.2.64)

$$\frac{1}{2|X|^2} = \left(h + \frac{p}{r}\right)^2 - \frac{\mu^2}{g^2}\,\overline{H}_s^2 . \tag{5.2.69}$$

As the asymptotic behaviour of $\overline{H}_s$ is the same as the one for the 't Hooft-Polyakov monopole, the condition imposed by asymptotic flatness still is $h^2 = 1 + \mu^2 g^{-2}$. Given this normalization, the asymptotic mass is

$$M = hp + \frac{\mu}{g^2} , \tag{5.2.70}$$

which should be positive for a physical solution. In this respect, we would like to point out that the product hp should be positive as otherwise the metrical factor would become negative or zero, should it coincide with the zero of $\overline{H}_s$, at a finite distance, ruining our interpretation of the metric as describing the outside of a regular black hole. This then implies that the mass is automatically positive. Finally, let us point out that neither the mass nor the modulus h depend on the Protogenov hair parameter s.

The metrical factor is clearly singular at $r = 0$, but given the interpretation of the metric this is not a problem as long as the geometry near $r = 0$, which corresponds to the near horizon geometry, is that of an $aDS_2 \times S^2$ space. This is the case if

$$S_{bh} \equiv \lim_{r \to 0} \frac{r^2}{2|X|^2} = p^2 - \frac{1}{g^2} , \tag{5.2.71}$$

is positive and can thence be identified with the entropy of the black hole.

The scalars for this solution are given by

$$Z^i = \frac{\mu}{g}\,\frac{r\overline{H}_s}{p + hr}\,n^i , \tag{5.2.72}$$

[11] One can consider the limiting solution for $s \to \infty$, the result of which was called a black hedgehog in Ref. [36]. This solution has, apart from not containing hyperbolic functions, no special properties and will not be considered seperately.

whose asymptotic behaviour is the same as for the 't Hooft-Polyakov monopole. Its behaviour near the horizon, *i.e.* near $r = 0$, is easily calculated to be

$$\lim_{r\to 0} Z^i = -\frac{1}{gp} n^i , \tag{5.2.73}$$

and does not depend on the moduli nor on the Protogenov hair, but only on the asymptotic charges. Observe, however, that since $\overline{H}_s = 0$ at some finite $r_s > 0$, there is a 2-sphere outside the horizon at which the scalars vanish, which is not a singularity for the scalars of this model.

(III) Coloured black holes

There is another particular solution to the $SO(3)$ Bogomol'nyi equation that has all the necessary properties, and this solution is given by

$$H = -P = \frac{1}{gr^2}\left[\frac{1}{1+\lambda^2 r}\right] . \tag{5.2.74}$$

This solution has the same $r \to 0$ behaviour as the hairy solutions, but is such that in the asymptotic regime it has no Higgs v.e.v. nor colour charge. Given the foregoing discussion, it is clear that this solution can be used to build a regular black hole solution, and we can and will be brief.

The regularity of the metric goes once again through the judicious election of h and p: the normalization condition implies that $|h| = 1$ which then also implies that the asymptotic mass of the solution is $M = |p|$. It may seem strange that the YM-configuration does not contribute to the mass, but it does so, at least for a regular black hole solution, in an indirect fashion: the condition for a regular horizon is clearly given by Eq. (5.2.71), which implies that $|p| > 1/g$. With these choices then, the scalars Z are regular for $r > 0$ and at the horizon they behave as in Eq. (5.2.73).

Non-maximal symmetry breaking in $SO(5)$ gauged $\overline{\mathbb{CP}}^{10}$

In Ref. [95], E. Weinberg presented an explicit solution for a spherically symmetric monopole solution that breaks the parent $SO(5)$ gauge group down to $U(2)$; in this section we will discuss the embedding of this solution into supergravity and also generalize it to a family of hairy black holes by introducing Protogenov hair[12].

The starting point of the derivation of Weinberg's monopole is the explicit embedding of an 't Hooft-Polyakov monopole into an $\mathfrak{so}(3)$ subalgebra of $\mathfrak{so}(5)$. In order to make this embedding paramount we take the generators of $\mathfrak{so}(5)$ to be J_i, $\overline{J}_i$

[12] In Ref. [100] the general equations for a spherically symmetric solution to the $SO(5)$ Bogomol'nyi equations were derived. This opens up the possibility of analysing the system along the lines of Ref. [97], but for the moment this has not lead to anything new.

$(i = 1, 2, 3)$ and P_a $(a = 1, \ldots, 4)$. These generators satisfy the following commutation relations

$$\begin{aligned} [J_i, J_j] &= \varepsilon_{ijk}\ J_k\,, & [J_i, P_a] &= P_c\ \Sigma_i{}^c{}_a\,, \\ [\bar{J}_i, \bar{J}_j] &= \varepsilon_{ijk}\ \bar{J}_k\,, & [\bar{J}_i, P_a] &= P_c\ \overline{\Sigma}_i{}^c{}_a\,, \\ [J_i, \bar{J}_j] &= 0\,, & [P_a, P_b] &= -2\ J_i\ \Sigma^i_{ab} - 2\ \bar{J}_i\ \overline{\Sigma}^i_{ab}\,, \end{aligned} \tag{5.2.75}$$

where we have introduced the 't Hooft symbols Σ_i^{ab} and $\overline{\Sigma}_i^{ab}$. The Σ (*resp.* $\overline{\Sigma}$) are self-dual (*resp.* anti-selfdual) 2-forms on $\mathbb{R}^4$ and satisfy the following relations

$$\begin{aligned} [\Sigma_i, \Sigma_j] &= \varepsilon_{ijk}\Sigma_k\,, & [\overline{\Sigma}_i, \overline{\Sigma}_j] &= \varepsilon_{ijk}\overline{\Sigma}_k\,, & [\Sigma_i, \overline{\Sigma}_j] &= 0\,, \\ \Sigma_i^2 &= -\tfrac{1}{4}\ 1_4\,, & \overline{\Sigma}_i^2 &= -\tfrac{1}{4}\ 1_4\,, & \Sigma_{iab}\overline{\Sigma}_j^{ab} &= 0\,. \end{aligned} \tag{5.2.76}$$

We would like to stress that $\overline{\Sigma}$ is not the complex nor the Hermitean conjugate of Σ.

Following Weinberg we make the following Ansatz for the $\mathfrak{so}(5)$-valued connection and Higgs field, taking T_A $(A = 1, \ldots, 10)$ to be the generators of $\mathfrak{so}(5)$,

$$\mathtt{A}_m \equiv A^A{}_m\ T_A = -\varepsilon_{mj}{}^i n^j \left[rP\ J_i + rB\ \bar{J}_i\right] + M_m{}^a\ P_a\,, \tag{5.2.77}$$

$$-\tfrac{1}{\sqrt{2}}\mathtt{I} \equiv -\tfrac{1}{\sqrt{2}}\mathcal{I}^A\ T_A = rH\ n^i J_i + rK\ n^i \bar{J}_i + \Omega^a\ P_a\,, \tag{5.2.78}$$

where P, B, H and K are functions of r only. M and Ω are determined by the criterion that we have an 't Hooft-Polyakov monopole in some $\mathfrak{so}(3)$-subalgebra, which we take to be generated by the J_i. One way of satisfying this criterion is by choosing

$$M_m{}^a = F\ \delta^a_m \quad , \quad \Omega^a = -F\ \delta^{a0}\,, \tag{5.2.79}$$

which implies that the Bogomol'nyi equation in the J_i sector reduce to Eqs. (5.2.60) and (5.2.61).

The analysis of the Bogomol'nyi equations in the remaining sectors impose the constraint that $K = -B$ and the differential equations[13]

$$2g\ F^2 = rK' + 2K + K(1 - gr^2 K)\,, \tag{5.2.80}$$

$$F' = \tfrac{1}{2}gr\ F\left[2P + H + K\right]\,. \tag{5.2.81}$$

[13]In order to go from Weinberg's notation [95] to ours one needs to change $A \to -rP$, $G \to -rB$, $H \to rH$, $K \to rK$, $e \to -g$ and also $F \to F/\sqrt{2}$.

The final ingredient, needed for the calculation of the metrical factor, consists of finding an expression for the $SO(5)$-invariant quantity $\mathcal{I}^A\mathcal{I}^A$: this is

$$\tfrac{1}{2}\,\mathcal{I}^A\mathcal{I}^A \;=\; r^2H^2 \;+\; r^2K^2 \;+\; 2\;F^2\;. \tag{5.2.82}$$

In conclusion, given a solution to Eqs. (5.2.60,5.2.61,5.2.80) and (5.2.81) we can discuss their embedding into the $SO(5)$-gauged $\overline{\mathbb{CP}}^{10}$-model by means of Eq. (5.2.82).

Weinberg's monopole in supergravity

The explicit form of Weinberg's monopole is given by the solution in Eq. (5.2.62) and

$$K(r) \;=\; -P(r)\;L(r;a) \;\equiv\; \frac{\mu}{gr}\,\overline{K}\;, \tag{5.2.83}$$

$$F(r) \;=\; \frac{\mu}{2g\;\cosh\left(\mu r/2\right)}\;L^{1/2}(r;a) \;\equiv\; \frac{\mu}{g}\,\overline{F}\;, \tag{5.2.84}$$

where the profile function L, given by

$$L(r;a) \;=\; \left[1 \;+\tfrac{\mu r}{2a}\coth\left(\mu r/2\right)\,\right]^{-1}\;, \tag{5.2.85}$$

depends on a positive parameter a called the *cloud parameter*. The cloud parameter a is a measure for the extention of the region in which the Higgs field in the $\overline{J}_i$- and the P_a-directions are active: in fact when $a=0$ the profile functions vanishes identically and we are dealing with an embedding of the 't Hooft-Polyakov monopole. The maximal extention is for $a\to\infty$ which then means that $L=1$.

As one can see from the definitions, K and F are positive semi-definite functions that asymptote exponentially to zero. This not only means that the gauge symmetry is asymptotically broken to $U(2)$, but also that K and F will not contribute to the asymptotic mass, nor to the normalization condition. Unlike the 't Hooft-Polyakov monopole or the degenerate Wilkinson-Bais $SU(3)$-monopole (F.0.11), however, the regularity of the solution does *not* imply that the Higgs field vanishes at $r=0$! In fact, near $r=0$ one finds that

$$\overline{F}\;\sim\;\tfrac{1}{2}\,\sqrt{\frac{a}{1+a}}\;+\ldots\quad,\quad \overline{K}\;\sim\;\frac{\mu a}{3!(a+1)}\;r\;+\ldots\;. \tag{5.2.86}$$

It is this behaviour that may pose a problem for creating a globally regular solution and is the reason for including it in this article.

Using Eqs. (5.2.58) and (5.2.82) and choosing as in Sec. (5.2.3) $p=0$, we can write the metrical factor as

$$\frac{1}{2|X|^2} \;=\; 1\;+\;\frac{\mu^2}{g^2}\left[1\;-\;\overline{H}^2\;-\;\overline{K}^2\;-\;2\overline{F}^2\right]\;, \tag{5.2.87}$$

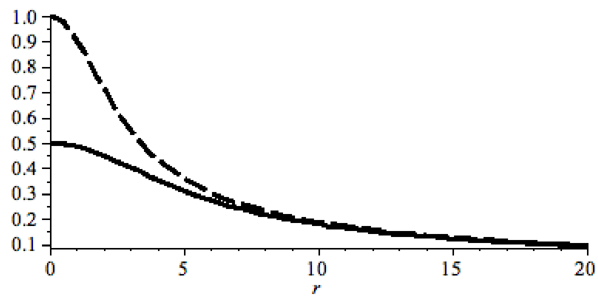

Figure 5.2.2: A plot of $1 - \overline{H}^2 - \overline{K}^2 - 2\overline{F}^2$: the dashed line corresponds to $a = 0$ and the solid line corresponds to the maximal cloud extention, *i.e.* $L = 1$.

where we already used the normalization condition $h^2 = 1 + \mu^2 g^{-2}$. As mentioned above, $\overline{K}$ and $\overline{F}$ asymptote exponentially to zero and cannot contribute to the mass, which is the one for the 't Hooft-Polyakov monopole, *i.e.* $M = \mu g^{-2}$.

Let us then investigate the behaviour of (5.2.87) at $r = 0$: a simple substitution shows that

$$\left. \frac{1}{2|X|^2} \right|_{r=0} = 1 + \frac{\mu^2}{g^2} \frac{2a+1}{2(a+1)} , \tag{5.2.88}$$

which is always positive so that the non-zero value of the Higgs field at the origin is no obstruction to the construction of a globally regular supergravity solution. The remaining question as far as the global regularity of the solution is concerned, is whether there are values of r for which the metrical factor (5.2.87) becomes negative. This however never happens as one can see from Fig. (1) which shows a plot of $1 - \overline{H}^2 - \overline{K}^2 - 2\overline{F}^2$ for the values of $a = 0$ and $a = \infty$.

Another hairy black hole

The introduction of Protogenov hair, *i.e.* a real and positive parameter s, in Weinberg's monopole solution is trivial and leads to the following solution

$$L_s(r;a) = \left[1 + \frac{\mu r}{2a} \coth\left(\frac{\mu r + s}{2} \right) \right]^{-1} , \tag{5.2.89}$$

$$F = \frac{\mu}{g} \overline{F}_s = \frac{\mu}{2g \cosh\left(\frac{\mu r + s}{2}\right)} L_s^{1/2} , \tag{5.2.90}$$

$$K = \frac{\mu}{gr} \overline{K}_s = \frac{\mu}{gr} \left[\frac{1}{\mu r} - \frac{1}{\sinh(\mu r + s)} \right] L_s . \tag{5.2.91}$$

A	G	H	$\mathsf{G} \circ \mathcal{V}$	$\mathsf{H} \circ \mathcal{X}^0$	$\mathsf{H} \circ \mathcal{X}^i$	$\mathsf{I}_3(\mathcal{X}^i)$	$\max(G)$
$\mathbb{R}$	$Sp(3;\mathbb{R})$	$U(3)$	$\mathbf{14}'$	$\mathbf{1}_{-3}$	$\mathbf{6}_{-1}$	$\det(\mathcal{X})$	
$\mathbb{C}$	$SU(3,3)$	$S[U(3) \otimes U(3)]$	$\mathbf{20}$	$(\mathbf{1},\mathbf{1})_{-3}$	$(\mathbf{3},\overline{\mathbf{3}})_{-1}$	$\det(\mathcal{X})$	$SU(3)_{diag}$
$\mathbb{Q}$	$SO^*(12)$	$U(6)$	$\mathbf{32}'$	$\mathbf{1}_{-3}$	$\mathbf{15}_{-1}$	$\mathrm{Pf}\,(\mathcal{X})$	$SU(4)$
$\mathbb{O}$	$E_{7(-25)}$	$E_6 \otimes SO(2)$	$\mathbf{56}$	$\mathbf{1}_3$	$\mathbf{27}_1$	$\mathrm{Tr}\left([\Omega\mathcal{X}]^3\right)/3!$	

Table 5.2.1: List of characteristics of Symmetric Special Geometries; all the names of the representations are the ones used by Slansky [101]. The meaning of the different columns is explained in the main text.

supplemented by the expression for H and P given in Eq. (5.2.67). As far as the limiting cases of this family is concerned, it is clear that Weinberg's monopole is obtained in the limit $s \to 0$; in the limit $s \to \infty$ we find that $F \to 0$ and the solution splits up into the direct sum of an $SO(3)$ black hedgehog, *i.e.* an $s \to \infty$ limit of (5.2.67), and an $SO(3)$ coloured black hole, Eq. (5.2.74).

As in the case of the hairy $SO(3)$ black holes, the introduction of the hair parameter s preserves the asymptotic behaviour of Weinberg's monopole and the solution is regular for $r > 0$. This immediately implies that the normalization condition for h once again reads $h^2 = 1 + \mu^2 g^{-2}$ and that the asymptotic mass of this solution is given by Eq. (5.2.70), which is positive with the usual proviso that $hp > 0$.

As in the case of the hairy black holes in the $SO(3)$-gauged $\overline{\mathbb{CP}}^3$-models, the regularity of the metric imposes the constraint that the entropy

$$S_{bh} \;=\; p^2 \;-\; \frac{2}{g^2}\;, \tag{5.2.92}$$

be positive. This positivity of the entropy also ensures that the physical scalars stay in their domain of definition at $r = 0$. Indeed, the physical scalars can be compactly written as

$$\mathsf{Z} \;=\; Z^A\,T_A \;=\; \frac{\mu}{g}\left[\frac{r\overline{H}_s}{p+hr}\;n^iJ_i \;-\; \frac{r\overline{K}_s}{p+hr}\;n^i\overline{J}_i \;+\; \frac{r\overline{F}_s}{p+hr}\;P_0\right]\;, \tag{5.2.93}$$

which are therefore regular for $r > 0$. Their value at $r = 0$ is

$$\mathsf{Z}|_{r=0} \;=\; -\frac{1}{gp}\;n^i\left(J_i \;+\; \overline{J}_i\right)\;, \tag{5.2.94}$$

which, as in the case of the $SO(3)$ solution, depend only on the asymptotic charges.

Non-Abelian solutions in Magic models

In this section we would like to discuss the embeddings of monopole solutions into the gauged Magic supergravity theories. We want to show that it is not always possible to construct, given a prepotential for a theory, a globally regular solution based on a given monopole solution. We would like to stress that this holds for a given prepotential, as the choice of symplectic section for a given gauged model is physical due to the breakdown of symplectic invariance.

To start looking for ways to embed monopoles into gauged magic supergravities, we must discuss first the possible gaugings of the magic models, which boils down to a group theory problem whose outcome is given in Table 5.2.1, which we are going to explain now.

The scalar manifolds of the magic models are based on symmetric coset spaces G/H, which are given in the second and the third column in the table. As the isometry-group of the scalar manifold, which for the magic models is isomorphic to G, acts on the symplectic section defining the model (see Appendix C.2), we should specify under what representation of G it transforms; this representation is given in the column denoted as $\mathsf{G} \circ \mathcal{V}$. The following 2 columns determine how the isotropy subgroup H acts on the complex scalars $Z^i = \mathcal{X}^i/\mathcal{X}^0$; the reason why this is important will be discussed presently.

As we are interested in monopoles, we shall restrict ourselves to *compact gauge groups* G, which implies that $G \subseteq \mathsf{H}$. Moreover, as we restricted ourselves to a specific class of gaugings, *i.e.* gaugings that satisfy Eq. (C.2.13), we should use a prepotential that is G-invariant. Manifestly H-invariant prepotentials for the magic models were given in Ref. [102]. These prepotentials are of the STU-type and have the form

$$\mathcal{F}(\mathcal{X}) = \frac{\mathsf{I}_3\left(\mathcal{X}^i\right)}{\mathcal{X}^0}\,, \tag{5.2.95}$$

where I_3 is a cubic H'-invariant[14], whose value for the specific magic model can be found in the seventh column of Table 5.2.1.

Another implication of our choice of possible gauge groups is that we can only consider $G \subseteq \mathsf{H}$ for which the branching of the H-representation of the $\mathcal{X}^i$ to G-representations contains only the adjoint representation and singlets. This is a very restrictive property and the maximal possibilities we found are listed in the last column of Table 5.2.1.

Having discussed the possible models, we must then start discussing the actual embedding of the magnetic monopoles. The first thing is to solve the stabilization equation to find $\mathcal{R}$ in terms of $\mathcal{I}$. This is a complicated question but luckily a general solution exists and was found by Bates and Denef [93]; this solution uses the fact that the generic entropy functions for these models are known. For our purposes, however, the full machinery is not needed. Instead, we shall consider the simpler setting of embedding a purely magnetic monopole in the matter sector and only turn on an electric component for the graviphoton. This means that we should solve the stabilization equations,

[14]By H' we mean H minus the $U(1)$-factors.

$$
\begin{aligned}
0 &= \Im\mathrm{m}\,\mathcal{L}^0 \quad , \quad \mathcal{I}_0 = -\Im\mathrm{m}\left[\mathsf{I}_3(\mathcal{L}^i)/(\mathcal{L}^0)^2\right] \, , \\
\mathcal{I}^i &= \Im\mathrm{m}\,\mathcal{L}^i \quad , \quad 0 = \Im\mathrm{m}\left[\partial_i \mathsf{I}_3(\mathcal{L}^i) \,/\mathcal{L}^0\right] \, ,
\end{aligned}
\tag{5.2.96}
$$

where we absorbed the function X into the $\mathcal{L}$'s. This system admits a solution

$$
\mathcal{R}^i = 0 \; , \; \mathcal{R}^0 = -\frac{\sqrt{\mathcal{I}_0 \; \mathsf{I}_3(\mathcal{I}^i)}}{\mathcal{I}_0} \quad \text{provided that } \mathcal{I}_0 \; \mathsf{I}_3(\mathcal{I}^i) > 0 \, . \tag{5.2.97}
$$

With this solution to the stabilization equation, it is then straightforward to use Eq. (5.2.39) to determine

$$
\frac{1}{2|X|^2} = 4\sqrt{\mathcal{I}_0 \; \mathsf{I}_3(\mathcal{I}^i)} \, . \tag{5.2.98}
$$

The $\mathbb{C}$-magic model

Let us then consider the $\mathbb{C}$-magic model, which allows an $SU(3)$ gauging. The reason why this is the case is easy to understand: as one can see from Table 5.2.1 the $\mathcal{L}$'s transform under $SU(3)\otimes SU(3)$ as a $(\mathbf{1},\mathbf{1})\oplus(\mathbf{3},\overline{\mathbf{3}})$ representation. Choosing to gauge the diagonal $SU(3)$ means identifying the left and the right $SU(3)$ actions so that w.r.t. the diagonal action the $\mathcal{L}$'s transform as $\mathbf{1}\oplus\mathbf{3}\otimes\overline{\mathbf{3}} = \mathbf{1}\oplus\mathbf{1}\oplus\mathbf{8}$, which is just what we wanted.

The spherically symmetric monopole solution to the $SU(3)$ Bogomol'nyi equations were found by Wilkinson and Bais in Ref. [103], and a discussion of these solutions is given in Appendix F. In order to discuss the embedding of the WB-monopole, we gather the components of the symplectic vector $\mathcal{I}$ into a 3×3 matrix, $\mathcal{I}^{\mathbf{1}\oplus\mathbf{8}}$, and as this matrix behaves as the sum of a singlet and the adjoint under the diagonal $SU(3)$, we must take it to be

$$
\mathcal{I}^{\mathbf{1}\oplus\mathbf{8}} = \tfrac{1}{\sqrt{2}}\left(\lambda\,\mathbf{I}_3 - 2\Phi\right) \, , \tag{5.2.99}
$$

where Φ is defined in Eq. (F.0.2) and

$$
\lambda = l + L/r \, , \tag{5.2.100}
$$

is a real and spherically symmetric harmonic function. If we then also conveniently redefine $\sqrt{2}\mathcal{I}_0 \equiv H$, where

$$
H = h + q/r \, , \tag{5.2.101}
$$

is another real harmonic function, we can express Eq. (5.2.98) as

$$
\frac{1}{2|X|^2} = \sqrt{H\,(\lambda - \phi_1)\,(\lambda - \phi_2 + \phi_1)(\lambda + \phi_2)} \, . \tag{5.2.102}
$$

Given the asymptotic behaviour of the WB solution, let us for clarity discuss the non-degenerate solution whose asymptotic behaviour is given in Eq. (F.0.10), we can normalize the solution to be asymptotically Minkowski by demanding that

$$1 \; = \; h \, \prod_{a=1}^{3} (l+\mu_a) \; . \tag{5.2.103}$$

Using this normalization, we can then extract the asymptotic mass which turns out to be

$$M \; = \; \tfrac{1}{4}\left[\frac{q}{h} \; + \; L \, \sum_{i=1}^{3} (l+\mu_i)^{-1} \; + \; 2\frac{\mu_3 - \mu_1}{(l+\mu_1)(l+\mu_3)} \right] \; , \tag{5.2.104}$$

and must be ensured to be positive.

Let us then look for a globally regular embedding of the WB-monopole by tuning the free parameters: as before, we shall take $q = L = 0$ in order to avoid the Coulomb singularities in the Abelian field strengths. The first obvious remark is that h is already fixed in terms of l and the μ_a due to Eq. (5.2.103), so that we need to discuss the possible values for l: a first constraint for l comes from the positivity of the mass. Using the facts that $\mu_1 < 0$ and $\mu_3 > 0$, which follow from the constraint and the chosen ordering, in the mass formula (5.2.104) we see that this implies

$$M \; = \; \frac{\mu_3 \; - \; \mu_1}{2(l+\mu_1)(l+\mu_3)} \; > \; 0 \quad \Longrightarrow \quad l < -\mu_3 \;\; \text{or} \;\; l > -\mu_1 \; . \tag{5.2.105}$$

As we are interested in finding globally regular embeddings, we should discuss the regularity of the metric at $r = 0$: as the ϕ_i's vanish at the origin we see that regularity implies that

$$h \; l^3 \; = \; \prod_a \left(1 + \frac{\mu_a}{l} \right)^{-1} \; > \; 0 \; . \tag{5.2.106}$$

It is not hard to see that the above holds for the 2 bounds on l derived in Eq. (5.2.105). At this point then, the real question is whether, given the constraints on h and l derived above, there are values for r other than $r = 0$ or $r = \infty$ for which the metrical factor in Eq. (5.2.102) vanishes; from the monotonicity of ϕ_1 and ϕ_2 it is clear that if this is to happen, then this is because the factor $\lambda - \phi_2 + \phi_1$ vanishes. Seeing, then, that the combination $\phi_1 - \phi_2$ takes values between $-\mu_3$ and $-\mu_1$, we see that Eq. (5.2.102) never vanishes if

$$\lambda \; > \; \max\left(|\mu_1|, |\mu_3|\right) \quad \text{or} \quad \lambda \; < \; -\max\left(|\mu_1|, |\mu_3|\right) \; . \tag{5.2.107}$$

In order to finish the discussion of the regularity, we must have a look at the physical scalars: for the above embedding they are schematically given by $Z^{\mathbf{1}\oplus\mathbf{8}} =$

$i\,\mathcal{I}^{\mathbf{1}\oplus\mathbf{8}}/\mathcal{R}^0$, where $\mathcal{R}^0$ is given in Eq. (5.2.97). The regularity then follows straightforwardly from the regularity of monopole solution and the metric.

The $\mathbb{Q}$-magic model

All the embeddings of YM monopoles discussed till now, share a common ingredient, namely the occurrence of additional Abelian fields, whose associated harmonic functions can be used to compensate for the vanishing of the Higgs field at $r = 0$. In the above example, this rôle is played by λ and $\mathcal{I}_0$ and in the $\overline{\mathbb{CP}}^n$ and $\mathcal{ST}[2,n]$-models by the graviphoton. In fact, a model in which no such a compensator exists is the $\mathbb{Q}$-magic model.

As displayed in Table 5.2.1, the $\mathcal{X}$ in the matter sector lie in the **15** of $SU(6)$, which corresponds to holomorphic 2-forms. As $SU(6)$ admits an $SO(6) \sim SU(4)$ as a singular subgroup for which the relevant branching is $\mathbf{15} \to \mathbf{15}$, we can try to embed an $SU(4)$ WB monopole [103]. This monopole is given, as in the $SU(3)$ case, by 3 functions ϕ_i $(i = 1, 2, 3)$ and their embedding into the $\mathbb{Q}$-model has $\mathsf{I}_3(\mathcal{I}) = \mathrm{Pf}(\mathcal{X}) = \phi_1\phi_2\phi_3$. The asymptotic behaviour can of course be compensated for by choosing $\mathcal{I}_0$ judiciously, but the real problem lies at $r = 0$. At the origin the ϕ_i vanish as $\phi_1 \sim r^3$, $\phi_2 \sim r^4$ and $\phi_3 \sim r^3$ [103], which means that at the origin we have $\mathsf{I}_3(\mathcal{I}) \sim r^7 + \ldots$ The only freedom we then have is to use the harmonic function $\mathcal{I}_0$, but it is straightforward to see that this is of no use whatsoever, meaning that the resulting spacetime, as well as the physical scalars, are singular at $r = 0$.

Growing hair on the $SU(3)$ WB-monopole

Let us then end this section, with a small discussion of the hairy black hole version of the $SU(3)$-monopole. As is discussed in Appendix (F.1), singular deformations of the $SU(3)$-monopole can be found with great ease, and is determined by constants β_a $(a = 1, 2, 3)$ whose sum is zero. The hard part is to determine the values for the β's for which the metrical factor (5.2.102) does not vanish for $r > 0$. In fact, lacking general statements about the behaviour of the ϕ's, or the Q's, for general β, we shall restrict ourselves to the minimal choice $\beta_a = s\mu_a$ for $s > 0$. For this choice of β's, seeing as we are only shifting the position of where the Q's vanish from $r = 0$ to $r = -s$, the Q are monotonic, positive definite functions on $\mathbb{R}^+$. If we then rewrite the ϕ's as

$$\phi_i(r) \;=\; -\partial_r \log(Q_i) + \frac{2}{r} \;=\; -\partial_r \log(Q_i) + \frac{2}{r+s} + \frac{2s}{r(s+r)} \;\equiv\; \varphi_i(r;s) + \frac{2s}{r(s+r)}\;, \tag{5.2.108}$$

where the φ_i are regular and vanish only at $r = -s$; in fact, they correspond to the monopole's Higgs field, and are therefore negative definite on $\mathbb{R}^+$. As pointed out in the appendix, the asymptotic behaviour of the ϕ_i's remain the same as in the monopole case, so that also the normalization condition (5.2.103) and the asymptotic mass of the object (5.2.104) remain the same.

The negativity of the φ_i brings us to the next point, namely the absence of zeroes of the metrical factor at non-zero r. This is best illustrated by having a look at the function H in Eq. (5.2.102): it is clear that if H is to have no zeroes for $r > 0$, then h and q must be either both positive or negative, as otherwise $H = 0$ at $|h|r = |q|$. Following this line of reasoning on all the individual building blocks of the metrical factor in Eq. (5.2.102), and choosing for convenience h and q to be positive, shows that we must take

$$\lambda \ > \ \max(|\mu_1|, |\mu_3|) \quad \text{and} \quad L \ > \ 2 \, , \tag{5.2.109}$$

which automatically implies that the mass, Eq. (5.2.104), is positive.

In order to show that this solution corresponds to the description of a black hole outside its horizon, we must show that the near origin geometry is that of a Robinson-Bertotti/$AdS_2 \times S^2$ spacetime. As the φ_i are regular at $r = 0$, the singularities in the Higgs field come from the $1/r$ terms in Eq. (5.2.108); it is then easy to see that the near-origin geometry is indeed of the required type and that the resulting black hole horizon has entropy

$$S_{bh} \ = \ \sqrt{\, q \ L \ (L^2 \ - \ 4) \,} \ . \tag{5.2.110}$$

Of course, also in this solution the attractor mechanism is at work as one can see by calculating the values of the scalar fields at $r = 0$, *i.e.*

$$\lim_{r \to 0} Z^{\mathbf{1} \oplus \mathbf{8}} \ = \ \frac{iq}{2 S_{bh}} \, \mathrm{diag} \, (\ L - 2 \ , \ L \ , \ L + 2 \ ) \ . \tag{5.2.111}$$

5.2.4 The null case

In the null case the two spinors ϵ_1, ϵ_2 are proportional and, following the same procedure as in Refs. [26, 27], we can write[15] $\epsilon_I = \phi_I \epsilon$ where the ϕ_Is are normalized $\phi_I \phi^I = 1$ and can be understood as a unit vector selection a particular direction in $SU(2)$ or, equivalently, in S^3. It is useful to project the equations in the $SU(2)$ directions parallel and perpendicular to ϕ_I. For the fermions supersymmetry transformation rules we obtain the following four equations:

[15] The scalars ϕ_I carry a -1 charge and the spinor ϵ a +1 charge, so ϵ_I is neutral. On the other hand, the ϕ_Is have zero Kähler weight and ϵ has Kähler weight 1/2.

$$\phi^I \delta_\epsilon \psi_{I\,\mu} = \tilde{\mathfrak{D}}_\mu \epsilon \,, \tag{5.2.112}$$

$$\phi_I \delta_\epsilon \lambda^{Ii} = i \not{\mathfrak{D}} Z^i \epsilon^* \,, \tag{5.2.113}$$

$$-\epsilon_{IJ}\phi^I \delta_\epsilon \lambda^{Ji} = [\not{G}^{i\,+} + W^i]\epsilon \,, \tag{5.2.114}$$

$$-\epsilon^{IJ}\phi_I \delta_\epsilon \psi_{J\,\mu} = T^{+}{}_{\mu\nu}\gamma^\nu \epsilon^* + \epsilon^{IJ}\phi_I \partial_\mu \phi_J \epsilon \,. \tag{5.2.115}$$

The first three equations are formally identical to the supersymmetry variations of the gravitino, chiralini and gaugini in a gauged $N = 1, d = 4$ supergravity theory with vanishing superpotential that one would get by projecting out the component $N = 2$ gravitini perpendicular to ϕ_I (last equation). This is no coincidence as we could use the Ansatz $\epsilon_I = \phi_I \epsilon$ to perform a truncation of the $N = 2, d = 4$ theory to an $= 1, d = 4$ theory[16]. Thus, the $N = 2$ null case reduces to an equivalent $N = 1$ case modulo some details (the presence of the fourth equation and the covariant derivative $\tilde{\mathfrak{D}}$) that will be discussed later. We shall benefit from this fact by using the results of Refs. [30, 106] in our analysis. We can also predict the absence of domain-wall solutions in this case, since they only occur in $N = 1, d = 4$ supergravity for non-vanishing superpotential.

Before proceeding, observe that the covariant derivative acting on the supersymmetry parameter ϵ in $\phi^I \delta_\epsilon \psi_{I\,\mu}$ is defined by

$$\tilde{\mathfrak{D}}_\mu \epsilon \equiv \{\nabla_\mu + \tfrac{i}{2}\tilde{\mathcal{Q}}_\mu\}\epsilon \,, \qquad \tilde{\mathcal{Q}}_\mu \equiv \hat{\mathcal{Q}}_\mu + \zeta_\mu \,, \tag{5.2.116}$$

where

$$\zeta_\mu \equiv -2i\phi^I \partial_\mu \phi_I \,, \tag{5.2.117}$$

is a real $U(1)$ connection associated to the remaining local $U(1)$ freedom that is unfixed by our normalization of ϕ_I. It can be shown, by comparing the integrability equations of the above KSEs with the KSIs as in Refs. ([26, 27, 37]), that this connection is flat[17] and can be eliminated by choosing the phase of ϵ appropriately. We will assume that this has been done and will ignore it from now on.

The KSEs in the null case are therefore Eqs. (5.2.112)-(5.2.115) equalled to zero. To analyze them we add to the system an auxiliary spinor η, with the same chirality as ϵ but with opposite $U(1)$ charges and normalized as

[16] The Ansatz of Refs. [104, 105] is recovered for the particular choice $\phi_I = \delta_I{}^1$.

[17] This can be understood as follows: except for ζ_μ, all the objects that appear in the KSEs are related to supergravity fields and, when working out the integrability conditions, they end up being related to the different terms of the different equations of motion. The terms derived from ζ_μ (components of its curvature) are unrelated to any fields and one quickly concludes that they must vanish.

$$\bar{\epsilon}\eta = -\bar{\eta}\epsilon = \tfrac{1}{2}\,. \tag{5.2.118}$$

This normalization condition will be preserved *iff* η satisfies

$$\mathfrak{D}_\mu \eta + a_\mu \epsilon = 0\,, \tag{5.2.119}$$

for some a_μ with $U(1)$ charges -2 times those of ϵ, *i.e.*

$$\mathfrak{D}_\mu a_\nu = (\nabla_\mu - i\hat{\mathcal{Q}}_\mu)a_\nu\,, \tag{5.2.120}$$

to be determined by the requirement that the integrability conditions of this differential equation be compatible with those of the differential equation for ϵ.

The introduction of η allows for the construction of a null tetrad

$$l_\mu = i\sqrt{2}\bar{\epsilon^*}\gamma_\mu \epsilon\,, \qquad n_\mu = i\sqrt{2}\bar{\eta^*}\gamma_\mu \eta\,, \qquad m_\mu = i\sqrt{2}\bar{\epsilon^*}\gamma_\mu \eta\,, \qquad m^*_\mu = i\sqrt{2}\bar{\epsilon}\gamma_\mu \eta^*\,. \tag{5.2.121}$$

l and n have vanishing $U(1)$ charges but m (m^*) has charge -1 ($+1$), so that the metric constructed using the tetrad

$$ds^2 = 2\hat{l}\otimes\hat{n} - 2\hat{m}\otimes\hat{m}^*\,, \tag{5.2.122}$$

is invariant.

The orientation of the null tetrad is important: we choose the complex null tetrad $\{e^u, e^v, e^z, e^{z^*}\} = \{\hat{l}, \hat{n}, \hat{m}, \hat{m}^*\}$ such that

$$\epsilon^{uvzz^*} = \epsilon_{uvzz^*} = +i\,, \qquad \gamma_5 \equiv -i\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^{uv}\gamma^{zz^*}\,. \tag{5.2.123}$$

We can also construct three independent selfdual 2-forms[18]:

$$\Phi^{(1)}{}_{\mu\nu} = \bar{\epsilon}\gamma_{\mu\nu}\epsilon = 2l_{[\mu}m^*_{\nu]}\,, \tag{5.2.124}$$

$$\Phi^{(2)}{}_{\mu\nu} = \bar{\eta}\gamma_{\mu\nu}\epsilon = [l_{[\mu}n_{\nu]} + m_{[\mu}m^*_{\nu]}]\,, \tag{5.2.125}$$

$$\Phi^{(3)}{}_{\mu\nu} = \bar{\eta}\gamma_{\mu\nu}\eta = -2n_{[\mu}m_{\nu]}\,, \tag{5.2.126}$$

or, in form language

[18] The expression of these 2-forms in terms of the vectors are found by studying the contractions between the 2-forms and vectors using the Fierz identities.

$$\hat{\Phi}^{(1)} = \hat{l} \wedge \hat{m}^*\,, \tag{5.2.127}$$

$$\hat{\Phi}^{(2)} = \tfrac{1}{2}[\hat{l} \wedge \hat{n} + \hat{m} \wedge \hat{m}^*]\,, \tag{5.2.128}$$

$$\hat{\Phi}^{(3)} = -\hat{n} \wedge \hat{m}\,. \tag{5.2.129}$$

Killing equations for the vector bilinears and first consequences

Let us first consider the algebraic KSEs Eqs. (5.2.113–5.2.115) from them one can immediately obtain

$$\mathfrak{D} Z^i = = A^i \hat{l} + B^i \hat{m}\,, \tag{5.2.130}$$

$$T^+ = \tfrac{1}{2}\phi\, \hat{\Phi}^{(1)}\,, \tag{5.2.131}$$

$$G^{i\,+} = \tfrac{1}{2}\phi^i\, \hat{\Phi}^{(1)} - \tfrac{1}{2}W^i \hat{\Phi}^{(2)}\,, \tag{5.2.132}$$

$$\epsilon^{IJ}\phi_I d\phi_J = \tfrac{i}{\sqrt{2}}\phi \hat{l}\,, \tag{5.2.133}$$

where ϕ, ϕ^i, A^i and B^i are complex functions to be determined.

The last equation combined with the vanishing of ζ_μ imply that

$$d\phi_I \sim \hat{l}\,, \qquad d\phi \sim \hat{l}\,. \tag{5.2.134}$$

The resulting vector field strengths $F^{\Lambda\,+}$ are of the form

$$F^{\Lambda\,+} = \tfrac{1}{2}\phi^\Lambda \hat{\Phi}^{(1)} - \tfrac{i}{2}\mathcal{D}^\Lambda \hat{\Phi}^{(2)}\,, \tag{5.2.135}$$

where the ϕ^Λ are complex functions related to ϕ and ϕ^i by

$$\phi^\Lambda = i\mathcal{L}^{*\Lambda}\phi + 2f^\Lambda{}_i \phi^i\,, \tag{5.2.136}$$

and we have defined

$$\mathcal{D}^\Lambda \equiv -2if^\Lambda{}_i W^i\,. \tag{5.2.137}$$

Observe that as

$$\mathcal{D}^\Lambda = -igf_{\Sigma\Omega}{}^\Lambda \mathcal{L}^\Omega \mathcal{L}^{*\Sigma} = \tfrac{1}{2}g\Im\mathrm{m}\mathcal{N}^{-1|\Lambda\Sigma}\mathcal{P}_\Sigma\,, \tag{5.2.138}$$

is real, we find that the field strengths are given by

$$F^{\Lambda} = -\tfrac{1}{2}(\phi^{*\Lambda}\hat{m} + \phi^{\Lambda}\hat{m}^{*}) \wedge \hat{l} - \tfrac{i}{2}\mathcal{D}^{\Lambda}\hat{m} \wedge \hat{m}^{*}\,. \tag{5.2.139}$$

Let us consider the differential KSE $\mathfrak{D}_{\mu}\epsilon = 0$ and the auxiliar KSE Eq. (5.2.119): a straightforward calculation results in

$$\mathfrak{D}_{\mu}l_{\nu} = \nabla_{\mu}l_{\nu} = 0\,, \tag{5.2.140}$$

$$\mathfrak{D}_{\mu}n_{\nu} = \nabla_{\mu}n_{\nu} = -a^{*}_{\mu}m_{\nu} - a_{\mu}m^{*}_{\nu}\,, \tag{5.2.141}$$

$$\mathfrak{D}_{\mu}m_{\nu} = (\nabla_{\mu} - i\hat{\mathcal{Q}}_{\mu})m_{\nu} = -a_{\mu}l_{\nu}\,. \tag{5.2.142}$$

The first of these equations implies that l^{μ} is a covariantly constant null Killing vector, Eq. (5.2.140), which tells us that the spacetime is a Brinkmann *pp*-wave [107]. Since l^{μ} is a Killing vector and $d\hat{l} = 0$ we can introduce the coordinates u and v such that

$$\hat{l} = l_{\mu}dx^{\mu} \equiv du\,, \tag{5.2.143}$$

$$l^{\mu}\partial_{\mu} \equiv \frac{\partial}{\partial v}\,. \tag{5.2.144}$$

We can also define a complex coordinate z by

$$\hat{m} = e^{U}dz\,, \tag{5.2.145}$$

where U may depend on z, z^{*} and u but not on v. Given the chosen coordinates, the most general form of $\hat{n}$ is

$$\hat{n} = dv + Hdu + \hat{\omega}\,, \qquad \hat{\omega} = \omega_{\underline{z}}dz + \omega_{\underline{z}^{*}}dz^{*}\,, \tag{5.2.146}$$

where all the functions in the metric are independent of v. Either H or the 1-form $\hat{\omega}$ could, in principle, be removed by a coordinate transformation, but we have to check that the tetrad integrability equations (5.2.140)-(5.2.142) are satisfied by our choices of e^{U}, H and $\hat{\omega}$.

With above choice of coordinates, Eq. (5.2.122) leads to the metric

$$ds^{2} = 2du(dv + Hdu + \hat{\omega}) - 2e^{2U}dzdz^{*}\,. \tag{5.2.147}$$

Let us then consider the tetrad integrability equations (5.2.140)-(5.2.142): the first equation is solved because the metric does not depend on v. The third equation, with the choice (5.2.145) for the coordinate z implies

$$\hat{a} = n^\mu[\partial_\mu U - i\hat{\mathcal{Q}}_\mu]\hat{m} + D\hat{l}\,, \tag{5.2.148}$$

$$0 = m^\mu[\partial_\mu U - i\hat{\mathcal{Q}}_\mu]\,, \tag{5.2.149}$$

$$0 = l^\mu A^\Lambda{}_\mu \Im\mathrm{m}\, \lambda_\Lambda\,, \tag{5.2.150}$$

where D is a function to be determined. The last equation can be solved by the gauge choice

$$l^\mu A^\Lambda{}_\mu = 0\,. \tag{5.2.151}$$

In this gauge the complex scalars Z^i are v-independent. The remaining components of the gauge field $A^\Lambda{}_\mu$ are also v-independent as is indicated by the absence of a $\hat{l}\wedge\hat{n}$, $\hat{m}\wedge\hat{n}$ or a $\hat{m}^*\wedge\hat{n}$ term in the vector field strength. This in its turn, implies the v-independence of all the components of the vector field strengths, of the functions ϕ^i and, finally, of A^i and B^i.

The above condition does not completely fix the gauge freedom of the system, since v-independent gauge transformations preserve it. We can use this residual gauge freedom to remove the $A^\Lambda{}_{\underline{u}}$ component of the gauge potential by means of a v-independent gauge transformation. This leaves us with only one complex independent component $A^\Lambda{}_{\underline{z}}(z,z^*,u) = (A^\Lambda{}_{\underline{z}^*})^*$ and

$$F^\Lambda{}_{\underline{uz}} = \partial_{\underline{u}} A^\Lambda{}_{\underline{z}} = \tfrac{1}{2}e^U\phi^\Lambda\,, \tag{5.2.152}$$

$$F^\Lambda{}_{\underline{zz}^*} = \partial_{\underline{z}} A^\Lambda{}_{\underline{z}^*} + \tfrac{1}{2}g f_{\Sigma\Omega}{}^\Lambda A^\Sigma{}_{\underline{z}} A^\Omega{}_{\underline{z}^*} - \text{c.c.} = -\tfrac{i}{2}e^{2U}\mathcal{D}^\Lambda\,. \tag{5.2.153}$$

We can then treat $F^\Lambda{}_{\underline{zz}^*} dz\wedge dz^*$ as a 2-dimensional YM field strength on the 2-dimensional space with Hermitean metric $2e^{2U}dzdz^*$, both of them depending on the parameter u. This implies that we can always write

$$F^\Lambda{}_{\underline{zz}^*} = 2i\partial_{\underline{z}}\partial_{\underline{z}^*} Y^\Lambda\,, \tag{5.2.154}$$

for some real $Y^\Lambda(z,z^*,u)$. In the Abelian, *i.e.* ungauged, case

$$A^\Lambda{}_{\underline{z}} = -i\partial_{\underline{z}} Y^\Lambda\,. \tag{5.2.155}$$

Using Eq (B.1.26) we can express the second of the tetrad conditions, Eq. (5.2.149), as

$$\partial_{\underline{z}^*}(U + \mathcal{K}/2) = -gA^\Lambda{}_{\underline{z}^*}\lambda_\Lambda\,. \tag{5.2.156}$$

In the ungauged case this equation (and its complex conjugate) can be immediately integrated to give $U = -\mathcal{K}/2 + h(u)$. The function $h(u)$ can be eliminated by a coordinate redefinition that does not change the form of the Brinkmann metric.

In the Abelian case of the pure $N = 1, d = 4$ theory, it is possible to have constant momentum maps (D-terms), as considered in Ref. [108], and $\lambda_\Lambda = -i\mathcal{P}_\Lambda$ and Eq. (5.2.155) would lead to

$$\partial_{\underline{z}^*}(U + \mathcal{K}/2 + gY^\Lambda\mathcal{P}_\Lambda) = 0\,, \tag{5.2.157}$$

which is solved by $U = -\mathcal{K}/2 - gY^\Lambda\mathcal{P}_\Lambda + h(u)$; $h(u)$ can still be eliminated by a coordinate transformation. In the $N = 2, d = 4$ theory, however, it is not possible to use constant momentum maps to gauge an Abelian symmetry and the situation is slightly more complicated. The integrability condition of Eq. (5.2.156) and its complex conjugate is solved by

$$A^\Lambda{}_{\underline{z}^*}\lambda_\Lambda = \partial_{\underline{z}^*}[R(z, z^*, u) + S^*(z^*, u)]\,, \tag{5.2.158}$$

where R is a real function and $S(z, u)$ a holomorphic function of z, which then implies

$$U = -\mathcal{K}/2 - g(R + S + S^*)\,. \tag{5.2.159}$$

Finally, the second tetrad integrability equation (5.2.141) implies

$$D = e^{-U}(\partial_{\underline{z}^*}H - \dot{\omega}_{\underline{z}^*})\,, \tag{5.2.160}$$

$$(d\omega)_{\underline{z}\underline{z}^*} = 2ie^{2U}n^\mu\hat{\mathcal{Q}}_\mu\,, \tag{5.2.161}$$

whence $\hat{a}$ is given by

$$\hat{a} = [\dot{U} - \tfrac{1}{2}e^{-2U}(d\omega)_{\underline{z}\underline{z}^*}]\hat{m} + e^{-U}(\partial_{\underline{z}^*}H - \dot{\omega}_{\underline{z}^*})\hat{l}\,. \tag{5.2.162}$$

Killing spinor equations

In the previous sections we have shown that supersymmetric configurations belonging to the null case must necessarily have a metric of the form Eq. (5.2.147), vector field strengths of the form Eq. (5.2.139), and scalar field strengths of the form Eq. (5.2.130); they must further satisfy Eqs. (5.2.133,5.2.149) and (5.2.161) for some $SU(2)$ vector ϕ_I. We now want to show that these conditions are sufficient for a field configuration $\{g_{\mu\nu}, A^\Lambda, F^\Lambda, \mathfrak{D}Z^i\}$ to be supersymmetric.

It takes little to no time to see that all the components of the KSEs are satisfied for constant Killing spinors (in the chosen gauge, frame, etc.) that obey the condition

$$\gamma^u\epsilon^I = 0\,. \tag{5.2.163}$$

This constraint, which is equivalent to $\gamma^z \epsilon^I = 0$, together with chirality, imply that the Killing spinors live in a complex 1-dimensional space, whence we can write $\epsilon^I = \xi^I \epsilon = 0$. Up to normalization, solving the KSEs requires that $\xi^I = \phi^I$, where the functions ϕ^I are given as part of the definition of the supersymmetric field configuration. As a result, the supersymmetric configurations of this theory preserve, generically, 1/2 of the 8 supercharges.

Observe that in order to prove the existence of Killing spinors it has not been necessary to impose the integrability conditions of the field strengths, i.e. the Bianchi identities of the vector field strengths etc., *nor* the integrability constraints of Eqs. (5.2.133), (5.2.149) and (5.2.161). We are however forced to do so in order to have well-defined field configurations in terms of the fundamental fields $\{g_{\mu\nu}, A^\Lambda, Z^i\}$. We will deal with these integrability conditions and the equations of motion in the next section.

Supersymmetric null solutions

Let us start by computing the Bianchi identities and Maxwell equations taking the expression for $F^{\Lambda\,+}$ in (5.2.135) as our starting point. We find

$$\begin{aligned} \mathfrak{D}F^{\Lambda\,+} &= \left\{\tfrac{1}{2}m^{*\,\mu}\mathfrak{D}_\mu\phi^\Lambda - \tfrac{i}{4}n^\mu\mathfrak{D}_\mu\mathcal{D}^\Lambda - \tfrac{i}{2}\mathcal{D}^\Lambda n^\mu[\partial_\mu U - i\hat{\mathcal{Q}}_\mu]\right\}\hat{l}\wedge\hat{m}\wedge\hat{m}^* \\ &\quad +\tfrac{i}{4}\left\{m^{*\,\mu}\mathfrak{D}_\mu\mathcal{D}^\Lambda\hat{l}\wedge\hat{n}\wedge\hat{m} + \text{c.c.}\right\}. \end{aligned} \tag{5.2.164}$$

Observe that the terms in the second line are purely imaginary, so that

$$\begin{aligned} \star\mathcal{B}^\Lambda &= -2\Re\mathrm{e}\,\mathfrak{D}F^{\Lambda\,+} \\ &= -i\left\{\Im\mathrm{m}(m^{*\,\mu}\mathfrak{D}_\mu\phi^\Lambda) - \tfrac{1}{2}n^\mu\mathfrak{D}_\mu\mathcal{D}^\Lambda - \mathcal{D}^\Lambda n^\mu\partial_\mu U\right\}\hat{l}\wedge\hat{m}\wedge\hat{m}^*. \end{aligned} \tag{5.2.165}$$

A similar calculation for F_Λ leads to

$$-\mathfrak{D}F_\Lambda \tag{5.2.166}$$

$$= -2\Re\mathrm{e}\,\mathfrak{D}(\mathcal{N}^*_{\Lambda\Sigma}F^{\Sigma\,+}) \tag{5.2.167}$$

$$\begin{aligned} &= -i\left\{\Im\mathrm{m}\,(m^{*\,\mu}\mathfrak{D}_\mu\phi_\Lambda) - \tfrac{1}{2}n^\mu\mathfrak{D}_\mu\Re\mathrm{e}\,\mathcal{D}_\Lambda - \Re\mathrm{e}\,\mathcal{D}_\Lambda n^\mu\partial_\mu U - \Im\mathrm{m}\,\mathcal{D}_\Lambda n^\mu\hat{Q}_\mu\right\}\hat{l}\wedge\hat{m}\wedge\hat{m}^* \\ &\quad +\Re\mathrm{e}\left[m^{*\,\mu}\mathfrak{D}_\mu\Im\mathrm{m}\,\mathcal{D}_\Lambda\hat{l}\wedge\hat{n}\wedge\hat{m}\right], \end{aligned} \tag{5.2.168}$$

where

$$\phi_\Lambda \equiv \mathcal{N}^*_{\Lambda\Sigma}\phi^\Sigma\,, \qquad \mathcal{D}_\Lambda \equiv \mathcal{N}^*_{\Lambda\Sigma}\mathcal{D}^\Sigma\,, \;\Rightarrow \Im\mathrm{m}\,\mathcal{D}_\Lambda = -\tfrac{1}{2}g\mathcal{P}_\Lambda\,. \tag{5.2.169}$$

Of course we can also calculate

$$\tfrac{1}{2}g\star\Re\mathrm{e}\,(k^*_{\Lambda\,i}\mathfrak{D}Z^i) = \tfrac{i}{2}g\Im\mathrm{m}\,(n^\mu\mathfrak{D}_\mu Z^i\partial_i\mathcal{P}_\Lambda)\hat{l}\wedge\hat{m}\wedge\hat{m}^* + \tfrac{1}{2}g\Re\mathrm{e}\,[m^{*\mu}\mathfrak{D}_\mu Z^i\partial_i\mathcal{P}_\Lambda\hat{l}\wedge\hat{n}\wedge\hat{m}]\,, \tag{5.2.170}$$

which means that the Maxwell equation can be expressed as

$$\begin{aligned}
\star\mathcal{E}_\Lambda &= -\mathfrak{D}F_\Lambda + \tfrac{1}{2}g\star\Re\mathrm{e}\,(k^*_{\Lambda\,i}\mathfrak{D}Z^i) \\
&= -i\Big\{\Im\mathrm{m}(m^{*\,\mu}\mathfrak{D}_\mu\phi_\Lambda) - \tfrac{1}{2}n^\mu\mathfrak{D}_\mu\Re\mathrm{e}\,\mathcal{D}_\Lambda - \Re\mathrm{e}\,\mathcal{D}_\Lambda n^\mu\partial_\mu U \\
&\qquad -\Im\mathrm{m}\,\mathcal{D}_\Lambda n^\mu\hat{Q}_\mu - \tfrac{1}{2}g\Im\mathrm{m}\,(n^\mu\mathfrak{D}_\mu Z^i\partial_i\mathcal{P}_\Lambda)\Big\}\,\hat{l}\wedge\hat{m}\wedge\hat{m}^*
\end{aligned} \tag{5.2.171}$$

In concordance with the KSIs, the Maxwell equations and Bianchi identities have only one non-trivial component, wherefore all the KSIs that involve them are automatically satisfied.

Finally, the only non-automatically satisfied component of the Einstein equations is

$$\mathcal{E}_{\underline{uu}} = R_{\underline{uu}} + 2\mathcal{G}_{ij^*}A^iA^{*\,j^*} - 2\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}\phi^\Lambda\phi^{*\Sigma} = 0\,. \tag{5.2.172}$$

Using our coordinate and gauge choices $l^\mu A^\Lambda{}_\mu = A^\Lambda{}_{\underline{v}} = 0$ and $n^\mu A^\Lambda{}_\mu = A^\Lambda{}_{\underline{u}} = 0$, we can rewrite the above Bianchi identities, Maxwell equations and Einstein equation as

$$\Im\mathrm{m}\,\mathfrak{D}_{\underline{z}}(e^U\phi^\Lambda) = -\tfrac{1}{2}\partial_{\underline{u}}(e^{2U}\mathcal{D}^\Lambda)\,, \tag{5.2.173}$$

$$\Im\mathrm{m}\,\mathfrak{D}_{\underline{z}}(e^U\phi_\Lambda) = -\tfrac{1}{2}\partial_{\underline{u}}(e^{2U}\Re\mathrm{e}\,\mathcal{D}_\Lambda) - \tfrac{1}{2}g\Im\mathrm{m}\,[\partial_{\underline{u}}Z^ie^{\mathcal{K}}\partial_i(e^{-\mathcal{K}}\mathcal{P}_\Lambda)]\,, \tag{5.2.174}$$

$$\begin{aligned}
\partial_{\underline{z}}\partial_{\underline{z}^*}H &= \partial_{\underline{z}}\dot{\omega}_{\underline{z}^*} + e^{2U}\{\partial_{\underline{u}} + [\dot{U} - \tfrac{1}{2}e^{-2U}(d\omega)_{\underline{zz}^*}]\}[\dot{U} - \tfrac{1}{2}e^{-2U}(d\omega)_{\underline{zz}^*}] \\
&\quad + e^{2U}\mathcal{G}_{ij^*}(A^iA^{*\,j^*} + 2\phi^i\phi^{*j^*}) + \tfrac{1}{2}e^{2U}|\phi|^2\,.
\end{aligned} \tag{5.2.175}$$

where we made used of

$$\begin{aligned}
\mathfrak{D}_{\underline{z}^*}(e^U\phi^\Lambda) &\equiv \partial_{\underline{z}^*}(e^U\phi^\Lambda) + gf_{\Sigma\Omega}{}^\Lambda A^\Sigma{}_{\underline{z}^*}e^U\phi^\Omega\,, &(5.2.176)\\
\mathfrak{D}_{\underline{z}^*}(e^U\phi_\Lambda) &\equiv \partial_{\underline{z}^*}(e^U\phi_\Lambda) + gf_{\Lambda\Sigma}{}^\Omega A^\Sigma{}_{\underline{z}^*}e^U\phi^\Omega\,. &(5.2.177)
\end{aligned}$$

To summarize our results, supersymmetric configurations have vector and scalar field strengths and metric given by Eqs. (5.2.139,5.2.130) and (5.2.147) and must satisfy the first-order differential Eqs. (5.2.161) and (5.2.156). We must also find ϕ_I and ϕ such that

$$\epsilon^{IJ}\phi_I \partial_{\underline{u}}\phi_J = \tfrac{i}{\sqrt{2}}\phi\,. \tag{5.2.178}$$

If a supersymmetric configuration satisfies the second-order differential Eqs. (5.2.173-5.2.175) then it satisfies all the classical equations of motion and is supersymmetric solutions.

u-independent supersymmetric null solutions

In the u-independent case the equations that we have to solve simplify considerably. First of all, since the complex scalars Z^i are u-independent, we have $A^i = 0$ and $(d\omega)_{zz^*} = 0$, whence we can take $\hat{\omega} = 0$. Furthermore, $\phi^\Lambda = 0$ (see Eq. (5.2.152)), which implies $\phi = \phi^i = 0$ (see Eq. (5.2.136)) and the constancy of ϕ_I, which is otherwise arbitrary. We need to solve Eq. (5.2.156), which is only possible if its integrability condition Eq. (5.2.158), which we repeat here for clarity,

$$A^\Lambda{}_{\underline{z}^*}\lambda_\Lambda = \partial_{\underline{z}^*}[R(z,z^*,u) + S^*(z^*,u)]\,, \tag{5.2.179}$$

is satisfied. Then, the solution is

$$U = -\mathcal{K}/2 - g(R + S + S^*)\,. \tag{5.2.180}$$

We also need to find covariantly-holomorphic functions $Z^i(z,z^*)$ by solving

$$\partial_{\underline{z}^*}Z^i + gA^\Lambda{}_{\underline{z}^*}k_\Lambda{}^i = 0\,, \tag{5.2.181}$$

which depends strongly on the model.

Finally, the only e.o.m. need to solve is the Einstein equation Eq. (5.2.175): in this case it reduces to the 2-dimensional Laplace equation and is solved by real harmonic functions H on $\mathbb{R}^2$.

In spite of the apparent simplicity of this system, we have not been able to find solutions different from those of the ungauged theory.

5.2.5 Summary

In this Chapter we found the general form of all the supersymmetric configurations of $N = 2, d = 4$ Einstein-Yang-Mills theories and have analyzed the conditions that fields have to satisfy in order to give rise to a supersymmetric solution. In the timelike case, we presented and analyzedd some spherically-symmetric solutions, which describe monopoles and hairy black holes. As the monopole solutions to the Bogomol'nyi equations are regular on $\mathbb{R}^3$, we investigated the question of whether this regularity can be extended to the full supergravity solution, which we called global regularity. This is a tricky question whose answer, perhaps disappointingly, but understandably, is that it depends on the model. As should be clear from the results of Section 5.2.3, the biggest obstruction to generating globally-regular supergravity solutions out of spherically-symmetric monopoles can also be one of its virtues, namely that at the origin the Higgs field vanishes; as long as the model we are using has extra Abelian fields, this 'problem' can be obviated, but otherwise, such as happens in the $SO^*(12)$ model, it is a real showstopper.

The hairy black holes were generated by the introduction of a parameter $s > 0$ called the *Protogenov hair*. The introduction of this parameter in the solutions is straightforward and basically consists of doing a coordinate shift in the exponential parts of the explicit expressions for the gauge connection and the Higgs field. The effect of this coordinate shift w.r.t.the monopole solution is to leave unchanged the asymptotic behaviour of the solution, but to change the behaviour of the solution at the origin. In fact, due to the positivity of s, the singularity is of Coulomb type and opens up the possibility of creating black holes similar to the ones occurring in Abelian theories. We analyzed how the attractor mechanism works for the supersymmetric non-Abelian black holes. The solutions we studied show that the asymptotic data needed to specify an $N = 2$ $d = 4$ sugra black hole (*i.e.* the asymptotic mass, the moduli and the asymptotic charges) are independent of the parameter s which is, however, needed in order to specify the black hole fully and demonstrates the failure of the no-hair theorem for gravity coupled to YM fields in an explicit and analytic manner.[19] More surprisingly, the hair parameters don't show up in other relevant quantities such as the entropy of the black hole or the attractor values for the scalars at the horizon: a general understanding of why this happens is lacking but needed.

The attractor mechanism that holds for the scalars of the Abelian black holes still works, but in a generalized way: the Higgs field is not gauge-invariant and one can only expect "attraction" up to gauge transformations. Gauge-invariant combinations of the scalar fields do have fixed points on the horizon. In the null case we did not find new supersymmetric solutions different from those of the ungauged theory.

[19]There can of course be more hairy parameters than just the Protogenov hair. In fact, the *cloud* parameter a in Eqs. (5.2.85) and (5.2.89) should also be considered as hair.

Chapter 6

Coupling of higher-dimensional objects to p-forms

6.1 1-brane solutions in $N = 2$ Supergravity

To describe the dynamics of p-dimensional extended objects, *p-branes*, one has to generalize the action of a massive point-particle. While a point-particle moves along a *worldline* in space-time, a p-brane sweeps out a $(p+1)$-dimensional surface, the *worldvolume*, parametrized by $p+1$ coordinates σ^i, $i = 0 \dots p$. If the particle carries an electric charge, its interaction with the electromagnetic field A_μ is described by the minimal coupling

$$q \int dX^\mu A_\mu(X(\tau)) \,, \tag{6.1.1}$$

where $\tau = \sigma^0$ is the worldline coordinate. In an analogous way, a p-brane which carries "charge" can couple minimally to an $(p+1)$-form antisymmetric tensor field C and the corresponding term in the action (called a Wess-Zumino term) takes the form

$$S_{WZ} = q \int dX^{\mu_1} \dots X^{\mu_{p+1}} C_{\mu_1 \dots \mu_{p+1}}(X(\sigma)). \tag{6.1.2}$$

The wordline action for a point-particle is of the form

$$S = m \int d\tau \sqrt{\frac{\partial X^\mu}{\partial \tau} \frac{\partial X^\nu}{\partial \tau} g_{\mu\nu}(X(\tau))} + q \int d\tau A_\mu \frac{\partial X^\mu}{\partial \tau} \,, \tag{6.1.3}$$

and its generalization to extended objects

$$S = T \int d^{p+1}\sigma \sqrt{|g_{ij}|} + q \int d\sigma^{i_1} \ldots d\sigma^{i_{p+1}} C_{\mu_1 \ldots \mu_{p+1}} \frac{\partial X^{\mu_1}}{\partial \sigma^{i_1}} \ldots \frac{\partial X^{\mu_{p+1}}}{\partial \sigma^{i_{p+1}}}\,, \tag{6.1.4}$$

where

$$g_{ij} = g_{\mu\nu} \frac{\partial X^{\mu}}{\partial \sigma^{i}} \frac{\partial X^{\nu}}{\partial \sigma^{j}} \tag{6.1.5}$$

is the world-volume metric induces by the space-time metric $g_{\mu\nu}$ and T the p-brane tension (which in case of a point-particle is just its mass). If one deals with D-branes, i.e. p-branes on which open strings can end, one has to take into account an additional world-volume vector gauge field, which is induced by the endpoints of the string moving along the brane). The dynamics then is described by the Dirac-Born-Infeld action

$$S = T \int d^{p+1}\sigma \sqrt{|g_{ij} + \mathcal{F}_{ij}|} + q \int C\,, \tag{6.1.6}$$

where $\mathcal{F}_{ij}$ is the generalized field strength of the gauge field A_i

$$\mathcal{F}_{ij} = 2\partial_{[i} A_{j]}(\sigma) + B_{ij} \tag{6.1.7}$$

and

$$B_{ij} = \partial_i X^{\mu} \partial_j X^{\nu} B_{\mu\nu} \tag{6.1.8}$$

the pullback of a space-time 2-form gauge field $B_{\mu\nu}$. The (electric) charge q of the p-brane can be calculated in d dimensions using the higher-dimensional version of Gauss's law $q = \int_{d-p-2} \star F_{p+2}$. Note that up to now we were only considering the purely bosonic action for a p-brane. When we are interested in supersymmetry, we also have to take into account fermions. Thus we have to extend the set of bosonic coordinates $X^{\mu}(\sigma)$ by a set of anti-commuting coordinates $\theta^{\alpha}(\sigma)$. A key role in the description of supersymmetric brane actions is played by a fermionic symmetry called *κ-symmetry*. This symmetry implies world-volume supersymmetry with equal number of bosonic and fermionic degrees of freedom, since half of the spinor degrees of freedom become redundant because they may be eliminated by a gauge choice. It further relates the brane tension T to its charge q, ensuring that the brane ground states are stable, i.e. they are BPS states.

In what follows we will study the extension of $N = 2$ four-dimensional supergravity including magnetic vector fields and 2-form potentials. In four dimensions 2-form potentials are dual to those scalars which parameterize the Noether currents. They couple electrically to 1-dimensional branes, just like the 8-form potentials of IIB supergravity play an important role when discussing the supersymmetry properties of 7-branes in ten dimensions [109–112]. It was shown in [113] that one cannot in general dualize just any scalar into a 2-form potential. The objects to dualize are those Noether currents associated with the isometries of the scalar sigma models which extend to be symmetries of the full theory. Dualizing the Noether currents one obtains as many 2-forms as there are isometries. In general the field strengths of these 2-forms satisfy constraints such that the number of 2-form degrees of freedom equals the number of scalar degrees of freedom which occur in the Noether currents. We explicitly construct the Noether currents for all the duality symmetries of ungauged $N = 2, d = 4$ supergravity coupled to both vector multiplets and hypermultiplets.

Via a straightforward dualization prescription we construct the 2-form potentials and prove that the supersymmetry algebra can be closed on them. Once we have found the explicit supersymmetry transformations for the 2-forms we proceed to construct the leading terms of a half-supersymmetric world-sheet effective action. Finally, we discuss in some detail the properties of the half-supersymmetric stringy cosmic string solutions. The above program is first performed for the duality symmetries associated with the scalars coming from the vector multiplets and then repeated for the duality symmetries associated with the scalars coming from the hypermultiplets.

6.1.1 The 1-forms

The $N = 2, d = 4$ supergravity theory coupled to n_V vector multiplets contains $n_V + 1$ 'fundamental' vector fields $A^\Lambda{}_\mu$ whose supersymmetry transformation rules are given in Eq. (2.2.33). The potentials $A^\Lambda{}_\mu$ couple electrically to charged particles. In the next Section we will construct the leading terms of the bosonic part of the κ-symmetric world-line effective actions for particles electrically charged under $A^\Lambda{}_\mu$.

As mentioned in Section 2.2, the equations of motion of the potentials $A^\Lambda{}_\mu$, Eqs. (2.2.7), can be understood as providing the Bianchi identities for a set of dual field strengths F_Λ, defined in Eq. (2.2.9). These equations imply the local on-shell existence of $n_V + 1$ dual potentials $A_{\Lambda\,\mu}$. The dual potentials $A_{\Lambda\,\mu}$ couple electrically to particles which are magnetically charged under the fundamental vector fields $A^\Lambda{}_\mu$. In this Section we will derive the supersymmetry transformation rules for the dual potentials $A_{\Lambda\,\mu}$. This result will then be used in the next Section to construct the leading terms of the bosonic part of the κ-symmetric world-line effective actions for particles electrically charged under the $A_{\Lambda\,\mu}$.

The fundamental potentials and their duals can be seen as, respectively, the upper and lower components of the symplectic vector $\mathcal{A}_\mu$ defined in Eq. (2.2.17). Electric-magnetic duality transformations act linearly on it. This behaviour under duality transformations suggests the following Ansatz for the supersymmetry transformation rule of $\mathcal{A}$:

$$\delta_\epsilon \mathcal{A}_\mu \;=\; \tfrac{1}{4}\mathcal{V}\,\epsilon_{IJ}\bar{\psi}^I_\mu \epsilon^J + \tfrac{i}{8}\mathfrak{D}_i\mathcal{V}\,\epsilon_{IJ}\bar{\lambda}^{Ii}\gamma_\mu\epsilon^J + \text{c.c.}\,. \tag{6.1.9}$$

This Ansatz agrees with the supersymmetry transformation rule of the fundamental potentials $A^\Lambda{}_\mu$ as given in Eq. (2.2.33) and with the fact that the symplectic vector of 1-forms $\mathcal{A}_\mu$ transform linearly under $Sp(2n_V + 2, \mathbb{R})$. The supersymmetry algebra closes on $\mathcal{A}_\mu$ with the above supersymmetry transformation rule. Indeed, we find for the commutator of two supersymmetries acting on $\mathcal{A}_\mu$,

$$[\delta_\eta, \delta_\epsilon]\mathcal{A}_\mu = \delta_{\text{g.c.t.}}(\xi)\mathcal{A}_\mu + \delta_{\text{gauge}}(\Lambda)\mathcal{A}_\mu \,. \tag{6.1.10}$$

The general coordinate transformation of $\mathcal{A}_\mu$ is given by

$$\delta_{\text{g.c.t.}}(\xi)\mathcal{A}_\mu = \pounds_\xi \mathcal{A}_\mu = \xi^\nu \partial_\nu \mathcal{A}_\mu + (\partial_\mu \xi^\nu)\mathcal{A}_\nu \,, \tag{6.1.11}$$

with $\pounds_\xi$ denoting the Lie derivative and where the infinitesimal parameter ξ^ρ is given by

$$\xi^\mu \;\equiv\; -\tfrac{i}{4}\bar{\eta}^I \gamma^\mu \epsilon_I + \text{c.c.}\,,$$

and $\delta_{\text{gauge}}(\Lambda)$ is a $U(1)$ gauge transformation with parameter Λ^Λ. The gauge transformation of $\mathcal{A}_\mu$ is given by

$$\delta_{\text{gauge}}(\Lambda)\mathcal{A}_\mu = \partial_\mu \Lambda \,, \tag{6.1.12}$$

where the gauge transformation parameter Λ is the symplectic-covariant generalization of Λ^Λ and is given by

$$\Lambda \equiv -\xi^\rho \mathcal{A}_\rho + \tfrac{1}{4}\left(\mathcal{V}\epsilon_{IJ}\bar{\eta}^I \epsilon^J + \text{c.c.}\right) \,. \tag{6.1.13}$$

6.1.2 World-line actions for 0-branes

In this Section we will construct the leading terms of the bosonic part of a κ-invariant world-line effective action for 0-branes that couple to the 1-form potentials $A^\Lambda{}_\mu$ and $A_{\Lambda\,\mu}$. In doing so we will take into account the symplectic structure of the theory. The actions will be invariant under symplectic transformations provided we also transform an appropriate set of the charges, in the spirit of Ref. [114].

It is clear that the 0-branes of $N = 2, d = 4$ supergravity coupled to n_V vector multiplets can carry both electric charges q_Λ and magnetic charges p^Λ with respect to the fundamental potentials $A^\Lambda{}_\mu$. The couplings of the magnetic 0-branes are, however, better described as electric couplings to the dual potentials $A_{\Lambda\,\mu}$. A 0-brane with symplectic charge vector

$$q \equiv \begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix} \,. \tag{6.1.14}$$

will couple electrically to the potential $\mathcal{A}$. The only symplectic-invariant coupling is $\langle q \mid \mathcal{A}\rangle$. We thus propose the following Wess–Zumino term

$$\int d\tau \,\langle\, q \mid \mathcal{A}_\mu \,\rangle\, \frac{dX^\mu}{d\tau} \,, \tag{6.1.15}$$

where τ is the world-line parameter and X^μ the embedding coordinate of the 0-brane. This Ansatz is clearly the only one satisfying the requirements of symplectic invariance and gauge invariance.

The corresponding kinetic term in the 0-brane action is not much more difficult to guess. Symplectic invariance requires that the charges q_Λ and p^Λ appear in a symplectic invariant combination with the scalars in the tension. The simplest combination is just the central charge

$$\mathcal{Z} = \langle\, q \mid \mathcal{V}\,\rangle\,, \tag{6.1.16}$$

whose asymptotic absolute value is known to give the mass of supersymmetric black holes of these theories. Then, the world-line effective action takes the form

$$S = \int d\tau\ |\mathcal{Z}|\ \sqrt{\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}g_{\mu\nu}(X)} + \int d\tau \langle\, q \mid \mathcal{A}_\mu\,\rangle \frac{dX^\mu}{d\tau}\,. \tag{6.1.17}$$

Using the supersymmetry transformations (2.2.32), (2.2.34) and (6.1.9) we find that the action (6.1.17) preserves half of the supersymmetries with the projector given by

$$\epsilon_I + i\frac{\mathcal{Z}}{|\mathcal{Z}|}\epsilon_{IJ}\frac{\gamma_\tau}{\sqrt{g_{\tau\tau}}}\epsilon^J = 0\,, \tag{6.1.18}$$

where the subindex τ means contraction of a space-time index μ with $dX^\mu/d\tau$. This is the same constraint that the Killing spinors of supersymmetric $N = 2, d = 4$ black holes satisfy [26, 87, 89, 115]. In the static gauge, $\dot{X}^\mu = dX^\mu/d\tau = \delta^\mu{}_t$, assuming a static metric, so that $\sqrt{g_{tt}} = e^0{}_t$ and denoting by $e^{i\alpha}$ the phase of the central charge $\mathcal{Z}$, the above projector takes the form

$$\epsilon_I + ie^{i\alpha}\epsilon_{IJ}\gamma_0\epsilon^J = 0\,. \tag{6.1.19}$$

This equation is satisfied for spinors of the form

$$\epsilon_I = |X|^{1/2}e^{\frac{i}{2}\alpha}\epsilon_{I\,0}\,, \qquad \epsilon_{I\,0} + i\epsilon_{IJ}\gamma_0\epsilon^{J\,0} = 0\,, \tag{6.1.20}$$

in which the $\epsilon_{I\,0}$ are constant spinors and with $|X|$ some real function.

6.1.3 The 2-forms: the vector case

In this Section we will construct the most general 2-forms associated to the isometries of the special Kähler manifold one can introduce in $N = 2, d = 4$ supergravity coupled to n_V vector multiplets and n_H hypermultiplets. The 2-forms associated to the isometries of the quaternionic Kähler manifold will be discussed in Section 6.1.5. For the subset of commuting isometries a similar program has been performed in [116] where also actions for the dualized scalars, which are part of so-called vector-tensor multiplets, are given.

The Noether current

As explained in Section 2.2 only the group G_V of isometries of the special Kähler manifold which can be embedded in $Sp(2n_V+2,\mathbb{R})$ are symmetries of the full set of equations of motion and Bianchi identities. Despite the fact that these duality transformations only leave invariant the equations of motion together with the Bianchi identities, it is possible to construct a conserved Noether current associated to this invariance [31]. This is because under variations of the scalars $\delta_Z\mathcal{L}+\delta_{Z^*}\mathcal{L}$ the Lagrangian is invariant up to the divergence of an anomalous current, denoted here and in [31] by $\hat{J}^\mu$. Hence, we have

$$\delta_Z\mathcal{L}+\delta_{Z^*}\mathcal{L} = -\partial_\mu(\sqrt{|g|}\hat{J}^\mu)\,. \tag{6.1.21}$$

In the case of p-brane actions coupled to supergravity the Noether current associated to the super-Poincaré invariance of the coupled system contains a similar anomalous contribution [117], which is known to give rise to central charges in the supersymmetry algebra.

Applying the Noether theorem we get

$$\partial_\mu\left(\delta Z^i\frac{\partial\mathcal{L}}{\partial(\partial_\mu Z^i)}+\delta Z^{*i^*}\frac{\partial\mathcal{L}}{\partial(\partial_\mu Z^{*i^*})}\right) = -\partial_\mu(\sqrt{|g|}\hat{J}^\mu)\,, \tag{6.1.22}$$

so that the Noether current

$$J_N^\mu = \delta Z^i\frac{1}{\sqrt{|g|}}\frac{\partial\mathcal{L}}{\partial(\partial_\mu Z^i)}+\delta Z^{*i^*}\frac{1}{\sqrt{|g|}}\frac{\partial\mathcal{L}}{\partial(\partial_\mu Z^{*i^*})}+\hat{J}^\mu\,, \tag{6.1.23}$$

is covariantly conserved, i.e. $\nabla_\mu J_N^\mu=0$. In this Subsection we will compute J_N^μ for the isometries of the Kähler metric $\mathcal{G}_{ij^*}$ which are embedded in $Sp(2n_V+2,\mathbb{R})$.

Infinitesimally, the symmetries under consideration act on the complex scalars as

$$\delta Z^i = \alpha^A k_A{}^i(Z)\,, \tag{6.1.24}$$

where the $k_A{}^i(Z)$ are $\dim G_V$ holomorphic Killing vectors[1] ($A=1,\cdots,\dim G_V$) and where α^A denotes a set of real infinitesimal parameters. The Lie brackets of the Killing vectors give the Lie algebra of G_V with structure constants $f_{AB}{}^C$,

$$[k_A,k_B] = -f_{AB}{}^C k_C\,, \tag{6.1.25}$$

where $k_A = k_A{}^i\partial_i + k_A{}^{*\,i^*}\partial_{i^*}$.

On the vector field strengths the symmetries act as an infinitesimal $Sp(2n_V+2,\mathbb{R})$ transformation

$$\delta\mathcal{F} = T\mathcal{F}\,, \tag{6.1.26}$$

[1] The holomorphicity of the components $k_A{}^i$ follows from the Killing equation.

where $T \in \mathfrak{sp}(2n_V+2,\mathbb{R})$, i.e. $T^T\Omega+\Omega T=0$. The matrix T can be expressed as a linear combination of the generators of the isometry group G_V of $\mathcal{G}_{ij^*}$ that is embedded in $\mathfrak{sp}(2n_V+2,\mathbb{R})$. In other words,

$$T=\alpha^A T_A\,,\qquad [T_A,T_B]=f_{AB}{}^C T_C\,,\qquad T_A\in\mathfrak{sp}(2n_V+2,\mathbb{R})\,. \tag{6.1.27}$$

On the other hand, if

$$T=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\,, \tag{6.1.28}$$

then, the condition $T^T\Omega+\Omega T=0$ implies

$$c^T=c\,,\quad b^T=b\,,\quad \text{and}\quad a^T=-d\,. \tag{6.1.29}$$

To find the current $\hat{J}^\mu$ we start by writing the Lagrangian of (1.2.7) in the following form

$$\mathcal{L}=\tfrac{1}{2}F^\Lambda{}_{\mu\nu}\frac{\partial\mathcal{L}}{\partial F^\Lambda{}_{\mu\nu}}+\mathcal{L}_{\rm inv}\,, \tag{6.1.30}$$

where

$$\mathcal{L}_{\rm inv}=\sqrt{|g|}\left[R+2\mathcal{G}_{ij^*}\partial_\mu Z^i\partial^\mu Z^{*j^*}\right]\,, \tag{6.1.31}$$

is the part of the Lagrangian that is invariant under (6.1.24) and where

$$\frac{\partial\mathcal{L}}{\partial F^\Lambda{}_{\mu\nu}}=-4\sqrt{|g|}\star F_\Lambda{}^{\mu\nu}\,. \tag{6.1.32}$$

Next we compute the variation of $\mathcal{L}$ with respect to the variation of the scalars

$$\delta_Z\mathcal{L}+\delta_{Z^*}\mathcal{L}=\delta\mathcal{L}-\delta_F\mathcal{L}\,, \tag{6.1.33}$$

where $\delta\mathcal{L}$ is the total variation and $\delta_F\mathcal{L}$ denotes the variation of $\mathcal{L}$ with respect to the field strength $F^\Lambda_{\mu\nu}$. The total variation of $\mathcal{L}$ under the transformations (6.1.24) and (6.1.26) is

$$\delta\mathcal{L}=\delta\left(-2\sqrt{|g|}F^\Lambda{}_{\mu\nu}\star F_\Lambda{}^{\mu\nu}\right)=-2\sqrt{|g|}\left[\star F_\Lambda{}^{\mu\nu}b^{\Lambda\Sigma}F_{\Sigma\,\mu\nu}+\star F^{\Lambda\,\mu\nu}c_{\Lambda\Sigma}F^\Sigma{}_{\mu\nu}\right]\,, \tag{6.1.34}$$

where we have used Eqs. (6.1.29). The variation, $\delta_F\mathcal{L}$, is

$$\delta_F\mathcal{L}=\delta F^\Lambda{}_{\mu\nu}\frac{\partial\mathcal{L}}{\partial F^\Lambda{}_{\mu\nu}}=-4\sqrt{|g|}\left[\star F_\Lambda{}^{\mu\nu}a^\Lambda{}_\Sigma F^\Sigma{}_{\mu\nu}+\star F_\Lambda{}^{\mu\nu}b^{\Lambda\Sigma}F_{\Sigma\,\mu\nu}\right]\,. \tag{6.1.35}$$

Using once again Eqs. (6.1.29) it then follows that

$$\delta\mathcal{L} - \delta_F\mathcal{L} = 2\sqrt{|g|}\langle \star\mathcal{F}^{\mu\nu} \mid T\mathcal{F}_{\mu\nu} \rangle \,. \tag{6.1.36}$$

The result Eq. (6.1.36) can be written as the divergence of an anomalous current $\hat{J}$ i.e. one can show, using Eqs. (2.2.4) and (2.2.7), that

$$-\partial_\mu(\sqrt{|g|}\hat{J}^\mu) = \delta\mathcal{L} - \delta_F\mathcal{L} \,, \tag{6.1.37}$$

where $\hat{J}^\mu$ is given by

$$\hat{J}^\mu = -4\langle \star\mathcal{F}^{\mu\nu} \mid T\mathcal{A}_\nu \rangle \,. \tag{6.1.38}$$

At the same time we have for the right hand-side of this equation

$$\delta\mathcal{L} - \delta_F\mathcal{L} = \delta_Z\mathcal{L} + \delta_{Z^*}\mathcal{L} = \partial_\mu\left(\delta Z^i \frac{\partial\mathcal{L}}{\partial(\partial_\mu Z^i)} + \delta Z^{*i^*}\frac{\partial\mathcal{L}}{\partial(\partial_\mu Z^{*i^*})}\right) \,, \tag{6.1.39}$$

so that the Noether current, J^μ_N, is given by

$$J^\mu_N = \delta Z^i \frac{1}{\sqrt{|g|}}\frac{\partial\mathcal{L}}{\partial(\partial_\mu Z^i)} + \delta Z^{*i^*}\frac{1}{\sqrt{|g|}}\frac{\partial\mathcal{L}}{\partial(\partial_\mu Z^{*i^*})} + \hat{J}^\mu \,, \tag{6.1.40}$$

with $\hat{J}^\mu$ given by Eq. (6.1.38), and satisfies

$$\partial_\mu\left(\sqrt{|g|}J^\mu_N\right) = 0 \,. \tag{6.1.41}$$

Under gauge transformations of the 1-form potentials $\mathcal{A}$ the anomalous current $\hat{J}^\mu$ and hence J^μ_N are not invariant: they transform as the divergence of an antisymmetric tensor. We will have to take this point into account in the next subsection when dualizing the Noether current into a 2-form.

It will be convenient to write the scalar part of the Noether current, i.e. the part $J_N - \hat{J}$, in terms of the symplectic sections $\mathcal{V}$ instead of the physical scalars since $\mathcal{V}$ transforms linearly under $Sp(2n_V + 2, \mathbb{R})$. This is achieved using

$$\delta\mathcal{V} = \delta Z^i\partial_i\mathcal{V} + \delta Z^{*i^*}\partial_{i^*}\mathcal{V} \,, \tag{6.1.42}$$

and Eqs. (C.0.1) and (C.0.3). We have

$$\delta Z^i \frac{\partial\mathcal{L}}{\partial(\partial_\mu Z^i)} = -2i\sqrt{|g|}\langle \delta\mathcal{V} \mid \mathfrak{D}^\mu\mathcal{V}^* \rangle \,. \tag{6.1.43}$$

Hence, the Noether current (6.1.40) can be expressed in terms of $\mathcal{V}$ as

$$J^\mu_N = -2i\langle \delta\mathcal{V} \mid \mathfrak{D}^\mu\mathcal{V}^* \rangle + \text{c.c.} + \hat{J}^\mu \,. \tag{6.1.44}$$

We continue to find an explicit expression for $\delta\mathcal{V}$. The symplectic sections transform under global $Sp(2n_V+2,\mathbb{R})$ and under local Kähler transformations. The Kähler potential transforms as

$$\delta_\alpha \mathcal{K} \equiv \pounds_{\alpha^A k_A} \mathcal{K} = \alpha^A \left(k_A{}^i \partial_i \mathcal{K} + k_A{}^{*\, i^*} \partial_{i^*} \mathcal{K} \right) = \lambda(Z) + \lambda^*(Z^*)\,, \tag{6.1.45}$$

where

$$\lambda(Z) = \alpha^A \lambda_A(Z)\,. \tag{6.1.46}$$

It can be shown that the functions $\lambda_A(Z)$ satisfy

$$k_A^i \partial_i \lambda_B - k_B^i \partial_i \lambda_A = -f_{AB}{}^C \lambda_C\,. \tag{6.1.47}$$

When $\lambda \neq 0$ all the objects of the theory with non-zero Kähler weight (in particular all the spinors and the symplectic section $\mathcal{V}$) will feel the effect of the symplectic transformation through a Kähler transformation. Infinitesimally one has

$$\delta_{\text{Kähler}} \mathcal{V} = -\tfrac{1}{2}(\lambda - \lambda^*)\mathcal{V}\,, \tag{6.1.48}$$

as follows from Eq. (C.0.12). Next we introduce the momentum map, denoted by $\mathcal{P}_A^0$ and defined by

$$\mathcal{P}_A^0 \equiv i k_A{}^i \partial_i \mathcal{K} - i\lambda_A\,. \tag{6.1.49}$$

One then readily shows that $\delta\mathcal{V}$, given via equations (6.1.42) and (6.1.24), can be written as

$$\delta\mathcal{V} = \alpha^A \left(k_A{}^i \mathcal{D}_i \mathcal{V} + i\mathcal{P}_A^0 \mathcal{V} - \tfrac{1}{2}(\lambda_A - \lambda_A^*)\mathcal{V} \right)\,. \tag{6.1.50}$$

Since $\mathcal{V}$ only transforms under symplectic and Kähler transformations we conclude[2] that we must have

$$\delta\mathcal{V} = T\mathcal{V} - \tfrac{1}{2}(\lambda - \lambda^*)\mathcal{V}\,, \qquad \text{where} \qquad T\mathcal{V} = \alpha^A \left(k_A{}^i \mathcal{D}_i \mathcal{V} + i\mathcal{P}_A^0 \mathcal{V} \right)\,, \tag{6.1.51}$$

where T is a generator of $\mathfrak{sp}(2n_V+2)$. Taking the product of the r.h.s. of the second equation with $\mathcal{V}$ we get the additional condition that the generators of G_V must satisfy:

$$\langle\, \mathcal{V} \mid T_A \mathcal{V} \,\rangle = 0\,. \tag{6.1.52}$$

[2] Actually, this is a consequence of requiring that the reparametrizations generated by the Killing vectors preserve not just the metric but the whole special Kähler geometry. This is what we are implicitly doing here and it is a condition necessary to have symmetries of the complete supergravity theory and not just of the bosonic equations of motion. We thank Patrick Meessen for a useful discussion on this point.

The set of generators T_A which satisfy the constraint (6.1.52) and which form a subgroup of $\mathfrak{sp}(2n_V+2,\mathbb{R})$ is sometimes referred to as the *duality symmetry Lie algebra* [118].
Since, on the other hand

$$\delta\mathcal{V} = \pounds_{\alpha^A k_A}\mathcal{V} = \alpha^A\left(k_A{}^i\partial_i\mathcal{V} + k_A{}^{*\,i^*}\partial_{i^*}\mathcal{V}\right)\,, \tag{6.1.53}$$

we can write

$$\pounds_{\alpha^A k_A}\mathcal{V} - T\mathcal{V} + \tfrac{1}{2}(\lambda-\lambda^*)\mathcal{V} = 0\,, \tag{6.1.54}$$

as the necessary and sufficient condition for the transformation to be a symmetry of the supergravity theory[3].

One verifies that the above way of writing the action of T on $\mathcal{V}$, see Eq. (6.1.51), satisfies Eq. (6.1.27). By decomposing $T\mathcal{V}$ into the complete basis $\{\mathcal{V},\mathfrak{D}_i\mathcal{V},\mathcal{V}^*,\mathfrak{D}_{i^*}\mathcal{V}^*\}$ for the space of symplectic sections (see Appendix C below Eq. (C.0.3)) we find

$$\mathcal{P}^0_A = -\langle\mathcal{V}\mid T_A\mathcal{V}^*\rangle\,,\qquad\text{and}\qquad k_A{}^i = -i\mathcal{G}^{ij^*}\partial_{j^*}\mathcal{P}^0_A\,. \tag{6.1.55}$$

Substituting (6.1.51) into expression (6.1.44) we obtain a manifestly symplectic-invariant expression for the Noether current

$$J_{N\mu} = 2i\langle\,\mathfrak{D}_\mu\mathcal{V}^*\mid T\mathcal{V}\,\rangle + \text{c.c.} - 4\langle\,\star\mathcal{F}_{\mu\nu}\mid T\mathcal{A}^\nu\,\rangle\,. \tag{6.1.56}$$

Dualizing the Noether current

In form notation the conservation of the Noether current 1-form J_N is just $d\star J_N = 0$. We can define a 3-form[4] $G = \star J_N$, which satisfies $dG = 0$, so that locally $G = dB$. Note that G is not gauge invariant because J_N is not, either, due to the term $\hat{J}$ ($\delta_{\text{gauge}}G = \delta_{\text{gauge}}\hat{J}$). We can write this term in the form

$$\star\hat{J} = -4\langle\,\mathcal{F}\mid T\mathcal{A}\,\rangle\,, \tag{6.1.57}$$

where the exterior product between the forms in the symplectic inner product is always assumed and as a result the 2-form B gauge transformation is given by

$$\delta_{\text{gauge}}B = d\Lambda_1 - 4\langle\,\mathcal{F}\mid T\Lambda\,\rangle\,, \tag{6.1.58}$$

where the symplectic vector Λ is defined through Eq. (6.1.12).

We can define the following gauge-invariant 2-form field strength

[3] This condition can be read in two different ways: the Lie derivative of the section $\mathcal{V}$ has to vanish up to symplectic and Kähler transformations or the symplectic- and Kähler-covariant Lie derivative of $\mathcal{V}$ has to vanish identically.

[4] Of course, we have $\dim G_V$ Noether currents and as many dual 3-forms G_A but it is convenient to work with $G = \alpha^A G_A$.

$$H = dB + 4\langle \mathcal{F} \mid T\mathcal{A} \rangle \,. \tag{6.1.59}$$

It is then clear that H is dual to the scalar part of the Noether current J_N,

$$H = \star(J_N - \hat{J}) \,. \tag{6.1.60}$$

The scalar part of the Noether current is proportional to the Killing vectors. At any given point there are only $2n_V$ (real) independent vectors. Thus, if we allow for Z^i-dependent coefficients, in general we will find linear combinations of scalar parts of the Noether currents. As a result, there will be as many constraints on the 2-form field strengths H_A and, at most there will be $2n_V$ independent real 2-forms.

The 2-form supersymmetry transformation

In the previous Subsection we have constructed a set of 2-forms associated to the isometries of the special Kähler manifold of ungauged $N = 2, d = 4$ supergravity and we have found their gauge transformations. Our goal in this Section is to find their supersymmetry transformations. The main requirement that the proposed supersymmetry transformation of the 2-form B must satisfy is that the commutator agrees with the universal local supersymmetry algebra of the theory given by

$$[\delta_\eta, \delta_\epsilon] = \delta_{\text{g.c.t.}}(\xi) + \delta_{\text{gauge}}(\Lambda) \,, \tag{6.1.61}$$

and which may be extended to include 2-forms to

$$[\delta_\eta, \delta_\epsilon] = \delta_{\text{g.c.t.}}(\xi) + \delta_{\text{gauge}}(\Lambda) + \delta_{\text{gauge}}(\Lambda_1) \,. \tag{6.1.62}$$

The expressions for ξ and Λ are given by Eqs. (6.1.12) and (6.1.13), respectively. The 2-form gauge transformation parameter Λ_1 is to be found in terms of η and ϵ.

Since B is defined by $dB = \star J_N$, the commutator of two supersymmetry variations on B must close into the algebra (6.1.62). We have

$$\delta_{\text{g.c.t.}}(\xi) B_{\mu\nu} = \pounds_\xi B_{\mu\nu} = \xi^\rho \partial_\rho B_{\mu\nu} + (\partial_\mu \xi^\rho) B_{\rho\nu} + (\partial_\nu \xi^\rho) B_{\mu\rho} = \xi^\rho (dB)_{\rho\mu\nu} - 2\partial_{[\mu}\left(\xi^\rho B_{\nu]\rho}\right) \,, \tag{6.1.63}$$

with $\pounds_\xi B_{\mu\nu}$ the Lie derivative of $B_{\mu\nu}$ with respect to ξ^ρ. Further, $\delta_{\text{gauge}}(\Lambda_1) B_{\mu\nu}$ is given in Eq. (6.1.58). Hence, the supersymmetry transformations of $B_{\mu\nu}$ must lead to the commutator

$$[\delta_\eta, \delta_\epsilon] B_{\mu\nu} = \xi^\rho \frac{1}{\sqrt{|g|}} \epsilon_{\rho\mu\nu\sigma} J_N{}^\sigma - 4\langle \mathcal{F}_{\mu\nu} \mid T\Lambda \rangle + 2\partial_{[\mu}\left(\Lambda_{\nu]} - \xi^\rho B_{\nu]\rho}\right) \,, \tag{6.1.64}$$

where we have substituted the duality relation, Eq. (6.1.60), for $(dB)_{\mu\rho\sigma}$ in (6.1.63).

We make the following Ansatz for the supersymmetry transformation of $B_{\mu\nu}$ (up to second order in fermions),

$$
\begin{aligned}
\delta_\epsilon B_{\mu\nu} &= a\langle\, \mathfrak{D}_i\mathcal{V} \mid T\mathcal{V}^* \,\rangle\, \bar\epsilon_I \gamma_{\mu\nu}\lambda^{iI} + \text{c.c.} \\
&\quad + b\langle\, \mathcal{V} \mid T\mathcal{V}^* \,\rangle\, \bar\epsilon^I \gamma_{[\mu}\psi_{I\nu]} + \text{c.c.} \\
&\quad + c\langle\, \mathcal{A}_{[\mu} \mid T\delta_\epsilon \mathcal{A}_{\nu]} \,\rangle\, .
\end{aligned}
\tag{6.1.65}
$$

This Ansatz is based on the requirement that all terms must have Kähler weight zero and that the 2-forms are real valued. The matrix T satisfies Eq. (6.1.52).

We evaluate the commutator as follows. First we perform standard gamma matrix manipulations, change the order of the spinors, evaluate the complex conjugated terms and use relations from special geometry. Exhausting all such operations using formulae from Appendices A and C leads to the following expression for the commutator

$$
\begin{aligned}
[\delta_\eta,\delta_\epsilon]B_{\mu\nu} &= 4ia\xi^\sigma \tfrac{1}{\sqrt{|g|}}\epsilon_{\sigma\mu\nu\rho}\,[\langle\, \mathfrak{D}^\rho\mathcal{V} \mid T\mathcal{V}^* \,\rangle - \langle\, \mathfrak{D}^\rho\mathcal{V}^* \mid T\mathcal{V} \,\rangle] \\
&\quad \Big[+4ia\langle\, \mathfrak{D}_i\mathcal{V} \mid T\mathcal{V}^* \,\rangle \mathcal{G}^{ij^*}\langle\, \mathfrak{D}_{j^*}\mathcal{V}^* \mid \mathcal{F}_{\mu\nu} \,\rangle \epsilon^{IJ}\bar\eta_I\epsilon_J \\
&\quad -2b\langle\, \mathcal{V} \mid T\mathcal{V}^* \,\rangle\langle\, \mathcal{V}^* \mid \mathcal{F}_{\mu\nu} \,\rangle \epsilon^{IJ}\bar\eta_I\epsilon_J + \text{c.c.} \Big] \\
&\quad -8a\xi_{[\nu}\partial_{\mu]}\langle\, \mathcal{V} \mid T\mathcal{V}^* \,\rangle + 4ib\langle\, \mathcal{V} \mid T\mathcal{V}^* \,\rangle \partial_{[\mu}\xi_{\nu]} + c\langle \mathcal{A}_{[\mu} \mid [\delta_\eta,\delta_\epsilon]\mathcal{A}_{\nu]} \,\rangle\, ,
\end{aligned}
\tag{6.1.66}
$$

where it has been assumed that a and ib are real parameters. The parameter ξ^ρ is given by (6.1.12). The notation $[\cdots + \text{c.c.}]$ means that one should take the complex conjugate of whatever is written on the left within the brackets. The parameter a has been chosen to be real in order to obtain the scalar part of the Noether current in the first line of (6.1.66). The parameter ib has been chosen to be real so that the Kähler connection 1-form $\mathcal{Q}_\mu$ appearing in $\delta_\epsilon \Psi_{I\,\mu}$ cancels when adding the complex conjugated terms. We then take $2b = 4ia$ so that the first and the second term of the third line of Eq. (6.1.66) combine into a 2-form gauge transformation parameter. Expression (6.1.66) is further manipulated using the completeness relation Eq. (5.2.23). This is the step where we impose the condition that T must satisfy Eq. (6.1.52). Using next the result for the 1-form commutator, Eq. (6.1.10), to write out the term proportional to c in (6.1.66), we obtain

$$
\begin{aligned}
[\delta_\eta, \delta_\epsilon] B_{\mu\nu} = {} & 4ia\xi^\sigma \frac{1}{\sqrt{|g|}} \epsilon_{\sigma\mu\nu\rho} \left[\langle \mathfrak{D}^\rho \mathcal{V} \mid T \mathcal{V}^* \rangle - \langle \mathfrak{D}^\rho \mathcal{V}^* \mid T \mathcal{V} \rangle \right] - 8a\partial_{[\mu} \left(\langle \mathcal{V} \mid T \mathcal{V}^* \rangle \xi_{\nu]} \right) \\
& + 16a \langle \mathcal{F}_{\mu\nu} \mid T \left(\Lambda + \xi^\rho A_\rho \right) \rangle - \tfrac{c}{8} \xi^\sigma \frac{1}{\sqrt{|g|}} \epsilon_{\sigma\mu\nu\rho} \hat{J}^\rho - c\partial_{[\mu} \langle \mathcal{A}_{\nu]} \mid T \left(\Lambda + \xi^\rho \mathcal{A}_\rho \right) \rangle \\
& + \tfrac{c}{2} \langle \mathcal{F}_{\mu\nu} \mid T \Lambda \rangle + c \langle \mathcal{F}_{\mu\nu} \mid T \xi^\rho \mathcal{A}_\rho \rangle \,,
\end{aligned}
\tag{6.1.67}
$$

where Λ is the 1-form gauge transformation parameter given in (6.1.13). This can be seen to be equal to the desired result, Eq. (6.1.64), for $c = -16a$ and $a = -1/2$. We thus obtain the following supersymmetry variation rule for $B_{\mu\nu}$

$$
\begin{aligned}
\delta_\epsilon B_{\mu\nu} \quad = \quad & -\tfrac{1}{2} \langle \mathfrak{D}_i \mathcal{V} \mid T \mathcal{V}^* \rangle \, \bar{\epsilon}_I \gamma_{\mu\nu} \lambda^{iI} + \text{c.c.} \\
& -i \langle \mathcal{V} \mid T \mathcal{V}^* \rangle \, \bar{\epsilon}^I \gamma_{[\mu} \psi_{I\nu]} + \text{c.c.} \\
& +8 \langle \mathcal{A}_{[\mu} \mid T \delta_\epsilon \mathcal{A}_{\nu]} \rangle \,.
\end{aligned}
\tag{6.1.68}
$$

The 1-form gauge transformation parameter Λ_μ is given by

$$
\Lambda_\mu = 2 \langle \mathcal{V} \mid T \mathcal{V}^* \rangle \xi_\mu - 4 \langle \mathcal{A}_\mu \mid T (\Lambda + \xi^\rho \mathcal{A}_\rho) \rangle + \xi^\rho B_{\mu\rho} \,. \tag{6.1.69}
$$

6.1.4 World-sheet actions: the vector case

In this Section we will construct the leading terms of the bosonic part of a κ-invariant world-sheet action for the stringy cosmic strings that couple to the 2-form potentials B that were constructed in Section 6.1.3. Just as in the 0-brane case of Section 6.1.2, we will construct actions which are manifestly symplectic invariant.

According to the results of the previous Sections we expect to have strings which carry charges with respect to each of the $\dim G_V$ 2-forms $B_{A\,\mu\nu}$ that one can define. We define a $\dim G_V$-dimensional charge vector q^A. Symplectic invariance suggests a world-sheet action with leading terms

$$
S = q^A \int d^2\sigma \, \langle \mathcal{V} \mid T_A \mathcal{V}^* \rangle \sqrt{|g_{(2)}|} + c q^A \int B_A \,, \tag{6.1.70}
$$

where $g_{(2)}$ and B_A are the pullbacks of the space-time metric and 2-forms onto the world-sheet, respectively and where c is some normalization constant that will be fixed later. The tension of the string is given by the momentum map $\mathcal{P}^0_A$ as given in Eq. (6.1.55).

The Wess–Zumino term of this action is, however, not gauge invariant under the gauge transformation (6.1.58) and it seems impossible to make it gauge invariant by adding additional terms to the Wess–Zumino term without adding more degrees of freedom to the 2-dimensional world-sheet theory.

Actually, the same problem arises in the construction of a κ-symmetric world-sheet action for the heterotic superstring in backgrounds with non-trivial Yang–Mills fields since the NSNS 2-form transforms under Yang–Mills gauge transformations similar to Eq. (6.1.58). In the 10-dimensional case of strings propagating in backgrounds with non-trivial Yang-Mills fields the solution to this puzzle lies in the addition of heterotic fermions to the world-sheet action whose gauge transformations cancel against the Yang–Mills part of the NSNS 2-form gauge transformation [119]. We suggest that a similar effect could be at work here.

If this is the case, then, in checking the invariance under supersymmetry transformations of the above world-sheet action we must ignore the term $\langle \mathcal{A}_{[\mu} \mid T\delta_\epsilon \mathcal{A}_{\nu]}\rangle$ in the 2-form supersymmetry transformation rule. This term should be cancelled by anomalous terms in the supersymmetry transformations of the world-sheet spinors. With this proviso we find that the above action preserves half of the supersymmetries with the projector

$$\tfrac{1}{2}(1+4c\gamma_{01})\epsilon_I = 0 \qquad \text{with} \qquad c = \tfrac{1}{4}\,. \tag{6.1.71}$$

We will see in the next Section that the stringy cosmic string solutions for which the above action provides the sources require in order to preserve half of the supersymmetries exactly the same condition to be satisfied by the Killing spinor.

6.1.5 Supersymmetric vector strings

Stringy cosmic string solutions of $N = 2, d = 4$ supergravity coupled to vector multiplets were found in [26][5]. They preserve half of the original supersymmetries and belong to the 'null class' of supersymmetric solutions characterized by the fact that the Killing vector that one can construct from their Killing spinors is null. Generically solutions in this class have Brinkmann-type metrics

$$ds^2 = 2du(dv + Hdu + \hat{\omega}) - 2e^{-\mathcal{K}(Z,Z^*)}dzdz^*\,, \tag{6.1.72}$$

where $\mathcal{K}$ is the Kähler potential of the vector scalar manifold and where $\hat{\omega}$ is determined from the equation

$$(d\hat{\omega})_{\underline{zz^*}} = 2ie^{-\mathcal{K}}\mathcal{Q}_{\underline{u}}\,, \tag{6.1.73}$$

with $\mathcal{Q}_\mu$ the pullback of the Kähler 1-form connection given in Eq. (B.0.3). The complex scalars Z^i are functions of u and z.

[5] Solutions related to these by dimensional reduction have been obtained in a 3-dimensional context in Ref. [120].

It is not easy to interpret physically these solutions for a generic dependence on the null coordinate u. When there is no dependence on u we can take $\hat{\omega} = 0$ and the metric is that of a superposition of cosmic strings (described by $\mathcal{K}$) lying in the direction $u-v$ and gravitational and electromagnetic waves (described by H) propagating along the same direction.

Setting $H = 0$ (which generically requires that we switch off all the electromagnetic fields) we obtain solutions that only describe cosmic strings. In order to study the behavior of these solutions under the symmetries of the theory, it is convenient to express them in an arbitrary system of holomorphic coordinates, which amounts to the introduction of an arbitrary holomorphic function $f(z)$ whose absolute value appears in the metric and whose phase appears in the Killing spinors of the solution

$$\left\{ \begin{array}{rcl} ds^2 & = & 2dudv - 2e^{-\mathcal{K}(Z,Z^*)}|f|^2 dzdz^*\,, \\ \\ Z^i & = & Z^i(z)\,, \qquad\qquad f = f(z)\,, \\ \\ \epsilon_I & = & (f/f^*)^{1/4}\epsilon_{I\,0}\,, \qquad\qquad \gamma_{z^*}\epsilon_{I\,0} = 0\,. \end{array} \right. \tag{6.1.74}$$

If we take $z = x_2 + ix_3$ then the condition $\gamma_{z^*}\epsilon_{I\,0} = 0$ is equivalent to Eq. (6.1.71).

The holomorphic functions $Z^i(z), f(z)$ are assumed to be defined on the Riemann sphere $\hat{\mathbb{C}}$, but, generically, they will not be single-valued on it due to the presence of branch cuts. These branch cuts are to be associated with the presence of cosmic strings just as was done in the particular case of the $SL(2,\mathbb{R})/U(1)$ special Kähler manifold studied in Refs. [109] and [110].

As a general rule bosonic fields must be single-valued unless they are subject to a gauge symmetry which forces us to identify as physically equivalent those configurations which are related by admissible gauge transformations. In the theories that we are considering the complex scalars $Z^i(z)$ do not transform under any gauge symmetry. Only the global group of isometries G_V of $\mathcal{G}_{ij^*}$ acts on them and only a discrete subgroup $G_V(\mathbb{Z}) \subseteq Sp(2n_V+2,\mathbb{Z})$ will be a global symmetry at the quantum level.

In the resulting theories two values of $Z^i(z)$ may be considered equivalent if they are related by a $G_V(\mathbb{Z})$ transformation. This enables one to construct solutions in which the scalars $Z^i(z)$ are multi-valued functions with branch cuts related to the elements of $G_V(\mathbb{Z})$. The source for a branch cut is provided by the Wess–Zumino term of a cosmic string. This is explained in detail for the 10-dimensional case of the 7-branes in [109].

Next we discuss the emergence of axions related to the presence of Killing vectors. For every Killing vector $\alpha^A k_A{}^i$ one can always find an adapted coordinate system $\{Z^i\}$ such that the metric $\mathcal{G}_{ij^*}$ does not depend on the real part of the coordinate Z^1, say. In this coordinate system $\alpha^A k_A{}^i \partial_i = \partial_1$ and the isometries generated by it act as constant shifts of Z^1 by a real constant:

$$\delta Z^1 = c \in \mathbb{R}\,. \tag{6.1.75}$$

This transformation only acts on the real part of Z^1, χ^1, which is, then, what it is sometimes meant by an axion: a real scalar field with no non-derivative couplings to the other scalars and with a shift symmetry[6]

It is clear that we can, in principle, define as many different axion fields as there are independent Killing vectors [7], i.e. $\dim G_V$, i.e. as many as 2-forms, which can be understood as their duals. Their (both those of the axions and 2-forms) equations of motion are not necessarily independent, though, and they will satisfy a number of constraints, as discussed before, and, at most, there can be $2n_V$ independent axions.

We now discuss the properties of the cosmic string solutions in a local neighborhood of the location z_0 in the transverse space of a cosmic string. Infinitesimally the transformation of the scalars Z^i when going around z_0 is given by Eq. (6.1.24). In some coordinate basis, the transformation will only be an axion shift.

Besides the scalars Z^i also the Killing spinors ϵ_I will undergo transformations when going around the cosmic string at z_0. This is because when the scalars transform as in Eq. (6.1.24) the Kähler potential transforms as

$$\mathcal{K}(Z', Z'^*) = \mathcal{K}(Z, Z^*) + \lambda_\alpha(Z) + \lambda^*_\alpha(Z^*)\,. \tag{6.1.76}$$

From the fact that the Killing spinor ϵ_I has Kähler weight 1/2 it then follows that

$$\epsilon_I(z) \to e^{\frac{1}{4}[\lambda_\alpha - \lambda^*_\alpha] + \frac{i}{2}\varphi_\alpha}\epsilon_I(z)\,, \tag{6.1.77}$$

when going around z_0. The phases φ_α relate to the fact that in general the spinors transform under the double cover of G_V[8]. The Killing spinor ϵ_I is defined in terms of the holomorphic function $f(z)$ via Eqs. (6.1.74). The monodromy of f when going around z_0 must be

$$f(z) \to e^{\lambda_\alpha[Z(z)] + i\varphi_\alpha} f(z)\,. \tag{6.1.78}$$

[6] A more precise definition would require χ^1 to be a pseudoscalar too. Actually, the real and imaginary parts of the complex scalars in $N = 2, d = 4$ vector supermultiplets have different parities, but, in a general model with arbitrary coordinates one should look at the couplings to the vector fields to determine the parity of χ^1.

On the other hand, the action of $N = 2, d = 4$ supergravity indicates that the axions must appear in $\Re\mathrm{e}\mathcal{N}_{\Lambda\Sigma}$, which couples to the parity-odd term $F^\Lambda \wedge F^\Sigma$. Under symplectic transformations $\begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}$ $\Re\mathrm{e}\mathcal{N}$ is shifted to $\Re\mathrm{e}\mathcal{N} + B$, as one expects from axions. This suggests another possible characterization of axions: χ^1 is an axion if its shifts are embedded in the Abelian subgroup of symplectic transformations of the form $\begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}$.

[7] However, they cannot be used simultaneously, since we can only use simultaneously adapted coordinates for commuting isometries.

[8] One can even include yet another phase factor in the transformation rule for the Killing spinors which incorporates the fact that ϵ_I may come back to itself up to a sign, i.e. one can include nontrivial spin structures.

The cosmic string solutions contain information about the moduli space of the theory, i.e. the space of inequivalent values for Z^i. The classical moduli space is defined by the requirement

$$\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma} < 0\,, \tag{6.1.79}$$

in order that the kinetic terms of the 1-forms have the right sign in the action (1.2.7). The zeros of the polynomial $\delta Z^i = \alpha^A k_A{}^i$ which belong to the space (6.1.79) (or possibly on the boundary thereof) are fixed points of the monodromy and therefore comprise the loci of the cosmic strings in the quantum moduli space:

$$\{Z^i \,|\, \mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma} < 0\}/\mathrm{G_V}(\mathbb{Z})\,. \tag{6.1.80}$$

Drawing from the analogy with the $SL(2,\mathbb{R})/U(1)$ case studied in [110] one can expect all physical properties of globally well-defined stringy cosmic string solutions to be mapped into geometrical properties of the space (6.1.80). Such properties are the total mass, possible deficit angles at the sites of the cosmic strings, orders of monodromy transformations (the number of times the same monodromy has to be applied in order to equal the identity), etc. Here we will not attempt to work out the global properties of these solutions, since they are strongly model-dependent.

In the $SL(2,\mathbb{R})/U(1)$ case one could have derived all geometrical properties of the quantum moduli space $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})/U(1)$ by studying the globally well-defined supersymmetric stringy cosmic string solutions. It is therefore natural to ask the question whether this is generally true, i.e. whether (some class of) quantum moduli spaces of Calabi–Yau reduced supergravities can be obtained by studying the properties of the stringy cosmic string solutions.

We leave this for a future investigation.

The 2-forms: the hyper case

If we consider $N = 2$, $d = 4$ supergravity with general matter couplings, we can have apart from the complex scalars in the vector multiplets $4n_H$ real scalars when coupling gravity to n_H hypermultiplets. In the following we repeat the program of introducing 2-forms in order to dualize the hyperscalars which parameterize the Noether currents of some isometry group of the quaternionic Kähler manifold. We first construct the Noether currents, dualize them and subsequently construct the supersymmetry transformation rule for the dual 2-forms. For the subset of commuting isometries a similar program has been performed in [121] where also actions for the dualized scalars are given.

6.1.6 The Noether current

The transformations we are dealing with are just the isometries of the quaternionic Kähler manifold that we write in the form

$$\delta q^u = \alpha^A k_A{}^u(q)\,, \tag{6.1.81}$$

where $k_A{}^u$ are the components of the Killing vectors $k_A = k_A{}^u\partial_u$ that generate the isometry group G_H of H_{uv}. The parameters α^A are real parameters.

Associated to each of the isometries we can define a momentum map[9] $\mathsf{P}_{AI}{}^J$ defined by the equation

$$\mathfrak{D}_u \mathsf{P}_{AI}{}^J = -\mathsf{J}_I{}^J{}_{uv} k_A{}^v\,, \tag{6.1.82}$$

where $\mathsf{J}_I{}^J{}_{uv}$ is the triplet complex structures of the quaternionic-Käher manifold.

Following [122] we write the triplet of complex structures $\mathsf{J}_I{}^J{}_{uv}$ in terms of the Quadbeins as follows

$$\mathsf{J}_I{}^J{}_{uv} = \tfrac{i}{2}(\sigma_x)_I{}^J\,\mathsf{J}^x{}_{uv} \qquad \text{with} \qquad \mathsf{J}^{x\,u}{}_v = -i\mathsf{U}^{\alpha I}{}_v(\sigma_x)_I{}^J\mathsf{U}_{\alpha J}{}^u\,, \tag{6.1.83}$$

where the σ_x, $x = 1, 2, 3$, are the three Pauli matrices. We will often write $\mathsf{P}_I{}^J \equiv \alpha^A \mathsf{P}_{AI}{}^J$.

The Noether current associated to the these isometries, which do not act on the vector fields, is just

$$J^\mu_N = \delta q^u \frac{1}{\sqrt{|g|}}\frac{\partial\mathcal{L}}{\partial(\partial_\mu q^u)} = 4\mathsf{H}_{uv}\partial^\mu q^v \delta q^u\,, \tag{6.1.84}$$

and satisfies $\nabla_\mu J^\mu_N = 0$.

Dualizing the Noether current

Since the isometries of the quaternionic Kähler manifold do not act on the vectors of the theory they are symmetries of the action and there will be no anomalous contribution to the Noether current such as $\hat{J}$ which we encountered when discussing the isometries of the special Kähler manifold. We can thus immediately define the gauge-invariant 3-form field strength H via

$$H = dB = \star J_N\,, \tag{6.1.85}$$

where $H = \alpha^A H_A$ and $B = \alpha^A B_A$.

[9] Momentum maps play a crucial role in the gauging of the isometries. It is therefore interesting to note that the mathematics which governs the 2-forms is similar to that used in gauged matter coupled $N = 2$, $d = 4$ supergravity.

The 2-form supersymmetry transformation

We know that, since B is defined by $dB = \star J_N$, the commutator of two supersymmetry variations on B must close into the algebra (6.1.62), i.e. it must lead to the commutator

$$[\delta_\eta, \delta_\epsilon] B_{\mu\nu} = \xi^\rho \frac{1}{\sqrt{|g|}} \epsilon_{\rho\mu\nu\sigma} J_N{}^\sigma + 2\partial_{[\mu} \left(\Lambda_{\nu]} - \xi^\rho B_{\nu]\rho} \right) . \qquad (6.1.86)$$

In order to achieve this, we make the following Ansatz for the supersymmetry variation of the 2-form (up to second order in fermions)

$$\begin{aligned} \delta_\epsilon B_{\mu\nu} &= a \mathsf{P}_I{}^J \bar{\epsilon}^I \gamma_{[\mu} \psi_{J|\nu]} + \text{c.c.} \\ &\quad + b \mathsf{U}_{\alpha J}{}^u \mathfrak{D}_u \mathsf{P}_I{}^J \bar{\epsilon}^I \gamma_{\mu\nu} \zeta^\alpha + \text{c.c.} \, , \end{aligned} \qquad (6.1.87)$$

where a and b are arbitrary complex constants.

Evaluating the commutator and assuming that a and ib are real parameters we obtain

$$\begin{aligned} [\delta_\eta, \delta_\epsilon] B_{\mu\nu} &= -\tfrac{3}{2} ib (\star dq^w)_{\mu\nu\rho} \xi^\rho \mathsf{H}_{vw} \delta q^v \\ &\quad + \tfrac{3}{2} ib \mathsf{J}_I{}^K{}_{vw} \delta q^v \partial_{[\nu} q^w X_{\mu]K}{}^I \\ &\quad + 2\partial_{[\mu} \left(\Lambda_{\nu]} - \xi^\rho B_{\nu]\rho} \right) - a \mathsf{J}_I{}^K{}_{vw} \delta q^v \partial_{[\nu} q^w X_{\mu]K}{}^I \, , \end{aligned} \qquad (6.1.88)$$

where we have defined the matrix of vector fields

$$X_{\mu I}{}^J \equiv -\bar{\eta}^J \gamma_\mu \epsilon_I - \bar{\eta}_I \gamma_\mu \epsilon^J \, , \qquad (6.1.89)$$

and where the gauge parameter Λ_μ is given by

$$\Lambda_\mu = -\tfrac{a}{2} X_J{}^I{}_\mu \mathsf{P}_I{}^J + \xi^\rho B_{\mu\rho} . \qquad (6.1.90)$$

Next we choose $a = \frac{3}{2} ib$ and we are left with

$$[\delta_\eta, \delta_\epsilon] B_{\mu\nu} = -\tfrac{3}{2} ib (\star dq^w)_{\mu\nu\rho} \xi^\rho \mathsf{H}_{vw} \delta q^v + 2\partial_{[\mu} \left(\Lambda_{\nu]} - \xi^\rho B_{\nu]\rho} \right) .$$

If we compare this expression with Eq. (6.1.64) using Eq. (6.1.84) we read off that $ib = -\frac{8}{3}$, so that $a = -4$.

The supersymmetry transformation of the 2-forms dual to the hyperscalars parameterizing the Noether current (6.1.84) is thus

$$\begin{aligned}\delta_\epsilon B_{\mu\nu} &= -4\mathsf{P}_I{}^J\bar{\epsilon}^I\gamma_{[\mu}\psi_{J|\nu]} + \text{c.c.}\\ &\quad +\tfrac{8i}{3}\mathsf{U}_{\alpha J}{}^u\mathfrak{D}_u\mathsf{P}_I{}^J\bar{\epsilon}^I\gamma_{\mu\nu}\zeta^\alpha + \text{c.c.}\,,\end{aligned} \tag{6.1.91}$$

and the 2-form gauge parameter Λ_μ is given by

$$\Lambda_\mu = 2X_J{}^I{}_\mu\mathsf{P}_I{}^J + \xi^\rho B_{\mu\rho}. \tag{6.1.92}$$

6.1.7 World-sheet actions: the hyper case

Stringy cosmic strings in the hyper case are strings electrically charged under the 2-forms B constructed in Section 6.1.5. In this Section we will construct the bosonic part of the string effective action, which preserves half of the supersymmetries of the theory. In analogy with the Ansatz that we made for the strings in the vector case we again express the tension of the string in terms of the momentum maps. We make the following Ansatz

$$S = \int d^2\sigma\mathcal{T}_1\sqrt{|g_{(2)}|} + c\,q^A\int B_A, \tag{6.1.93}$$

where c is some real number which will be fixed later. The tension is given by

$$\mathcal{T}_1 = \sqrt{(\mathsf{P}^x)^2} \quad \text{where} \quad \mathsf{P}^x = \alpha^A\mathsf{P}^x{}_A \quad \text{with} \quad \mathsf{P}_I{}^J = \tfrac{i}{2}\mathsf{P}^x(\sigma_x)_I{}^J \tag{6.1.94}$$

and in taking the square we sum over $x = 1, 2, 3$.

Performing a supersymmetry variation of the action (6.1.93) using the transformation rules (2.2.32), (2.2.35) and (6.1.91) we find that the string action preserves half of the supersymmetries with a projector given by

$$\Pi_I{}^J = \tfrac{1}{2}(\delta_I{}^J - \tfrac{8ci}{\sqrt{(\mathsf{P}^x)^2}}\mathsf{P}_I{}^J\gamma_{01}), \qquad \Pi_I{}^J\epsilon^I = 0, \qquad \text{where } c = -\tfrac{1}{4}. \tag{6.1.95}$$

An important distinction with the analogous string action constructed in Section 6.1.4 is that in the present case the Wess–Zumino term is gauge invariant up to a total derivative whereas in the case of strings coupled to 2-forms dual to vector scalars the Wess–Zumino term is not by itself gauge invariant, cf. the discussion below Eq. (6.1.70). In fact one may consider the action (6.1.93) as the first example of a 1/2 BPS $(d-3)$-brane action which is well-defined (at the bosonic level) for all possible $(d-2)$-form potentials. In the $d = 10$-dimensional situation only the brane actions related to the D7-branes are well understood. For the other 8-forms which couple to the Q7-branes of [109] there are still open problems regarding a proper understanding of the world-volume dynamics. The fact that in the particular case of the hyperstrings we can construct well-defined actions supports the idea that in general one can treat all isometries of any scalar sigma model in any supergravity on an equal footing (provided they pertain to be discrete isometries of the quantum moduli space). This suggests that in order to find the full spectrum of 1/2 BPS states one best considers the same supergravity theory in various coordinate systems in which these isometries take on a simple form.

6.1.8 Supersymmetric hyperstrings

In Ref. [27] it was shown that the c-map transforms supersymmetric stringy cosmic string solutions of the vector scalar manifold into supersymmetric stringy cosmic string solutions of the hyperscalar manifold. The latter belong to the timelike class of supersymmetric solutions characterized by the fact that the Killing vector that one can construct from the Killing spinors of the solution is timelike. The metric for this class of solutions (for vanishing vector multiplets) takes the following form

$$ds^2 = dt^2 - \gamma_{\underline{mn}} dx^m dx^n \,. \tag{6.1.96}$$

The 3-dimensional spatial metric $\gamma_{\underline{mn}}$ (or its Dreibeins $V^x{}_{\underline{m}}$) is related to the hyperscalars $q^u(x)$ by two conditions. The first condition is

$$V_x{}^{\underline{m}}\, \partial_{\underline{m}} q^u \; \mathsf{U}^{\alpha J}{}_u \, (\sigma_x)_J{}^I \;=\; 0\,, \tag{6.1.97}$$

and the second condition reads, in a given $SU(2)$ and Lorentz gauge,

$$\varpi_{\underline{m}}{}^{xy} = \varepsilon^{xyz} \mathsf{A}^z{}_u \; \partial_{\underline{m}} q^u \,, \tag{6.1.98}$$

where $\varpi_{\underline{m}}{}^{xy}$ is the spin connection 1-form of the 3-dimensional metric and $\mathsf{A}^z{}_u \partial_{\underline{m}} q^u$ is the pullback of the $SU(2)$ connection of the quaternionic-Kähler manifold parameterized by the scalars q^u. In the gauge in which Eq. (6.1.98) holds the Killing spinors take the form

$$\epsilon_I = \epsilon_{I\,0}, \qquad \Pi^x{}_I{}^J \; \epsilon_{J\,0} = 0 \qquad \text{with} \qquad \Pi^x{}_I{}^J \;\equiv\; \tfrac{1}{2}[\; \delta_I{}^J \;-\; \gamma^{0(x)}\; (\sigma_{(x)})_I{}^J \;] \tag{6.1.99}$$

where the notation (x) in (6.1.99) means that x is not summed over so the constraints are imposed for each non-vanishing component of the $SU(2)$ connection.

We now repeat for the hyperscalars parameterizing a quaternionic Kähler manifold with isometry group G_H the discussion of Section 6.1.5. The fields will only depend on two spatial coordinates (x^1 and x^2, say, that can always be combined into a complex coordinate z) which parameterize the transverse space of the cosmic string. The metric will take the form

$$ds^2 = dt^2 - (dx^3)^2 - 2e^{\Phi(z,z^*)} dz dz^* \,, \tag{6.1.100}$$

and the hyperscalars will be real functions $q^u(z,z^*)$. A convenient Dreibein basis is

$$\hat{V}^3 = dx^3\,, \qquad \hat{V}^z = V dz\,, \qquad \hat{V}^{z^*} = V^* dz^*\,, \qquad |V|^2 = e^{\Phi(z,z^*)}\,. \tag{6.1.101}$$

In this Dreibein basis the supersymmetry conditions Eqs. (6.1.97) and (6.1.98) take the respective form

$$\mathsf{U}^{\alpha 2}{}_u \partial_{\underline{z}} q^u = \mathsf{U}^{\alpha 1}{}_u \partial_{\underline{z}^*} q^u \quad = \quad 0\,, \tag{6.1.102}$$

$$\varpi_{\underline{z}}{}^{zz^*} \quad = \quad \mathsf{A}^3{}_u\ \partial_{\underline{z}} q^u\,, \tag{6.1.103}$$

$$\mathsf{A}^1{}_u\ \partial_{\underline{m}} q^u = \mathsf{A}^2{}_u\ \partial_{\underline{m}} q^u \quad = \quad 0\,. \tag{6.1.104}$$

The Killing spinors of these solutions, in this basis, are given by

$$\epsilon_I = \epsilon_{I\,0}\,, \qquad \Pi^3{}_I{}^J\ \epsilon_{J\,0} = 0\,. \tag{6.1.105}$$

It can be shown that in this gauge the pullbacks of the complex structures J^1 and J^2 vanish while J^3 remains nonzero and one recovers the projection operator Eq. (6.1.95). As in the case of the vector scalars, it is convenient to work in a more general coordinate system in which the metric takes the form

$$ds^2 = dt^2 - (dx^3)^2 - 2e^{\Phi(z,z^*)}|f|^2 dz dz^*\,, \tag{6.1.106}$$

where $f(z)$ is a holomorphic function. The supersymmetry conditions, Eqs. (6.1.102) and (6.1.104), do not change and Eq. (6.1.103) is still satisfied *with the old spin connection*. If the new spin connection is computed with respect to the new frame

$$\hat{V}^3 = dx^3\,, \qquad \hat{V}^z = V f^* dz\,, \qquad \hat{V}^{z^*} = V^* f dz^*\,, \tag{6.1.107}$$

then, we find that

$$\varpi_{\underline{z}}{}^{zz^*} = \varpi_{\underline{z}}{}^{zz^*}{}_{\text{old}} + \partial_{\underline{z}} \log f\,, \tag{6.1.108}$$

and then the Killing spinors take the form

$$\epsilon_I = e^{\frac{1}{2}\log(f/f^*)\gamma^{03}} \epsilon_{I\,0}\,, \tag{6.1.109}$$

the constant spinor $\epsilon_{I\,0}$ obeying the same constraints as above, Eqs. (6.1.105). These same constraints allow us to rewrite it in the equivalent form

$$\epsilon_I = \exp\left\{\tfrac{1}{2}\log(f/f^*)\sigma_3\right\}_I{}^J \epsilon_{J\,0}\,. \tag{6.1.110}$$

The multi-valuedness of the Killing spinors ϵ_I of these solutions is related to the $U(1) \subset SU(2)$ gauge transformation where the $U(1)$ subgroup is associated to the non-vanishing component $\mathsf{A}^3{}_u \partial_{\underline{z}} q^u$ of the $SU(2)$ connection pulled back on the space-time. The transformations of the Killing spinors determine the monodromy properties of the holomorphic function f similarly to what happens in the case of the vector scalars.

6.2 Possible couplings of the $N = 1$ hierarchy p-forms to $(p-1)$-branes

Some, but not all, of the p-forms in the hierarchy may be associated to dynamical supersymmetric branes. In order to construct a κ-symmetric action for a $(p-1)$-brane that couples to a certain p-form, two necessary conditions are that the p-form transforms under no Stückelberg shift and that under supersymmetry it transforms into a gravitino multiplied by some scalars may couple to branes. In $N = 1, d = 4$ supergravity the p-forms that satisfy this condition are the (subset) of 2-forms $B_{\mathbf{a}}$ whose gauge transformations are massless. These are the 2-forms whose field strengths are dual to ungauged isometry currents. From the analysis of [28, 30] we know that these couple to strings (one-branes that have been referred to as stringy cosmic strings). Another form which satisfies the criteria is the 3-form C' which is a natural candidate to describe couplings to domain walls. We note that there are no 1-forms and 4-forms that can couple to a massive brane. There are thus no 1/2 BPS black holes in the theory and no 1/2 BPS space-time filling branes. The latter fact may be qualitatively understood from the fact that one cannot truncate the minimal $N = 1, d = 4$ supersymmetry algebra to a supersymmetry algebra with half of the original supercharges.

Chapter 7

Summary

This thesis deals with four-dimensional Supergravity theories and solutions thereto. In Chapter 1 we gave an overview of the main motivations for studying Supersymmetry, Supergravity and Superstring Theory. We shortly described how Supersymmetry might help to address some problems the Standard Model of Particle Physics seems to suffer from and we summarized the most important properties of Superstring theory and its low-energy limit, Supergravity. In Chapter 2 we introduced the theories we were going to work with in this thesis, i.e. four-dimensional Supergravities with four and eight supercharges, respectively. There we described these theories, ignoring possible gaugings. The problem of gauging was considered in Chapter 3. We saw how the introduction of the most general gaugings this is using electric and magnetic vector fields as gauge fields, implies the existence of a tensor hierarchy of higher degree p-forms. In Chapter 4 we applied the obtained results to $N = 1$ and $N = 2$ Supergravity. In Chapter 5 we found and classified the supersymmetric solutions to $N = 2$ four-dimensional Supergravity, using the tequnique as described in the introduction of this thesis, Chapter 1. In Chapter 6.1 we studied the coupling of extended solutions to $N = 2$ $d = 4$ Supergravity, taking into account the "predictions" of the four-dimensional tensor hierarchy found in Chapter 3.

This thesis is based on the publications which are listed in Appendix G.

Chapter 8

Resumen

En esta tesis hemos estudiado teorías de Supergravedad en cuatro dimensiones y soluciones de las mismas. En elcapítulo 1 hemos dado una visión general sobre las motivaciones principales para estudiar Supersimetría, Supergravedad y finalmente la Teoría de Supercuerdas. Hemos descrito brevemente como Supersimetría puede facilitar soluciones a varios "problemas" que parece padecer el Modelo Estándar de las Partículas Elementales y resumido las propriedades más importantes de la Teoría de Supercuerdas y de su límite de bajas energías: la teoría de Supergravedad. En el capítulo 2 hemos introducido las teorías estudiandas en esta tesis, es decir las Supergravedades cuatridimensionales con cuatro y ocho supercargas. En él hemos descrito dichas teorías ignorando posibles gaugeos. El problema de los gaugeos lo hemos considerado en el capítulo 3. Vimos como la introducción de los gaugeos más generales, es decir utilizando tanto campos vectoriales eléctricos como magnéticos como campos gauge, implica la existencia de una jerarquía de tensores con grados más altos. En el capítulo 4 aplicamos los resultados obtenidos anteriormente a las Supergravedades $N = 1$ y $N = 2$. En el capítulo 5 encontramos y clasificamos las soluciones supersímetricas de Supergravedad $N = 2$ cuatridimensional, utilizando el procedimiento descrito en la introducción de esta tesis, capítulo 1. En el capítulo 6.1 estudiamos el acoplo de soluciones extendidas de la teoria de Supergravedad $N = 2$ a p-formas, teniendo en cuenta las "predicciones" de la jerarquía general cuatridimensional hallada en el capítulo 3.

Esta tesis está basada en las publicaciones que están listadas en el Apéndice G.

Chapter 9

Conclusions

In this thesis we studied $N = 1$ and $N = 2$ Supergravity in four dimensions.

In Chapter 3 we studied the most general gaugings of four-dimensional Supergravity theories. To do so, we introduced the *embedding tensor formalism*. We showed how the second-order p-form equations of motion and the projected scalar equations of motion of general $d = 4$ gauged supergravity theories can be derived from a *duality hierarchy*, i.e. a set of first-order duality relations between p-form curvatures. Our starting point was the complete *tensor hierarchy* of the embedding tensor formalism, which we used to derive the off-shell gauge algebra for a set of p-form potentials, not including the scalars nor the metric tensor. Next, in a second step we put the tensor hierarchy on-shell by introducing duality relations between the curvatures of the tensor hierarchy, which leads to the desired equations of motion. In a third and final step, we constructed a gauge-invariant action for all the fields of the tensor hierarchy.

Whilst up to this point the tensor hierarchy was studied in the most general way, i.e. without specifying which four-dimensional Supergravity is being dealt with, the next step was the study of the gaugings of $N = 1, 2$ Supergravity in Chapter 4. When studying the most general gaugings of $N = 1$ four-dimensional Supergravity, we were led to considering the full hierarchy of p-form fields realized in this theory. We constructed the supersymmetric tensor hierarchy of N=1, d=4 supergravity and found some differences with the general bosonic construction of 4-dimensional gauged supergravities: the extension of $N = 1$ $d = 4$ Supergravity involves additional 3- and 4-forms which are not predicted by the general hierarchy. It turned out that the additional 3-form is dual to the superpotential, thus not associated to any gauge symmetry. We studied the closure of the supersymmetry algebra on all the bosonic p-form fields of the hierarchy up to duality relations. It turned out that in order to close the supersymmetry algebra without the use of duality relations, one must construct the hierarchy in terms of supermultiplets.

The solutions to four-dimensional Supergravity were studied in Chapter 5. In

Chapter 5.1 we found the complete classification of the supersymmetric solutions of $N = 2$ $d = 4$ ungauged supergravity coupled to an arbitrary number of vector- and hypermultiplets. We found that in the timelike case the hypermultiplets cause the constant-time hypersurfaces to be curved with an $SU(2)$ holonomy induced by the quaternionic structure of the hyperscalar manifold. The solutions have the same structure as without hypermultiplets but now depend on functions which are harmonic w.r.t. the curved 3-dimensional space. We discussed an example obtained from a hyper-less solution via the c-map. In the null case we found that the hyperscalars can only depend on the null coordinate and the solutions are essentially those of the hyper-less case.

In Chapter 5.2 we found the general form of all the supersymmetric configurations and solutions of $N = 2, d = 4$ Einstein-Yang-Mills theories. In the timelike case the solutions to the full supergravity equations could be constructed from known flat spacetime solutions of the Bogomol'nyi equations. This allowed the regular, sometimes globally regular, supersymmetric embedding in supergravity of regular monopole solutions (such as 't Hooft-Poyakov's, Weinberg's, Wilkinson and Bais's) but also embeddings of non-regular solutions to the Bogomol'nyi equations, which turned out to be regular black holes with different forms of non-Abelian hair. We found that the attractor mechanism is realized in a gauge-covariant way. In the null case we determined the general equations that supersymmetric configurations and solutions must satisfy.

In the last chapter, we studied the coupling of the one-dimensional solutions to $N = 2$ $d = 4$ Supergravity, found in Chapter 5.1, to 2-forms as predicted by the general four-dimensional tensor hierarchy. These 2-forms couple electrically to strings which we refer to as *stringy cosmic strings*. The 1/2 BPS bosonic world-sheet actions for these strings were constructed and its implications discussed.

Chapter 10

Conclusiones

En esta tesis hemos estudiado Supergravedad $N = 1$ y $N = 2$ en cuatro dimensiones.

En el capítulo 3 hemos estudiado los gaugeos más generales de teorías de Supergravedad cuatridimensionales. Para ello hemos introducido primero el formalismo del *embedding tensor*. Hemos mostrado cómo las ecuaciones de movimiento de las p-formas, las cuales son ecuaciones de segundo orden, y la proyección de las ecuaciones de movimiento de los escalares de Supergravedad general cuatridimensional pueden derivarse de una jerarquía de dualidades, es decir de un conjunto de relaciones de dualidad de primer orden entre las curvaturas de las p-formas. Nuestro punto de partida ha sido la jerarquía completa de tensores del formalismo del *embedding tensor*, el cual hemos utilizado para derivar el álgebra gauge *off-shell* para un conjunto de potenciales, p-formas, sin incluir los escalares ni el tensor métrico. En segundo lugar hemos puesto la jerarquía de tensores *on-shell* introduciendo relaciones de dualidad entre las curvaturas de la jerarquía de tensores, lo que nos llevó a las ecuaciones de movimiento deseadas. En un tercer paso hemos construido una acción invariante gauge para todos los campos de la jerarquía de tensores.

Mientras hasta este punto la jerarquía de tensores fue estudiada de la manera más general, es decir sin especificar de que teoría de Supergravedad cuatridimensional se trata, el paso siguente ha sido el estudio de los gaugeos de $N = 1, 2$ Supergravedad en cuatro dimensiones en el capítulo 4. Al estudiar los gaugeos más generales de la Supergravedad $N = 1$ cuatridimensional, fuimos llevados a considerar la jerarquía de tensores completa, realizada en esta teoría. Hemos construido la jerarquía supersimétrica de tensores de la Supergravedad $N = 1$, $d = 4$ y encontrado algunas diferencias con la construcción general bosónica de Supergravedades gaugeadas cuatridimensionales. Hemos estudiado el cierre del álgebra de supersimetría en todas las p-formas bosónicas de la jerarquía salvo relaciones de dualidad. Resultó que, para cerrar el álgebra de supersimetría sin usar relaciones de dualidad, es necesario construir la jerarquía en términos de supermultipletes.

Las soluciones de Supergravedad cuatridimensional fueron estudiadas en el capítulo 5. En el capítulo 5.1 hemos hallado la clasificación completa de las soluciones supersimétricas de $N = 2$ $d = 4$ Supergravedad sin gaugear, acoplada a un número arbitrario de vector- e hipermultipletes. Hemos encontrado que en el caso tipo tiempo los hipermultipletes causan la curvatura de las hipersuperficies de tiempo constante con holonomía $SU(2)$ inducida por la estructura quaterniónica de la variedad de los hiperescalares. Las soluciones tienen la misma estructura que sin hipermultipletes pero ahora dependen de funciones que son harmónicas con respecto al espacio curvo tridimensional. En el caso nulo hemos encontrado que los hiperescalares sólo dependen en la coordenada nula y las soluciones son esencialmente las del caso sin hipermultipletes.

En el capítulo 5.2 hemos encontrado la forma general de todas las configuraciones y soluciones de teorías $N = 2$ $d = 4$ de tipo Einstein-Yang-Mills. En el caso tipo tiempo las soluciones de las ecuaciones enteras de Supergravedad podían construirse partiendo de soluciones de espacio plano de las ecuaciones de Bogomol'nyi. Esto nos permitió el embebimiento supersimétrico regular, en algunos casos regular globalmente, dentro de Supergravedad de soluciones regulares de tipo monopolo (tales como los de 't Hooft-Poyakov, Weinberg, Wilkinson and Bais), pero tambien el embebimiento de soluciones singulares de las ecuaciones de Bogomol'nyi, que resultan ser agujeros negros regulares con diferentes formas de pelo no-Abeliano. Hemos encontrado que la realización del mecanismo del atractor es covariante gauge. En el caso nulo hemos determinado las ecuaciones generales que tienen que satisfacer las configuraciones y soluciones supersimetricas.

En el último capítulo hemos estudiado el acoplo de soluciones unidimensionales de la Supergravedad $N = 2$ $d = 4$, encontradas en el capítulo 5.1, a 2-formas como predice la jerarquía tensorial general cuatridimensional. Estas 2-formas se acoplan eléctricamente a cuerdas, las cuales etiquetamos como *stringy cosmic strings*. Las acciones bosónicas 1/2 BPS en la superficie de universo para estas cuerdas fueron construidas y discutidas sus implicaciones.

Chapter 11

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[1] alias "El α-jefe" o simplemente "El Alpha"
[2] a.k.a. "El β-jefe" or simply "El Beta"

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Appendix A

Conventions

In this thesis we use basically the notation of Ref. [82] and the conventions of Ref. [38], to which we have adapted the formulae of Ref. [82]. The main differences between the conventions of those two references are the signs of spin connection, the completely antisymmetric tensor ϵ^{abcd} and γ_5. Thus, chiralities are reversed and self-dual tensors are replaced by anti-self-dual tensors and vice-versa. The curvatures are identical. Finally, the normalization of the 2-form components differs by a factor of 2: for us

$$F = dA = \tfrac{1}{2} F_{\mu\nu} dx^{\mu} \wedge dx^{\nu} \Rightarrow F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]} \,, \tag{A.0.1}$$

which amounts to a difference of a factor of 2 in the vectors supersymmetry transformations. Further, all fermions and supersymmetry parameters from Ref. [82] have been rescaled by a factor of $\frac{1}{2}$, which introduces additional factors of $\frac{1}{4}$ in all the bosonic fields supersymmetry transformations.

The meaning of the different indices used in this paper is explained in Table A.0.1. We use the shorthand $\bar{n} \equiv n + 1$.

Type	Associated structure
$\mu,\, \nu,\, \ldots$	Curved space
$a,\, b,\, \ldots$	Tangent space
$m, n, \ldots$	Cartesian $\mathbb{R}^3$-indices
$i, j, \ldots;\, i^*, j^*, \ldots$	Complex scalar fields and their conjugates. There are n of them.
$\Lambda, \Sigma, \ldots$	$\mathfrak{sp}(\bar{n})$ indices ($\bar{n} = n + 1$)
$I, J, \ldots$	$N = 2$ spinor indices

Table A.0.1: Meaning of the indices used in this paper.

To make this paper as self-contained as possible, we proceed to review our conventions in detail.

A.1 Tensors

We use Greek letters $\mu, \nu, \rho, \ldots$ as (*curved*) tensor indices in a coordinate basis and Latin letters $a, b, c \ldots$ as (*flat*) tensor indices in a tetrad basis. Underlined indices are always curved indices. We symmetrize () and antisymmetrize [] with weight one (i.e. dividing by $n!$). We use mostly minus signature $(+---)$. η is the Minkowski metric and a general metric is denoted by g. Flat and curved indices are related by tetrads $e_a{}^\mu$ and their inverses $e^a{}_\mu$, satisfying

$$e_a{}^\mu e_b{}^\nu g_{\mu\nu} = \eta_{ab}\,, \qquad e^a{}_\mu e^b{}_\nu \eta_{ab} = g_{\mu\nu}\,. \tag{A.1.1}$$

∇ is the total (general- and Lorentz-) covariant derivative, whose action on tensors and spinors (ψ) is given by

$$\begin{aligned} \nabla_\mu \xi^\nu &= \partial_\mu \xi^\nu + \Gamma_{\mu\rho}{}^\nu \xi^\rho\,, \\ \nabla_\mu \xi^a &= \partial_\mu \xi^a + \omega_{\mu b}{}^a \xi^b\,, \\ \nabla_\mu \psi &= \partial_\mu \psi - \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\psi\,, \end{aligned} \tag{A.1.2}$$

where γ_{ab} is the antisymmetric product of two gamma matrices (see next section), $\omega_{\mu b}{}^a$ is the spin connection and $\Gamma_{\mu\rho}{}^\nu$ is the affine connection. The respective curvatures are defined through the Ricci identities

$$\begin{aligned} [\nabla_\mu, \nabla_\nu]\ \xi^\rho &= R_{\mu\nu\sigma}{}^\rho(\Gamma)\,\xi^\sigma + T_{\mu\nu}{}^\sigma \nabla_\sigma \xi^\rho\,, \\ [\nabla_\mu, \nabla_\nu]\ \xi^a &= R_{\mu\nu b}{}^a(\omega)\xi^b\,, \\ [\nabla_\mu, \nabla_\nu]\ \psi &= -\tfrac{1}{4}R_{\mu\nu}{}^{ab}(\omega)\gamma_{ab}\psi\,. \end{aligned} \tag{A.1.3}$$

and given in terms of the connections by

$$\begin{aligned} R_{\mu\nu\rho}{}^\sigma(\Gamma) &= 2\partial_{[\mu}\Gamma_{\nu]\rho}{}^\sigma + 2\Gamma_{[\mu|\lambda}{}^\sigma \Gamma_{\nu]\rho}{}^\lambda\,, \\ R_{\mu\nu a}{}^b(\omega) &= 2\partial_{[\mu}\,\omega_{\nu]a}{}^b - 2\omega_{[\mu|a}{}^c\,\omega_{|\nu]c}{}^b\,. \end{aligned} \tag{A.1.4}$$

These two connections are related by the tetrad postulate

$$\nabla_\mu e_a{}^\mu = 0\,, \tag{A.1.5}$$

by

$$\omega_{\mu a}{}^b = \Gamma_{\mu a}{}^b + e_a{}^\nu \partial_\mu e_\nu{}^b\,, \tag{A.1.6}$$

which implies that the curvatures are, in turn, related by

$$R_{\mu\nu\rho}{}^{\sigma}(\Gamma) = e_\rho{}^a e^\sigma{}_b R_{\mu\nu a}{}^b(\omega)\,. \tag{A.1.7}$$

Finally, metric compatibility and torsionlessness fully determine the connections to be of the form

$$\begin{aligned} \Gamma_{\mu\nu}{}^{\rho} &= \tfrac{1}{2} g^{\rho\sigma}\left\{\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right\}, \\ \omega_{abc} &= -\Omega_{abc} + \Omega_{bca} - \Omega_{cab}\,, \qquad \Omega_{ab}{}^c = e_a{}^\mu e_b{}^\nu \partial_{[\mu} e^c{}_{\nu]}\,. \end{aligned} \tag{A.1.8}$$

The 4-dimensional fully antisymmetric tensor is defined in flat indices by tangent space by

$$\epsilon^{0123} = +1\,, \qquad \Rightarrow \epsilon_{013} = -1\,, \tag{A.1.9}$$

and in curved indices by

$$\epsilon^{\mu_1\cdots\mu_3} = \sqrt{|g|}\, e^{\mu_1}{}_{a_1} \cdots e^{\mu_3}{}_{a_3} \epsilon^{a_3\cdots a_3}\,, \tag{A.1.10}$$

so, with upper indices, is independent of the metric and has the same value as with flat indices.

We define the (Hodge) dual of a completely antisymmetric tensor of rank k, $F_{(k)}$ by

$$\star F_{(k)}{}^{\mu_1\cdots\mu_{(d-k)}} = \frac{1}{k!\sqrt{|g|}} \epsilon^{\mu_1\cdots\mu_{(d-k)}\mu_{(d-k+1)}\cdots\mu_d} F_{(k)\mu_{(d-k+1)}\cdots\mu_d}\,. \tag{A.1.11}$$

Differential forms of rank k are normalized as follows:

$$F_{(k)} \equiv \tfrac{1}{k!} F_{(k)\,\mu_1\cdots\mu_k} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_k}\,. \tag{A.1.12}$$

For any 4-dimensional 2-form, we define

$$F^{\pm} \equiv \tfrac{1}{2}(F \pm i \star F)\,, \qquad \pm i \star F^{\pm} = F^{\pm}\,. \tag{A.1.13}$$

For any two 2-forms F, G, we have

$$F^{\pm}{}_{\mu\nu} G^{\mp\,\mu\nu} = 0\,, \qquad F^{\pm}{}_{[\mu}{}^{\rho} G^{\mp}{}_{\nu]\rho} = 0\,. \tag{A.1.14}$$

Given any 2-form $F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$ and a non-null 1-form $\hat{V} = V_\mu dx^\mu$, we can express F in the form

$$F = -V^{-2}[E \wedge \hat{V} - \star(B \wedge \hat{V})]\,, \qquad E_\mu \equiv V^\nu F_{\nu\mu}\,, \qquad B_\mu \equiv \star V^\nu F_{\nu\mu}\,. \tag{A.1.15}$$

For the complex combinations $F^{\pm}$ we have

$$F^{\pm} = -V^{-2}[C^{\pm} \wedge \hat{V} \pm i \star (C^{\pm} \wedge \hat{V})] , \qquad C^{\pm}{}_{\mu} \equiv V^{\nu} F^{\pm}{}_{\nu\mu} . \tag{A.1.16}$$

If we have a (real) null vector l^{μ}, we can always add three more null vectors $n^{\mu}, m^{\mu}, m^{*\,\mu}$ to construct a complex null tetrad such that the local metric in this basis takes the form

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{A.1.17}$$

with the ordering (l, n, m, m^*). For the local volume element we obtain $\epsilon^{lnmm^*} = i$. With the dual basis of 1-forms $\left(\hat{l}, \hat{n}, \hat{m}, \hat{m}^*\right)$ we can construct three independent complex self-dual 2-forms that we choose to normalize as follows:

$$\begin{aligned} \hat{\Phi}^{(1)} &= \hat{l} \wedge \hat{m}^* , \\ \hat{\Phi}^{(2)} &= \tfrac{1}{2}[\hat{l} \wedge \hat{n} + \hat{m} \wedge \hat{m}^*] , \\ \hat{\Phi}^{(3)} &= -\hat{n} \wedge \hat{m} . \end{aligned} \tag{A.1.18}$$

Any self-dual 2-form F^+ can be written as a linear combination of these, with complex coefficients:

$$F^+ = c_i \hat{\Phi}^{(i)} . \tag{A.1.19}$$

The coefficients c_i can be found by contracting F^+ with $l^{\mu}, n^{\mu}, m^{\mu}, m^{*\mu}$:

$$\begin{aligned} l^{\nu} F^{+}{}_{\nu\mu} &= -\tfrac{1}{2} c_2 l_{\mu} - c_3 m_{\mu} , \\ n^{\nu} F^{+}{}_{\nu\mu} &= c_1 m^*_{\mu} + \tfrac{1}{2} c_2 n_{\mu} , \\ m^{\nu} F^{+}{}_{\nu\mu} &= c_1 l_{\mu} + \tfrac{1}{2} c_2 m_{\mu} , \\ m^{*\nu} F^{+}{}_{\nu\mu} &= -\tfrac{1}{2} c_2 m^*_{\mu} - c_3 n_{\mu} . \end{aligned} \tag{A.1.20}$$

A.2 Gamma matrices and spinors

We work with a purely imaginary representation

$$\gamma^{a\,*} = -\gamma^{a} , \tag{A.2.1}$$

and our convention for their anticommutator is

$$\{\gamma^a, \gamma^b\} = +2\eta^{ab}\,. \tag{A.2.2}$$

Thus,

$$\gamma^0\gamma^a\gamma^0 = \gamma^{a\,\dagger} = \gamma^{a\,-1} = \gamma_a\,. \tag{A.2.3}$$

The chirality matrix is defined by

$$\gamma_5 \equiv -i\gamma^0\gamma^1\gamma^2\gamma^3 = \tfrac{i}{4!}\epsilon_{abcd}\gamma^a\gamma^b\gamma^c\gamma^d\,, \tag{A.2.4}$$

and satisfies

$$\gamma_5{}^\dagger = -\gamma_5{}^* = \gamma_5\,, \qquad (\gamma_5)^2 = 1\,. \tag{A.2.5}$$

With this chirality matrix, we have the identity

$$\gamma^{a_1\cdots a_n} = \frac{(-1)^{[n/2]}i}{(4-n)!}\epsilon^{a_1\cdots a_n b_1\cdots b_{4-n}}\gamma_{b_1\cdots b_{4-n}}\gamma_5\,. \tag{A.2.6}$$

Our convention for Dirac conjugation is

$$\bar\psi = i\psi^\dagger\gamma_0\,. \tag{A.2.7}$$

Using the identity Eq. (A.2.6) the general $d = 4$ Fierz identity ($p = +1$ for *commuting* spinors and $p = -1$ for *commuting* spinors) takes the form

$$\begin{aligned} p(\bar\lambda M\chi)(\bar\psi N\varphi) \quad = \quad & \tfrac{1}{4}(\bar\lambda MN\varphi)(\bar\psi\chi) + \tfrac{1}{4}(\bar\lambda M\gamma^a N\varphi)(\bar\psi\gamma_a\chi) - \tfrac{1}{8}(\bar\lambda M\gamma^{ab}N\varphi)(\bar\psi\gamma_{ab}\chi) \\ & -\tfrac{1}{4}(\bar\lambda M\gamma^a\gamma_5 N\varphi)(\bar\psi\gamma_a\gamma_5\chi) + \tfrac{1}{4}(\bar\lambda M\gamma_5 N\varphi)(\bar\psi\gamma_5\chi)\,. \end{aligned} \tag{A.2.8}$$

We use 4-component chiral spinors. In the $N = 1$ theory the chirality of all spinors is negative

$$\gamma_5\psi_\mu = -\psi_\mu\,, \qquad \gamma_5\lambda^\Lambda = -\lambda^\Lambda\,, \qquad \gamma_5\chi^i = -\chi^i\,, \qquad \gamma_5\epsilon = -\epsilon\,, \tag{A.2.9}$$

and is reversed by complex conjugation:

$$\gamma_5\psi^*_\mu = \psi^*_\mu\,, \qquad \gamma_5\lambda^{*\Lambda} = \lambda^{*\Lambda}\,, \qquad \gamma_5\chi^{*i^*} = \chi^{*i^*}\,, \qquad \gamma_5\epsilon^* = \epsilon^*\,, \tag{A.2.10}$$

In the $N = 2$ theory the chirality of the spinors is related to the position of the $SU(2)$ index or $Sp(2m)$ index as follows::

$$\gamma_5 \psi_{I\,\mu} = -\psi_{I\,\mu}\,, \qquad \gamma_5 \lambda^{Ii} = +\lambda^{Ii}\,, \qquad \gamma_5 \zeta_\alpha = -\zeta_\alpha\,, \qquad \gamma_5 \epsilon_I = -\epsilon_I\,. \tag{A.2.11}$$

Both (chirality and position of the index) are reversed under complex conjugation:

$$\gamma_5 \psi^I{}_\mu = \psi^I{}_\mu\,, \qquad \gamma_5 \lambda_I{}^{i^*} = -\lambda_I{}^{i^*}\,, \qquad \gamma_5 \zeta^\alpha = +\zeta^\alpha\,, \qquad \gamma_5 \epsilon^I = \epsilon^I\,. \tag{A.2.12}$$

We take this fact into account when Dirac-conjugating chiral spinors:

$$\bar{\epsilon}^I \equiv i(\epsilon_I)^\dagger \gamma_0\,, \qquad \bar{\epsilon}^I \gamma_5 = +\bar{\epsilon}^I\,, \quad \text{etc.} \tag{A.2.13}$$

Appendix B

Kähler geometry

A Kähler manifold $\mathcal{M}$ is a complex manifold on which there exist complex coordinates Z^i and $Z^{*\,i^*} = (Z^i)^*$ and a function $\mathcal{K}(Z, Z^*)$, called the *Kähler potential*, such that the line element is

$$ds^2 = 2\mathcal{G}_{ii^*}\ dZ^i dZ^{*\,i^*}\,, \tag{B.0.1}$$

with

$$\mathcal{G}_{ii^*} = \partial_i \partial_{i^*} \mathcal{K}\,. \tag{B.0.2}$$

The *Kähler (connection) 1-form* $\mathcal{Q}$ is defined by

$$\begin{aligned}
\mathcal{Q} \quad &\equiv \quad \tfrac{1}{2i}(dZ^i \partial_i \mathcal{K} - dZ^{*\,i^*} \partial_{i^*} \mathcal{K}) && \text{(B.0.3)}\\
&= \quad \tfrac{1}{2i}(\partial - \bar{\partial})\mathcal{K}\,, && \text{(B.0.4)}
\end{aligned}$$

and the *Kähler 2-form* $\mathcal{J}$ is its exterior derivative

$$\begin{aligned}
\mathcal{J} \quad &\equiv \quad d\mathcal{Q} = i\mathcal{G}_{ii^*} dZ^i \wedge dZ^{*\,i^*} && \text{(B.0.5)}\\
&= \quad i\partial\bar{\partial}\mathcal{K}\,. && \text{(B.0.6)}
\end{aligned}$$

Note that this yields immediately that the Kähler 2-form is closed:[1]

$$d\mathcal{J} = 0. \tag{B.0.9}$$

The Levi-Cività connection on a Kähler manifold is given by

[1] Actually there is an alternative way to define a Kähler manifold:
Definition: A Kähler manifold is an Hermitean manifold whose Kähler form is closed.

$$\Gamma_{jk}{}^{i} = \mathcal{G}^{ii^*}\partial_j \mathcal{G}_{i^*k}\,, \qquad \Gamma_{j^*k^*}{}^{i^*} = \mathcal{G}^{i^*i}\partial_{j^*}\mathcal{G}_{k^*i}\,. \tag{B.0.10}$$

The Riemann curvature tensor has as only non-vanishing components $R_{ij^*kl^*}$, but we will not need their explicit expression. The Ricci tensor is given by

$$R_{ii^*} = \partial_i \partial_{i^*}\left(\tfrac{1}{2}\log \det \mathcal{G}\right)\,, \tag{B.0.11}$$

and the Ricci 2-form by

$$\mathcal{R} = iR_{ii^*}dz^i \wedge dz^{*i^*}. \tag{B.0.12}$$

The Kähler potential is not unique: it is defined only up to *Kähler transformations* of the form

$$\mathcal{K}'(Z,Z^*) = \mathcal{K}(Z,Z^*) + f(Z) + f^*(Z^*)\,, \tag{B.0.13}$$

where $f(Z)$ is any holomorphic function of the complex coordinates Z^i. Under these transformations, the Kähler metric and Kähler 2-form are invariant, while the components of the Kähler connection 1-form transform according to

$$\mathcal{Q}'_i = \mathcal{Q}_i - \tfrac{i}{2}\partial_i f\,. \tag{B.0.14}$$

By definition, objects X with Kähler weight $(q,\bar{q})$ transform under the above Kähler transformations like:

$$X' = Xe^{-(qf+\bar{q}f^*)/2} \tag{B.0.15}$$

and the Kähler-covariant derivative $\mathcal{D}$ acting on them is given by

$$\mathcal{D}_i \equiv \nabla_i + iq\mathcal{Q}_i\,, \qquad \mathcal{D}_{i^*} \equiv \nabla_{i^*} - i\bar{q}\mathcal{Q}_{i^*}\,, \tag{B.0.16}$$

where ∇ is the standard covariant derivative associated to the Levi-Cività connection on $\mathcal{M}$.

This then implies

$$\begin{aligned} d\mathcal{J} &= (\partial+\bar{\partial})i\mathcal{G}_{ii^*}dz^i \wedge dz^{*\,i^*} \\ &= i\partial_j \mathcal{G}_{ii^*}dz^j \wedge dz^i \wedge dz^{*\,i^*} + i\partial_{j^*}\mathcal{G}_{ii^*}dz^{*\,j^*} \wedge dz^i \wedge dz^{*\,i^*} \\ &= \tfrac{i}{2}(\partial_j \mathcal{G}_{ii^*} - \partial_i \mathcal{G}_{ji^*})dz^j \wedge dz^i \wedge dz^{*\,i^*} + \tfrac{i}{2}(\partial_{j^*}\mathcal{G}_{ii^*} - \partial_{i^*}\mathcal{G}_{ij^*})dz^{*j^*} \wedge dz^i \wedge dz^{*\,i^*}\,, \end{aligned}$$

leading to the following relations

$$\partial_j \mathcal{G}_{ii^*} = \partial_i \mathcal{G}_{ji^*}\,, \qquad \partial_{j^*}\mathcal{G}_{ii^*} = \partial_{i^*}\mathcal{G}_{ij^*}\,, \tag{B.0.7}$$

whose solutions is (locally) given by

$$\mathcal{G}_{ii^*} = \partial_i \partial_{i^*}\mathcal{K}, \tag{B.0.8}$$

and the converse is also true locally (see definition above).

The Ricci identity for this covariant derivative is, on objects without vector indices and Kähler weight $(q, \bar{q})$

$$[\mathcal{D}_i, \mathcal{D}_{j^*}] = -\tfrac{1}{2}(q - \bar{q})\mathcal{G}_{ij^*}\,. \tag{B.0.17}$$

When $(q, \bar{q}) = (1, -1)$, this defines a complex line bundle $L^1 \to \mathcal{M}$ over the Kähler manifold $\mathcal{M}$ whose first, and only, Chern class equals the Kähler 2-form $\mathcal{J}$. A complex line bundle with this property is known as a *Kähler-Hodge (KH) manifold* and provides the formal starting point for the definition of a special Kähler manifold[2] that is explained in the next Appendix. These are the manifolds parametrized by the complex scalars of the chiral multiplets of $N = 1, d = 4$ supergravity. Furthermore, objects such as the sueprpotential and all the spinors of the theory have a well-defined Kähler weight. The manifolds parametrized by the complex scalars of the vector multiplets of $N = 2, d = 4$ supergravity are also KH manifolds but must satisfy further constraints that define what is known as *special Kähler geometry*, described in Appendix C.

We will often use the spacetime pullback of the Kähler-covariant derivative on tensor fields with Kähler weight $(q, -q)$ (weight q, for short) for which it takes the simple form

$$\mathfrak{D}_\mu = \nabla_\mu + iq\mathcal{Q}_\mu\,, \tag{B.0.18}$$

where ∇_μ is the standard spacetime covariant derivative plus possibly the pullback of the Levi-Cività connection on $\mathcal{M}$; $\mathcal{Q}_\mu$ is the pullback of the Kähler 1-form, *i.e.*

$$\mathcal{Q}_\mu = \tfrac{1}{2i}(\partial_\mu Z^i \partial_i \mathcal{K} - \partial_\mu Z^{*\,i^*} \partial_{i^*} \mathcal{K})\,. \tag{B.0.19}$$

Note that for a Kähler manifold the torsion vanishes, and since it is proportional to the exterior derivative of the Ricci 2-form $\mathcal{R}$ defined in Eq. (B.0.12), $\mathcal{R}$ is closed and hence a representative of $H^{(1,1)}$ and the *first Chern class* of a Kähler manifold is given by

$$c_1(M) = \tfrac{1}{2\pi}[\mathcal{R}]. \tag{B.0.20}$$

B.1 Gauging holomorphic isometries of Kähler-Hodge manifolds

We are now going to review some basics of the gauging of holomorphic isometries of Kähler-Hodge manifolds that occur in $N = 1$ and $N = 2, d = 4$ supergravities. We will first study the general problem in complex manifolds. This is enough for purely bosonic theories in which only the complex structure is relevant. The Kähler-Hodge structure is necessary in presence of fermions and only those transformations

[2] Some basic references for this material are [123–125] and the review [83]. The definition of special Kähler manifold was made in Ref. [126], formalizing the original results of Ref. [80].

that preserve it will be symmetries of the full theory that can be gauged. We will study this problem next. The special-Kähler structure is necessary in $N = 2, d = 4$ supergravity and, again, only those transformations that preserve it are symmetries that can be gauged. This problem will be studied in Appendix C.2, after which we define special-Kähler manifolds.

B.1.1 Complex manifolds

We start by assuming that the Hermitean metric $\mathcal{G}_{ij^*}$ (we will use the Kähler-Hodge structure later) admits a set of Killing vectors[3] $\{K_\Lambda = k_\Lambda{}^i\partial_i + k^*_\Lambda{}^{i^*}\partial_{i^*}\}$ satisfying the Lie algebra

$$[K_\Lambda, K_\Sigma] = -f_{\Lambda\Sigma}{}^\Omega K_\Omega \,, \tag{B.1.1}$$

of the group G_V that we want to gauge.

Hermiticity implies that the components $k_\Lambda{}^i$ and $k^*_\Lambda{}^{i^*}$ of the Killing vectors are, respectively, holomorphic and antiholomorphic and satisfy, separately, the above Lie algebra. Once (anti-) holomorphicity is taken into account, the only non-trivial components of the Killing equation are

$$\tfrac{1}{2}\pounds_\Lambda \mathcal{G}_{ij^*} = \nabla_{i^*} k^*_{\Lambda\, j} + \nabla_j k_{\Lambda\, i^*} = 0 \,, \tag{B.1.2}$$

where $\pounds_\Lambda$ stands for the Lie derivative w.r.t. K_Λ.

The standard σ-model kinetic term $\mathcal{G}_{ij^*}\partial_\mu Z^i \partial^\mu Z^{*j^*}$ is automatically invariant under infinitesimal reparametrizations of the form

$$\delta_\alpha Z^i = \alpha^\Lambda k_\Lambda{}^i(Z) \,, \tag{B.1.3}$$

if the α^Λs are constants. If they are arbitrary functions of the spacetime coordinates $\alpha^\Lambda(x)$ we need to introduce a covariant derivative using as connection the vector fields present in the theory. The covariant derivative is

$$\mathfrak{D}_\mu Z^i = \partial_\mu Z^i + g A^\Lambda{}_\mu k_\Lambda{}^i \,, \tag{B.1.4}$$

and transforms as

$$\delta_\alpha \mathfrak{D}_\mu Z^i = \alpha^\Lambda(x)\partial_j k_\Lambda{}^i \mathfrak{D}_\mu Z^j = -\alpha^\Lambda(x)(\pounds_\Lambda - K_\Lambda)\mathfrak{D}_\mu Z^j \,, \tag{B.1.5}$$

provided that the gauge potentials transform as

$$\delta_\alpha A^\Lambda{}_\mu = -g^{-1}\mathfrak{D}_\mu \alpha^\Lambda \equiv -g^{-1}(\partial_\mu \alpha^\Lambda + g f_{\Sigma\Omega}{}^\Lambda A^\Sigma{}_\mu \alpha^\Omega) \,. \tag{B.1.6}$$

The gauge field strength is given by

[3] The index Λ always takes values from 1 to n_V ($\bar{n} = n_V + 1$) in $N = 1$ ($N = 2$) supergravity , but some (or all) the Killing vectors may be zero.

$$F^{\Lambda}{}_{\mu\nu} = 2\partial_{[\mu}A^{\Lambda}{}_{\nu]} + gf_{\Sigma\Omega}{}^{\Lambda}A^{\Sigma}{}_{[\mu}A^{\Omega}{}_{\nu]}\,, \tag{B.1.7}$$

and transforms under gauge transformations as

$$\delta_{\alpha}F^{\Lambda}{}_{\mu\nu} = -\alpha^{\Sigma}(x)f_{\Sigma\Omega}{}^{\Lambda}F^{\Omega}{}_{\mu\nu}\,. \tag{B.1.8}$$

Now, to make the σ-model kinetic term gauge invariant it is enough to replace the partial derivatives by covariant derivatives

$$\mathcal{G}_{ij^*}\partial_{\mu}Z^{i}\partial^{\mu}Z^{*j^*} \quad\longrightarrow \mathcal{G}_{ij^*}\mathfrak{D}_{\mu}Z^{i}\mathfrak{D}^{\mu}Z^{*j^*}\,. \tag{B.1.9}$$

For any tensor field Φ (spacetime $\mu,\nu,\ldots$, gauge $\Lambda,\Sigma,\ldots$ and target space tensor $i,i^*,\ldots$ indices are not explicitly shown) transforming covariantly under gauge transformations, i.e. tranforming as

$$\delta_{\alpha}\Phi = -\alpha^{\Lambda}(x)(\mathbb{L}_{\Lambda} - K_{\Lambda})\Phi\,, \tag{B.1.10}$$

where we have defined the *Lie covariant derivative*[4]

$$\mathbb{L}_{\Lambda} \equiv \pounds_{\Lambda} - \mathcal{S}_{\Lambda}\,, \tag{B.1.11}$$

and $\mathcal{S}_{\Lambda}$ represents a symplectic rotation, the gauge covariant derivative is given by

$$\mathfrak{D}_{\mu}\Phi = \{\nabla_{\mu} + \mathfrak{D}_{\mu}Z^{i}\Gamma_{i} + \mathfrak{D}_{\mu}Z^{*i^*}\Gamma_{i^*} - gA^{\Lambda}{}_{\mu}(\mathbb{L}_{\Lambda} - K_{\Lambda})\}\Phi\,. \tag{B.1.12}$$

In particular, on $\mathfrak{D}_{\mu}Z^{i}$

$$\mathfrak{D}_{\mu}\mathfrak{D}_{\nu}Z^{i} = \nabla_{\mu}\mathfrak{D}_{\nu}Z^{i} + \Gamma_{jk}{}^{i}\mathfrak{D}_{\mu}Z^{j}\mathfrak{D}_{\nu}Z^{k} + gA^{\Lambda}{}_{\mu}\partial_{j}k_{\Lambda}{}^{i}\mathfrak{D}_{\nu}Z^{j}\,, \tag{B.1.13}$$

$$[\mathfrak{D}_{\mu},\mathfrak{D}_{\nu}]\,Z^{i} = gF^{\Lambda}{}_{\mu\nu}k_{\Lambda}{}^{i}\,. \tag{B.1.14}$$

An important case is that of the fields Φ which only depend on the spacetime coordinates through the complex scalars Z^{i} and their complex conjugates so that $\nabla_{\mu}\Phi = \partial_{\mu}\Phi = \partial_{\mu}Z^{i}\partial_{i}\Phi + \partial_{\mu}Z^{*i^*}\partial_{i^*}\Phi$. Φ is an *invariant field* if[5]

$$\mathbb{L}_{\Lambda}\Phi \equiv (\pounds_{\Lambda} - \mathcal{S}_{\Lambda})\Phi = 0\,. \tag{B.1.15}$$

Only if all the fields that occur in the theory are invariant fields, the theory can be gauged. Only in that case $\nabla_{\mu}\Phi = \partial_{\mu}\Phi = \partial_{\mu}Z^{i}\partial_{i}\Phi + \partial_{\mu}Z^{*i^*}\partial_{i^*}\Phi$ can be

[4] We will extend this definition to fields with non-zero Kähler weight after we study the symmetries of the Kähler structure. For the moment we only consider tensors of the Hermitean space with metric $\mathcal{G}_{ij^*}$, possibly with gauge and spacetime indices.

[5] Alternatively, we could say that it is a field invariant under reparametrizations up to rotations.

true irrespectively of gauge transformations. These fields transform under gauge transformations according to

$$\delta_\alpha \Phi = -\alpha^\Lambda (\mathbb{L}_\Lambda - K_\Lambda)\Phi = \alpha^\Lambda K_\Lambda \Phi \,, \tag{B.1.16}$$

and their covariant derivative is given by

$$\mathfrak{D}_\mu \Phi = \{\partial_\mu + \mathfrak{D}_\mu Z^i \Gamma_i + \mathfrak{D}_\mu Z^{*i^*} \Gamma_{i^*} + g A^\Lambda{}_\mu K_\Lambda\}\Phi \,, \tag{B.1.17}$$

and is always the covariant pullback of the target covariant derivative:

$$\mathfrak{D}_\mu \Phi = \mathfrak{D}_\mu Z^i \nabla_i \Phi + \mathfrak{D}_\mu Z^{*i^*} \nabla_{i^*} \Phi \,. \tag{B.1.18}$$

Let us consider, for instance, the holomorphic *kinetic matrix* $f_{\Lambda\Sigma}(Z)$ in $N = 1, d = 4$ supergravity or the *period matrix* $\mathcal{N}_{\Lambda\Sigma}(Z, Z^*)$ in $N = 2, d = 4$ supergravity, both of which are symmetric matrices that codify the couplings between the complex scalars and the vector fields. These matrices transform under global rotations of the vector fields

$$\delta_\alpha A^\Lambda{}_\mu = -\alpha^\Sigma f_{\Sigma\Omega}{}^\Lambda A^\Omega{}_\mu \,, \tag{B.1.19}$$

according to

$$\delta_\alpha f_{\Lambda\Sigma} \equiv -\alpha^\Omega \mathcal{S}_\Omega f_{\Lambda\Sigma} = 2\alpha^\Omega f_{\Omega(\Lambda}{}^\Pi f_{\Sigma)\Pi} \,, \tag{B.1.20}$$

(analogously for $\mathcal{N}_{\Lambda\Sigma}$) and under the reparametrizations of the complex scalars Eq. (B.1.3).

$$\delta_\alpha f_{\Lambda\Sigma} = -\alpha^\Omega \pounds_\Omega f_{\Lambda\Sigma} - \alpha^\Omega k_\Omega{}^i \partial_i f_{\Lambda\Sigma} \,. \tag{B.1.21}$$

These transformations will only be a symmetry of the theory if their values coincide, i.e. if

$$(\pounds_\Omega - \mathcal{S}_\Omega) f_{\Lambda\Sigma} = \mathbb{L}_\Omega f_{\Lambda\Sigma} = 0 \,, \tag{B.1.22}$$

i.e. only if $f_{\Lambda\Sigma}(Z)$ is an invariant field according to the above definition. Its covariant derivative is given by

$$\mathfrak{D}_\mu f_{\Lambda\Sigma} = \mathfrak{D}_\mu Z^i \partial_i f_{\Lambda\Sigma} \,, \tag{B.1.23}$$

on account of its holomorphicity.

B.1.2 Kähler-Hodge manifolds

A Kähler manifold is a *Hodge-Kähler manifold* if and only if there exists a line bundle $\mathcal{L} \longrightarrow \mathcal{M}$ such that its first Chern class equals the cohomology class of the Kähler 2-form $\mathcal{J}$:

$$c_1(\mathcal{L}) \;=\; [\,\mathcal{J}\,] \tag{B.1.24}$$

In local terms this means that there is a holomorphic section $\Omega(z)$ such that we can write [70]

$$\mathcal{J} \;=\; i\,\mathcal{G}_{ij^\star}\,dz^i \wedge d\bar{z}^{j^\star} \;=\; i\,\bar{\partial}\,\partial \log \parallel \Omega(z) \parallel^2 . \tag{B.1.25}$$

Let us now assume that the scalar manifold is not just Hermitean but Kähler-Hodge. Let us study how the Kähler structure is preserved, first.

The transformations generated by the Killing vectors will preserve the Kähler structure if they leave the Kähler potential invariant up to Kähler transformations, i.e., for each Killing vector K_Λ

$$\pounds_\Lambda \mathcal{K} \equiv k_\Lambda{}^i \partial_i \mathcal{K} + k^*_\Lambda{}^{i^*} \partial_{i^*} \mathcal{K} = \lambda_\Lambda(Z) + \lambda^*_\Lambda(Z^*)\,. \tag{B.1.26}$$

From this condition it follows that

$$\pounds_\Lambda \lambda_\Sigma - \pounds_\Sigma \lambda_\Lambda = -f_{\Lambda\Sigma}{}^\Omega \lambda_\Omega\,. \tag{B.1.27}$$

On the other hand, the preservation of the Kähler structure implies the conservation of the Kähler 2-form $\mathcal{J}$

$$\pounds_\Lambda \mathcal{J} = 0\,. \tag{B.1.28}$$

The closedness of $\mathcal{J}$ implies that $\pounds_\Lambda \mathcal{J} = d(i_{k_\Lambda} \mathcal{J})$ and therefore the preservation of the Kähler structure implies the existence of a set of real 0-forms $\mathcal{P}_\Lambda$ known as *momentum maps* such that

$$i_{k_\Lambda} \mathcal{J} = d\mathcal{P}_\Lambda\,. \tag{B.1.29}$$

A local solution for this equation is provided by

$$i\mathcal{P}_\Lambda = k_\Lambda{}^i \partial_i \mathcal{K} - \lambda_\Lambda\,, \tag{B.1.30}$$

which, on account of Eq. (B.1.26) is equivalent to

$$i\mathcal{P}_\Lambda = -(k^*_\Lambda{}^{i^*} \partial_{i^*} \mathcal{K} - \lambda^*_\Lambda)\,, \tag{B.1.31}$$

or

$$\mathcal{P}_\Lambda = i_{k_\Lambda} \mathcal{Q} - \tfrac{1}{2i}(\lambda_\Lambda - \lambda^*_\Lambda)\,. \tag{B.1.32}$$

The momentum map can be used as a prepotential from which the Killing vectors can be derived:

$$k_{\Lambda\, i^*} = i\partial_{i^*}\mathcal{P}_\Lambda\,. \tag{B.1.33}$$

This is whay they are sometimes called *Killing prepotentials*.

The momentum maps are defined, in principle, up to an additive real constant. In $N=1, d=4$ theories (but not in $N=2, d=4$) it is possible to have non-vanishing, constant, momentum maps with $i\mathcal{P}_\Lambda = -\lambda_\Lambda$ for vanishing Killing vectors. In this case no isometry is gauged. Instead, it is the $U(1)$ symmetry associated to Kähler transformations (in Kähler-Hodge manifolds) that is gauged. These constant momentum maps are called *D*- or *Fayet-Iliopoulos* terms and appear as in the supersymmetry transformation rules of gaugini, in the potential and in the covariant derivatives of sections that we are going to discuss.

Using Eqs. (B.1.1),(B.1.26) and (B.1.27) one finds

$$\pounds_\Lambda \mathcal{P}_\Sigma = 2ik_{[\Lambda}{}^{i} k^*_{\Sigma]}{}^{j^*}\mathcal{G}_{ij^*} = -f_{\Lambda\Sigma}{}^{\Omega}\mathcal{P}_\Omega\,. \tag{B.1.34}$$

This equation fixes the additive constant of the momentum map in directions in which a non-Abelian group is going to be gauged.

The gauge transformation rule a section Φ of Kähler weight (p,q) is[6]

$$\delta_\alpha\Phi = -\alpha^\Lambda(x)(\mathbb{L}_\Lambda - K_\Lambda)\Phi\,, \tag{B.1.35}$$

where $\mathbb{L}_\Lambda$ stands for the symplectic and Kähler-covariant Lie derivative w.r.t. K_Λ and is given by

$$\mathbb{L}_\Lambda\Phi \equiv \{\pounds_\Lambda - [\mathcal{S}_\Lambda - \tfrac{1}{2}(p\lambda_\Lambda + q\lambda^*_\Lambda)]\}\Phi\,, \tag{B.1.36}$$

where the $\mathcal{S}_\Lambda$ are $\mathfrak{sp}(2\bar{n})$ matrices that provide a representation of the Lie algebra of the gauge group G_V acting on the section Φ:

$$[\mathcal{S}_\Lambda, \mathcal{S}_\Sigma] = +f_{\Lambda\Sigma}{}^{\Omega}\mathcal{S}_\Omega\,. \tag{B.1.37}$$

The gauge covariant derivative acting on these sections is given by

$$\begin{aligned}\mathfrak{D}_\mu\Phi \;\;=\;\; & \{\nabla_\mu + \mathfrak{D}_\mu Z^i\Gamma_i + \mathfrak{D}_\mu Z^{*i^*}\Gamma_{i^*} + \tfrac{1}{2}(pk_\Lambda{}^i\partial_i\mathcal{K} + qk^*_\Lambda{}^{i^*}\partial_{i^*}\mathcal{K}) \\ & + gA^\Lambda{}_\mu[\mathcal{S}_\Lambda + \tfrac{i}{2}(p-q)\mathcal{P}_\Lambda - (\pounds_\Lambda - K_\Lambda)]\}\Phi\,.\end{aligned} \tag{B.1.38}$$

Invariant sections are those for which

$$\mathbb{L}_\Lambda\Phi = 0\,, \quad\Rightarrow\quad \pounds_\Lambda\Phi = [\mathcal{S}_\Lambda - \tfrac{1}{2}(p\lambda_\Lambda + q\lambda^*_\Lambda)]\Phi\,, \tag{B.1.39}$$

[6]Again, spacetime and target space tensor indices are not explicitly shown. Symplectic indices are not shown, either.

and their gauge covariant derivatives are, again, the covariant pullbacks of the Kähler-covariant derivatives:

$$\mathfrak{D}_\mu \Phi = \mathfrak{D}_\mu Z^i \mathcal{D}_i \Phi + \mathfrak{D}_\mu Z^{*i^*} \mathcal{D}_{i^*} \Phi \,. \tag{B.1.40}$$

The prime example of invariant field is the covariantly holomorphic section $\mathcal{L}(Z, Z^*)$ of the $N = 1, d = 4$ theories. This is a Kähler weight $(1, -1)$ section related to the holomorphic superpotential $W(Z)$ by

$$\mathcal{L}(Z, Z^*) \equiv W(Z) e^{\mathcal{K}/2} \,, \tag{B.1.41}$$

and its covariant holomorphicity follows from the holomprphicity of W:

$$\mathcal{D}_{i^*} \mathcal{L} = (\partial_{i^*} + i\mathcal{Q}_{i^*})\mathcal{L} = e^{\mathcal{K}/2} \partial_{i^*} (e^{-\mathcal{K}/2} \mathcal{L}) = e^{\mathcal{K}/2} \partial_{i^*} W = 0 \,. \tag{B.1.42}$$

In order for the global transformation Eq. (B.1.3) to be a symmetry of the full theory that we can gauge $\mathcal{L}$ must be an invariant section, that is

$$\mathbb{L}_\Lambda \mathcal{L} = \{\pounds_\Lambda + \tfrac{1}{2}(\lambda_\Lambda - \lambda^*_\Lambda)\}\mathcal{L} = 0 \,, \;\; \Rightarrow \;\; K_\Lambda \mathcal{L} = -\tfrac{1}{2}(\lambda_\Lambda - \lambda^*_\Lambda)\mathcal{L} \,. \tag{B.1.43}$$

Then, under gauge transformations it will transform according to

$$\delta_\alpha \mathcal{L} = -\tfrac{1}{2}\alpha^\Lambda(x)(\lambda_\Lambda - \lambda^*_\Lambda)\mathcal{L} \,, \tag{B.1.44}$$

and its covariant derivative will be given by

$$\mathfrak{D}_\mu \mathcal{L} = (\partial_\mu + i\hat{\mathcal{Q}}_\mu)\mathcal{L} = \mathfrak{D}_\mu Z^i \mathcal{D}_i \mathcal{L} \,, \tag{B.1.45}$$

where we have defined

$$\hat{\mathcal{Q}}_\mu \equiv \mathcal{Q}_\mu + gA^\Lambda{}_\mu \mathcal{P}_\Lambda \,. \tag{B.1.46}$$

Observe that this 1-form is, in general, different from the "covariant pullback" of the Kähler 1-form:

$$\tfrac{1}{2i}\mathfrak{D}_\mu Z^i \partial_i \mathcal{K} + \text{c.c.} \,. \tag{B.1.47}$$

The difference between this and the correct one is

$$\tfrac{1}{2i}\mathfrak{D}_\mu Z^i \partial_i \mathcal{K} + \text{c.c.} - \hat{\mathcal{Q}}_\mu = gA^\Lambda{}_\mu \Im\text{m}\lambda_\Lambda \,, \tag{B.1.48}$$

and only vanishes when the isometries that have been gauged leave the Kähler potential exactly invariant (i.e. $\lambda_\Lambda = 0$).

It should be evident that $\mathcal{D}_i \mathcal{L}$ is also an invariant field and, therefore the part of the $N = 1, d = 4$ supergravity potential that depends on the superpotential

$$-24|\mathcal{L}|^2 + 8\mathcal{G}^{ij^*}\mathcal{D}_i\mathcal{L}\mathcal{D}_{j^*}\mathcal{L}^* \,. \tag{B.1.49}$$

is automatically exactly invariant.

On the other hand Eq. (B.1.34) proves that the momentum map itself is an invariant field. Then,

$$\begin{aligned}
\delta_\alpha \mathcal{P}_\Lambda &= -\alpha^\Sigma(x) f_{\Sigma\Lambda}{}^\Omega \mathcal{P}_\Omega \,, \\
\mathfrak{D}_\mu \mathcal{P}_\Lambda &= \partial_\mu \mathcal{P}_\Lambda + g f_{\Lambda\Sigma}{}^\Omega A^\Sigma{}_\mu \mathcal{P}_\Omega \,, \\
\mathfrak{D}_\mu \mathcal{P}_\Lambda &= \mathfrak{D}_\mu Z^i \partial_i \mathcal{P}_\Lambda + \mathfrak{D}_\mu Z^{*i^*} \partial_{i^*} \mathcal{P}_\Lambda \,,
\end{aligned} \tag{B.1.50}$$

and the part of the $N = 1, d = 4$ supergravity potential that depends on it

$$+\tfrac{1}{2} g^2 (\Im\mathrm{m}\, f)^{-1|\Lambda\Sigma} \mathcal{P}_\Lambda \mathcal{P}_\Sigma \,, \tag{B.1.51}$$

is also automatically invariant.

Finally, let us consider the spinor of the theory. They are not invariant fields, as they do not depend only on te Z^i. They have a non-vanishing Kähler weight which is $(-1/2, 1/2)$ times their chirality. For instance, for the gravitino of the $N = 1, d = 4$ theories we have

$$\begin{aligned}
\delta_\alpha \psi_\mu &= -\tfrac{1}{4}\alpha^\Lambda(x)(\lambda_\Lambda - \lambda^*_\Lambda)\psi_\mu \,, \\
\mathfrak{D}_\mu \psi_\nu &= \{\nabla_\mu + \tfrac{i}{2}\hat{\mathcal{Q}}\}\psi_\nu \,.
\end{aligned} \tag{B.1.52}$$

B.2 Kähler weights of certain frequently used objects appearing in $N = 2$ $d = 4$ Supergravity

The Kähler weights $(q, \bar{q})$ of an object as defined in Eq. (B.0.15):

	ϵ_I	ϵ^I	$\bar{\epsilon}_I$	$\bar{\epsilon}^I$	λ^{Ii}	$\psi_{I\mu}$	ϵ	η
q	$1/2$	$-1/2$	$1/2$	$-1/2$	$-1/2$	$1/2$	$1/2$	$-1/2$
$\bar{q}$	$-1/2$	$1/2$	$-1/2$	$1/2$	$1/2$	$-1/2$	$-1/2$	$1/2$

Table B.2.1: Kähler weights of certain fermionic fields

	Z^i	F^Λ	G^{i+}	T^+	$\mathcal{V}$	$\mathcal{U}_i$	$\mathcal{T}^i{}_\Lambda$	$\mathcal{T}_\Lambda$	$\mathcal{N}_{\Lambda\Sigma}$	$\mathfrak{D}_i\mathcal{U}_j$	$\mathfrak{D}_{i^*}\mathcal{U}_j$	$\mathcal{C}_{ijk}$	Ω
q	0	0	-1	1	1	1	-1	1	0	1	1	2	2
$\bar{q}$	0	0	1	-1	-1	-1	1	-1	0	-1	-1	-2	0

Table B.2.2: Kähler weights of certain bosonic fields

Appendix C

Special Kähler geometry

In this appendix we shall discuss the geometric structure underlying the couplings of vector supermultiplets in $N = 2$ $d = 4$ supergravity, which has received the name of *special Kähler geometry*.

Having discussed the coordinate independent formulation of special geometry, we shall make contact to the original formulation of Lauwers and De Wit in appendix (C.1) by means of a function called the *prepotential*. Appendix (C.2) we shall discuss the topic of isometries in special geometry and how this is used in order to construct gauged supergravities. Finally, in appendices C.4) and (C.4) we shall discuss some specific examples of special geometries.

The formal definition of special geometry starts off as follows: consider a flat $2\bar{n}$-dimensional vector bundle $E \rightarrow \mathcal{M}$ with structure group $Sp(\bar{n};\mathbb{R})$, and take a section $\mathcal{V}$ of the product bundle $E \otimes L^1 \rightarrow \mathcal{M}$ and its complex conjugate $\overline{\mathcal{V}}$, which formally is a section of the bundle $E \otimes L^{-1} \rightarrow \mathcal{M}$. Then, a *special Kähler manifold*[1], is a bundle $E \otimes L^1 \rightarrow \mathcal{M}$, for which there exists a section $\mathcal{V}$ such that

$$\mathcal{V} = \left(\begin{array}{c} \mathcal{L}^\Lambda \\ \mathcal{M}_\Lambda \end{array} \right) \;\; \rightarrow \;\; \left\{ \begin{array}{lcl} \langle \mathcal{V} \mid \mathcal{V}^* \rangle & \equiv & \mathcal{L}^{*\,\Lambda}\mathcal{M}_\Lambda - \mathcal{L}^\Lambda \mathcal{M}^*_\Lambda \equiv -i\,, \\ \mathfrak{D}_{i^*}\mathcal{V} & = & (\partial_{i^*} + \tfrac{1}{2}\partial_{i^*}\mathcal{K})\mathcal{V} = 0\,, \\ \langle \mathfrak{D}_i \mathcal{V} \mid \mathcal{V} \rangle & = & 0\,. \end{array} \right. \tag{C.0.1}$$

If we then define

$$\mathcal{U}_i \equiv \mathfrak{D}_i \mathcal{V} = \left(\begin{array}{c} f^\Lambda{}_i \\ h_{\Lambda\, i} \end{array} \right)\,, \;\; \mathcal{U}^*{}_{i^*} = (\mathcal{U}_i)^*\,, \tag{C.0.2}$$

[1]Some basic references for this material are [123–125] and the review [83]. The definition of special Kähler manifold was made in Ref. [126], formalizing the original results of Ref. [80].

then it follows from the basic definitions that

$$\begin{aligned} \mathfrak{D}_{i^*}\,\mathcal{U}_i &= \mathcal{G}_{ii^*}\,\mathcal{V} & \langle\mathcal{U}_i \mid \mathcal{U}^*{}_{i^*}\rangle &= i\mathcal{G}_{ii^*}\,, \\ \langle\mathcal{U}_i \mid \mathcal{V}^*\rangle &= 0\,, & \langle\mathcal{U}_i \mid \mathcal{V}\rangle &= 0\,. \end{aligned} \tag{C.0.3}$$

Taking the covariant derivative of the last identity $\langle\mathcal{U}_i \mid \mathcal{V}\rangle = 0$ we find immediately that $\langle\mathfrak{D}_i\mathcal{U}_j \mid \mathcal{V}\rangle = -\langle\,\mathcal{U}_j \mid \mathcal{U}_i\rangle$. It can be shown that the r.h.s. of this equation is antisymmetric while the l.h.s. is symmetric, so that

$$\langle\mathfrak{D}_i\mathcal{U}_j \mid \mathcal{V}\rangle = \langle\mathcal{U}_j \mid \mathcal{U}_i\rangle = 0\,. \tag{C.0.4}$$

The importance of this last equation is that if we group together $\mathcal{E}_\Lambda = (\mathcal{V},\mathcal{U}_i)$, we can see that $\langle\mathcal{E}_\Sigma \mid \mathcal{E}^*{}_\Lambda\rangle$ is a non-degenerate matrix. This then allows us to construct an identity operator for the symplectic indices, such that for a given section of $\mathcal{A} \ni \Gamma\,(E,\mathcal{M})$ we have

$$\mathcal{A} = i\langle\mathcal{A} \mid \mathcal{V}^*\rangle\mathcal{V} - i\langle\mathcal{A} \mid \mathcal{V}\rangle\;\mathcal{V}^* + i\langle\mathcal{A} \mid \mathcal{U}_i\rangle\mathcal{G}^{ii^*}\;\mathcal{U}^*{}_{i^*} - i\langle\mathcal{A} \mid \mathcal{U}^*{}_{i^*}\rangle\mathcal{G}^{ii^*}\mathcal{U}_i\,. \tag{C.0.5}$$

Using $\{\mathcal{E}_\Sigma, \mathcal{E}^*{}_\Lambda\}$ as a basis for the space of symplectic sections we obtain the following completeness relation

$$i\mathbb{1} = -\mid\mathcal{V}^*\rangle\langle\mathcal{V}\mid + \mid\mathcal{V}\rangle\langle\mathcal{V}^*\mid - \mathcal{G}^{ii^*}\mid\mathfrak{D}_i\mathcal{V}\rangle\langle\mathfrak{D}_{i^*}\mathcal{V}^*\mid + \mathcal{G}^{ii^*}\mid\mathfrak{D}_{i^*}\mathcal{V}^*\rangle\langle\mathfrak{D}_i\mathcal{V}\mid\,. \tag{C.0.6}$$

As we have seen $\mathfrak{D}_i\mathcal{U}_j$ is symmetric in i and j, but what more can be said about it: as one can easily see, the inner product with $\mathcal{V}^*$ and $\mathcal{U}^*{}_{i^*}$ vanishes due to the basic properties. Let us then define the Kähler-weight 2 object

$$\mathcal{C}_{ijk} \equiv \langle\mathfrak{D}_i\;\mathcal{U}_j \mid \mathcal{U}_k\rangle \;\;\rightarrow\;\; \mathfrak{D}_i\;\mathcal{U}_j = i\mathcal{C}_{ijk}\mathcal{G}^{kl^*}\mathcal{U}^*{}_{l^*}\,, \tag{C.0.7}$$

where the last equation is a consequence of Eq. (C.0.5). Since the $\mathcal{U}$'s are orthogonal, however, one can see that $\mathcal{C}$ is completely symmetric in its 3 indices. Furthermore one can show that

$$\mathfrak{D}_{i^*}\;\mathcal{C}_{jkl} = 0\,, \qquad \mathfrak{D}_{[i}\;\mathcal{C}_{j]kl} = 0\,. \tag{C.0.8}$$

Observe that these equations imply the existence of a function $\mathcal{S}$, such that

$$\mathcal{C}_{ijk} = \mathfrak{D}_i\mathfrak{D}_j\mathfrak{D}_k\;\mathcal{S}\,. \tag{C.0.9}$$

The function $\mathcal{S}$ is given by [127]

$$\mathcal{S} \sim \mathcal{L}^\Lambda\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}\mathcal{L}^\Sigma\,, \tag{C.0.10}$$

where $\mathcal{N}$ is the period or monodromy matrix. This matrix is defined by the relations

$$\mathcal{M}_\Lambda = \mathcal{N}_{\Lambda\Sigma}\mathcal{L}^\Sigma\,, \qquad h_{\Lambda\, i} = \mathcal{N}^*{}_{\Lambda\Sigma} f^\Sigma{}_i\,. \tag{C.0.11}$$

The relation $\langle \mathcal{U}_i \mid \overline{\mathcal{V}}\rangle = 0$ then implies that $\mathcal{N}$ is symmetric, which then also trivializes $\langle \mathcal{U}_i \mid \mathcal{U}_j\rangle = 0$.

From the properties, Eqs. (C.0.1), one concludes that $\mathcal{V}$ transforms under Kähler transformations as

$$\mathcal{V} \to e^{-\frac{1}{2}(\lambda-\lambda^*)}\mathcal{V}\,. \tag{C.0.12}$$

From the other basic properties in (C.0.3) we find

$$\mathcal{L}^\Lambda \Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}\mathcal{L}^{*\,\Sigma} \;=\; -\tfrac{1}{2}\,, \tag{C.0.13}$$

$$\mathcal{L}^\Lambda \Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma} f^\Sigma{}_i \;=\; \mathcal{L}^\Lambda \Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma} f^{*\,\Sigma}{}_{i^*} = 0\,, \tag{C.0.14}$$

$$f^\Lambda{}_i\, \Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma} f^{*\,\Sigma}{}_{i^*} \;=\; -\tfrac{1}{2}\mathcal{G}_{ii^*}\,. \tag{C.0.15}$$

Further identities that can be derived are

$$(\partial_i \mathcal{N}_{\Lambda\Sigma})\mathcal{L}^\Sigma \;=\; -2i\Im\mathrm{m}(\mathcal{N})_{\Lambda\Sigma}\; f^\Sigma{}_i\,, \tag{C.0.16}$$

$$\partial_i \mathcal{N}^*{}_{\Lambda\Sigma}\; f^\Sigma{}_j \;=\; -2\mathcal{C}_{ijk}\mathcal{G}^{kk^*}\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma} f^{*\,\Sigma}{}_{k^*}\,, \tag{C.0.17}$$

$$\mathcal{C}_{ijk} \;=\; f^\Lambda{}_i f^\Sigma{}_j \partial_k \mathcal{N}^*_{\Lambda\Sigma}\,, \tag{C.0.18}$$

$$\mathcal{L}^\Sigma \partial_{i^*}\mathcal{N}_{\Lambda\Sigma} \;=\; 0\,, \tag{C.0.19}$$

$$\partial_{i^*}\mathcal{N}^*{}_{\Lambda\Sigma}\; f^\Sigma{}_i \;=\; 2i\mathcal{G}_{ii^*}\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}\mathcal{L}^\Sigma\,. \tag{C.0.20}$$

An important identity one can derive, and that will be used various times in the main text, is given by

$$U^{\Lambda\Sigma} \equiv f^\Lambda{}_i \mathcal{G}^{ii^*} f^{*\,\Sigma}{}_{i^*} = -\tfrac{1}{2}\Im\mathrm{m}(\mathcal{N})^{-1|\Lambda\Sigma} - \mathcal{L}^{*\,\Lambda}\mathcal{L}^\Sigma\,, \tag{C.0.21}$$

whence $(U^{\Lambda\Sigma})^* = U^{\Sigma\Lambda}$.

We can define the graviphoton and matter vector projectors

$$\mathcal{T}_\Lambda \equiv 2i\mathcal{L}_\Lambda = 2i\mathcal{L}^\Sigma \Im\mathrm{m}\,\mathcal{N}_{\Sigma\Lambda}\,, \tag{C.0.22}$$

$$\mathcal{T}^i{}_\Lambda \equiv -f^*{}_\Lambda{}^i = -\mathcal{G}^{ij^*} f^*{}^\Sigma{}_{j^*} \Im\mathrm{m}\,\mathcal{N}_{\Sigma\Lambda}\,. \tag{C.0.23}$$

Using these definitions and the above properties one can show the following identities for the derivatives of the period matrix:

$$\begin{aligned} \partial_i \mathcal{N}_{\Lambda\Sigma} &= 4\mathcal{T}_{i(\Lambda}\mathcal{T}_{\Sigma)}\,, \\ \partial_{i^*}\mathcal{N}_{\Lambda\Sigma} &= 4\mathcal{C}^*{}_{i^*j^*k^*}\mathcal{T}^{i^*}{}_{(\Lambda}\mathcal{T}^{j^*}{}_{\Sigma)}\,. \end{aligned} \tag{C.0.24}$$

For further details and identities, the interested reader can consult the basic references [82,123–125], the review [83] or Ref. [26,38] whose conventions and results we follow.

C.1 Prepotential: Existence and more formulae

Let us start by introducing the explicitly holomorphic section $\Omega = e^{-\mathcal{K}/2}\mathcal{V}$, which allows us to rewrite the system Eqs. (C.0.1) as

$$\Omega = \begin{pmatrix} \mathcal{X}^\Lambda \\ \mathcal{F}_\Sigma \end{pmatrix} \rightarrow \begin{cases} \langle \Omega \mid \Omega^* \rangle \equiv \mathcal{X}^{*\Lambda}\mathcal{F}_\Lambda - \mathcal{X}^\Lambda \mathcal{F}^*_\Lambda = -i\, e^{-\mathcal{K}}\,, \\ \partial_{i^*}\Omega = 0\,, \\ \langle \partial_i \Omega \mid \Omega \rangle = 0\,. \end{cases} \tag{C.1.1}$$

Observe that the first of Eqs. (C.1.1) together with the definition of the period matrix $\mathcal{N}$ imply the following expression for the Kähler potential:

$$e^{-\mathcal{K}} = -2\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}\mathcal{X}^\Lambda \mathcal{X}^{*\Sigma}\,. \tag{C.1.2}$$

If we now assume that $\mathcal{F}_\Lambda$ depends on Z^i through the $\mathcal{X}$'s, then from the last equation we can derive that

$$\partial_i \mathcal{X}^\Lambda \left[2\mathcal{F}_\Lambda - \partial_\Lambda \left(\mathcal{X}^\Sigma \mathcal{F}_\Sigma\right)\right] = 0\,. \tag{C.1.3}$$

If $\partial_i\mathcal{X}^\Lambda$ is invertible as an $n \times \bar{n}$ matrix, then we must conclude that

$$\mathcal{F}_\Lambda = \partial_\Lambda \mathcal{F}(\mathcal{X})\,, \tag{C.1.4}$$

where $\mathcal{F}$ is a homogeneous function of degree 2, called the *prepotential*.

Making use of the prepotential and the definitions (C.0.11), we can calculate

$$\mathcal{N}_{\Lambda\Sigma} = \mathcal{F}^*_{\Lambda\Sigma} + 2i\frac{\Im\mathrm{m}\mathcal{F}_{\Lambda\Lambda'}\mathcal{X}^{\Lambda'}\Im\mathrm{m}\mathcal{F}_{\Sigma\Sigma'}\mathcal{X}^{\Sigma'}}{\mathcal{X}^{\Omega}\Im\mathrm{m}\mathcal{F}_{\Omega\Omega'}\mathcal{X}^{\Omega'}}\,. \tag{C.1.5}$$

Having the explicit form of $\mathcal{N}$, we can also derive an explicit representation for $\mathcal{C}$ by applying Eq. (C.0.19). One finds

$$\mathcal{C}_{ijk} = e^{\mathcal{K}}\partial_i\mathcal{X}^{\Lambda}\partial_j\mathcal{X}^{\Sigma}\partial_k\mathcal{X}^{\Omega}\mathcal{F}_{\Lambda\Sigma\Omega}\,, \tag{C.1.6}$$

so that the prepotential really determines all structures in special geometry.

A last remark has to be made about the existence of a prepotential: clearly, given a holomorphic section Ω a prepotential need not exist. It was shown in Ref. [125], however, that one can always apply an $Sp(\bar{n},\mathbb{R})$ transformation such that a prepotential exists. Clearly the $N=2$ SUGRA action is not invariant under the full $Sp(\bar{n},\mathbb{R})$, but the equations of motion and the supersymmetry equations are. This means that for the purpose of this article we can always, even if this is not done, impose the existence of a prepotential.

C.2 Gauging holomorphic isometries of special Kähler manifolds

By hypothesis (preservation of the special Kähler structure), the canonical weight $(1,-1)$ section $\mathcal{V}$ is an invariant section

$$K_{\Lambda}\mathcal{V} = [\mathcal{S}_{\Lambda} - \tfrac{1}{2}(\lambda_{\Lambda} - \lambda^*_{\Lambda})]\mathcal{V}\,, \tag{C.2.1}$$

and its gauge covariant derivative is given by

$$\mathfrak{D}_{\mu}\mathcal{V} = \mathfrak{D}_{\mu}Z^i\mathcal{D}_i\mathcal{V} = \mathfrak{D}_{\mu}Z^i\mathcal{U}_i\,. \tag{C.2.2}$$

Using the covariant holomorphicity of $\mathcal{V}$ one can write

$$K_{\Lambda}\mathcal{V} = k_{\Lambda}{}^i\mathcal{U}_i - i\mathcal{P}_{\Lambda}\mathcal{V} - \tfrac{1}{2}(\lambda_{\Lambda} - \lambda^*_{\Lambda})\mathcal{V}\,. \tag{C.2.3}$$

Comparing with Eq. (C.2.1) we get

$$k_{\Lambda}{}^i\mathcal{U}_i(\mathcal{S}_{\Lambda} + i\mathcal{P}_{\Lambda})\mathcal{V}\,, \tag{C.2.4}$$

and taking the symplectic product with $\mathcal{V}^*$, we find another expression for the momentum map

$$\mathcal{P}_{\Lambda} = \langle\, \mathcal{V}^* \mid \mathcal{S}_{\Lambda}\mathcal{V}\,\rangle\,, \tag{C.2.5}$$

which leads, via Eq. (B.1.33) to another expression for the Killing vectors

$$k_\Lambda{}^i = i\partial^i \mathcal{P}_\Lambda = i\langle\, \mathcal{V} \mid \mathcal{S}_\Lambda \mathcal{U}^{*i}\,\rangle\,. \tag{C.2.6}$$

If we take the symplectic product with $\mathcal{V}$ instead, we get the following condition

$$\langle\, \mathcal{V} \mid \mathcal{S}_\Lambda \mathcal{V}\,\rangle = 0\,. \tag{C.2.7}$$

Using the same identity and $\mathcal{G}_{ij^*} = -i\langle \mathcal{U}_i \mid \mathcal{U}^*_{j^*}\,\rangle$ one can also show that

$$k_\Lambda{}^i k^*_\Sigma{}^{j^*} \mathcal{G}_{ij^*} = \mathcal{P}_\Lambda \mathcal{P}_\Sigma - i\langle\, \mathcal{S}_\Lambda \mathcal{V} \mid \mathcal{S}_\Sigma \mathcal{V}^*\,\rangle\,. \tag{C.2.8}$$

It follows that

$$\langle\, \mathcal{S}_{[\Lambda} \mathcal{V} \mid \mathcal{S}_{\Sigma]} \mathcal{V}^*\,\rangle = -\tfrac{1}{2} f_{\Lambda\Sigma}{}^\Omega \mathcal{P}_\Omega. \tag{C.2.9}$$

The gauge covariant derivative of $\mathcal{U}_i$ is

$$\mathfrak{D}_\mu \mathcal{U}_i = \mathfrak{D}_\mu Z^j \mathcal{D}_j \mathcal{U}_i + \mathfrak{D}_\mu Z^{*j^*} \mathcal{D}_{j^*} \mathcal{U}_i = i\mathcal{C}_{ijk} \mathcal{U}^{*j} \mathfrak{D}_\mu Z^k + \mathcal{G}_{ij^*} \mathcal{V} \mathfrak{D}_\mu Z^{*j^*}\,. \tag{C.2.10}$$

On the supersymmetry parameters ϵ_I, which have $(1/2,-1/2)$ weight

$$\mathfrak{D}_\mu \epsilon_I = \left\{ \nabla_\mu + \tfrac{i}{2} \hat{\mathcal{Q}}_\mu \right\} \epsilon_I\,, \tag{C.2.11}$$

where $\hat{\mathcal{Q}}$ is defined in Eq. (B.1.46).

The formalism developed thus far, applies to any group G_V of isometries. However, we will restrict ourselves to those for which the matrices

$$\mathcal{S}_\Lambda = \begin{pmatrix} a_\Lambda{}^\Omega{}_\Sigma & b_\Lambda{}^{\Omega\Sigma} \\ & \\ c_{\Lambda\Omega\Sigma} & d_{\Lambda\Omega}{}^\Sigma \end{pmatrix}, \tag{C.2.12}$$

have $b = c = 0$. The symplectic transformations with $b \neq 0$ are not symmetries of the action and the gauging of symmetries with $c \neq 0$ leads to the presence of complicated Chern-Simons terms in the action. The matrices a and d are

$$a_\Lambda{}^\Omega{}_\Sigma = f_{\Lambda\Sigma}{}^\Omega\,, \qquad d_{\Lambda\Omega}{}^\Sigma = -f_{\Lambda\Omega}{}^\Sigma\,. \tag{C.2.13}$$

These restrictions lead to additional identities. First, observe that the condition Eq. (C.2.7) takes the form

$$f_{\Lambda\Sigma}{}^\Omega \mathcal{L}^\Sigma \mathcal{M}_\Omega = 0\,, \tag{C.2.14}$$

and the covariant derivative of Eq. (C.2.7) $\langle\, \mathcal{V} \mid \mathcal{S}_\Lambda \mathcal{U}_i\,\rangle = 0$

$$f_{\Lambda\Sigma}{}^\Omega (f^\Sigma{}_i \mathcal{M}_\Omega + h_{\Omega\, i} \mathcal{L}^\Sigma) = 0\,. \tag{C.2.15}$$

Then, using Eqs. (C.2.5) and (C.2.6) and Eqs. (C.2.7),(C.2.14) and (C.2.15) we find that

$$\mathcal{L}^{\Lambda}\mathcal{P}_{\Lambda} \;=\; 0\,, \tag{C.2.16}$$

$$\mathcal{L}^{\Lambda}k_{\Lambda}{}^{i} \;=\; 0\,, \tag{C.2.17}$$

$$\mathcal{L}^{*\Lambda}k_{\Lambda}{}^{i} \;=\; -if^{*\Lambda\,i}\mathcal{P}_{\Lambda}\,. \tag{C.2.18}$$

From the first two equations it follows that

$$\mathcal{L}^{\Lambda}\lambda_{\Lambda} = 0\,. \tag{C.2.19}$$

Some further equations that can be derived and are extensively used in the calculation throughout the text are explicit versions of Eqs. (C.2.5) and (C.2.6), *i.e.*

$$\mathcal{P}_{\Lambda} \;=\; 2f_{\Lambda\Sigma}{}^{\Gamma}\Re\mathrm{e}\left(\mathcal{L}^{\Sigma}\mathcal{M}^{*}_{\Gamma}\right)\,, \qquad k_{\Lambda\,i^{*}} \;=\; if_{\Lambda\Sigma}{}^{\Gamma}\left(f^{*\Sigma}_{i^{*}}M_{\Gamma} + \mathcal{L}^{\Sigma}h^{*}_{\Gamma i^{*}}\right)\,. \tag{C.2.20}$$

Finally, notice the identity

$$k_{\Lambda\,i^{*}}\mathfrak{D}Z^{*i^{*}} - k^{*}_{\Lambda i}\mathfrak{D}Z^{i} = i\mathfrak{D}\mathcal{P}_{\Lambda} = i(d\mathcal{P}_{\Lambda} + f_{\Lambda\Sigma}{}^{\Omega}A^{\Sigma}\mathcal{P}_{\Omega})\,. \tag{C.2.21}$$

The absolutely last comment in this appendix is the following: if we start from the existence of a prepotential $\mathcal{F}(\mathcal{X})$, then Eq. (C.2.7) implies

$$0 \;=\; f_{\Lambda\Sigma}{}^{\Gamma}\;\mathcal{X}^{\Sigma}\partial_{\Gamma}\;\mathcal{F}\;, \tag{C.2.22}$$

the meaning of which is that one can gauge only the invariances of the prepotential. To put it differently: if you want to construct a model having $\mathfrak{g}$ as the gauge algebra, you need to pick a prepotential that is $\mathfrak{g}$-invariant.

C.3 Some examples of quadratic prepotentials

In this subsection we are going to discuss some special geometries that appear in the main text.

The minimal special Kähler manifold

The minimal special Kähler manifold is not really a manifold as its main aim is to reduce the general framework of vector coupled $N = 2$ $d = 4$ sugra to the minimal version comprising only of the gravity supermultiplet; that is to say that there are no scalars, whence no Kähler space.

Having said this, consider the simple prepotential[2]

$$\mathcal{F} \;=\; -\frac{\alpha}{4}\;(\mathcal{X})^2 \qquad\quad (\alpha \in \mathbb{C}/0)\;. \tag{C.3.1}$$

As there are no scalars in this setting, we take the corresponding Kähler potential to vanish, *i.e.* $\mathcal{K} = 0$, so that the normalisation condition in Eq. (C.1.1) together with the usual moduli fixing $\mathcal{X} = 1$ leads to

$$\mathrm{Im}(\alpha) \;=\; 1\;. \tag{C.3.2}$$

As we are dealing with a model having a prepotential, we can calculate the 1×1-matrix $\mathcal{N}$ using Eq. (C.1.5), which leads to

$$\mathcal{N} \;=\; -\tfrac{\alpha}{2} \qquad\quad \longrightarrow \qquad \mathrm{Im}\,(\mathcal{N}) \;=\; -\tfrac{1}{2}\;, \tag{C.3.3}$$

so that as announced $\mathrm{Im}\,(\mathcal{N})$ is a negative definite matrix. As one can see from Eq. (2.2.1), the real part of α corresponds to a θ-term for the maxwell field; since this is a surface term we can put $\mathrm{Re}(\alpha) = 0$ at the cost of losing manifest EM-duality in the action. The equations of motion are however invariant under EM-duality transformations.

Plugging the above 'geometry', together with vanishing hyperscalars, into the action (2.2.1) we obtain the, up normalisation, the standard Einstein-Maxwell action

$$S \;=\; \int d^4x\;\sqrt{|g|}\,\left[R \;-\; F^2\right]\;, \tag{C.3.4}$$

which is invariant under the following supersymmetry transformations

$$\delta_\epsilon \Psi_{\mu I} \quad=\quad \nabla_\mu \epsilon_I \;+\; \tfrac{i}{4}\;\not{F}\;\gamma_\mu \varepsilon_{IJ}\epsilon^J\;, \tag{C.3.5}$$

$$\delta_\epsilon e_\mu{}^a \quad=\quad \tfrac{1}{2i}\;\mathrm{Re}\left(\;\overline{\psi}{}^I_\mu \gamma^a \epsilon_I\;\right)\;. \tag{C.3.6}$$

The $\overline{\mathbb{CP}}^n$ models

The $\overline{\mathbb{CP}}^n$ models are special in that the scalar manifold is a homogeneous space $SU(1,n)/U(n) \sim \overline{\mathbb{CP}}^n$, which is a non-compact version of $\mathbb{CP}^n = SU(n+1)/U(n)$. It is defined by a specific quadratic prepotential, namely

$$\mathcal{F} \;=\; \frac{1}{4i}\;\mathcal{X}^T\eta\mathcal{X} \qquad \text{with} \qquad \eta \;=\; \mathrm{diag}\,(+\;,\;[-]^n)\;. \tag{C.3.7}$$

Using the choice $\mathcal{X}^0 = 1$ and $\mathcal{X}^i = Z^i$ $(i = 1,\ldots,n)$, we find that the Kähler potential is given by

$$e^{-\mathcal{K}} \;=\; 1 \;-\; |Z|^2\;, \tag{C.3.8}$$

[2] As there is only one symplectic coordinate, namely $\mathcal{X}^0$, we shall not write its symplectic index and just put $\mathcal{X}^0 = \mathcal{X}$.

which not only implies that $0 \leq |Z|^2 \leq 1$, but also that the Kähler metric is the 'standard' Fubini-Study metric

$$\mathcal{G}_{i\bar{\jmath}} \;=\; \frac{\delta_{i\bar{\jmath}}}{1-|Z|^2} + \frac{Z^j\overline{Z}^{\bar{\imath}}}{(1-|Z|^2)^2} \quad\longrightarrow\quad \mathcal{G}^{i\bar{\jmath}} \;=\; \left(1-|Z|^2\right)\,\left[\delta^{i\bar{\jmath}} \;-\; \overline{Z}^{\bar{\imath}}\; Z^j\right] \;. \tag{C.3.9}$$

Also, introducing the notations $\mathcal{X}_\Lambda \equiv \eta_{\Lambda\Sigma}\mathcal{X}^\Sigma$ and $\mathcal{X}\cdot\mathcal{X} = \mathcal{X}_\Lambda\mathcal{X}^\Lambda$, we can express the monodromy matrix as

$$\mathcal{N}_{\Lambda\Sigma} \;=\; \frac{i}{2}\left(\,\eta_{\Lambda\Sigma} \;-\; 2\,\frac{\mathcal{X}_\lambda\mathcal{X}_\Sigma}{\mathcal{X}\cdot\mathcal{X}}\,\right)\;. \tag{C.3.10}$$

The imaginary part of the monodromy matrix then satisfies

$$\mathrm{Im}\,(\mathcal{N})_{\Lambda\Sigma} \;=\; \tfrac{1}{2}\left(\eta_{\Lambda\Sigma} \;-\; \frac{\mathcal{X}_\Lambda\mathcal{X}_\Sigma}{\mathcal{X}\cdot\mathcal{X}} \;-\; \frac{\overline{\mathcal{X}}_\Lambda\overline{\mathcal{X}}_\Sigma}{\overline{\mathcal{X}}\cdot\overline{\mathcal{X}}}\right)\,, \tag{C.3.11}$$

$$\mathrm{Im}\,(\mathcal{N})^{-1|\;\Lambda\Sigma} \;=\; 2\,\left(\,\eta^{\Lambda\Sigma} \;-\; \frac{\mathcal{X}^\Lambda\overline{\mathcal{X}}^\Sigma \;+\; \overline{\mathcal{X}}^\Lambda\mathcal{X}^\Sigma}{\mathcal{X}\cdot\overline{\mathcal{X}}}\,\right)\;. \tag{C.3.12}$$

Since we are dealing with a quadratic prepotential, the Yukawa couplings ($\mathcal{C}_{ijk}$) vanish identically.

The explicit solution to the stabilisation equation reads

$$\left.\begin{array}{rcl} \mathcal{R}^\Lambda & = & -2\eta^{\Lambda\Sigma}\,\mathcal{I}_\Sigma \\ \mathcal{R}_\Lambda & = & \tfrac{1}{2}\eta_{\Lambda\Sigma}\,\mathcal{I}^\Sigma \end{array}\right\} \;\rightarrow\; \frac{1}{2|X|^2} \;=\; \tfrac{1}{2}\,\eta_{\Lambda\Sigma}\,\mathcal{I}^\Lambda\mathcal{I}^\Sigma \;-\; 2\,\eta^{\Lambda\Sigma}\,\mathcal{I}_\Lambda\mathcal{I}_\Sigma\;. \tag{C.3.13}$$

Cobining Eq. (C.2.22) with Eq. (C.3.7), we see that in the $\overline{\mathbb{CP}}^n$ models we can gauge an arbitrary $\overline{n} = n+1$ dimensional subgroup of $SO(1,n)$.

C.4 The $\mathcal{ST}[2,n]$ models

The $\mathcal{ST}[2,n]$ models have as their Kähler geometry the homogeneous space $\frac{SU(1,1)}{U(1)}\times\frac{SO(2,n)}{SO(2)\otimes SO(n)}$, which is of complex-dimension $n{+}1$, and must therefore be embedded into $Sp(n{+}1;\mathbb{R})$. As we are mainly interested in the solution to the stabilization equations, which for this model were solved in Ref. [128], and also in the gaugeability of the model, it is convenient to start with the parametrization of the symplectic section for which no prepotential exists. One advantage of this parametrization is that the $SO(2,n)$ symmetry is obvious as one can see from

$$\mathcal{V}^T \;=\; \left(\mathcal{L}^\Lambda\;,\;\eta_{\Lambda\Sigma}\;\mathrm{S}\mathcal{L}^\Sigma\right) \;\;\text{where}\;\; \eta = \mathrm{diag}\left([+]^2,[-]^n\right) \;\;\text{and}\;\; \mathcal{L}^T\eta\mathcal{L} \;=\; 0\,, \tag{C.4.1}$$

where the constraint is necessary to ensure the correct number of degrees of freedom. Also, and for want of a better place to say so, we take the symplectic indices to run over $\Lambda = (\underline{1}, 0, \ldots, n)$.

In order to declutter the solution to the stabilization equation $\mathcal{I} = \Im\mathrm{m}\,(\mathcal{V}/X)$, we absorb the X into the $\mathcal{L}$ and introduce the abbreviations $p^{\Lambda} = \mathcal{I}^{\Lambda}$ and $q_{\Lambda} = \mathcal{I}_{\Lambda}$. If we then also use η to raise and lower the indices, we can write the stabilization equation as

$$2i\, p^{\Lambda} \;=\; \mathcal{L}^{\Lambda} - \mathcal{L}^{*\Lambda} \;\;,\;\; 2i\, q^{\Lambda} \;=\; \mathrm{S}\,\mathcal{L}^{\Lambda} - \mathrm{S}^{*}\,\mathcal{L}^{*\Lambda} \;\;\longrightarrow\;\; \mathcal{L}^{\Lambda} \;=\; \frac{q^{\Lambda} \;-\; \mathrm{S}^{*}\, p^{\Lambda}}{\Im\mathrm{m}\,\mathrm{S}}\;. \tag{C.4.2}$$

The function S is then easily found by solving the constraint $\mathcal{L}_{\Lambda}\mathcal{L}^{\Lambda} = 0$, and gives

$$\mathrm{S} \;=\; \frac{p\cdot q}{p^2} \;-i\; \frac{\sqrt{\,p^2q^2 \;-\; (p\cdot q)^2\,}}{p^2}\;, \tag{C.4.3}$$

so that we have the constraint $p^2q^2 > (p\cdot q)^2$; the sign of $\Im\mathrm{m}\,\mathrm{S}$ is fixed by the positivity of the metrical function, which with the above sign reads

$$\frac{1}{2|X|^2} \;=\; 2\sqrt{\,p^2q^2 \;-\; (p\cdot q)^2\,}\;. \tag{C.4.4}$$

We would like to stress that this solution is manifestly $SO(2,n)$ (co/in)variant and automatically solves the constraint $\mathcal{L}^{T}\eta\mathcal{L} = 0$, without any constraints on p^{Λ} nor on q_{Λ}.

For our applications, namely the regularity of the embeddings of monopoles and the attractor mechanism, it is important to to know the expression of the moduli in terms of $(n+1)$ unconstrained fields, one of which should be S as it corresponds to the axidilaton. This means that we should have n unconstrained fields Z^a ($a = 0, 1, \ldots, n-1$) and express them in terms of p's and q's.

One way of doing this is through the introduction of so-called Calabi-Visentini coordinates which means that ($a = 1, \ldots, n$)

$$\mathcal{L}^{\underline{1}} \;=\; \tfrac{1}{2}\, Y^0 \,\left(1 + \vec{Z}^2\right)\;,\; \mathcal{L}^{0} \;=\; \tfrac{i}{2}\, Y^0\, \left(\vec{Z}^2 - 1\right)\;,\; \mathcal{L}^{a} \;=\; Y^0\, Z^a\;, \tag{C.4.5}$$

which after solving for Y^0 means that the scalar fields are given by

$$Z^a \;=\; \frac{q^a \;-\; \mathrm{S}^{*}\, p^a}{q^{\underline{1}} + iq^0 \;-\; \mathrm{S}^{*}\,(p^{\underline{1}} + ip^0)}\;, \tag{C.4.6}$$

and S is given by expression (C.4.3). Observe that in this parametrization the $SO(n)$ invariance is manifest.

In order to discuss the possible groups that can be gauged in these models, let us recall that a given compact simple Lie algebra $\mathfrak{g}$ of a group G is a subalgebra of

$\mathfrak{so}(\dim(\mathfrak{g}))$ and furthermore the latter's vector representation branches into $\mathfrak{g}$'s adjoint representation. This then implies that in an $\mathcal{ST}[2,n]$-model one can always gauge a group G as long as $n \geq \dim(\mathfrak{g})$.

In Section 5.2.3 the explicit details are given for the $\overline{\mathbb{CP}}^n$ models, but at least as far as the embedding of the monopoles are concerned, the embedding into the $\mathcal{ST}$-models is similar. In order to show that this is the case, consider the case of a purely magnetic solution, so that $q^a = 0$, and take furthermore $q_0 = p^{\underline{1}} = 0$ and normalize $q_{\underline{1}} = 1$. Using this Ansatz in Eq. (C.4.4) we obtain

$$\frac{1}{2|X|^2} \;=\; 2\sqrt{\;p^2\;} \;=\; 2\sqrt{\;(p^0)^2 \;-\; (p^a)^2\;}\;, \tag{C.4.7}$$

which, apart from the $\sqrt{\;}$, is just the same expression as obtained in the $\overline{\mathbb{CP}}^n$-models and leads to the same conditions for the global regularity of the metric. Using the same Ansatz in Eq. (C.4.6) for the scalars, one finds

$$Z^a \;=\; -i\frac{\sqrt{p^2}}{p^2 \;+\; p^0\sqrt{p^2}}\; p^a\;. \tag{C.4.8}$$

This then means that as long as $p^0 > 0$ and p^2 is regular and positive definite, as is the case for the solutions in section (5.2.3), the embeddings of the monopoles is a globally regular supergravity solution.

Appendix D

Quaternionic Kähler geometry

A *quaternionic Kähler manifold* is, to start with, a real $4m$-dimensional Riemannian manifold HM endowed with a triplet of complex structures $\mathsf{J}^x : T(\mathsf{HM}) \to T(\mathsf{HM})$, $(x = 1, 2, 3)$ that satisfy the quaternionic algebra

$$\mathsf{J}^x \mathsf{J}^y = -\delta^{xy} + \varepsilon^{xyz} \mathsf{J}^z , \tag{D.0.1}$$

and with respect to which the Riemannian metric, denoted by H, is Hermitean:

$$\mathsf{H}(\ \mathsf{J}^x X,\ \mathsf{J}^x Y\ ) = \mathsf{H}(X, Y) , \qquad \forall X, Y \in T(\mathsf{HM}),\ \ x = 1, 2, 3 . \tag{D.0.2}$$

This implies the existence of a triplet of 2-forms $\mathsf{K}^x(X, Y) \equiv \mathsf{H}(X,\ \mathsf{J}^x Y)$ globally known as the $\mathfrak{su}(2)$-valued *hyperKähler 2-forms*, with components $\mathsf{K}^x{}_{uv} = \mathsf{J}^x{}_{uv} = \mathsf{H}_{uw} \mathsf{J}^{x\,w}{}_v$.

The structure of quaternionic Kähler manifold also requires an $SU(2)$ bundle to be constructed over HM with connection 1-form A^x with respect to which the hyperKähler 2-form is covariantly constant[1], i.e.

$$\mathsf{D}_u \mathsf{K}^x{}_{vw} \ \equiv\ \nabla_u \mathsf{K}^x{}_{vw} + \varepsilon^{xyz}\ \mathsf{A}^y{}_u \mathsf{K}^z{}_{vw} \ =\ 0 , \tag{D.0.3}$$

where ∇_u is the standard, torsionless, Riemannian covariant derivative in HM.

Then, depending on whether the curvature of this bundle

$$\mathsf{DDK}^x = \varepsilon^{xyz} \mathsf{F}^y \wedge \mathsf{K}^z , \qquad \mathsf{F}^x \ \equiv\ d\mathsf{A}^x \ +\ \tfrac{1}{2} \varepsilon^{xyz}\ \mathsf{A}^y \wedge \mathsf{A}^z , \tag{D.0.4}$$

is zero or proportional to the hyperKähler 2-form

[1] Not just covariantly closed.

$$\mathsf{F}^x \;=\; \varkappa\, \mathsf{K}^x\,, \qquad \varkappa \in \mathbb{R}_{/\{0\}}\,, \tag{D.0.5}$$

the manifold is a *hyperKähler manifold* or a *quaternionic Kähler manifold*, respectively.

The $SU(2)$ connection acts on objects with vectorial $SU(2)$ indices, such as the chiral spinors in this article, as follows:[2]

$$\begin{aligned} \mathsf{D}\xi_I &\equiv d\xi_I + \mathsf{A}_I{}^J\xi_J \quad , \quad \mathsf{F}_I{}^J = d\mathsf{A}_I{}^J + \mathsf{A}_I{}^K \wedge \mathsf{A}_K{}^J\,, \\ \mathsf{D}\chi^I &\equiv d\chi^I + \mathsf{B}^I{}_J\chi^J \quad , \quad \mathsf{G}^I{}_J = d\mathsf{B}^I{}_J + \mathsf{B}^I{}_K \wedge \mathsf{B}^K{}_J\,. \end{aligned} \tag{D.0.7}$$

Consistency with the raising and lowering of vector $SU(2)$ indices by means of the εs, as specified in footnote (2), then implies that

$$\mathsf{B}^I{}_J \;=\; -\mathsf{A}^I{}_J \;\equiv\; -\varepsilon^{IK}\; \mathsf{A}_K{}^L\; \varepsilon_{LJ}\,, \tag{D.0.8}$$

whereas compatibility with the raising of indices due to complex conjugation implies

$$\mathsf{B}^I{}_J = (\mathsf{A}_I{}^J)^*\,. \tag{D.0.9}$$

Taking these two things together, means that $\mathsf{A}_I{}^J$ is an anti-Hermitean matrix whence we expand

$$\mathsf{A}_I{}^J \;=\; \tfrac{i}{2}\, \mathsf{A}^x\, (\sigma^x)_I{}^J \quad \text{and} \quad \mathsf{B}^I{}_J \;=\; -\tfrac{i}{2}\, \mathsf{A}^x\, (\sigma^x)^I{}_J\,, \tag{D.0.10}$$

where for the σ-matrices the indices are raised and lowered with ε. At this point, there remains a question about the normalisation of the Pauli matrices, which is readily fixed by imposing that

$$\mathsf{F}_I{}^J \;=\; \tfrac{i}{2}\, \mathsf{F}^x\, (\sigma^x)_I{}^J\,, \tag{D.0.11}$$

which means that

$$(\sigma^x\sigma^y)_I{}^J \;=\; \delta^{xy}\; \delta_I{}^J \;-\; i\varepsilon^{xyz}\; (\sigma^z)_I{}^J\,. \tag{D.0.12}$$

It is convenient to use a Vielbein on HM having as "flat" indices a pair (αI) consisting of one $SU(2)$-index I and one $Sp(m)$-index $\alpha = 1, \cdots, 2m$

$$\mathsf{U}^{\alpha I} \;=\; \mathsf{U}^{\alpha I}{}_u\; dq^u\,, \tag{D.0.13}$$

where $u \;=\; 1, \ldots, 4m$ and from now on we shall refer to this object as the Quadbein. This Quadbein is related to the metric H_{uv} by

[2] On objects with adjoint $SU(2)$ indices, such as the hyperKähler structure, it is defined above. Furthermore, we adopt the following convenion for raising and lowering vector $SU(2)$ indices:

$$\chi^I = \chi_J\varepsilon^{JI}\,, \qquad \xi_I = \varepsilon_{IJ}\xi^J\,. \tag{D.0.6}$$

$$\mathsf{H}_{uv} \;=\; \mathsf{U}^{\alpha I}{}_{u}\; \mathsf{U}^{\beta J}{}_{v}\; \varepsilon_{IJ}\mathbb{C}_{\alpha\beta}\,, \tag{D.0.14}$$

where $\mathbb{C}_{\alpha\beta}$ is the $2m \times 2m$ antisymmetric symplectic metric, and $\mathbb{C}^{\alpha\beta}$ is the same matrix[3], so

$$\mathbb{C}^{\gamma\alpha}\mathbb{C}_{\gamma\beta} = \delta^{\alpha}{}_{\beta}\,. \tag{D.0.16}$$

From this definition, it follows that

$$2\;\mathsf{U}^{\alpha I}{}_{(u}\;\mathsf{U}^{\beta J}{}_{v)}\;\mathbb{C}_{\alpha\beta} = \mathsf{H}_{uv}\varepsilon^{IJ}\,. \tag{D.0.17}$$

Furthermore, it is required that

$$\mathsf{U}_{\alpha I\,u} \equiv (\mathsf{U}^{\alpha I}{}_{u})^{*} \;=\; \varepsilon_{IJ}\mathbb{C}_{\alpha\beta}\;\mathsf{U}^{\beta J}{}_{u}\,. \tag{D.0.18}$$

The inverse Quadbein $\mathsf{U}^{u}{}_{\alpha I}$ satisfies

$$\mathsf{U}_{\alpha I}{}^{u}\;\mathsf{U}^{\alpha I}{}_{v} = \delta^{u}{}_{v}\,, \tag{D.0.19}$$

and, therefore,

$$\mathsf{U}_{\alpha I}{}^{u} = \mathsf{H}^{uv}\;\varepsilon_{IJ}\mathbb{C}_{\alpha\beta}\;\mathsf{U}^{\beta J}{}_{v}\,. \tag{D.0.20}$$

The Quadbein satisfies a *Vielbein postulate*, *i.e.* they are covariantly constant with respect to the standard Levi-Cività connection $\Gamma_{uv}{}^{w}$, the $SU(2)$ connection $\mathsf{B}_{u}{}^{I}{}_{J}$ and the $Sp(m)$ connection $\Delta_{u}{}^{\alpha}{}_{\beta}$:

$$\mathsf{D}_{u}\;\mathsf{U}^{\alpha I}{}_{v} \;=\; \partial_{u}\mathsf{U}^{\alpha I}{}_{v} \;-\; \Gamma_{uv}{}^{w}\;\mathsf{U}^{\alpha I}{}_{w} \;+\; \mathsf{B}_{u}{}^{I}{}_{J}\;\mathsf{U}^{\alpha J}{}_{v} \;+\; \Delta_{u}{}^{\alpha}{}_{\beta}\;\mathsf{U}^{\beta I}{}_{v} = 0\,. \tag{D.0.21}$$

This postulate relates the three connections and the respective curvatures, leading to the statement that the holonomy of a quaternionic Kähler manifold is contained in $Sp(1)\cdot Sp(m)$, *i.e.*

$$R_{ts}{}^{uv}\;\mathsf{U}^{\alpha I}{}_{u}\;\mathsf{U}^{\beta J}{}_{v} \;=\; -\mathsf{G}^{IJ}_{ts}\;\mathbb{C}^{\alpha\beta} \;-\; \overline{\mathsf{R}}_{ts}{}^{\alpha\beta}\;\varepsilon^{IJ} \;=\; \mathsf{F}^{IJ}_{ts}\;\mathbb{C}^{\alpha\beta} \;-\; \overline{\mathsf{R}}_{ts}{}^{\alpha\beta}\;\varepsilon^{IJ}\;, \tag{D.0.22}$$

where

$$\overline{R}_{ts}{}^{\alpha}{}_{\beta} \;=\; 2\partial_{[t}\Delta_{s]}{}^{\alpha}{}_{\beta} + 2\Delta_{[t|}{}^{\alpha}{}_{\gamma}\;\Delta_{|s]}{}^{\gamma}{}_{\beta}\,, \tag{D.0.23}$$

is the curvature of the $Sp(m)$ connection.

[3]We adopt the following convenion for raising and lowering vector $Sp(m)$ indices:

$$\chi^{\alpha} = \chi_{\beta}\mathbb{C}^{\beta\alpha}\,, \qquad \xi_{\alpha} = \mathbb{C}_{\alpha\beta}\xi^{\beta}\,. \tag{D.0.15}$$

The covariant constancy of the Pauli matrices and symplectic metric together with the covariant constancy of the Quadbeins suggests that it should be possible to express the hyperKähler 2-forms in terms of them. One can check that

$$\mathsf{K}^{x}{}_{uv} = -i\sigma^{x}{}_{IJ}\mathsf{U}^{\alpha I}{}_{u}\mathsf{U}^{\beta J}{}_{v}\mathbb{C}_{\alpha\beta}\,, \qquad \sigma^{x}{}_{IJ} \equiv \sigma^{x}{}_{I}{}^{K}\varepsilon_{JK}\,, \tag{D.0.24}$$

satisfies the quaternionic algebra Eq. (D.0.1) and is covariantly constant, as required. This leads to

$$\mathsf{U}^{\alpha I}{}_{u}\;\mathsf{U}^{\beta J}{}_{v}\;\mathbb{C}_{\alpha\beta} = \tfrac{1}{2}\mathsf{H}_{uv}\varepsilon^{IJ} - \tfrac{i}{2}\mathsf{K}^{x}{}_{uv}\sigma^{x\,IJ}\,, \qquad \sigma^{x\,IJ} \equiv \varepsilon^{KI}\sigma^{x}{}_{K}{}^{J}\,. \tag{D.0.25}$$

The symmetric part of this equation is just Eq. (D.0.17) and the antisymmetric part of this equation leads to

$$K^{IJ}{}_{uv} = \tfrac{i}{2}\mathsf{K}^{x}{}_{uv}\sigma^{x\,IJ} = -\mathsf{U}^{\alpha I}{}_{[u}\;\mathsf{U}^{\beta J}{}_{v]}\;\mathbb{C}_{\alpha\beta}\,, \tag{D.0.26}$$

from which we get the useful relation

$$\mathsf{F}_{\mu\nu}{}^{IJ} = -\varkappa\,\mathbb{C}_{\alpha\beta}\mathsf{U}^{\alpha I}{}_{u}\mathsf{U}^{\beta J}{}_{v}\partial_{[\mu}q^{u}\partial_{\nu]}q^{v}\,. \tag{D.0.27}$$

D.1 Gauging isometries of quaternionic Kähler manifolds

We start by assuming that the metric H_{uv} admits Killing vectors $\mathsf{k}_{\Lambda}{}^{u}$ satisfying the Lie algebra

$$[k_{\Lambda}, k_{\Sigma}] = -f_{\Lambda\Sigma}{}^{\Omega}k_{\Omega}\,, \tag{D.1.1}$$

where, as in previous cases, for certain values of Λ the vectors and the structure constants can vanish. The metric and the ungauged sigma model are invariant under the global transformations

$$\delta_{\alpha}q^{u} = \alpha^{\Lambda}\mathsf{k}_{\Lambda}{}^{u}(q)\,. \tag{D.1.2}$$

In order to make this global invariance local, we just have to replace the standard derivatives of the scalars by the covariant derivatives

$$\mathfrak{D}_{\mu}q^{u} \equiv \partial_{\mu}q^{u} + gA^{\Lambda}{}_{\mu}\mathsf{k}_{\Lambda}{}^{u}\,, \tag{D.1.3}$$

which will transform according to

$$\delta_{\alpha}\mathfrak{D}_{\mu}q^{u} = \alpha^{\Lambda}(x)\partial_{v}\mathsf{k}_{\Lambda}{}^{u}\mathfrak{D}_{\mu}q^{v}\,, \tag{D.1.4}$$

provided that the gauge potentials transform in the standard form Eq. (B.1.6).

This is enough to gauge the global symmetry of the scalars' kinetic term. However, the isometries of the metric need not be global symmetries of the full supergravity theory. They have to preserve the quaternionic-Kähler structure as well, and not just the metric. In order to discuss the preservation of this structure, we need to define $SU(2)$-covariant Lie derivatives.

Let $\psi^x(q)$ be a field on HM transforming under infinitesimal local $SU(2)$ transformations according to

$$\delta_\lambda \psi^x = -\varepsilon^{xyz}\lambda^y\psi^z \,. \tag{D.1.5}$$

Its $SU(2)$ covariant derivative is given by

$$\mathsf{D}\psi^x = d\psi^x + \varepsilon^{xyz}\mathsf{A}^y\psi^z \,, \tag{D.1.6}$$

where the $SU(2)$ connection 1-form transforms as

$$\delta_\lambda \mathsf{A}^x = \mathsf{D}\lambda^x \,. \tag{D.1.7}$$

To define an $SU(2)$-covariant Lie derivative with respect to the Killing vector k_Λ $\mathbb{L}_\Lambda$, we add to the standard one $\pounds_\Lambda$ a local $SU(2)$ transformation whose transformation parameter is given by the compensator field ${\mathsf{W}_\Lambda}^x$:

$$\mathbb{L}_\Lambda \psi^x \equiv \pounds_\Lambda \psi^x + \varepsilon^{xyz}{\mathsf{W}_\Lambda}^y\psi^z \,, \tag{D.1.8}$$

which is such that

$$\delta_\lambda {\mathsf{W}_\Lambda}^x = \pounds_\Lambda \lambda^x - \varepsilon^{xyz}\lambda^y{\mathsf{W}_\Lambda}^z = \mathbb{L}_\Lambda\lambda^x \,. \tag{D.1.9}$$

$\mathbb{L}_\Lambda$ is clearly a linear operator which satisfies the Leibnitz rule for scalar and vector products of SU(2) vectors. The Lie derivative must also satisfy

$$[\mathbb{L}_\Lambda,\, \mathbb{L}_\Sigma] = \mathbb{L}_{[\mathsf{k}_\Lambda,\, \mathsf{k}_\Sigma]} \tag{D.1.10}$$

which implies the Jacobi identity. This requires

$$\pounds_\Lambda {\mathsf{W}_\Sigma}^x - \pounds_\Sigma {\mathsf{W}_\Lambda}^x + \varepsilon^{xyz}{\mathsf{W}_\Lambda}^y\mathsf{W}^z_\Sigma = -f_{\Lambda\Sigma}{}^\Gamma{\mathsf{W}_\Gamma}^x \,, \tag{D.1.11}$$

where, due to the assumed linearity of W_Λ on k_Λ, $\mathsf{W}_{[\mathsf{k}_\Lambda,\, \mathsf{k}_\Sigma]} = -f_{\Lambda\Sigma}{}^\Gamma\mathsf{W}_\Gamma$.

In order to satisfy equation (D.1.11) we introduce another SU(2) vector

$${\mathsf{W}_\Lambda}^x \equiv {\mathsf{k}_\Lambda}^u{\mathsf{A}^x}_u - {\mathsf{P}_\Lambda}^x \,, \tag{D.1.12}$$

which has to satisfy the equivariance condition

$$\mathsf{D}_\Lambda{\mathsf{P}_\Sigma}^x - \mathsf{D}_\Sigma{\mathsf{P}_\Lambda}^x - \varepsilon^{xyz}{\mathsf{P}_\Lambda}^y{\mathsf{P}_\Sigma}^z - \varkappa\, {\mathsf{k}_\Lambda}^u{\mathsf{k}_\Sigma}^v\, {\mathsf{K}^x}_{uv} = -f_{\Lambda\Sigma}{}^\Gamma{\mathsf{P}_\Gamma}^x \,, \tag{D.1.13}$$

where $\mathsf{D}_\Lambda \equiv \mathsf{k}_\Lambda{}^u \mathsf{D}_u$ and we have used Eq. (D.0.5). $\mathsf{P}_\Lambda{}^x$ is going to be the *triholomorphic momentum map* when we impose the preservation of the hyperKähler structure K^x by the global transformations Eq. (D.1.2) and this compensating $SU(2)$ transformation with parameter W_Λ. This condition is expressed using $\mathbb{L}$:

$$\begin{aligned}
\mathbb{L}_\Lambda \mathsf{K}^x{}_{uv} &= \pounds_\Lambda \mathsf{K}^x{}_{uv} + \varepsilon^{xyz}(\mathsf{k}_\Lambda{}^w \mathsf{A}^y{}_w - \mathsf{P}_\Lambda{}^y)\mathsf{K}^z{}_{uv} && \text{(D.1.14)} \\
&= -2\mathsf{D}_{[u|}(\mathsf{k}_\Lambda{}^w \mathsf{K}^x{}_{w|v]}) - \varepsilon^{xyz}\mathsf{P}_\Lambda{}^y \mathsf{K}^z{}_{uv} && \text{(D.1.15)} \\
&= 0\,. && \text{(D.1.16)}
\end{aligned}$$

Using the covariant constancy of the hyperKähler structure, this condition can be rewritten in the form

$$2(\nabla_{[u|}\mathsf{k}_\Lambda{}^w)\mathsf{K}^x{}_{w|v]} - \varepsilon^{xyz}\mathsf{P}_\Lambda{}^y \mathsf{K}^z{}_{uv} = 0\,, \tag{D.1.17}$$

and, contracting the whole equation with $\mathsf{K}^{y\,uv}$ we find

$$\mathsf{K}^{x\,uv}\nabla_u \mathsf{k}_{\Lambda\,v} = -2m\mathsf{P}_\Lambda{}^x\,. \tag{D.1.18}$$

Acting on both sides of this equations with D_w and using the Killing vector identity $\nabla_w\nabla_u \mathsf{k}_{\Lambda\,v} = R_{wruv}\mathsf{k}_\Lambda{}^r$ we get

$$\mathsf{k}_\Lambda{}^r R_{wruv}\mathsf{K}^{x\,uv} = -2m\mathsf{D}_w \mathsf{P}_\Lambda{}^x\,. \tag{D.1.19}$$

Finally, using Eqs. (D.0.24) in Eq. (D.0.22) we get

$$R_{wruv}\mathsf{K}^{x\,uv} = -2m\mathsf{F}^x{}_{wr} = -2m\varkappa\, \mathsf{K}^x{}_{wr}\,, \tag{D.1.20}$$

and substituting above, we arrive at

$$\mathsf{D}_u \mathsf{P}_\Lambda{}^x = \varkappa\, \mathsf{K}^x{}_{uv}\mathsf{k}_\Lambda{}^v\,, \tag{D.1.21}$$

which can be taken as the equation that defines the triholomorphic momentum map. From this equation we find

$$\mathsf{D}_\Sigma \mathsf{P}_\Lambda{}^x = \varkappa\, \mathsf{k}_\Sigma{}^u \mathsf{k}_\Lambda{}^v \mathsf{K}^x{}_{uv}\,, \tag{D.1.22}$$

and, substituting directly in Eq. (D.1.13) we get

$$\mathbb{L}_\Lambda \mathsf{P}_\Sigma{}^x = \mathsf{D}_\Lambda \mathsf{P}_\Sigma{}^x - \varepsilon^{xyz}\mathsf{P}_\Lambda{}^y \mathsf{P}_\Sigma{}^z + f_{\Lambda\Sigma}{}^\Omega \mathsf{P}_\Omega{}^x = 0\,, \tag{D.1.23}$$

which says that the triholomprhic momentum map is an invariant field and

$$\varepsilon^{xyz}\mathsf{P}_\Lambda{}^y \mathsf{P}_\Sigma{}^z - \varkappa\, \mathsf{k}_\Lambda{}^u \mathsf{k}_\Sigma{}^v \mathsf{K}^x{}_{uv} = f_{\Lambda\Sigma}{}^\Omega \mathsf{P}_\Omega{}^x\,. \tag{D.1.24}$$

Now, for a field Φ (possibly with spacetime, quaternionic, $SU(2)$ or gauge indices) which under Eq. (D.1.2) transforms according to

$$\delta_\alpha \Phi = -\alpha(\mathbb{L}_\Lambda - \mathsf{k}_\Lambda)\Phi\,, \tag{D.1.25}$$

we define the gauge covariant derivative

$$\mathfrak{D}_\mu \Phi \equiv \{\nabla_\mu + \mathfrak{D}_\mu q^u \Gamma_u - g A^\Lambda{}_\mu(\mathbb{L}_\Lambda - \mathsf{k}_\Lambda) + \mathfrak{D}_\mu q^u \mathsf{A}^x{}_u\}\Phi\,. \tag{D.1.26}$$

For the triholomorphic momentum map, we have, on account of Eq. (D.1.23), which we can rewrite in the form

$$k_\Lambda{}^u \partial_u \mathsf{P}_\Sigma{}^x = -\varepsilon^{xyz}(k_\Lambda{}^u \mathsf{A}^y{}_u - \mathsf{P}_\Lambda{}^y)\mathsf{P}_\Sigma{}^z - f_{\Lambda\Sigma}{}^\Omega \mathsf{P}_\Omega{}^x\,, \tag{D.1.27}$$

the following expressions for its gauge covariant derivative

$$\mathfrak{D}_\mu \mathsf{P}_\Lambda{}^x = \partial_\mu \mathsf{P}_\Lambda{}^x + \varepsilon^{xyz}\hat{\mathsf{A}}^y{}_\mu \mathsf{P}_\Lambda{}^z + f_{\Lambda\Sigma}{}^\Omega A^\Sigma{}_\mu \mathsf{P}_\Omega{}^x\,, \tag{D.1.28}$$

$$\mathfrak{D}_\mu \mathsf{P}_\Lambda{}^x = \mathfrak{D}_\mu q^u \mathsf{D}_u \mathsf{P}_\Lambda{}^x\,, \tag{D.1.29}$$

where we have defined

$$\hat{\mathsf{A}}^x{}_\mu \equiv \partial_\mu q^u \mathsf{A}^x{}_u + g A^\Lambda{}_\mu \mathsf{P}_\Lambda{}^x\,. \tag{D.1.30}$$

Under Eq. (D.1.2), spinors with $SU(2)$ indices undergo the following transformation

$$\delta_\alpha \psi_I = -\alpha^\Lambda \mathsf{W}_\Lambda{}^x \tfrac{i}{2}\sigma^x{}_I{}^J \psi_J\,. \tag{D.1.31}$$

Then, using the general formula, their covariant derivative is given by

$$\mathfrak{D}_\mu \psi_I = \nabla_\mu \psi_I + \hat{\mathsf{A}}^x{}_\mu\, \tfrac{i}{2}\sigma^x{}_I{}^J \psi_J\,. \tag{D.1.32}$$

If we take into account their Kähler weight and possible gaugings of the isometries of the special-Kähler manifold, we have for the supersymmetry parameters of $N = 2, d = 4$ supergravity

$$\mathfrak{D}_\mu \epsilon_I = \{\nabla_\mu + \tfrac{i}{2}\hat{\mathcal{Q}}_\mu\}\epsilon_I + \hat{\mathsf{A}}^x{}_\mu \tfrac{i}{2}\sigma^x{}_I{}^J \epsilon_J\,. \tag{D.1.33}$$

D.2 All about the C-map

The c-map is a manifestation of the T-duality between the type IIA and IIB theories, compactified on the same Calabi-Yau 3-fold. Since T-duality in supergravity theories is implemented by dimensional reduction, to be told that the c-map is derived by dimensionally reducing an $N = 2$ $d = 4$ SUGRA coupled to n vector- and m hypermultiplets to $d = 3$, and dualizing every vector field into a scalar field, should not come as too big a surprise.

D.2.1 Dual-Quaternionic metric and its symmetries

In order to derive the c-map, consider the, rather standard, KK-Ansatz

$$\begin{array}{lclclcl} \hat{e}^a & = & e^{-\phi}\, e^a & ; & \hat{e}^{\underline{y}} & = & e^{\phi}\,(dy+A)\ , \\ \hat{A}^\Lambda & = & B^\Lambda + C^\Lambda\,(dy+A) & \rightarrow & \hat{F}^\Lambda & = & F^\Lambda\ + dC^\Lambda\wedge(dy+A)\ , \\ F^\Lambda & = & dB^\Lambda\ +\ C^\Lambda\ F & , & F & = & dA\ , \end{array} \tag{D.2.1}$$

and use it on the *ungauged* action (2.2.1); the resulting action reads

$$\begin{aligned} \mathcal{S}_{(3)} \ = \ & \int d^3\sqrt{g}\left[\tfrac{1}{2}R\ + d\phi^2\ - e^{-2\phi}\mathrm{Im}(\mathcal{N})_{\Lambda\Sigma}\ dC^\Lambda dC^\Sigma\ + \mathcal{G}_{i\bar{\jmath}}dZ^i d\overline{Z}^{\bar{\jmath}}\ + \mathsf{H}_{uv}dq^u dq^v\right] \\ & + \int_3 \left(\tfrac{1}{2}\mathfrak{F}^T\ M\wedge\ *\mathfrak{F}\ +\ \mathfrak{F}^T\ \wedge\ Qd\mathfrak{C}\right)\ , \end{aligned} \tag{D.2.2}$$

where we have defined the $(\overline{n}+1)$-vectors $\mathfrak{F}^T = (dB^\Lambda, dA)$ and $\mathfrak{C}^T = (C^\Lambda, 0)$. Furthermore the $(\overline{n}+1)\times(\overline{n}+1)$-matrices M and Q are given by

$$M\ =\ 2e^{2\phi}\begin{pmatrix} \mathrm{Im}(\mathcal{N}) & \mathrm{Im}(\mathcal{N})\cdot C \\ C^T\cdot\mathrm{Im}(\mathcal{N}) & C^T\cdot\mathrm{Im}(\mathcal{N})\cdot C - \frac{e^{2\phi}}{4} \end{pmatrix}\ ;\ Q\ =\ 2\begin{pmatrix} \mathrm{Re}(\mathcal{N}) & 0 \\ C^T\cdot\mathrm{Re}(\mathcal{N}) & 0 \end{pmatrix}\ . \tag{D.2.3}$$

The field strengths can then be integrated out by adding to the above action a Lagrange multiplier term $\mathfrak{F}^T\wedge d\mathfrak{L}$, imposing the Bianchi identity $d\mathfrak{F}=0$. $\mathfrak{F}$ can then be integrated out by using its equation of motion $*\mathfrak{F} = M^{-1}(d\mathfrak{L}\ + Q\ d\mathfrak{C})$, resulting in 3d gravity coupled to a sigma model describing two disconnected quaternionic manifolds, one with metric $\mathsf{H}_{uv}dq^u dq^v$, and the other one coming from the gravity- and vector multiplets. Taking $\mathfrak{L}^T = (T_\Lambda, \theta)$ we can write the metric of this $4\overline{n}$-dimensional quaternionic manifold as

$$
\begin{aligned}
ds^2_{DQ} \;&=\; d\phi^2 - e^{-2\phi}\mathrm{Im}(\mathcal{N})_{\Lambda\Sigma}\; dC^\Lambda dC^\Sigma \;+ e^{-4\phi}\left(d\theta - C^\Lambda dT_\Lambda\right)^2 \;+ \mathcal{G}_{i\bar{\jmath}}\; dZ^i\; d\overline{Z}^{\bar{\jmath}} \\
&\quad -\tfrac{1}{4}e^{-2\phi}\mathrm{Im}(\mathcal{N})^{-1|\Lambda\Sigma}\;\left(dT_\Lambda + 2\mathrm{Re}(\mathcal{N})_{\Lambda\overline{\Lambda}}dC^{\overline{\Lambda}}\right)\left(dT_\Sigma + 2\mathrm{Re}(\mathcal{N})_{\Sigma\overline{\Sigma}}dC^{\overline{\Sigma}}\right)
\end{aligned} \tag{D.2.4}
$$

The fact that this metric is indeed quaternionic was proven in [69]. This kind of quaternionic manifolds is, for an obvious reason, called *dual quaternionic manifolds*, and is generically characterized by the existence of at least $2(\overline{n}+1)$-translational isometries [129], about which more in a few lines.

Anyway, seeing as this dual quaternionic manifold comes from a special geometry it is nice, and even possible, to write it in a manifestly $Sp(2\overline{n};\mathbb{R})$ covariant manner: this is achieved by doing the coordinate transformations $T_\Lambda \to -2T_\Lambda$ and $\theta \to \theta - C^\Lambda T_\Lambda$ and introducing the real symplectic vector $\mathcal{Y}^T \equiv (\mathcal{Y}^\Lambda, \mathcal{Y}_\Lambda) = (C^\Lambda, T_\Lambda)$, resulting in

$$
ds^2_{DQ} \;=\; d\phi^2 + \mathcal{G}_{i\bar{\jmath}}dZ^i d\overline{Z}^{\bar{\jmath}} + e^{-4\phi}\left(d\theta - \langle\mathcal{Y}|d\mathcal{Y}\rangle\right)^2 + e^{-2\phi}\; d\mathcal{Y}^T\; \mathfrak{M}\; d\mathcal{Y}\;, \tag{D.2.5}
$$

where $\mathfrak{M}$ is the $2\overline{n}\times 2\overline{n}$-matrix

$$
\begin{aligned}
\mathfrak{M} \;&=\; -\begin{pmatrix} \mathrm{Im}(\mathcal{N}) \;+\; \mathrm{Re}(\mathcal{N})\mathrm{Im}(\mathcal{N})^{-1}\mathrm{Re}(\mathcal{N}) & -\mathrm{Re}(\mathcal{N})\mathrm{Im}(\mathcal{N})^{-1} \\ -\mathrm{Im}(\mathcal{N})^{-1}\mathrm{Re}(\mathcal{N}) & \mathrm{Im}(\mathcal{N})^{-1}\end{pmatrix} && \text{(D.2.6)} \\
&=\; 2\Omega\;\mathrm{Re}\left(\mathcal{V}\;\mathcal{V}^\dagger \;+\; \mathcal{U}_i\;\mathcal{G}^{i\bar{\jmath}}\;\mathcal{U}^\dagger_j\right)\Omega^T\;, && \text{(D.2.7)}
\end{aligned}
$$

where Ω is the inner product left invariant by $Sp(2\overline{n};\mathbb{R})$. Moreover, $\mathfrak{M}$ is positive definite and has the correct and obvious properties [130] to make the metric $Sp(2\overline{n},\mathbb{R})$-covariant.

As mentioned above, the Dual-quaternionic metric always has $2(\bar{n}+1)$ translational isometries and introducing $\partial_\Lambda \equiv \partial_{\mathcal{Y}^\Lambda}$ and $\partial^\Lambda \equiv \partial_{\mathcal{Y}_\Lambda}$ the Killing vectors for these, obvious, isometries are given by

$$
\begin{array}{lcllclcl}
U &=& \partial_\phi + \mathcal{Y}_\Lambda\partial^\Lambda + \mathcal{Y}^\Lambda\partial_\Lambda + 2\theta\;\partial_\theta & ; & V &=& \partial_\theta\;, \\
X^\Lambda &=& \partial^\Lambda \;+\; \mathcal{Y}^\Lambda\partial_\theta & , & X_\Lambda &=& \partial_\Lambda \;-\; \mathcal{Y}_\Lambda\partial_\theta\;.
\end{array} \tag{D.2.8}
$$

These vector fields satisfy the commutation relation of a Heisenberg algebra, *i.e.*

$$
\begin{array}{lclclcl}
\left[U, X^\Lambda\right] &=& -X^\Lambda & , & \left[U, X_\Lambda\right] &=& -X_\Lambda\;, \\
\left[U, V\right] &=& -2\;V & , & \left[X^\Lambda, X_\Sigma\right] &=& -2\delta^\Lambda{}_\Sigma\;V\;.
\end{array} \tag{D.2.9}
$$

The automorphism group of this Heisenberg algebra is $Sp(\overline{n},\mathbb{R})$, as was to be expected.

As discussed in Appendix (C.2), the special geometry can also have isometries and we must then ask ourselves how these manifest themselves in the Dual Quaternionic geometry. The key to finding out how these isometries act, lies in Eq. (D.2.6) and Eq. (C.2.1), which allows one to derive

$$\pounds_{\mathtt{K}}\mathfrak{M} \;=\; -S_{\mathtt{K}}^{T}\;\mathfrak{M} \;-\; \mathfrak{M}\;S_{\mathtt{K}}\;. \tag{D.2.10}$$

This transformation the means that the lift of the special geometry Killing vector $\mathtt{K}$ to the Dual Quaternionic metric is given by

$$\mathtt{K} \;=\; \mathtt{K}^{i}\partial_{i} \;+\; \mathtt{K}^{\bar{\imath}}\partial_{\bar{\imath}} \;+\; f_{\Lambda\Sigma}{}^{\Omega}\left[\mathcal{Y}^{\Sigma}\partial_{\Omega} \;-\; \mathcal{Y}_{\Omega}\partial^{\Sigma}\right] \;=\; \mathtt{K}^{i}\partial_{i} \;+\; \mathtt{K}^{\bar{\imath}}\partial_{\bar{\imath}} + \mathcal{Y}^{T}\;S^{T}\;\partial^{\natural}\;, \tag{D.2.11}$$

where we have defined the symplectic vector $\partial^{\natural} \;=\; (\partial_{\Lambda},\partial^{\Lambda})^{T}$. The advantage of writing the Killing vector like this, becomes clear when we want to confirm that the commutation relation in Eq. (D.1.1) holds for the lifted Killing vectors. In fact, using the identity $\partial^{\natural}\;\mathcal{Y}^{T} = \mathbb{I}_{2n}$ this calculation is a trifle. Of course, we can also introduce the symplectic vector of generators

$$X^{\natural} \;=\; \left(X_{\Lambda},X^{\Lambda}\right)^{T} \;=\; \partial^{\natural} \;+\; \Omega\mathcal{Y}\;V \quad;\quad X^{\flat} \;\equiv\; \Omega^{-1}\;X^{\natural}\;, \tag{D.2.12}$$

then one can see that

$$\left[\mathtt{K}_{\Lambda},X^{\natural}\right] \;=\; -S_{\Lambda}^{T}\;X^{\natural} \quad,\quad \left[\mathtt{K}_{\Lambda},X^{\flat}\right] \;=\; S_{\Lambda}\;X^{\flat}\;, \tag{D.2.13}$$

And the rest of the actions of $\mathtt{K}$ vanish. And just in case you were wondering, you can see that this action satisfies the Jacobi identity. A useful relation is

$$\imath_{\mathtt{K}}\;d\mathcal{Y} \;=\; S_{\mathtt{K}}\;\mathcal{Y}\;. \tag{D.2.14}$$

D.2.2 The universal qK-space

Let us first have a look at the case when we c-map the minimal theory. In that case a Quadbein is easily found to be

$$\mathsf{U}^{\alpha I} \;=\; \begin{pmatrix} E^{0} & F^{0} \\ -\overline{F^{0}} & \overline{E^{0}} \end{pmatrix} \;\text{ with }\; \left\{\begin{array}{lcl} \sqrt{2}\;E^{0} & = & d\phi \;+\; ie^{-2\phi}\left[d\theta - \langle\mathcal{Y}|d\mathcal{Y}\rangle\right] \\ F^{0} & = & e^{-\phi}\;\langle d\mathcal{Y}\;|\;\mathcal{V}\rangle \end{array}\right. \tag{D.2.15}$$

where we have chosen to keep $\mathcal{V}$ for future convenience. The needed $\mathfrak{su}(2)$ connection can easily be found by using Eq. (D.0.21) and leads to

$$\left.\begin{array}{lcl} \mathsf{A}^{1} & = & 2\sqrt{2}\;\mathrm{Im}\left(F^{0}\right) \\ \mathsf{A}^{2} & = & -2\sqrt{2}\;\mathrm{Re}\left(F^{0}\right) \\ \mathsf{A}^{3} & = & \sqrt{2}\;\mathrm{Im}\left(E^{0}\right) \end{array}\right\} \;\text{ and }\; \Delta^{\alpha}{}_{\beta} \;=\; -\tfrac{3i}{\sqrt{2}}\;\mathrm{Im}\left(E^{0}\right)\;\sigma^{3\;\alpha}{}_{\beta}\;. \tag{D.2.16}$$

The field strengths for the above $\mathfrak{su}(2)$ connection can be compared with the triple-Kähler structures defined in Eq. (D.0.24), which can be calculated straightforwardly to give

$$\mathsf{K}^1 \;=\; -2\mathrm{Im}\left(E^0\wedge\overline{F^0}\right) \;\;,\;\; \mathsf{K}^2 \;=\; -2\mathrm{Re}\left(E^0\wedge\overline{F^0}\right) \;\;,\;\; \mathsf{K}^3 \;=\; -\mathrm{Im}\left(E^0\wedge\overline{E^0} \;-\; F^0\wedge\overline{F^0}\right) \;. \tag{D.2.17}$$

Said comparison then shows that the connection and the triple-Kähler structure satisfy Eq. (D.0.5) with $\varkappa = -2$, in concordance with the results obtained from the KSIs and can be seen as a further check on the consistency of the determination of $\varkappa$.

D.2.3 Quadbein, $\mathfrak{su}(2)$-connection and momentum maps

In the foregoing section we derived the $\mathfrak{su}(2)$ connection for the simplest of dual quaternionic spaces, and in this section we shall determine it for the general DQ-spaces in Eq. (D.2.5). The first thing to do is to write down a suitable Quadbein and find the the As. A convenient way to do this is by looking at the example in the foregoing section and asking oneself what: can possibly change in the connection? Most of the objects that enter in the general case have index properties that arrise from special geometry and, seeing as we kept everything as symplectic invariant as possible, we should expect the $\mathfrak{su}(2)$ connection to be as covariant as possible. This basically means that only the Kähler connection, $\mathcal{Q}$, can appear. In fact, it must appear as F^0 has a non-vanishing Kähler weight.

In order to advance, spilt the $Sp(2m)$-index α as $(\Lambda\ \bar{\alpha})$, with $\bar{\alpha} = 1,2$ and $\Lambda = 0,1,\ldots,n$, where n is the number of vector multiplets before applying the c-map. This then enables us to write down a putative Quadbein and use it to calculate the triple-Kähler forms, *i.e.*

$$\mathsf{U}^{(\Lambda\bar{\alpha})I} \;=\; \begin{pmatrix} E^\Lambda & F^\Lambda \\ -\overline{F^\Lambda} & \overline{E^\Lambda}\end{pmatrix} \;\text{ and }\; \left\{\begin{array}{lcl} \mathsf{K}^1 &=& -2\mathrm{Im}\left(E^\Lambda\wedge\overline{F^\Lambda}\right) \\[2ex] \mathsf{K}^2 &=& -2\mathrm{Re}\left(E^\Lambda\wedge\overline{F^\Lambda}\right) \\[2ex] \mathsf{K}^3 &=& -\mathrm{Im}\left(E^\Lambda\wedge\overline{E^\Lambda} \;-\; F^\Lambda\wedge\overline{F^\Lambda}\right)\end{array}\right. \tag{D.2.18}$$

where of course the expressions for the $\Lambda = 0$ components are the ones given in Eq. (D.2.15). Introducing the Vielbein $E_i{}^a$ $(i,a = 1,\ldots,n)$ and the tangent object $\mathcal{U}^{\bar{a}}$ through the definitions

$$E_i{}^a\overline{E}_{\bar{\jmath}}{}^{\bar{a}} \;\equiv\; \mathcal{G}_{i\bar{\jmath}} \;\;,\;\; \mathcal{U}^{\bar{a}} \;\equiv\; \mathcal{U}_i\,\overline{E}^{i\;\bar{a}} \;, \tag{D.2.19}$$

we see that imposing Eq. (D.0.5) with $\varkappa = -2$ and the choice

$$\mathsf{A}^1 \;=\; 2\sqrt{2}\,\mathrm{Im}\left(F^0\right) \quad , \quad \mathsf{A}^2 \;=\; -2\sqrt{2}\,\mathrm{Re}\left(F^0\right) \quad , \quad \mathsf{A}^3 \;=\; \sqrt{2}\,\mathrm{Im}\left(E^0\right) \,+\, \mathcal{Q}\;, \tag{D.2.20}$$

which is dictated by the $\Lambda = 0$ sector, implies that

$$\sqrt{2}E^\Lambda \;=\left\{\begin{array}{lcl} E^0 &=& d\phi \;+\; ie^{-2\phi}\left[d\theta - \langle\mathcal{Y}|d\mathcal{Y}\rangle\right] \\ \overline{E}^{\bar{a}} &=& \overline{E}^{\bar{a}}_{\bar{\imath}}\; d\overline{Z}^{\bar{\imath}} \end{array}\right. \quad \text{and} \;\; F^\Lambda \;=\left\{\begin{array}{lcl} F^0 &=& e^{-\phi}\langle d\mathcal{Y}|\mathcal{V}\rangle \\ F^{\bar{a}} &=& -e^{-\phi}\langle d\mathcal{Y}|\mathcal{U}^{\bar{a}}\rangle \end{array}\right. \tag{D.2.21}$$

So even though it might seem strange, the index λ splits as $\Lambda = 0, \bar{a}$ and the minus-sign in the definition of $F^{\bar{a}}$ in *not* a typo, but is necessary.

Having found the connection and the triple-Kähler forms, we are all set to start finding the momentum maps corresponding to the isometries (D.2.8) and (D.2.11). Let us start with the easiest ones: U and V. Their momentum maps are readily found to be

$V\;:\quad \mathsf{P}^1 = 0$ $\mathsf{P}^2 = 0$ $\mathsf{P}^3 = e^{-2\phi}$	$U\;:\quad \mathsf{P}^1 = 2\sqrt{2}\;e^{-\phi}\;\mathrm{Im}\left(\langle\mathcal{Y}|\mathcal{V}\rangle\right)$ $\mathsf{P}^2 = -2\sqrt{2}\;e^{-\phi}\;\mathrm{Re}\left(\langle\mathcal{Y}|\mathcal{V}\rangle\right)$ $\mathsf{P}^3 = 2e^{-2\phi}\;\theta$
$X^\Lambda\;:\quad \mathsf{P}^{\Lambda\;1} = 2\sqrt{2}\;e^{-\phi}\;\mathrm{Im}\left(\mathcal{V}^\Lambda\right)$ $\mathsf{P}^{\Lambda\;2} = -2\sqrt{2}\;e^{-\phi}\;\mathrm{Re}\left(\mathcal{V}^\Lambda\right)$ $\mathsf{P}^{\Lambda\;3} = 2e^{-2\phi}\;\mathcal{Y}^\Lambda$	$X_\Lambda\;:\quad \mathsf{P}^1_\Lambda = -2\sqrt{2}\;e^{-\phi}\;\mathrm{Im}\left(\mathcal{V}_\Lambda\right)$ $\mathsf{P}^2_\Lambda = 2\sqrt{2}\;e^{-\phi}\;\mathrm{Re}\left(\mathcal{V}_\Lambda\right)$ $\mathsf{P}^3_\Lambda = -2e^{-2\phi}\;\mathcal{Y}_\Lambda$

(D.2.22)

This then concludes the discussion of the momentum maps for the ever-present Heisenberg isometries of the DQ-spaces; what remains to be done however is to find the momentum maps for the isometries inherited from the Special Geometry, namely the isometries displayed in Eq. (D.2.11). This can of course be calculated and results in

$$\begin{array}{lcl} \mathsf{P}^1_\Lambda &=& -2\sqrt{2}e^{-\phi}\;\mathrm{Im}\left(\langle\mathcal{Y}|S_\Lambda\mathcal{V}\rangle\right) \\ \mathsf{P}^2_\Lambda &=& 2\sqrt{2}e^{-\phi}\;\mathrm{Re}\left(\langle\mathcal{Y}|S_\Lambda\mathcal{V}\rangle\right) \\ \mathsf{P}^3_\Lambda &=& \mathsf{P}_\Lambda \;-\; e^{-2\phi}\;\langle\mathcal{Y}|S_\Lambda\mathcal{Y}\rangle\;, \end{array} \tag{D.2.23}$$

where $\mathtt{P}_\Lambda$ is the $U(1)$-momentum map defined in Eq. (B.1.29).

Let us end this appendix with a small remark: we derived the c-map through dimensional reduction over a spacelike circle. Similarly one can dimensionally reduce the action over a timelike circle, resulting in a space of signature $(2\overline{n}, 2\overline{n})$ and whose holonomy is contained in $Sp(1, \mathbb{R}) \cdot Sp(\overline{n})$. In the rigid limit, *i.e.* when $\lambda = 0$, one recovers the $(1, 2)$/para-hyperKähler structure discussed in *e.g.* [131, 132] The para-universal para-quaternionic manifold, *i.e.* the manifold one obtains by the timelike c-map from minimal $N = 2$ $d = 4$ SUGRA, can be seen to be $SU(1, 2)/U(1, 1)$.

Appendix E

Projectors, field strengths and gauge transformations of the $4d$ tensor hierarchy

E.1 Projectors of the $d = 4$ tensor hierarchy

The 4-dimensional hierarchy's field strengths are defined in terms of the invariant tensors $Z^{MA}, Y_{AM}{}^{B}, W_{C}{}^{MAB}, W_{CNPQ}{}^{M}, W_{CNP}{}^{EM}$ which act as projectors. In this appendix we collect their definitions and the properties that they satisfy.

The projectors are defined by

$$Z^{PA} \equiv -\tfrac{1}{2}\Omega^{NP}\vartheta_{N}{}^{A} = \begin{cases} +\tfrac{1}{2}\vartheta^{\Lambda A}\,, \\ \\ -\tfrac{1}{2}\vartheta_{\Lambda}{}^{A}\,, \end{cases} \,, \tag{E.1.1}$$

$$Y_{AM}{}^{C} \equiv \vartheta_{M}{}^{B} f_{AB}{}^{C} - T_{A\,M}{}^{N}\vartheta_{N}{}^{C}\,, \tag{E.1.2}$$

$$W_{C}{}^{MAB} \equiv -Z^{M[A}\delta_{C}{}^{B]}\,, \tag{E.1.3}$$

$$W_{CNPQ}{}^{M} \equiv T_{C\,(NP}\delta_{Q)}{}^{M}\,, \tag{E.1.4}$$

$$W_{CNP}{}^{EM} \equiv \vartheta_{N}{}^{D} f_{CD}{}^{E}\delta_{P}{}^{M} + X_{NP}{}^{M}\delta_{C}{}^{E} - Y_{CP}{}^{E}\delta_{N}{}^{M}\,. \tag{E.1.5}$$

They satisfy the orthogonality relations

$$Z^{MA} Y_{AN}{}^{C} = \tfrac{1}{2} \Omega^{PM} Q_{PN}{}^{C} = 0\,, \tag{E.1.6}$$

$$Y_{AM}{}^{C} W_{C}{}^{MAB} = Y_{AM}{}^{C} W_{CNPQ}{}^{M} = Y_{AM}{}^{C} W_{CNP}{}^{EM} = 0\,. \tag{E.1.7}$$

Taking the variation of the relations between constraints Eqs. (3.2.10), (3.2.13) and (3.2.16) we find

$$Q^{AB} Y_{BP}{}^{E} - \tfrac{1}{2} Z^{NA} Q_{NP}{}^{E} = 0\,, \tag{E.1.8}$$

$$Q_{(MN)}{}^{A} - 3 L_{MNP} Z^{PA} - 2 Q^{AB} T_{BMN} = 0\,. \tag{E.1.9}$$

Differentiating these identities with respect to the embedding tensor, using Eqs. (E.2.7)-(E.2.9) we also find the following relations among the W tensors:

$$\begin{aligned} & W_{C}{}^{MAB} Y_{BP}{}^{E} - \tfrac{1}{2} Z^{NA} W_{CNP}{}^{EM} \\ & -\frac{1}{4} Q^{M}{}_{P}{}^{E} \delta_{C}^{A} + Q^{AB} \left[\delta_{P}^{M} f_{BC}{}^{E} - T_{BP}{}^{M} \delta_{C}^{E} \right] = 0\,, \end{aligned} \tag{E.1.10}$$

$$W_{C(MN)}{}^{AQ} - 3 W_{CMNP}{}^{Q} Z^{PA} - \tfrac{3}{2} L_{MN}{}^{Q} \delta_{C}{}^{A} - 2 W_{C}{}^{QAB} T_{B\,MN} = 0\,. \tag{E.1.11}$$

E.2 Properties of the W tensors

The W tensors defined in Eqs. (3.2.61)-(3.2.63) satisfy the following properties, which relate them to the embedding tensor constraints:

$$\Theta_{M}{}^{C} W_{C}{}^{MAB} = 2 Q^{AB}\,, \tag{E.2.1}$$

$$\Theta_{M}{}^{C} W_{CNPQ}{}^{M} = L_{NPQ}\,, \tag{E.2.2}$$

$$\Theta_{M}{}^{C} W_{CNP}{}^{EM} = 2 Q_{NP}{}^{E}\,, \tag{E.2.3}$$

$$\frac{\partial Q^{AB}}{\partial \Theta_M{}^C} = W_C{}^{MAB}\,, \tag{E.2.4}$$

$$\frac{\partial L_{NPQ}}{\partial \Theta_M{}^C} = W_{CNPQ}{}^M\,, \tag{E.2.5}$$

$$\frac{\partial Q_{NP}{}^E}{\partial \Theta_M{}^C} = W_{CNP}{}^{EM}\,. \tag{E.2.6}$$

Under variations we have

$$\delta\Theta_M{}^C W_C{}^{MAB} = \Theta_M{}^C \delta W_C{}^{MAB} = \tfrac{1}{2}\delta(\Theta_M{}^C W_C{}^{MAB}) = \delta Q^{AB}\,, \tag{E.2.7}$$

$$\delta\Theta_M{}^C W_{CNPQ}{}^M = \delta L_{NPQ}\,, \tag{E.2.8}$$

$$\delta\Theta_M{}^C W_{CNP}{}^{EM} = \Theta_M{}^C \delta W_{CNP}{}^{EM} = \tfrac{1}{2}\delta(\Theta_M{}^C W_{CNP}{}^{EM}) = \delta Q_{NP}{}^E\,, \tag{E.2.9}$$

where Q^{AB}, $Q_{NP}{}^E$ and L_{NPQ} are the quadratic and linear constraints Eqs. (3.2.10), (3.2.13) and (3.2.16) imposed on the embedding tensor and where we have not used the constraints themselves.

E.3 Transformations and field strengths in the $D = 4$ tensor hierarchy

The gauge transformations of the different fields of the tensor hierarchy are

$$
\begin{aligned}
\delta_h A^M &= -\mathfrak{D}\Lambda^M - Z^{MA}\Lambda_A\,, && \text{(E.3.1)}\\
\delta_h B_A &= \mathfrak{D}\Lambda_A + 2T_{A\,NP}[\Lambda^N F^P + \tfrac{1}{2}A^N\wedge\delta_h A^P] - Y_{AM}{}^C\Lambda_C{}^M\,, && \text{(E.3.2)}\\
\delta_h C_A{}^M &= \mathfrak{D}\Lambda_A{}^M - F^M\wedge\Lambda_A - \delta_h A^M\wedge B_A - \tfrac{1}{3}T_{A\,NP}A^M\wedge A^N\wedge\delta_h A^P + \Lambda^M H_A\\
&\quad -W_A{}^{MAB}\Lambda_{AB} - W_{ANPQ}{}^M\Lambda^{NPQ} - W_{ANP}{}^{EM}\Lambda_E{}^{NP}\,,\\
\delta_h D_{AB} &= \mathfrak{D}\Lambda_{AB} + \alpha B_{[A}\wedge Y_{B]P}{}^E\Lambda_E{}^P + \mathfrak{D}\Lambda_{[A}\wedge B_{B]} - 2\Lambda_{[A}\wedge H_{B]}\\
&\quad +2T_{[A|NP}[\Lambda^N F^P - \tfrac{1}{2}A^N\wedge\delta_h A^P]\wedge B_{|B]}\,, && \text{(E.3.3)}\\
\delta_h D_E{}^{NP} &= \mathfrak{D}\Lambda_E{}^{NP} - [F^N - \tfrac{1}{2}(1-\alpha)Z^{NA}B_A]\wedge\Lambda_E{}^P\\
&\quad +C_E{}^P\wedge\delta_h A^N + \tfrac{1}{12}T_{EQR}A^N\wedge A^P\wedge A^Q\wedge\delta_h A^R + \Lambda^N G_E{}^P\,, && \text{(E.3.4)}\\
\delta_h D^{NPQ} &= \mathfrak{D}\Lambda^{NPQ} - 2A^{(N}\wedge dA^P\wedge\delta_h A^{Q)} - \tfrac{3}{4}X_{RS}{}^{(N}A^{P|}\wedge A^R\wedge A^S\wedge\delta_h A^{|Q)}\\
&\quad -3\Lambda^{(N}F^P\wedge F^{Q)}\,, && \text{(E.3.5)}
\end{aligned}
$$

and their gauge-covariant field strengths are

$$
\begin{aligned}
F^M &= dA^M + \tfrac{1}{2}X_{[NP]}{}^M A^N\wedge A^P + Z^{MA}B_A\,, && \text{(E.3.6)}\\
H_A &= \mathfrak{D}B_A + T_{A\,RS}A^R\wedge[dA^S + \tfrac{1}{3}X_{NP}{}^S A^N\wedge A^P] + Y_{AM}{}^C C_C{}^M\,, && \text{(E.3.7)}\\
G_C{}^M &= \mathfrak{D}C_C{}^M + [F^M - \tfrac{1}{2}Z^{MA}B_A]\wedge B_C + \tfrac{1}{3}T_{C\,SQ}A^M\wedge A^S\wedge dA^Q\\
&\quad +\tfrac{1}{12}T_{C\,SQ}X_{NT}{}^Q A^M\wedge A^S\wedge A^N\wedge A^T\\
&\quad +W_C{}^{MAB}D_{AB} + W_{CNPQ}{}^M D^{NPQ} + W_{CNP}{}^{EM}D_E{}^{NP}\,, && \text{(E.3.8)}
\end{aligned}
$$

These field strengths are related by the following hierarchical Bianchi identities

$$\mathfrak{D}F^M = Z^{MA}H_A\,, \tag{E.3.9}$$

$$\mathfrak{D}H_A = Y_{AM}{}^C G_C{}^M + T_{A\,MN}F^M \wedge F^N\,. \tag{E.3.10}$$

E.4 Gauge transformations in the $D = 4$ duality hierarchy and action

In hierarchy variables, the total action takes the form

$$\begin{aligned} S \;=\; \int \big\{ & \star R - 2\mathcal{G}_{ij^*}\mathfrak{D}Z^i \wedge \star\mathfrak{D}Z^{*\,j^*} + 2F^\Sigma \wedge G_\Sigma - \star V \\ & -4Z^{\Sigma A}B_A \wedge \big(F_\Sigma - \tfrac{1}{2}Z_\Sigma{}^B B_B\big) - \tfrac{4}{3}X_{[MN]\Sigma}A^M \wedge A^N \wedge \big(F^\Sigma - Z^{\Sigma B}B_B\big) \\ & -\tfrac{2}{3}X_{[MN]}{}^\Sigma A^M \wedge A^N \wedge \big(dA_\Sigma - \tfrac{1}{4}X_{[PQ]\Sigma}A^P \wedge A^Q\big) \\ & -2\mathfrak{D}\vartheta_M{}^A \wedge (C_A{}^M + A^M \wedge B_A) + 2Q_{NP}{}^E(D_E{}^{NP} - \tfrac{1}{2}A^N \wedge A^P \wedge B_E) \\ & +2Q^{AB}D_{AB} + 2L_{NPQ}D^{NPQ}\big\}\,, \end{aligned} \tag{E.4.1}$$

A general variation of this action is given by

$$\begin{aligned} \delta S \;=\; \int \Bigg\{ & \delta g^{\mu\nu}\frac{\delta S}{\delta g^{\mu\nu}} + \left(\delta Z^i \frac{\delta S}{\delta Z^i} + \text{c.c.}\right) - \delta A^M \wedge \star\frac{\delta S}{\delta A^M} + 2\delta B_A \wedge \star\frac{\delta S}{\delta B_A} \\ & -2\mathfrak{D}\vartheta_M{}^A \wedge \delta C_A{}^M + 2Q_{NP}{}^E\delta D_E{}^{NP} + 2Q^{AB}\delta D_{AB} + 2L_{NPQ}\delta D^{NPQ} \\ & +\delta\vartheta_M{}^A\frac{\delta S}{\delta\vartheta_M{}^A}\Bigg\}\,, \end{aligned} \tag{E.4.2}$$

where

$$\frac{\delta S}{\delta g^{\mu\nu}} = \star\mathbb{I}\Big\{G_{\mu\nu} + 2\mathcal{G}_{ij^*}[\mathfrak{D}_\mu Z^i \mathfrak{D}_\nu Z^{*\,j^*} - \tfrac{1}{2}g_{\mu\nu}\mathfrak{D}_\rho Z^i \mathfrak{D}^\rho Z^{*\,j^*}] - G^M{}_{(\mu|}{}^\rho \star G_{M|\nu)\rho} + \tfrac{1}{2}g_{\mu\nu}V\Big\}\,, \tag{E.4.3}$$

$$\tfrac{1}{2}\frac{\delta S}{\delta Z^i} = \mathcal{G}_{ij^*}\mathfrak{D}\star\mathfrak{D}Z^{*\,j^*} - \partial_i G_M{}^+ \wedge G^{M+} - \star\tfrac{1}{2}\partial_i V\,, \tag{E.4.4}$$

$$\begin{aligned}-\tfrac{1}{4}\star\frac{\delta S}{\delta A^M} = {} & \mathfrak{D}F_M - \tfrac{1}{4}\vartheta_M{}^A \star j_A - \tfrac{1}{3}dX_{[PQ]M}\wedge A^P \wedge A^Q + \tfrac{1}{2}Q_{MP}{}^E C_E{}^P - \tfrac{1}{2}Q_{(NM)}{}^E A^N \wedge B_E \\ & - L_{MNP}A^N \wedge \big(dA^P + \tfrac{3}{8}X_{[RS]}{}^P A^R \wedge A^S\big) + \tfrac{1}{8}Q_{NP}{}^E T_{EQM} A^N \wedge A^P \wedge A^Q \\ & - d(F_M - G_M) - X_{[MN]}{}^P A^N \wedge (F_P - G_P) + \tfrac{1}{2}\mathfrak{D}\vartheta_M{}^A \wedge B_A\,,\end{aligned} \tag{E.4.5}$$

$$\star\frac{\delta S}{\delta B_A} = \vartheta^{PA}(F_P - G_P) + Q^{AB}B_B - \mathfrak{D}\vartheta_M{}^A \wedge A^M - \tfrac{1}{2}Q_{NP}{}^A A^N \wedge A^P\,, \tag{E.4.6}$$

$$\begin{aligned}\tfrac{1}{2}\frac{\delta S}{\delta \vartheta_M{}^A} = {} & (G_A{}^M - \tfrac{1}{2}\star\partial V/\partial\vartheta_M{}^A) - A^M \wedge (H_A + \tfrac{1}{2}\star j_A) \\ & + \tfrac{1}{2}T_{ANP}A^M \wedge A^N \wedge (F^P - G^P) - (F^M - G^M)\wedge B_A\,,\end{aligned} \tag{E.4.7}$$

and vanishes, up to total derivatives, for the gauge transformations

$$\delta_a \vartheta_M{}^A = 0\,, \tag{E.4.8}$$

$$\delta_a Z^i = \Lambda^M \vartheta_M{}^A k_A{}^i\,, \tag{E.4.9}$$

$$\delta_a A^M = \delta_h A^M\,, \tag{E.4.10}$$

$$\delta_a B_A = \delta_h B_A - 2T_{ANP}\Lambda^N (F^P - G^P)\,, \tag{E.4.11}$$

$$\delta_a C_A{}^M = \delta_h C_A{}^M + \Lambda_A \wedge (F^M - G^M) - \Lambda^M (H_A + \tfrac{1}{2}\star j_A)\,, \tag{E.4.12}$$

$$\delta_a D_{AB} = \delta_h D_{AB} + 2\Lambda_{[A} \wedge (H_{B]} + \tfrac{1}{2} \star j_{B]}) - 2T_{[A|\,NP}\Lambda^N (F^P - G^P) \wedge B_{|B]}\,, \tag{E.4.13}$$

$$\delta_a D_E{}^{NP} = \delta_h D_E{}^{NP} - \Lambda^N (G_E{}^P - \tfrac{1}{2} \star \partial V/\partial\vartheta_P{}^E) + 2(F^N - G^N) \wedge \Lambda_E{}^P\,, \tag{E.4.14}$$

$$\begin{aligned}\delta_a D^{NPQ} = {} & \delta_h D^{NPQ} - 3\delta A^{(N} \wedge A^P \wedge (F^{Q)} - G^{Q)}) + 6\Lambda^{(N} F^P \wedge (F^{Q)} - G^{Q)}) \\ & -3\Lambda^{(N}(F^P - G^P) \wedge (F^{Q)} - G^{Q)})\,,\end{aligned} \tag{E.4.15}$$

Appendix F

The Wilkinson-Bais monopole in $SU(3)$

In Ref. [103], Bais and Wilkinson derived the general spherically symmetric monopoles to the $SU(N)$ Bogomol'nyi equations. In this case we are going to discuss their monopole for the case of $SU(3)$ as it can be embedded into the $\overline{\mathbb{CP}}^8$, $ST[2,8]$ and the $SU(3,3)/S[U(3)\otimes U(3)]$ model.

The derivation is best done using Hermitean generators and in the fundamental, which means that we use the definitions

$$\mathfrak{D}\Phi \;=\; d\Phi - i\,[A,\Phi] \quad , \quad F \;=\; dA \;-\; i\;A\wedge A\;, \tag{F.0.1}$$

where A and Φ are $\mathfrak{su}(3)$-valued, and we have taken $g=1$.

The maximal form of the fields compatible with spherical symmetry are given by

$$\Phi \;=\; \tfrac{1}{2}\mathrm{diag}\left[\phi_1(r)\;;\;\phi_2(r)-\phi_1(r)\;;\;-\phi_2(r)\right]\;, \tag{F.0.2}$$

$$A \;=\; J_3\;\cos(\theta)d\varphi \;+\; \tfrac{i}{2}\left[C-C^\dagger\right]\;d\theta \;+\; \tfrac{1}{2}\left[C+C^\dagger\right]\;\sin(\theta)d\varphi\;, \tag{F.0.3}$$

where $J_3=\mathrm{diag}(1;0;-1)$ and C is the real and upper-triangular matrix

$$C \;=\; \begin{pmatrix} 0 & a_1(r) & 0 \\ 0 & 0 & a_2(r) \\ 0 & 0 & 0 \end{pmatrix}\;. \tag{F.0.4}$$

Plugging the above Ansätze into the Bogomol'nyi equation $\mathfrak{D}\Phi=\star F$, leads to the following equations ($i=1,2$)

$$r^2\partial_r\phi_i \;=\; a_i^2 - 2 \;\;,\;\; 2\partial_r a_1 \;=\; a_1\,(2\phi_1-\phi_2) \;\;,\;\; 2\partial_r a_2 \;=\; a_2\,(2\phi_2-\phi_1)\;. \qquad \text{(F.0.5)}$$

Following Wilkinson and Bais [103], we solve the equations for the a_i by defining new functions $Q_i(r)$ through

$$\phi_i \;=\; -\partial_r \log Q_i + \tfrac{2}{r} \;\;,\;\; a_1 \;\equiv\; \frac{r\sqrt{Q_2}}{Q_1} \;\;,\;\; a_2 \;\equiv\; \frac{r\sqrt{Q_1}}{Q_2}\;, \qquad \text{(F.0.6)}$$

after which the remaining equations are

$$Q_2 \;=\; \partial_r Q_1\partial_r Q_1 \;-\; Q_1\partial_r^2 Q_1 \;\;,\;\; Q_1 \;=\; \partial_r Q_2\partial_r Q_2 \;-\; Q_2\partial_r^2 Q_2 \qquad \text{(F.0.7)}$$

The solution found by Wilkinson & Bais for $SU(3)$ then given by

$$\left.\begin{array}{rcl} Q_1 &=& \sum_{a=1}^3\; A_a\; e^{\mu_a r} \\ && \\ Q_2 &=& \sum_{a=1}^3\; A_a\; e^{-\mu_a r} \end{array}\right\} \longleftarrow \left\{\begin{array}{rcl} 0 &=& \sum_{a=1}^3\;\mu_a \\ && \\ A_1 &=& -A_2A_3\;\left(\mu_2-\mu_3\right)^2 \\ A_2 &=& -A_3A_1\;\left(\mu_3-\mu_1\right)^2 \\ A_3 &=& -A_1A_2\;\left(\mu_1-\mu_2\right)^2 \end{array}\right. \;. \qquad \text{(F.0.8)}$$

The solution to the above equations is

$$A_a \;=\; \prod_{b\neq a}\left(\mu_a-\mu_b\right)^{-1}\;. \qquad \text{(F.0.9)}$$

Defining the useful quantity $V_n \equiv \sum_{a=1}^3\; A_a\mu_a^n$, we can see by direct inspection that $V_0 = V_1 = V_3 = 0$ and that $V_1 = 1$. Using these quantities one can see that around $r=0$ we see that $Q_i \sim r^2/2 + \mathcal{O}(r^3)$, which means that the $\phi_i \sim -V_4/3!\; r + \mathcal{O}(r^2)$, implying that the solution is completely regular on $\mathbb{R}^3$. Furthermore, one can show that the Q are monotonic, positive semi-definite functions on $\mathbb{R}^+$ that vanish only at $r=0$, at which point also its derivative vanishes. This furthermore implies that the ϕ_i are negative semi-definite functions on $\mathbb{R}^+$.

The asymptotic behaviour of the Higgs field is easily calculated and, choosing $\mu_1<\mu_2<\mu_3$, is readily seen to be

$$\lim_{r\to\infty}\Phi \;=\; -\tfrac{1}{2}\,\mathrm{diag}\,(\;\mu_3\;;\;\mu_2\;;\;\mu_1\;)\;+\;\frac{1}{r}\;J_3\;+\ldots \qquad \text{(F.0.10)}$$

from which the breaking of $SU(3)\to U(1)^2$ is paramount.

The above solution does not admit the possibility of having degenerate μ's, but as emphasised by Wilkinson & Bais, such a solution can be obtained as a limiting solution. For this, define $\mu_1 = -2$, $\mu_2 = 1-\delta$ and $\mu_3 = 1+\delta$, for $\delta > 0$, and calculate the solution. This solution admits a non-singular $\delta \to 0$ limit, which is

$$Q_1 \;=\; \tfrac{1}{9}\left[e^{-2r} \;+\; (3r-1)e^{r}\right] \;\;,\;\; Q_2 \;=\; \tfrac{1}{9}\left[e^{2r} \;-\; (3r+1)e^{-r}\right] \;. \tag{F.0.11}$$

The symmetry breaking pattern in this degenerate case is $SU(3) \to U(2)$ as becomes clear from the asymptotic behaviour of the Higgs field, *i.e.*

$$\lim_{r\to\infty} \Phi \;=\; -\mathtt{Y} \;+\; \frac{1}{r}\,\mathtt{Y} \;\text{ where }\; \mathtt{Y} \;=\; \tfrac{1}{2}\,\mathrm{diag}\,(1\;,\;1\;,\;-2)\;. \tag{F.0.12}$$

F.1 A hairy deformation of the W&B monopole

The foregoing derivation of Wilkinson & Bais's monopole was cooked up to give a regular solution, and we would like to have a hairy version of this monopole. This is easily achieved by applying the Protogenov trick, which calls for adding constants in the exponential parts of the monopole fields; in this case, we simply extend the Ansatz for the Q_i's to

$$Q_1 \;=\; \sum_{a=1}^{3}\; A_a\; e^{\mu_a r+\beta_a} \;\;,\;\; Q_2 \;=\; \sum_{a=1}^{3}\; A_a\; e^{-\mu_a r-\beta_a} \;, \tag{F.1.1}$$

and plug it into Eq. (F.0.7). Obviously this leads to a solution if $\sum\;\mu_a = \sum\beta_a = 0$ and A_a is once again given by Eq. (F.0.9). Furthermore, it is clear that the asymptotic behaviour does not change and it is the one in Eq. (F.0.10); what does change is the behaviour of the solution at $r = 0$, which is singular except when $\beta_a = 0$.

Using the above expression we can also create a hairy version of the degenerate monopole: we have to make the same Ansatz as the one used in the derivation of Eq. (F.0.11), and also define $\beta_2 = s+\delta\gamma/3$, $\beta_3 = s-\delta\gamma/3$ and $\beta_1 = -2s$, which is the maximal possibility compatible with a regular limit. Taking then the limit $\delta \to 0$ we find

$$Q_1 \;=\; \tfrac{1}{9}\left[e^{-2(r+s)} \;+\; (3r+\gamma-1)e^{r+s}\right] \;\;,\;\; Q_2 \;=\; \tfrac{1}{9}\left[e^{2(r+s)} \;-\; (3r+\gamma+1)e^{-(r+s)}\right] \;. \tag{F.1.2}$$

which leads to ϕ_i's that are singular at $r = 0$ but with the asymptotic behaviour displayed in Eq. (F.0.12).

Appendix G

Publications

- M. Hübscher, P. Meessen and T. Ortín,

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