Universidad Autónoma de Madrid
Facultad de Ciencias
Departamento de Física Teórica

Solitons, Vacua and Gauge Duals in Supergravity

Memoria de Tesis Doctoral realizada por
D. Ernesto Lozano Tellechea,
presentada ante el Departamento de Física Teórica
de la Universidad Autónoma de Madrid
para la obtención del Título de Doctor en Ciencias.

Tesis Doctoral dirigida por
D. Tomás Ortín Miguel,
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Introducción

Encontrar la formulación de una única teoría que describa y prediga todo lo que observamos en la Naturaleza es probablemente el objetivo más ambicioso de la Física Teórica. Una forma en la que creemos que dicho fin puede llevarse a cabo es mediante una formulacion matemática tanto para los constituyentes últimos de la materia como para la forma en la que éstos interaccionan entre sí. Actualmente, tenemos una teoría que encaja perfectamente con esta última idea: el Modelo Estándar de las partículas elementales. Esta teoría tiene la virtud de ser conceptualmente sencilla, en el sentido de que sólo contiene un número relativamente pequeño de partículas elementales diferentes y, por otro lado, porque describe la dinámica de las mismas en términos de tan solo tres interacciones fundamentales distintas. Por otro lado, las predicciones del Modelo Estándar muestran un acuerdo sorprendentemente bueno con todos los experimentos de Altas Energías que hasta la fecha han sido llevados a cabo. Sin embargo, sabemos la descripción de la Naturaleza que nos proporciona el Modelo Estándar no es completa, dado que éste no incluye a la interacción gravitatoria.

Por otro lado, es cierto que tenemos una teoría clásica de la gravitación: la Relatividad General. La misma se trata de una teoría muy diferente de cualquier otra que conocemos: de acuerdo con la misma, la propia geometría del espaciotiempo es algo dinámico, esto es, podemos imaginar al espacio y al tiempo como una superficie que se deforma ante la presencia de materia o energía. Esto puede parecer algo extraño, pero todas las predicciones realizadas por la Relavidad General que han podido ser puestas a prueba experimentalmente han sido confirmadas con gran éxito. El problema con esta teoría proviene de su naturaleza clásica, pues nadie, hasta el momento, ha conseguido formularla de una forma que la haga compatible con la Mecánica Cuántica. Creemos que conseguir esto se trata de algo muy importante por dos razones: por un lado, existen numerosas pistas que apuntan que, a distancias suficientemente pequeñas (distancias a las que, por otra parte, no hemos sido aun capaces de comprobar experimentalmente la Relatividad General), el comportamiento de cualquier sistema físico tiene que quedar descrito por una teoría cuántica. Por otro lado, una teoría cuántica de la gravedad parece ser el único camino posible para unificar la gravitavión con el resto de las interacciones fundamentales.

Actualmente, nuestra mejor propuesta para una teoría que cumpla ambos requisitos (una teoría unificada de todas las interacciones y una teoría cuántica de la gravedad) es la Teoría de Cuerdas. Aunque, históricamente, el descubrimiento de la misma no tuvo nada que ver con éste propósito [1], tal y como la entendemos hoy en día esta teoría puede entenderse como algo que surge a partir de una idea muy sencilla: considerar a los constituyentes últimos de la Naturaleza no como partículas puntuales, sino como objetos que tienen una dimensión ("cuerdas") [2]. El hecho de que las partículas elementales que conocemos se comporten como puntos se puede explicar fácilmente si estas cuerdas son lo son suficientemente pequeñas. Por otro lado, partículas que vemos como distintas no se corresponderían con distintos tipos de cuerdas, sino con la "misma" cuerda oscilando de formas diferentes [3, 4].

Las teorías de cuerdas tienen algunas propiedades que las hacen muy interesantes desde un punto de vista teórico. En primer lugar, las cuerdas pueden ser cuantizadas de una forma matemáticamente consistente. Por otro lado, si uno intenta formular una teoría cuántica de cuerdas que interaccionan entre sí, la propia consistencia matemática de la teoría implica que la misma ha de incluir a la gravedad, constituyendo, de esta forma, una teoría cuántica de la gravitación. Finalmente, la física de la teoría de cuerdas resulta ser lo suficientemente rica como para poder incluir, en principio, al Modelo Estándar [5–9]. Estas propiedades son exactamente las que esperaríamos encontrar en una teoría unificada.

El problema de la Teoría de Cuerdas es que todas sus predicciones están fuera del alcance de los experimentos actuales. Aun así, si en los últimos treinta años se han dedicado grandes esfuerzos al estudio de la Teoría de Cuerdas, esto no se debe sino a su gran consistencia matemática. Es esto lo que ha permitido un enorme progreso en este campo a pesar de la total ausencia de confirmaciones (o refutaciones) experimentales de la teoría.

Teorías de Supercuerdas

La cuestión es, pues, cuáles son las posibles teorías de cuerdas que podemos considerar. Si sólo consideramos las llamadas teorías de cuerdas "críticas"[1], exigir la ausencia de taquiones y la presencia de fermiones en el espectro de la cuerda nos deja con sólo cinco teorías de cuerdas posibles [7, 10]. Éstas son las Teorías de Supercuerdas Tipo IIA, Tipo IIB, Tipo I, Heterótica-$SO(32)$ y Heterótica-$E_8 \times E_8$. Todas ellas son supersimétricas y todas ellas necesitan de un espaciotiempo de diez dimensiones. De esta manera, vemos que imponer estas dos condiciones (por otro lado, bastante naturales) sobre el espectro de la cuerda exigen la presencia de supersimetría y la idea original de Kaluza-Klein de un espaciotiempo de más de cuatro dimensiones [11]. Estas propiedades son muy buenas, dado que permiten muchas posibilidades para poder explicar, de una forma unificada, toda la física observada a bajas energías [12]. Pero entonces la pregunta natural es: ¿de qué manera surge el Universo tal y como lo vemos (no supersimétrico y con cuatro dimensiones) de uno supersimétrico y con diez dimensiones? Este problema es el de la elección de vacío de la Teoría de Cuerdas y, hasta el momento, no tiene una solución definitiva.

Una forma de conseguir pistas sobre cuál debe ser la solución de este problema puodría venir de una aproximaxión "fenomenológica". Con esto nos referimos a tratar de encontrar, "a mano", un estado fundamental en Teoría de Cuerdas, de modo que encontremos uno (al menos) que reproduzca lo mejor posible la Física que observamos [13–15]. Tal objetivo ha de resultar posible si la Teoría de Cuerdas es correcta, y creemos que obtener resultados satisfactorios en esta dirección es algo extremadamente importante. Por un lado, esto podría proporcionarnos predicciones propias de Teoría de Cuerdas que sea posible comprobar en futuros experimentos. Por otro lado, algo así podría, quizá, darnos pistas acerca de cuál es sobre el verdadero vacío de la Teoría de Cuerdas, sobre el mecanismo que selecciona dicho vacío y, en general, acerca de la estructura completa de la teoría.

Otra posibilidad para encarar este problema es intentar avanzar, a partir de primeros principios, en nuestro conocimiento de la estructura completa, no perturbativa, de la teoría. La Teoría de Cuerdas resulta ser una teoría extremadamente complicada, y se puede decir que aún no conocemos su

[1]Las cuerdas no críticas resultan ser de una extraoirdinaria compejidad y no tienen una interpretación clara.

formulación definitiva. Un conocimiento completo de la misma debería de darnos una respuesta a problemas aún abiertos, como el de la constante cosmológica, la ruptura de supersimetría, o cómo el Modelo Estándar y la Relatividad General surgen a partir de la Teoría de Cuerdas. Los resultados que presentamos en esta tesis deben considerarse, en líneas generales, como esfuerzos en esta dirección, esto es, como esfuerzos dedicados a revelar la estructura matemática *formal* de Teoría de Cuerdas.

Dualidad, Branas y Teoría M

En la última década se ha descubierto y desarrollado una pieza fundamental que concierne a la estructura formal de la Teoría de Cuerdas. Esta pieza es lo que hoy llamamos "dualidades" [16–20]. Este avance ha venido acompañado del descubrimiento de que las teorías de cuerdas no sólo contien cuerdas, sino que, además, su espectro no perturbativo incluye objetos extensos de carácter solitónico [21]. Estas configuraciones se denominan genéricamente "branas", y han jugado un papel crucial en todos los avances recientes en Teoría de Cuerdas [22].

Las dualidades son fundamentales en nuestra concepción actual de lo que debe ser la estructura completa de la Teoría de Cuerdas. Las mismas constituyen simetrías muy especiales: en general, no son simetrías físicas de un sistema dado, sino que relacionan bien diferentes teorías o bien diferentes "backgrounds" o regímenes dentro de una misma teoría. Muchas veces las utilizamos para trasladar un problema difícil en una teoría dada a otro más sencillo en otra teoría distinta, o bien para formular el mismo en términos de una situación física diferente en la cual es más fácil de resolver. Pero las dualidades no son solamente "herramientas útiles": también contienen implicaciones profundas en nuestra forma de concebir la Teoría de Cuerdas tal y como la conocemos. Gracias a esto, hoy sabemos que las cinco Teorías de Supercuerdas están relacionadas entre sí, de una forma u otra, mediante transformaciones de dualidad. Esto cambia drásticamente la visión que teníamos de las cinco diferentes Supercuerdas, dado que motiva fuertemente la posibilidad de que estas teorías, en apariencia distintas, sean en realidad diferentes expansiones perturbativas de una única teoría que las subyace. Esta hipotética teoría se conoce bajo el nombre de "Teoría M" [19, 20].

Las dualidades implican, por otro lado, que la Teoría de Cuerdas contiene un espectro completo de estados que no aparecen en el espectro perturbativo de la cuerda. Los mismos son D-branas [22] y otros muchos estados relacionados de tipo solitónico [21]. Las principales propiedades de este tipo de objetos son, por un lado, su carácter extenso, y por otro, el hecho de que los mismos contienen teorías gauge en su "worldvolume". Esta propiedad es el origen de muchos nuevos intentos de introducir el Modelo Estándar en Teoría de Cuerdas [14], y también es el origen de las profundas relaciones entre teorías gauge y teorías de cuerdas que han sido encontradas y desarrolladas en los últimos años [23].

Supergravedad

Esta tesis está dedicada al estudio de la Teoría de cuerdas mediante el uso, para ello, de Supergravedad. Como explicaremos en el Capítulo 1, Supergravedad aparece como la descripción efectiva a bajas energías de la Teoría de Cuerdas [24, 25]. Además, la primera supone el límite clásico de la segunda. Tratándose de un límite particular, Supergravedad no puede describir toda la física que la Teoría de Cuerdas contiene, pero veremos que muchos aspectos no perturbativos de la Teoría de Cuerdas se pueden observar también en la aproximación de Supergravedad. Si esto es posible es principalmente debido a la existencia de supersimetría y a la existencia de dualidades. Supersimetría hace que, muchas veces, los resultados obtenidos en el límite de Supergravedad resulten fiables. Por otro lado, las dualidades son una propiedad de la Teoría de cuerdas que también se ve y puede explotarse en Supergravedad, dado que aquí aparecen como simetrías de las acciones y/o de las ecuaciones de movimiento [26]. En particular, todos los objetos solitónicos extensos, cuya existencia las dualidades predicen y que consideramos como elementales[2] aparecen también en Supergravedad. Todos estos son aspectos no perturbativos de la Teoría de Cuerdas que, como hemos destacado, son una parte muy importante de la teoría que todavía no entendemos bien.

Supergravedad también nos proporciona una buena descripción de lo que se cree que es otra "esquina" de la teoría M y que no es ninguna de las cinco Teorías de Supercuerdas conocidas. Esta "esquina" extra es la teoría de Supergravedad en once dimensiones. La misma se conoce desde hace mu-

[2] Con esto queremos decir que no se consideran objetos compuestos de otros solitones más fundamentales.

chos años [27], y es especial es única. Once resulta ser el número máximo de dimensiones en las que es posible formular consistentemente una teoría supersimétrica y, por otro lado, no hay varias posibilidades para distintas teorías en once dimensiones: sólo hay una. La misma está, por otro lado, muy conectada con Teoría de Cuerdas: la reducción de Kaluza-Klein de esta teoría en un círculo no es más que la acción efectiva de la cuerda Tipo IIA [28], y la reducción en un segmento proporciona el límite a bajas energías de la teoría Heterótica-$E_8 \times E_8$ [29]. Teniendo en cuenta que las teorías de Supergravedad en diez dimensiones describen la física a bajas energías de las supercuerdas, es bastante lógico suponer que Supergravedad en once dimensiones también proporciona el límite a bajas energías de alguna teoría cuántica aún por descubrir. Dicha teoría podría muy bien ser la propia Teoría M.

En esta tesis afrontamos varios aspectos de las teorías de Supergravedad. Las teorías de Supergravedad son interesantes por sí mismas, pero nuestro enfoque aquí es considerarlas como una descripción efectiva de las Teorías de Cuerdas. En este sentido, consideramos Supergravedad como una herramienta para extraer información sobre la Teoría de Cuerdas. Por lo tanto, todos los resultados contenidos en las siguientes páginas intentan suponer alguna utilidad si se los interpreta en el contexto de Teoría de Cuerdas. Este es el motivo por el cual hemos intentado, a lo largo de los capítulos introductorios, hacer manifiestas las relaciones entre ciertos aspectos concretos de Supergravedad y de Teoría de Cuerdas.

Resumen de Contenidos

El contenido de los siguientes capítulos se ha dividido en tres Partes diferentes, teniendo en cuenta su naturaleza común.

La **Parte I** está relacionada con soluciones de Supergravedad de las que puede argumentarse que están relacionadas con estados no perturbativos de cuerdas. En el Capítulo 3 consideraremos las posibles soluciones tipo agujero negro de la Teoría de Supergravedad con $N = 4$, $d = 4$. En particular, estudiaremos qué posibles soluciones físicas de una teoría dada están incluidas en las órbitas de su grupo de dualidades, tanto en los casos supersimétricos como en los que no lo son. En el Capítulo 4 consideraremos esta idea aplicada a diez y once dimensiones. Encontraremos una familia completa de soluciones solitónicas de Supergravedad 1/2 BPS que argumentaremos que deben corresponder a estados elementales de cuerdas que no aparecen en el espectro no perturbativo de cuerdas conocido. También mostraremos cómo sus cargas centrales asociadas pueden incluirse en el superálgebra correspondiente.

La **Parte II** está relacionada con soluciones de Supergravedad que se pueden entender como vacios de cuerdas supersimétricos. En el Capítulo 6 mostraremos como todos los vacíos máximamente supersimétricos conocidos con ocho supercargas en cuatro, cinco y seis dimensiones se relacionan mediante reducción dimensional. En particular, mostraremos que la existencia, en cinco dimensiones, de una familia continua de vacíos máximamente supersimétricos (que interpolan entre $AdS_2 \times S^3$ y $AdS_3 \times S^2$) admite una explicación sencilla en función de dualidad electromagnética en cuatro dimensiones. En el Capítulo 7 presentaremos una nueva construcción para encontrar los espinores de Killing en espaciotiempos homogéneos máximamente supersimétricos. Mostraremos cómo la supersimetría de estos espacios está codificada en su descripción en términos de teoría de grupos. Además, también mostraremos cómo las superálgebras correspondientes de estos espacios coset se pueden calcular usando esta sencilla construcción geométrica. En el Capítulo 8 presentaremos la descripción como coset del límite la solución en $d = 5$ que describe el límite cercano al horizonte del agujero negro extremo con rotatación. El mismo era el único espaciotiempo máximamente supersimétrico que no tenía previamente una descripción conocida en términos de un espacio homogéneo.

La **Parte III** está relacionada con los duales en Supergravedad de ciertas teorías gauge. En el Capítulo 10 encontraremos dos geometrías distintas que son duales a la misma teoría gauge: $N = 4$ (ocho supercargas), $d = 3$ SYM. Una configuración describe branas fraccionarias en un cierto orbifold, y la otra nos da el álgebra producida por una configuración de D-branas enrolladas en ciclos supersimétricos. Se supone que ambos sistemas de branas están relacionados en cierto límite, y estudiaremos este aspecto desde el punto de vista de sus soluciones de supergravedad. También comprobaremos sus respectivas predicciones para la función beta a un loop de la teoría gauge.

Los **Capítulos 1, 2, 5 y 9** son capítulos introductorios. En el Capítulo 1 resumimos la conexión entre Supergravedad y Teoría de Cuerdas y la forma en la que la primera durge de la segunda. El resto de capítulos son una introducción al tema general que se estudia en la parte a la que pertenecen. en ellos hemos tratado de explicar lo mejor posible los conceptos que consideramos necesarios para entender el tema general correspondiente a cada Parte. Nuestra intención ha sido, asimismo, motivar de la mejor manera posible las partes más técnicas de este trabajo.

Los **Capítulos 3, 4, 6, 7, 8 y 10** se corresponden, respectivamente, con los resultados publicados en [30–35].

Introduction

An ambitious aim of Theoretical Physics is to find a single theory able to describe and predict everything we see in Nature. A way in which we think that this can be done is to find a mathematical formulation for the fundamental building blocks of matter and the way in which they interact with each other. At present, we have a very precise theory that exactly fits into this idea: the Standard Model of elementary particles. It has the virtue of being conceptually simple, at least in the sense that it just contains a relatively small number of different elementary particles, and also in the sense that their dynamics is described in terms of only three different fundamental interactions. In addition, its predictions agree extremely well with all experiments. However, the description of Nature provided by the Standard Model is not complete, because it does not include gravity.

We also have a classical theory of gravitation, which is General Relativity. This theory is rather different from any other. According to it, the geometry of spacetime is dynamical: spacetime can be seen as a surface whose shape gets modified by the presence of any matter or energy density. This may look strange, but all the predictions made by General Relativity that we have been able to test with some experiment have been successfully confirmed. The problem with it arises because it is a classical theory, and no one has succeeded in making it compatible with Quantum Mechanics. On the one hand, there is strong evidence that at short enough distances (distances at which we have not been able to test General Relativity with experiments), the behaviour of any physical system must be described by a quantum theory. On the other hand, the only way in which it seems possible to unify gravity with the remaining interactions is via a quantum theory of gravity.

At present, the most promising candidate we have for a theory that fulfills both requirements (i.e. a unified theory of all interactions, and also a theory of quantum gravity) is String Theory. Although historically it was not discovered with this purpose in mind [1], the way in which we nowadays understand String Theory makes it arise from a very simple idea: namely, that the fundamental constituents in Nature are not pointlike particles, but one-dimensional objects ("strings") [2]. The fact that all elementary particles we know behave as points can be explained if the strings are sufficiently small, and different particles would not correspond to different kinds of strings, but to the "same" string oscillating in different ways [3, 4].

String theories have some properties which make them very appealing from a theoretical point of view. To start with, strings can be consistently quantized. Moreover, if one tries to formulate a quantum theory of interacting strings, mathematical consistency requires that such a theory must necessarily include gravity, hence providing a quantum theory of it. In addition, their physics is rich enough to contain, in principle, the full Standard Model [5–9]. These are exactly the features that we would expect to find in a unified theory.

The problem of String Theory is that all its predictions are beyond the reach of present experiments. If in the last thirty years many efforts have been devoted to the study of String Theory it is because of its strong mathematical consistency. This has allowed to make a huge progress in this field in spite of the absolute lack of experimental tests of it.

Superstring Theories

The question is then what are the possible string theories that we can consider. If we restrict ourselves to the so-called "critical" string theories[3], then requiring the absence of tachyons and the presence of fermions in the string spectrum leaves us with just five possible consistent theories of interacting strings [7, 10]. These are the Type IIA, Type IIB, Type I, Heterotic-$SO(32)$ and Heterotic-$E_8 \times E_8$ Superstring Theories. All of them are supersymmetric, and all of them require spacetime to be ten dimensional. We see therefore that imposing these two (quite natural) conditions on the string spectrum implies both Supersymmetry and the original idea of Kaluza-Klein of a higher

[3]Noncritical strings are extremely difficult to deal with and admit no clear interpretation.

dimensional spacetime [11]. These are very good properties because they allow for many possibilities to explain in a unified way all the low energy observed Physics [12]. But then the natural question is: how does the Universe as we see it (non-supersymmetric and four-dimensional) arise from a supersymmetric, ten-dimensional one? This is the question of the choice of vacuum in String Theory and, at present, it has no definite answer.

A way to get a hint about the answer to this question could be a "phenomenological" approach. By this we mean to choose by hand a String Theory ground state, in such a way that we find one (at least) that reproduces the observed Physics [13–15]. This may look insatisfactory from a theoretical point of view, but such a thing has to be possible if String Theory is correct, and we think that obtaining positive results in these directions is extremely important. On the one hand, this could eventually provide us with genuine String Theory predictions concerning future experiments. On the other hand, it could provide us some essential hints about what the true String Theory vacuum is, the mechanism that selects it and, in general, the full structure of the theory.

Another possibility to face this problem is to try to gain insight, from first principles, into the full nonperturbative structure of the theory. String Theory turns out to be an extremely complicated theory, and it can be said that the whole theory is not known yet. A full knowledge of the theory should provide us with the answer to questions concerning key open problems, such as the cosmological constant, supersymmetry braking, or how the Standard Model and General Relativity emerge from String Theory. The results reported in this thesis must be considered, in general lines, as efforts in this direction, i.e. as efforts devoted to uncover the *formal* mathematical structure of String Theory.

Duality, Branes and M-Theory

In the last decade, a fundamental piece concerning the formal structure of String Theory was discovered and developed: what today we call "dualities" [16–20]. This progress came together with the discovery that String Theory contains not only strings, but also many other solitonic extended objects in its nonperturbative spectrum [21]. These are generally referred to as "branes" and they have played an essential role in all recent developments in String Theory [22].

Duality symmetries are fundamental in the present understanding that we have about the full structure of String Theory. They are very special symmetries: in general, they are not physical symmetries of a given system, but rather they relate different theories or different backgrounds or physical regimes of the same theory. Many times they are used to map a difficult problem of a given theory into an easier problem of another theory, or else to map it into a different physical situation in which it becomes easy to solve. But they are not only "useful tools": they also have deep implications about the way in which we should think about String Theory. It turns out that all of the five Superstring Theories that we know are related, in one way or another, by means of a duality transformation. This changes drastically the picture that we had concerning the different Superstrings, because it strongly suggests the possibility that all these different-looking theories could be just different perturbative expansions of a unique underlying theory. This hypothetical theory is referred to as "M-theory" [19, 20].

Duality also implies that String Theory contains a whole spectrum of states which do not arise in the perturbative string spectrum. These are D-branes [22] and many other related solitonic states [21]. The main properties of these objects is that they are extended and that they have gauge theories living in their worldvolume. This is at the origin of many novel attempts to embed the Standard Model into String Theory [14], and also at the origin of the deep relations between gauge theories and string theories found and developed in the last years [23].

Supergravity

This thesis is devoted to the study of String Theory by means of Supergravity. As explained in Chapter 1, Supergravity arises as the low energy effective description of string physics [24, 25]. Moreover, Supergravity is a classical limit of String Theory. Being a particular limit, it does certainly not describe all the physics contained within the full String Theory, but we will see that many nonperturbative aspects of String Theory can be seen from the Supergravity approach. If this is possible it is mainly because of the existence of supersymmetry and duality. Supersymmetry ensures many times reliability of the Supergravity approximation. Duality is a String Theory property that can still be seen and exploited in the Supergravity limit, since it translates into global symmetries of the Supergravity actions and/or equations of motion [26]. In particular, all supersymmetric solitonic extended objects predicted by duality which are thought to be elementary[4] have a Supergravity counterpart. All these are nonperturbative aspects of String Theory which, as emphasized, are a most important part of the theory which still remains to be fully understood.

Supergravity also provides us with a well description of what is thought to be another "corner" of M-theory, one which is not any of the five known Superstring Theories. This extra "corner" is eleven-dimensional Supergravity. This theory was discovered many years ago [27], and it is very special because it is unique. Eleven is the maximal number of dimensions in which it is possible formulate a consistent supersymmetric theory. Furthermore, there are not several possibilities for different eleven-dimensional theories: there is just one. Eleven-dimensional supergravity is also closely linked to String Theory: its Kaluza-Klein reduction on a circle yields the Type IIA effective action [28], and reduction on a segment gives the low energy limit of the Heterotic-$E_8 \times E_8$ theory [29]. Taking into account the fact that ten-dimensional supergravities describe the low energy physics of superstrings, it is very natural to suppose that eleven-dimensional supergravity also represents the low energy limit of some, so far undiscovered, quantum theory. Such a quantum theory could very well be M-theory itself.

In this thesis we address several aspects of Supergravity theories. Supergravity theories are interesting by themselves, but the point of view adopted here will be to look at them as an effective description of String Theory. In this sense, Supergravity is to be understood here as a tool to extract String Theory information. Therefore all the results reported in the pages that will follow attempt to be of some use when interpreted in a String Theory context. This is why in the introductory Chapters we have tried to explain the relation between some specific aspects of Supergravity and their String Theory counterpart.

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[4] By this we mean that they are not understood as composites of other, more elementary solitons.

Summary of Contents

The contents of the forthcoming Chapters have been separated into three different Parts according to their common nature.

Part I deals with Supergravity solutions which can be argued to correspond to nonperturbative string sates. In Chapter 3 we focus on the possible black hole solutions of the $N = 4, d = 4$ supergravity theory. Here we address the question of what possible physical solutions of a given theory are contained in the orbits of its full duality group, both in the supersymmetric cases and in the non-supersymmetric ones. In Chapter 4 we pursue this idea but now applied to ten and eleven dimensions. There we find a whole family of 1/2 BPS solitonic supergravity solutions which we argue that should correspond to elementary (in the sense explained above) string states that would be missing from the known nonperturbative superstring spectrum. We also show how their associated central charges can appear in the corresponding superalgebra.

Part II deals with supergravity solutions which can be understood as supersymmetric string vacua. In Chapter 6 we show how all known maximally supersymmetric vacua with eight supercharges in four, five and six dimensions are related by uplifting and dimensional reduction. In particular, it is shown that the existence, in five dimensions, of a continuous family of maximally supersymmetric vacua (which interpolates between $AdS_2 \times S^3$ and $AdS_3 \times S^2$) admits a simple explanation in terms of four-dimensional electric-magnetic duality. In Chapter 7 a novel construction for finding the Killing spinors in maximally supersymmetric homogeneous spacetimes is presented. We show how the supersymmetry of these spaces is encoded in their group-theoretical description, and we also show how the corresponding superalgebras of these coset spaces can be computed using this very simple geometrical construction. In Chapter 8 we present the coset description of the near-horizon limit of the extreme rotating black hole in $d = 5$. This was the only maximally supersymmetric spacetime for which a description in terms of a homogeneous space was not previously known.

Part III deals with the Supergravity duals of certain gauge theories. In Chapter 10 we find two different geometries which are dual to the same gauge theory, namely $N = 4$ (eight supercharges), $d = 3$ SYM. One supergravity configuration describes fractional branes in a certain orbifold, and the other one gives the geometry produced by a configuration of D-branes wrapping supersymmetric cycles. Both brane systems are supposed to be related in a certain limit, and we explore this fact from the point of view of their supergravity solutions. We also check the independent predictions that they are supposed to give for the one-loop gauge theory beta function.

Chapters 1, 2, 5 and 9 are introductory chapters. In Chapter 1 we review the connection between Supergravity and String Theory and the way in which the former emerges from the latter. The remaining Chapters are an introduction to the general topic to which the corresponding Part is devoted. In them, we have tried to explain as best as possible the different concepts that we believe are needed for the general understanding of the corresponding subject. Our aim has also been to motivate at best the technical parts of this work.

Chapters 3, 4, 6, 7, 8 and 10 correspond, respectively, to the results reported in [30–35].

Chapter 1

Supergravity and String Theory

1.1 Supergravity and Effective String Theories

There are five ten dimensional supergravities which are considered as the corresponding low energy effective theories of the five known consistent superstring theories. What is the precise meaning of this? Moreover, supergravity theories are field theories. How does this limit emerge from String Theory?

Any superstring theory contains, in its perturbative spectrum, a finite number of massless excitations and an infinite number of massive states. The masses squared of the massive states are proportional to the string tension

$$M^2 \sim \frac{1}{\alpha'}, \tag{1.1.1}$$

where the coefficient is given by the oscillator number of the corresponding string state. Since this oscillator number can take arbitrarily high integer values, an infinite set of massive modes is contained within the string spectrum. Strings have not been seen yet, and so, if they really exist, the fundamental constant α' has to be a very small quantity, since it also sets the string length, which is of order $l_s \equiv \sqrt{\alpha'}$ [1]. Then even the first excited string state will be a very massive one, and, as a consequence, it makes sense to consider the

[1] Of course, α' being dimensionful, when we say "small" we mean "small when measured in appropriate units". Very small when, for example, measured in units of the inverse top mass squared.

(presumably very wide) energy regime in which the string physics is governed just by the light modes of the spectrum –energies much lower than the mass of the first excited string state.

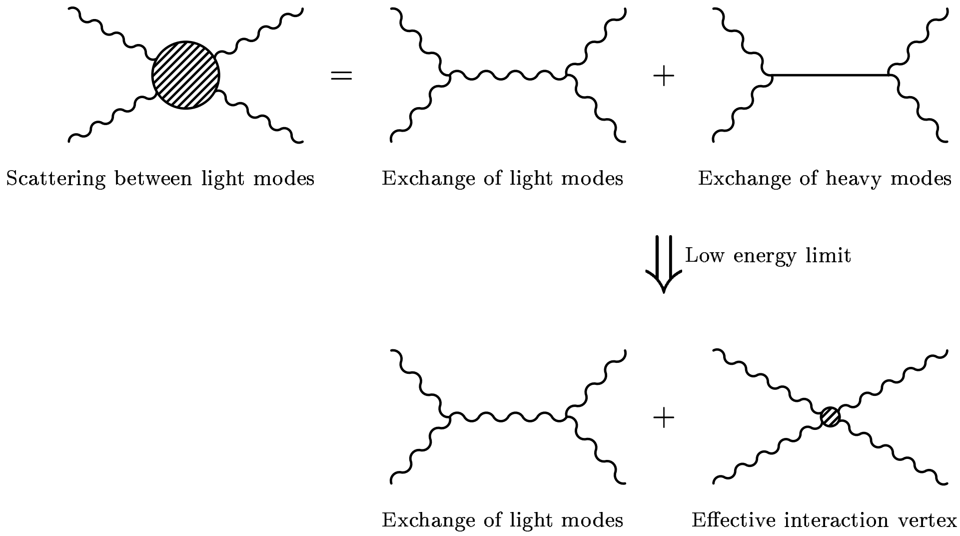
String Theory provides us with an algorithm for computing scattering amplitudes[2]. This algorithm is perturbative in the string coupling constant g_s and, therefore, computations can only be carried out if we *assume* that g_s is small. In particular, we will be interested in string S-matrix elements that only involve massless states in the external legs, which at small enough energies will be the only allowed processes. Also, in the computation of these S-matrix elements, we must decide at which order in perturbation theory we will stop. We will decide to stop at tree level.

String amplitudes are defined as a sum over all possible worldsheet topologies, each topology being a single string diagram. A consequence of this is that every single diagram takes automatically into account the exchanges of all possible string states, both the light and the heavy modes. What we mean by a *tree level effective action* is an action that:

- just includes the light modes of the string spectrum, but
- it reproduces the above mentioned tree level string amplitudes *in their low energy limit.*

The exchange of light modes in the corresponding string amplitudes will still be present in the Feynman diagrams of the effective theory. However, what in the string amplitudes was described by an exchange of heavy modes will translate into the effective action as modified interaction vertices for the light modes. This has to be so since no heavy modes are included in the effective action. Schematically:

[2]We recall that, at present, there is no Lagrangian formulation from which the computation rules for evaluating string amplitudes can be derived.



We emphasize the following: although an effective action built in this way only contains the low energy dynamics of the massless string spectrum, it is *not* an "amputated String Theory", obtained by simply removing all the massive states and all the physics arising from them. It is, instead, a physical low energy limit of String Theory that *does* take account of the physics of heavy modes. Only, since these have been integrated out, all the physics coming from their propagation is "summarized" in effective interaction vertices. Of course, such a "summary" is only valid at low energies.

We have not argued yet why such an effective theory should be a *field* theory, while String Theory is not. This has to do with the way in which the above low energy limit must be defined. The key point is that String Theory has only one dimensionful constant: α', the inverse string tension. This enforces *both* the mass scale set by (1.1.1) and, as we also mentioned, the string length to be related. In fact:

$$\begin{aligned} \text{mass scale} \quad &\sim \quad m_s \equiv \frac{1}{\sqrt{\alpha'}}\,, \\ \text{string length} \quad &\sim \quad l_s \equiv \sqrt{\alpha'}\,. \end{aligned} \tag{1.1.2}$$

The low energy limit we have been talking about is, obviously, the limit of small external momenta when measured in units of the string mass:

$$\frac{p_{\text{ext}}}{m_s} \ll 1\,.$$

But this just means that the distance scale Δx below which our effective theory will no longer describe accurately the physics is

$$\Delta x \gg l_s \,,$$

(i.e., much more than the string length), since resolving smaller distances would require energies at which our effective theory is no longer valid. Our effective theory arises then as the properly taken $\alpha' \to 0$ limit of String Theory (the so-called "zero-slope limit"), and this is a *field theory* limit because, if we are at energies such that strings can be considered to have negligible size (or, equivalently, infinite tension), then they can be considered as particles. It is in this sense in which it can be said that String Theory is an extension of Field Theory, and that Field Theory is contained within String Theory just as a certain limit. In a similar sense as, for example, it can be said that Classical Mechanics arises as the $\hbar \to 0$ limit of Quantum Mechanics.

The perturbative expansion we mentioned before deserves a further comment. As we said, we stopped our perturbative series in the string coupling constant at *tree* level, and what this means is that supergravity theories are the *classical* limit of String Theory. A particular feature of String Theory is that the string coupling, defined as the quantity which appears once for each interaction vertex occurring in a string amplitude (the quantity that "counts" string loops) is not an independent parameter, but it is instead given by the expectation value of one of the massless modes present in the spectrum of all string theories: the dilaton field ϕ. The relation between both is

$$g_s = e^{<\phi>} \,, \tag{1.1.3}$$

and hence the string coupling is a local quantity that should be, in principle, determined by the theory itself[3]. We said that we needed to assume that the string coupling is small to justify our perturbative expansion. Now we see that this assumption will be justified whenever we are able to argue that the expectation value of the dilaton field is small. The fact that supergravity actions are tree level actions is reflected in the dilaton factor $e^{-2\phi}$ that appears in supergravity actions when written in the string frame.

[3]Unfortunately, no mechanism within String Theory fixing the value of the string coupling is known yet.

Supergravity actions are string effective actions in the sense explained above: they reproduce the tree level, low energy S-matrix elements of the massless string spectrum in the field theory limit of String Theory (classical references are [2, 36]). The resulting actions are (supersymmetric) generalizations of General Relativity. The procedure used to construct such an effective action defines the regime of validity of Supergravity:

- *Weak coupling*, since string amplitudes can only be computed perturbatively. We will always be implicitly assuming that the string coupling is small and, therefore, for a supergravity background to be reliable, the value of the dilaton field must always be under control. We will see, however, that both supersymmetry and duality (in particular S-duality, which relates weak and strong coupling) will make many times possible to trust strong coupling results.

- *Low energies*, since, by construction, we have integrated out all massive string modes. If we were to use supergravity as an effective action to compute scattering amplitudes[4], we should always take into account that our results would fail if we considered processes that involve energies of the order of the string mass or higher. Also, a supergravity background should never describe a region of spacetime in which the energy density is of the order of the one given by the string tension. This seems quite reasonable on physical grounds, but it must be said that finding a precise, quantitative test for this statement will be very hard (if not impossible) in most cases, since in a curved spacetime energy is not well defined locally.

- *Long distances*, since we took the field theory limit and we must stay at distances at which strings still "behave as particles". Quantitatively, this translates into the constraint that, for a given supergravity background to be reliable, the curvature of the spacetime must be small

[4]It is well known that supergravity theories are nonrenormalizable, a fact that tell us that they cannot constitute, by themselves, a candidate for any ultimate quantum theory. But just as any *effective* field theory (as, e.g., the Pauli theory of weak interactions), they could be used to get quantitative information of scattering processes. The difference between a renormalizable and a nonrenormalizable theory is just that the latter have, from the very beginning, both the energy regime of applicability and the precision of the results bounded by the natural cutoff of the theory (in this case, the string mass). But below this cutoff they are perfectly valid, at least as a tool to get quantitative information.

when measured in string units, i.e., $R\alpha' \ll 1$. This could also be seen, at least sometimes, as a possible criterion of stability in terms of energy density in the sense commented in the last paragraph, since the curvature of spacetime will be determined by its energy content.

All this can be summarized by saying that supergravity actions are a double expansion, both in g_s (first condition) and in α' (second and third conditions). They are, in fact, given by the lowest contributions in both parameters, although higher order corrections can be considered (see e.g. [37] for a discussion about α' corrections and [38] and references therein for a discussion about string loop corrections to the string effective action). These corrections are thus String Theory predictions that modify (in particular) General Relativity.

By a supergravity background we mean a field configuration which solves the supergravity equations of motion, i.e., nothing but *classical* solutions of a *classical* field theory. Whenever such a background does not fulfill one of the requirements above, it cannot be said anymore that it may contain any reliable information about the underlying string physics, since we would be taking the supergravity approximation out of its own regime of validity (it must also be said that all conditions above are necessary, although, in general, not sufficient). But, as will be explained later on, there will be many situations under which it can be argued that a lot of String Theory information is encoded in classical solutions of supergravity theories.

1.2 Supergravity Backgrounds and Conformal String Backgrounds

So far we have introduced Supergravity as arising from a certain *physical* limit of String Theory. There is, however, a formal *consistency* requirement in String Theory which also leads, in a completely different way, to the same equations of motion that the ones we would get from the supergravity actions obtained as explained in the previous Section. This is the requirement of conformal symmetry. To illustrate some specific formulas we will focus here, for simplicity, in the case of the closed bosonic string, but the general results also apply to the superstring case.

The Polyakov action for a free, closed, bosonic string is given by

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{\gamma}\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \,, \tag{1.2.1}$$

where $\gamma_{\alpha\beta}$ is the worldsheet metric (with worldsheet coordinates σ^α) and the X^μ are the embedding coordinates of the string in spacetime. (α, β) are worldsheet indices, and (μ, ν) are d-dimensional spacetime indices. $\eta_{\mu\nu}$ is the d-dimensional Minkowski metric, and so the above action describes the propagation of a string that moves freely in flat spacetime. The action (1.2.1) can also be seen as a two dimensional field theory for the massless scalar fields X^μ coupled to the worldsheet metric. The physics of this worldsheet metric is something that we now discuss.

Together with other symmetries (like spacetime and worldsheet reparametrization invariance), the above action has a very important classical symmetry: Weyl invariance. Precisely because it is two dimensional it is invariant under local rescalings of the metric ("conformal" or "Weyl" rescalings). This is really a key symmetry because of a number of reasons, but here we will explain its importance in the following way. Among other properties, Weyl invariance allows to gauge away *all* the degrees of freedom contained within the worldsheet metric $\gamma_{\alpha\beta}$. This is very important, because if we are interested in formulating an action describing the physics of a relativistic string, such an action should *only* contain the degrees of freedom associated to the motion of the string, and these (vibrational, translational and rotational modes) are all already contained within the X^μ. We must thank then conformal invariance for making "vanish" all extra physics that could, in principle, arise from the degrees of freedom described by the worldsheet metric. This is why it is called an "auxiliary" field and this is why it was introduced: because it allows for a quadratic action (and hence easy to quantize, since the Nambu-Goto action is not), but it introduces *no* new degrees of freedom in the theory. If we were interested in possible two dimensional field theories this would not be, of course, a problem at all. But if we are interested in a theory of strings (by this meaning *just* strings) we should care about having at our disposal a formulation which describes the strings physics and nothing else.

What happens upon quantization of the Polyakov action? In general, a quantized theory may spoil some of the symmetries present in the classical theory. This is a well-known phenomenon, and when it happens we say that the theory has an anomaly. Often anomalies are harmless: they simply tell us, for example, that a classical conservation law is violated in the quantum theory, and, *a priori*, there is no reason due to which such a thing should render the quantum theory inconsistent. This is the case of *global* anomalies: a quantum theory is not inconsistent if a classical, global symmetry is violated quantum mechanically. The case is completely different, however, when *gauge* anomalies occur. Gauge anomalies are inadmissible if the quantum theory is to make sense, and this is so precisely because it is the gauge symmetry what is usually needed to get rid of the unphysical degrees of freedom of a theory. A simple example is that of gauge symmetry in electrodynamics: it is this symmetry what allows to gauge away the unphysical, longitudinal polarization mode of the photon (in four dimensions, for example, a photon is described in its Lagrangian formulation by a vector field A_μ with four real components, although we know that a massless non-scalar particle only has two physical degrees of freedom). Another example is conformal invariance in String Theory: this is what allows to gauge away the unphysical degrees of freedom of the worldsheet metric.

In view of this, we must care about what happens to the conformal symmetry when quantizing the Polyakov action, since it should not be anomalous. One then finds that the cancellation of the conformal anomaly is precisely one of the reasons due to which the spacetime dimension must be fixed, in the case of the bosonic string, to $d = 26$[5]. So by taking $d = 26$ we are sure of having a consistent theory of quantum bosonic strings which is still conformal invariant. What else can be done?

We can now analyze its spectrum, and we see that the massless spectrum of the theory consists of three states: a spin 2 field $g_{\mu\nu}$, which we identify with the graviton; a 2-index antisymmetric tensor field $B_{\mu\nu}$; and a scalar field ϕ, the dilaton. Now one could wonder how the physics of a test string should be when propagating not in flat space as in (1.2.1), but in a "condensate" or "coherent state" made by the massless excitations of the strings around. We can think in such a massless condensate as a nontrivial, stable field configuration arising from nontrivial vacuum expectation values of the massless modes. To consider such a situation the action (1.2.1) is clearly no longer

[5]This can be seen to happen in this way in the covariant quantization scheme, and so, in this case, $d = 26$ appears as a *consistency* requirement. However, in other quantization procedures $d = 26$ emerges differently. For example, in the light-cone quantization this condition must be *chosen* only if we do not want to lose Lorentz invariance and, also, if we want the states in the spectrum to fit in well-defined irreducible representations of the Lorentz group.

valid, and the correct action to consider now can be shown to be [39, 40]:

$$
\begin{aligned}
S &= -\frac{1}{4\pi\alpha'}\int d^2\sigma \left(\sqrt{\gamma}\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}(X) + \epsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu}(X)\right) + \\
&\quad + \frac{1}{4\pi}\int d^2\sigma \sqrt{\gamma} R^{(2)} \phi(X)\,, \qquad (1.2.2)
\end{aligned}
$$

where $\epsilon^{\alpha\beta}$ is the antisymmetric symbol in two dimensions and $R^{(2)}$ is the worldsheet curvature scalar, the one built from $\gamma_{\alpha\beta}$. In two dimensions its integral over the manifold is a topological quantity: the Euler characteristic of the worldsheet manifold. This can be seen, when considering string interactions (and assuming constant dilaton within the worldsheet), to be the reason for the relation (1.1.3) between the string coupling and the expectation value of the dilaton field.

Now the question of conformal invariance arises again, since it can be seen that for a string propagating in background fields, the condition $d = 26$ is no longer enough to ensure the cancellation of the conformal anomaly. This is evident even at the classical level, since the last term in (1.2.2) is not, by itself, conformal invariant[6]. Viewed as a two dimensional field theory action, we see that the fields $g_{\mu\nu}$, $B_{\mu\nu}$ and ϕ play in (1.2.2) the role of local, functional couplings. A conformal theory is scale invariant, and this means that the theory is insensitive to long or short distances, or to low or high energies. A consequence of this is that all beta functions vanish. They vanish because, in a scale invariant theory, there should be no need to choose an energy scale at which any coupling must be defined. Therefore, if Weyl invariance is to be a symmetry of the quantized theory the beta functions should vanish[7]. However, if we compute the beta functions for the theory defined by the nonlinear sigma model above (which is a very hard field theory exercise) in

[6] This will not worry us because of two reasons. First, note that this term carries a different power in α', and so it can be considered as an α' correction to the remaining part of the action. In this sense it is not so strange that it breaks conformal invariance. But also, and most importantly, we will see that conformal symmetry will be fully restored at the quantum level under certain conditions.

[7] This seems compelling. However, the actual issue is *not* whether the beta function of a two dimensional field theory vanishes or not, but if the theory given by (1.2.2) is invariant, once it is quantized, under local rescalings of the worldsheet metric or not. It can be shown, however, that at least at one loop the vanishing of the beta functions *implies* Weyl invariance.

$d = 26$, what we get is [40]:

$$\begin{cases} \beta_g &= R_{\mu\nu} - 2\nabla_\mu \partial_\nu \phi + \frac{1}{4} H_{\mu\rho\sigma} H_\nu{}^{\rho\sigma}\,, \\ \beta_B &= \nabla_\rho H^{\rho\mu\nu} - 2H^{\mu\nu\rho}\partial_\rho \phi\,, \\ \beta_\phi &= \nabla^2\phi - (\partial\phi)^2 - \frac{1}{4}R - \frac{1}{4\cdot 3!}H^2\,, \end{cases} \tag{1.2.3}$$

where $H \equiv 3\partial B$, and $R_{\mu\nu}$, R and ∇_μ are the usual expressions for the curvatures and (functional) covariant derivative with respect to the metric $g_{\mu\nu}$.

It is worth commenting a little bit on the computation of these beta functions. As always in field theory, they can only be computed perturbatively. The details are rather cumbersome, but by looking at (1.2.2) it is not hard to convince oneself that a number that will measure the size of such couplings is $\sqrt{\alpha'}$. This is, in fact, the parameter around which we will be doing perturbation theory[8]. This is an important thing: sigma model perturbation theory gives us the α' expansion. The beta functions obtained above are the lowest contributions in α', which are the one loop contributions from the $g_{\mu\nu}$ and $B_{\mu\nu}$ terms and the tree level contribution from the dilaton term (this is due to the different α' dependence of the latter)[9]. Considering higher sigma model loops would yield α' corrections[10]. The beta functions obtained above are therefore computed in the $\alpha' \to 0$ limit –what we called "field theory limit" in the preceding section.

According to (1.2.3) the field theory under consideration will not be, in general, conformally invariant. But if we place the string in a $(g_{\mu\nu}, B_{\mu\nu}, \phi)$-background such that all the above beta functions vanish:

$$\beta_g = \beta_B = \beta_\phi = 0\,, \tag{1.2.4}$$

[8]Of course, it makes no sense to talk about a perturbative expansion around a dimensionful parameter, since a dimensionful quantity cannot be said to be neither small nor large –this is simply a matter of the units we choose to measure it. It can be seen, however, that the true perturbative parameter is $\sqrt{\alpha'}/r$, r being a typical length of order the "curvature radius" ($r^2 \sim 1/R$ when R is nonvanishing) of the 26 dimensional spacetime manifold.

[9]Out of the critical dimension $d = 26$ there is, however, a leading order term in the α' expansion ($\sim 1/\alpha'$) arising from a (tree level) contribution due precisely to the dilaton coupling term of (1.2.2). This term appears as a constant in the beta functions and becomes a cosmological constant term in the spacetime action (see below).

[10]Notice that this has nothing to do with a *string* loop expansion, which would mean g_s corrections. We will comment on this below.

then conformal invariance will be achieved. Hence, from now on, we will require any string background to satisfy (1.2.4) in order to have a conformally invariant String Theory.

We must not forget that equations (1.2.4) are functional equations: from the two dimensional field theory viewpoint, they are functionals of the bosonic fields X^μ. However, we can also give them a *spacetime* interpretation, since the X^μ are also spacetime coordinates, the spacetime coordinates telling us where and how the string is placed. If we adopt this spacetime interpretation, the above equations are nothing but second order differential equations which turn out to coincide with the Euler-Lagrange equations of the following spacetime action:

$$S = \frac{1}{16\pi G_N^{(26)}} \int d^{26}x \sqrt{|g|} e^{-2\phi} \left(R - 4(\partial\phi)^2 + \frac{1}{12} H^2 \right) . \tag{1.2.5}$$

$G_N^{(26)}$ is the Newton constant in 26 dimensions, which has dimensions of $(\text{mass})^{-24}$ and must be put in front of the action to give it the right units. Surprisingly, this is precisely the low energy effective action that we would have obtained, by following the procedure explained in Section 1.1, for the case of the closed bosonic string. Eq. (1.2.5) describes gravity coupled to the bosonic fields $B_{\mu\nu}$ and ϕ. In the case of the superstring one obtains is a corresponding supergravity action.

This approach to obtain the effective string action was carried out in [40] for the case of both the bosonic string and the superstring, where some considerations concerning background fermions where also made. Older related references are e.g. [5, 41]. A different approach in the case of the superstring, based on the requirement of fermionic κ-symmetry (instead of conformal invariance), was carried out in [42]. The agreement between the beta functions and the equations of motion derived from the S-matrix generating functional beyond the leading order in α' was considered in [43].

Note that the approximations made in the computation of the beta functions are the same as the ones we made when computing the string effective action. First, this is a low energy approximation, since a stable condensate of massless string modes is not conceivable if we are at energies at which massive states can be excited. This is made explicit in the fact that we are working in the field theory limit (pointlike strings): we took α' to be small when computing perturbatively the beta functions (1.2.3), since there α' was the expansion parameter. Finally, it remains to justify that (1.2.5) is also a tree level expansion in the string coupling. This fact is much less obvious to see here, because it is not implicit in the computation of the beta functions: such a computation provides us with the UV behaviour of the two dimensional theory, which is a local property, while the issue of how many string loops are we considering is completely different, since it is related to the topology of the worldsheet manifold. An indication of the fact that we are at tree level is provided by the dilaton power in front of (1.2.5), which is the right power for a string tree level action. The fact that the above computation was tree level can be seen to be implicit in the sigma model action that we used: when considering one loop string amplitudes one generically finds divergencies, and counterterms in the original sigma model Lagrangian must be added in order to cancel them [44, 45]. We will comment further on string loop corrections to string effective actions in Section 5.2.3.

We would like to stress that all the considerations made so far are fully perturbative, both in the two dimensional field theory on the worldsheet (i.e. perturbative in α') and in the string coupling constant g_s. However, we will see in the following Chapters that one is sometimes able to get nonperturbative results from these perturbative effective actions. This can happen when duality arguments or appropriate nonrenormalization theorems (concerning sigma model or string perturbation theory) can be invoked. The latter are many times due to supersymmetry. This is why both supersymmetry and duality play a central role in string physics.

We have seen that Supergravity arises from String Theory as the low energy effective action for the massless modes, but also that Supergravity equations of motion must be obeyed by any string background in order to get a conformal String Theory defined on it. This is why supergravity solutions are important for String Theory: if some String Theory effects are to be seen in its low energy limit, those should be encoded in a certain supergravity solution. For example, we will be many times looking at supergravity solutions as describing the long range fields produced by nonperturbative string states or providing us with solutions which we will identify with string vacua. The basic support for this identification are the reasons enumerated above.

Part I

Supergravity Solitons

Chapter 2

Supergravity Backgrounds and String States

One of the main uses of Supergravity concerning its application to String Theory is the identification between supergravity solutions and the long range fields produced by nonperturbative string states. By "nonperturbative" we mean string configurations that are neither the single excitations of a single string nor can be obtained when treating these perturbatively. The best known example of such an identification is that of *supergravity* p-brane solutions and *string* D-branes [22, 46, 47]. The study of supergravity configurations related to string states has proven to be extremely useful.

Here we attempt to explain why certain nonperturbative string states should have a supergravity solution counterpart, how the identification between both works, and why such an identification can be made.

2.1 A Simple Example

Let us consider the Maxwell theory of classical electromagnetism with *no* sources. In its Lagrangian formulation it is given by the action:

$$S = -\frac{1}{4}\int d^4x\, F^2\,, \tag{2.1.1}$$

with equation of motion an Bianchi identity given by

$$d^\star F = 0\,, \qquad dF = 0\,. \tag{2.1.2}$$

A possible solution of these is provided by

$$F_{tr} = -\frac{q}{r^2}\,, \tag{2.1.3}$$

where q is any constant and r is the radial coordinate in spherical coordinates.

The theory defined by (2.1.1) just describes the electromagnetic field. However, we always say that (2.1.3) is the solution "corresponding to a point particle of charge q". Why are we allowed to make such an identification if our starting theory was not a theory including charged matter?

First of all, the solution given above describes a nontrivial field. So it is natural to suppose that there could be something in space (something that we will call *source*) distorting the trivial vacuum field configuration ($\equiv F = 0$). The question is: are we able to describe the properties of such a source just by means of the solution given by (2.1.3)?

First we observe that our solution is perfectly regular everywhere except at $r = 0$, where it is singular and hence no longer valid. Since our theory just describes an electromagnetic field, this field must be the only thing existing in the region of space where our solution is well behaved. Therefore, if a source is to be present, it can only be placed where (2.1.3) fails. This place is a point, and hence our source can only be pointlike. Secondly, we have at our disposal the Gauss law, and so we are able to compute the net electric charge contained within a region of space just from the knowledge of the electromagnetic field on the boundary of that region. If we apply the Gauss law here, we will discover that there is a net charge given by q precisely at $r = 0$ (and nowhere else). This is what enables us to say that our solution describes the electric field emitted by a *pointlike charge* of value q.

These will be the same kind of questions that we will made ourselves whenever we find a supergravity solution describing nontrivial field configurations. The way in which we will extract information of string states from certain supergravity solutions is technically more involved, but the principles that rule the matching between one thing and another are the same as those explained here.

Branes in supergravity should be regarded as nothing much more complicated than generalized electrons (elementary sources) or monopoles of generalized electromagnetic fields[1]. The fact that they must be extended (as opposed to pointlike) is simply due to the fact that in higher dimensions

[1]Monopole configurations are nonsingular and therefore they are not "sources". How-

gauge fields can be p-forms of higher rank. The presence of gravity, the spacetime dimension and many other things make things much more complicated technically (with the beautiful consequence that Physics is also much richer), but with this example we want to make clear that, from a supergravity perspective, they are conceptually very simple.

2.1.1 Addition of Sources

We have seen that a source-free theory can be enough to point towards the existence of sources and to elucidate some of their properties. However, the fact that a given field configuration like (2.1.3) is indeed the one created by a pointlike charge could be established in a much more solid way if we had at our disposal a precise theory of pointlike charges. If we were able to consistently couple the above action to that of a pointlike charge, solve the full equations of motion and getting at the end the same result, that would really be a strong support to our assumptions (we must not forget that, regardless the quite compelling arguments given above, we were inferring the existence of an object in a place where our solution completely breaks down). In electromagnetism this can be done, and one can consistently couple a charged current to the Maxwell theory.

This is also important for another reason. When we only care about solving the source-free equations of motion, one finds that the charge q is an arbitrary number (i.e. a modulus in the space of solutions). It is just an integration constant, and no criterion in order to fix its value is provided by the source-free theory. To solve this problem what one needs is a precise model for, say, electrons, in which their charge is a precise, fixed number. Then one can couple the source-free action to the current of an electron (in whose definition its actual charge will be included), and this will force q to equal the physical charge of our particle. However, once we know that a particular source action exists, and that it can be consistently coupled to the source-free theory, there is no real need to solve the whole system of equations of motion more than once. If we know which sources can be consistently coupled to the "bulk" action, and somehow we know the physical charges of the objects our solution is to describe, we are always entitled to solve the source-free equations of motion and set by hand q to the requested physical value.

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ever, throughout this Chapter we will use the word "source" in a generalized sense to refer to all those configurations carrying some "elementary charge" with respect to some gauge potential or its dual.

In supergravity, the analogue of the source action describing a pointlike charge will be the worldvolume actions of the fundamental and solitonic extended objects present in String Theory. We emphasize this issue because having a well defined theory of elementary sources is of capital conceptual importance if we want to argue, on solid grounds, that a supergravity field configuration can be identified with a known String Theory state. For example, the nowadays well-established connection between the supergravity p-branes first found by Horowitz and Strominger [46] and string D-branes first considered by Dai, Leigh and Polchinski [47], would have never been possible without a String Theory computation showing that the latter share all the properties seen in the supergravity solutions: RR-charged extended objects breaking half of the supersymmetries [22].

Sometimes, however, there will be supergravity solutions for which a string description is not known. For example, it is known that the stringy description of a Dirichlet brane is an hyperplane where open strings are attached, but an analogous microscopic description of the NS5-brane (a very well known supergravity solution [48, 49]) in terms of string excitations does not exist, and thus the most natural way to infer its existence and properties is from the knowledge of the supergravity solution. This will also be the case for the results reported in Chapter 4. We see in this way that situations in which a supergravity solution lacks a microscopic string description can be interesting by themselves, since these solutions might be pointing towards the existence of other, so far unexplored, nonperturbative string states whose presence in String Theory may be difficult to guess by other means.

2.2 The Supergravity Description of String Sources

Macroscopic nonperturbative string states charged with respect to any of the massless fields present in supergravity theories should be seen in Supergravity, i.e. there should be a supergravity solution reproducing the nontrivial behaviour of the fields emitted by that kind of sources. That this should be so seems quite obvious once one takes into account both facts explained in Chapter 1 concerning the relation between Supergravity and String Theory.

By a "macroscopic state" we mean here a "heavy enough" or "charged enough" one, so that the long range (massless) fields produced by it are also strong enough to distort spacetime at distances bigger than the string scale –the distance scales described by supergravity. On the other hand, we will be mainly interested in time-independent (i.e., static or stationary) supergravity solutions[2]. As a consequence, we will demand from such a state to be dynamically stable. This is a reason due to which it is very likely to be, in addition, "nonperturbative" (in the sense explained at the beginning of this Chapter): one could think of an extremely massive single string excitation as being also "macroscopic" (in the sense explained here), but such states are expected to be quantum mechanically unstable against decay into light modes. On the contrary, a nonperturbative state has a chance for being stable if there is a topological reason or a conservation law protecting it against decay.

It so happens that String Theory *does* contain such stable, nonperturbative macroscopic states. We can look at this in two complementary ways:

- From String Theory to Supergravity: String Theory predicts the existence of states whose mass density goes like an inverse power of the string coupling. If we consider the case of D-branes, for example, their tensions go like g_s^{-1}. So it is conceivable that, precisely in the weak coupling regime, they will be heavy enough to have an associated supergravity solution. It could be argued, however, that even for a small but finite value of the coupling constant they could still be too light to produce a strong gravitational field. But it is a fact that these are BPS states (we will develop this shortly), and so as many of them as we wish could be placed together to constitute a stable and heavy enough state (in fact, this argument can be applied to all BPS states, regardless of how their masses scale with the string coupling). Also, since D-branes carry the quantum unit of RR charge [22] and have the minimal mass allowed for such a carrier[3], their stability against decay is ensured by a simple charge conservation argument.

- From Supergravity to String Theory: whenever a nontrivial field configuration with the characteristics of being produced by a source is found as a supergravity solution it makes sense to wonder about its microscopic description, i.e. about the source itself. Such a microscopic description is very likely to be found within String Theory.

[2]Time dependent backgrounds are usually interpreted as cosmological solutions.
[3]This is related to the fact that they saturate all BPS bounds. See Section 2.4.1.

Which properties of the source can be read from a supergravity solution? Well, exactly as in the example of Section 2.1, these are its spatial extension and its charges with respect to the supergravity fields. Bosonic fields of supergravity include gauge fields and the metric tensor, so we will be able to compute, in principle, the gauge charge(s) and the "spacetime" or "geometrical" charges (mass, angular momentum, etc.) of the source.

The spatial extension is usually read from the region where our solution becomes singular. This is in principle so for the case of "electric" configurations associated to some gauge potential. "Magnetic" ones (charged with respect to a dual potential, i.e. the analogues of a Dirac monopole), like the NS5-brane for example, can be regular field configurations everywhere, but they always have an associated "core" or "energy locus" which is also read from the behaviour of the supergravity solution. There are refinements to this procedure, however, arising from the fact that singularities in a solution can be physical singularities or just coordinate singularities, from the fact that due to the presence of gravity we can find special regions like event horizons, and one can also find even more exotic situations, as that of the Kaluza-Klein monopole[4]. But away from possible subtleties, the general rule to determine the "size" of the source is the one given here, and the reason for this is exactly the same as the one given in Section 2.1 when there we inferred that the source had to be pointlike.

The supergravity solution will always tell us under which gauge fields the source is charged, and these charges can always be properly computed by the corresponding generalization of the Gauss law.

Finally, the "spacetime" charges like the mass are more problematic, since in theories with gravity one must go to infinity to compute, say, the *total* mass-energy of the whole spacetime, which in general is only defined globally. A first indication that this has to be so comes from the fact that, in a curved space, the on-shell equation

$$\nabla_\mu T^{\mu\nu} = 0\,, \tag{2.2.1}$$

does not imply any *local* conservation law for the matter energy-momentum tensor. A conservation law could only be inferred from a continuity equation

[4]In that case, what could look like a worldvolume direction will be interpreted at the end as a transverse, isometric one, since it is argued that it has to be compact. This last property is not seen in the metric, and hence a naive interpretation based on the coordinate dependence of the solution leads to a wrong conclusion about its spatial extension.

of the kind

$$\partial_\mu \left(\sqrt{|g|} T^{\mu\nu} \right) = 0 \,, \tag{2.2.2}$$

which will never be satisfied in the case of a curved spacetime, because Eq. (2.2.1) makes the term $-\Gamma_{\rho\sigma}{}^{\nu} T^{\rho\sigma}$ appear in the r.h.s. of (2.2.2). This can also be understood form the fact that, in a curved spacetime, the needed symmetries under which a conservation law and a well defined notion of mass can be established are only realized asymptotically. For example, in asymptotically flat spacetimes this gives rise to the notion of ADM mass [50][5], which is what we usually will be identifying with the mass of the source. Other spacetime charges such as angular momenta can also be associated to the source in stationary spacetimes. And also, as in the case of the mass, these can be defined and computed only asymptotically. The reason is, again, that these conserved charges are always associated to isometries that only the asymptotics of our solution has. This is a particular feature of all spacetime charges. On the contrary, note that reaching infinity is not necessary to compute the gauge charge, since gauge symmetry is perfectly preserved everywhere.

In supergravity theories, however, there is another property associated to the source that we will be able to compute and that we have not mentioned so far: the number of residual supersymmetries and the corresponding supercharges. This kind of charges also fall into the class of what we called spacetime charges (this is not so strange, since supersymmetry enlarges the Poincaré group, not the gauge symmetry group), and the same considerations concerning the asymptotics apply to them. The preserved supersymmetries and its implications will turn out to be of capital importance. We will develop this issue more carefully in Section 2.3.

[5] The definition and computation of conserved charges in spacetimes with arbitrary asymptotic behaviour was developed in [51].

2.2.1 Asymptotic Behaviour: Sources as Vacuum Perturbations

The preceding discussion about spacetime charges has a sort of "nontrivial implication" or "hidden assumption": the concept of vacuum. At least, a concept of "vacuum" arising as opposed to the concept of source itself.

As explained, to compute the mass associated to a source one can only do it *with respect to the asymptotic spacetime* and use, to define a conserved mass associated to a conservation law, the asymptotic isometries (which, in turn, become *global* symmetries). It could be said that, to compute the mass, one must first go very far away from the source and, only then, apply the corresponding "gravitational generalization" of the Gauss law. From a certain viewpoint this is not so strange, since gravity itself contributes to the total energy. In electromagnetism, the only charge given by the Gauss law is the one contained within the integration surface, so the latter must enclose the whole source (in the case of a charged sphere, for example) if we want to compute its *total* charge. With the mass one must do something similar, and one must go to the asymptotic region to have first an "isolated system", because the fields emitted by a source also contribute to the total energy. Of course, this is nothing but an analogy and should not be taken further: the crucial difference is that, while in the case of a gauge charge the computation can be carried anyway (one could compute the charge of "half a sphere") because a conservation law for the charge holds *locally*, in General Relativity energy and its conservation can only be defined globally.

Hence we see that some properties of the source can only be defined with respect to those of the asymptotic region, and it is this asymptotic region what we are *defining* here as "vacuum". It is in this sense in which a given source solution can be interpreted as a localized (i.e., a configuration with a core) "perturbation of the vacuum". We call the source a "perturbation" simply because it will not be seen from a big enough distance from the core. For example, a solution with the properties of being describing a source and which is asymptotically Minkowski spacetime will be considered as a perturbation of the Minkowski vacuum. Note that the only reason for calling it vacuum is nothing else that it can be considered as an "empty spacetime" ("empty" in the sense that it has no core) enjoying, in addition, a lot of (super) symmetries, much more than the full solution.

All possible supergravity field configurations could be classified into equivalence classes according to their asymptotic behaviour. Therefore, all spacetimes falling into the same equivalence class (i.e., all spacetimes sharing a given asymptotics, e.g. Minkowski) could be considered, in principle, as being describing all possible macroscopic sources that can exist in a given vacuum –the one defined as the common asymptotic spacetime shared by all those solutions.

Spacetimes which "asymptote to themselves" are homogeneous spacetimes, and have no core where one could think that there is something like a source. If one uses the Abbott and Deser formalism [51] to compute the total energy of such spacetimes one gets, by construction, zero. This may be taken as another reason for calling them vacua[6]. The fact that one can *wonder*, e.g., about the mass of a given spacetime and getting a finite result seems very much intrinsically related to the concept of source itself and to the idea of a source embedded in a given vacuum. We will investigate further this idea of vacuum in Chapter 5, and we will deal with spacetimes that can be considered as vacua in the sense explained here in Part II.

2.3 Nonrenormalization and BPS Condition

So far we have explained the relation between string states and supergravity backgrounds, and we have sketched how the identification between both goes. But we did not say *why* such an identification should work. To answer this question one must understand first the problem of why it could fail.

As emphasized in Chapter 1, Supergravity is a *classical* limit of String Theory. Therefore, all the information we will be able to extract is tree level. If we attempt to properly obtain some properties of the states of a quantum theory, we must then be sure that the information we will get from supergravity is not spoiled by quantum corrections. As mentioned, one is in principle able to assign, from the knowledge of the fields of a supergravity solution, a mass and a charge to something that we identify with a quantum stringy state. The question is if these values, as computed from supergravity, are stable or not against quantum corrections, i.e., if they get renormalized or not in the quantum theory. We see therefore that nonrenormalization of the quantum numbers of the source is needed for the reliability of the supergravity background, simply because the latter just provides us with tree level information.

[6]We will consider this more carefully in Chapter 5.

We will be mostly concerned with the issue of whether the *ratio* between mass and charge obtained from supergravity still remains valid beyond tree level or not. As we will see, this (the charge-mass ratio) is a very important piece of information. One must also realize that it is exactly this one the true question to be raised in a supergravity context, and that a fixed charge-mass ratio beyond tree level is the only requirement which makes sense to demand in order to trust a supergravity solution. As explained, in supergravity these numbers arise from integration constants, and the actual values of integration constants do not constitute an issue: as long as their ratio is fixed, different values will just differ by an immaterial overall normalization. Of course, the actual values of *physical* charges or masses do matter, but to compute them one must go to the full String Theory and perform the proper calculation. It is only then when the question about the renormalization of their individual values, and not only their ratio, applies and matters.

The problem was essentially solved by Witten and Olive [52] in the context of supersymmetric gauge theories. Although one can certainly not make an exact parallel of their argument in the case of a theory with local supersymmetry, the very last reason leading to nonrenormalization also holds in a supergravity theory. As we will see, an absolute condition for applying the reasoning leading to nonrenormalization of the mass-charge ratio of the source is that the supergravity solution must be describing a quantum BPS state. We now explain what the meaning of this is, and next we sketch the argument itself in the case of a supergravity theory.

2.3.1 Global Supersymmetry Algebras and Central Charges

Let us consider all possible spacetimes falling into a given equivalence class as defined in Section 2.2.1 (all spacetimes with the same asymptotics or, equivalently, all the macroscopic excitations of a given vacuum). To this equivalence class one can associate something which is called its "global supersymmetry algebra" or its "superalgebra".

For given asymptotics, the corresponding superalgebra is, by definition, the finite-dimensional algebra of spacetime (super) symmetries enjoyed by the asymptotic spacetime, i.e. the vacuum superisometry algebra. The reason for making this definition is the following. We will be interested in using this superalgebra to get some information (things like a BPS bound, for example) from the conserved charges assigned to a certain source-solution with a given asymptotic behaviour. For this to be possible, what one needs to consider are precisely the isometries of the vacuum, since only with respect to them conserved geometrical (super) charges can be defined. Also, these charges will be identified, later on, with the tree level quantum numbers of the quantum stringy state describing the source, and then we will need to have at our disposal something that we could call, for example, a "mass operator". To match this with the classical supergravity picture, such a mass operator should be well defined also in the classical theory.

Consequently, whenever we write down a certain superalgebra we should always keep in mind that it is the superalgebra *associated to a given background* –a background such that spacetime charges can be defined with respect to it[7]. For example, if such background is Minkowski spacetime, the corresponding superalgebra will be an extension of the Poincaré algebra including fermionic generators. For simplicity, in what follows we will always work in the case of a super-Poincaré algebra, although other superalgebras (like AdS ones) referring to different asymptotic behaviours exist and could of course be considered.

The generic expression for the anticommutator of two fermionic generators Q_α and Q_β is (we will always suppress numerical factors which depend on the conventions chosen):

$$\left\{Q^\alpha, Q^\beta\right\} = (\gamma^a \mathcal{C}^{-1})^{\alpha\beta} P_a \,, \tag{2.3.1}$$

where $\mathcal{C}$ is the charge conjugation matrix and the P_a are the translation generators. The indices α and β are spinorial indices, since the different Q-generators will always be arranged in sets that transform as spinors under the Lorentz group. It is known that the above algebras can be consistently[8]

[7]A different thing is the *local* algebra obeyed by the infinitesimal generators of the symmetry transformations of our *theory*. This is obtained by computing the commutators of every infinitesimal symmetry transformations (under which our classical Lagrangian is on-shell invariant) when acting on the fields of our theory. Of course, a needed consistency requirement is that such a local algebra closes (at least on-shell), and that it is equally realized on all the fields of the theory. But this is of no use to our purposes because, in general, no conserved charges can be associated to the local symmetry generators.

[8]By this we mean that the corresponding super-Jacobi identities are still obeyed in the central-extended case.

enlarged by including *central charges*, henceforth denoted generically by $\mathcal{Z}$, which always occur in the anticommutator of two supercharges[9]. We can talk about two kinds of central charges: those having R-symmetry indices and those having spacetime indices[10]. The former may appear in theories of extended supersymmetry in which the Q-generators arrange themselves in $N > 1$ sets of spinors. In that case one labels the generators as $Q^{i\alpha}$, where $i, j = 1, ..., N$, and the central-extended algebra takes the generic form

$$\left\{Q^{i\alpha}, Q^{j\beta}\right\} = \delta^{ij}(\gamma^a\mathcal{C}^{-1})^{\alpha\beta}P_a + (\mathcal{C}^{-1})^{\alpha\beta}\mathcal{Z}^{ij} \,. \tag{2.3.2}$$

In the case of a Poincaré superalgebra the different $\mathcal{Z}^{ij}$ commute with every other generator, hence the name. In the four dimensional case, such kind of superalgebras were shown in [53] to be the maximal central extensions allowed compatible with Poincaré invariance. However, if one does not require Poincaré invariance, more terms, now with Lorentz indices, can be consistently added to the anticommutator of two supercharges. These terms generically appear in the anticommutators as

$$\left\{Q^{\alpha}, Q^{\beta}\right\} = (\gamma^a\mathcal{C}^{-1})^{\alpha\beta}P_a + (\gamma^{a_1\cdots a_p}\mathcal{C}^{-1})^{\alpha\beta}\mathcal{Z}_{a_1\cdots a_p} \,. \tag{2.3.3}$$

Although the $\mathcal{Z}_{a_1\cdots a_p}$ are usually also called "central charges" (and we will keep that name for them), actually they are not central elements of the algebra, since the super-Jacobi identities imply that they must have non-vanishing commutators with the Lorentz group generators. Central charges with Lorentz indices where shown in [54] to appear in theories containing extended objects.

We have added elements to a symmetry algebra. Therefore, these new elements should be also symmetry generators. What are, then, the symmetries generated by the central charge operators? In the case of the central charges $\mathcal{Z}^{ij}$ with R-symmetry indices, it can be seen that the corresponding symmetry is that of global rotations mixing the N gravitini that will be present in our theory as a consequence of the N-extended supersymmetry. In the case of the central charges $Z_{a_1\cdots a_p}$ with Lorentz indices, the associated symmetry

[9] Of course, the elements entering in the algebra are all *operators*: the generators of the corresponding symmetry transformations. However, it is customary to call the Q-generators "supercharges" and the $\mathcal{Z}$-generators "central charges" since, in the case of a global superalgebra, each one has an associated conserved charge.

[10] They can also have both. For simplicity, we consider here each case separately.

is that of gauge transformations of the $(p+1)$-form gauge fields contained in the theory. In fact, it so happens that the allowed central charges always correspond, in one way or another, with the gauge fields (and their Hodge duals) of the theory under consideration, and thus they should also be in correspondence with the elementary charged configurations with respect to the different gauge fields [54]. This also explains the breakdown of Lorentz invariance: in theories containing $(p+1)$-form gauge fields we know that the states charged under these are p-extended objects, and it is their presence what will break the Lorentz symmetry of the vacuum. Moreover, the fact that the symmetry generated by the central charge operators is gauge symmetry implies that the corresponding conserved charge will be the gauge charge. The identification between central charges and gauge charges will be essential in what follows. The fact that central charges have Lorentz indices takes into account the fact that, for extended objects, the electric or magnetic charge is oriented in space and is to be proportional to some volume form –the one describing the worldvolume of the extended object.

2.3.2 Supersymmetric Backgrounds and BPS Bounds

In the classical limit represented by Supergravity, local supersymmetry is realized on the fields of the theory by means of algebraic or first order differential transformations that mix the fermionic and the bosonic fields. The infinitesimal action of the supersymmetry generators Q on the supergravity fields can be written schematically as

$$\left\{ \begin{array}{llllllll} Q: & \phi & \rightarrow & (1+\epsilon Q)\phi & = & \phi + \delta_\epsilon \phi\,, & \delta_\epsilon \phi & = \psi\epsilon\,. \\ Q: & \psi & \rightarrow & (1+\epsilon Q)\psi & = & \psi + \delta_\epsilon \psi\,, & \delta_\epsilon \psi & = \partial\epsilon + \phi\epsilon\,. \end{array} \right. \tag{2.3.4}$$

Here ϕ denotes a bosonic field, ψ a fermionic one and $\epsilon = \epsilon(x)$ is the local parameter of the infinitesimal transformation. This parameter is a Grassmann number, i.e. a classical spinor. In general, a given solution $S = \{\phi(x), \psi(x)\}$ will transform into another one $S' = \{\phi'(x), \psi'(x)\}$ if we apply on it the transformations given by (2.3.4). But, for certain nontrivial solutions, there may be some orbits in the space of the supersymmetry transformations that leave it invariant ($S' = S$). When this happens we say that the solution S is supersymmetric, or that it has "residual supersymmetries". We will always be interested in bosonic backgrounds, for which $\psi(x) = 0$, and therefore the requirement for preserved supersymmetry is the existence of a nontrivial spinor field $\epsilon(x)$ satisfying the first order differential equation:

$$\partial\epsilon + \phi\epsilon = 0\,. \tag{2.3.5}$$

These equations are customarily called "Killing spinor equations", and their solutions are called "Killing spinors". Since there will be a Grassmann parameter per supersymmetry generator, the number of independent solutions of the above equation coincides with the number of independent supersymmetry transformations which can act on S leaving the field configuration invariant. This number is the amount of supercharges preserved by the bosonic field configuration.

It turns out that, for *any* background (supersymmetric or not), representation theory of its corresponding global superalgebra *always* implies that one or several inequalities of the form

$$M^2 \geq |Z|^2 \tag{2.3.6}$$

hold. Here M is the eigenvalue of the suitably defined mass operator built from the translation generators of the global superalgebra (P_0 for a massive particle in the rest frame, for example). This value will coincide with the ADM mass of the supergravity background. With Z we denote any of the eigenvalues of the central charge operators, and each of them will be given in general by a combination of the gauge charges of the supergravity solution. This kind of inequalities are called "BPS bounds", and the solutions for which the equality is obeyed are called "BPS-saturated". The above inequalities follow from the positive-definite nature of operators of the kind[11]

$$(Q^\alpha \pm Q^\beta)^2 \geq 0\,, \tag{2.3.7}$$

and are a generalization, to the case of supersymmetry algebras that include central charges, of the usual theorems on positive-energy representations implied by Supersymmetry . For a classical background, however, this should

[11]One must be careful, though, since this positive-definiteness is not a consistency requirement for a *classical* symmetry algebra (classical states need not be in unitary representations). One can however prove this kind of inequalities going, if necessary, to the appropriate representation. On the contrary, in the quantum theory, the fact that these operators are positive-definite arises from unitarity.

be supplemented by a careful analysis concerning the conserved charges of the supergravity solution, to prove that the eigenvalues of the operators involved indeed coincide with the classical charges[12] [56].

For which kind of backgrounds is the BPS bound saturated? It can be shown that the equality holds if and only if the solution is supersymmetric, i.e. if there are some (linear combinations of) supersymmetry transformations that leave it invariant. This follows from the kind of expressions symbolized by (2.3.3) and (2.3.7). We see therefore that supersymmetric solutions obey a certain minimal-mass condition. BPS-saturated solutions turn out to be of capital importance, and have deep implications that we explore next.

2.3.3 Nonrenormalization of BPS States

Suppose that we have a solution of the field equations of supergravity and that we interpret it as describing the fields produced by a stringy source. This source will be described quantum mechanically by a quantum state that we denote by $|S\rangle$. Suppose also that we were able to compute, from the supergravity solution, the mass and the charge attributed to that source. These values will be proportional to the mass and the charge of $|S\rangle$ when computed at tree level.

Let us consider the case in which our classical solution is BPS. As usual, we will of course *impose* the classical global superalgebra to be the spacetime symmetry algebra also of the quantum theory. This will imply that the quantum state is annihilated by some of the fermionic charges

$$Q|S\rangle = 0\,, \tag{2.3.8}$$

at least at tree level, since the analogous of (2.3.8) holds for the classical background. If this is so, a saturated-BPS bound of the kind of (2.3.6) will be satisfied by the quantum numbers of the source $|S\rangle$. The reason why all this should remain valid *beyond tree level* is the original argument

[12] A very interesting phenomenon is that, when we are dealing with solutions describing black holes or black p-branes (solutions with horizons), the above inequalities coincide with those imposed by the "cosmic censorship" that forbids the existence of naked singularities. In those cases, the inequality is saturated only by certain extremal configurations and so, for those supergravity solutions, assuming the cosmic censorship would imply a proof for (2.3.6) or, vice versa, Supersymmetry in Nature could be a proof of the cosmic censorship conjecture [55].

pointed out by Witten and Olive [52], which is the following. Representation theory of the (massive) states saturating a BPS bound, i.e. those for which $M = |Z|$, tell us that these lie in representations with strictly *less* states than those of the states for which $M > |Z|$[13]. The assumption is, then, that perturbative quantum corrections arising from higher loops renormalization of the mass or charge will not spoil the *relation* $M = |Z|$, because this would imply that the state $|S\rangle$ suddenly "jumps" into a higher dimensional representation containing more states. Although not a proof, it is difficult to think how a perturbative renormalization (carried out by a *continuous* beta function) could originate such a discrete jump in the number of states, even in String Theory.

The above argument does not say anything about the renormalization of the individual values of masses and charges. One may somehow have a tree level computation (which of course could never be a Supergravity computation alone) yielding classical values M and Z for the mass and some charges. These values could of course be shifted to M_R and Z_R when taking into account quantum radiative corrections[14]. But the above argument applies also to the renormalized values, and therefore we expect the relation $M_R = |Z_R|$ to hold in any case. We refer to the beginning of the present Section for a discussion about the reliability of a supergravity solution anyway as far as the charge-mass ratios are fixed.

We see therefore that supergravity solutions preserving some fraction of supersymmetry can be trusted regardless the fact that they are classical solutions. We emphasize that the existence of an argument, like the one given here, ensuring their reliability is an *absolute requirement* if one's purpose is to use Supergravity to get information from String Theory. The essential keys in the reasoning exposed above are the identification between central charges and gauge charges and representation theory of BPS states.

We want to stress, however, the following. The arguments given here *only* ensure (when applicable) that a given supergravity solution is reliable regardless the fact that it is a classical solution, and that some *necessary*

[13] In a superalgebra with no central charges (or even in the simple Poincaré algebra), massive representations are also bigger than the massless ones. The BPS case explained here is an analogous phenomenon, and it can be seen as an "intermediate" possibility that can arise in superalgebras with central charges.

[14] However, in theories with enough supersymmetry, like for example $N \geq 4$ in four dimensions, even these individual renormalizations do not occur.

requirements are satisfied in order to give it the interpretation of a description for the long wavelength limit of the fields emitted by some quantum stringy state. But let us consider the case in which we have a very well behaved supersymmetric supergravity solution that satisfies all the criteria enumerated above, and let us also consider that, furthermore, we have at our disposal some known, well defined BPS string state $|S\rangle$ that shares all the properties (charges, mass, supersymmetries, etc.) of the supergravity solution. This state will really be, for sure, a strong candidate for a quantum description of the supergravity source. But even in such a case all this constitutes, in a strict sense, no proof at all that the quantum state $|S\rangle$ is the stringy source that indeed produces, in the low energy limit, the field configuration described by the supergravity solution. All arguments given along this Section are just *necessary conditions* that allow for a possible match between both, but they *only* concern the classical charges and the quantum numbers of the source. The fact that the classical solution saturates a BPS bound is strong evidence that it is reliable, because there should be no dangerous quantum corrections and hence it will describe a possible, meaningful physical configuration. If one whishes, this evidence could even be promoted to a proof of *reliability*, but never to a proof of *correspondence* with any known quantum state. Such a proof would need of a sort of uniqueness theorem that enforces a one-to-one match, or else a full analysis at the quantum level of the behaviour of the massless fields emitted by the quantum source. In the case of D-branes this could be, in principle, possible, since one has a precise prescription for constructing $|S\rangle$. Such an attempt, with successful results, was carried out in [57], where the asymptotic behaviour of the classical supergravity p-brane solutions was reproduced from boundary state computations.

The matching between the String Theory description of nonperturbative string states and their Supergravity description has been the main character of many important achievements of String Theory in the last years. Two of them are of particular relevance: one is the microscopic explanation of the black hole entropy in a String Theory context [58], which has its roots in the two different, complementary descriptions of D-branes provided by Supergravity and String Theory. The other one is the the gauge/gravity CArrespondence. Although *a posteriori* one could certainly say that the resulting equivalence between Type IIB String Theory on $AdS_5 \times S^5$ and $N = 4$ SYM has nothing to do with D-branes, it was originally motivated [23] from the interplay between the String Theory and Supergravity pictures of D-branes.

2.4 Properties of BPS States

We have seen that BPS-saturated states play a very special role in Supergravity and String Theory: it is in the BPS limit where both pictures seem to be allowed to match. Given their importance, we now comment on some special properties enjoyed by supersymmetric solutions of Supergravity/BPS states of String Theory.

2.4.1 Stability

To start with, let us consider their dynamical stability. As mentioned at the beginning of Section 2.2, we expect the stringy state corresponding to a given supergravity solution to be stable against decay if the latter describes a static or stationary spacetime. We also said in Section 2.3.2 that a supersymmetric background saturates one or several BPS bounds of the kind of (2.3.6). The number of *different* existing bounds will be given by the number of different nonzero eigenvalues of the central charge matrix (roughly, the number of different charges that will be "switched on" in our solution). Furthermore, it turns out that the number of *saturated* bounds is related to the number of supersymmetries preserved by the supergravity solution. To focus on a concrete example, let as consider an hypothetical case in which we have two nonzero eigenvalues Z_1 and Z_2 of the central charge operator. The whole set of BPS bounds implies that

$$M \geq \max\Big(|Z_1|, |Z_2|\Big) \,. \tag{2.4.1}$$

A solution for which $|Z_1| > |Z_2|$ will only have one BPS bound saturated, and will only preserve, for example, 1/4 of the available supersymmetries[15]. But it so happens that for a supersymmetric solution having $|Z_1| = |Z_2|$, and that hence saturates *all* BPS bounds, the preserved supersymmetry is always *one half* of the total amount. This is a general property regardless the number of BPS bounds. Therefore, these states obey a true minimal-mass condition, and so their stability against decay is ensured both classically and quantum-mechanically. This is the case of D-branes, for example.

One-half BPS-saturated states are very important in String Theory, since one can always argue that they are very likely to be "elementary" states

[15] A precise counting of the preserved supersymmetries will depend on the explicit theory we are working on and on the explicit solution we are dealing with, though.

(as opposed to bound states of simpler configurations, for example) on the grounds of the above argument. All known supergravity and string extended objects which are considered to be elementary sources –because they have no known description in which they could be seen as composed objects[16]– preserve, in fact, one half of the supersymmetries.

A different (although less rigorous) semiclassical argument concerning the stability of certain supersymmetric states can be applied to the case of p-branes which are the extremal limit of a certain black p-brane solution. As already mentioned (see footnote 12), for this kind of solutions the BPS bound coincides with the one imposed by the cosmic censorship, and saturated configurations correspond to extremal ones. If one calculates the Hawking temperature of those extremal configurations one gets $T = 0$, which supports the idea of their stability. Of course, the reliability of the semiclassical computation (precisely in the extremal limit) underlies the validity of the present argument.

2.4.2 No-Force Condition

Another, very interesting property of supersymmetric solutions is that they always obey a no-force condition. By this we mean that if one has a supersymmetric solution describing, for example, a brane localized at some point in the transverse space, multi-pole static solutions describing several identical parallel branes located at different points in transverse space also exist. The fact that such configurations solve the equations of motion and, moreover, are static, means that they are classically stable. They will feel attracted to each other by gravitational forces, and they will feel repelled from each other due to Coulomb-like forces, but a explicit calculation shows that these contributions exactly cancel no matter what the distance between the branes

[16] A quite exotic counterexample is provided by fractional branes, since a regular D-brane can actually be considered as a superposition of fractional ones. However, the situation under which such a thing can happen is also quite exotic, because fractional branes only exist at the fixed points of orbifold theories, and this is something that drastically modifies all the discussion concerning asymptotics, vacua, conserved charges, etc. that we have developed along the present Chapter (the asymptotic spacetime is no longer a manifold, to start with). In any case, and also in an orbifold theory, a "bulk" D-brane is really elementary, at least in the sense that no description of it exists in terms of other string states. A review on fractional branes will be presented in Chapter 9. We will deal with them in Chapter 10.

is. In the case of D-branes, where a description in terms of string states is available, and hence a quantum computation can explicitly be done, the no-force condition can be seen from a computation of the cylinder diagram. In the closed string channel (tree level), one can explicitly check that the NS-NS and RR sector contributions exactly cancel.

This no-force condition is important because it allows for "bound" states where many parallel branes can be piled up at the same point in transverse space. The fact that such configurations exist and are indeed as stable as a single brane (the only difference is that the mass and the charge of the configuration are those of a single brane multiplied by the number of branes) is important for many applications.

2.4.3 Classical BPS Solutions as Solitons

We end by commenting on a very interesting analogy between some classical BPS supergravity solutions and quantum field theory solitons. In a quantum field theory with central charges, the classical solutions that saturate the BPS bound are usually solitonic configurations [52]: classical, nonsingular, finite-energy field configurations that spatially interpolate between two different vacua of the field theory under consideration. In fact, the addition of central charges in the supersymmetry algebra was shown by Witten and Olive to be *needed* in a field theory that has multiple vacua that label topological sectors. In [52] they referred to these central charges as "topological charges", precisely because their value, when acting on a soliton state, coincides with the topological charge carried by the soliton.

In Supergravity a parallel cannot be made, simply because there is *no* space of vacua: there is no potential, and no fixed criterion to find something like the "minimal energy configurations" exists (not even to talk about a vacuum selection mechanism). This is one of the major problems of String Theory. It is true that some spacetimes can be *considered* as vacua, and in fact we call them like that. We gave some heuristic arguments for making such a definition in Section 2.2.1. We stress that this is nothing but an *analogy* with the true vacua of quantum field theory[17], and, as such, it must be recognized that this definition lacks the needed rigor. However, it is a quite appealing analogy. A further argument in support of it comes from the behaviour of many supergravity solutions that saturate a BPS bound:

[17] A related discussion can be found at the beginning of Chapter 5.

it is a fact that, at least some of them, possess all the properties that one would usually attribute to a field theory soliton, *provided* that one takes the definition of a vacuum spacetime pointed out here as correct. In particular, the M2- and M5-brane solution of eleven-dimensional Supergravity, the D3-brane solution of Type IIB supergravity and the NS5-brane solution of all ten-dimensional supergravities interpolate in space between the following spacetimes [59]:

M2	$\mathcal{M}_{11}$	$\longleftrightarrow$	$AdS_4 \times S^7$
M5	$\mathcal{M}_{11}$	$\longleftrightarrow$	$AdS_7 \times S^4$
D3	$\mathcal{M}_{10}$	$\longleftrightarrow$	$AdS_5 \times S^5$
NS5	$\mathcal{M}_{10}$	$\longleftrightarrow$	$\mathcal{M}_7 \times S^3$

Table 2.1: Eleven and ten-dimensional solutions that interpolate between different vacua. $\mathcal{M}_d$ stands for d-dimensional Minkowski spacetime.

They are all perfectly regular everywhere, and all the spacetimes between which they interpolate fall into the class of spacetimes that can be considered as vacua. There are other very well-known supergravity solutions (like the other Dp-brane solutions) that do not fit so well in the above classification, in the sense that they do not interpolate between any two vacua: at least from a supergravity viewpoint, they are singular at their core. However, they are customarily called "solitons" too, because, if one pursues the field theory analogy, it is true that all of them saturate a BPS bound and have a well defined and finite "topological" charge.

From a quite optimistic point of view, all this might shed some light on (or provide some hints to) the problem of what is a String Theory vacuum. But the question of why should be ten dimensional Minkowski spacetime $\mathcal{M}_{10}$ disfavoured with respect to e.g. $AdS_5 \times S^5$ or $\mathcal{M}_4 \times T^6$ still remains completely obscure.

2.5 Nonperturbative String States and String Duality

Concerning the investigation of nonperturbative states in String Theory and Supergravity, there is a very important issue that we have not developed so far: string dualities. Here we do not attempt to give an exhaustive review on this topic (reviews with references are e.g. [60–63]), but we would like to explain their importance for nonperturbative string physics.

Duality in String Theory can be understood as a "generalized symmetry" that maps a given situation into a different one. It can relate weak and strong coupling (S-duality), different backgrounds (T-duality) or even different superstring theories. As explained in Chapter 1, String Theory (and Supergravity, as a limit of it) is only defined perturbatively. The importance of duality symmetries is that, many times, they can be used to relate perturbative and nonperturbative regimes, so the tools we have at our disposal (perturbation theory) may suffice to gain insight into nonperturbative string physics. In particular, duality has been one strong argument pointing towards the existence of nonperturbative states such as D-branes [47].

2.5.1 Duality in String Theory

Let us recall first what T- and S-duality are from the String Theory point of view.

T-Duality

T-duality arises as a symmetry relating different compactifications of String Theory. In the simplest case of compactification on a circle, T-duality tells us that a closed String Theory compactified on a circle of radius R is equivalent to (a maybe different) String Theory compactified on a circle of "dual" radius $\tilde{R}$ given by the inverse of R in string units

$$R \xrightarrow{T} \tilde{R} = \frac{\alpha'}{R}$$

upon further exchange of winding and Kaluza-Klein modes [16]. This equivalence is seen perturbatively both in the spectrum and in string amplitudes, and hence it is the statement of equivalence of string physics on different radii of compactification. If we consider compactifications on higher dimensional torii, the T-duality group will be enlarged to take into account the internal symmetries of the compact space. Note that T-duality is exclusively a String Theory phenomenon, since only strings (or, at least, extended objects) can have winding modes associated to a compactification. It can also be seen as a manifestation of the existence of a minimal length in String Theory: at least when talking about compactification, physics below the fundamental string unit of length $l_s = \sqrt{\alpha'}$ has a dual picture (a different description that describes the same physics) above that scale[18]. At the level of superstring theories, Type IIA and Type IIB are T-dual to each other, the same as Heterotic-$SO(32)$ and Heterotic-$E_8 \times E_8$. What this means is that both pairs of theories describe the same physics when compactified on circles of dual radii.

S-Duality

S-duality is completely different since, to start with, it is intrinsically non-perturbative. On general grounds, it is the statement of equivalence under the inversion of the string coupling g_s [17]

$$g_s \xrightarrow{S} \tilde{g}_s = \frac{1}{g_s},$$

at least under certain circumstances. How this is to be implemented in practice and which are its consequences depends on the particular theory we are considering. For example, it is believed that the Type IIB theory is self-S-dual, and that Type I and Heterotic-$SO(32)$ are S-dual to each other. Many times S-duality can be understood as a generalized electric-magnetic duality in the spirit of [64].

In general compactified theories, both dualities can be embedded into a larger one which is customarily called U-duality [18]. This is, in general, bigger than the "direct product" of S- and T-dualities, because additional symmetries (which do not come neither from reparametrization invariance of the internal manifold nor have a higher dimensional origin) may appear in the compactified case.

[18] The so-called self-dual radius $R_s = \sqrt{\alpha'}$ is in fact special: compactification on a circle of this size exhibits enhanced gauge symmetry, because new light degrees of freedom appear at this point.

2.5.2 Duality in Supergravity

One of the things that makes Supergravity extremely useful for the study of nonperturbative string physics is that string dualities are also seen in the limit represented by Supergravity. In general, dualities are seen in Supergravity as *global symmetries* of supergravity actions or supergravity equations of motion. Many of these global symmetries in Supergravity were already known before the discovery of string dualities, but they simply "were there". An explanation for them is thus found within String Theory (see e.g. [65]). These can be symmetries of a single supergravity theory (e.g. S-duality in the Type IIB effective action) or they can relate two different theories (e.g. T-duality between the Type IIA and Type IIB theories). In the forthcoming Chapters we will be mainly concerned with S- and T-dualities of Type II theories. At the level of Supergravity, the rules implementing these string dualities were found in [26]. They are given in terms of a map between the different supergravity fields, and hence they can be used to relate dual supergravity backgrounds[19].

The fact that, in Supergravity, duality transformations of the fields are either symmetries of the equations of motion or they map solutions of a theory into solutions of a different one has very useful consequences. This allows to use them as solution-generating techniques to get new solutions from a known one, simply by acting on it with the considered duality group.

2.5.3 Duality and the Nonperturbative Spectrum of String Theory

A consequence of string duality that has been a key element for many important developments in the last years is that the *states* of the nonperturbative String Theory spectrum are related by different dualities. For example, all

[19]Many times we will be talking about T-duality as relating two different *noncompact* ten dimensional backgrounds (as e.g. the D-brane solutions of the Type IIA and IIB theories, which can be –correctly– interpreted as describing no compact directions at all). But if we talk about T-duality this has to be interpreted carefully, since a compactification on a circle must always be implicitly assumed. In fact, to establish T-duality between two different backgrounds one always needs to have, in each of them, an isometric direction (the direction "along which we T-dualize") to allow for a compactification and subsequent T-dualization. T-duality-related backgrounds of noncompact, ten-dimensional spacetime must be interpreted as a "decompactification limit" of the actual (compactified) T-dual backgrounds.

Dp-branes of different p are related by T-duality. This fact becomes manifest also in Supergravity, since one can see that the corresponding supergravity *backgrounds* describing the different extended objects of String Theory are also related by the corresponding duality transformations as they appear in the supergravity actions. This makes duality a powerful tool to explore the nonperturbative String Theory spectrum, and the fact that this tool can be used at the Supergravity level makes it particularly easy to use. It is a tool because, as explained, the duality-transformation rules are known in Supergravity, and hence we can act with them on a given background to get, at the end, a different one[20].

Given the fact that string dualities can make us "jump" from some solution describing a certain nonperturbative state into a different one, a natural question arises: which are all possible nonperturbative string states that are contained within the orbits of the transformations of the duality groups ?

This is the main question that we will address in Chapters 3 and 4. In Chapter 3 we will face this issue in a simple context: $N = 4$ Supergravity in four dimensions. This is a theory that arises from a compactification of the ten dimensional Heterotic string on a six-torus, but with the interesting property that it exhibits both S- and T-dualities (which, as we will see, become global symmetries of the supergravity action of this theory). Hence one can wonder about which is the biggest family of solutions of this theory with well-defined physical charges that one can get by acting with these symmetries. Such a set of solutions will be, by definition, invariant (as a family) under the whole duality group of the theory. We will see that we are able to find such an invariant family even for the non-supersymmetric cases. A big subclass of the solutions that we will find describe black holes. This is important because, for black holes, the "no-hair theorem" tells us which should be *all* possible physical configurations, and we will also see that all possible black hole solutions (BPS or not) are indeed included in the family of solutions that one can get from the duality symmetries of the theory. We thus find an example, namely black holes of four dimensional $N = 4$ Supergravity, in which all possible physical configurations are predicted by duality.

[20] All this is of course limited by the reliability of the Supergravity approximation. At the level of Supergravity, the procedure sketched here can certainly be applied as long as Supergravity is supposed to provide us with reliable information about the corresponding string state, but there are some cases (like those of stable non-BPS states [66]) for which a supergravity description is not even known.

In Chapter 4 we will apply this idea to the much more interesting case of ten and eleven-dimensional supergravities. There we will argue that the "standard" ten dimensional spectrum of elementary, nonperturbative string states is not enough to provide a "full representation" of the S- and T-duality groups. By exploring the T-duals of the S-dual of the D7-brane (and new solutions originating from them) we will find a whole family of new ten dimensional solutions, all of them describing 1/2 BPS states charged under a single gauge field. We will argue that these solutions should then describe new, elementary, nonperturbative string states that would be missing from the known ten dimensional string spectrum.

Chapter 3

The General, Duality Invariant Family of Non-BPS Black-Hole Solutions of $N = 4, d = 4$ Supergravity

Introduction

The low-energy effective action of the heterotic string compactified on T^6 is that of pure $N = 4, d = 4$ Supergravity coupled to $N = 4$ super Yang-Mills. It is possible to truncate consistently this theory to the simpler pure supergravity theory. From the string theory point of view the truncation consists in introducing always equal numbers of Kaluza-Klein and winding modes for each cycle. The truncated theory still exhibits S and T dualities and, thus, pure $N = 4, d = 4$ Supergravity provides a simple framework in which to study classical solutions which still can be considered as solutions of the full effective String Theory. The bosonic sector of this theory is also known in the literature as "Dilaton-Axion Gravity" or as "Einstein-Maxwell Dilaton-Axion Theory" when only a single vector field is considered.

Perhaps the most interesting solutions of the 4-dimensional string effective action are the black-hole type ones[1] since they constitute the best testing ground for the Quantum Gravity theory contained in String Theory. It is believed that a good Quantum Gravity theory should be able to explain

[1] For a review of black holes in toroidally compactified string theory see e.g. [67] and [68].

in terms of microscopic degrees of freedom the values of the macroscopic thermodynamical quantities found classically and semiclassically. There has been some success in this respect for supersymmetric ("BPS-saturated") and near-supersymmetric black holes although the results are to be interpreted carefully since the supersymmetric limit is singular in many respects.

A great deal of effort has been put in finding the most general families of black-hole solutions whose thermodynamical properties should exhibit also invariance (or, rather, covariance) under the duality symmetries of the theory and covering the supersymmetric and non-supersymmetric cases and, further, covering stationary (not static) cases.

The first two examples of this kind of families of solutions were found in Ref. [69][2]. The first family of solutions corresponds to non-supersymmetric, static black-hole solutions and the second to supersymmetric, static, multi-black-hole solutions of $N = 4, d = 4$ supergravity. Under the dualities of the theory, solutions of each family transform into other solutions of the same family, with the same functional form. Thus, only the values of the charges and moduli transform. The supersymmetric solutions are given in terms of two constrained complex harmonic functions.

Different extensions and properties of these solutions in the context of Dilaton-Axion Gravity were later obtained in Refs. [74–84].

A main step forward was given in Ref. [85] where it was realized that the form of the above supersymmetric solutions was dictated the special geometry of the associated $N = 2, d = 4$ Supergravity theory. The two complex harmonic functions are associated to coordinates and certain components of the metric are associated to the Kähler potential and the holomorphic vector. It was found that similar Ansatzs could be used in other $N = 2, d = 4$ Supergravity theories with different matter multiplets and Kähler potentials.

Finally, in Refs. [86,87] the most general supersymmetric black-hole-type solutions of pure $N = 4, d = 4$ Supergravity (SWIP solutions) where found. The only difference with those of Ref. [69] is that the complex harmonic functions are now completely arbitrary and unconstrained. This automatically allows for the introduction of angular momentum and NUT charge in the solutions. In fact the constraint simply meant that these charges were not allowed. The generating solution for regular, supersymemtric, $N = 8$ supergravity black hole solutions has been found in Ref. [88].

[2]In the much simpler context of pure $N = 2, d = 4$ Supergravity the IWP solutions of Refs. [70–72] also have this property.

Similar supersymmetric solutions were later found for other $N = 2, d = 4$ Supergravity theories [89] with vector multiplets[3].

For the non-supersymmetric solutions of Ref. [69] the story has been different since no clear relation with the underlying special geometry was established. In Ref. [91] a general recipe for obtaining non-supersymmetric solutions from supersymmetric solutions in $N = 2, d = 4$ Supergravity theories, previously used in other contexts, was shown to work for *static* black holes: one simply has to deform the metric with the introduction of a non-supersymmetry (non-extremality) function.

What has to be done in more general cases (stationary, for instance) is far from clear and general duality-invariant families of stationary non-supersymmetric solutions are not available in the literature and no recipe to build them is known.

In this Chapter we present such a general duality-invariant family of stationary non-supersymmetric solutions of pure $N = 4, d = 4$ Supergravity characterized by completely independent electric and magnetic charges, mass, angular momentum and NUT charge plus the asymptotic values of the scalar fields[4].

The rest of the Chapter is organized as follows: in Section 3.1 we describe the bosonic sector of $N = 4, d = 4$ Supergravity theory. In Section 3.2 we give and study the general family of solutions we relate it to others already known. In Section 3.3, we focus our attention in the black hole type subfamily of metrics and calculate the explicit values for their entropy and the temperature, showing that also these quantities can be put in a manifestly duality- invariant form. Section 3.4 contains our conclusions. The Appendices contain the definitions of the different charges we use and their duality-invariant combinations.

[3]For a review on supersymmetric black hole solutions of supergravity theories see e.g. Ref. [90].

[4]When we talk about general solutions we are implicitly excluding the possibility of having primary scalar hair. Solutions with primary scalar hair are in all known cases (see, e.g. [92]), singular (providing evidence for the never proven "no-hair theorem") and, being interested in true black holes with event horizons covering all the physical singularities, these cases are not important for us and in the solutions which we are going to present the scalar charges are always completely determined by the $U(1)$ charges. Nevertheless, it should be pointed out that more general solutions (some of them supersymmetric) which include primary scalar hair must exist and should be related to the ones given here by formal T duality in the time direction [93].

3.1 $N=4,\ d=4$ Supergravity

3.1.1 Description of the System and Equations of Motion

The bosonic sector of pure $N=4, d=4$ Supergravity contains two real scalar fields (axion a and dilaton ϕ), six Abelian vector fields $A^{(n)}_\mu$ (which we generalize to an arbitrary number N) and the metric $g_{\mu\nu}$. The action reads[5]

$$\begin{aligned} S \;=\; & \tfrac{1}{16\pi}\int d^4x\sqrt{|g|}\,\Bigg\{R + 2(\partial\phi)^2 + \tfrac{1}{2}e^{4\phi}(\partial a)^2 - e^{-2\phi}\sum_{n=1}^{N} F^{(n)}F^{(n)} \\ & + a\sum_{n=1}^{N} F^{(n)}\,{}^\star F^{(n)}\Bigg\}\,. \end{aligned} \tag{3.1.1}$$

The axion and dilaton are combined into a single complex scalar field, the *axidilaton* λ:

$$\lambda = a + ie^{-2\phi}\,. \tag{3.1.2}$$

For each vector field we can also define its $SL(2,\mathbb{R})$-dual, which with our conventions will be given by:

$$\tilde{F}^{(n)}{}_{\mu\nu} \equiv e^{-2\phi}\,{}^\star F^{(n)}{}_{\mu\nu} + aF^{(n)}{}_{\mu\nu}\,. \tag{3.1.3}$$

The equations of motion derived from the action (3.1.1) plus the Bianchi identities for the vector fields can be written as follows:

$$\nabla_\mu{}^\star\tilde{F}^{(n)\,\mu\nu} = 0\,,$$

$$\nabla_\mu{}^\star F^{(n)\,\mu\nu} = 0\,,$$

$$\nabla^2\phi - \tfrac{1}{2}e^{4\phi}(\partial a)^2 - \tfrac{1}{2}e^{-2\phi}\textstyle\sum_{n=1}^{N} F^{(n)}F^{(n)} = 0\,,$$

$$\nabla^2 a + 4\partial_\mu\phi\,\partial^\mu a - e^{-4\phi}\textstyle\sum_{n=1}^{N} F^{(n)}\,{}^\star F^{(n)} = 0\,,$$

$$R_{\mu\nu} + 2\partial_\mu\phi\partial_\nu\phi + \tfrac{1}{2}e^{4\phi}\partial_\mu a\partial_\nu a - 2e^{-2\phi}\textstyle\sum_{n=1}^{N}\left(F^{(n)}{}_{\mu\rho}F^{(n)}{}_\nu{}^\rho - \tfrac{1}{4}g_{\mu\nu}F^{(n)}F^{(n)}\right) = 0\,.$$

[5] Our conventions coincide with those of Ref. [94]. In particular, we use mostly minus signature and Hodge duals are defined such that ${}^\star F^{(n)\,\mu\nu} = \frac{1}{2\sqrt{|g|}}\epsilon^{\mu\nu\rho\sigma}F^{(n)}_{\rho\sigma}$ with $\epsilon^{0123} = +1$.

Observe that we have written the Maxwell equations as the Bianchi equations for the $SL(2,\mathbb{R})$ duals. Therefore N dual vector potentials $\tilde{A}^{(n)}_{\mu}$ defined by

$$\tilde{F}^{(n)}_{\mu\nu} = \partial_{\mu}\tilde{A}^{(n)}_{\nu} - \partial_{\nu}\tilde{A}^{(n)}_{\mu}\,, \tag{3.1.5}$$

exist locally.

The axidilaton parametrizes an $SL(2,\mathbb{R})/SO(2)$ coset [95], the equations of motion being invariant under global $SL(2,\mathbb{R})$ ("S duality") transformations. If Λ is an $SL(2,\mathbb{R})$ matrix

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad - bc = 1\,, \tag{3.1.6}$$

then the vector fields and their duals transform as doublets

$$\begin{pmatrix} \tilde{F}^{(n)}{}_{\mu\nu} \\ \\ F^{(n)}{}_{\mu\nu} \end{pmatrix} \longrightarrow \Lambda \begin{pmatrix} \tilde{F}^{(n)}{}_{\mu\nu} \\ \\ F^{(n)}{}_{\mu\nu} \end{pmatrix}, \tag{3.1.7}$$

and the axidilaton transforms according to

$$\lambda \longrightarrow \frac{a\lambda + b}{c\lambda + d}\,. \tag{3.1.8}$$

This is an electromagnetic duality rotation that acts on the dilaton. From the point of view of String Theory, this is the 4-dimensional string coupling constant. Hence the name S duality.

Furthermore the N (6 in the SUGRA theory) vector fields can be $SO(N)$-rotated. These are "T duality" transformations (perturbative from the String Theory point of view). The full duality group is, then $SL(2,\mathbb{R}) \otimes SO(N)$.

3.2 The General Solution

We now present the family of solutions. All the fields in our solutions may be expressed in terms of two *fixed* complex harmonic functions of the three dimensional Euclidean space, $\mathcal{H}_1$ and $\mathcal{H}_2$, a set of N complex constants $k^{(n)}$, a "non-extremality" function W and a background 3-dimensional metric ${}^{(3)}\gamma_{ij}$. In all of them appear the physical constants defined in Appendix A.2. Only

Υ, the axidilaton charge, is not independent. The harmonic functions are

$$\left\{ \begin{array}{rcl} \mathcal{H}_1 & = & \frac{1}{\sqrt{2}} e^{\phi_0} e^{i\beta} \left(\lambda_0 + \dfrac{\lambda_0 \mathfrak{M} + \bar{\lambda}_0 \Upsilon}{\tilde{\rho}} \right) , \\ \mathcal{H}_2 & = & \frac{1}{\sqrt{2}} e^{\phi_0} e^{i\beta} \left(1 + \dfrac{\mathfrak{M} + \Upsilon}{\tilde{\rho}} \right) , \end{array} \right. \tag{3.2.1}$$

where $\tilde{\rho}^2 \equiv x^2 + y^2 + (z + i\alpha)^2$ is the usual complex radial coordinate, and β is an arbitrary, unphysical real number related to the duality transformation of these functions under $SL(2,\mathbb{R})$ (see the explanation in Section 3.2.1). The complex constants are

$$k^{(n)} = -\tfrac{1}{\sqrt{2}} e^{-i\beta} \frac{\mathfrak{M}\Gamma^{(n)} + \overline{\Upsilon\Gamma^{(n)}}}{|\mathfrak{M}|^2 - |\Upsilon|^2} . \tag{3.2.2}$$

In supersymmetric cases (e.g. Ref. [87]) it is useful to introduce oblate spheroidal coordinates which are related to the ordinary Cartesian ones by:

$$\left\{ \begin{array}{rcl} x & = & \sqrt{r^2 + \alpha^2}\ \sin\theta \cos\varphi , \\ y & = & \sqrt{r^2 + \alpha^2}\ \sin\theta \sin\varphi , \\ z & = & r \cos\theta . \end{array} \right. \tag{3.2.3}$$

The three dimensional Euclidean metric is written in these coordinates in the following way:

$$d\vec{x}^2 = \frac{r^2 + \alpha^2 \cos^2\theta}{r^2 + \alpha^2} dr^2 + \left(r^2 + \alpha^2 \cos^2\theta\right) d\theta^2 + \left(r^2 + \alpha^2\right) \sin^2\theta \, d\varphi^2 . \tag{3.2.4}$$

In terms of (3.2.3) the radial coordinate $\tilde{\rho}$ that appears in (3.2.1) may be expressed as $\tilde{\rho} = r + i\alpha \cos\theta$. Furthermore, in these new coordinates, the "non-extremality" function has a simple form:

$$W = 1 - \frac{{r_0}^2}{r^2 + \alpha^2 \cos^2\theta} , \tag{3.2.5}$$

where r_0, given by

$$r_0^2 = |\mathfrak{M}|^2 + |\Upsilon|^2 - \sum_{n=1}^{N} |\Gamma^{(n)}|^2 . \tag{3.2.6}$$

is usually called "extremality parameter" in the static cases. In stationary cases, though, $r_0 = 0$ means that the solution is supersymmetric but in general it is not an extreme black hole (nor a black hole). Thus, a more appropriate name is *supersymmetry parameter*. The *extremality parameter* will be $R_0^2 = r_0^2 - \alpha^2$.

Finally, the last ingredient is the background metric ${}^{(3)}\gamma_{ij}$

$$\begin{aligned} d\vec{x}^{\,2} &= {}^{(3)}\gamma_{ij}dx^i dx^j = \frac{r^2+\alpha^2\cos^2\theta - r_0^2}{r^2+\alpha^2-r_0^2}\,dr^2 + \left(r^2+\alpha^2\cos^2\theta - {r_0}^2\right)d\theta^2 \\ &\quad + \left(r^2+\alpha^2-{r_0}^2\right)\sin^2\theta d\varphi^2\,, \end{aligned} \tag{3.2.7}$$

which differs from (3.2.4) in non-supersymmetric ($r_0 \neq 0$) cases and is not flat[6]. This is an important qualitative difference between the usual supersymmetric IWP-type [70, 71] metrics (e.g. those of Refs. [72, 86, 87]) and our solution.

We can now describe the solutions. They take the form

$$\left\{ \begin{aligned} ds^2 &= e^{2U}W\left(dt+\omega_\varphi d\varphi\right)^2 - e^{-2U}W^{-1}\ {}^{(3)}\gamma_{ij}dx^i dx^j\,, \\ A^{(n)}{}_t &= 2e^{2U}\,\Re\mathrm{e}\left(k^{(n)}\mathcal{H}_2\right)\,, \\ \tilde{A}^{(n)}{}_t &= 2e^{2U}\,\Re\mathrm{e}\left(k^{(n)}\mathcal{H}_1\right)\,, \\ \lambda &= \frac{\mathcal{H}_1}{\mathcal{H}_2}\,, \end{aligned} \right. \tag{3.2.8}$$

where

$$e^{-2U} = 2\,\Im\mathrm{m}\left(\mathcal{H}_1\bar{\mathcal{H}}_2\right) = 1 + 2\Re\mathrm{e}\left(\frac{\mathfrak{M}}{r+i\alpha\cos\theta}\right) + \frac{|\mathfrak{M}|^2 - |\Upsilon|^2}{r^2+\alpha^2\cos^2\theta}\,,$$

and where

$$\begin{aligned} \omega_\varphi &= 2n\cos\theta + \alpha\sin^2\theta\left(e^{-2U}W^{-1}-1\right) \\ &= \frac{2}{r^2+\alpha^2\cos^2\theta - r_0^2}\times \\ &\quad \times\left\{n\cos\theta\left(r^2+\alpha^2-r_0^2\right) + \alpha\sin^2\theta\left[mr + \tfrac{1}{2}\left({r_0}^2 + |\mathfrak{M}|^2 - |\Upsilon|^2\right)\right]\right\}\,. \end{aligned}$$

[6]In (3.2.8) as well as in (3.2.7) the x^i label the coordinates r, θ and φ for $i = 1, 2, 3$ respectively.

ω_φ can be interpreted as the unique non-vanishing covariant component of a 1-form $\omega = \omega_i dx^i$ which for $\alpha \neq 0$ is a solution of the following equation in 3-dimensional space:

$$^\star d\omega - e^{-2U\star} d\mu - W^{-1}\Re\mathrm{e}\left(\mathcal{H}_1 d\bar{\mathcal{H}}_2 - \bar{\mathcal{H}}_2 d\mathcal{H}_1\right) = 0\,, \tag{3.2.9}$$

where μ is also a 1-form whose only non-vanishing component μ_φ is given by

$$\mu_\varphi = \frac{r_0^2}{\alpha}\,\frac{r^2+\alpha^2-r_0^2}{r^2+\alpha^2\cos^2\theta - r_0^2}\,, \tag{3.2.10}$$

and where the 3-dimensional background metric ${}^{(3)}\gamma_{ij}$ has to be used in the Hodge duals.

For $\alpha = 0$ the μ term in Eq. (3.2.9) has to be eliminated and the equation tales the form

$$^\star d\omega - W^{-1}\Re\mathrm{e}\left(\mathcal{H}_1 d\bar{\mathcal{H}}_2 - \bar{\mathcal{H}}_2 d\mathcal{H}_1\right) = 0\,. \tag{3.2.11}$$

We can also write our solution in the standard form used to describe general rotating black holes, which will be useful to describe the structure of the singularities:

$$\begin{aligned} ds^2 \;=\;& \frac{\Delta - \alpha^2\sin^2\theta}{\Sigma}\,dt^2 + 2\alpha\sin^2\theta\,\frac{\Sigma + \alpha^2\sin^2\theta - \Delta}{\Sigma}\,dt d\varphi - \\ & -\frac{\Sigma}{\Delta}\,dr^2 - \Sigma\,d\theta^2 - \frac{\left(\Sigma + \alpha^2\sin^2\theta\right)^2 - \Delta\alpha^2\sin^2\theta}{\Sigma}\sin^2\theta\,d\varphi^2\,, \end{aligned} \tag{3.2.12}$$

where

$$\begin{aligned} \Delta &= r^2 - {R_0}^2 = r^2 + \alpha^2 - {r_0}^2\,, \\ \Sigma &= (r+m)^2 + (n + \alpha\cos\theta)^2 - |\Upsilon|^2\,. \end{aligned} \tag{3.2.13}$$

This completes the description of the general solution. Now we are going to describe its properties.

3.2.1 Duality Properties

We can study the effect of duality transformations in two ways which are fully equivalent in this family of solutions: we can study the effect of the transformations of the fields or simply the effect of the transformations on the physical constants. One of the main features of our family of solutions is precisely this equivalence: we can simply transform the physical constants (adding "primes") because the functional form of the solutions always will remain invariant.

Let us, then, study the effect of $SL(2,\mathbb{R})$ transformations of the charges $\mathfrak{M}, \Gamma^{(n)}, \Upsilon$ and moduli λ_0 in Appendix A.2.

Both the complex harmonic functions $\mathcal{H}_{1,2}$ and the complex constants $k^{(n)}$ are defined up to a phase: if we multiply $\mathcal{H}_{1,2}$ by a constant phase and the $k^{(n)}$'s by the opposite one, the solution remains unchanged. We have made this fact explicit by including the arbitrary angle β in their definition.

As it can take any value, in particular we can require it to change in the following way when performing an $SL(2,\mathbb{R})$ rotation:

$$e^{i\beta} \longrightarrow e^{i\arg(c\lambda_0+d)}e^{i\beta}\,. \tag{3.2.14}$$

With this choice the $k^{(n)}$'s are left invariant, while the pair $\mathcal{H}_{1,2}$ transforms as a doublet:

$$\begin{pmatrix} \mathcal{H}_1' \\ \mathcal{H}_2' \end{pmatrix} = \Lambda \begin{pmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{pmatrix}. \tag{3.2.15}$$

$\mathcal{H}_{1,2}$ appear only through two invariant combinations: e^{-2U} and $\Re\mathrm{e}\left(\mathcal{H}_1 d\bar{\mathcal{H}}_2 -\bar{\mathcal{H}}_2 d\mathcal{H}_1\right)$. The remaining building blocks of the solution are μ and W which are invariant if the supersymmetry parameter r_0 is invariant. This (first proven in Ref. [96]) is shown in Appendix A.3.

Under $SO(N)$ the $k^{(n)}$'s transform as vectors, as they should, and everything else is invariant.

The relation of the form of these solutions to $N=2$ special geometry is the same as in the supersymmetric case [85, 87] and we will not repeat here that discussion. The only difference is the introduction of the background metric ${}^{(3)}\gamma$, and the functions μ, W which "deform" the supersymmetric solution but have no special meaning from the special geometry point of view.

3.2.2 Reduction to Other Known Solutions

We can now relate our solution to those less general found in the literature. We can consider two types of solutions: supersymmetric and non-supersymmetric. The most general family of supersymmetric black-hole type solutions of $N=4, d=4$ Supergravity (SWIP solutions) was found in Refs. [86, 87]. Due to the existence of supersymmetry, the family contains two arbitrary harmonic functions. To describe point-like solutions one chooses harmonic functions with a single pole. Our solutions reduce precisely to these when the supersymmetry parameter vanishes: $r_0 = 0$. As shown in Appendix A.3 in the $r_0 = 0$ limit at least one of the two possible Bogomol'nyi bounds of $N = 4$ Supergravity are saturated. In this case $W = 1, \mu_\varphi = 0$, ${}^{(3)}\gamma$ becomes flat (in spheroidal coordinates) and we recover the structure of the SWIP solutions. The SWIP solutions always saturate one bound due to the constraints that the constants $k^{(n)}$ satisfy

$$\begin{aligned} \sum_{n=1}^{N} (k^{(n)})^2 &= 0\,, \\ \sum_{n=1}^{N} |k^{(n)}|^2 &= \tfrac{1}{2}\,. \end{aligned} \tag{3.2.16}$$

while, in our case there is no constraint on the charges (apart from the one on the scalar charge, associated to the non-hair theorem)

$$\begin{aligned} \sum_{n=1}^{N} (k^{(n)})^2 &= \frac{-\mathfrak{M}\overline{\Upsilon}}{(|\mathfrak{M}|^2 - |\Upsilon|^2)^2}\, {r_0}^2\,, \\ \sum_{n=1}^{N} |k^{(n)}|^2 &= \tfrac{1}{2}\left(1 - \frac{|\mathfrak{M}|^2 + |\Upsilon|^2}{(|\mathfrak{M}|^2 - |\Upsilon|^2)^2}\, {r_0}^2\right). \end{aligned} \tag{3.2.17}$$

In Ref. [87] it is shown how this solution reduces to supersymmetric solutions with angular momentum, NUT charge etc. Only some of the static ones (those with 1/4 of the supersymmetries unbroken) are black holes with a regular horizon. These include extreme Reissner-Nordström black holes and their axion-dilaton generalizations [55, 96–99]. The rest have naked singularities.

As for the non-supersymmetric solutions, the non-extreme Taub-NUT axion-dilaton solutions of Ref. [73] are clearly covered by our general solution. Further, in Ref. [75] were found general point-like solutions for a theory with only one vector field ("axion-dilaton gravity"). We can see that our solutions reduce to these ones by setting $N = 1$. The principal difference is that, in this particular case, we can fit the analogous of expression (A.2.10) in the definition (3.2.6) for the extremality parameter, giving

$$r_{0,N=1}^2 = \left(|\mathfrak{M}| - |\Upsilon|\right)^2 , \tag{3.2.18}$$

and inserting this into the metric (3.2.8) we get exactly Eqs. (31-35) of Ref. [75] up to a shift in the radial coordinate. Although one can immediately see that the functional form of the metric (3.2.8) does not change very much from that found in [75], somewhat different results appear when considering multiple vector fields, due to the constraints that the physical parameters obey when only one vector field is present. This analysis was already done in detail in [87], and we refer to this paper for further discussion.

A generalization of the non-extreme solutions of [75] for the same theory, but with an arbitrary number of vector fields (*i.e.*, the same theory we are treating), was found in [82]. However, the solutions reported there concern only the *static* case, and therefore the total number of independent physical parameters is $2N+4$. As it was shown in that paper, the metric for the static case is of the "Reissner-Nordström-type", but with a variable mass factor. It can be seen that, taking the static ($\alpha = 0$) limit of our metric (3.2.8), and shifting the radial coordinate by a quantity $m + \sqrt{|\Upsilon|^2 - n^2}$, we recover the same solution of [82] (Eq. (7.6) of that reference) up to redefinitions in the different constants parametrizing the solution.

Finally, we observe that setting the axidilaton charge equal to zero (which can be done with appropriate combinations of electric and magnetic charges) in Eqs. (3.2.13-3.2.12), we recover the Kerr-Newman solution in Boyer-Lindquist coordinates (but with a constant shift equal to the mass in the radial coordinate. See, *e.g.*, Refs. [100, 101]).

3.3 Black-Hole-Type Solutions

3.3.1 Singularities

We now carry out the analysis of the structure of our solutions. First, we proceed to study the different types of singularities of the metric. Due to the standard form of $g_{\mu\nu}$ in terms of Δ and Σ, the singularities in terms of these functions are those of all Kerr-type metrics, *i.e.*, we have coordinate singularities at

$$\Delta = 0\,, \qquad \theta = 0\,, \tag{3.3.1}$$

and a curvature singularity at

$$\Sigma = 0\,. \tag{3.3.2}$$

The first of Eqs. (3.3.1) gives the possible horizons. To study the different cases, let us shift the radial coordinate to recover the Boyer-Lindquist coordinates in which this kind of solutions are usually given. If we perform the following rescaling:

$$r \longrightarrow r - m$$

then Δ and Σ of (3.2.13) become

$$\begin{aligned} \Delta &= (r-m)^2 - {R_0}^2\,, \\ \Sigma &= r^2 + \alpha^2\cos^2\theta - |\Upsilon|^2\,, \end{aligned} \tag{3.3.4}$$

where we also have made the NUT charge n equal to zero in order to obtain black-hole-type solutions. In studying the singularities given by $\Delta = 0$ we have three cases to consider:

a) $R_0^2 < 0\,,\ \ ({r_0}^2 < \alpha^2)\,.$

Here $\Delta = 0$ has no real solutions, we have a naked singularity at $\Sigma = 0$ and no true black hole interpretation is possible. This is the case of supersymmetric ($r_0 = 0$) rotating ($\alpha \neq 0$) "black holes".

b) $R_0^2 > 0\,,\ \ ({r_0}^2 > \alpha^2)\,.$

In this case we have two horizons placed at

$$r_\pm = m \pm R_0\,. \tag{3.3.5}$$

To see if in this case we have a true black hole we must verify that the singularity is always hidden by the event horizon. The region where the singularity is placed is given by the following equation:

$$\Sigma = 0 \ \Leftrightarrow\ r_{\text{sing}}^2 = |\Upsilon|^2 - \alpha^2\cos^2\theta\,. \tag{3.3.6}$$

This is not the usual "ring singularity", but a more complicated 2-dimensional *surface* in general. Depending on the values of the charges, this can have the topology of the surface of a torus (maybe degenerate in certain cases to the surfaces of two concentric ellipsoids). Whatever its shape is, it is always confined in the region

$$r_{\rm sing}^2 \leq |\Upsilon|^2 \,, \tag{3.3.7}$$

while, on the other hand, the would-be event horizon

$$r_+ = m + R_0 > m \,, \tag{3.3.8}$$

and it will cover the singularity if $m > |\Upsilon|$. Using the value of $|\Upsilon|$ in terms of the other charges it is easy to prove

$$\left(|\mathfrak{M}| - |\Upsilon|\right)^2 > \alpha^2 \,. \tag{3.3.9}$$

We can now distinguish two cases:

i) $|\mathfrak{M}| - |\Upsilon| > |\alpha| \,.$

In this case (setting $n = 0$) the horizon covers the singularity and the object is a true black hole. Using the expressions in Appendix A.3 it is possible to prove that this happens when both

$$|\mathfrak{M}| > |\mathcal{Z}_{1,2}| \,, \tag{3.3.10}$$

which is the case allowed by supersymmetry (but not supersymmetric).

ii) $|\mathfrak{M}| - |\Upsilon| < |\alpha| \,.$

In this case there are naked singularities. This is the case forbidden by supersymmetry since one can show that in it both Bogomol'nyi bounds are simultaneously violated

$$|\mathfrak{M}| < |\mathcal{Z}_{1,2}| \,. \tag{3.3.11}$$

c) $R_0^2 = 0 \,, \quad ({r_0}^2 = \alpha^2) \,.$

This is the extremal case, and here we have a single would-be horizon placed at

$$r_\pm = m \,. \tag{3.3.12}$$

Again, we can distinguish two cases

i) $|\mathfrak{M}| - |\Upsilon| > |\alpha| \,, \quad |\mathfrak{M}| > |\mathcal{Z}_{1,2}| \,.$

In this case the singularity is inside the horizon and we have a true extreme rotating black hole. This is the case allowed by supersymmetry (not supersymmetric unless $\alpha = 0$).

i) $|\mathfrak{M}| - |\Upsilon| < |\alpha| \,, \quad |\mathfrak{M}| < |\mathcal{Z}_{1,2}| \,.$

The singularity is outside the "horizon" and this is not a black hole.

3.3.2 Entropy and temperature

We can now calculate the physical quantities associated to the true black-hole-type solutions. The entropy of the BH can be worked out by a straightforward computation of the area of the event horizon. This gives the following result:

$$A_{horizon} = 4\pi \left({r_+}^2 + \alpha^2 - |\Upsilon|^2 \right) , \tag{3.3.13}$$

so that for the Bekenstein-Hawking entropy of the black hole we get, in units such that $G = \hbar = c = 1$

$$S = \pi \left(2m^2 + 2mR_0 - I_2 \right) , \tag{3.3.14}$$

where I_2 is the quadratic duality invariant defined in Eq. (A.1.7). It is useful to have the expression of the entropy in terms of the mass and supersymmetry central charges

$$S = \pi \left\{ (m^2 - |\mathcal{Z}_1|^2) + (m^2 - |\mathcal{Z}_2|^2) + 2\sqrt{(m^2 - |\mathcal{Z}_1|^2)(m^2 - |\mathcal{Z}_2|^2) - J^2} \right\} .$$

For vanishing angular momentum $J = 0$, this expression can be further simplified to

$$S = \pi \left[\left(m^2 - |\mathcal{Z}_1|^2 \right)^{1/2} + \left(m^2 - |\mathcal{Z}_2|^2 \right)^{1/2} \right]^2 , \tag{3.3.15}$$

which means that, if we believe the extrapolation of this formula to all extreme cases, the entropy vanishes if and only if both Bogomol'nyi bounds are saturated and 1/2 of the supersymmetries are unbroken [55].

When any one of the two possible Bogomol'nyi bounds is saturated (for $J = 0$) the entropy is proportional to the difference between the modulus of the two central charges, which is proportional to the quartic duality invariant I_4 defined in Eq. (A.1.8), which is moduli-independent.

The temperature can be calculated imposing the regularity of the metric near the event horizon in imaginary time. Following the standard prescription [102], we must shift the time t and the rotation parameter α to the values $t \to i\tau$ and $\alpha \to i\tilde{\alpha}$ respectively. This yields the Euclidean section of the metric, and the absence of conical singularities at the event horizon in imaginary time requires the identification $(\tau, \varphi) \sim (\tau + \beta_H, \varphi - \tilde{\Omega}_H \beta_H)$, where $\tilde{\Omega}_H$ is the Euclidean angular velocity of the event horizon and β_H is the inverse Hawking temperature. For the (real) angular velocity of the horizon we have the following result:

$$\Omega_H = \frac{\alpha}{r_+^2 + \alpha^2 - |\Upsilon|^2}\,, \tag{3.3.16}$$

and so we obtain, in a perfectly straightforward way, the value for the Hawking temperature of the black hole:

$$T_H = \frac{R_0}{2S}\,. \tag{3.3.17}$$

For $J = 0$ the temperature always vanishes in the supersymmetric limit, except in the case in which $1/2$ of the supersymmetries are going to be left unbroken. In that case the limit is simply not well defined.

3.4 Conclusions

We have given a new set of solutions of pure $N = 4$, $d = 4$ supergravity which are beyond the BPS limit (in both directions) and which constitute the most general stationary point-like solution of this theory, since all the conserved charges are present in our solution, and all of them can take completely arbitrary values[7]. These solutions include black holes as well as Taub-NUT spacetimes, BHs being non-extremal in the general case. We have also shown that our family of solutions, and also the thermodynamic quantities associated to the BHs, are duality-invariant.

From a more technical point of view, we hope that the Ansatz providing the solution (basically characterized by the introduction of the 'non-extremality' function W and the non-flat three-dimensional metric (3.2.7) as "background" space) will prove helpful for the task of finding more non-extreme black holes s in other models, in particular, in those arising from more realistic compactifications of string theory, like compactifications on Calabi-Yau spaces, orbifolds, etc.

[7] The only possible addition would be primary scalar hair, but we are not interested in that kind of solutions.

Chapter 4

7-Branes and Higher Kaluza-Klein Branes

Introduction

In the last few years there has been a lot of interest in discovering classical solutions of effective superstring theories (supergravity theories) with such properties that one could argue that they represent the fields produced by solitonic objects present in the superstring spectrum. The interplay between the knowledge of the superstring spectrum and the knowledge of classical solutions has been very fruitful since each of them has contributed to the increase of the other. The two most important tools used in this field have been supersymmetry and duality. Unbroken supersymmetry ensures in many cases the absence of corrections of the classical solutions and the lack of quantum corrections to the mass of the corresponding objects in the string theory spectrum. Hence, more effort has been put in finding supersymmetric (i.e. admitting Killing spinors) solutions, associated to BPS string states. Duality transformations preserve in general supersymmetry, relating different states in dual theories. In general [18], but not always [26] duality relations between different higher-dimensional theories manifest themselves as non-compact global symmetries of the compactified supergravity theory that leave invariant its equations of motion so one can use them to transform known solutions into new solutions, preserving their supersymmetry properties.

Thus, it so happens that most classical solutions of superstring effective field theories belong to chains or families of solutions related by duality transformations. The best known chain of solutions is that of the Dp-branes, with $p = 0, \ldots, 8$ in 10 dimensions. They belong to two different theories: 10-dimensional type IIA for p even and 10-dimensional type IIB for p odd. All of them preserve 1/2 of the supersymmetries available, represent objects with p spatial worldvolume dimensions and $9 - p$ transverse dimensions (Dirichlet branes), carrying charge associated to the RR $(p+1)$-form $\hat{C}^{(p+1)}$ whose existence was discovered by Polchinski [22], and are related by generalized Buscher type II T duality transformations [26, 94].

Sometimes it is possible to find families of solutions that are, by themselves, representations of the duality group in the sense that they are invariant, as families, under the full duality group. This is the case, for instance, of the SWIP solutions of $N = 4, d = 4$ supergravity constructed in Ref. [30, 87]. In that case one can argue that all the solitonic objects of a given type (charged, stationary, black holes) and preserving a certain amount of supersymmetry are described by particular solutions, with particular values of the parameters of that general family. More interesting cases are $N = 8$, $N = 4$ with 22 vector multiplets and general $N = 2$ theories, all in $d = 4$, but fully general solutions in their duality-invariant form are not available. A great deal is, however, known of the solitonic spectrum of the 4-dimensional theories due to our knowledge of their duality groups (the so-called U duality group in the $N = 8$ case). All these theories can be obtained from 10-dimensional theories by compactification (toroidal or more general) and the compactification of the solitonic 10-dimensional objects gives rise to 4-dimensional solitonic objects of different kinds, depending on how the 10-dimensional objects are wrapped in the internal dimensions and one can study if these objects fill 4-dimensional duality multiplets. It has been realized that this is not the case if one considers only the standard 10-dimensional solitons: the Dp-branes, KK monopole, gravitational wave (W), fundamental string (F1) and solitonic 5-brane (S5) [104–107]. More 10-dimensional solitons are needed to give rise to all the 4-dimensional solitons predicted by duality and some of the properties they should exhibit, in particular the dependence of the mass in the radii of the internal dimensions and the coupling constant, have been deduced.

In this Chapter we present candidates for some of the missing 10 dimensional solitons and study them. The key to their construction is the realization that there are 4-dimensional duality symmetries which are neither present in 10 dimensions nor are a simple consequence of reparametrization invariance in the internal coordinates. These are, in general, S duality (i.e. electric-magnetic) transformations which only exist in certain dimensions and that enable us to use the mechanism *reduction-S dualization-oxidation* to generate new solutions in higher dimensions.

Let us consider a familiar example: 5 dimensional gravity compactified in a circle. The 4-dimensional theory has electric-magnetic duality and one expects an S duality symmetric spectrum. However, if we only considered the 5-dimensional plane wave solution we would only find electrically charged 4-dimensional solitons. To find the magnetically charged ones we S dualize these and, oxidizing the solutions to 5 dimensions we find the Kaluza-Klein (KK) monopole [108, 109]. In principle, this is a solution one would not expect in 5 dimensions since it has one dimension necessarily compactified in a circle.

The solutions we present can be generated in a similar fashion, exploiting S dualities present in dimensions lower than 10 and 11 and have similar properties: there are dimensions that cannot be decompactified. Somehow this is consistent with the fact that they are generated using dualities that only exist if some of the dimensions are compact.

One of the problems raised by the need to consider new 10- and 11-dimensional solutions was that fact that the 10- and 11-dimensional supersymmetry algebras did not contain central charges associated to those possible new objects. In our opinion the predictive power of the supersymmetry algebras has been overestimated and we will propose a way to include in them these new objects.

The rest of this Chapter is organized as follows: in Section 4.1 we present our family of T duality-related solutions whose construction via the *reduction-S dualization-oxidation* mechanism is explained in Section 4.2. In Section 4.3 we find other duality-related solutions in 10 and 11 dimensions. In Section 4.4 we calculate the dependence of the masses of these objects on compactification radii and coupling constants and in Section 4.5 we calculate the Killing spinors of all the solutions we have presented. Our conclusions are in Section 4.6. In Appendix B.2 we derive the $SL(2,\mathbb{R})/SO(2)$ sigma model from toroidal compactification and explain how $SL(2,\mathbb{R})$ is broken to $SL(2,\mathbb{Z})$ and in Appendix B.1 we briefly review holomorphic $(d-3)$-brane solutions of the $SL(2,\mathbb{R})/SO(2)$ sigma model to clarify certain points.

4.1 The Basic Family of Solutions

The basic family of solutions are solutions of the type II supergravity theories in $d=10$ and are a sort of deformation of the family of Dp-brane solutions for $0 \leq p \leq 7$. As such, they have $p+1$ worldvolume coordinates $t, \vec{y}_p = (y^1, \ldots, y^p)$ and $9-p$ transverse coordinates. We combine two of them into the complex coordinate ω and the remaining $7-p$ we denote by $\vec{x}_{7-p} = (x^1, \ldots, x^{7-p})$. The solutions can collectively be written in the string-frame metric in the form[1]

$$\left\{\begin{array}{rcl} d\hat{s}_s^2 & = & \left(\frac{H}{\mathcal{H}\bar{\mathcal{H}}}\right)^{-1/2} \left(dt^2 - d\vec{y}_p^{\,2}\right) - \left(H\mathcal{H}\bar{\mathcal{H}}\right)^{1/2} d\omega d\bar{\omega} - \left(\frac{H}{\mathcal{H}\bar{\mathcal{H}}}\right)^{1/2} d\vec{x}_{7-p}^{\,2}\,, \\ \hat{C}^{(p+1)}{}_{ty^1\cdots y^p} & = & (-1)^{\left[\frac{(p+1)}{2}\right]} \left(\frac{H}{\mathcal{H}\bar{\mathcal{H}}}\right)^{-1}\,, \\ \hat{C}^{(7-p)}{}_{x^1\ldots x^{7-p}} & = & -\frac{A}{\mathcal{H}\bar{\mathcal{H}}}\,, \\ e^{\hat{\phi}} & = & \left(\frac{H}{\mathcal{H}\bar{\mathcal{H}}}\right)^{\frac{3-p}{4}}\,, \end{array}\right. \tag{4.1.1}$$

where we function $\mathcal{H} = \mathcal{H}(\omega)$ is a complex, holomorphic, function of ω, i.e. $\partial_{\bar{\omega}}\mathcal{H} = 0$ with the behavior $\mathcal{H} \sim \frac{1}{2\pi i}\log\omega$ around $\omega = 0$, where we assume the object is placed. Its real and imaginary parts are

$$\mathcal{H} = A + iH\,. \tag{4.1.2}$$

These solutions have the same form as the standard Dp-brane solutions if we delete everywhere the combination $\mathcal{H}\bar{\mathcal{H}}$, but they are clearly different. In particular we can understand them as having $7-p$ extra isometric directions that should be considered compact[2]. Our goal will be to understand how

[1] For convenience, we give the form of the potential to which the p-brane naturally couples $\hat{C}^{(p+1)}$ and the dual one $\hat{C}^{(7-p)}$. In the $p=3$ case, these are the two non-vanishing sets of components of the 4-form potential with self-dual field strength. (Our conventions are those of Ref. [94] whose type II T duality rules, generalizing those of Ref. [26], we use.) Since the solutions we will be dealing with are not asymptotically flat, we do not write explicitly the asymptotic values of the scalars (for example, $\hat{\phi}_0$ for the dilaton).

[2] It seems difficult (it is perhaps impossible) to extend the dependence of the function

they arise, their M theoretic origin and their supersymmetry properties and explore the implications of it all. We will also find other solutions related by dualities with them or belonging to the same class. Since we will find that all these solutions preserve a half of the symmetries, we are going to argue that they describe the long range fields of elementary, non-perturbative objects of string theory and we will calculate their masses.

4.2 Construction of the Solutions

The solutions (4.1.1) can be obtained by successive T duality transformations in worldvolume directions of the $p=7$ solution. The $p=7$ solution is nothing but the type IIB solitonic 7-brane (S7) that was obtained by S duality from the D7-brane and called $Q7$-brane in Ref. [94]. The worldvolume directions are transformed into transverse isometric directions that should be considered compact[3]. Thus, we obtain a chain of T dual solutions of both type II theories.

There is an alternative way of constructing these solutions that also helps to understand them. Let us consider a piece of the 10-dimensional type II supergravity theories in which we only keep the metric, the dilaton and the field strength $\hat{G}^{(8-p)}$ of the RR $(7-p)$-form $\hat{C}^{(7-p)}$. The action is

$$\hat{S} = \int d^{10}\hat{x}\sqrt{|\hat{g}|}\,\left\{e^{-2\hat{\phi}}\left[\hat{R} - 4(\partial\hat{\phi})^2\right] + \tfrac{(-1)^{7-p}}{2\cdot(8-p)!}\left(\hat{G}^{(8-p)}\right)^2\right\}\,. \tag{4.2.1}$$

Now, let us compactify it over a $(7-p)$-torus using a simplified Kaluza-Klein Ansatz that only takes into account the volume modulus of the internal torus, the dilaton (both rewritten in terms of two convenient scalars φ and η), the internal volume mode of the RR $(7-p)$-form, a and the $(3+p)$-dimensional

$\mathcal{H}$ to those coordinates. Furthermore, the construction procedure *reduction-S dualization-oxidation* and the dependence of the masses on the radii of those dimensions that we are going to calculate later on suggest that those coordinates should be compactified on a torus.

[3]This is somewhat analogous to what happens in the well-known duality between the solitonic fivebrane S5 and the KK monopole in which a transverse direction of the S5 is T dualized into an isometric, compact, direction of the KK monopole.

Einstein metric $g_{\mu\nu}$:

$$\left\{\begin{array}{rcl} d\hat{s}^2 & = & e^{\frac{1}{2}\varphi+\frac{1}{2}\sqrt{\frac{7-p}{p+1}}\eta} g_{\mu\nu}dx^\mu dx^\nu - e^{-\frac{1}{2}\varphi+\frac{1}{2}\sqrt{\frac{p+1}{7-p}}\eta} d\vec{x}_{7-p}^{\,2} \,, \\ \hat{C}^{(7-p)}{}_{x^1\ldots x^{7-p}} & = & a\,, \\ e^{\hat{\phi}} & = & e^{\frac{p-3}{4}\varphi+\frac{7-p}{4}\sqrt{\frac{p+1}{7-p}}\eta}\,. \end{array}\right. \tag{4.2.2}$$

After some straightforward calculations one obtains, in all cases, the reduced action

$$S = \int d^{p+3}x\sqrt{|g|}\,\left\{R + \tfrac{1}{2}\frac{\partial\tau\partial\bar{\tau}}{(\Im\mathrm{m}\tau)^2} + \tfrac{1}{2}(\partial\eta)^2\right\}\,, \tag{4.2.3}$$

where

$$\tau = a + ie^{-\varphi}\,, \tag{4.2.4}$$

i.e. gravity coupled to an $SL(2,\mathbb{R})/SO(2)$ sigma model parametrized in the standard form by the complex scalar (sometimes known as *axidilaton* although here this name could be misleading since in some cases ($p=3$) the string dilaton simply does not contribute to it) τ and another scalar, η, decoupled from τ. In the $p=7$ case ($d=10$) this is the well-known piece of the type IIB supergravity action. In lower dimensions, it is integrated in much bigger sigma models associated to much bigger U-duality groups[4] but it is a most interesting part of it.

There is a very general solution of this model

$$\left\{\begin{array}{rcl} ds^2 & = & dt^2 - d\vec{y}_p^{\,2} - Hd\omega d\bar{\omega}\,, \\ \tau & = & \mathcal{H}\,, \\ \eta & = & 0\,, \end{array}\right. \tag{4.2.5}$$

with $\partial_{\bar{\omega}}\mathcal{H}=0$. In $d=10$ ($p=7$) this is just the general D7-brane solution. Choosing $\mathcal{H}\sim\log\omega$ we get the single D7-brane solution. In lower dimensions, these solutions are just compactifications of the standard general Dp-brane solution in which we have assumed that the harmonic function only depends on two transverse directions (ω) and we have dualized the RR

[4] In $d=6$ dimensions, this model was studied in Ref. [110] and in $d=8$ it was studied in Ref. [111].

$(p+1)$-potential, giving rise to the real part of $\mathcal{H}$. Thus, this is a well-known solution.

We can now perform an $SL(2,\mathbb{R})$ duality rotation of this solution[5] $\tau \to -1/\tau$, since this is a symmetry of the dimensionally reduced action[6] that leaves the Einstein metric invariant. This is not a symmetry of the 10-dimensional action and one really needs extra compact dimensions to establish it. The resulting solutions[7]

$$\left\{ \begin{array}{rcl} ds^2 & = & dt^2 - d\vec{y}_p^{\,2} - H d\omega d\bar{\omega}\,, \\ \\ \tau & = & -1/\mathcal{H}\,, \\ \\ \eta & = & 0\,, \end{array} \right. \tag{4.2.6}$$

are nothing but the solutions Eqs. (4.1.1) reduced according to the above KK Ansatz.

What we are doing here is similar to what one does in standard KK theory: reducing to 4 dimensions the 5-dimensional pp wave one obtains the electric, extreme KK black hole. Since the $d=4$ theory has electric-magnetic duality as a symmetry, one can find the magnetic, extreme KK black hole and then uplift it to $d=5$ to find the KK monopole [108, 109] that has a special isometric direction that cannot be decompactified. The symmetry between the pp wave and the KK monopole cannot be established without assuming one compact direction. It is only natural, by analogy, to consider here that the dimensions that we have compactified cannot be decompactified after the duality transformations. We will support this assumption not by geometrical arguments but calculating the masses of these objects and finding its dependence on the radii of those dimensions.

[5]Continuous duality symmetries are usually broken to their discrete subgroups, for instance $SL(2,\mathbb{R})$ is usually broken to $SL(2,\mathbb{Z})$. This can be clearly seen in the case in which the $SL(2,\mathbb{R})/SO(2)$ sigma model originates in a toroidal compactification and is explained in Appendix B.2. In other cases one has to study the quantization of charges to arrive to the same conclusion. We will loosely use the continuous of the discrete form of the duality group in the understanding that in some contexts only the discrete one is really a symmetry of the theory.

[6]In general, it is only a symmetry of the equations of motion of the complete, untruncated, type II theory.

[7]In Appendix B.1 we discuss these general solutions and in which sense they are new. We stress that we are considering only the choice holomorphic function $\mathcal{H} \sim \frac{1}{2\pi i}\log\omega$.

4.3 Duality-related Solutions and M-theoretic Origin

Since we are dealing with many new solutions, we first propose to denote them by "Dp_i" where "$p+1$" is the worldvolume and "i" is the number of isometric directions. According to this notation, the solutions described by Eq. (4.1.1) are in the $p=7$ case $D7_0$ (the type IIB S dual of the D7-brane, called Q7 in Ref. [94]), $D6_1$ for $p=6$, and $D5_2$, $D4_3$, $D3_4$, $D2_5$, $D1_6$, $D0_7$ for the remaining cases.

For all the type IIB solutions in the class (4.1.1) we can find an S dual using the 10-dimensional type IIB S duality symmetry. While in the $p=7$ case the S dual solution is just the well-known D7-brane, and in the $p=3$ case the solution is self-dual, in the $p=5,1$ cases we find genuinely new solutions. For $D5_2$ we get a solution which is a deformation of the solitonic fivebrane, and we call $S5_2$

$$
\mathbf{S5_2}\quad \left\{
\begin{array}{rcl}
d\hat{s}_s^2 & = & dt^2 - d\vec{y}_5{}^2 - H d\omega d\bar{\omega} - \dfrac{H}{\mathcal{H}\bar{\mathcal{H}}} d\vec{x}_2{}^2 \,, \\[2ex]
\hat{\mathcal{B}}_{x^1x^2} & = & -\dfrac{A}{\mathcal{H}\bar{\mathcal{H}}}\,, \\[2ex]
\hat{\tilde{\mathcal{B}}}_{\sqcup\dagger\infty\cdots\dagger\triangledown} & = & \left(\dfrac{\mathcal{H}}{\mathcal{H}\bar{\mathcal{H}}}\right)^{-\infty}\,, \\[2ex]
e^{\hat{\phi}} & = & \left(\dfrac{H}{\mathcal{H}\bar{\mathcal{H}}}\right)^{\frac{1}{2}}\,,
\end{array}
\right. \tag{4.3.1}
$$

and for $D1_6$, we get a sort of deformation of the fundamental string solution that we call $F1_6$

$$\mathbf{F1_6}\begin{cases} d\hat{s}_s^2 &= \left(\frac{H}{\mathcal{H}\bar{\mathcal{H}}}\right)^{-1}(dt^2 - dy^2 - Hd\omega d\bar{\omega}) - d\vec{x}_6^2\,, \\ \hat{\mathcal{B}}_{ty} &= -\left(\frac{H}{\mathcal{H}\bar{\mathcal{H}}}\right)^{-1}\,, \\ \hat{\tilde{\mathcal{B}}}_{\S\infty\cdots\S/} &= \frac{\mathcal{A}}{\mathcal{H}\bar{\mathcal{H}}}\,, \\ e^{\hat{\phi}} &= \left(\frac{H}{\mathcal{H}\bar{\mathcal{H}}}\right)^{-\frac{1}{2}}\,. \end{cases} \tag{4.3.2}$$

These two solutions only have non-trivial common sector NSNS fields and therefore they are also solutions of the heterotic string effective field theory. We can also understand these solutions by appealing to the existence in both cases of a reduced action of the form Eq. (4.2.3) that arises from the 10-dimensional actions

$$\hat{S} = \int d^{10}\hat{x}\sqrt{|\hat{g}|}\, e^{-2\hat{\phi}}\left[\hat{R} - 4(\partial\hat{\phi})^2 + \tfrac{1}{2\cdot 3!}\hat{H}^2\right]\,, \tag{4.3.3}$$

and

$$\hat{S} = \int d^{10}\hat{x}\sqrt{|\hat{g}|}\,\left\{e^{-2\hat{\phi}}\left[\hat{R} - 4(\partial\hat{\phi})^2\right] + \tfrac{1}{2\cdot 7!}e^{2\hat{\phi}}\hat{\tilde{H}}^2\right\}\,, \tag{4.3.4}$$

where $\hat{H}$ is the NSNS 3-form field strength and $\hat{\tilde{H}} = e^{2\hat{\phi}}\,{}^{\star}\hat{H}$ is the dual 7-form field strength. Reducing the first action to 8 dimensions with the Ansatz

$$\begin{cases} d\hat{s}^2 &= e^{\frac{1}{\sqrt{3}}\eta}g_{\mu\nu}dx^\mu dx^\nu - e^{-\varphi}d\vec{x}_2^{\,2}\,, \\ \hat{B}_{x^1x^2} &= a\,, \\ e^{\hat{\phi}} &= e^{\frac{\sqrt{3}}{2}\eta - \frac{1}{2}\varphi}\,, \end{cases} \tag{4.3.5}$$

and the second action down to 4 dimensions with the Ansatz

$$\begin{cases} d\hat{s}^2 &= e^{\varphi}g_{\mu\nu}dx^\mu dx^\nu - e^{\frac{1}{\sqrt{3}}\eta}d\vec{x}_6^{\,2}\,, \\ \hat{\tilde{B}}_{x^1\cdots x^6} &= -a\,, \\ e^{\hat{\phi}} &= e^{\frac{1}{2}\varphi + \frac{\sqrt{3}}{2}\eta}\,, \end{cases} \tag{4.3.6}$$

we get in both cases Eq. (4.2.3) in 8 and 4 dimensions.

As for the M-theoretic origin of the type IIA solutions, they can be derived from the following 11-dimensional solutions through compactification of the 11th dimension (z): a *pp* wave with 7 extra isometries

$$\mathbf{WM_7}\quad d\hat{\hat{s}}^2 = -2dtdz - \frac{H}{\mathcal{H}\bar{\mathcal{H}}}dz^2 - \mathcal{H}\bar{\mathcal{H}}d\omega d\bar{\omega} - d\vec{x}_7^{\,2}\,, \tag{4.3.7}$$

a deformation of the M2-brane

$$\mathbf{M2_6}\quad \begin{cases} d\hat{\hat{s}}^2 = \left(\frac{H}{\mathcal{H}\bar{\mathcal{H}}}\right)^{-2/3}(dt^2 - d\vec{y}_2^{\,2}) - H^{1/3}\left(\mathcal{H}\bar{\mathcal{H}}\right)^{2/3}d\omega d\bar{\omega} - \left(\frac{H}{\mathcal{H}\bar{\mathcal{H}}}\right)^{1/3}d\vec{x}_6^{\,2}\,, \\ \hat{\hat{C}}_{ty^1y^2} = -\left(\frac{H}{\mathcal{H}\bar{\mathcal{H}}}\right)^{-1}\,, \\ \hat{\hat{\tilde{C}}}_{x^1\cdots x^6} = \frac{A}{\mathcal{H}\bar{\mathcal{H}}}\,, \end{cases} \tag{4.3.8}$$

a deformation of the M5-brane

$$\mathbf{M5_3}\quad \begin{cases} d\hat{\hat{s}}^2 = \left(\frac{H}{\mathcal{H}\bar{\mathcal{H}}}\right)^{-1/3}(dt^2 - d\vec{y}_5^{\,2}) - H^{2/3}\left(\mathcal{H}\bar{\mathcal{H}}\right)^{1/3}d\omega d\bar{\omega} - \left(\frac{H}{\mathcal{H}\bar{\mathcal{H}}}\right)^{2/3}d\vec{x}_3^{\,2}\,, \\ \hat{\hat{\tilde{C}}}_{ty^1\cdots y^5} = -\left(\frac{H}{\mathcal{H}\bar{\mathcal{H}}}\right)^{-1}\,, \\ \hat{\hat{C}}_{x^1x^2x^3} = -\frac{A}{\mathcal{H}\bar{\mathcal{H}}}\,, \end{cases} \tag{4.3.9}$$

and the KK monopole (with no dependence on the 11th dimension)

$$\mathbf{KK7M}\quad d\hat{\hat{s}}^2 = dt^2 - d\vec{y}_6^{\,2} - H(d\omega d\bar{\omega} + dz^2) - H^{-1}\left(dy^7 - Adz\right)^2\,. \tag{4.3.10}$$

In these four cases we can also trace the origin of the solution to the existence of a sector like that in Eq. (4.2.3) in the reduced action of 11-dimensional supergravity. In the purely gravitational cases, the action (4.2.3) can be derived from the dimensional reduction of the Einstein term alone as shown in detail in Appendix B.2. In the second and third cases, one needs the 6-form or the 3-form dual potential respectively.

In some cases the dimensional reduction of these 11-dimensional solutions in isometric directions different from z produce new 10-dimensional solutions. In particular, we get two purely gravitational solutions

$$\mathbf{W_6} \quad d\hat{s}^2 = -2dtdz - \frac{H}{\mathcal{H}\bar{\mathcal{H}}}dz^2 - \mathcal{H}\bar{\mathcal{H}}d\omega d\bar{\omega} - d\vec{x}_6{}^2\,, \tag{4.3.11}$$

and the Kaluza-Klein monopole with no dependence in z

$$\mathbf{KK6} \quad d\hat{s}^2 = dt^2 - d\vec{y}_5{}^2 - H(d\omega d\bar{\omega} + dz^2) - H^{-1}\left(dy^7 - Adz\right)^2\,. \tag{4.3.12}$$

In all cases (see Figure 4.1) we see that whenever we reduce the same 11-dimensional solution over 2 directions to 9 dimensions and we do it in different order, we get a pair of 9-dimensional solutions that form an $SL(2,\mathbb{R})$ ($SL(2,\mathbb{Z})$) doublet and also originate from a type IIB $SL(2,\mathbb{R})$ ($SL(2,\mathbb{Z})$) doublet as it must [26].

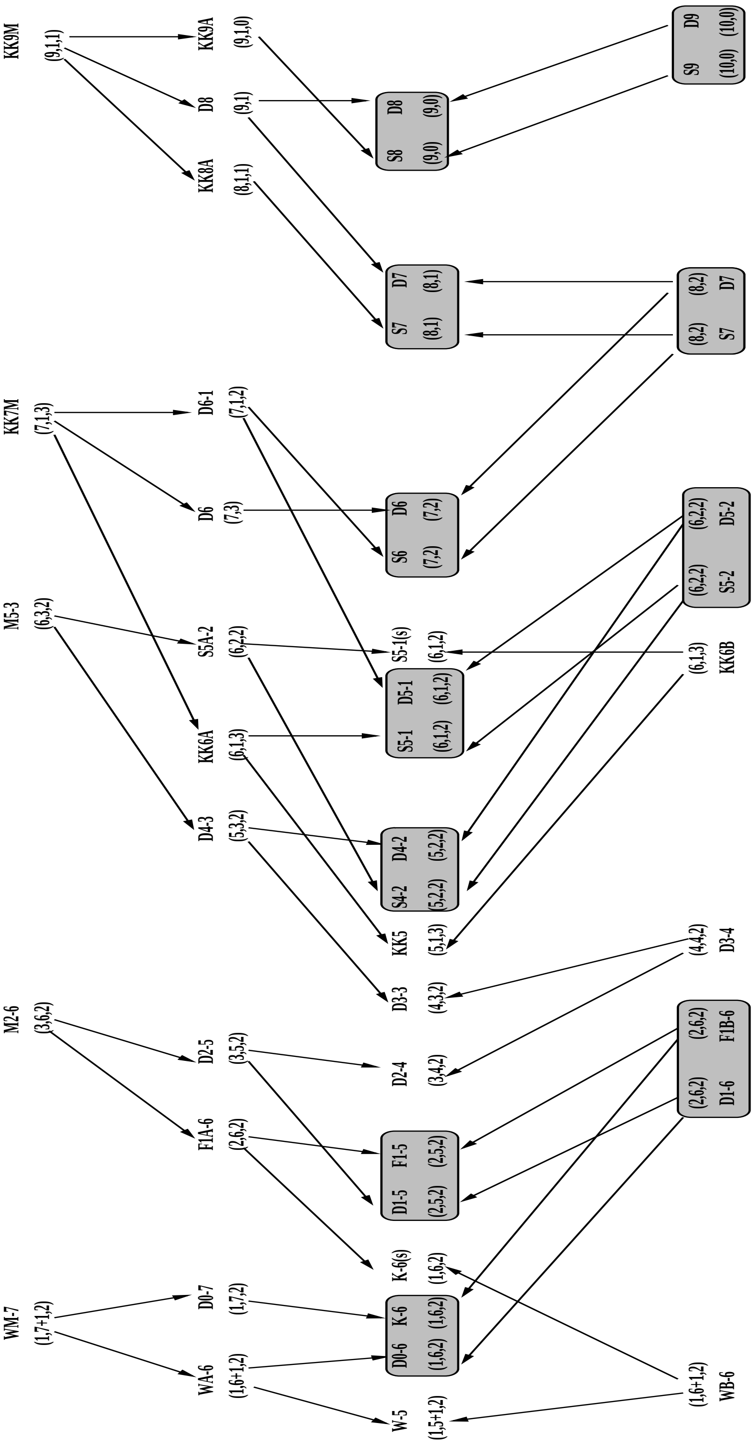

Figure 4.1: Duality relations between KK branes. The numbers in parenthesis represent the worldvolume dimension, isometric and transverse directions. The arrows indicate dimensional reduction in the corresponding kind of direction. In the upper row we represent M-theory KK branes, below 10-dimensional type IIA branes and below them 9-dimensional branes. Type IIB KK branes are in the bottom row. Pairs of branes in boxes are S duality doublets. They are always related to reductions from 11 to 9 dimensions of the same object in two different orders. Sometimes there is an third object with the same numbers as those in a doublet, but transforming as a singlet and we denote it with (s).

4.4 Masses

The mass of the Dp_i solutions can be calculated using S and T duality rules from the standard $D7$-brane and can be written in a general formula:

$$M_{Dp_i} = \frac{R_3 \ldots R_{p+2}(R_{p+3} \ldots R_9)^2}{g^3 \ell_s^{p+2i+1}} . \tag{4.4.1}$$

The masses of the NSNS solutions found by S duality from the $D5_2$ and the $D1_6$ are

$$\begin{aligned} M_{S5_2} &= \frac{R_3 \ldots R_7 (R_8 R_9)^2}{g^2 \ell_s^{10}} , \\ M_{F1_6} &= \frac{R_3 (R_4 \ldots R_9)^2}{g^4 \ell_s^{14}} . \end{aligned} \tag{4.4.2}$$

The masses of the 11-dimensional objects from which the type IIA objects can be derived can be calculated using the relations between the 11-dimensional Planck length $\ell_{\text{Planck}}^{(11)}$ and the radius of the 11th dimension[8] R_{10} and the type IIA string coupling constant g_A and the string length ℓ_s ${\ell_{\text{Planck}}}^{(11)} = 2\pi \ell_s g_A^{1/3}$ and $R_{10} = \ell_s g_A$:

$$\begin{aligned} M_{M2_6} &= \frac{R_3 R_4 (R_5 \ldots R_{10})^2}{(\ell_{\text{Planck}}^{(11)})^{15}} , \\ M_{M5_3} &= \frac{R_3 \ldots R_6 (R_7 R_8 R_9)^2 R_{10}}{(\ell_{\text{Planck}}^{(11)})^{12}} . \end{aligned} \tag{4.4.3}$$

where $\ell_{\text{Planck}}^{(11)}$ is the reduced 11-dimensional Planck length $\ell_{\text{Planck}}^{(11)} = \ell_{\text{Planck}}^{(11)}/2\pi$.

These expressions should be compared with the well-known expression of the mass of the 11-dimensional KK monopole $KK7M$ when the special isometric direction is x^{10}

$$M_{KK7M} = \frac{R_4 \ldots R_9 R_{10}^2}{(\ell_{\text{Planck}}^{(11)})^9} , \tag{4.4.4}$$

or the 10-dimensional KK monopole $KK6$ (A or B) when the special isometric direction is x^9

$$M_{KK6} = \frac{R_4 \ldots R_8 R_9^2}{g^2 \ell_s^8} . \tag{4.4.5}$$

[8] R_{11} is the conventional name in the literature. Here we use R_m for the radius of the coordinate x^m.

In both cases the mass is not simply proportional to the volume of the brane which is assumed wrapped on a torus but depends quadratically on the radius of the special isometric direction. The same happens to the masses of all the Dp_i branes: they depend quadratically on the radii of the directions that we have argued are isometric, which supports our assumption.

Apart from the dependence on the radii we see that in general these objects are highly non-perturbative since their masses are proportional to g^{-3} and g^{-4} except for $S5_2$, whose mass goes like g^{-2}, as for any standard solitonic object.

The momentum of the WM_7 solution is

$$M_{WM_7} = \frac{(R_3 \ldots R_9)^2 R_{10}^3}{(\ell_{\text{Planck}}^{(11)})^{18}} . \tag{4.4.6}$$

4.5 Killing Spinors and Unbroken Supersymmetries

It is important to find the amount of supersymmetry preserved by our solutions since, if they preserve less than one half of the total supersymmetry available, one could argue that they correspond to composite objects. Since all these solutions are related by S and T duality transformations to the $D7$-brane, which preserves exactly $1/2$ of the supersymmetries, it is to be expected that they will do so as well. Nevertheless, a direct calculation of the Killing spinors should always be performed since it will confirm our expectations and it will also provide us with projectors that will help us to associate the solutions to central charges in the supersymmetry algebra and therefore to identify them with supersymmetric states in the string spectrum.

We first calculate the Killing spinors of the Dp_i family of solutions with the obvious choice for the Vielbein basis[9]

$$e_{\underline{i}}{}^{i} = \left(\frac{\mathcal{H}\bar{\mathcal{H}}}{H}\right)^{1/4} , \quad e_{\underline{m}}{}^{m} = \left(\frac{\mathcal{H}\bar{\mathcal{H}}}{H}\right)^{-1/4} , \quad e_{\underline{8}}{}^{8} = e_{\underline{9}}{}^{9} = \left(\frac{\mathcal{H}\bar{\mathcal{H}}}{H}\right)^{1/4} H^{1/2} . \tag{4.5.1}$$

For the Type IIA solutions we use the supersymmetry transformation rules for the gravitino and dilatino which, in the purely bosonic background

[9]Underlined indices are world indices and non-underlined indices are tangent space indices. They take values in the ranges $i = 0, 1, \ldots, p$, $m = p+1, \ldots, 7$.

we are considering, take the form[10]

$$\left\{\begin{array}{rcl} \delta_{\hat\epsilon}\hat\psi_{\hat\mu} &=& \left[\partial_{\hat\mu} - \frac{1}{4}\,\not{\hat\omega}_{\hat\mu} + \frac{i}{8}\frac{1}{(8-p)!}e^{\hat\phi}\,\not{\hat G}^{(8-p)}\hat\Gamma_{\hat\mu}(-\hat\Gamma_{11})^{\frac{8-p}{2}}\right]\hat\epsilon\,, \\ \delta_{\hat\epsilon}\hat\tau &=& \left[\not\partial\hat\phi + \frac{i}{4}\frac{p-3}{(8-p)!}e^{\hat\phi}\,\not{\hat G}^{(8-p)}(-\hat\Gamma_{11})^{\frac{8-p}{2}}\right]\hat\epsilon\,, \end{array}\right. \tag{4.5.2}$$

Imposing the vanishing of dilatino transformation rule we obtain the following constraint in the Killing spinor:

$$\left[1 - i\hat\Gamma^{p+1}\cdots\hat\Gamma^{8}\hat\Gamma^{9}(-\hat\Gamma_{11})^{\frac{8-p}{2}}\right]\hat\epsilon = 0\,, \tag{4.5.3}$$

or, equivalently

$$\left[1 - (-1)^{[p/2]}i\hat\Gamma^{0}\cdots\hat\Gamma^{p}(-\hat\Gamma_{11})^{\frac{10-p}{2}}\right]\hat\epsilon = 0\,. \tag{4.5.4}$$

This constraint automatically sets to zero the worldvolume (t, y^i) and transverse, isometric (x^m) components of the supersymmetry variation of the gravitino. The remaining transverse components (x^8, x^9) give in *all* cases, the following coupled partial differential equations

$$\left\{\begin{array}{rcl} \delta_{\hat\epsilon}\hat\psi_{\underline{8}} &=& \left[\partial_{\underline{8}} - \frac{1}{4}\hat\Gamma^{8}\hat\Gamma^{9}\,\partial_{\underline{9}}\log(\mathcal{H}\bar{\mathcal{H}}) + \frac{1}{8}\,\partial_{\underline{8}}\log\left(\frac{H}{\mathcal{H}\bar{\mathcal{H}}}\right)\right]\hat\epsilon = 0\,, \\ \delta_{\hat\epsilon}\hat\psi_{\underline{9}} &=& \left[\partial_{\underline{9}} - \frac{1}{4}\hat\Gamma^{9}\hat\Gamma^{8}\,\partial_{\underline{8}}\log(\mathcal{H}\bar{\mathcal{H}}) + \frac{1}{8}\,\partial_{\underline{9}}\log\left(\frac{H}{\mathcal{H}\bar{\mathcal{H}}}\right)\right]\hat\epsilon = 0\,. \end{array}\right. \tag{4.5.5}$$

Now, using the Cauchy-Riemann equations for the holomorphic function $\mathcal{H}$, i.e.:

$$\partial_{\underline{8}}A = +\partial_{\underline{9}}H\,, \qquad \partial_{\underline{9}}A = -\partial_{\underline{8}}H\,, \tag{4.5.6}$$

we can express $\partial_{\underline{8}}\log(\mathcal{H}\bar{\mathcal{H}})$ and $\partial_{\underline{9}}\log(\mathcal{H}\bar{\mathcal{H}})$ in the following way:

$$\partial_{\underline{9}}\log(\mathcal{H}\bar{\mathcal{H}}) = -2\partial_{\underline{8}}(\arg\mathcal{H})\,, \qquad \partial_{\underline{8}}\log(\mathcal{H}\bar{\mathcal{H}}) = +2\partial_{\underline{9}}(\arg\mathcal{H})\,, \tag{4.5.7}$$

and the Killing spinor equations are easily seen to be solved by

$$\left\{\begin{array}{l} \left[1 - (-1)^{[p/2]}i\hat\Gamma^{0}\cdots\hat\Gamma^{p}(-\hat\Gamma_{11})^{\frac{10-p}{2}}\right]\hat\epsilon_0 = 0\,, \\ \hat\epsilon = e^{-\frac{1}{2}\arg(\mathcal{H})\,\hat\Gamma^{8}\hat\Gamma^{9}}\left(\frac{\mathcal{H}\bar{\mathcal{H}}}{H}\right)^{1/8}\hat\epsilon_0\,. \end{array}\right. \tag{4.5.8}$$

[10]Our type IIA spinors are full 32-component Majorana spinors.

$\hat{\epsilon}_0$ being any constant spinor satisfying the above constraint.

In the type IIB cases we use the relevant supersymmetry transformation laws[11]

$$\left\{\begin{array}{lcl} \delta_{\hat{\varepsilon}}\hat{\zeta}_{\hat{\mu}} & = & \left[\partial_{\hat{\mu}} - \frac{1}{4}\not{\hat{\omega}}_{\hat{\mu}} + \frac{1}{8}\frac{1}{(8-p)!}e^{\hat{\varphi}}\,\not{\hat{G}}^{(8-p)}\hat{\Gamma}_{\hat{\mu}}\mathcal{P}_{\frac{9-p}{2}}\right]\hat{\varepsilon}\,, \\ \delta_{\hat{\varepsilon}}\hat{\chi} & = & \left[\not{\partial}\hat{\varphi} + \frac{1}{4}\frac{3-p}{(8-p)!}e^{\hat{\varphi}}\,\not{\hat{G}}^{(8-p)}\mathcal{P}_{\frac{9-p}{2}}\right]\hat{\varepsilon}\,, \end{array}\right. \tag{4.5.9}$$

where $\mathcal{P}_n$ is

$$\mathcal{P}_n \left\{\begin{array}{ll} \sigma^1\,, & n \text{ even}\,, \\ i\sigma^2\,, & n \text{ odd}\,. \end{array}\right.$$

Proceeding as in the type IIA case, we find the Killing spinors

$$\left\{\begin{array}{l} \left[1 + (-1)^{[p/2]}\hat{\Gamma}^0\cdots\hat{\Gamma}^p\mathcal{P}_{\frac{p+1}{2}}\right]\hat{\varepsilon}_0 = 0\,, \\ \hat{\varepsilon} = e^{-\frac{1}{2}\arg(\mathcal{H})\,\hat{\Gamma}^8\hat{\Gamma}^9}\left(\frac{\mathcal{H}\bar{\mathcal{H}}}{H}\right)^{1/8}\hat{\varepsilon}_0\,, \end{array}\right. \tag{4.5.10}$$

where, now, $\hat{\varepsilon}_0$ is any pair of constant positive-chirality Majorana-Weyl spinors satisfying the above constraint.

The Killing spinors of the $S5_2$ and the $F1_6$ can be found in a similar fashion and are, respectively

$$\left\{\begin{array}{l} \left[1 - \hat{\Gamma}^6\hat{\Gamma}^7\hat{\Gamma}^8\hat{\Gamma}^9\sigma^3\right]\hat{\varepsilon}_0 = 0\,, \\ \hat{\varepsilon} = e^{-\frac{1}{2}\arg(\mathcal{H})\,\hat{\Gamma}^8\hat{\Gamma}^9}\hat{\varepsilon}_0\,, \end{array}\right.$$

and

$$\left\{\begin{array}{l} \left[1 + \hat{\Gamma}^0\hat{\Gamma}^1\sigma^3\right]\hat{\varepsilon}_0 = 0\,, \\ \hat{\varepsilon} = e^{-\frac{1}{2}\arg(\mathcal{H})\,\hat{\Gamma}^8\hat{\Gamma}^9}\left(\frac{\mathcal{H}\bar{\mathcal{H}}}{H}\right)^{1/4}\hat{\varepsilon}_0\,, \end{array}\right.$$

[11]Our type IIB spinors are pairs (whose indices 1,2 are not explicitly shown of 32-component, positive chirality, Majorana-Weyl spinors. Pauli matrices act on the indices not shown.

Before discussing these results it is worth finding the Killing spinors of the 11-dimensional solutions. The only relevant supersymmetry transformation rule is that of the gravitino, which with our conventions is:

$$\delta_{\hat{\hat{\epsilon}}}\hat{\hat{\psi}}_{\hat{\hat{\mu}}} = \left[2\partial_{\hat{\hat{\mu}}} - \tfrac{1}{2}\,\hat{\hat{\not{\omega}}}_{\hat{\hat{\mu}}} + \tfrac{i}{144}\left(\hat{\hat{\Gamma}}^{\hat{\hat{\alpha}}\hat{\hat{\beta}}\hat{\hat{\gamma}}\hat{\hat{\delta}}}{}_{\hat{\hat{\mu}}} - 8\hat{\hat{\Gamma}}^{\hat{\hat{\beta}}\hat{\hat{\gamma}}\hat{\hat{\delta}}}\hat{\hat{\eta}}_{\hat{\hat{\mu}}}{}^{\hat{\hat{\alpha}}}\right)\hat{\hat{G}}_{\hat{\hat{\alpha}}\hat{\hat{\beta}}\hat{\hat{\gamma}}\hat{\hat{\delta}}}\right]\hat{\hat{\epsilon}}\,. \tag{4.5.11}$$

In the obvious Vielbein basis we find, for the WM_7 solution

$$\begin{cases} \left[1 - \hat{\hat{\Gamma}}^{0}\hat{\hat{\Gamma}}^{10}\right]\hat{\hat{\epsilon}}_0 = 0\,, \\ \hat{\hat{\epsilon}} = e^{-\frac{1}{2}\arg(\mathcal{H})\hat{\hat{\Gamma}}^{8}\hat{\hat{\Gamma}}^{9}}\left(\dfrac{\mathcal{H}\bar{\mathcal{H}}}{H}\right)^{1/4}\hat{\hat{\epsilon}}_0\,. \end{cases} \tag{4.5.12}$$

for the $M2_6$ solution

$$\begin{cases} \left[1 + i\hat{\hat{\Gamma}}^{0}\hat{\hat{\Gamma}}^{1}\hat{\hat{\Gamma}}^{2}\right]\hat{\hat{\epsilon}}_0 = 0\,, \\ \hat{\hat{\epsilon}} = e^{-\frac{1}{2}\arg(\mathcal{H})\hat{\hat{\Gamma}}^{8}\hat{\hat{\Gamma}}^{9}}\left(\dfrac{\mathcal{H}\bar{\mathcal{H}}}{H}\right)^{1/6}\hat{\hat{\epsilon}}_0\,, \end{cases} \tag{4.5.13}$$

for the $M5_3$ solution

$$\begin{cases} \left[1 - \hat{\hat{\Gamma}}^{0}\cdots\hat{\hat{\Gamma}}^{4}\hat{\hat{\Gamma}}^{10}\right]\hat{\hat{\epsilon}}_0 = 0\,, \\ \hat{\hat{\epsilon}} = e^{-\frac{1}{2}\arg(\mathcal{H})\hat{\hat{\Gamma}}^{8}\hat{\hat{\Gamma}}^{9}}\left(\dfrac{\mathcal{H}\bar{\mathcal{H}}}{H}\right)^{1/12}\hat{\hat{\epsilon}}_0\,, \end{cases} \tag{4.5.14}$$

and for the $KK7M$ solution, as it is well known, the Killing spinor is any constant spinor $\hat{\hat{\epsilon}}_0$ satisfying the constraint

$$\left[1 + i\hat{\hat{\Gamma}}^{0}\cdots\hat{\hat{\Gamma}}^{6}\right]\hat{\hat{\epsilon}}_0 = 0\,. \tag{4.5.15}$$

In all cases one can see that these solutions preserve one half of the supersymmetries.

4.6 Conclusions

In this Chapter we have presented new 10-dimensional solutions of the type II theories that can be thought of as having a certain number of isometric, compact, dimensions, that cannot be decompactified (one could say that these are really solutions of lower-dimensional theories) and which we have referred generically to as "KK-branes". We have described how they can be obtained via the *reduction-S dualization-oxidation* which could explain why some of the directions have to be compactified in circles since S duality only exists in the compactified theory. Furthermore, we have computed the masses of these solutions and we have found that they depend on the square of the radii of the directions that we have identified as compact, just as it happens in the KK monopole case, which is consistent with our identification. The mass formula are also coincident with what is needed to complete the U duality invariant spectrum of $N = 8, d = 4$ supergravity [105–107]. It has also been recently argued that the presence of certain KK-branes is necessary to explain from the M theory point of view the existence of some massive/gauged type II supergravities in lower dimensions [112].

Perhaps the only element that does not seem to fit in the picture we are putting forward is the supersymmetry algebra since there seems to be no place in it for the new objects. For the sake of concreteness we will focus in the 11-dimensional supersymmetry algebra ("M algebra") but the problems and the solutions we propose can be applied in the obvious way to other cases.

The M algebra is usually written, up to convention-dependent numerical factors c, c_n, in the form[12]

$$\{Q^\alpha, Q^\beta\} = c\left(\Gamma^a \mathcal{C}^{-1}\right)^{\alpha\beta} P_a + \tfrac{c_2}{2}\left(\Gamma^{a_1 a_2}\mathcal{C}^{-1}\right)^{\alpha\beta} \mathcal{Z}^{(2)}_{a_1 a_2} + \tfrac{c_5}{5!}\left(\Gamma^{a_1\cdots a_5}\mathcal{C}^{-1}\right)^{\alpha\beta} \mathcal{Z}^{(5)}_{a_1\cdots a_5}\,.$$

A lightlike component of the momentum is then associated to the gravitational waves moving in that direction, the spatial components of $\mathcal{Z}^{(2)}$ and $\mathcal{Z}^{(5)}$ are associated respectively to $M2$- and $M5$-branes wraped in those directions. The timelike components have more complicated interpretations: in the $\mathcal{Z}^{(5)}$ case, they are associated to the KK monopole in a complicated way and in the $\mathcal{Z}^{(2)}$ case they are associated to an object that we would call the $KK9$-brane of which we only know that it should give the $D8$-brane upon

[12] See e.g. [113].

dimensional reduction. All these objects break (preserve) a half of the available supersymmetries and strict relations between their masses and charges can be derived from the algebra.

Clearly the M algebra contains a good deal of information about the solitons of the theory that realizes it (11-dimensional supergravity or M theory). However, it is clear that it does not contain all the information about them. To start with, it does not tell us why some branes are fundamental and some are solitonic, it does not tell us why some objects exist in the uncompactified theory (the wave, $M2$ and $M5$) while other objects only exist when one dimension is compactified in a circle (the KK monopole and the $KK9$-brane). Furthermore, all solitonic objects should be associated to spacelike components of central charges: that is the result we would always get if we performed the calculation. All this is not so surprising: the M algebra is not derived from the theory and their solutions but just by imposing consistency of the possible central charges. If we were able to derive the algebra from M theory and its solitonic solutions, the central charges would be associated to specific objects and we would know whether they have compact dimensions or not. Since we do know many things about the solitonic solutions, we can try to reflect what we know in a form of the M algebra mathematically consistent and then we can check if the results are consistent with dualities.

To start with, we consider the M algebra with the most general central extensions allowed:

$$\left\{Q^{\alpha}, Q^{\beta}\right\} = c\left(\Gamma^{a}\mathcal{C}^{-1}\right)^{\alpha\beta} P_{a} + \sum_{n=2,5,6,9,10} \tfrac{c_n}{n!}\left(\Gamma^{a_1\cdots a_n}\mathcal{C}^{-1}\right)^{\alpha\beta} \mathcal{Z}^{(n)}_{a_1\cdots a_n}\,. \quad (4.6.1)$$

We know the wave is associated to P, the $M2$-brane to $\mathcal{Z}^{(2)}$ and the M5-brane to $\mathcal{Z}^{(5)}$. We also know [114] that the KK monopole is a sort of 6-brane with one of the 4 possible transverse dimensions wrapped in a circle. We are going to reflect this fact by writing, instead of just the $\mathcal{Z}^{(6)}$ term as above, the term

$$\tfrac{c_6}{6!}\left(\Gamma^{a_1\cdots a_6}\mathcal{C}^{-1}\right)^{\alpha\beta} \mathcal{Z}^{(7)}_{a_1\cdots a_6 a_7} k^{a_7}\,, \quad (4.6.2)$$

where k^{a} is a vector pointing in the compact direction.

We also know that the $KK9$-brane (or $M9$-brane) [115] has 9 spacelike worldvolume dimensions one of which is always wrapped on a circle. We reflect this fact by writing, instead of just the $\mathcal{Z}^{(9)}$ term as above, the term

$$\tfrac{c_9}{9!}\left(\Gamma^{a_1\cdots a_9}\mathcal{C}^{-1}\right)^{\alpha\beta} \mathcal{Z}^{(8)}_{a_1\cdots a_8} l_{a_9}\,, \quad (4.6.3)$$

where l_a is a vector pointing in the direction around which the $KK9$-brane is wrapped.

We do not know of any brane associated to $\mathcal{Z}^{(10)}$ and so we will not consider it in the M algebra, which takes the form

$$\begin{aligned}\{Q^\alpha, Q^\beta\} &= c\,(\Gamma^a\mathcal{C}^{-1})^{\alpha\beta}\,P_a + \tfrac{c_2}{2}\,(\Gamma^{a_1a_2}\mathcal{C}^{-1})^{\alpha\beta}\,\mathcal{Z}^{(2)}_{a_1a_2} + \tfrac{c_5}{5!}\,(\Gamma^{a_1\cdots a_5}\mathcal{C}^{-1})^{\alpha\beta}\,\mathcal{Z}^{(5)}_{a_1\cdots a_5}\\ &\quad + \tfrac{c_6}{6!}\,(\Gamma^{a_1\cdots a_6}\mathcal{C}^{-1})^{\alpha\beta}\,\mathcal{Z}^{(7)}_{a_1\cdots a_6a_7}k^{a_7} + \tfrac{c_9}{9!}\,(\Gamma^{a_1\cdots a_9}\mathcal{C}^{-1})^{\alpha\beta}\,\mathcal{Z}^{(8)}_{a_1\cdots a_8}l_{a_9}\,.\end{aligned}$$

We could certainly write more general central charges by allowing more vectors to be present in the algebra, meaning allowing objects with more isometric directions such as the $M2_6$ or the $M5_3$ branes presented in this Chapter. However, considering objects with just one special isometry will be enough to present our ideas.

Let us now reduce this algebra in one dimension. From each of the standard central charges we get two central charges in one dimension less, namely $P, \mathcal{Z}^{(0)}$ from P, $\mathcal{Z}^{(1)}, \mathcal{Z}^{(2)}$ from $\mathcal{Z}^{(2)}$ and $\mathcal{Z}^{(4)}, \mathcal{Z}^{(5)}$ from $\mathcal{Z}^{(5)}$, corresponding to the known reductions of M theory solitons: wave and $D0$-brane from the wave, $F1$ and $D2$-brane from the $M2$-brane and $D4$- and $S5$-brane from the $M5$-brane. From each of the new charges we have introduced we get instead three lower dimensional central charges: from the contraction $\mathcal{Z}^{(7)}k$ associated to the KK monopole we get a $\mathcal{Z}^{(6)}$ associated to the $D6$-brane when k points in the direction we are reducing, we get a contraction $\mathcal{Z}^{(6)}k$ associated to the type IIA KK monopole ($KK6A$) if we reduce on the KK monopole worldvolume and we get a $\mathcal{Z}^{(7)}k$ associated to the $D6_1$ (called $KK7A$ in Ref. [94], also studied in Ref. [116]) if we reduce in a transverse direction. From the product $\mathcal{Z}^{(8)}l$ we get a $\mathcal{Z}^{(8)}$, associated to the $D8$-brane when we reduce the KK9-brane in the isometric direction l points to, we get a product $\mathcal{Z}^{(7)}l$ associated to an object with the same features of the M theory $KK9$-brane but in one dimension less and a product $\mathcal{Z}^{(8)}l$ associated to a type IIA spacetime filling $KK9$-brane referred to as $NS-9A$-brane in Ref. [117]. The result is the following form of the type IIA supersymmetry algebra:

$$
\begin{aligned}
\{Q^\alpha, Q^\beta\} \;=\;& c\,(\Gamma^a\mathcal{C}^{-1})^{\alpha\beta}\, P_a + \textstyle\sum_{n=0,1,4,8} \frac{c_n}{n!}\,(\Gamma^{a_1\cdots a_n}\Gamma_{11}\mathcal{C}^{-1})^{\alpha\beta}\, \mathcal{Z}^{(n)}_{a_1\cdots a_n} \\
& + \textstyle\sum_{n=2,5,6} \frac{c_n}{n!}\,(\Gamma^{a_1\cdots a_n}\mathcal{C}^{-1})^{\alpha\beta}\, \mathcal{Z}^{(n)}_{a_1\cdots a_n} \\
& + \tfrac{c_5}{5!}\,(\Gamma^{a_1\cdots a_5}\Gamma_{11}\mathcal{C}^{-1})^{\alpha\beta}\, \mathcal{Z}^{(6)}_{a_1\cdots a_5 a_6} k^{a_6} + \tfrac{c_6}{6!}\,(\Gamma^{a_1\cdots a_6}\mathcal{C}^{-1})^{\alpha\beta}\, \mathcal{Z}^{(7)}_{a_1\cdots a_6 a_7} l^{a_7} \\
& + \tfrac{c_8}{8!}\,(\Gamma^{a_1\cdots a_8}\mathcal{C}^{-1})^{\alpha\beta}\, \mathcal{Z}^{(7)}_{a_1\cdots a_7} m_{a_8} + \tfrac{c_9}{9!}\,(\Gamma^{a_1\cdots a_9}\mathcal{C}^{-1})^{\alpha\beta}\, \mathcal{Z}^{(8)}_{a_1\cdots a_8} n_{a_9}\,.
\end{aligned}
$$

Every known solitonic solution of the type IIA supergravity theory has an associated charge in this algebra. If we now reduce again to nine dimensions we will get the algebra of the massive 9-dimensional theories presented in Ref. [94] with $SL(2,\mathbb{Z})$ covariance. This is possible only because we have allowed for charges corresponding to KK-branes in 11 dimensions. To get the same algebra from the type IIB side a charge has to be introduced for the $S7$ brane which, even though it does not carry any $SO(2)$ R-symmetry indices, is not invariant but is interchanged with the $D7$-brane charge under S duality.

Part II

Supergravity Vacua

Chapter 5

Supergravity Vacua and Homogeneous Spacetimes

In Part I we dealt with a special kind of supergravity solutions: solutions such that we could argue that they were describing the long range fields emitted by stringy sources. The general features of this kind of supergravity backgrounds were studied in detail in Chapter 2. Now we will develop some results concerning a rather different class of spacetimes, namely those that we defined as *vacua* in Section 2.2.1. They were introduced at that moment because we needed to clarify the notion of conserved charges in theories with general covariance. We refer to Section 2.2 for a discussion about the topic of spacetime or geometrical charges, and to Sections 2.2.1 and 2.4.3 for related discussions about the concept of vacuum as will be considered here. Here we simply recall the idea used there to motivate a possible definition of vacuum. For a given background $\mathcal{M}$ to which one can attribute conserved charges, the notion of "its" vacuum seems intuitive: its asymptotic spacetime $\mathcal{V}$. Such a correspondence between the spacetime $\mathcal{V}$ and the vacuum state should be natural, since to compute the geometrical charges of $\mathcal{M}$ we always do it with respect to those of $\mathcal{V}$. If we understand $\mathcal{M}$ as a local perturbation of $\mathcal{V}$, the procedure to compute the energy of $\mathcal{M}$ just mimics what one always does in Field Theory to compute the energy of any state, which only has a meaning in terms of a relative shift with respect to the vacuum energy.

Supergravity Vacua and Field Theory Vacua

There is, however, a crucial difference between supergravity vacua and vacua as they appear in Field Theory, on which we would like to comment. In Field Theory a vacuum state is a *ground state* of the theory. This means that it is a solution of the equations of motion of the effective field theory action that *minimizes the energy.* There may be a unique solution to this problem or there may be infinitely many, and a vacuum state may enjoy all the symmetries of the theory or it may not, hence giving rise to the phenomenon of spontaneous symmetry breaking. But uniqueness or degree of symmetry do not play any role in its definition –what it must be is a ground state[1]. The main point is the following: while in Field Theory this is a well defined problem, in Supergravity it is not. As in Field Theory, every asymptotic spacetime $\mathcal{V}$ that we will call vacuum in Supergravity will be a solution of the equations of motion of some supergravity theory. But the essential difference arises from the fact that the energies of two vacua $\mathcal{V}_1$ and $\mathcal{V}_2$ cannot be *compared*, because each of them carries its own prescription to compute the energy. One can use these prescriptions to calculate the total energy of a spacetime $\mathcal{M}_{1(2)}$ which is asymptotically $\mathcal{V}_{1(2)}$, but if one uses them to compute their own total energies one gets, by construction, zero. In this way, the difference between the total energies of e.g. two different Anti de Sitter spacetimes with two different cosmological constants cannot be evaluated. Therefore, the most one can argue is, for example, that Minkowski spacetime can be considered as the ground state within the set of all spacetimes which are asymptotically Minkowski[2]. But there is no criterion at all to say that Minkowski spacetime should be a ground state within the set of *all* possible spacetimes. And of course, this also means there is no criterion to determine if Minkowski is a better "vacuum" state than AdS with cosmological constant Λ_1, or if the latter is better than AdS with cosmological constant Λ_2.

So the question of which are the possible true String Theory or Supergravity vacua (the classical ground states of the String Theory effective action) is not even well defined in general: simply, there is no moduli space of vacua.

[1] It is true that ground states always turn out to exhibit, in addition, a very high degree of symmetry. But this is a common property of all solutions to the mathematical problem (in any context) of finding extrema, and has nothing to do with its actual definition.

[2] Provided that, in addition, some positive-energy theorem for excitations in Minkowski applies. We will comment on this in Section 5.2.

This is a major problem of String Theory. We will however keep the name of "supergravity vacua" for a certain class of asymptotic spacetimes.

We would like to address two issues next. First, in view that the reasoning leading us to a definition of vacuum was the idea of "vacuum of (or associated to) a source", we will try to investigate a more intrinsic definition of this kind of spacetimes: a definition which does not require starting with some given spacetime and extract afterwards its asymptotics to find something that we can call vacuum. This will make us consider homogeneous spacetimes. Secondly, we will address some aspects concerning the stability of these vacuum spacetimes.

5.1 Homogeneous spacetimes

Let us consider the case in which we have a spacetime $\mathcal{V}$ which is a solution of the supergravity field equations. Such a spacetime will be given by a metric $g_{\mu\nu}$ and, in general, by other bosonic fields which may be nontrivial[3]. We will denote the whole set of these bosonic fields by ϕ, and we will write our spacetime as $\mathcal{V} = \{g_{\mu\nu}, \phi\}$. We want to find a criterion to determine whether $\mathcal{V}$ can be considered as a vacuum spacetime or not.

We already have some intuition about this: this spacetime should be "purely asymptotic", so that *any other* (physically inequivalent) spacetime $\mathcal{M}$ which is asymptotically $\mathcal{V}$ could be regarded as a *perturbation* of $\mathcal{V}$, in the sense that a computation of the geometrical charges of $\mathcal{M}$ will always yield a finite result. "Purely asymptotic" just means that *all* points in $\mathcal{V}$ must be physically equivalent, i.e. that $\mathcal{V}$ must have no special points in any physical sense. This implies that a necessary condition to be satisfied by $\mathcal{V}$ is that it cannot have any physical "core" or energy locus, which seems to fit quite well with the idea of $\mathcal{V}$ as a spacetime with no excitations. Can we map these intuitive conditions into a precise, covariant, mathematical formulation?

The answer is affirmative. A spacetime with no preferred points is called *homogeneous*, and the mathematical requirement is as follows:

[3]We will always demand from $\mathcal{V}$ to solve the equations of motion of our theory and, in general, this will not be achieved if the only nontrivial field is $g_{\mu\nu}$. On the other hand, we will only consider, as usual, the case of purely bosonic backgrounds.

$\mathcal{V}$ is homogeneous iff for *any* two points p and q in $\mathcal{V}$ there exists an *isometry* that takes p into q[4].

The fact that the transformation relating p and q is an isometry[5] means that the only difference between being at p or being at q can be, at most, an unphysical coordinate reparametrization. If this happens for all p and q in $\mathcal{V}$, then all points will be physically equivalent. Given the importance of this kind of spacetimes in the Chapters that will follow, we proceed next to develop a bit on their definition as well as on some other technical aspects.

However, we want to make clear the following: we are *not* claiming a one-to-one correspondence between all possible candidates to a Supergravity or String Theory vacuum and all homogeneous spacetimes. We simply pursued an appropriate mathematical definition that fits the best with the physical consequences that we were able to extract from the notion of vacuum as introduced here. There are in fact lots of string constructions (for example Calabi-Yau or K3 compactifications, orbifold compactifications, etc) which assume a string theory ground state which does not admit any homogeneous spacetime description.

5.1.1 The Reparametrization Symmetry Algebra

Let us consider $\mathcal{V} = \{g_{\mu\nu}, \phi\}$, and let us consider the effect on it of infinitesimal general coordinate transformations

$$x^\mu \longrightarrow x^{\mu\,\prime} \simeq x^\mu + \sigma\xi^\mu(x)\,, \tag{5.1.1}$$

where σ is a constant, infinitesimal parameter and ξ^μ is the vector field that generates the transformation. Under this infinitesimal local translation, the functional change of the bosonic fields describing the solution $\mathcal{V}$ is given by (minus) their Lie derivatives along ξ, denoted here as $\mathcal{L}_\xi$.

Let us define the *symmetry group* K of reparametrizations of $\mathcal{V}$. This is the finite dimensional group generated by all the transformations of the

[4]This is a definition which is suitable for Riemannian manifolds (see the comments in Section 5.1.3), and is the one which is usually introduced in Cosmology (see e.g. [101], p. 92).

[5]If $\mathcal{V}$ consists of more fields apart from the metric we will require the corresponding transformation to be, in addition, a symmetry of the full solution. See below.

kind of (5.1.1) that leave all the fields in $\mathcal{V}$ invariant. That is, the g.c.t.'s generated by all those k_I ($I = 1, ..., \dim K$) satisfying[6]:

$$\mathcal{L}_{k_I} g_{\mu\nu} = \mathcal{L}_{k_I} \phi = 0 \,. \tag{5.1.2}$$

These are the Killing vectors that leave invariant not only the metric but also all other bosonic fields ϕ entering in the solution given by $\mathcal{V}$ (hence K will be, in general, a subgroup of the full isometry group of $g_{\mu\nu}$). Since the infinitesimal action of the k_I on the metric and on the other bosonic fields of the theory is given by their Lie derivatives $\mathcal{L}_{k_I}$, the Lie algebra of K is the one satisfied by the differential operators $\mathcal{L}_{k_I}$. These obey the property:

$$[\mathcal{L}_{k_I}, \mathcal{L}_{k_J}] = \mathcal{L}_{[k_I, k_J]} \,, \tag{5.1.3}$$

where $[k_I, k_J]$ is the *Lie bracket* of the vector fields k_I and k_J, defined as[7]:

$$[k_I, k_J]^\mu \equiv k_I^\nu \partial_\nu k_J^\mu - k_J^\nu \partial_\nu k_I^\mu \,. \tag{5.1.4}$$

The fact that the commutator of two Lie derivatives obeys (5.1.3), together with the fact that (in the absence of torsion) the Lie bracket of two vector fields is also a vector, immediately implies that $[k_I, k_J]$ is a symmetry generator if k_I and k_J are, and so the symmetry algebra closes. Associating an operator P_I to every Killing vector k_I in the symmetry algebra (which on bosonic fields will be represented as $P_I = -\mathcal{L}_{k_I}$), the Lie algebra of K can be written as:

$$[P_I, P_J] = f_{IJ}{}^K P_K \,, \tag{5.1.5}$$

where the $f_{IJ}{}^K$ are the corresponding structure constants. The above algebra is the Lie algebra of K, the reparametrization symmetry algebra of the solution given by $\mathcal{V}$.

---

[6]There will be many more symmetry generators, of course. In particular, all those generating internal symmetries of the fields and all those generating supersymmetries. The whole set of symmetry generators will give the full global superalgebra associated to $\mathcal{V}$. But here we are only concerned with the *reparametrizations* that leave invariant the full solution.

[7]This gives the functional change of the vector field k_J under an infinitesimal coordinate transformation generated by k_I, i.e. $[k_I, k_J] = -\mathcal{L}_{k_I} k_J$.

5.1.2 Homogeneous Spacetimes

We will denote by k a generic element in the symmetry group K. Infinitesimally, they will be given by:

$$k \simeq 1 + \sigma^I P_I \,, \tag{5.1.6}$$

where the σ^I are infinitesimal parameters. K acts on the fields of our theory by means of the above expression with the P_I in the appropriate representation (the corresponding Lie derivative). But, moreover, K also has an action on points of the spacetime manifold (henceforth denoted by $M_{\mathcal{V}}$): the one given by (5.1.1),

$$x^\mu \longrightarrow x^{\mu\prime} \simeq x^\mu + \sigma^I k_I^\mu \,. \tag{5.1.7}$$

Observe that such an action can be interpreted both a group of ("active") motions on $M_{\mathcal{V}}$ and as a group of ("passive") diffeomorphisms. We are now ready to define what an homogeneous spacetime is.

Definition: the spacetime $\mathcal{V}$ is said to be *homogeneous* if K acts *transitively* on the spacetime manifold.

"Transitively" means what we already advanced at the beginning of this Section: that *any* two points will always be related by a symmetry transformation generated by K, whose infinitesimal action on the spacetime manifold is given by (5.1.7). This means that one can "recover" the whole spacetime by acting with K on a single point. Notice that for a spacetime to be homogeneous, it is enough to admit a *subgroup* $G \subset K$ that acts transitively on it. We refer to the beginning of the present Section for a discussion concerning the relation between homogeneous spacetimes and vacuum spacetimes.

Simple examples of homogeneous spaces are given by Minkowski spacetimes, AdS spacetimes or spheres in any dimension. In each of these manifolds any point can be reached from any other by means of an isometry. In fact, these spaces have no special points or no "center" in any sense. All spacetimes listed in the r.h.s. of Table 2.1, as well as those of the l.h.s of Table 6.1 are also homogeneous.

A simple example of a spacetime which is *not* homogeneous is Schwarzschild. If, in usual spherical coordinates, we choose two infinitesimally near points along the radial direction:

$$p = r_0 \,, \quad q = r_0 + \delta r \,,$$

one can see that there is no transformation of the kind of (5.1.7) taking p into q, since $k_r^\mu = \delta_r^\mu$ is not a Killing vector of the Schwarzschild solution. On physical grounds, it is very clear why the Schwarzschild spacetime is not homogeneous: it has a very well defined physical core. The same happens, for example, to all p-brane solutions in Supergravity, which are not homogeneous spaces.

5.1.3 Homogeneous Spacetimes and Coset Spaces

A very strong mathematical property concerning homogeneous spacetimes is their close relation with group theory (general references are [120]). It can be shown that any homogeneous manifold $M_{\mathcal{V}}$ is always diffeomorphic to the coset space

$$M_{\mathcal{V}} \cong G/H \,, \tag{5.1.8}$$

where G is any subgroup of K acting transitively on $M_{\mathcal{V}}$ and $H \subset G$ is its isotropy subgroup (the subgroup of G that leaves fixed a given point of $M_{\mathcal{V}}$). Here G/H stands for the set of equivalence classes in G defined under right multiplication by elements of H, which is usually written as

$$G/H = \{gH\} \ , \quad \text{for } g \in G \,.$$

Notice that

$$\dim M_{\mathcal{V}} = \dim G/H = \dim G - \dim H \,. \tag{5.1.9}$$

Statement (5.1.8) has the implicit assumption that G/H can be given the structure of a differentiable manifold. In fact, this is the case if G is *any* Lie group and $H \subset G$ is any subgroup of G[8]. Then G/H is a manifold on which G has a natural transitive action, and H turns out to be the isotropy subgroup of G. If further technical requirements are achieved, it is possible to put a Riemannian metric on this manifold, and this metric will always contain G in its isometry group[9]. However, the full isometry group of the resulting metric can be larger, and this the reason why we made from the beginning the distinction between the isometry group, the symmetry group of the full solution (what we denoted by K) and a subgroup G of K that

[8] The only requirement is that H has to be topologically closed.

[9] The conditions needed to put a metric on G/H, as well as the standard procedure to obtain it, are explained in Chapter 7.

acts transitively on $M_{\mathcal{V}}$[10].

Note that the procedure that we have just sketched in the preceding paragraph (from a Lie group G to a Riemannian manifold endowed with a metric which contains G in its isometry group) is the *opposite* to the one we have been developing along this Section: we started with a bosonic solution of Supergravity, i.e. a spacetime manifold equipped with a Riemannian metric from the beginning. We did it in this way because we were pursuing a definition of homogeneous spacetimes motivated by physical considerations. In fact, the definition of homogeneous spacetime as introduced here (recall that the group needed to fulfill the transitivity condition was required to be contained within the *isometry* group of the metric) is only suitable for Riemannian manifolds. In general, a topological space M (not even a manifold) is called homogeneous simply if there is a group G that has a transitive action on it. In Supergravity, however, we will only be interested in those cases in which, at the end, the space M can be given the structure of a differentiable manifold (in that case it will always be diffeomorphic to G/H, H being the isotropy subgroup of G) and, further, a G-invariant Riemannian metric can be put on M. In that case, the definition of homogeneous spacetime as given here applies.

We will deal with the construction of homogeneous spacetimes as originating from a coset description in Chapters 7 and 8, where we explain how the metric is obtained. The metrics of the coset spaces that we will consider there will always coincide with the metric of some supergravity solution, and we will see how the underlying coset structure of these homogeneous (vacuum) spacetimes plays a very important role in all their (super) symmetry structure. Of particular physical importance are the global superalgebras of vacuum spacetimes (see the discussion in Section 2.3), and we will show how the superalgebras of these vacua are, in principle, also encoded in their group theory coset description.

[10] Here we will be interested in *coset* spaces. A very interesting but different case is that of *group* manifolds. These can be seen as trivial cosets, arising when the only element in the isotropy subgroup is the identity. In this case, the construction mentioned above (with slight differences) can serve to put a metric on the group G itself. For example, S^3 can be seen as the $SU(2)$ group manifold. This has important applications in String Theory, because now the manifold is endowed with a richer structure, namely that of a group (we recall that a generic coset G/H does not admit a group structure unless H is a normal subgroup of G).

We have introduced the notion of homogeneous spacetimes motivated by the Physics of vacua. We have recalled their close relation with coset spaces and group theory because this is the mathemathical tool needed to obtain and understand the results reported in the following Chapters.

5.2 Stability

In Chapter 2 we commented on some generic aspects concerning the quantum and dynamical stability of Supergravity backgrounds describing massive and charged configurations of String Theory. These questions arose from the fact that Supergravity is an approximation to String Theory, and hence we were wondering about the goodness and reliability of our approximation in those cases. Of course, the same kind of questions apply to the case of vacuum solutions. Although the general principles concerning the reliability of a Supergravity solution were already introduced in Chapter 2, there are some significant differences when one is interested in investigating these aspects in the case of a vacuum solution. Here we will content ourselves with some brief comments concerning these differences.

5.2.1 Dynamical Stability

Given a certain vacuum solution $\mathcal{V}$, a natural question to pose is about its possible decay into another state. This is very likely to be so if we do not have at hand a sort of "positive-energy" theorem for excitations in $\mathcal{V}$ (see e.g. [121]) because, in such a case, the argument saying that $\mathcal{V}$ can be considered as the ground state within the class of spacetimes which approach it asymptotically would have no support (see the introduction to this Chapter).

Dynamical stability has been analyzed classically for different vacua other than Minkowski. Such analyses are based on the behaviour of small fluctuations around the classical solution. For example, Abbot and Deser [51] considered the stability of both dS and AdS spacetimes. They found that they are stable in the framework of Einstein gravity. A semiclassical analysis concerning vacuum stability was made in [121] for the case of the original Kaluza-Klein vacuum $\mathcal{M}_4 \times S^1$. This was made by considering the amplitude of the decay process into another state (with a sort of "instanton computation"), and the KK-vacuum was found to be an unstable "false vacuum". As usual, the presence of supersymmetry simplifies many times stability considerations. As another example, the classical stability of Freund-Rubin compactifications of eleven-dimensional supergravity (4-dim. Minkowski times a 7-dim. homogeneous space) was studied in [122]. There they found that all supersymmetric vacua are automatically stable, but also that some others with no supersymmetries at all are stable against small fluctuations.

The true question is, of course, if dynamical stability holds quantum mechanically. One way to prove this would be the derivation of a BPS bound of the kind of (2.3.6) for vacua other than Minkowski. This would imply the absence of tachyons if the vacuum superisometry algebra is promoted to a quantum symmetry. However, to derive such a bound one needs to know how the global superalgebra of a given vacuum is with full knowledge of its central extensions, which is not a straightforward task. A partial approach to this problem will be presented in Chapter 7.

5.2.2 Accuracy in the α' Expansion

As said in Chapter 1, the Supergravity effective action is a double expansion in both the string tension and the string coupling, and it is given by the lowest nontrivial contributions in both parameters. Another natural question to be raised is, then, about the accuracy of a given solution in the α' expansion since, in principle, every supergravity solution is to be α'-corrected. At this respect there is a particular kind of vacuum spacetimes, namely *pp*-waves, which play a very special role, since it has been shown that certain classes of *pp*-waves are exact solutions to all orders in α' (i.e., to all orders in sigma-model perturbation theory). The usual argument is that higher α' corrections to the effective action involve higher order curvature terms, all of which vanish for certain *pp*-wave spacetimes [123]. In some cases this has also been explicitly checked to all orders in sigma-model perturbation theory, and even at the full nonperturbative level for some cases in which a certain *pp*-wave spacetime geometry makes the two-dimensional theory an integrable model [124]. For further references concerning the exactness of wave backgrounds and their importance for string quantization see [125].

5.2.3 Stability Against Quantum Corrections

It remains the question about if *string* loops could correct the spacetime geometry, which as a supergravity solution is just string tree level. This is a really important question, because string radiative corrections could render a supergravity vacuum solution completely meaningless[11] (see the discussion in Section 2.3). For supergravity backgrounds describing massive and charged objects we had a non-renormalization argument for states saturating a BPS-bound, which roughly speaking is mass=charge (this was explained in Section 2.3.3). What happens in the case of a vacuum spacetime, with no mass and no charge?

One could invoke a similar argument, but this time concerning representation theory of massless (instead of BPS) states. But we would be cheating in that case, because we would be using the superisometry algebra associated to a classical spacetime whose reliability is precisely what we are questioning. Of course, the present objection underlies the nonrenormalization argument of Section 2.3.3. At that moment, we completely skipped the problem of the stability of the asymptotic spacetime and took it for granted: there we only faced the problem of the string quantum corrections to the stringy *source* quantum numbers.

Quantum stability of the vacuum is one of the most important problems one can face in Physics and, of course, also in String Theory. One can find in fact highly nontrivial effects. For example, Fischler and Susskind showed in [45] that, taking into account one-loop effects in the bosonic string, a cosmological constant term is generated in the spacetime equations of motion (the one-loop corrected sigma-model beta functions), and hence 26-dimensional Minkowski space turns out to be *unstable* against string quantum corrections[12]. The only way to deal with string loops effects to analyze vacuum stability seems to be to compute the string loop corrected spacetime effective action. For the case of the bosonic string, a full analysis was done in [129] by requiring sigma-model conformal invariance and, again, a cos-

[11]We emphasize that this is *not* the case for α'-corrections, at least if one assumes convergence of the series in α'. α'-corrections can distort the classical spacetime picture at short distances but, *a priori*, do not render the classical solution "impossible", at least up to a certain resolution scale.

[12]This was achieved by adding counterterms to the two dimensional sigma-model action in order to cancel divergencies in one-loop string amplitudes (see the comments at the end of Section 1.2). Quite surprisingly, the appearance of a cosmological constant in the bosonic string was shown in [45] to be due to the presence of dilaton tadpoles, not to the presence of a tachyon in the spectrum. The cosmological constant in the bosonic string is thus a one-loop effect because conformal invariance ensures vanishing tadpoles on the sphere (a nice argument for this is given e.g. in [126]). Previous investigations concerning the cosmological constant in String Theory were presented in [127] and [128].

mological constant term in the resulting spacetime equations of motion was found. There, one-loop corrected version of the effective action (1.2.5) was found to be

$$S = \frac{1}{16\pi G_N^{(26)}} \int d^{26}x \sqrt{|g|} \left[e^{-2\phi} \left(R - 4(\partial\phi)^2 + \frac{1}{12} H^2 \right) + \Lambda \right], \qquad (5.2.1)$$

i.e., the correction is precisely the addition of a cosmological constant term. The fact that it is a one-loop correction can be guessed from the different dilaton power in front of it.

The situation drastically changes when considering the superstring case. The conclusion in this case can be roughly summarized by saying that if (enough) supersymmetry is preserved by a classical vacuum solution, no corrections to it arise from string loops. For example, Martinec proved in [130] the stability, among other supersymmetric backgrounds, of ten-dimensional Minkowski spacetime. Later on, Dine and Seiberg showed in [131], with very general arguments, that any superstring compactification down to four dimensions (and also its corresponding ten dimensional decompactification limit) preserving at least four supercharges (i.e. $N = 1$ supersymmetry in four dimensions), suffers no quantum corrections to all orders in the string loop expansion. String loop corrections to supergravity actions were considered in [132, 133].

Chapter 6

On $d = 4, 5, 6$ Vacua with 8 Supercharges

Introduction

There is currently a renewed interest on maximally supersymmetric vacua stemming from the discovery, and re-discovery of previously overlooked, maximally supersymmetric H*pp*-wave solutions [134–136]. These solutions have very interesting properties: they are not only supergravity solutions (i.e. solutions of the lowest-order superstring effective action) but it can be argued that they are exact solutions of superstring theory to all orders and therefore good vacua on which superstrings can be quantized [124, 125, 137] and the D-branes can be discussed [138, 139]. Further, these solutions can be obtained by a limiting procedure that preserves (or increases) the number of unbroken (super)symmetries [140–142] (for a review see, e.g. Ref. [143]), a feature which has given rise to the Hpp/CFT correspondence (See *e.g.* [144]).

It is the standard lore that maximally supersymmetric vacua (other than products of Minkowski spacetime by circles) of higher-dimensional supergravity theories cannot be dimensionally reduced preserving all their unbroken supersymmetries (See e.g. [145, 146] and references therein): in general, the Killing spinors of these vacua depend on all coordinates. This dependence complicates its compactification and dimensional reduction. First, only for certain radii of the compact direction the Killing spinors will have the right periodicity and, thus, only for those radii the compactified solutions preserve the same amount of supersymmetry as the uncompactified one. Second, unless the Killing spinors are independent of the compact coordinates (or have a very special dependence on them, as in some generalized dimensional reductions [147]), the components of the Killing spinor that do depend on the compact coordinate have to be projected out of the dimensionally reduced theory [148], leading to less supersymmetry. Since T duality of classical solutions involves their dimensional reduction it should not come as a surprise that the supersymmetry of the maximally supersymmetric vacua is not preserved by T duality either [149, 150].

In this Chapter we are going to show that the known maximally supersymmetric $d = 4, 5, 6$ vacua of theories with 8 supercharges ($N = 2$ or $N = (2, 0)$ theories) can be dimensionally reduced/oxidized preserving all their unbroken supersymmetries because in all the $d = 5, 6$ cases it is possible to choose coordinates in which the Killing spinor is independent of the coordinate we use for dimensional reduction.

That the coordinate choice that preserves all supersymmetry in dimensional reduction is always possible looks highly non-trivial. However, thinking in terms of oxidation of the $d = 4, 5$ theories it is evident that all unbroken supersymmetry should be preserved: these theories can be obtained by standard dimensional reduction of the $d = 6, 5$ ones supplemented by a truncation of the matter multiplets that appear in the reduction. It is, therefore, guaranteed that, if we have a solution of the $d = 4, 5$ theories that preserves all 8 supersymmetries, it comes from some $d = 5, 6$ solution that also preserves those 8 supersymmetries and therefore has to be one of the essentially unique maximally supersymmetric vacua of the theory [1]. Thus, the maximally supersymmetric vacua of these theories must be related by dimensional reduction/oxidation and we are going to show exactly how this happens. The independence of the Killing spinors of the compact coordinates is an implicit automatic consequence of the above arguments.

Let us now briefly review the known maximally supersymmetric vacua of these theories:

[1]To the best of our knowledge, though, no theorem proving the uniqueness of the maximally supersymmetric vacua we are dealing with exists for the $d = 6$, $N = (2, 0)$ theory. A classification of the spacetimes admitting Killing spinors in four dimensions was given in [72], and a complete classification of supersymmetric solutions in $d = 5$, $N = 2$ supergravity has recently appeared [151].

$N = (2,0), d = 6$:

1. The 1-parameter family of Kowalski-Glikman (KG) H*pp*-wave solutions found in Ref. [152] that we will denote by $KG6(2,0)$.
2. The 1-parameter family of solutions with $AdS_3 \times S^3$ geometry found in Ref. [153] as the near-horizon limit of the self-dual string solution.

$N = 2, d = 5$:

1. The 1-parameter family of KG solutions solutions found in Ref. [152] that we will denote by $KG5$.
2. The 1-parameter family of solutions with $AdS_3 \times S^2$ geometry found in Ref. [153] as near-horizon limit of the extreme string solution.
3. The 1-parameter family of solutions with $AdS_2 \times S^3$ geometry found in Ref. [154] as near-horizon limit of the extreme black hole solution.
4. The 2-parameter family of $N = 2, d = 5$ solutions found in Ref. [155] as the near-horizon limit of the supersymmetric rotating black hole solution.

The third family is contained in the fourth and corresponds to a vanishing rotation parameter. We will show that the second family is also contained in the fourth and corresponds to the value 1 of the rotation parameter.

$N = 2, d = 4$:

1. The 1-parameter[2] family of KG solutions solutions found in Ref. [156] that we will denote by $KG4$.
2. The 2-parameter family of electric/magnetic $N = 2, d = 4$ Robinson-Bertotti solutions [157] that have the geometry $AdS_2 \times S^2$.

[2]Electric-magnetic duality rotations only change the polarization plane of an electromagnetic wave and their effect on this family of solutions can be undone by a rotation that leaves the form of the metric invariant.

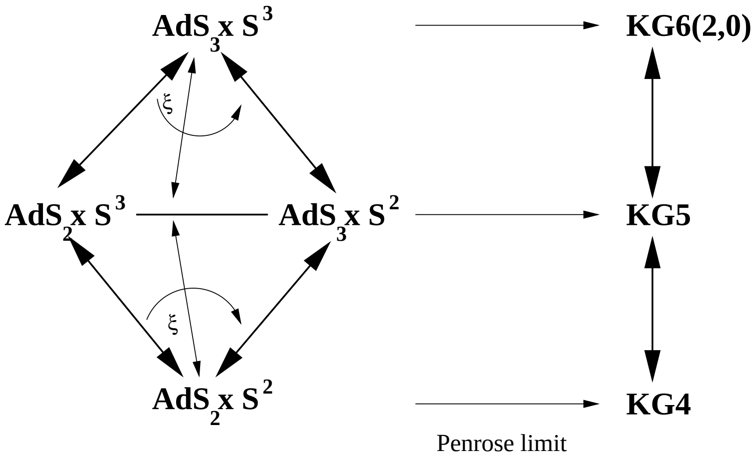

Figure 6.1: Relations between the $d = 4, 5, 6$ vacua with 8 supercharges.

The connections between these vacua that we have found are summarized in Figure 6.1[3]. The relations between the KG solutions are straightforward. The $AdS_3 \times S^3$ can be dimensionally reduced in the direction of the S^1 Hopf fiber of the 3-sphere and then we get $AdS_3 \times S^2$. It can also be reduced in the S^1 fiber of the AdS_3, giving $AdS_2 \times S^3$. Finally, we can rotate these fibers an angle ξ and reduce, getting the maximally supersymmetric solution that is the near-horizon limit of the rotating 5-dimensional extreme black hole. The angular momentum parameter j is essentially $\sin \xi$. Thus, this 2-parameter family of 5-dimensional vacua interpolates (in parameter space) between the $AdS_2 \times S^3$ and the $AdS_3 \times S^2$ vacua. The reduction of any member of this family in the remaining fiber gives an electric/magnetic Robinson-Bertotti solution where $\sin \xi$ is the ratio between the electric and the magnetic fields.

This Chapter is organized as follows: in Section 6.1 we study how the $KG6(2,0)$ and $KG5$ solutions can be dimensionally reduced preserving all the supersymmetry after describing briefly the general form of *pp*-wave solutions and their sources in Section 6.1.1. In Section 6.2 we study how the $AdS_m \times S^n$-type vacua of these theories are related by oxidizing them. Section 6.3 contains our conclusions and some discussion.

[3]The left hand side of the relations of the relations were discussed in [158]. We thank K. Skenderis for pointing this out to us.

6.1 Dimensional Reduction of Maximally Supersymmetric *Hpp*-Waves

Before we study the reduction of *KG* solutions it is worth studying briefly general supergravity *pp*-wave solutions.

6.1.1 General *pp*-Wave Solutions

pp-waves spacetimes are those whose metric admits a covariantly constant null vector. A metric with this property can always be put in the form

$$ds^2 = 2du(dv + Kdu + \mathcal{A}_a dx^a) + \tilde{g}_{ab}dx^a dx^b\,, \tag{6.1.1}$$

where the functions $K, \mathcal{A}_a, \tilde{g}_{ab}$ depend only on the wave-front coordinates x^a and on the null coordinate u. $\mathcal{A}_a$ is known as the *Sagnac connection* [159] and can always be set to zero by means of a coordinate transformation.

In supergravity theories it is natural to look for *pp*-wave solutions of the system

$$S_a = \frac{1}{16\pi G_N^{(d)}} \int d^dx\sqrt{|g|}\left[R + \tfrac{1}{2}(\partial\varphi)^2 + \tfrac{(-1)^{p+1}}{2(p+2)!}e^{-2a\varphi}F^2_{(p+2)}\right]\,, \tag{6.1.2}$$

where $F_{(p+2)} = dA_{(p+1)}$, of the form

$$\left\{\begin{array}{rcl} ds^2 & = & 2du(dv + Kdu) + \tilde{g}_{ab}dx^a dx^b\,, \\ F_{(p+2)} & = & du \wedge C\,, \end{array}\right. \tag{6.1.3}$$

where C is a $(p+1)$-form on the wave-front space and, as all the other fields in this Ansatz, it is independent of v.

A general solution is provided by a Ricci-flat wave-front metric $\tilde{g}_{ab}$ which must also satisfy

$$\tilde{\nabla}_a\left(\tilde{g}^{bc}\partial_u\tilde{g}_{bc}\right) - \tilde{\nabla}_b\left(\tilde{g}^{bc}\partial_u\tilde{g}_{ac}\right) = 0\,,$$

a harmonic $(p+1)$-form C in wave-front space

$$\tilde{d}C = \tilde{d}^{\star}C = 0\,,$$

with arbitrary u-dependence, an arbitrary function $\varphi(u)$; and a function $K(u, x^a)$ satisfying the equation

$$\tilde{\nabla}^2 K + \tfrac{1}{4}\partial_u \tilde{g}^{ab}\partial_u \tilde{g}_{ab} + \tfrac{1}{2}\tilde{g}^{ab}\partial_u^2 \tilde{g}_{ab} + \tfrac{1}{2}(\partial_u \varphi)^2 + \tfrac{(-1)^{p+1}}{2(p+1)!} e^{-2a\varphi} C^2 = 0\,.$$

The simplest choice of Ricci-flat wave-front space leads to the solutions

$$\begin{aligned} \tilde{g}_{ab} &= -\delta_{ab}\,, \qquad C = C(u)\,, \qquad \varphi = \varphi(u)\,, \qquad K = H + A\,, \\ A &\equiv A_{ab}(u) x^a x^b = -\tfrac{1}{4}\left[(\partial_u \varphi)^2 + \tfrac{(-1)^{p+1}}{(p+1)!} e^{-2a\varphi} C^2\right] (\mathrm{tr} M)^{-1} M_{ab}\, x^a x^b\,, \end{aligned} \tag{6.1.4}$$

where $H = H(x^a)$ is an arbitrary harmonic function in wave-front space, M_{ab} is a constant symmetric matrix and C and φ are just arbitrary functions of u.

The function K that contains all the information has, therefore, two pieces: the harmonic function $H(x)$, independent of the gauge field and dilaton (i.e. purely gravitational), and the matrix $A_{ab}(u)$ that depends on the gauge field and dilaton. One can argue that H represents excitations over a vacuum that consists of a self-supported (source-less) gauge field and dilaton and a metric described by $A_{ab}(u)$. For instance, one can try to match the above solution with a charged, mass-less, p-brane source with effective action

$$\begin{aligned} S_p[X^\mu, \gamma_{ij}] &= -T_p \int d^{p+1}\xi \sqrt{|\gamma|}\, e^{-2b\varphi}\, \gamma^{ij} \partial_i X^\mu \partial_j X^\nu g_{\mu\nu} + \\ &\quad + \tfrac{(-1)^{p+1}\mu_p}{(p+1)!} \int d^{p+1}\xi\, A_{(p+1)\,\mu_1 \ldots \mu_{p+1}} \partial_{i_1} X^{\mu_1} \ldots \partial_{i_{p+1}} X^{\mu_{p+1}} \epsilon^{i_1 \ldots i_{p+1}}\,. \end{aligned} \tag{6.1.5}$$

The following Ansatz[4]

$$U(\xi) = 0\,, \qquad V(\xi) = \alpha\, \xi^0\,, \qquad X^a(\xi) = 0\,, \qquad \sqrt{|\gamma|}\, \gamma^{00} = 1\,, \tag{6.1.6}$$

α being some constant and ξ^0 being the worldvolume time coordinate (plus the above values Eq. (6.1.4) for the spacetime fields) representing the brane moving in a direction transverse to its worldvolume reduces all the equations

[4] Such an Ansatz was also discussed in Ref. [139] for probing the type IIB's KG wave, but was found to be inconsistent. Here such trouble is avoided because, contrary to Ref. [139], a massless p-brane is used.

of motion to only one which is not automatically satisfied,[5] *i.e.*

$$\partial_a \partial_a H = 16\pi G_N^{(d)} T_p e^{-2b\varphi(0)} \alpha^2 \delta(u) \delta(\vec{x}_8) \,. \tag{6.1.7}$$

Thus, only H feels the source and the gauge field seems to be self-supported. The solutions with $H = 0$ can be interpreted as vacua and can be described as homogeneous spaces [135, 160] (Hpp-waves). Actually, the presence of a covariantly constant null vector ensures that at least half of the supersymmetries will always be unbroken if we embed the above solutions in a supergravity theory (even for $H \neq 0$) but in some cases (the Kowalski-Glikman solutions [134,136,152,156]) there are Hpp-wave solutions that preserve all the supersymmetries. See [161] for a discussion on waves that preserve fractions of the supersymmetry.

It was recently shown [146] that the maximal amount of supersymmetry that can be preserved in a circle compactification of the KG10 solution [136] is 3/4 and the same thing holds for the 11-dimensional KG wave [134]. Although one would expect the same to happen in the $N = 2$ $d = 6, 5, 4$ KG-solutions, we are going to show that they are related by dimensional reduction. First of all, the susy preserving dimensional reduction is possible after a change of coordinates in which the dependence on the compact coordinate is removed at the expense of introducing a non-vanishing Sagnac connection. It turns out that in the new coordinates the Killing spinors are independent of the compact coordinates so that dimensional reduction will preserve all of them. Furthermore, the Sagnac connection becomes a KK vector that combines in the right way with the other vector fields present to cancel the matter multiples that arise in the two dimensional reductions involved.

6.1.2 Reduction of $KG6(2,0)$ to $KG5$

$N = (2,0), d = 6$ supergravity[6] consists of the metric $\hat{e}^{\hat{a}}{}_{\hat{\mu}}$, 2-form field $\hat{B}^-_{\hat{\mu}\hat{\nu}}$ with anti-self-dual field strength $\hat{H}^- = 3\partial \hat{B}^-$ and positive-chirality symplec-

[5] For $p = 0$ the charge μ_0 has to be set to zero in order to satisfy the equation of motion for that gauge field, but in the other cases the value of μ_p does not play any role.

[6] Our conventions are essentially those of Ref. [162] with some changes in the normalizations of the fields. In particular $\gamma_7 = \gamma_0 \cdots \gamma_5$, $\gamma_7^2 = +1$, $\epsilon^{012345} = +1$, $\gamma^{a_1 \cdots a_n} = \frac{(-1)^{[n/2]}}{(6-n)!} \epsilon^{a_1 \cdots a_n b_1 \cdots b_{6-n}} \gamma_{b_1 \cdots b_{6-n}} \gamma_7$. Positive and negative chiralities are defined by $\gamma_7 \hat{\psi}^\pm = \pm \hat{\psi}^\pm$.

tic Majorana-Weyl gravitino $\hat{\psi}^+_{\hat{\mu}}$. The bosonic equations of motion can be derived from the action

$$\hat{S} = \int d^6\hat{x}\sqrt{|\hat{g}|}\,[\hat{R} + \tfrac{1}{12}\hat{H}^2]\,, \tag{6.1.8}$$

imposing afterwards the anti-self-duality constraint ${}^\star\hat{H}^- = -\hat{H}^-$. The gravitino supersymmetry transformation rule is (for zero fermions)

$$\delta_{\hat{\epsilon}^+}\hat{\psi}^+_{\hat{a}} = \left(\hat{\nabla}_{\hat{a}} - \tfrac{1}{48}\,\hat{\not{H}}^-\hat{\gamma}_{\hat{a}}\right)\hat{\epsilon}^+\,. \tag{6.1.9}$$

This can be reduced to $N = 2, d = 5$ supergravity (metric $e^a{}_\mu$, *graviphoton* vector field $\mathcal{V}_\mu$ and symplectic-Majorana gravitino ψ_μ) coupled to a vector multiplet consisting of a gaugino (the 6th component of the 6-dimensional gravitino, a real scalar (the KK one) and a vector field $\mathcal{W}_\mu$. The vector fields $\mathcal{V}_\mu$ and $\mathcal{W}_\mu$ are combinations of scalars, the KK vector field that comes from the 6-dimensional metric A_μ and the vector field that comes from the 6-dimensional 2-form B_μ. The identification of the right combinations will be made by imposing consistency of the truncation.

Using the same techniques as in the reduction of $N = 2B, d = 10$ supergravity on a circle Ref. [94] one gets the 5-dimensional action

$$S = \int d^5x\sqrt{|g|}\,k\,[R - \tfrac{1}{4}k^2F^2(A) - \tfrac{1}{4}k^{-2}F^2(B) + \tfrac{\epsilon}{8\sqrt{|g|}}k^{-1}F(A)F(B)B]\,. \tag{6.1.10}$$

The truncation to pure supergravity involves setting $k = 1$ consistently, i.e. in such a way that its equation of motion is always satisfied. The k equation of motion with $k = 1$ (upon use of Einstein's equation) implies the constraint

$$F^2(B) = 2F^2(A)\,. \tag{6.1.11}$$

Let us introduce two linear combinations $\mathcal{F}(\mathcal{V}), \mathcal{G}(\mathcal{W})$ of the vector field strengths

$$\left\{\begin{array}{rcl}\mathcal{F}(\mathcal{V}) & = & \alpha F(A) + \beta F(B)\,,\\[2ex] \mathcal{G}(\mathcal{W}) & = & -\beta F(A) + \alpha F(B)\,,\end{array}\right. \tag{6.1.12}$$

with $\alpha^2 + \beta^2 = 1$. Substituting them into the above constraint we see that it is automatically satisfied with $\mathcal{G}(\mathcal{W}) = 0$ and $\beta^2 = 2\alpha^2$, so $\alpha = s(\alpha)/\sqrt{3}$ and $\beta = s(\beta)\sqrt{2}/\sqrt{3}$. These conditions reduce the equations of motion of A

and B to a single equation for $\mathcal{V}$. This equation and the resulting Einstein equation can be derived from the action

$$S = \int d^5x\sqrt{|g|}\,[R - \tfrac{1}{4}\mathcal{F}^2 + s(\alpha)\tfrac{\epsilon}{12\sqrt{3}\sqrt{|g|}}\mathcal{F}\mathcal{F}\mathcal{V}]\,, \tag{6.1.13}$$

which is that of the bosonic sector of $N = 2, d = 5$ supergravity [163]. The relative sign of α and β will be fixed by supersymmetry: using the decomposition

$$\begin{aligned} \hat{\gamma}^a &= \gamma^a \otimes \sigma^1\,, \qquad a = 0,1,2,3,4\,,\\ \hat{\gamma}^5 &= \mathbb{I} \otimes i\sigma^2\,, \\ \hat{\gamma}_7 &= \hat{\gamma}_0 \cdots \hat{\gamma}_5 = \mathbb{I} \otimes \sigma^3\,, \end{aligned} \tag{6.1.14}$$

where the γ^as are 5-dimensional gamma matrices satisfying $\gamma_0 \cdots \gamma_4 = \mathbb{I}$, using chirality, we can split the gravitino supersymmetry transformation rule into

$$\left\{\begin{aligned} \delta_\epsilon \hat{\psi}_a &= \left\{\nabla_a - \tfrac{1}{8\sqrt{2}}k^{-1}\, \not{F}(B)\gamma_a - \tfrac{1}{4}k\, \not{F}_a(A)\right\}\epsilon\,,\\ \delta_\epsilon \hat{\psi}_w &= \left\{\partial_w + \tfrac{1}{2}\, \not{\partial}\log k + \tfrac{1}{8}k\, \not{F}(A) - \tfrac{1}{8\sqrt{2}}k^{-1}\, \not{F}(B)\right\}\epsilon\,. \end{aligned}\right. \tag{6.1.15}$$

$\hat{\psi}_w$ is the 5-dimensional gaugino and its supersymmetry transformation has to be identically zero. This can be achieved by taking ϵ independent of w and identifying $s(\alpha) = s(\beta)$ so

$$\mathcal{G} \equiv s(\alpha)\left(\tfrac{1}{\sqrt{3}}F(B) - \sqrt{\tfrac{2}{3}}F(A)\right) \equiv 0\,. \tag{6.1.16}$$

It only remains the supersymmetry transformation law of $\hat{\psi}_a$ that becomes the 5-dimensional gravitino. Expressed in terms of the surviving vector field, it takes the right form[7] [163]

$$\delta_\epsilon \psi_a = \left\{\nabla_a - s(\alpha)\tfrac{1}{8\sqrt{3}}(\gamma^{bc}\gamma_a + 2\gamma^b g^c{}_a)\mathcal{F}_{bc}\right\}\epsilon\,. \tag{6.1.17}$$

[7] Actually, either the sign of the Chern-Simons term or the $\mathcal{F}$ term in the supersymmetry transformation rule in Ref. [163] is wrong. Choosing the sign of α we can make either of them coincide with those in Eqs. (6.1.10) and (6.1.17), but not both at the same time. A further check of these signs is provided by the reduction to $d = 4$: the consistency conditions for the truncation to pure $N = 2, d = 4$ supergravity coming from the action and the gaugino supersymmetry transformation rule are incompatible with the signs of Ref. [163] but fully compatible with ours.

The relation between 6- and 5-dimensional pure supergravity fields is

$$\left|\begin{array}{rcl} \hat{g}_{\underline{ww}} & = & -1\,, \\ \\ \hat{g}_{\mu\underline{w}} & = & \frac{s(\alpha)}{\sqrt{3}}\mathcal{V}_{\mu}\,, \end{array}\right. \qquad \left|\begin{array}{rcl} \hat{B}^{-}_{\mu\underline{w}} & = & \frac{s(\alpha)}{\sqrt{3}}\mathcal{V}_{\mu}\,, \\ \\ \hat{g}_{\mu\nu} & = & g_{\mu\nu} - \frac{1}{3}\mathcal{V}_{\mu}\mathcal{V}_{\nu}\,, \end{array}\right. \tag{6.1.18}$$

while the $\hat{B}^{-}_{\mu\nu}$ components can be found imposing anti-self-duality.

Now, let us consider the $KG6(2,0)$ solution [152] in canonical coordinates with $\hat{B}^{-}$ in a convenient gauge

$$KG6(2,0):\left\{\begin{array}{rcl} d\hat{s}^2 & = & 2du[dv + \frac{\lambda_6^2}{8}\vec{x}^{\,2}_{(4)}du] - d\vec{x}^{\,2}_{(4)}\,, \qquad \vec{x}_{(4)} \equiv (x,y,z,w)\,, \\ \\ \hat{B}^{-} & = & \lambda_6 du\wedge(zdw - xdy)\,. \\ \\ \hat{\epsilon} & = & \left[1 - \frac{\lambda_6}{4}\hat{\gamma}^{+23}\vec{x}_{(4)}\cdot\vec{\hat{\gamma}}\right]\exp\left(\frac{u\lambda_6}{4}\hat{\gamma}^{+23}\hat{\gamma}^{-}\right)\,\hat{\epsilon}^{(0)}\,, \end{array}\right. \tag{6.1.19}$$

Performing the coordinate transformations

$$\left\{\begin{array}{rcl} z & = & \cos\left(\frac{\lambda_6}{2}u\right)z' + \sin\left(\frac{\lambda_6}{2}u\right)w'\,, \\ \\ w & = & -\sin\left(\frac{\lambda_6}{2}u\right)z' + \cos\left(\frac{\lambda_6}{2}u\right)w'\,, \\ \\ v & = & v' + \frac{\lambda_6}{2}z'w'\,, \end{array}\right. \tag{6.1.20}$$

the solution takes the w'-independent form

$$KG6(2,0):\left\{\begin{array}{rcl} \hat{s}^2 & = & 2du[dv' + \frac{\lambda_6^2}{8}(x^2+y^2)du + \lambda_6 z'dw'] - d\vec{x}'^{\,2}_{(4)}\,, \\ \\ \hat{B}^{-} & = & \lambda_6 du\wedge(z'dw' - xdy)\,. \\ \\ \hat{\epsilon} & = & \left[1 + \frac{\lambda_6}{4}\hat{\gamma}^{+23}\left\{x\hat{\gamma}_2 + y\hat{\gamma}_3\right\}\right]\exp\left(\frac{u\lambda_6}{4}\left\{\hat{\gamma}^{45} + \hat{\gamma}^{+23}\hat{\gamma}^{-}\right\}\right)\hat{\epsilon}^{(0)}\,. \end{array}\right. \tag{6.1.21}$$

with $\vec{x}'_{(4)} \equiv (x,y,z',w')$.

It is easy to see that it satisfies the truncation conditions

$$\hat{g}_{\underline{ww}} = -1\,, \quad \hat{B}^{-}_{\mu\underline{w}} = \hat{g}_{\mu\underline{w}}\,, \quad \partial_{\underline{w}}\hat{\epsilon}^{+} = 0\,, \tag{6.1.22}$$

and, thus, it can be reduced to a solution of pure $N=2, d=5$ supergravity that turns out to be the maximally supersymmetric $KG5$ solution [152]:

$$KG5: \left\{ \begin{array}{rcl} ds^2 & = & 2du[dv' + \frac{\lambda_5^2}{24}(4z'^2 + x^2 + y^2)du] - d\vec{x}'^2_{(3)}\,, \\ \mathcal{F} & = & \lambda_5 du \wedge dz'\,, \qquad \lambda_5 = -s(\alpha)\sqrt{3}\lambda_6\,. \end{array} \right. \tag{6.1.23}$$

with $\vec{x}'_{(3)} \equiv (x, y, z')$.

6.1.3 Reduction of $KG5$ to $KG4$

The action Eq. (6.1.10) can be straightforwardly reduced to $d=4$ dimensions giving the action of $N=2, d=4$ supergravity (consisting of the metric, the *graviphoton* vector field V_μ and a gravitino) coupled to a vector multiplet (consisting of a vector W_μ and two real scalars k, l plus a gaugino) [164]. The two vectors will be combinations of the KK vector A_μ that comes from the metric and the vector B_μ that comes from the 5-dimensional vector $\mathcal{V}_\mu$. To determine the right combinations, we study the consistency of the truncation of the fields that belong for sure to the matter multiplet $k=1, l=0$ and the gaugino.

The action for the 4-dimensional bosonic fields is

$$\begin{aligned} S \; = & \int d^4x \sqrt{|g|}\, k\, \left\{ R + \tfrac{1}{2}k^{-2}(\partial l)^2 - \tfrac{1}{4}k^2 F^2(A) - \tfrac{1}{4}[F(B) + lF(A)]^2 \right. \\ & \left. + s(\alpha)\frac{k^{-1}l}{4\sqrt{3}\sqrt{|g|}}\epsilon[F(B) + lF(A) - 2A\partial l]^2 \right\}. \end{aligned} \tag{6.1.24}$$

Setting $k=1, l=0$ in the equations of motion of k and l we get two constraints:

$$\left\{ \begin{array}{rcl} 3F^2(A) + F^2(B) & = & 0\,, \\ \sqrt{3}F(A) - s(\alpha)^\star F(B) & = & 0\,. \end{array} \right. \tag{6.1.25}$$

The second constraint implies the first and is actually sufficient to identify the graviphoton and the matter vector field strengths[8] up to global,

[8] For $k=1, l=0$ only.

irrelevant, signs, that we fix arbitrarily

$$\left\{ \begin{array}{lcl} F(V) & = & \frac{1}{2}{}^{\star}F(A) - s(\alpha)\frac{\sqrt{3}}{2}F(B)\,, \\ F(W) & = & -\frac{\sqrt{3}}{2}{}^{\star}F(A) - s(\alpha)\frac{1}{2}F(B)\,. \end{array} \right. \tag{6.1.26}$$

Setting $F(W) = 0$ (which is consistent with the W_μ equation of motion) we get the action of (the bosonic sector of) pure $N = 2, d = 4$ supergravity (the Einstein-Maxwell action)

$$S = \int d^4x \sqrt{|g|}\,[R - \tfrac{1}{4}F^2(V)]\,. \tag{6.1.27}$$

We can see that this truncation is consistent with the supersymmetry transformation rules. The 5-dimensional matrices $\hat{\gamma}^{\hat{a}}$ decompose into 4-dimensional matrices as follows:

$$\hat{\gamma}^a = \gamma^a\,, \quad a = 0,1,2,3\,, \hat{\gamma}^4 = -i\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3\,. \tag{6.1.28}$$

The 5-dimensional symplectic-Majorana spinors are a pair of ordinary 4-component Dirac spinors related by the symplectic-Majorana constraint. Thus, in $d = 4$ we simply keep one of them, which will be unconstrained and decomposable, if necessary, into a pair of 4-dimensional Majorana spinors.

Now if the supersymmetry parameter is independent of the compactification direction y and we set $k = 1, l = 0$ in the y component of the gravitino transformation rule (which should become the gaugino transformation rule), we find that

$$\delta_\epsilon \hat{\psi}_y = \frac{i}{4\sqrt{3}}\, \not{F}(W)\epsilon\,. \tag{6.1.29}$$

and, so, the truncation $F(W) = 0$ is consistent with setting the gaugino to zero. The supersymmetry transformation rule of the surviving gravitino is

$$\delta_\epsilon \psi_a = \left[\nabla_a + \tfrac{1}{8}\, \not{F}(V)\gamma_a\right]\epsilon\,. \tag{6.1.30}$$

The relation between 5-dimensional and 4-dimensional fields that satisfy the truncation condition is

$$\left| \begin{array}{rcl} \hat{g}_{\underline{yy}} & = & -1\,, \\ 2\partial_{[\mu}\hat{g}_{\nu]\underline{y}} & = & \frac{-1}{4\sqrt{|g|}}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}(V)\,, \\ \hat{g}_{\mu\nu} & = & g_{\mu\nu} - \hat{g}_{\mu\underline{y}}\hat{g}_{\nu\underline{y}}\,, \end{array} \right| \begin{array}{rcl} \hat{\mathcal{V}}_{\underline{y}} & = & 0\,, \\ \hat{\mathcal{V}}_\mu & = & -s(\alpha)\frac{\sqrt{3}}{2}V_\mu\,, \\ & & \end{array} \tag{6.1.31}$$

Now, to apply these results to the $KG5$ solution Eq. (6.1.23) we first perform the change of coordinates

$$\left\{\begin{array}{rcl} x & = & \cos\left(\frac{\lambda_5}{2\sqrt{3}}u\right)x' + \sin\left(\frac{\lambda_5}{2\sqrt{3}}u\right)y'\,, \\ \\ y & = & -\sin\left(\frac{\lambda_5}{2\sqrt{3}}u\right)x' + \cos\left(\frac{\lambda_5}{2\sqrt{3}}u\right)y'\,, \\ \\ v' & = & v'' + \frac{\lambda_5}{2\sqrt{3}}x'y'\,, \end{array}\right. \tag{6.1.32}$$

that puts the $KG5$ solution in the y'-independent form

$$KG5: \left\{\begin{array}{rcl} ds^2 & = & 2du[dv'' + \frac{\lambda_5^2}{6}z'^2 du + \frac{\lambda_5}{\sqrt{3}}x'dy'] - d\vec{x}'^2_{(3)}\,, \quad \vec{x}'_{(3)} \equiv (x', y', z')\,, \\ \\ \mathcal{F} & = & \lambda_5 du \wedge dz'\,. \end{array}\right. \tag{6.1.33}$$

In this form, the $KG5$ solution just happens to satisfy the truncation condition that allows us to reduce it to a pure $N=2, d=4$ supergravity solutions that turns out to be the $KG4$ maximally supersymmetric spacetime [156], as promised

$$KG4: \left\{\begin{array}{rcl} ds^2 & = & 2du[dv'' + \frac{\lambda_4^2}{8}\vec{x}'^{\,2}_{(2)}\, du] - d\vec{x}'^2_{(2)}\,, \qquad \vec{x}'_{(2)} \equiv (x', z')\,, \\ \\ F & = & \lambda_4 du \wedge dz'\,, \qquad \lambda_4 = s(\alpha)\frac{2}{\sqrt{3}}\lambda_5\,. \end{array}\right. \tag{6.1.34}$$

At first sight it is surprising that in all cases the truncation condition can be satisfied, at least in a certain gauge. Actually, it is easy to see that it must happen by thinking in terms of oxidation of the lower-dimensional solutions: Since the $N=2, d=5$ theory can be reduced to $N=2, d=4$ supergravity coupled to a vector multiplet that can be consistently truncated, any solution of pure $N=2, d=4$ supergravity can be uplifted to a solution of the $N=2, d=5$ theory with the same, or bigger, amount of supersymmetry. Therefore, the $KG4$ solution can be uplifted to a maximally supersymmetric solution of the $N=2, d=5$ theory which turns out to be the $KG5$ solution in non-canonical coordinates. Essentially the same mechanism works in the oxidation of the $KG5$ solution to a maximally supersymmetric solution of $N=(2,0), d=6$ that turns out to be the $KG6(2,0)$.

Now it is clear that the same should happen in all cases: all solutions of pure $N=2, d=4$ supergravity must be related via dimensional reduction/oxidation to pure $N=2, d=5$ and $N=(2,0), d=6$ supergravity solutions that preserve the same amount of supersymmetry. In particular, maximally supersymmetric solutions of these three theories should be related. We have seen that this is true for the KG spacetimes and now we are going to study the $AdS_n \times S^m$ spacetimes.

6.2 Oxidation of Maximally Supersymmetric $d = 4, 5, 6$ $AdS_n \times S^m$ Spacetimes

6.2.1 Oxidation of the Robinson-Bertotti Solution

The Robinson-Bertotti solution [157] can be obtained either as a particular member of the Majumdar-Papapetrou family of solutions of the Einstein-Maxwell equations [165] or as the near-horizon limit of the extreme Reissner-Nordström black hole solution [166] and is given in its electric and magnetic versions by

$$\left\{ \begin{array}{rcl} ds^2 & = & R_2^2\, d\Pi^2_{(2)} - R_2^2\, d\Omega^2_{(2)}\,, \\ \\ F_{\chi\phi} & = & -2R_2 \mathrm{ch}\chi\,, \\ \\ F_{\theta\varphi} & = & 2R_2 \sin\theta\,, \end{array} \right. \tag{6.2.1}$$

with

$$\left\{ \begin{array}{rcl} d\Pi^2_{(2)} & \equiv & \mathrm{ch}^2\chi\, d\phi^2 - d\chi^2\,, \\ \\ d\Omega^2_{(2)} & \equiv & d\theta^2 + \sin^2\theta d\varphi^2\,, \end{array} \right. \tag{6.2.2}$$

The metric is that of the direct product of that of AdS_2 with radius R_2 in global coordinates $\phi \in [0, 2\pi)$, $\chi \in [0, \infty)$ and that of S^2 with radius R_2 in standard spherical coordinates $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi)$. It is known to be maximally supersymmetric in $N = 2, d = 4$ supergravity [166, 167] in both the electric and magnetic cases, since the whole $N = 2, d = 4$ supergravity is invariant under chiral/dual transformations.

Electric Case

Following the rules found in the previous section (with $s(\alpha) = +1$) and, further, *assuming that the compact coordinate* $y \in [0, 4\pi R_2)$ and using instead

$\psi = y/R_2$, we find the d=5 solution

$$\left\{ \begin{array}{rcl} d\hat{s}^2 & = & R_2^2\, d\Pi^2_{(2)} - (2R_2)^2\, d\Omega^2_{(3)}\,, \\ \\ \hat{\mathcal{F}}_{\chi\phi} & = & \sqrt{3}R_2\, \mathrm{ch}\chi\,, \end{array} \right. \tag{6.2.3}$$

with

$$d\Omega^2_{(3)} \equiv \tfrac{1}{4}\left[d\Omega^2_{(2)} + (d\psi + \cos\theta d\varphi)^2\right]\,, \tag{6.2.4}$$

which is the direct product of AdS_2 with radius R_2 in global coordinates $\phi \in [0, 2\pi)$, $\chi \in [0, \infty)$ and that of S^3 with radius $2R_2$ in Euler-angle coordinates $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi)$, $\psi \in [0, 4\pi)$. This solution is the near-horizon limit of a 5-dimensional extreme black hole and it is maximally supersymmetric [168].

Magnetic Case

Straightforward application of the oxidation rules, now with the compact coordinate in the range $y \in [0, 2\pi R_2)$ with $\eta = y/R_2$ and rescaling $\chi \to \chi/2$ leads us to

$$\left\{ \begin{array}{rcl} d\hat{s}^2 & = & (2R_2)^2\, d\Pi^2_{(3)} - R_2^2\, d\Omega^2_{(2)}\,, \\ \\ \hat{\mathcal{F}}_{\theta\varphi} & = & -\sqrt{3}R_2\, \sin\theta\,, \end{array} \right. \tag{6.2.5}$$

with

$$d\Pi^2_{(3)} \equiv \tfrac{1}{4}\left[d\Pi^2_{(2)} - (d\psi + \mathrm{sh}(\chi/2)d\phi)^2\right]\,, \tag{6.2.6}$$

the metric of AdS_3 in a form that suggests that this spacetime can be understood as an S^1 fibration over AdS_2.

The above 5-dimensional solution has the metric of $AdS_3 \times S^2$ which is the near-horizon limit of the extreme $d = 5$ string [153].

It has been observed many times that $N = 2, d = 5$ supergravity (its action and field content) is a theory that resembles very much $N = 1, d = 11$ supergravity [163]. One additional similarity is the presence of two maximally supersymmetric vacua ($AdS_4 \times S^7$ and $AdS_7 \times S^4$) which are respectively the near-horizon limits of the solutions that describe the extended objects of the theory: black hole and string in $d = 5$ and M2 and M5 branes in $d = 11$. However, it should be clear that we can obtain new $d = 5$ vacua from new $d = 4$ vacua, if they exist. As a matter of fact they do exist: the dyonic RB solutions which have both electric and magnetic components of the electromagnetic field and share the same $AdS_2 \times S^2$ metric.

Dyonic Case

The dyonic RB solution is given by

$$\left\{\begin{array}{rcl} ds^2 & = & R_2^2\, d\Pi^2_{(2)} - R_2^2\, d\Omega^2_{(2)}\,, \\ \\ F & = & -\dfrac{2}{R_2}\cos\xi dr\wedge dt + 2R_2\sin\xi\sin\theta d\theta\wedge d\varphi\,, \end{array}\right. \tag{6.2.7}$$

where now, for convenience, we use the following AdS_2 metric:

$$R_2^2\, d\Pi^2_{(2)} = \left(\frac{r}{R_2}\right)^2 dt^2 - \left(\frac{R_2}{r}\right)^2 dr^2\,. \tag{6.2.8}$$

This family of solutions, that includes the purely electric and magnetic cases that we have just seen, has another parameter apart from the radius R_2: the duality rotation angle ξ.

Following the oxidation rules we find a 5-dimensional family of maximally supersymmetric solutions

$$\left\{\begin{array}{rcl} d\hat{s}^2 & = & \left(\dfrac{r}{R_2}\right)^2 dt^2 - \left(\dfrac{R_2}{r}\right)^2 dr^2 - \\ & & -\left(dy - \dfrac{r}{R_2}\sin\xi dt + R_2\cos\xi\cos\theta d\varphi\right)^2 - R_2^2 d\Omega^2_{(2)}\,, \\ \\ \hat{\mathcal{F}} & = & \dfrac{\sqrt{3}}{R_2}\cos\xi\ dr\wedge dt - \sqrt{3}R_2\sin\xi\sin\theta\ d\theta\wedge d\varphi\,, \end{array}\right. \tag{6.2.9}$$

The explicit form of the Killing spinors in this case reads

$$\epsilon\ =\ \exp\left(-X\ \log\left(r\right)\right)\exp\left(t\ Y\right)\exp\left(\theta\ Z\right)\exp\left(-\tfrac{\varphi}{2}\ \gamma^{34}\right)\epsilon_{(0)}\ , \tag{6.2.10}$$

where

$$X\ =\ \tfrac{1}{2}\left[\sin\left(\xi\right)\ \gamma^{02}\ +\ \cos\left(\xi\right)\ \gamma^{0}\right]\ , \tag{6.2.11}$$

$$Y\ =\ \tfrac{1}{2}\left[\sin\left(\xi\right)\ \gamma^{12}\ +\ \cos\left(\xi\right)\ \gamma^{1}\ -\ \gamma^{01}\right]\ , \tag{6.2.12}$$

$$Z\ =\ \tfrac{1}{2}\left[\cos\left(\xi\right)\ \gamma^{24}\ +\ \sin\left(\xi\right)\ \gamma^{4}\right]\ , \tag{6.2.13}$$

After the coordinate redefinitions

$$\cos(\xi)\; t \to t\,, \qquad \frac{y}{R_2 \cos(\xi)} \to \psi\,, \tag{6.2.14}$$

takes the form

$$\left\{\begin{array}{rcl} d\hat{s}^2 & = & \left[\frac{r}{R_2} dt + R_2 \sin\xi\; (d\psi + \cos\theta d\varphi)\right]^2 - \left(\frac{R_2}{r}\right)^2 dr^2 - (2R_2)^2 d\Omega^2_{(3)}\,, \\ \\ \hat{\mathcal{F}} & = & \frac{\sqrt{3}}{R_2}\, dr \wedge dt - \sqrt{3} R_2 \sin\xi \sin\theta\; d\theta \wedge d\varphi\,. \end{array}\right. \tag{6.2.15}$$

If we set $R_2 = 1/2$, $\sin\xi = j$, $2t \to t$ and $r \to r^2$ we recover a solution that describes the near-horizon limit of the supersymmetric [154] rotating $d = 5$ black hole, given in [155, 169]. While it was known that in the zero-rotation limit $j = 0$ this solution has the metric of $AdS_2 \times S^3$, the result in the limiting case $j \to 1$ was unknown since it is a singular limit. However, by means of the inverse of the above coordinate transformations, the limit can be taken in such a way that the limiting metric, at $\xi = \pi/2$, is regular: $AdS_3 \times S^2$. Thus, the near-horizon limit of the $j = 1$ supersymmetric rotating black hole and the near-horizon limit of the string are identical.

Finally, let us mention the superalgebras associated to these vacua. As was pointed out in Ref. [155], the superalgebra associated to the solution (6.2.15), is $su(1,1|2) \oplus u(1)$ when $0 < j < 1$ and gets enhanced to $su(1,1|2) \oplus su(2)$ when $j = 0$. Combining this with the smooth $\xi = \pi/2$ limit for the family (6.2.9), one sees that the superalgebra associated to the $AdS_3 \times S^2$ has to be $su(1,1|2) \oplus sl(2,\mathbb{R})$ and not $sl(2,\mathbb{R}) \rtimes su(1,1|2)$ as was hinted at in Ref. [170].

6.2.2 Oxidation to $d = 6$

The oxidation of Eq. (6.2.9) gives, after rotation of the two isometric coordinates y, w by the angle associated to the 4-dimensional electric-magnetic duality ξ

$$\left\{\begin{array}{rcl} w & = & \cos\xi\eta + R_2 \sin\xi\psi\,, \\ \\ y & = & -\sin\xi\eta + R_2 \cos\xi\psi\,, \end{array}\right. \tag{6.2.16}$$

one recovers the solution

$$\left\{\begin{array}{lcl} d\hat{s}^2 & = & (2R_2)^2\, d\Pi^2_{(3)} - (2R_2)^2\, d\Omega^2_{(3)}\,, \\ \\ \hat{B}^- & = & \frac{r}{R_2} d\eta \wedge dt - R_2^2 \cos\theta d\varphi \wedge d\psi\,, \end{array}\right. \tag{6.2.17}$$

whose metric is that of $AdS_3 \times S^3$, the maximally supersymmetric solution which is the near-horizon limit of a self-dual string [153]. It is known that the uplifting of the near-horizon limit of the rotating $d = 5$ black hole gives, for any value of the rotation parameter, $AdS_3 \times S^3$ [171].

6.3 Conclusions

In this Chapter we have shown that the known supersymmetric vacua of the $d = 6$ $N = (2, 0)$, $d = 5$ $N = 2$ and $d = 4$ $N = 2$ supergravity are linked by dimensional reduction. Although this may come as a bit of a surprise when thinking in terms of dimensional reduction, it is quite obvious from the oxidation point of view: since all three theories have 8 supercharges and oxidation cannot reduce the number of preserved supersymmetries, a lower dimensional maximally supersymmetric solution must lift to a maximally supersymmetric solution.

From the supergravity point of view, the relations can hold because the dimensionally reduced theories can be truncated consistently to the minimal $N = 2$ supergravity, *i.e.* without any matter couplings. A subtle point in the dimensional reduction is that for the Killing spinors to survive the dimensional reduction, the Killing spinors must be independent of the compact coordinates. In a coordinate independent way, this means that there must be a Killing vector whose action on the Killing spinor vanishes, or put differently, there is a bosonic generator in the superalgebra associated to the solution [172], that is represented trivially on the supercharges. Actually, it is not difficult to see that from the superalgebra point of view, the relation between the $N = 2$ vacua was going to hold.

For definiteness let us consider the superalgebras associated to the $AdS_p \times S^q$ spacetimes (See table (6.1)), the analogous results for the KG-waves being obtainable by a Inönü-Wigner contraction on their $AdS \times S$ counterparts [173].[9] It is clear that the way to preserve supersymmetry is by em-

[9]The exception is of course the family of metrics in Eq. (6.2.9), when $\xi \neq 0, \pi/2$, since

Space	Theory	Solution	Superalgebra
$AdS_3 \times S^3$	$N = (2,0)$ $d = 6$	(6.2.17)	$su(1,1\|2) \oplus sl(2,\mathbb{R}) \oplus su(2)$
$AdS_3 \times S^2$	$N = 2$ $d = 5$	(6.2.5)	$su(1,1\|2) \oplus sl(2,\mathbb{R})$
$AdS_2 \times S^3$	$N = 2$ $d = 5$	(6.2.3)	$su(1,1\|2) \oplus su(2)$
Dyonic	$N = 2$ $d = 5$	(6.2.9)	$su(1,1\|2) \oplus u(1)$
$AdS_2 \times S^2$	$N = 2$ $d = 4$	(6.2.1)	$su(1,1\|2)$

Table 6.1: Solutions and their associated superalgebras.

bedding the generator of translations in the compactification direction, in the non-$su(1,1|2)$ part of the superalgebra. For the dimensional reduction from $d = 6$ to $d = 5$, there are basically 3 choices, corresponding to the 3 5-dimensional solutions given in Eqs. (6.2.3,6.2.5,6.2.9). For a further reduction to $d = 4$ there is basically one way to embed such a translation generator. Note that the chain of relations exposed in this letter is quite unique among the vacua: Had we considered for example the $AdS_3 \times S^3$ solution in the $d = 6$ $N = (4,0)$ supergravity, we would have had to conclude that, since the associated superalgebra is $su(1,1|2) \oplus su(1,1|2)$, there is no way to preserve the 16 supercharges in a circle compactification.

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its Penrose contraction has 2 more isometries.

Chapter 7

Geometric Construction of Killing Spinors and Supersymmetry Algebras in Homogeneous Spacetimes

Introduction

In theories with local supersymmetry (supergravity and superstring theories), the maximally supersymmetric solutions are usually identified as vacua, although vacua with less unbroken supersymmetry can also be interesting. The vacuum supersymmetry algebra, together with Wigner's theorem, determine which fields can be defined on it, their conserved (quantum) numbers, the particle spectrum etc. Thus, the supersymmetry algebra is a very important piece of information.

The calculation of the supersymmetry algebra of a solution (see, for example, Ref. [174]), involves the calculation of its Killing vectors and Killing spinors, and the computation of bilinears and Lie derivatives of the Killing spinors which can sometimes be difficult or tedious, since their functional form has no manifest geometrical meaning.

However, most known maximally supersymmetric solutions have the spacetime metric of some symmetric space that can be identified with a coset G/H main result is that, quite generally, the Killing spinor equation in maximally supersymmetric solutions can be put in the form

$$(d + u^{-1}du)\kappa = 0\,, \tag{7.0.1}$$

which, written in the form $u^{-1}d(u\kappa) = 0$ tells us that the Killing spinors are given by

$$\kappa = u^{-1}\kappa_0\,, \tag{7.0.2}$$

where κ_0 is a constant Killing spinor. u is a coset representative in the spinorial representation. Then, the bilinears $\bar{\kappa}\gamma^\mu\kappa$ can be easily decomposed into Killing vectors and the Lie-Lorentz derivative of the Killing spinors with respect to the Killing vectors are also easily computed. This simplifies dramatically the calculation of the supersymmetry algebras of these maximally supersymmetric solutions.

In Section 7.1 we give a extremely sketchy review of the theory of symmetric spaces needed to prove the above general result in the examples that will follow. In Section 7.2 we use the machinery just introduced to give a construction of the metric of several well-known maximally supersymmetric supergravity solutions (all of them corresponding to symmetric, but, in general, not maximally symmetric spacetimes) and to show how the general rule for the construction of the Killing spinors works in practice. We start with the simplest non-trivial example: AdS_4 in $N = 1, d = 4$ (AdS) supergravity (Section 7.2.1). Then we consider the next non-trivial example: the Robinson-Bertotti solution with geometry $AdS_2 \times S^2$ (Section 7.2.2) which we then generalize to other known maximally supersymmetric solutions with geometries of the type $AdS \times S$ (Section 7.2.3). Finally, we consider in Section 7.2 the last kind of known maximally supersymmetric solutions: the Kowalski-Glikman solutions with Hpp-wave geometries. Section 7.3 contains our conclusions perspectives for future work.

7.1 Symmetric Spaces

Let us consider[1] the $(p+q)$-dimensional Lie group G, its p-dimensional subgroup H and the q-dimensional space of right cosets $G/H = \{gH\}$. The Lie algebra $\mathfrak{g}$ of G is spanned by the generators T_I ($I = 1, \cdots, p+q$) with Lie algebra

$$[T_I, T_J] = f_{IJ}{}^K T_K\,. \tag{7.1.1}$$

[1] Two Physics-oriented general references are [175] and [176].

The Lie algebra of H is generated by the subalgebra $\mathfrak{h} \subset \mathfrak{g}$ spanned by the generators M_i $(i = 1, \cdots, p)$ with Lie brackets

$$[M_i, M_j] = f_{ij}{}^k M_k \,. \tag{7.1.2}$$

The vector subspace spanned by the remaining generators, denoted by P_a, $(a, b = 1, \cdots, q)$ is denoted by $\mathfrak{k}$ and, as vector spaces we have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{h}$. Exponentiating the generators of $\mathfrak{k}$ we can construct a coset representative $u(x) = u(x^1, \cdots, x^q)$. We will always construct the coset representative as a product of generic elements of the 1-dimensional subgroups generated by the P_as:

$$u(x) = e^{x^1 P_1} \cdots e^{x^q P_q} \,. \tag{7.1.3}$$

Under a left transformation $g \in G$ u transforms into another element of G which only becomes a coset representative $u(x')$ after a right transformation with an element $h \in H$, which is a function of g and x:

$$g u(x) = u(x') h \,. \tag{7.1.4}$$

The adjoint representation of $\mathfrak{g}$ has as representation space $\mathfrak{g}$ and can be defined by its action on its generators: for any $T \in \mathfrak{g}$

$$\Gamma_{\mathrm{Adj}}(T)(T_I) \equiv [T, T_I] \,, \quad \Rightarrow \Gamma_{\mathrm{Adj}}(T_I)^K{}_J = f_{IJ}{}^K \,. \tag{7.1.5}$$

Exponentiating the generators of the Lie algebra $\mathfrak{g}$ in the adjoint representation, we get the adjoint representation of the group G

$$\Gamma_{\mathrm{Adj}}(g(x)) = \exp\{x^I \Gamma_{\mathrm{Adj}}(T_I)\} \,. \tag{7.1.6}$$

that acts on the Lie algebra generators

$$T'_J = T_L \Gamma_{\mathrm{Adj}}(g)^L{}_J \,. \tag{7.1.7}$$

Actually, in any representation r, the adjoint action of G on $\mathfrak{g}$ is given by

$$\Gamma_r(g)\Gamma_r(T_I)\Gamma_r(g^{-1}) = \Gamma_r(T_J)\Gamma_{\mathrm{Adj}}(g)^J{}_I \,. \tag{7.1.8}$$

The *Killing metric* K_{IJ} is defined by

$$K_{IJ} \equiv \mathrm{Tr}[\Gamma_{\mathrm{Adj}}(T_I)\Gamma_{\mathrm{Adj}}(T_J)] \,, \tag{7.1.9}$$

and by construction it is invariant under the adjoint action of G, due to the cyclic property of the trace.

The homogeneous space G/H can be used to construct a symmetric space if the pair $(\mathfrak{k}, \mathfrak{h})$ is a *symmetric pair* satisfying

$$\begin{aligned} [\mathfrak{h}, \mathfrak{h}] &\subset \mathfrak{h}\,, \\ [\mathfrak{k}, \mathfrak{h}] &\subset \mathfrak{k}\,, \\ [\mathfrak{k}, \mathfrak{k}] &\subset \mathfrak{h}\,. \end{aligned} \tag{7.1.10}$$

The first condition is always satisfied for homogeneous spaces since $\mathfrak{h}$ is a subalgebra. The second condition says that $\mathfrak{k}$ is a representation of H. The two components of a symmetric pair are mutually orthogonal with respect to the Killing metric which is block-diagonal.

The first step is the construction of the left-invariant Lie-algebra valued *Maurer-Cartan 1-form V*

$$V \equiv -u^{-1}du = e^a P_a + \vartheta^i M_i\,, \tag{7.1.11}$$

that we have decomposed in *horizontal* e^a and *vertical* components ϑ^i. By construction, V satisfies the *Maurer-Cartan equations*

$$dV - V\wedge V = 0\,, \quad \Rightarrow \left\{ \begin{array}{lcl} de^a - \vartheta^i \wedge e^b f_{ib}{}^a & = & 0\,, \\ d\vartheta^i - \frac{1}{2}\vartheta^j \wedge \vartheta^k f_{jk}{}^i - \frac{1}{2} e^a \wedge e^b f_{ab}{}^i & = & 0\,. \end{array} \right. \tag{7.1.12}$$

The horizontal components e^a provide us with a co-frame for G/H. Under left multiplication by a constant element $g \in G$ $u(x') = gu(x)h^{-1}$, which implies for the Maurer-Cartan 1-form components

$$\left\{ \begin{array}{lcl} e^a(x') & = & (he(x)h^{-1})^a = \Gamma_{\mathrm{Adj}}(h)^a{}_b e^b(x)\,, \\ \vartheta^i(x') & = & (h\vartheta(x)h^{-1})^i + (h^{-1}dh)^i\,. \end{array} \right. \tag{7.1.13}$$

The second step to construct a symmetric space is the construction of the metric. With a symmetric bilinear form B_{ab} in $\mathfrak{k}$ we can construct a Riemannian metric

$$ds^2 \sim B_{ab} e^a \otimes e^b\,, \tag{7.1.14}$$

that will be invariant under the left action of G if B is:

$$f_{i(a}{}^{c}B_{b)c} = 0\,. \tag{7.1.15}$$

This is guaranteed if $B_{mn} = K_{mn}$, the projection on $\mathfrak{k}$ of the Killing metric, but sometimes this is singular and another one has to be used . The resulting Riemannian metric contains G in its isometry group (which could be bigger) and must admit $p+q$ Killing vector fields $k_{(I)}$. The Killing vectors $k_{(I)}$ and the so-called *H-compensator* $W_I{}^i$ are defined through the infinitesimal version of the transformation rule $gu(x) = u(x')h$ with

$$\begin{aligned} g &= 1+\sigma^I T_I\,,\\ h &= 1-\sigma^I W_I{}^i M_i\,, \\ x^{\mu\prime} &= x^\mu + \sigma^I k_{(I)}{}^\mu\,. \end{aligned} \tag{7.1.16}$$

Using the above equations into

$$u(x+\delta x) = u(x) + \sigma^I k_{(I)} u\,, \tag{7.1.17}$$

we get

$$T_I u = k_I u - u W_I{}^i M_i\,. \tag{7.1.18}$$

Acting with u^{-1} on the left and using the definitions of the adjoint action and the Maurer-Cartan 1-forms , we get

$$T_J \Gamma_{\mathrm{Adj}}(u^{-1})^J{}_I = -k_{(I)}{}^a P_a - (k_{(I)}{}^\mu \vartheta^i{}_\mu + W_I{}^i) M_i\,, \tag{7.1.19}$$

which, projected on the horizontal and vertical subspaces gives the following expressions for the tangent space components of the Killing vector fields and the H-compensator

$$k_{(I)}{}^a = -\Gamma_{\mathrm{Adj}}(u^{-1}(x))^a{}_I\,, \tag{7.1.20}$$

$$W_I{}^i = -k_{(I)}{}^\mu \vartheta^i{}_\mu - \Gamma_{\mathrm{Adj}}(u^{-1}(x))^i{}_I\,. \tag{7.1.21}$$

H-Covariant Derivatives

According to the second of Eqs. (7.1.13) the vertical components θ^i transform as an $\mathfrak{h}$-valued connection. In fact, comparing the Maurer-Cartan equation for the horizontal components e^a with the Cartan structure equation for the co-frame and (torsionless) spin connection

$$de^a - \omega^a{}_b \wedge e^b = 0\,, \tag{7.1.22}$$

we find that the spin connection is given by

$$\omega^a{}_b = \vartheta^i f_{ib}{}^a = \vartheta^i \Gamma_{\text{Adj}}(M_i)^a{}_b\,. \tag{7.1.23}$$

We use these results to define the *H-covariant derivative* that acts on any object that transforms contravariantly $\phi' = \Gamma_r(h)\phi$ or covariantly $\psi' = \psi\Gamma_r(h^{-1})$ (for instance, $u(x)$ itself) in the representation r of H:

$$\mathcal{D}_\mu\phi \equiv \partial_\mu\phi - \vartheta_\mu{}^i\Gamma_r(M_i)\phi\,, \qquad \mathcal{D}_\mu\psi \equiv \partial_\mu\psi + \psi\vartheta_\mu{}^i\Gamma_r(M_i) \tag{7.1.24}$$

In particular, the Maurer-Cartan equations tell us that

$$\mathcal{D}_{[\mu}e^a{}_{\nu]} = 0\,. \tag{7.1.25}$$

By definition, the Levi-Cività connection is given by

$$\Gamma_{\mu\nu}{}^a \equiv \mathcal{D}_\mu e^a{}_\nu\,. \tag{7.1.26}$$

Finally, let us introduce the *H-covariant Lie derivative with respect to the Killing vectors* $k_{(I)}$[2] on objects that transform contravariantly (ϕ) or covariantly y (ψ) in the representation r of H:

$$\mathbb{L}_{k_{(I)}}\phi \equiv \mathcal{L}_{k_{(I)}}\phi + W_I{}^i\Gamma_r(M_i)\phi\,, \quad \mathbb{L}_{k_{(I)}}\psi \equiv \mathcal{L}_{k_{(I)}}\psi - \psi W_I{}^i\Gamma_r(M_i)\,. \tag{7.1.27}$$

[2] H-covariant Lie derivatives can be defined with respect to any vector, although the Lie bracket property Eq. (7.1.28) is only satisfied for Killing vectors. The spinorial Lie derivative [177–179] or the Lie-Lorentz derivative that naturally appear in calculations of supersymmetry algebras [174,180] can actually be seen as particular examples of this more general operator (see e.g. Ref. [181]), and, actually, are identical objects when acting on Killing spinors of maximally supersymmetric spacetimes, as we are going to show.

This Lie derivative has, among other properties

$$[\mathbb{L}_{k_{(I)}}, \mathbb{L}_{k_{(J)}}] = \mathbb{L}_{[k_{(I)}, k_{(J)}]}, \tag{7.1.28}$$

$$\mathbb{L}_{k_{(I)}} e^a = 0 \tag{7.1.29}$$

$$\mathbb{L}_{k_{(I)}} u = \mathcal{L}_{k_{(I)}} u - u W_I{}^i M_i = T_I u, \tag{7.1.30}$$

where the last property follows from Eqs. (7.1.21) and (7.1.18).

7.2 Killing Spinors in Symmetric Spacetimes

Most maximally supersymmetric solutions of supergravity theories have the metric of some symmetric spacetime. In some cases (Minkowski and AdS) the spacetime is also maximally symmetric but in other cases ($AdS \times S$ and KG spacetimes) it is not, but we can always use the procedure explained in the previous section to construct the metric, spin connection and Killing vectors. We are going to see, example by example, that, when we construct in that way the metric, the Killing spinor equation always takes the form Eq. (7.0.1). It is, nevertheless, convenient to give a brief overview of how we arrive to the general result. Then, we are going to show how the general result can be exploited to calculate the commutators of the supersymmetry algebra.

In all supergravity theories, the Killing spinor equation is of the the form

$$(\nabla_\mu + \Omega_\mu)\kappa = 0, \tag{7.2.1}$$

where the form of Ω depends on specific details of the theory. Multiplying by dx^μ, it takes the form

$$\left(d - \tfrac{1}{4}\omega_{ab}\gamma^{ab} + \Omega\right)\kappa = 0. \tag{7.2.2}$$

If we construct the symmetric space as in the previous section, then the spin connection 1-form ω_{ab} is given by Eq. (7.1.23) and takes values in the vertical Lie subalgebra $\mathfrak{h}$. Further,

$$\Gamma_s(M_i) \equiv \tfrac{1}{4} f_{ia}{}^b \gamma_b{}^a, \tag{7.2.3}$$

provides a (spinorial) representation of $\mathfrak{h}$ and the Killing spinor equation becomes

$$\left(d - \vartheta^i \Gamma_s(M_i) + \Omega\right) \kappa = 0\,. \tag{7.2.4}$$

In all the cases that we are going to examine

$$\Omega = -e^a \Gamma_s(P_a)\,, \tag{7.2.5}$$

where the matrices $\Gamma_s(P_a)$ are products of a number of Dirac gamma matrices (and, possibly, of other matrices in extended supergravities). Thus, on account of the definition of the Maurer-Cartan 1-forms Eq. (7.1.11), the Killing spinor equation can be written in the form Eq. (7.0.1)

$$\left(d - e^a \Gamma_s(P_a) - \vartheta^i \Gamma_s(M_i)\right) \kappa = \left(d + \Gamma_s(u^{-1}) d\Gamma_s(u)\right) \kappa = 0\,, \tag{7.2.6}$$

with

$$\Gamma_s(u) = e^{x^1 \Gamma_s(P_1)} \cdots e^{x^q \Gamma_s(P_q)}\,, \tag{7.2.7}$$

and the solution can be written in the form

$$\kappa^\alpha = \Gamma_s(u^{-1})^\alpha{}_\beta \kappa_0{}^\beta\,, \tag{7.2.8}$$

for an arbitrary constant spinor $\kappa_0{}^\beta$ (we have written explicitly the spinor indices here). Since there will be as many independent Killing spinors as components has a real spinor[3], we can use a spinorial index α to label a basis of Killing spinors:

$$\kappa_{(\alpha)}{}^\beta = \Gamma_s(u^{-1})^\beta{}_\alpha\,. \tag{7.2.9}$$

Killing spinors and Killing vectors are used to find the supersymmetry algebra of supergravity backgrounds (see, e.g. [155,174,180]). Killing spinors are related to supercharges and Killing vectors to bosonic charges. The anticommutator of two supercharges gives bosonic charges and, correspondingly the bilinears $-i\bar{\kappa}\gamma^\mu\kappa$ of Killing spinors are Killing vectors. To calculate the anticommutator of any two supercharges $\{Q_{(\alpha)}, Q_{(\beta)}\}$ associated to the Killing spinors $\kappa_{(\alpha)}$ we have to decompose the bilinears into linear combinations of the Killing vectors $k_{(I)}$

$$-i\bar{\kappa}_{(\alpha)}\gamma^\mu \kappa_{(\beta)} \partial_\mu = c_{\alpha\beta}{}^I k_{(I)}\,, \tag{7.2.10}$$

[3] We are considering only Majorana spinors.

finding the coefficients $c_{\alpha\beta}{}^{I}$. Now, using the above general form of the Killing spinors, the bilinears take the form

$$-i\bar{\kappa}_{(\alpha)}\gamma^{\mu}\kappa_{(\beta)}\partial_{\mu} = -i\Gamma_s(u^{-1})_{\alpha}{}^{\gamma}\mathcal{C}_{\gamma\delta}(\gamma^a)^{\delta}{}_{\epsilon}\Gamma_s(u^{-1})^{\epsilon}{}_{\beta}\,, \tag{7.2.11}$$

where $\mathcal{C}$ is the charge conjugation matrix $\mathcal{C}^{-1}\gamma^{a\,T}\mathcal{C} = -\gamma^a$. Now, in most cases[4],the matrices γ^a happen to be proportional to the the dual[5] P^a of a Lie algebra generator P_a $\Gamma_s(P^a)$

$$\gamma^a = \mathcal{S}\Gamma_s(P^a)\,, \tag{7.2.12}$$

for some matrix $\mathcal{S}$ that depends on the case we are considering. The combination $\tilde{\mathcal{C}} \equiv \mathcal{C}\mathcal{S}$ acts as a charge conjugation matrix in the subspace spanned by the horizontal generators in the spinorial representation[6]

$$\tilde{\mathcal{C}}^{-1}\Gamma_s(P^a)^T\tilde{\mathcal{C}} = -\Gamma_s(P^a)\,, \tag{7.2.13}$$

so

$$\Gamma_s(u^{-1})^T\mathcal{C}\gamma^a = \Gamma_s(u^{-1})^T\tilde{\mathcal{C}}\Gamma_s(P^a) = \tilde{\mathcal{C}}\Gamma_s(u)\Gamma_s(P^a)\,. \tag{7.2.14}$$

and, thus,

$$-i\bar{\kappa}_{(\alpha)}\gamma^{\mu}\kappa_{(\beta)}\partial_{\mu} = -i\tilde{\mathcal{C}}_{\alpha\gamma}\Gamma_s(u)^{\gamma}{}_{\delta}\Gamma_s(P^a)^{\delta}{}_{\epsilon}\Gamma_s(u^{-1})^{\epsilon}{}_{\beta}e_a\,. \tag{7.2.15}$$

In this expression we can recognize $uP^a u^{-1}$ in the spinorial representation, which is the coadjoint action of the coset element u on P^a

$$-i\bar{\kappa}_{(\alpha)}\gamma^{\mu}\kappa_{(\beta)}\partial_{\mu} = -i\tilde{\mathcal{C}}_{\alpha\gamma}\Gamma_s(T^I)^{\gamma}{}_{\beta}\Gamma_{\text{Adj}}(u^{-1})^a{}_I e_a = -i\tilde{\mathcal{C}}_{\alpha\gamma}\Gamma_s(T^I)^{\gamma}{}_{\beta}k_{(I)}\,,$$

where we have used Eq. (7.1.20). The superalgebra structure constants $c_{\alpha\beta}{}^{I}$ can be readily identified with $-i\tilde{\mathcal{C}}_{\alpha\gamma}\Gamma_s(T^I)^{\gamma}{}_{\beta}$.

To complete all the commutation relations of the supersymmetry algebra, we need the commutators of the bosonic charges and the supercharges, which are determined by the spinorial or Lie-Lorentz derivative of the Killing vectors on the Killing spinors $\mathbb{L}_{k_{(I)}}\kappa_{(\alpha)}$ [174, 180], since this operation preserves the

[4]The exception seems to be the Kowalski-Glikman H*pp*-wave spacetimes.

[5]It is always possible to find the dual of a representation that uses (unitary) gamma matrices.

[6]We thank P. Meessen for pointing this out to us.

supercovariant derivative (at least in the ungauged supergravities that we are going to consider) and transforms Killing spinors into Killing spinors

$$\mathbb{L}_{k_{(I)}}\kappa_{(\alpha)} = c_{\alpha I}{}^{\beta}\kappa_{(\beta)}\,, \quad \Rightarrow [Q_{(\alpha)}, P_{(I)}] = c_{\alpha I}{}^{\beta}Q_{(\beta)}\,. \tag{7.2.16}$$

The Lie-Lorentz derivative acting on a (contravariant) spinor ψ is given by [178, 179]

$$\mathbb{L}_{k_{(I)}}\psi = k_{(I)}{}^{\mu}\nabla_{\mu}\psi + \tfrac{1}{4}\nabla_{a}k^{b}_{(I)}\gamma^{a}{}_{b}\psi\,. \tag{7.2.17}$$

On a symmetric space G/H,

$$\begin{aligned} k_{(I)}{}^{\mu}\nabla_{\mu}\psi &= k_{(I)}{}^{\mu}\partial_{\mu}\psi - k_{(I)}{}^{\mu}\vartheta^{i}{}_{\mu}\Gamma_{s}(M_{i})\psi\,, \\ \nabla_{\mu}k^{b}_{(I)} &= \partial_{\mu}k^{b}_{(I)} - \vartheta^{i}{}_{\mu}f_{ic}{}^{b}k^{c}_{(I)}\,. \end{aligned} \tag{7.2.18}$$

Furthermore

$$\begin{aligned} \partial_{\mu}k_{(I)}{}^{b} &= -\partial_{\mu}\Gamma_{\mathrm{Adj}}(u^{-1})^{b}{}_{I} \\ &= \Gamma_{\mathrm{Adj}}(u^{-1})^{b}{}_{J}\partial_{\mu}\Gamma_{\mathrm{Adj}}(u)^{J}{}_{K}\Gamma_{\mathrm{Adj}}(u^{-1})^{K}{}_{I} \\ &= -V^{J}{}_{\mu}f_{JK}{}^{b}\Gamma_{\mathrm{Adj}}(u^{-1})^{K}{}_{I} \\ &= -e^{a}{}_{\mu}f_{ai}{}^{b}\Gamma_{\mathrm{Adj}}(u^{-1})^{i}{}_{I} - \vartheta^{i}{}_{\mu}f_{ic}{}^{b}\Gamma_{\mathrm{Adj}}(u^{-1})^{c}{}_{I} \\ &= e^{a}{}_{\mu}f_{ia}{}^{b}\Gamma_{\mathrm{Adj}}(u^{-1})^{i}{}_{I} + \vartheta^{i}{}_{\mu}f_{ic}{}^{b}k_{(I)}{}^{c}\,, \end{aligned} \tag{7.2.19}$$

so

$$\tfrac{1}{4}\nabla_{a}k^{b}_{(I)}\Gamma^{a}{}_{b} = \tfrac{1}{4}f_{ia}{}^{b}\Gamma_{\mathrm{Adj}}(u^{-1})^{i}{}_{I}\gamma^{a}{}_{b} = -\Gamma_{\mathrm{Adj}}(u^{-1})^{i}{}_{I}\Gamma_{s}(M_{i})\,, \tag{7.2.20}$$

and

$$\begin{aligned} \mathbb{L}_{k_{(I)}}\psi &= k_{(I)}{}^{\mu}\partial_{\mu}\psi - k_{(I)}{}^{\mu}\vartheta^{i}{}_{\mu}\Gamma_{s}(M_{i})\psi - \Gamma_{\mathrm{Adj}}(u^{-1})^{i}{}_{I}\Gamma_{s}(M_{i})\psi \\ &= \mathcal{L}_{k_{(I)}}\psi + W^{i}_{I}\Gamma_{s}(M_{i})\psi\,. \end{aligned} \tag{7.2.21}$$

Then, the Lie-lorentz derivative coincides withe the H-covariant Lie derivative. On the inverse coset representative

$$\mathbb{L}_{k_{(I)}}\Gamma_{s}(u^{-1}) = -\Gamma_{s}(u^{-1})[\mathbb{L}_{k_{(I)}}\Gamma_{s}(u)]\Gamma_{s}(u^{-1}) = -\Gamma_{s}(u^{-1})\Gamma_{s}(T_{I})\,, \tag{7.2.22}$$

on account of Eq. (7.1.30), which implies the commutators

$$[Q_{(\alpha)}, T_{I}] = -Q_{(\beta)}\Gamma_{s}(T_{I})^{\beta}{}_{\alpha}\,. \tag{7.2.23}$$

7.2.1 AdS_4 in $N=1, d=4$ AdS Supergravity

AdS_4 is the maximally supersymmetric vacuum of $N=1, d=4$ AdS supergravity: the integrability conditions of the Killing spinor equations vanish identically, which implies that 4 independent solutions exist. They are not hard to find (see e.g. Ref [182]), but the expressions one gets in most coordinate systems are difficult to make sense of and they are difficult to work with to find supersymmetry algebras.

AdS_4 can be identified with the coset $SO(2,3)/SO(1,3)$. We introduce $SO(2,3)$ indices $\hat{a}, \hat{b}, \cdots = -1, 0, 1, 2, 3$. The metric is $\hat{\eta}^{\hat{a}\hat{b}} = \mathrm{diag}(++---)$ and $\mathfrak{g} = so(2,3)$ the Lie algebra of $SO(2,3)$ can be written in the general form

$$\left[\hat{M}_{\hat{a}\hat{b}}, \hat{M}_{\hat{c}\hat{d}}\right] = -\hat{\eta}_{\hat{a}\hat{c}}\hat{M}_{\hat{b}\hat{d}} - \hat{\eta}_{\hat{b}\hat{d}}\hat{M}_{\hat{a}\hat{c}} + \hat{\eta}_{\hat{a}\hat{d}}\hat{M}_{\hat{b}\hat{c}} + \hat{\eta}_{\hat{b}\hat{c}}\hat{M}_{\hat{a}\hat{d}}\,. \tag{7.2.24}$$

We can now perform a $1+4$ splitting of the indices $\hat{a} = (-1, a)$, $a = 0, 1, 2, 3$ and define a new basis

$$\hat{M}_{ab} = M_{ab}\,, \qquad \hat{M}_{a-1} = -g^{-1}P_a\,, \tag{7.2.25}$$

where we have introduced the dimensionful parameter g related to the AdS_4 radius R and to the cosmological constant Λ by

$$R = 1/g = \sqrt{-3/\Lambda}\,. \tag{7.2.26}$$

In terms of the new basis, the $so(2,3)$ algebra reads

$$\begin{aligned} [M_{ab}, M_{cd}] &= -\eta_{ac}M_{bd} - \eta_{bd}M_{ac} + \eta_{ad}M_{bc} + \eta_{bc}M_{ad}\,, \\ [P_c, M_{ab}] &= -2P_{[a}\eta_{b]c}\,, \qquad [P_a, P_b] = -g^2 M_{ab}\,. \end{aligned} \tag{7.2.27}$$

The M_{ab}s generate the subalgebra $\mathfrak{h} = so(1,3)$ of the Lorentz subgroup. The complement is $\mathfrak{k} = \{P_a\}$ and the above commutation relations tell us that we have a symmetric pair. Following the general recipe, we construct the coset representative

$$u(x) = e^{x^3 P_3} e^{x^2 P_2} e^{x^1 P_1} e^{x^0 P_0}\,, \tag{7.2.28}$$

and the Maurer-Cartan 1-forms e^a, that we are going to use as Vierbeins are[7]

$$\begin{array}{ll} e^0 = -dx^0 \,, & e^2 = -\cos x^0 \mathrm{ch}\, x^1 dx^2 \,, \\ e^1 = -\cos x^0 dx^1 \,, & e^3 = -\cos x^0 \mathrm{ch}\, x^1 \,\mathrm{ch}\, x^2 dx^2 \,. \end{array} \tag{7.2.30}$$

and using the Killing metric $(+---)$ we get the AdS_4 metric in somewhat unusual coordinates

$$ds^2 = (dx^0)^2 - \cos^2 x^0 \{(dx^1)^2 + \mathrm{ch}^2 x^1 [(dx^2)^2 + \mathrm{ch}^2 x^1 (dx^3)^2\} \,. \tag{7.2.31}$$

We do not need the explicit form of the vertical 1-forms ϑ^{ab}, but we need to know how they enter the spin connection. According to the general formula Eq. (7.1.23)

$$\omega^a{}_b = \tfrac{1}{4} e^{cd} f_{cd\,-1b}{}^{-1a} = \tfrac{1}{2} e \vartheta^{ac} \eta_{cb} \,. \tag{7.2.32}$$

The Killing spinor equation is

$$(d - \tfrac{1}{4}\omega_{ab}\gamma^{ab} - \tfrac{ig}{2}\gamma_a e^a)\kappa = 0 \,, \tag{7.2.33}$$

and takes immediately the form of Eq. (7.0.1) with

$$\Gamma_s(P_a) = \tfrac{ig}{2}\gamma_a \,, \qquad \Gamma_s(M_{ab}) = \tfrac{1}{2}\gamma_{ab} \,, \tag{7.2.34}$$

and the Killing spinors are of the general form $\kappa_{(\alpha)}{}^\beta = (u^{-1})^\beta{}_{(\alpha)}$.

We define the dual generators $\Gamma_s(P^a)$ by

$$\begin{array}{lll} \mathrm{Tr}\,[\Gamma_s(P^a)\Gamma_s(P_b)] = \delta^a{}_b \,, & \Rightarrow & \Gamma_s(P^a) = \frac{-i}{2g}\gamma^a \,, \\ \mathrm{Tr}\,[\Gamma_s(M^{ab})\Gamma_s(P_{cd})] = \delta^{ab}{}_{cd} \,, & \Rightarrow & \Gamma_s(M^{ab}) = -\frac{1}{2}\gamma^{ab} \,. \end{array} \tag{7.2.35}$$

The bilinears are, then $(\mathcal{S} = 1)$

$$\begin{aligned} -i\bar{\kappa}_{(\alpha)}\gamma^a \kappa_{(\beta)} e_a &= 2g\Gamma_s(u^{-1})^T \mathcal{C}\Gamma_s(P^a)\Gamma_s(u^{-1})e_a \\ &= g\mathcal{C}\Gamma_s(\hat{M}^{\hat{b}\hat{c}})\Gamma_{\mathrm{Adj}}(u^{-1})^a{}_{\hat{b}\hat{c}} e_a \\ &= g\mathcal{C}\Gamma_s(\hat{M}^{\hat{b}\hat{c}}) k_{(\hat{b}\hat{c})} \,, \end{aligned} \tag{7.2.36}$$

[7]In this and similar calculations one has to use the formula

$$e^{xX} Y e^{-xX} = \cos xY + \sin xZ \,, \tag{7.2.29}$$

where $[X,Y] = Z \,,\ \ [Y,Z] = X \,,\ \ [Z,X] = Y$.

and the anticommutator of the supercharges takes the well-known form

$$\{Q_{(\alpha)}, Q_{(\beta)}\} = g[\mathcal{C}\Gamma_s(\hat{M}^{\hat{a}\hat{b}})]_{\alpha\beta}\hat{M}_{\hat{a}\hat{b}} = -i(\mathcal{C}\gamma^a)_{\alpha\beta}P_a - \tfrac{g}{2}(\mathcal{C}\gamma^{ab})_{\alpha\beta}M_{ab}\,,$$

that reduces to the Poincaré supersymmetry algebra in the $g \to 0$ limit.

The commutators $[Q_{(\alpha)}, \hat{M}_{\hat{a}\hat{b}}]$ are given by the general formula (7.2.23):

$$[Q_{(\alpha)}, \hat{M}_{\hat{a}\hat{b}}] = -Q_{(\beta)}\Gamma_s(\hat{M}_{\hat{a}\hat{b}})^{\beta}{}_{\alpha}\,. \tag{7.2.37}$$

The generalization to higher dimensions[8] and to spheres, described as cosets $SO(n+1)/SO(n)$ is evident. As a matter of fact, the coset structure underlies the calculation of Killing spinors in S^n of Ref. [182] but only after this is realized the calculation of bilinears etc. becomes really simple.

7.2.2 The Robinson-Bertotti Solution in $N = 2, d = 4$ Supergravity

The Robinson-Bertotti solution of $N = 2, d = 4$ supergravity [157] can be obtained as the near-horizon limit of the extreme Reissner-Nordström black hole and is known to be maximally supersymmetric [166, 167], although, to the best of our knowledge, no explicit expression of its 8 real Killing spinors is available in the literature. The metric is that of the direct product of that of AdS_2 with radius R_2 and that of S^2 with radius R_2

$$\left\{\begin{array}{rcl} ds^2 & = & R_2^2\, d\Pi^2_{(2)} - R_2^2\, d\Omega^2_{(2)}\,, \\ \\ F & = & -\frac{2}{R_2}\omega_{AdS_2}\,, \end{array}\right. \tag{7.2.38}$$

where $d\Pi^2_{(2)}$ stands for the metric of the AdS_2 spacetime of unit radius, $d\Omega^2_{(2)}$ for the metric of the unit 2-sphere S^2 and ω_{AdS_2} for the volume 2-form of radius R_2. Both AdS_2 and S^2 are symmetric spacetimes $SO(2,1)/SO(2)$ and $SO(3)/SO(2)$ and we can construct them using the procedure explained in Section 7.1.

The Lie algebra of $SO(2,1)$ can be written in the form

$$[T_I, T_J] = -\epsilon_{IJK}\mathsf{Q}^{KL}T_L\,, \quad I, J, \cdots = 1, 2, 3,\,, \quad \mathsf{Q} = \mathrm{diag}\,(++-)\,, \tag{7.2.39}$$

[8] Maximally supersymmetric AdS vacua arise in gauged supergravities in $d \leq 7$.

and the Lie algebra of $SO(3)$ can be written in the form

$$[\tilde{T}_I, \tilde{T}_J] = -\epsilon_{IJK}\tilde{T}_K\,, \quad I, J, \dots = 1,2,3,\,. \tag{7.2.40}$$

We choose the subalgebra $\mathfrak{h}$ to be generated by T_1 and $\tilde{T}_3$ so $\mathfrak{k}$ is generated by T_2, T_3 and $, \tilde{T}_1, \tilde{T}_2$. We perform the following redefinitions

$$\begin{aligned} T_1 &= M_1\,, & \tilde{T}_1 &= R_2 P_3\,, \\ T_2 &= R_2 P_1\,, & \tilde{T}_2 &= R_2 P_2\,, \\ T_3 &= R_2 P_0\,, & \tilde{T}_3 &= M_2\,, \end{aligned} \tag{7.2.41}$$

and the coset representative is the product of two mutually commuting coset representatives $u, \tilde{u}$ with

$$u = e^{R_2\phi P_0} e^{R_2\chi P_1}\,, \qquad \tilde{u} = e^{R_2\varphi P_3} e^{R_2(\theta - \frac{\pi}{2})P_2}\,. \tag{7.2.42}$$

We get

$$\begin{aligned} e^0 &= -R_2 \mathrm{ch}\,\chi\, d\phi\,, & e^2 &= -R_2 d\theta\,, \\ e^1 &= -R_2 d\chi\,, & e^3 &= -R_2 \sin\theta d\varphi\,, \\ \vartheta^1 &= -\mathrm{sh}\,\chi\, d\phi\,, & \vartheta^2 &= -\cos\theta d\varphi\,, \end{aligned} \tag{7.2.43}$$

that lead, using $B = \mathrm{diag}\,(+ - --)$ to the above $AdS_2 \times S^2$ metric with

$$\left\{ \begin{aligned} d\Pi^2_{(2)} &\equiv \mathrm{ch}^2\chi\, d\phi^2 - d\chi^2\,, \\ d\Omega^2_{(2)} &\equiv d\theta^2 + \sin^2\theta d\varphi^2\,. \end{aligned} \right. \tag{7.2.44}$$

Contracting with dx^μ the $N = 2, d = 4$ Killing spinor equation

$$(\nabla_\mu + \tfrac{1}{8}\, \not{F} \gamma_\mu \sigma^2)\kappa = 0\,, \tag{7.2.45}$$

we immediately see that it takes the form

$$[d + (u\tilde{u})^{-1} d(u\tilde{u})] = 0\,, \tag{7.2.46}$$

where the Lie algebra generators are represented by

$$\begin{array}{rclrcl} \Gamma_s(P_0) & = & \frac{1}{2R_2}\gamma^1\sigma^2\,, & \Gamma_s(P_2) & = & \frac{1}{2R_2}\gamma^0\gamma^1\gamma^3\sigma^2\,, \\ \Gamma_s(P_1) & = & -\frac{1}{2R_2}\gamma^0\sigma^2\,, & \Gamma_s(P_3) & = & \frac{1}{2R_2}\gamma^0\gamma^1\gamma^2\sigma^2\,, \\ \Gamma_s(M_1) & = & \frac{1}{2}\gamma^0\gamma^1\,, & \Gamma_s(M_2) & = & \frac{1}{2}\gamma^2\gamma^3\,. \end{array} \tag{7.2.47}$$

The Killing spinors are, then

$$\kappa = (u\tilde{u})^{-1}\kappa_0 = e^{-\frac{1}{2}\phi\gamma^1\sigma^2}e^{\frac{1}{2}\chi\gamma^0\sigma^2}e^{-\frac{1}{2}\varphi\gamma^0\gamma^1\gamma^2\sigma^2}e^{-\frac{1}{2}(\theta-\frac{\pi}{2})\gamma^0\gamma^1\gamma^3\sigma^2}\kappa_0\,. \tag{7.2.48}$$

Let us now consider the bilinears $-i\bar{\kappa}\gamma^\mu\kappa$ and define the duals $\Gamma_s(P^a)$ by

$$\mathrm{Tr}[\Gamma_s(P^a)\Gamma_s(P_b)] = \delta^a{}_b\,. \tag{7.2.49}$$

Then

$$\gamma^a = -\tfrac{4}{R_2}\mathcal{S}\Gamma_s(P^a)\,, \qquad \mathcal{S} = \gamma^0\gamma^1\sigma^2\,, \tag{7.2.50}$$

and we can see that the modified charge conjugation matrix $\tilde{\mathcal{C}} = \mathcal{C}\mathcal{S}$ has the required property

$$\tilde{\mathcal{C}}^{-1}\Gamma_s(P_a)^T\tilde{\mathcal{C}} = -\Gamma_s(P_a)\,, \;\; \Rightarrow (u\tilde{u})^{-1\,T}\tilde{\mathcal{C}} = \tilde{\mathcal{C}}u\tilde{u}\,, \tag{7.2.51}$$

that allows us to express the bilinears in the form

$$-i\bar{\kappa}_{(\alpha i)}\gamma^a\kappa_{(\beta j)} = \tfrac{4i}{R_2}\left\{\tilde{\mathcal{C}}[\Gamma_s(T^I)k_{(I)} + \Gamma_s(\tilde{T}^I)\tilde{k}_{(I)}]\right\}_{(\alpha i\,\beta j)}\,, \tag{7.2.52}$$

where the $k_{(I)}$s are the Killing vectors of AdS_2 and the $\tilde{k}_{(I)}$s are those of S^2. This translates into the anticommutator

$$\{Q_{(\alpha i)}, Q_{(\beta j)}\} = -i\delta_{ij}(\mathcal{C}\gamma^a)_{\alpha\beta}P_a + \tfrac{i}{R_2}\mathcal{C}_{\alpha\beta}\epsilon_{ij}M_1 + \tfrac{1}{R_2}(\mathcal{C}\gamma_5)_{\alpha\beta}\epsilon_{ij}M_2\,. \tag{7.2.53}$$

The commutators of the supercharges and the bosonic generators are given by the general formula (7.2.23).

7.2.3 Other $AdS \times S$ Solutions

There are some other maximally supersymmetric vacua of supergravity theories with metrics which are the direct product of AdS_n and S^m spacetimes. They typically arise in the near-horizon limit of p-brane solutions that preserve only a half of the supersymmetries [59] and can be used in Freund-Rubin compactifications [183], with S^m as internal space, to get gauged supergravities in n dimensions with gauge group $SO(n+1)$. The known cases are $AdS_4 \times S^7$ and $AdS_7 \times S^4$in $N = 1, d = 11$ supergravity, $AdS_5 \times S^5$ in $N = 2B, d = 10$ supergravity, $AdS_3 \times S^3$ in $N = 2, d = 6$ supergravity, $AdS_2 \times S^3$ [168] and $AdS_3 \times S^2$ [153] in $N = 2, d = 5$ supergravity and the Robinson-Bertotti solution $AdS_2 \times S^2$ in $N = 2, d = 4$ that we have just studied and that can be taken as prototype.

The Killing spinors of all these solutions can be obtained in similar forms. The only complications that arise are due to the symplectic-Majorana nature of supergravity spinors in $4 < d < 8$. We are going to see next how the Killing spinors and vectors the supersymmetry algebras of $AdS_4 \times S^7$ and $AdS_7 \times S^4$in $N = 1, d = 11$ supergravity and $AdS_5 \times S^5$ in $N = 2B, d = 10$ supergravity can be quickly obtained.

$AdS_4 \times S^7$ in $N = 1, d = 11$ Supergravity

This solution is given by

$$\left\{ \begin{array}{rcl} ds^2 & = & R_4^2\, d\Pi^2_{(4)} - (2R_4)^2\, d\Omega^2_{(7)}\,, \\ \\ G & = & \frac{3}{R_4}\omega_{AdS_4}\,, \quad \Rightarrow G_{0123} = \frac{3}{R_4}\,, \end{array} \right. \tag{7.2.54}$$

where $d\Pi^2_{(4)}$ stands for the metric of the AdS_4 spacetime of unit radius, $d\Omega^2_{(7)}$ for the metric of the unit 7-sphere S^7 and ω_{AdS_4} for the volume 4-form of radius R_4.

We construct AdS_4 as in Section 7.2.1 with $g = 1/R_4$ and this gives us the first four Elfbeins e^a associated to the generators P_a $a = 0, 1, 2, 3$ and the first 6 1-forms $\vartheta^{ab} = -\vartheta^{ba}$ associated to the first 4 generators of the 11-dimensional Lorentz group M_{ab} $a, b = 0, 1, 2, 3$. The detailed expressions of these 1-forms is really not necessary.

To construct the sphere of radius $2R_4$ we split the $SO(8)$ Lie algebra generators

$$\left[\tilde{M}_{\tilde{a}\tilde{b}}, \tilde{M}_{\tilde{c}\tilde{d}}\right] = \delta_{\tilde{a}\tilde{c}}\tilde{M}_{\tilde{b}\tilde{d}} + \delta_{\tilde{b}\tilde{d}}\tilde{M}_{\tilde{a}\tilde{c}} - \delta_{\tilde{a}\tilde{d}}\tilde{M}_{\tilde{b}\tilde{c}} - \delta_{\tilde{b}\tilde{c}}\tilde{M}_{\tilde{a}\tilde{d}}\,. \tag{7.2.55}$$

into

$$\tilde{M}_{8i} = 2R_4 P_i\,, \qquad \tilde{M}_{ij} = M_{i+3\,j+3}\,, \quad i,j = 1,\cdots 7\,, \tag{7.2.56}$$

and provide the last 7 P_a's and Lorentz generators M_{ab} $a,b = 4,\ldots,8$. The standard procedure also gives us the associated 7 Elfbeins e^a and 1-forms ϑ^{ab} $a,b = 4,\ldots,8$. Again, the detailed expressions are not necessary. The metric in Eq. (7.2.54) is obtained using the Killing metric of both factors $(+-\cdots-)$.

The general arguments given at the beginning of this section ensure that

$$dx^\mu \nabla_\mu = d - \sum_{a<b} \vartheta^{ab}\Gamma_s(M_{ab})\,, \qquad \Gamma_s(M_{ab}) = \tfrac{1}{2}\Gamma_{ab}\,, \tag{7.2.57}$$

and a straightforward calculation gives for the second piece of the Killing spinor equation

$$\tfrac{i}{288}\left(\Gamma^{abcd}{}_f e^f - 8\Gamma^{abc}e^d\right) G_{abcd} = -e^a \Gamma_s(P_a)\,, \tag{7.2.58}$$

where

$$\Gamma_s(P_a) = \begin{cases} \frac{i}{2R_4}\Gamma^{0123}\Gamma_a\,, & a \leq 3\,, \\ -\frac{i}{4R_4}\Gamma^{0123}\Gamma_a\,, & a > 3\,. \end{cases} \tag{7.2.59}$$

The Killing spinor equation takes the general form Eq. (7.0.1) and is solved as usual. The specific form of the solution depends on the specific choice of coset representative, but it is unimportant in what follows.

Now, let us consider the bilinears $-i\bar{\kappa}_{(\alpha)}\Gamma^a\kappa_{(\beta)}$. Let us define generators $\Gamma_s(P^a)$ dual to the $\Gamma_s(P_a)$ that are exponentiated to construct the coset representative

$$\mathrm{Tr}\left[\Gamma_s(P^a)\Gamma_s(P_b)\right] = \delta^a{}_b\,. \tag{7.2.60}$$

They are given by

$$\begin{aligned} \Gamma_s(P^a) &= \begin{cases} -\frac{iR_4}{16}\Gamma^{0123}\Gamma^a\,, & a \leq 3\,, \\ -\frac{iR_4}{8}\Gamma^{0123}\Gamma^a\,, & a > 3\,. \end{cases}\,, \\ \Gamma_s(M^{ab}) &= -\tfrac{1}{16}\Gamma^{ab}\,. \end{aligned} \tag{7.2.61}$$

The gamma matrices that appear in the bilinears are related to these by

$$\begin{aligned} \Gamma^a &= \tfrac{-16i}{R_4}\mathcal{S}\Gamma_s(P^a)\,, \qquad a \leq 3\,, \\ \Gamma^a &= \tfrac{-8i}{R_4}\mathcal{S}\Gamma_s(P^a)\,, \qquad a > 3\,, \end{aligned} \qquad \mathcal{S} = \Gamma^{0123}\,, \tag{7.2.62}$$

and, since the modified charge conjugation matrix $\tilde{\mathcal{C}} = \mathcal{C}\mathcal{S}$ has the required property

$$\tilde{\mathcal{C}}^{-1}\Gamma_s(P^a)^T\tilde{\mathcal{C}} = -\Gamma_s(P^a)\,, \tag{7.2.63}$$

the bilinears can be written in the form (suppressing the indices α, β)

$$-i\bar{\kappa}\Gamma^a\kappa = \tfrac{-8}{R_4}\tilde{\mathcal{C}}[\Gamma_s(\hat{M}^{\hat{a}\hat{b}})k_{(\hat{a}\hat{b})} + \tfrac{1}{2}\Gamma_s(\tilde{M}^{\tilde{a}\tilde{b}})k_{(\tilde{a}\tilde{b})}]\,, \tag{7.2.64}$$

where hatted generators and Killing vectors belong to the AdS_4 factor and the tilded ones to the S^7 factor. The anticommutator of two supercharges can be immediately read in this expression and the commutator of supercharges and bosonic charges is given by the general formula Eq. (7.2.23).

$AdS_7 \times S^4$ in $N = 1, d = 11$ Supergravity

This solution is given by

$$\left\{\begin{array}{lcl} ds^2 & = & R_7^2\, d\Pi^2_{(7)} - (R_7/2)^2\, d\Omega^2_{(4)}\,, \\ \\ G & = & \frac{6}{R_7}\omega_{S^4}\,, \quad \Rightarrow G_{789\,10} = \frac{6}{R_7}\,, \end{array}\right. \tag{7.2.65}$$

where we use the same notation as in the preceding cases and ω_{S^4} stands for the volume of the sphere of radius $R_7/2$. The definitions of the P_a and M_{ab} generators and the construction of the Elfbeins etc. is almost identical to that of the preceding case and we immediately arrive at

$$dx^\mu\nabla_\mu = d - \sum_{a<b}\vartheta^{ab}\Gamma_s(M_{ab})\,, \qquad \Gamma_s(M_{ab}) = \tfrac{1}{2}\Gamma_{ab}\,. \tag{7.2.66}$$

The 1-forms ϑ^{ab} have a different form now, but we do not need to know it. The second piece of the Killing spinor equation takes the form

$$\tfrac{i}{288}\left(\Gamma^{abcd}{}_f e^f - 8\Gamma^{abc}e^d\right)G_{abcd} = -e^a\Gamma_s(P_a)\,, \tag{7.2.67}$$

where now

$$\Gamma_s(P_a) = \left\{\begin{array}{ll} \frac{i}{2R_7}\Gamma^{789\,10}\Gamma_a\,, & a \leq 6\,, \\ \\ -\frac{i}{R_4}\Gamma^{789\,10}\Gamma_a\,, & a > 6\,. \end{array}\right. \tag{7.2.68}$$

The Elfbeins are also different, but, yet again, we do not need to know their detailed expressions. The dual generators are defined as usual and are given by

$$\Gamma_s(P^a) = \begin{cases} -\frac{iR_7}{16}\Gamma^{789\,10}\Gamma^a\,, & a \leq 6\,, \\ -\frac{iR_7}{32}\Gamma^{789\,10}\Gamma^a\,, & a > 6\,. \end{cases}, \tag{7.2.69}$$

$$\Gamma_s(M^{ab}) = -\frac{1}{16}\Gamma^{ab}\,.$$

and

$$\begin{aligned} \Gamma^a &= \frac{16i}{R_7}\mathcal{S}\Gamma_s(P^a)\,, \qquad a \leq 6\,, \\ \Gamma^a &= \frac{-32i}{R_7}\mathcal{S}\Gamma_s(P^a)\,, \qquad a > 6\,, \end{aligned} \qquad \mathcal{S} = \Gamma^{789\,10}\,. \tag{7.2.70}$$

The modified charge conjugation matrix has the property Eq. (7.2.63) and we get, suppressing again $\alpha\beta$ indices

$$-i\bar{\kappa}\Gamma^a\kappa = \frac{-8}{R_7}\tilde{\mathcal{C}}[\Gamma_s(\hat{M}^{\hat{a}\hat{b}})k_{(\hat{a}\hat{b})} - 2\Gamma_s(\tilde{M}^{\tilde{a}\tilde{b}})k_{(\tilde{a}\tilde{b})}]\,, \tag{7.2.71}$$

where hatted generators and Killing vectors belong to the AdS_7 factor and the tilded ones to the S^4 factor. Again, the anticommutator of two supercharges can be immediately read in this expression and the commutator of supercharges and bosonic charges is given by the general formula Eq. (7.2.23).

$AdS_5 \times S^5$ in $N = 2B, d = 10$ Supergravity

The solution is given in the string frame by

$$\begin{cases} ds^2 = R_5^2 d\Pi^2_{(5)} - R_5^2 d\Omega^2_{(5)}\,, \\ G^{(5)} = \frac{4e^{-\varphi_0}}{R_5}(\omega_{AdS_5} + \omega_{S^5})\,, \quad \Rightarrow G^{(5)}_{01234} = G^{(5)}_{56789} = \frac{4e^{-\varphi_0}}{R_5}\,, \\ \varphi = \varphi_0\,. \end{cases} \tag{7.2.72}$$

This case is exactly analogous to the previous ones. The normalization in the splitting of the generators of $SO(2,4)$ and $SO(6)$ is now, respectively:

$$\begin{aligned} \hat{M}_{a-1} &= -R_5 P_a\,, \quad (a = 0, ..., 4) \\ \tilde{M}_{6a} &= R_5 P_a\,, \quad (a = 5, ..., 9)\,. \end{aligned} \tag{7.2.73}$$

Once again we do not need to know the explicit form of the Zehnbeins. From the covariant derivative term in the gravitino supersymmetry transformation (the variation of the dilatino vanishes automatically) we get the generators of $SO(1,4)$ and $SO(5)$ in the spinor representation. From the remaining piece:

$$-\tfrac{1}{16\cdot 5!}e^{\varphi_0}G^{(5)}_{bcdef}\Gamma^{bcdef}\Gamma_a i\sigma^2 = -e^a\Gamma_s(P_a)\,, \tag{7.2.74}$$

we read the spinor representation for the generators P_a[9]

$$\Gamma_s(P_a) = \begin{cases} \frac{i}{2R_5}\sigma^2\Gamma^{01234}\Gamma_a\,, & (a=0,\cdots,4) \\ -\frac{i}{2R_5}\sigma^2\Gamma^{01234}\Gamma_a\,. & (a=5,\cdots,9) \end{cases} \tag{7.2.75}$$

The dual generators are

$$\Gamma_s(P^a) = \tfrac{iR_5}{32}\sigma^2\Gamma^{01234}\Gamma^a \quad \Rightarrow \Gamma^a = \tfrac{32i}{R_5}\mathcal{S}\Gamma_s(P^a)\,, \quad \mathcal{S} = \sigma^2\Gamma^{01234}\,, \tag{7.2.76}$$

and the modified charge conjugation matrix has the required property Eq. (7.2.63) that leads to

$$-i\bar{\kappa}\Gamma^a\kappa = \tfrac{32}{R_5}\tilde{\mathcal{C}}[\Gamma_s(\hat{M}^{\hat{a}\hat{b}})k_{(\hat{a}\hat{b})} + \Gamma_s(\tilde{M}^{\tilde{a}\tilde{b}})k_{(\tilde{a}\tilde{b})}]\,. \tag{7.2.77}$$

7.2.4 H*pp*-wave Spacetimes and the $KG4,5,6,10,11$ Solutions

Although maximally supersymmetric *pp*-wave solutions were discovered long time ago by Kowalski-Glikman in $N=2, d=4$ and $N=1, d=11$ supergravity [134, 156], only recently they have received wide attention. This renewed interest has been accompanied with the discovery of new maximally supersymmetric solutions of the same kind (henceforth KG solutions) in $N=2B, d=10$ supergravity [136] and in $N=2, d=5,6$ supergravities [152], and by the realization that they can be obtained by taking a Penrose limit [140, 141] of the known $AdS\times S$ maximally supersymmetric solutions [142, 143].

The KG solutions are particular examples of homogeneous *pp*-wave spacetimes (H*pp*-waves), symmetric spacetimes to which we can apply our formalism. Let us review briefly the coset construction that leads to them [135, 160].

[9] Here there is another (completely equivalent, since we are dealing with chiral spinors) possibility, consisting in replacing Γ^{01234} by $-\Gamma^{56789}$.

The generators of $\mathfrak{g}$ in Hpp-wave spacetimes are $\{T_-, T_+, T_i, T_{\star i}\}$ $i = 1, \cdots, d-2$ and their non-vanishing Lie brackets are

$$[T_-, T_i] = T_{\star i}\,, \quad [T_-, T_{\star i}] = A_{ij} T_j\,, \quad [T_i, T_{\star j}] = A_{ij} T_+\,, \quad A_{ij} = A_{ji}\,. \quad (7.2.78)$$

T_+ is central in this Lie algebra. The subalgebra $\mathfrak{h}$ is generated by the $T_{\star i} \equiv M_i$ and $\mathfrak{k}$ is generated by $T_- \equiv P_-\,,\ T_+ \equiv P_+\,,\ T_i \equiv P_i$, and the coset representative is chosen to be

$$u = e^{x^- P_-} e^{x^+ P_+} e^{x^i P_i}\,, \qquad (7.2.79)$$

which lead to the Maurer-Cartan 1-form

$$V = u^{-1} du = -dx^- P_- - (dx^+ + \tfrac{1}{2} x^i x^j A_{ij} dx^-) P_+ - dx^i P_i - x^i dx^- M_i\,. \quad (7.2.80)$$

Since $\mathfrak{g}$ is not semisimple, its Killing metric is singular and cannot be used to construct a G-invariant metric. Instead, we choose[10]

$$B_{+-} = 1\,, \qquad B_{ij} = +\delta_{ij}\,, \qquad (7.2.81)$$

and we get the general Hpp-wave metric

$$ds^2 = 2dx^-(dx^+ + \tfrac{1}{2} x^i x^j A_{ij} dx^-) + dx^i dx^i\,. \qquad (7.2.82)$$

Different Hpp-wave metrics are characterized by the matrix A_{ij} up to $SO(d-2)$ rotations. On the other hand (and this is an important difference with the previous cases), the Hpp-wave metric can have more isometries: all possible rotations of the x^i that preserve the matrix A_{ij}. These rotations do not belong to $\mathfrak{g}$ and the corresponding Killing vectors cannot be found by applying Eq. (7.1.20).

Let us now consider the $KG11$ solution. Its metric is of the above general Hpp-wave form, with A_{ij} and the 4-form field strength given by

$$G_{-123} = \lambda\,, \qquad A_{ij} = \begin{cases} -\frac{1}{9}\lambda^2 \delta_{ij} & i,j = 1,2,3\,, \\ -\frac{1}{36}\lambda^2 \delta_{ij} & i,j = 4,\cdots,9\,. \end{cases} \qquad (7.2.83)$$

This solution is additionally invariant under rotations in the subspaces parametrized by x^1, x^2, x^3 and $x^4, \cdots, x^9$. Let us now consider the Killing

[10] The metric $B_{+-} = 1\,,\ B_{ij} = -\delta_{ij}$ is not invariant under the action of $\mathfrak{h}$ on $\mathfrak{k}$. Thus, we are forced to work with mostly plus signature in this section.

spinor equation. According to the general construction, we only need to compute the Ω part that involves the 4-form field strength. This can be written in the form $-e^a\Gamma_s(P_a)$ with

$$\begin{aligned}
\Gamma_s(P_-) &= \tfrac{\lambda}{12}(\Gamma^-\Gamma^+ + 1)\Gamma^{123}\,,\\
\Gamma_s(P_+) &= 0\,,\\
\Gamma_s(P_i) &= \begin{cases} -\tfrac{\lambda}{6}\Gamma^{-123}\Gamma_i\,, & i=1,2,3\,,\\ -\tfrac{\lambda}{12}\Gamma^{-123}\Gamma_i\,, & i=4,\cdots,9\,,\end{cases}
\end{aligned} \tag{7.2.84}$$

and the Killing spinor has the same form as usual, the only difference being that one of the P_a generators (P_+)is represented by zero and does not contribute to the coset representative u. The general formula Eq. (7.2.23) can be used to calculate the commutators of supercharges and bosonic generators. We see that P_+ is a central charge also in the superalgebra [135]. The calculation of the anticommutators of supercharges is more complicated, though, basically because we can construct duals

$$\Gamma_s(P^a) \sim \Gamma^{+123}\Gamma^a\,, \tag{7.2.85}$$

but the matrix Γ^{+123} is singular and the relation cannot be inverted. This is related to the existence of the extra rotational Killing vectors $k_{(ij)}$ that do appear in the bilinear $-i\bar{\kappa}_{(\alpha)}\Gamma^a\kappa e_a$ [135]. The above equation can in fact be used to relate the Killing vectors $k_{(I)}$ to some of all the possible bilinears. The additional Killing vectors $k_{(ij)}$ appear in the other bilinears (associated to the anticommutators $\{Q_-, Q_-\}$ in the notation of Ref. [135]).

7.3 Conclusions

In this Chapter we have checked in almost all known maximally supersymmetric backgrounds that the Killing spinor equation can be set in the form Eq. (7.0.1) and we have shown how this can be exploited to calculate their supersymmetry algebras using results from the theory of symmetric spaces. There is one exceptional case: the KG spaces, for which it is not easy to compute all the possible anticommutators $\{Q_{(\alpha)}, Q_{(\beta)}\}$.

The obvious extension of this work is to backgrounds with less supersymmetry, like those that can be obtained by replacing the sphere in $AdS \times S$ solutions by another homogeneous space with the right curvature [184–186].

Chapter 8

The Near-Horizon Limit of the Extreme Rotating $d = 5$ Black Hole as a Homogeneous Spacetime

Introduction

The vast majority of the known maximally supersymmetric solutions of supergravity theories seem to be symmetric spaces: Minkowski or AdS spacetimes, products of AdS spacetimes and spheres $AdS_m \times S^n$ or Hpp-wave spacetimes. Their Killing vectors and spinors and their relations that determine their supersymmetry algebras can be found by simple geometrical methods [33].

The only exception seems to be the near-horizon limit of the extreme rotating $d = 5$ black holes [155, 159, 171, 187]. This solution can be written in the form [32]

$$\left\{ \begin{array}{rcl} ds^2 & = & R^2 d\Pi^2_{(2)} - R^2 d\Omega^2_{(2)} - R^2(d\psi + \cos\alpha\cos\theta d\varphi - \sin\alpha\,\mathrm{sinh}\chi d\phi)^2\,, \\ & & \\ F & = & \sqrt{3}R\cos\alpha\,\mathrm{cosh}\chi d\chi \wedge d\phi - \sqrt{3}R\sin\alpha\sin\theta\, d\theta \wedge d\varphi\,, \end{array} \right. \tag{8.0.1}$$

where

$$
\begin{aligned}
d\Pi^2_{(2)} &= \cosh^2\chi \, d\phi^2 - d\chi^2 \,, \\
d\Omega^2_{(2)} &= d\theta^2 + \sin^2\theta \, d\varphi^2 \,,
\end{aligned}
\tag{8.0.2}
$$

are respectively the metrics of the unit radius AdS_2 spacetime and the unit radius 2-sphere S^2. The rotation parameter j is here $\cos\alpha$.

The metric of this solution looks like a sort of twisted product $AdS_3 \times S^3$ in which the sphere and the AdS spacetime share a common direction parametrized by ψ. Actually, when $\cos\alpha = 1$ (the purely electric solution), the dimension ψ belongs only to the sphere and the metric is exactly that of $AdS_2 \times S^3$ and, when $\cos\alpha = 0$, the dimension ψ belongs entirely to the AdS spacetime and the metric is exactly that $AdS_3 \times S^2$. These are singular limits, though, because the isometry group is the 7-dimensional $SO(2,1) \times SO(3) \times SO(2)$ for generic values of $\cos\alpha$ but becomes the 9-dimensional $SO(2,1) \times SO4)$ or $SO(2,2) \times SO(3)$ in the two limits

Not surprisingly, the solution can be obtained by dimensional reduction of the $AdS_3 \times S^3$ solution of $N = 2, d = 6$ supergravity along a direction which is a linear combination of the two S^1 fibers of the Hopf fibrations $AdS_3 \stackrel{S^1}{\rightarrow} AdS_2$ and $S^3 \stackrel{S^1}{\rightarrow} S^2$ [32]. It can also be obtained by dimensional oxidation of the dyonic Robinson-Bertotti solution [157] of $N = 2, d = 4$ supergravity [32], (whose metric is that of $AdS_2 \times S^2$ and is also maximally supersymmetric [166,167]) and these dimensional relations give us very important clues about the geometry of the solution and how to find a coset construction of its metric [176].

In fact, these relations immediately suggest that the metric could be constructed as an invariant metric over the coset $\frac{SO(2,1)\times SO(3)}{SO(2)_j}$, in which the subgroup $SO(2)_j$ is a combination of he two $SO(2)$ subgroups of $SO(2,1)$ and $SO(3)$, that is: the group manifold $SO(2,1)$ equipped with the bi-invariant metric can be identified with the AdS_3 spacetime and the coset $SO(2,1)/SO(2)$ with the left-invariant metric can be identified (locally) with the AdS_2 spacetime. Analogously, the group manifold $SO(3)$ equipped with the bi-invariant metric can be identified (locally) with the S^3 spacetime and the coset $SO(3)/SO(2)$ with the left-invariant metric can be identified with the S^2 spacetime. In the product $AdS_3 \times S^3$ there are two $SO(2)$ subgroups available for taking the quotient (which is equivalent to dimensional reduction) and one choice gives, in $d = 5$ $AdS_2 \times S^3$ and the other $AdS_3 \times S^2$. One could also take the quotient over the $SO(2)_j$ subgroup generated by a linear combination of the generators of the two above-mentioned $SO(2)$ subgroups and the left-invariant metric should be the one in Eq. (8.0.1).

There is another $SO(2)$ subgroup present, generated by the orthogonal linear combination. This $SO(2)$ commutes with the other one and belongs to its normalizer, which is $SO(2) \times SO(2)$. It is a well-known fact [176] that the isometry group of the left-invariant metric over a coset G/H is, generically $G \times N(H)/H$, where $N(H)$ is the normalizer of H and $N(H)/H$ is the right isometry group. Here $N(H)/H = SO(2)_j$ and then the full isometry group should be the 7-dimensional $SO(2,1) \times SO(3) \times SO(2)$, as we want. In the two singular limits, there is enhancement of the isometry group as explained above.

In this Chapter we are going to prove that our proposal is indeed correct by explicitly constructing first the metric in Eq. (8.0.1) as a left-invariant metric over the coset[1] $\frac{SO(2,1)\times SO(3)}{SO(2)_j}$. The spacetime, is, thus, homogeneous, but it is not symmetric. Secondly, we are then going to use this construction to find the Killing vectors and spinors, although we will find difficulties to relate them, due to the fact that in our construction we will not use the Killing metric, but instead we will use the Minkowski metric, which is also $SO(2)$-invariant: the Killing metric of the real form $so(2,1) \times so(3)$ has the signature $(--+---)$, i.e. the $so(2,1)$ part has the wrong signature in our conventions (mostly minus signature), but this can not be corrected by means of analytic continuation (one gets complex metrics or metrics with wrong signature). Fortunately, the Minkowski metric has the necessary properties.

8.1 Construction of the Metric and Killing Vectors

The Lie algebra of $SO(2,1)$ can be written in the form

$$[T_i, T_j] = -\epsilon_{ijk}\mathsf{Q}^{kl}T_l\,, \quad i,j,\dots = 1,2,3,\,, \qquad \mathsf{Q} = \mathrm{diag}\,(++-)\,, \quad (8.1.1)$$

and its Killing metric is $K = 2\mathrm{diag}\,(++-)$. To construct AdS_2, one has to take the coset $SO(2,1)/SO(2)$ where the subgroup $SO(2)$ is generated indistinctly by T_1 or T_2. We will choose for the sake of definiteness T_1. The projection of the Killing metric on the orthogonal subspace generated

[1] Our identification of the near-horizon limit of the rotating extreme black hole and the coset space is only local. We will not be concerned with global issues here.

by T_2, T_3 diag $(+-)$ has the right signature to give AdS_2. Actually, the signature is the opposite to our mostly minus conventions, but a global factor is immaterial and the time coordinate, compact, is associated to T_3 (the $-$ sign in the Killing metric).

It is important to observe that there is no real form of this algebra with Killing metric $K = \text{diag}\,(--+)$. Also, we are forced to associate the time coordinate with T_3.

The Lie algebra of $SO(3)$ can be written in the form

$$[\tilde{T}_i, \tilde{T}_j] = -\epsilon_{ijk}\tilde{T}_k\,, \quad i, j, \dots = 1, 2, 3, \,, \tag{8.1.2}$$

and its Killing metric is $K = 2\text{diag}\,(---)$. To construct S^2, one has to take the coset $SO(3)/SO(2)$ where the subgroup $SO(2)$ is generated by any of the generators $\tilde{T}_i$. We will choose T_3 for definiteness. Observe that there is no real form with Killing metric $K = 2\text{diag}\,(+++)$.

The subgroup $SO(2)$ that we will use will be the one generated by the combination

$$M \equiv \cos\alpha T_1 + \sin\alpha \tilde{T}_3\,. \tag{8.1.3}$$

We now make the following redefinitions

$$\begin{aligned} &P_0 = \tfrac{1}{R}T_3\,, \quad P_1 = \tfrac{1}{R}T_2\,, \quad P_2 = \tfrac{1}{R}\tilde{T}_1\,, \quad P_3 = \tfrac{1}{R}\tilde{T}_2\,, \\ &P_4 = -\tfrac{\sin\alpha}{R}T_1 + \tfrac{\cos\alpha}{R}\tilde{T}_3\,. \end{aligned} \tag{8.1.4}$$

The subalgebra $\mathfrak{h}$ is generated by M and the orthogonal subspace $\mathfrak{k}$ by the P_as. The non-vanishing commutators

$$\begin{array}{lll} [M, P_0] = \cos\alpha P_1\,, & [P_4, P_0] = -\tfrac{\sin\alpha}{R}P_1\,, & [P_0, P_1] = \tfrac{\cos\alpha}{R^2}M - \tfrac{\sin\alpha}{R}P_4\,, \\ [M, P_1] = \cos\alpha P_0\,, & [P_4, P_1] = -\tfrac{\sin\alpha}{R}P_0\,, & [P_2, P_3] = -\tfrac{\sin\alpha}{R^2}M - \tfrac{\cos\alpha}{R}P_4\,, \\ [M, P_2] = -\sin\alpha P_3\,, & [P_4, P_2] = -\tfrac{\cos\alpha}{R}P_3\,, & \\ [M, P_3] = \sin\alpha P_2\,, & [P_4, P_3] = -\tfrac{\cos\alpha}{R}P_2\,, & \end{array}$$

indicate that $[\mathfrak{k}, \mathfrak{h}] \subset \mathfrak{k}$ (reductivity) but $[\mathfrak{k}, \mathfrak{k}] \not\subset \mathfrak{h}$, so we do not have a symmetric pair and we will not have a symmetric space.

The Killing metric of the product group manifold $SO(2,1)\times SO(3)$ in the new basis $\{P_a, M\}$ $a=0,\cdots,4$ is

$$(K_{IJ})=\frac{2}{R^2}\begin{pmatrix} -1 & & & & & \\ & +1 & & & & \\ & & -1 & & & \\ & & & -1 & & \\ & & & & \sin^2\alpha-\cos^2\alpha & -2R\sin\alpha\cos\alpha \\ & & & & -2R\sin\alpha\cos\alpha & R^2(\cos^2\alpha-\sin^2\alpha) \end{pmatrix},$$

but we are not going to use it to construct the left-invariant metric. Instead, we will use the 5-dimensional Minkowski metric η_{ab}, which gives a metric invariant under the left action of G since

$$f_{M(a}{}^{c}\eta_{b)c}=0\,. \tag{8.1.5}$$

The coset representative is chosen to be

$$u(x)=e^{x^0P_0}\cdots e^{x^4P_4}\,, \tag{8.1.6}$$

and the left-invariant Maurer-Cartan 1-form $V=-u^{-1}du$ is

$$-V=T_I\Gamma_{\rm Adj}(e^{-x^4P_4})^I{}_J\Gamma_{\rm Adj}(e^{-x^1P_1})^I{}_{P_0}dx^0+T_I\Gamma_{\rm Adj}(e^{-x^4P_4})^I{}_{P_1}dx^1$$

$$+T_I\Gamma_{\rm Adj}(e^{-x^4P_4})^I{}_J\Gamma_{\rm Adj}(e^{-x^3P_3})^I{}_{P_2}dx^2+T_I\Gamma_{\rm Adj}(e^{-x^4P_4})^I{}_{P_3}dx^3+P_4dx^4\,,$$

and, with the definitions

$$V=e^aP_a+\vartheta M\,, \tag{8.1.7}$$

leads to the Fünfbeins e^a and to the H-connection ϑ

$$\begin{aligned}
-e^0 &= \cosh\left(\tfrac{x^1}{R}\right)\cosh\left(\tfrac{\sin\alpha}{R}x^4\right)dx^0+\sinh\left(\tfrac{\sin\alpha}{R}x^4\right)dx^1\,,\\
-e^1 &= \cosh\left(\tfrac{x^1}{R}\right)\sinh\left(\tfrac{\sin\alpha}{R}x^4\right)dx^0+\cosh\left(\tfrac{\sin\alpha}{R}x^4\right)dx^1\,,\\
-e^2 &= \cos\tfrac{x^3}{R}\cos\left(\tfrac{\cos\alpha}{R}x^4\right)dx^2-\sin\left(\tfrac{\cos\alpha}{R}x^4\right)dx^3\,,\\
-e^3 &= \cos\tfrac{x^3}{R}\sin\left(\tfrac{\cos\alpha}{R}x^4\right)dx^2+\cos\left(\tfrac{\cos\alpha}{R}x^4\right)dx^3\,,\\
-e^4 &= -\sin\alpha\sinh\left(\tfrac{x^1}{R}\right)dx^0-\cos\alpha\sin\tfrac{x^3}{R}dx^2+dx^4\,,\\
-\vartheta &= \tfrac{\cos\alpha}{R}\sinh\left(\tfrac{x^1}{R}\right)dx^0-\sin\alpha\sin\tfrac{x^3}{R}dx^2\,.
\end{aligned} \tag{8.1.8}$$

Redefining the coordinates

$$x^0/R = \phi\,, \quad x^1/R = \chi\,, \quad x^2/R = \varphi\,, \quad x^3/R = \theta + \pi/2\,, \quad x^4/R = \psi\,, \tag{8.1.9}$$

it is easy to see that the metric

$$ds^2 = \eta_{ab} e^a \otimes e^b\,, \tag{8.1.10}$$

is precisely that of Eq. (8.0.1).

According to the general results on homogeneous spaces the Killing vectors $k_{(I)}$ associated to the left isometry group $G = SO(2,1) \times SO(3)$ are given by

$$k_{(I)} = \Gamma_{\rm Adj}(u^{-1})^a{}_I e_a\,. \tag{8.1.11}$$

Their explicit expressions are

$$\begin{aligned}
k_{(P_0)} &= -\partial_{x^0}\,,\\
k_{(P_1)} &= {\rm tgh}(x^1/R)\sin(x^0/R)\partial_{x^0} - \cos(x^0/R)\partial_{x^1} - \sin\alpha\frac{\sin(x^0/R)}{\cosh(x^1/R)}\partial_{x^4}\,,\\
k_{(P_2)} &= -\partial_{x^2}\,,\\
k_{(P_3)} &= -\tan(x^3/R)\sin(x^2/R)\partial_{x^2} - \cos(x^2/R)\partial_{x^3} - \cos\alpha\frac{\sin(x^2/R)}{\cos(x^3/R)}\partial_{x^4}\,,
\end{aligned}$$

$$\begin{aligned}
k_{(P_4)} &= \sin\alpha\left[{\rm tgh}(x^1/R)\cos(x^0/R)\partial_{x^0} + \sin(x^0/R)\partial_{x^1}\right]\\
&\quad - \cos\alpha\left[\tan(x^3/R)\cos(x^2/R)\partial_{x^2} - \sin(x^2/R)\partial_{x^3}\right]\\
&\quad - \left\{\frac{\cos(x^0/R)}{\cosh(x^1/R)} - \cos^2\alpha\left[\frac{\cos(x^2/R)}{\cos(x^3/R)} - \frac{\cos(x^0/R)}{\cosh(x^1/R)}\right]\right\}\partial_{x^4}\,,\\
k_{(M)} &= -R\cos\alpha\left[{\rm tgh}(x^1/R)\cos(x^0/R)\partial_{x^0} + \sin(x^0/R)\partial_{x^1}\right]\\
&\quad -R\sin\alpha\left[\tan(x^3/R)\cos(x^2/R)\partial_{x^2} - \sin(x^2/R)\partial_{x^3}\right]\\
&\quad -R\sin\alpha\cos\alpha\left[\frac{\cos(x^2/R)}{\cos(x^3/R)} - \frac{\cos(x^0/R)}{\cosh(x^1/R)}\right]\partial_{x^4}\,.
\end{aligned}$$

The right isometry group is given by the vectors dual to the Maurer-Cartan 1-forms e_a associated to the generators of $N(H)/H$, and commute with the left Killing vectors. In this case, the generator of $N(H)/H$ is P_4 and the associated Killing vector denoted $k_{(N)}$ turns out to be

$$k_{(N)} = e_4 = -\partial_{x^4}\,. \tag{8.1.12}$$

8.2 Construction of the Killing Spinors and the Superalgebra

The Killing spinor equation of $N=2, d=5$ Supergravity is (choosing $s(\alpha) = +1$) [32,163]

$$\left\{\nabla_a - \tfrac{1}{8\sqrt{3}}(\gamma^{bc}\gamma_a + 2\gamma^b g^c{}_a)\mathcal{F}_{bc}\right\}\kappa = 0\,. \tag{8.2.1}$$

κ is an unconstrained Dirac spinor (one component of a pair of symplectic-Majorana spinors). We contract this equation with the Maurer-Cartan 1-forms e^a to write is in the form:

$$\left\{d - \tfrac{1}{4}\omega^a{}_b\gamma_a{}^b - \tfrac{1}{8\sqrt{3}}(\gamma^{bc}\mathcal{F}_{bc}\gamma_a + 2\gamma^b\mathcal{F}_{ba})e^a\right\}\kappa = 0\,. \tag{8.2.2}$$

In homogeneous spaces, the spin connection is given by

$$\omega^a{}_b = \vartheta^i f_{ib}{}^a + \tfrac{1}{2}e^c f_{cb}{}^a\,, \tag{8.2.3}$$

and we obtain a spinorial representation of the vertical generators M_i

$$\Gamma_s(M_i) = \tfrac{1}{4} f_{ib}{}^a\gamma_a{}^b\,. \tag{8.2.4}$$

In symmetric spaces the structure constants $f_{cb}{}^a = 0$ and the contribution of the spin connection to the Killing spinor equation is just $-\vartheta^i\Gamma_s(M_i)$ [33], but in this case we have extra terms

$$\begin{aligned} -\tfrac{1}{4}\omega^a{}_b\gamma_a{}^b &= -\vartheta\Gamma_s(M) - \tfrac{1}{8}e^c f_{cb}{}^a\gamma_a{}^b\,, \\ \Gamma_s(M) &\equiv \tfrac{1}{2}(\cos\alpha\gamma^{01} - \sin\alpha\gamma^{23})\,. \end{aligned} \tag{8.2.5}$$

The extra terms do not give by itself $-e^a\Gamma_s(P_a)$, but it can be checked that, combined with the terms that depend on the vector field strength, they do, and the Killing spinor equation takes the form

$$\begin{aligned} \{d - \vartheta\Gamma_s(M) - e^a\Gamma_s(P_a)\}\,\kappa &= 0\,, \\ \Gamma_s(P_a) &= -\tfrac{1}{2R}(\cos\alpha\gamma^{01} - \sin\alpha\gamma^{23})\gamma_a\,, \end{aligned} \tag{8.2.6}$$

which leads to

$$\kappa = \Gamma_s(u^{-1})\kappa_0 \,, \tag{8.2.7}$$

where κ_0 is a constant spinor. The matrix $\Gamma_s(u^{-1})^\beta{}_\alpha$ can be used as a basis of Killing spinors $\kappa_{(\alpha)}{}^\beta$ to which we associate supercharges $Q_{(\alpha)}$.

The commutators of the bosonic generators P_a, M, N of the superalgebra (associated to the Killing vectors) with the supercharges is given immediately by the spinorial Lie-Lorentz derivative of the Killing spinor with respect to the associated Killing spinors [174,180]. For the generators associated to the left isometry group $\{T_I\} = \{P_a, M\}$ we can use Eq. (2.23) of Ref. [33]

$$\mathbb{L}_{k_{(I)}}\Gamma_s(u^{-1}) = -\Gamma_s(u^{-1})[\mathbb{L}_{k_{(I)}}\Gamma_s(u)]\Gamma_s(u^{-1}) = -\Gamma_s(u^{-1})\Gamma_s(T_I) \,, \tag{8.2.8}$$

which implies the commutators

$$[Q_{(\alpha)}, T_I] = -Q_{(\beta)}\Gamma_s(T_I)^\beta{}_\alpha \,. \tag{8.2.9}$$

The other commutators with N are trivial.

Finally, let us consider the anticommutators of two supercharges. These are associated to the decomposition in Killing vectors the bilinears $-i\bar{\kappa}_\alpha\gamma^a\kappa_{(\beta)}e_a$. To find this decomposition is crucial to relate the contravariant gamma matrices γ^a with the bosonic generators in the spinorial representation $\Gamma_s(P_a)$. In this case, it is convenient to proceed as follows. First, we find the relation

$$\gamma^a = \eta^{ab}\gamma_b = -2R\mathcal{S}\Gamma_s(P_a) \,, \qquad \mathcal{S} = (\cos\alpha\gamma^{01} + \sin\alpha\gamma^{23}) \,, \tag{8.2.10}$$

and substitute into the bilinear

$$-i\bar{\kappa}_{(\alpha)}\gamma^a\kappa_{(\beta)}e_a = -i\Gamma_s(u^{-1})^\dagger\mathcal{D}\mathcal{S}\Gamma_s(P_b)\Gamma_s(u^{-1})\eta^{ba}e_a \,, \tag{8.2.11}$$

where $\mathcal{D} = i\gamma^0$ is th Dirac conjugation matrix. It can be checked that

$$\Gamma_s(u^{-1})^\dagger\mathcal{D}\mathcal{S} = \mathcal{D}\mathcal{S}\Gamma_s(u) \,, \tag{8.2.12}$$

and, recognizing the adjoint action of u on the $\Gamma_s(P_b)$ we have

$$-i\bar{\kappa}_\alpha\gamma^a\kappa_{(\beta)}e_a = -i\mathcal{D}\mathcal{S}\Gamma_s(T_I)\Gamma_{\text{Adj}}(u)^I{}_b\eta^{ba}e_a \,. \tag{8.2.13}$$

Now we use the following general property: for any $g \in G$, (if the Killing metric is nonsingular, as here)

$$\Gamma_{\text{Adj}}(g)^I{}_J = K_{JK}\Gamma_{\text{Adj}}(g^{-1})^K{}_L K^{LI} \,, \tag{8.2.14}$$

and the definition of the dual generators $T^I = K^{IJ}T_J$

$$-i\bar{\kappa}_\alpha \gamma^a \kappa_{(\beta)} e_a = -i\mathcal{D}\mathcal{S}\Gamma_s(T^I)\Gamma_{\mathrm{Adj}}(u^{-1})^J{}_I K_{Jb}\eta^{ba} e_a \,. \tag{8.2.15}$$

Since the Killing metric and the Minkowski metric are different, the r.h.s. of this expression does not give the Killing vectors of the left isometry group. We have to use a non-trivial property of $\Gamma_s(u^{-1})$. Let us define the matrix η^{IJ}

$$(\eta^{IJ}) = \begin{pmatrix} 1 & & & & & \\ & -1 & & & & \\ & & -1 & & & \\ & & & -1 & & \\ & & & & -1 & \\ & & & & & -R^{-2} \end{pmatrix} , \tag{8.2.16}$$

and, with it and the Killing metric, the matrix

$$R^I{}_J = \eta^{IK} K_{KJ} \,. \tag{8.2.17}$$

It can be checked that

$$\begin{gathered} R^I{}_J \Gamma_{\mathrm{Adj}}(u^{-1})^J{}_K = \Gamma_{\mathrm{Adj}}(u^{-1})^I{}_L R^L{}_K \,, \\ \Downarrow \\ \Gamma_{\mathrm{Adj}}(u^{-1})^J{}_I K_{Jb}\eta^{ba} = \Gamma_{\mathrm{Adj}}(u^{-1})^a{}_L R^L{}_I \,, \end{gathered} \tag{8.2.18}$$

and

$$-i\bar{\kappa}_\alpha \gamma^a \kappa_{(\beta)} e_a = -i\left[\mathcal{D}\mathcal{S}\Gamma_s(T^I)\right]_{\alpha\beta} R^L{}_I k_{(L)} \,, \tag{8.2.19}$$

that gives the anticommutators

$$\{Q_{(\alpha)}, Q_{(\beta)}\} = -i\left[\mathcal{D}\mathcal{S}\Gamma_s(T^I)\right]_{\alpha\beta} \left(R^a{}_I P_a + R^M{}_I M\right) . \tag{8.2.20}$$

Part III

Supergravity Duals

Chapter 9

The Gauge/Gravity Correspondence

In Parts I and II we have studied different aspects of classical solutions of Supergravity from a Supergravity perspective alone. The interest in doing this is theoretical: we expect that a full knowledge of the nonperturbative String Theory states and vacua will help us in understanding the nonperturbative structure of the theory. From this point of view, Supergravity must be seen as a *tool* to get this kind of information from String Theory. The reason why Supergravity can be used with such purposes is because it is a *physical* limit of String Theory, i.e. there is a certain limit in which string physics is (or we expect it to be) described by the Supergravity approximation.

We turn now to an application of Supergravity which is very different in spirit. First, the purpose will no longer be to extract information from String Theory but from *gauge theories*. Secondly, the relation between both things can be purely understood in terms of a *map* between them –what usually is called a duality. There is no direct physical limit in which Supergravity becomes a gauge theory nor viceversa. Instead, what we have is a well defined proposal for a mathematical correspondence between the physics of certain gauge theories and string (and hence also Supergravity) physics. Such a correspondence can be interpreted as an holographic map, and is referred to with the generic name of *gauge/gravity* or AdS/CFT *correspondence* [23,188, 189].

9.1 Type IIB Strings and $\mathcal{N}$=4 SYM

We will start by recalling the correspondence as originally proposed by Maldacena [23]. The statement is as follows: Type IIB String Theory on the maximally supersymmetric background $AdS_5 \times S^5$ with N units of RR five-form flux is dual to the four dimensional $\mathcal{N} = 4$ Super Yang-Mills gauge theory with gauge group $U(N)$. By Type IIB on $AdS_5 \times S^5$ one must understand the Type IIB Superstring quantized on that background. This means that, locally, one may have any possible process involving perturbative or nonperturbative, low or high energy string physics. These might distort locally the spacetime geometry, but spacetime will always approach $AdS_5 \times S^5$ *asymptotically*. The word "dual" asserts the existence of an exact map between any possible string phenomenon taking place in that background and any possible phenomenon occurring in the gauge theory.

9.1.1 The Correspondence from Black Hole and D-Brane Physics

The relation between gauge theories and theories of strings can be traced back to the work of 't Hooft many years ago [190] and to the origins of String Theory itself, when it was proposed as a dual model for hadrons. We will certainly not review all these ideas leading to the connection between strings and gauge theories, but we will try to explain the nowadays widely used argument to motivate the precise relation between $\mathcal{N} = 4$ SYM and Type IIB strings. This is based on the complementary descriptions of D-branes provided both by String Theory and Supergravity. We will review this argument in order to fix some ideas, since a related procedure will be used in Chapter 10 to set the relation between a specific gauge theory and a certain supergravity background.

At least qualitatively, this correspondence can be motivated from the holographic principle [191, 192] when applied to black hole physics and embedded in String Theory. To obtain the precise formulation of the correspondence one must consider the specific system of N parallel D3-branes. The argument requires the identification between string states and supergravity backgrounds that we developed in Chapter 2.

Let us motivate it first qualitatively. A line of thought which could be used is the following.

- Black hole thermodynamics and holography: as realized many years ago by Bekenstein [193], there are strong similarities between black hole physics and the laws of thermodynamics. In particular, an entropy can be assigned to a black hole, given by $S_{BH} = \frac{A_H}{4G_N}$, where A_H is the area of the event horizon. This idea received a strong support once Hawking discovered that, in fact, black holes radiate with a blackbody spectrum [194]. Then, if the black hole entropy is to be interpreted as an entropy as it appears anywhere else in Physics, a natural question arises: what are the microscopic degrees of freedom of a black hole? On the other hand, very general arguments led 't Hooft to the so-called holographic principle [191]: namely, that in a quantum theory of gravity the degrees of freedom of a given system should be stored at the boundary of the region containing it. In particular, the formula for the black hole entropy may be pointing out to us that the information in a black hole could be stored at the event horizon.

- Embedding of the problem of black hole entropy in String Theory: in this respect, a major success was achieved by Strominger and Vafa [58]. They were able to reproduce the semiclassical result of Bekenstein and Hawking for specific configurations of D-branes which are black holes in their supergravity description. The crucial point is that, in their computation of the entropy, the degrees of freedom they were looking at were those of the worldvolume of the branes which act as the source of the gravitational configuration. The counting of *these* d.o.f. is what yields the Bekenstein-Hawking entropy.

- The particular case of D3-branes: the supergravity description of N parallel D3 branes is a ten dimensional, extended analogue of an extremal black hole with a regular (nonzero area) horizon. The result of Maldacena [23] can be motivated from the facts mentioned above when applied to this particular system. First, the d.o.f. should be seen when "looking at the horizon"[1]. Secondly, these d.o.f. should *also* be encoded in the worldvolume description of the source. The worldvolume theory of N parallel D3-branes is, in a certain limit, four dimensional $\mathcal{N} = 4$ SYM with gauge group $U(N)$. In the gravitational description, the near-horizon region is $AdS_5 \times S^5$, and the theory "living" there is Type IIB. Hence both theories should be somehow related.

[1] The precise meaning of this is crucial. See below.

We stress that what one must consider is the *near-horizon* region. This is the key observation in [23]: the relevant region of spacetime we have to look at is the “infinite throat” $AdS_5 \times S^5$. There, an heuristic reason for the correspondence is given by considering the flux of energy from the near-extremal black p-brane configuration into the bulk[2], and further observing that these (low-energy) excitations will be retained within the throat region by the gravitational potential. If we assume unitarity along the emission process, the information of the black p-brane should be carried by these excitations. At the end of the emission process, when the extremal state is reached and the non-BPS branes become BPS, we end up with two systems which therefore could be identified: excitations in the troat (that will be described by closed Type IIB strings) and four dimensional $\mathcal{N} = 4$ SYM on the D-branes.

All this has to do with the limit in which the theory on the branes decouples from the bulk, leaving just $\mathcal{N} = 4$ SYM on the branes. This is a properly taken low energy $\alpha' \to 0$ limit.

Let us see this in a more precise way. The Type IIB supergravity solution describing N parallel D3-branes at the same point in transverse space is given by:

$$\left\{ \begin{array}{rcl} ds^2 &=& H^{-1/2}(dt^2 - d\vec{y}_3^2) - H^{1/2}(dr^2 + r^2 d\Omega_5^2) \\ G^{(5)} &=& F^{(5)} + {}^\star F^{(5)}\,, \qquad F^{(5)} = dH^{-1} \wedge dt \wedge dy_1 \wedge dy_2 \wedge dy_3 \\ e^\phi &=& g_s\,, \end{array} \right. \tag{9.1.1}$$

where H is the harmonic function given by

$$H = 1 + \frac{R_5^4}{r^4}\,, \qquad R_5^4 \equiv 4\pi g_s \alpha'^2 N\,, \tag{9.1.2}$$

and $\vec{y}_3$ are the spacelike worldvolume coordinates, r is the radial coordinate in transverse space and $d\Omega_5^2$ is the line element in the unit five-sphere. The value of the dilaton (the closed string coupling) is constant for this background, and remains as a free parameter. R_5 is an arbitrary integration constant of

[2]The non-extremal configuration, unlike the extremal one, has nonzero Hawking temperature and is thus radiating.

the supergravity solution but, as explained in Section 2.1.1, we have set it by hand so that the RR-charge of this solution is that of N D3-branes.

This solution looks singular at $r = 0$. However, this is not the case, since it can be seen that at $r = 0$ the surface in transverse space is not a point, but a five-sphere of finite radius R_5. In fact, the region described by the surface $r = 0$ is an *horizon*[3]. How does the solution (9.1.1) look like in the near-horizon region? If we take the limit

$$\frac{r}{R_5} \ll 1 \tag{9.1.3}$$

the metric becomes

$$ds^2 = \frac{r^2}{R_5^2}(dt^2 - d\vec{y}_3^2) - \frac{R_5^2}{r^2}dr^2 - R_5^2 d\Omega_5^2\,, \tag{9.1.4}$$

which is $AdS_5 \times S^5$. The whole solution in this limit still has RR flux, and it is a maximally supersymmetric background of Type IIB Supergravity. Its expression is given in (7.2.72)[4].

One can look differently at the near-horizon limit. If we write the metric in terms of the variable

$$\mu = \frac{r}{\alpha'}\,,$$

(which has units of energy) and *then* we take the formal limit $\alpha' \to 0$, we recover (9.1.4) written in terms of μ. An important thing is that this is assuming that *all* remaining parameters entering in the harmonic function H (g_s, N and, in particular, also μ) are fixed when taking the limit. The observation in [23] is that taking μ fixed as $\alpha' \to 0$ corresponds, from the point of view of the worldvolume theory, to decouple the open strings from the bulk and suppressing all higher derivative corrections, leaving just $\mathcal{N} = 4$ SYM on the D3-branes.

[3]This is exactly analogous to what happens to the four dimensional extreme Reissner-Nordström black hole when written in the so-called "isotropic" coordinates (the whole solution is in fact an exact ten-dimensional analogue of the RN black hole). Here, the horizon looks like a point simply because of the coordinates we are using, but near $r = 0$ the r^{-2} dependence of $H^{1/2}$ cancels with r^2 in the area factor and gives a finite result for the area of the horizon.

[4]Note that the solution given by (9.1.1) is asymptotically ($r \to \infty$) Minkowski. This is an explicit example of a supergravity solution that smoothly interpolates between two vacua (see Section 2.4.3).

This needs an explanation. μ has units of mass but, as an energy scale, it has no special meaning at all from the point of view of the supergravity solution. However, form the worldvolume point of view, this rescaled version of the radial coordinate does have a very clear interpretation: it is the expectation value of the worldvolume Higgs fields that correspond to transverse displacements of the branes. Hence, it is perfectly meaningful as an energy scale in the theory. This radial/energy relation is one of the most important pieces in the gauge/gravity "dictionary", and we see that it is at the heart of the correspondence.

9.1.2 The Large N Limit and the Supergravity Approximation

One must check the reliability of the near-horizon approximation. As said in Chapter 1, a supergravity solution can be trusted as long as curvatures are small when measured in units of the string length. It is true that the Ricci scalar of the spacetime (9.1.4) vanishes everywhere. But since this spacetime describes a product space, what we should require is that the curvatures of both the AdS_5 factor and the S^5 factor are small in string units. This means that the radius R_5 must satisfy:

$$\frac{R_5^4}{\alpha'^2} = 4\pi g_s N \gg 1 \,. \tag{9.1.5}$$

Since we also need $g_s \ll 1$ to trust the supergravity approximation, this means that the latter is valid in the *large N limit.* It is in this limit in which we expect Supergravity to provide a dual description of the gauge theory. Also, since the 't Hooft coupling of the gauge theory in terms of string parameters turns out to be given by $g_{\rm YM}^2 N = 2\pi g_s N$, we see that Supergravity is valid both in the large N and large 't Hooft coupling limit.

We stress, however, that the correspondence is supposed to work for any N or any $g_{\rm YM}$, since the full Type IIB String Theory on the $AdS_5 \times S^5$ background is supposed to be dual to $\mathcal{N} = 4$ SYM. Only the region in parameter space where Supergravity is valid (and where we will be able to do calculations) corresponds to the strongly coupled, large N limit of the gauge theory.

9.1.3 The Map Between the Gauge Theory and the Supergravity Side

Once the correspondence is established, what one needs is to know how to proceed in practice to relate both theories. Finding the "AdS/CFT dictionary" is one of the most important issues of the correspondence.

We already have two pieces of it: one is the relation between the units of RR five-form flux and the gauge group of the gauge theory, and the other one is the radial-coordinate/energy-scale relation. We have also mentioned how the string coupling and $g_{\rm YM}$ are related. A further match between both that we have not mentioned so far comes from the spacetime symmetries of both theories. The $AdS_5 \times S^5$ vacuum has 32 supercharges. $\mathcal{N} = 4$ SYM turns out to be superconformal, and both superalgebras can be shown to be isomorphic. Moreover, the isometry group of the five-sphere ($SO(6) \simeq SU(4)$) coincides with the global R-symmetry group of the gauge theory.

A big step concerning the relation between both theories was made in [188] and [189], where a precise recipe to compute gauge theory correlators from the supergravity side was put forward. This implies a match between supergravity *fields* and gauge theory *operators*. In [189] such a match is explained from the following holographic interpretation of the correspondence: the gauge theory can be thought as living at the conformal boundary of the $AdS_5 \times S^5$ spacetime, which is effectively four dimensional. In the coordinates of Eq. (9.1.4) this boundary is at $r \to \infty$. One can see that, in this limit, the radial part of the metric vanishes and that the four "worldvolume" directions become infinitely dilated, while the five-sphere remains at finite size.

Finally, let us observe that the *final* "output" of the correspondence has nothing to do with D-branes: it relates Type IIB String Theory on a certain vacuum (with no branes) with $\mathcal{N} = 4$ SYM. The use of D3-branes to relate their near-horizon geometry with their worldvolume theory can be seen as a trick to derivate the correspondence. However, as a recipe it is extremely useful, because it allows to establish a *precise* correspondence. The observation that there should be a connection between gauge and string theories is based on very general principles [5] and has received much attention for many years. However, a big problem has always been to find the appropriate string dual. Identifying the worldvolume theories of branes with their geometry is a huge step in this direction, and it has been shown to work in many other cases. This kind of identification is the one we will be using in Chapter 10.

9.2 Nonconformal Extensions

One of the main attempts in later works concerning the gauge/gravity correspondence has been to extend it to other gauge theories. In the case of $\mathcal{N} = 4$ SYM the correspondence has been proven to work extremely well, and all checks show a perfect agreement between the supergravity and the gauge theory predictions. However, one is eventually interested in finding the string dual to more realistic theories, nonconformal and with less supersymmetries. The problem turns out to be much more involved technically, and the dual geometries are often very complicated.

Here we will just review two approaches to this problem, which are the ones on which the results reported in Chapter 10 are based.

9.2.1 Gauge Theories from Wrapped Branes

The general framework was first put forward in [195], and it is based on the results found in [196]. The idea is as follows.

Wrapped D-Branes and Twisted Worldvolume Theories

The authors of [196] considered, from a String Theory point of view, how the worldvolume theories of D-branes wrapping nontrivial cycles should be[5]. On the one hand, D-branes are BPS states. On the other hand, they may be wrapping a nontrivial cycle which supports no covariant Killing spinors at all. In order to reconcile both things, they were led to the conclusion that the worldvolume theory of the branes has to be partially "topologically twisted". We now briefly explain what this means.

Let us consider a supersymmetric gauge theory in flat space. In general, it will have as global symmetries the Lorentz group L plus some R-symmetry group. A topological twist [197] can be seen as an exotic realization of this global symmetry group, arising from what could be called a "misidentification" of L and R: that is, by embedding a subgroup of R into L. Different embeddings will give rise to different possible twists. This has the consequence of effectively changing the spins of the fields carrying R-charge. In

[5]In general, the wrappings we will be considering do not involve all worldvolume directions of the D-brane. Only certain worldvolume directions may be wrapping a cycle, the remaining ones being flat.

particular, it will have the consequence of a rearrangement of the supercharges, with the corresponding modification of the supersymmetry of the theory.

An alternative way of looking at this (see e.g. [198] and references therein) is to consider a field theory on a generic curved manifold, hence coupled (minimally) to gravity. The twisting is then achieved by gauging a certain subgroup of the global R-symmetry group. Why such a thing could serve to implement a twist (i.e., an effective change of the spins of the fields carrying R-charge) can be seen with the following example. As always, the gauging of the R-symmetry group will make appear an extra gauge field A_μ^R, the connection in R. Let us take the case of a spinor ψ charged under A_μ^R. Its *total* (Lorentz plus gauge) covariant derivative can be schematically written as:

$$\tilde{\nabla}_\mu \psi = \left(\partial_\mu + \tfrac{1}{4}\,\not{\omega}_\mu + A_\mu^R\right)\psi \tag{9.2.1}$$

(the prefactors will depend on the specific theory we are considering). Let us suppose now that we give a background value to A_μ^R so that it cancels the spin connection term above[6]. In such a case, the total covariant derivative of the spinor will become

$$\tilde{\nabla}_\mu \psi = \partial_\mu \psi\,, \tag{9.2.2}$$

i.e., that of a scalar. One could say that the original "Lorentz"+gauge charge is reinterpreted as a different "Lorentz" charge.

What was shown in [196] is that, in the case of D-branes wrapping cycles, the twist of the worldvolume gauge theory is "automatically done": the needed extra gauge field A_μ^R has to appear in the worldvolume theory, and it is the connection associated to the nontrivial normal bundle of the curved manifold. This can be thought as an effective way of finding the massless open string spectrum (and also the corresponding gauge theory at low energies) in the case of non flat D-branes.

In these cases, the reason why the twist is what makes supersymmetry to be preserved can be now easily understood. Let us suppose, for example,

[6]Note that A_μ^R can be nonabelian, but the group R will not coincide, in general, with the spin group. When we say that a background value of A_μ^R could cancel the spin connection term, we mean a cancellation between certain *components* of both connections –those associated to a certain *common* subgroup of the respective groups (this is the subgroup of R that we will be gauging). This is how the embedding of a subgroup of R into L referred above is implemented in this picture. The identification between different common subgroups translates then into different twists of the theory.

that the curved part of the worldvolume manifold is a surface that admits no covariantly constant spinors. This means that the equation

$$\left(\partial_\mu + \tfrac{1}{4}\,\not\omega_\mu\right)\epsilon = 0\,, \tag{9.2.3}$$

(where now ϵ is the spinorial parameter of a supersymmetry transformation) has no nontrivial solutions. But if we can (or, as pointed out in [196], we are forced to) apply the procedure sketched above to (9.2.3), we see that, after the twist, finding covariantly constant spinors simply means finding constant spinors.

These main idea of all this can thus be represented as follows:

$$\begin{array}{ccc} \text{Flat D-brane} & \longleftrightarrow & \text{Ordinary SYM} \\ \Big| & & \Big| \\ \textit{wrapping} & & \textit{twisting} \\ \Big\downarrow & & \Big\downarrow \\ \text{Wrapped D-brane} & \longleftrightarrow & \text{Twisted SYM} \end{array}$$

In general, this will lead to reduced supersymmetry in the D-brane worldvolume (although some supersymmetry will always be preserved because of the above argument), and the resulting theory may be conformal or not. The particular case we will consider in Chapter 10 provides a nonconformal example.

The Corresponding Supergravity Solutions

In [195] Maldacena and Núñez took this phenomenon into account in order to study the gravity duals of gauge theories with less supersymmetry. As a direct consequence of the facts explained above, they noticed that considering the geometries produced by D-branes wrapping nontrivial cycles should provide us with gravity duals of theories with reduced supersymmetry and, in some cases, also nonconformal. Let us consider wrappings on some compact cycle that only involve certain worldvolume directions, the remaining ones being flat. In the limit in which the massive KK-modes decouple from the theory (low enough energies in the worldvolume theory) we will be left with an effective gauge theory that lives in a *flat* space of lower dimensions than that of the whole D-brane worldvolume. Since the (partial) breaking of supersymmetry and the possible loss of conformal invariance are global worldvolume aspects, this flat, lower-dimensional SYM gauge theory will have, in principle, less than sixteen supercharges and may be nonconformal, too.

Of course, the problem is how one should proceed in order to find supergravity solution associated to a *wrapped* D-brane. In this respect, they observed that, since this is equivalent to twisting the "original" gauge theory of the flat D-brane, the gravity dual should be described by some *gauged* supergravity. Let us motivate why this should be so.

Gauged supergravity theories can be obtained in two ways. One possibility is to consider these theories as the gauge theories of an AdS-supergroup (in the same spirit as ordinary supergravities can be considered as gauge theories of a super-Poincaré group, or that General Relativity can be considered as the gauge theory of the Poincaré group). Another possibility is to start with an extended Poincaré supergravity. Then, gauging the (global) R-symmetry group (in the same way as one always does to gauge a global symmetry of a given action) yields a gauged supergravity theory. These theories always have a cosmological constant term, and their vacua are AdS spacetimes. This is obvious if one thinks in terms of the first construction.

The parallel between the latter construction of gauged supergravities (by gauging the R-symmetry group) and the procedure to twist gauge theories explained above is manifest. One should not be confused, though: when considering the twist of the D-brane gauge theory, all the fields involved there are *worldvolume* fields, i.e. open strings. But when considering a gauged supergravity theory, all the fields of that theory (and, in particular, also the corresponding R-symmetry gauge field[7]) are *bulk* fields, i.e. closed strings. The use of gauged supergravity must be understood in the sense of the gauge/gravity correspondence [195]: from a match of symmetries in the gauge and gravity sides and in terms of an identification between supergravity *fields* and gauge theory *operators*.

[7] This field is in fact interpreted as a KK mode of a sphere compactification of some ten-dimensional, ungauged supergravity. See below.

The final recipe to find the near-horizon limit of a supergravity solution describing a Dp-brane wrapped on some q-cycle ($q < p$) can be summarized in the following way. First, we must go to a $d = (p + 2)$-dimensional gauged supergravity theory, since these are the gravity theories thet have all the fields that correspond to the twisted gauge theory operators. Then one finds a domain-wall solution in the gauged supergravity theory. That domain wall, with a $(p+1)$-dimensional worldvolume, will be identified at the end with the Dp-brane. Then we implement both the wrapping and the associated twist by choosing the appropriate configuration for the metric, the R-symmetry gauge field and the remaining fields of the gauged supergravity theory. By taking appropriate boundary conditions on the metric we define the geometry of the q-cycle on which we wish to wrap the branes. By choosing which components of the gauge field are nonvanishing[8] and its boundary conditions (to enforce it to equal the spin connection on the cycle, as in the gauge theory side) we implement a particular twist. Finally, it so happens that these $d = (p + 2)$ gauged supergravity theories turn out to be sphere compactifications of ten and eleven dimensional supergravities [199]. Therefore, after uplifting the gauged supergravity solution just found, one ends up with the desired ten or eleven dimensional supergravity solution describing the wrapped branes. This solution directly gives the near-horizon limit of the wrapped brane configuration, and it is supposed to be the geometry which is dual to the *flat* $(p - q + 1)$-dimensional gauge theory, i.e. dual to the D/M-brane gauge theory, but precisely in the low energy limit in which the $(p + 1)$-dimensional theory becomes effectively $(p - q + 1)$-dimensional.

Notice that the sphere on which one compactifies ten dimensional supergravity to get the corresponding $d = (p+2)$ gauged supergravity theory is *not* the cycle on which the D-branes are wrapped. The latter must be thought as a cycle of the resulting ten dimensional "ambient" space (a cycle inside a Calabi-Yau manifold, for example). The fact that the above procedure is supposed to yield directly the near-horizon geometry makes it insensitive to the global details of such ambient geometry (in the chosen example, the full CY-geometry), only the description of the particular cycle is needed.

[8] Again, we have to "turn on" only the appropriate subgroup of the R-symmetry group of gauged supergravity. This will be dictated by the normal bundle of the q-cycle and by the required twist.

A trivial application (but an illustrative exercise) of the procedure sketched above is to consider the vacuum solution of five dimensional gauged supergravity. This theory comes from a five-sphere compactification of Type IIB and, according to the explanation above, should be the right theory to consider if one whishes to describe wrapped D3-branes. Its trivial vacuum configuration is AdS_5. Further uplift to ten dimensions yields $AdS_5 \times S^5$, which is the near-horizon limit of flat D3-branes.

9.2.2 Gauge Theories from Fractional Branes

Another way to obtain supergravity backgrounds that are dual to less supersymmetric and nonconformal gauge theories is to find solutions corresponding to fractional D-branes. Both the supergravity description and their gauge duals are studied in [200, 201]. Let us first briefly recall first what fractional branes are.

Fractional Branes

Fractional branes are String Theory objects that one can consider when the spacetime is taken to be an orbifold. For concreteness, we will focus on the orbifold $\mathbb{C}^2/\mathbb{Z}_2$ of Type II theories, which is the one we will be considering in Chapter 10. Fractional branes can only exist at the orbifold fixed point, and they can be understood as a generalization of the familiar Dirichlet branes when the very particular consequences of being at the orbifold fixed point are taken into account.

As it is well known, in an orbifold theory of closed strings the spectrum contains an untwisted sector (the closed string states that survive the orbifold projection) and a twisted one (arising from "open" strings whose ends are at points that are identified only up to the action of the orbifold group). The twisted states can only exist and propagate along the orbifold fixed point, while the untwisted ones can propagate also into the bulk.

Fractional branes admit a very simple interpretation. As D-branes, they can be considered as hyperplanes contained within the orbifold fixed point where open strings are attached, but with the particular property that they are charged under the *twisted* sector of closed strings (i.e., they are the "twisted" version of regular D-branes). They turn out to carry untwisted RR-charge as well, and the unit of RR-charge carried by a fractional D-brane turns out to be one half of that of a regular D-brane.

The worldvolume theories of fractional branes will always have less supersymmetry than those of regular D-branes. This is easy to understand, since the orbifold action already breaks some bulk supersymmetries. In addition, their worldvolume theories are nonconformal. This may seem less intuitive, but we can think of it as a consequence of the loss of translational invariance implied by the orbifold action along the orbifolded directions. If we take into account the radial/energy relation that we explained in Section 9.1, we can easily guess that the worldvolume theory should not be energy scale invariant if the geometry is not translational invariant. However, to use this argument one should find, for each particular system, which is the proper radial coordinate. This is something quite easy in flat space[9], but in some complicated geometries it may become a nontrivial task. However, for the cases we will consider, the above reasoning holds. We see then that finding their associated supergravity solutions is of interest for the study of more realistic gauge theories.

The Corresponding Supergravity Solutions

Finally, we want to summarize how one can manage to find fractional brane geometries[10]. The way in which this can be done (at least for simple orbifolds like the one we are considering here, $\mathbb{C}^2/\mathbb{Z}_2$) is very simple both in spirit and in practice. The idea is to first find an effective action for the massless closed strings states of both the twisted and the untwisted sector of the orbifold theory. This will be a sum of two actions: one defined on the bulk, including the massless untwisted sector, and another one restricted to the orbifold fixed points which will just include the massless states of the twisted sector. Once we have this effective action at hand, we are able to find any classical solution of its equations of motion. In particular, we can find those that we will associate with fractional branes, in the same way as we find regular D-brane solutions in non-orbifold theories.

The key point to construct the spacetime effective action of an orbifold theory is to consider the orbifold space as the singular limit of its corresponding blow-up. The latter is a smooth spacetime which always contains some nontrivial cycles, and in the limit in which these cycles shrink to zero

[9] Up to the determination of the proportionality constant, something which is *not* unimportant for certain purposes [202].

[10] Classical solutions corresponding to fractional branes were constructed in [203].

size the spacetime becomes singular and the orbifold geometry is recovered. In a sense, what one does to find the effective action on the orbifold is very similar to ordinary Kaluza-Klein reduction[11]: one decomposes all fields of the ordinary Type II theory in directions along the cycles and transverse to them. For example, in the case of $\mathbb{C}^2/\mathbb{Z}_2$, in which the blown-up geometry just contains a two-cycle, a two-form field $\mathcal{B}$ would be decomposed as

$$\mathcal{B} = B + b \wedge \omega \,,$$

where B is again a two form (the "unwrapped part"), b will become a scalar degree of freedom and ω denotes the two-form dual to the two-cycle.

So we take the ordinary Type II effective action in ten dimensional space, decompose all the bosonic fields according to the above prescription, and finally integrate over the cycle. The form ω dual to the cycle has the property that its integral over the cycle remains fixed even in the singular limit. So we can think in doing the above decomposition and subsequent integration also in the limit in which the cycle has collapsed, and the final result will be valid in the singular limit that describes the orbifold geometry.

The new fields that we interpret as the "wrapped" parts of the ordinary ten dimensional ones (the field b in the above example) are found to exactly coincide with those of the twisted sector of the massless string spectrum. Once we integrate along the directions of the cycle, the terms of the full action that contain the twisted fields are found to be fixed point valued actions. An explicit example of this procedure can be found in Appendix C.1.2.

Looking at the orbifold space as the singular limit of its blown-up geometry has the following consequence: fractional Dp-branes can be seen as regular D$(p+2)$-branes wrapped on the corresponding cycle of the blow-up space [204–206], again in its singular limit. A natural question arises, then, concerning the possible relations between the construction we sketched in the preceding Section 9.2.1 and the one outlined here. This is what we will try to explore in the next Chapter, both from the point of view of the dual gauge theory and from the point of view of the supergravity backgrounds.

[11] This is just an analogy concerning only the procedure: the compact cycle is not a compact direction in spacetime, and hence a vanishing cycle is *not* a dimensional reduction.

Chapter 10

Gauge Theories from Wrapped and Fractional Branes

Introduction

The gauge/gravity correspondence has its origin on the fact that, on the one hand, D-branes are classical solutions of the low-energy string effective action and, on the other hand, they have a gauge theory living in their world-volume. This means that the low-energy dynamics of D-branes can be used to determine the properties of the gauge theory and vice-versa.

The most successful realization of this correspondence is the Maldacena conjecture [23], confirmed by all subsequent studies, according to which ten-dimensional type IIB string theory compactified on $AdS_5 \times S^5$ is dual to $\mathcal{N} = 4$ super Yang–Mills theory in four-dimensional Minkowski spacetime.

However, $\mathcal{N} = 4$ super Yang–Mills in four dimensions is a rather special theory due to its conformal properties and its high amount of supersymmetry. Therefore, a lot of effort has been recently devoted to find possible extensions of the Maldacena duality to non conformal and less supersymmetric gauge theories, or at least to use the low-energy brane dynamics to extract information on the properties of such more realistic theories.

Two approaches to this problem have been largely pursued, based respectively on the study of:

- D-branes *wrapped* on supersymmetric cycles [195];
- *Fractional* D-branes on orbifolds [204–207] [1].

After $\mathcal{N}=4$ super Yang–Mills, the next natural system to consider is $\mathcal{N}=2$ super Yang–Mills theory in four dimensions, also with matter. This system has been studied, on the one hand, by considering fractional D3-branes on orbifolds [200, 212, 213] and systems made of fractional D3/D7-branes [214, 215], and, on the other hand, by considering D5-branes wrapped on a two-cycle inside a Calabi–Yau two-fold [216, 217]. The low-energy dynamics of wrapped branes has also been recently used to study other gauge theories [218–224].

The use of fractional and wrapped branes presents some interesting similarities that are not surprising since fractional branes on orbifolds can be seen as D-branes wrapped on cycles that are vanishing in the orbifold limit of the ALE space which corresponds to the blow-up of the orbifold space [204–206]. In fact, by probing the supergravity solutions describing the two types of systems, one is able to recover all the relevant *perturbative* information on the Coulomb branch of the gauge theory living on the branes, namely the running coupling constant and the metric on the moduli space of the theory.

These two approaches, however, have not been able to provide information on the *nonperturbative* features of the gauge theories, as for instance on the instanton contribution to the moduli space of $\mathcal{N}=2$ super Yang–Mills in four dimensions. This is related to the existence at short distance of an enhançon [225] where the supergravity solution becomes inconsistent, because at this distance the probe brane becomes tensionless, signalling the appearance of new massless degrees of freedom. This means that the supergravity approximation is not valid anymore and that the region inside the enhançon is excised. This fact prevents to get information on the strong coupling regime of the gauge theory living in the world-volume of the branes, that is in fact determined by what happens inside the enhançon. To overcome this problem one must presumably also include the new massless degrees of freedom, as attempted for instance in Ref. [226][2].

[1] Apart from the two approaches that we consider in this Chapter, other interesting ones are based on the study of fractional D-branes on conifolds (see Ref. [208] and Ref.s therein) and of M-branes wrapped on Riemann surfaces [209–211].

[2] For recent developments concerning the physics of the enhançon see for instance Ref.s [227–229].

In this Chapter we will apply both approaches discussed above to the study of $\mathcal{N}=4$, $D=2+1$ super Yang–Mills theory, which is a theory with 8 Poincaré supercharges. The interest in this theory resides mainly in the fact that its properties, perturbative and nonperturbative, are well known [230, 231]. This is also the theory where the enhançon was first found [225] using a different approach based on the study of D6-branes wrapped on $K3$ surfaces.

We will first consider a system made up of N D4-branes wrapped on a two-cycle inside a Calabi–Yau two-fold. The crucial property of this system, as of any other system of branes wrapped on supersymmetric cycles, is that the geometrical structure of the background forces the gauge theory living on the world-volume of the branes to be partially *topologically twisted* [196] and this allows to preserve the desired amount of supersymmetry. To find the supergravity solution describing the D4-branes, we will use the techniques introduced in Ref. [195], which amount to find a solution of a lower dimensional gauged supergravity and then uplift it to ten or eleven dimensions. We will then use the uplifted solution in a probe computation in order to extract information on the Coulomb branch of the gauge theory which lives on the flat three-dimensional part of the world-volume of the brane, which is pure $\mathcal{N}=4$, $D=2+1$ super Yang–Mills with gauge group $SU(N)$.

Then, we will consider a system made of N fractional D2-branes and M D6-branes on the orbifold $\mathbb{R}^4/\mathbb{Z}_2$ and, solving explicitly the equations of motion of type IIA supergravity, we will find the corresponding classical solution. The probe computation will give us information on the same three-dimensional theory, now also coupled to M hypermultiplets in the fundamental representation of the gauge group.

In both approaches we find that, as in other cases, the probe analysis correctly reproduces the perturbative part of the moduli space, giving the exact running coupling constant of $\mathcal{N}=4$ super Yang–Mills in three dimensions, but is unable to give the instanton contribution. This analysis allows us to make some comments on the relation between the two solutions and to see that in both cases the gauge coupling constant can be obtained from a common expression representing the "stringy volume" of the two-cycle on which the branes are wrapped. Moreover, in both cases the locus where the "stringy volume" vanishes corresponds to the point where the Calabi–Yau two-fold in which the cycle is embedded manifests an enhanced gauge symmetry, which is the origin of the enhançon mechanism.

The structure of this Chapter is as follows. Section 10.1 and 10.2 are organized in an entirely parallel way, and can be read independently from each other. They describe the two different brane systems that we study in order to get information on $\mathcal{N}=4$, $D=2+1$ super Yang–Mills theory from supergravity, namely a system of N D4-branes wrapped on S^2 (in section 10.1) and a system of N fractional D2-branes and M D6-branes on the orbifold $\mathbb{R}^4/\mathbb{Z}_2$ (in section 10.2). In section 10.3, we discuss and comment the results of the previous two sections. Many details of the various computations are given in the appendices. In appendix C.1 we fix the conventions and discuss in detail how the two supergravity solutions were found. In appendix C.2 we discuss the world-volume actions for fractional branes. Finally, in appendix C.3 we give some details about the perturbative computation of the running coupling constant of the gauge theory that we consider.

10.1 D4-branes wrapped on S^2

10.1.1 Setup

In this section we are going to consider a system made of N D4-branes with two longitudinal directions wrapped on a two-sphere.

As discussed in Ref. [196], the gauge theory living on the world-volume of wrapped branes has to be topologically twisted. In this subsection we want to determine the topological twist that is needed in order to obtain at low-energy on the flat part of the world-volume of the D4-branes $\mathcal{N}=4$ super Yang–Mills theory in three space-time dimensions, that is a theory with 8 supercharges. The twist which preserves 8 supercharges is exactly the one imposed by the geometrical structure of the background when the two-sphere is seen as a nontrivial two-cycle inside a Calabi–Yau two-fold.

The configuration that we are going to study is schematically shown in the following table, where the symbols $-$, $\frown$ and $\cdot$ represent respectively unwrapped world-volume directions, wrapped world-volume directions and transverse directions:

				CY						
	$\mathbb{R}^{1,2}$			S^2		N_2		$\mathbb{R}^3$		
D4	$-$	$-$	$-$	$\frown$	$\frown$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$

In flat space, the presence of the D4-brane breaks spacetime Lorentz invariance in the following way: $SO(1,9) \longrightarrow SO(1,4) \times SO(5)_R$. The fact that the D4-brane is wrapped on S^2 introduces an additional breaking of $SO(1,4)$ into $SO(1,2) \times SO(2)_{S^2}$. The twist is then introduced by breaking the R-symmetry group $SO(5)_R$ into $SO(2)_G \times SO(3)$ and by identifying $SO(2)_G$ with $SO(2)_{S^2}$. In conclusion our configuration breaks the original $SO(1,4) \times SO(5)_R$ into $SO(1,2) \times SO(2)_{S^2} \times SO(2)_G \times SO(3)$ with the two $SO(2)$ groups identified. The fields of the gauge theory living on the wrapped D4-branes transform according to the following representations of the above groups:

	$SO(1,4)$	$\rightarrow$	$SO(1,2) \times SO(2)_{S^2}$	$SO(5)_R$	$\rightarrow$	$SO(2)_G \times SO(3)$
Vector	$\mathbf{5}$	$\rightarrow$	$(\mathbf{3},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2})$	$\mathbf{1}$	$\rightarrow$	$(\mathbf{1},\mathbf{1})$
Scalars	$\mathbf{1}$	$\rightarrow$	$(\mathbf{1},\mathbf{1})$	$\mathbf{5}$	$\rightarrow$	$(\mathbf{1},\mathbf{3}) \oplus (\mathbf{2},\mathbf{1})$
Fermions	$\mathbf{4}$	$\rightarrow$	$(\mathbf{2},+) \oplus (\mathbf{2},-)$	$\mathbf{4}$	$\rightarrow$	$(+,\mathbf{2}) \oplus (-,\mathbf{2})$

Since we are interested in the three-dimensional theory living on the *flat* part of the world-volume at very low energies, we must keep only the massless states, which are the ones transforming as singlets under $SO(2)_D \equiv (SO(2)_{S^2} \times SO(2)_G)_{\text{diag}}$:

	$SO(1,2) \times SO(2)_D \times SO(3)$
Vector	$(\mathbf{3},\mathbf{1},\mathbf{1})$
Scalars	$(\mathbf{1},\mathbf{1},\mathbf{3})$
Fermions	$2 \times (\mathbf{2},\mathbf{1},\mathbf{2})$

These states form exactly the vector multiplet of $\mathcal{N} = 4$, $D = 2+1$ super Yang–Mills theory.

10.1.2 The supergravity solution

In this subsection we will construct a supergravity solution describing the system just introduced, made of N D4-branes with two world-volume directions wrapped on S^2. One could in principle work in ten-dimensional type IIA supergravity, write a suitable Ansatz for such a system, then solve the equations of motion and find the solution. This is, however, not a simple task because it is not easy to implement directly in ten dimensions the topological twist that we have discussed in the previous subsection from the point of view of the gauge theory living on the brane. One has to proceed in a longer, but more straightforward way that has been introduced in Ref. [195]. Instead of working directly in the ten dimensional theory one starts by considering, for the case of a p-brane, a $(p+2)$-dimensional gauged supergravity theory that is obtained by compactifying the original D-dimensional theory (where of course $D = 10$ for the case of a D-brane or NS5-brane and $D = 11$ for the case of an M-brane) on S^{D-p-2}. The isometry group $SO(D-p-1)$ of S^{D-p-2} corresponds to the R-symmetry group that we discussed in the previous subsection. In gauged supergravity the R-symmetry group is gauged so that the theory contains $SO(D-p-1)$ gauge fields. In this theory one looks for a domain wall solution that preserves the desired amount of supersymmetry and breaks the original R-symmetry group in a way that implements the correct twist. In fact, in gauged supergravity the supersymmetry preserving condition contains also the gauge fields and can schematically be written as $(\partial_\mu + \omega_\mu + A_\mu)\,\epsilon = 0$. The discussed twist corresponds to the identification of some of the gauge fields with the spin connection of the manifold around which the brane is wrapped, $A_\mu = -\omega_\mu$, so that the request of finding covariantly constant spinors is equivalent to that of just finding constant spinors. Once the solution with the correct properties has been found, the last step is to uplift it to D dimensions by using the formulas given in Ref.s [199, 232].

In the following, in order to avoid many new calculations, we do not use directly a 6-dimensional gauged supergravity as it would be natural for a D4-brane. We will instead proceed in slightly different way by exploiting the fact that the solution of seven-dimensional gauged supergravity corresponding in eleven dimensions to an M5-brane wrapped on S^2 and preserving 8 supercharges has already been constructed [195]. Therefore we proceed as follows. We start from the solution of 7-dimensional gauged supergravity given in Ref. [195], and uplifting it to eleven dimensions using the formulas found in Ref. [232] we obtain a solution of 11-dimensional supergravity describing N M5-branes wrapped on S^2. Finally, upon compactification to ten dimensions we get the desired solution describing N D4-branes wrapped on S^2. The details of this procedure are given explicitly in Appendix C.1.1. Here we write directly the ten-dimensional solution in the string frame, which reads[3]:

[3]This solution was partially given in appendix 7.4 of Ref. [195].

$$
\begin{aligned}
ds^2_{\rm st} &= \left(\frac{R_A}{R_0}\right)^3 \Delta^{1/2} e^{3\rho}\, \eta_{\alpha\beta} d\xi^\alpha d\xi^\beta + \frac{R_A^3}{R_0}\Delta^{1/2} e^\rho (e^{2\rho} - \tfrac{1}{4})(d\tilde\theta^2 + \sin^2\tilde\theta d\tilde\varphi^2) \\
&+ \frac{R_A^3}{4R_0}\Delta^{-1/2} e^\rho \left(\frac{4\Delta}{e^{5\lambda}} d\rho^2 + \Delta d\chi^2 + \cos^2\chi (d\theta^2 + \sin^2\theta d\varphi^2)\right. \\
&\left. + e^{5\lambda}\sin^2\chi \left(d\psi + \cos\tilde\theta d\tilde\varphi\right)^2\right) , \qquad (10.1.1a)
\end{aligned}
$$

$$
e^{2\phi} = \left(\frac{R_A}{R_0}\right)^3 \Delta^{1/2} e^{3\rho} , \qquad (10.1.1b)
$$

$$
\begin{aligned}
C_3 &= \frac{R_A^3}{8}\, \frac{e^{5\lambda}\cos^3\chi \cos\theta \sin\tilde\theta}{\Delta}\, d\tilde\theta \wedge d\tilde\varphi \wedge d\varphi \\
&+ \frac{R_A^3}{8}\, \frac{e^{5\lambda}(\Delta + 2)\cos^2\chi \sin\chi \cos\theta}{\Delta^2}\, d\chi \wedge d\varphi \wedge \left(d\psi + \cos\tilde\theta d\tilde\varphi\right) \\
&+ \frac{R_A^3}{8}\, \frac{\partial_\rho (e^{5\lambda})\cos^3\chi \sin^2\chi \cos\theta}{\Delta^2}\, d\rho \wedge d\varphi \wedge \left(d\psi + \cos\tilde\theta d\tilde\varphi\right) , \qquad (10.1.1c)
\end{aligned}
$$

where the functions $e^{5\lambda}$ and Δ entering the solution are given by:

$$
e^{5\lambda} = \frac{e^{2\rho} + k e^{-2\rho} - \frac{1}{2}}{e^{2\rho} - \frac{1}{4}} , \qquad (10.1.2a)
$$

$$
\Delta = e^{5\lambda}\cos^2\chi + \sin^2\chi . \qquad (10.1.2b)
$$

Before proceeding let us give a short explanation of the various coordinates and constants appearing in eq.s (10.1.1)-(10.1.2):

- $\xi^{\alpha,\beta}$ $(\alpha, \beta = 0, 1, 2)$ are the coordinates along the unwrapped brane world-volume;
- $\tilde\theta\,, \tilde\varphi$ are the coordinates along the wrapped world-volume;
- ρ is a radial coordinate transverse to the brane;
- $\chi, \theta, \varphi, \psi$ parameterize the "twisted" four-sphere transverse to the brane;
- R_A is the radius of the AdS_7 space appearing in the near horizon geometry of the usual "flat" M5-brane solution (see appendix C.1.1), which is given in terms of ten dimensional quantities by $R_A = 2\sqrt{\alpha'}(\pi g_s N)^{1/3}$;

- R_0 is an arbitrary integration constant with dimension of a length that we will show to set the scale of the radius of the S^2 on which the D4-branes are wrapped;
- k is a dimensionless integration constant.

All coordinates are dimensionless except ξ^α which have dimension of a length.

A D4-brane is coupled naturally to a 5-form potential while the solution given above contains a RR 3-form potential. However, the latter is related to C_5 by the duality relation $dC_5 = {}^\star dC_3$ (in the string frame). By using it we get:

$$\begin{aligned} C_5 = &\frac{R_A^6}{R_0^4}\,\Delta e^{4\rho}\left(e^{2\rho}-\tfrac{1}{4}\right)\sin\tilde{\theta}\,d\xi^0\wedge d\xi^1\wedge d\xi^2\wedge d\tilde{\theta}\wedge d\tilde{\varphi} \\ &-\frac{R_A^6}{R_0^4}\,\frac{1}{2}e^{4\rho}\sin^2\chi\;d\xi^0\wedge d\xi^1\wedge d\xi^2\wedge d\rho\wedge\left(d\psi+\cos\tilde{\theta}d\tilde{\varphi}\right) \\ &-\frac{R_A^6}{R_0^4}\,\frac{1}{8}e^{4\rho}e^{5\lambda}\sin(2\chi)\;d\xi^0\wedge d\xi^1\wedge d\xi^2\wedge d\chi\wedge\left(d\psi+\cos\tilde{\theta}d\tilde{\varphi}\right)\,. \end{aligned} \tag{10.1.3}$$

A change of coordinates

The supergravity solution for the D4-branes wrapped on S^2 as given in eq. (10.1.1) is written in a way in which the role of the different coordinates and factors is not immediately clear. The first thing that we can do in order to clarify the role of the various terms appearing in the solution is to extract the warp factors for the longitudinal and transverse part of the metric in the string frame. They are given in terms of a function H that for a D4-brane is related to the dilaton through the following relation:

$$H = e^{-4\phi} = \left(\frac{R_0}{R_A}\right)^6 \Delta^{-1} e^{-6\rho}\,. \tag{10.1.4}$$

Using the previous definition of H, one can immediately see that the dependence on H of the four longitudinal unwrapped directions of the metric is the one corresponding to four "flat" world-volume directions: $H^{-1/2}\eta_{\alpha\beta}d\xi^\alpha d\xi^\beta$, as expected. We also expect three transverse directions (θ, φ and a suitable combination of ρ and χ) to be flat, apart from the usual warp factor $H^{1/2}$.

This can be seen to be correct[4] by using instead of the coordinates ρ and χ the following new coordinates:

$$\begin{cases} r = \frac{R_A^3}{2R_0^2} e^{2\rho} \cos\chi \\ \\ \sigma = \frac{R_A^3}{2R_0^2} \left[e^{2\rho} \left(e^{2\rho} - \frac{1}{4} \right) e^{5\lambda} \right]^{1/2} \sin\chi \end{cases} \tag{10.1.5}$$

which have dimensions of a length. In terms of the new coordinates in eq. (10.1.5), the solution for the metric, dilaton and R-R 5-form becomes:

$$\begin{aligned} ds^2_{\text{st}} = {} & H^{-1/2} \left[\eta_{\alpha\beta} d\xi^\alpha d\xi^\beta + \mathcal{Z} R_0^2 (d\tilde{\theta}^2 + \sin^2\tilde{\theta} d\tilde{\varphi}^2) \right] \\ & + H^{1/2} \left[dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2 \right) + \frac{1}{\mathcal{Z}} \left(d\sigma^2 + \sigma^2 \left(d\psi + \cos\tilde{\theta} d\tilde{\varphi} \right)^2 \right) \right] , \end{aligned} \tag{10.1.6a}$$

$$e^\phi = H^{-1/4} , \tag{10.1.6b}$$

$$C_5 = d\xi^0 \wedge d\xi^1 \wedge d\xi^2 \wedge \left[\frac{1}{H} \mathcal{Z} R_0^2 \sin\tilde{\theta} \; d\tilde{\theta} \wedge d\tilde{\varphi} - \frac{1}{\mathcal{Z}} \sigma d\sigma \wedge \left(d\psi + \cos\tilde{\theta} d\tilde{\varphi} \right) \right] , \tag{10.1.6c}$$

where the functions H and $\mathcal{Z}$ are (implicitly) defined as:

$$\begin{aligned} H(r,\sigma) &= \left(\frac{R_0}{R_A} \right)^6 \Delta^{-1}(r,\sigma) e^{-6\rho(r,\sigma)} , \\ \mathcal{Z}(r,\sigma) &= e^{-2\rho(r,\sigma)} \left(e^{2\rho(r,\sigma)} - \frac{1}{4} \right) . \end{aligned} \tag{10.1.7}$$

In the form given in eq. (10.1.6) the structure of the solution is much clearer. First of all one can clearly distinguish the trivial "flat" part of the solution from the nontrivial part coming from the internal directions of the four-dimensional Calabi–Yau space. In this sense, the coordinates r and σ that we have introduced represent two radial directions, respectively in the "flat" transverse space and in the space N_2 transverse to the brane but nontrivially fibered on the two-cycle on which the brane is wrapped. Moreover, the function $\mathcal{Z}$ represents the "running volume" of the two-cycle, with the constant

[4] See also Ref.s [233, 234] for related discussions.

R_0 being the radius of the S^2 when $\mathcal{Z} = 1$, while in the part of the metric containing σ and ψ we can easily see the twist which, as we have seen in section 10.1.1, is required for having a supersymmetric gauge theory living on the brane. Finally, also the R-R potential has a quite standard part (H^{-1} times the volume form of the longitudinal space), plus an additional part due to the twist.

Another change of coordinates can be implemented to extract some additional piece of information about the solution. If we define a new coordinate z and a function $\widetilde{\mathcal{Z}}$ as follows:

$$\begin{cases} z = R_0 \left(1 + \frac{\sigma^2}{R_0^2}\right)^{1/4} \\ \\ \widetilde{\mathcal{Z}} = \mathcal{Z} \left(1 + \frac{\sigma^2}{R_0^2}\right)^{-1/2} \end{cases} \tag{10.1.8}$$

the metric in eq. (10.1.6a) becomes:

$$\begin{aligned} ds^2_{\rm st} \; = \; & H^{-1/2} \left\{ \eta_{\alpha\beta} d\xi^\alpha d\xi^\beta + \widetilde{\mathcal{Z}} \; z^2 (d\tilde{\theta}^2 + \sin^2\tilde{\theta} d\tilde{\varphi}^2) \right\} \\ & + H^{1/2} \left\{ dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right) \right. \\ & \left. + \frac{1}{\widetilde{\mathcal{Z}}} \left[4 \left(1 - \frac{R_0^4}{z^4}\right)^{-1} dz^2 + z^2 \left(1 - \frac{R_0^4}{z^4}\right) \left(d\psi + \cos\tilde{\theta} d\tilde{\varphi}\right)^2 \right] \right\} . \end{aligned} \tag{10.1.9}$$

The metric we have obtained on the four-dimensional space spanned by the coordinates $\{\tilde{\theta}, \tilde{\varphi}, z, \psi\}$ is that of a "warped" Eguchi–Hanson space [235]. This fact provides additional evidence of the geometrical structure of the background: the D4-branes are wrapped on the two-sphere, of radius R_0, inside the simplest ALE space (which corresponds to the blow-up of an $\mathbb{R}^4/\mathbb{Z}_2$ orbifold)[5].

[5] As an aside, notice that, using eq.s (10.1.4)-(10.1.5) and eq. (10.1.8), also the M5-brane solution (C.1.5) (from which we derived the D4-brane solution) can be brought into a form analogous to the one given in eq. (10.1.6) or in eq. (10.1.9). These forms for the classical solutions describing wrapped branes seem indeed to be quite general. For instance, changes of coordinates similar to the ones implemented here can be used to put in these forms also the solution found in Ref. [216] for D5-branes wrapped on S^2.

10.1.3 Probing the wrapped brane solution

In order to extract information on the gauge theory living on the D4-branes, we will study the dynamics of a probe D4-brane wrapped on S^2 in the geometry generated by the solution found in the previous subsection (see Ref. [236] for a review on the probe technique). This will allow us to study the Coulomb branch of pure $\mathcal{N} = 4$ SYM theory in $2+1$ dimensions with gauge group $SU(N+1) \longrightarrow SU(N) \times U(1)$. The world-volume action for a single D4-brane in the string frame in the static gauge is given by:

$$\begin{aligned} S_{\text{probe}} &= -\frac{T_4}{\kappa}\int d^3\xi d\tilde{\theta} d\tilde{\varphi} e^{-\phi}\sqrt{-\det\left[G_{ab} + 2\pi\alpha' F_{ab}\right]} \\ &\quad + \frac{T_4}{\kappa}\int_{\mathcal{M}_5}\left(C_5 + 2\pi\alpha' C_3 \wedge F\right), \end{aligned} \tag{10.1.10}$$

where $a, b = \{0, 1, 2, \tilde{\theta}, \tilde{\varphi}\}$ and all the bulk fields are understood to be pull-backs onto the brane world-volume.

Let us first compute the static potential between the probe and the stack of N D4-branes, simply by substituting the solution (10.1.6) into eq. (10.1.10). The contribution of the Dirac–Born–Infeld part is given by:

$$e^{-\phi}\sqrt{-\det G_{ab}} = \sin\tilde{\theta}\frac{\mathcal{Z}R_0^2}{H}\left(1 + \frac{\sigma^2 H}{\mathcal{Z}^2 R_0^2}\right)^{1/2} . \tag{10.1.11}$$

Adding to it the Wess–Zumino part, whose contribution is computed using the expression (10.1.6c) of the R-R 5-form, we get the following expression for the static potential:

$$S_{\text{pot}} = -\frac{T_4}{\kappa}\int d^3\xi d\tilde{\theta} d\tilde{\varphi}\sin\tilde{\theta}\frac{\mathcal{Z}R_0^2}{H}\left[\left(1 + \frac{\sigma^2 H}{\mathcal{Z}^2 R_0^2}\right)^{1/2} - 1\right] . \tag{10.1.12}$$

We see that in general there is a force between the branes, and this means that the configuration is not supersymmetric. This had to be expected in some way because we are allowing the probe brane to move in *all* its transverse directions, including also the ones which are inside the Calabi–Yau space. If instead we allow the probe brane to move only in the "flat" part of the transverse space spanned by $\{r, \theta, \varphi\}$, keeping it fixed at the locus $\sigma = 0$ in the "internal" transverse space, we see that the potential (10.1.12) vanishes, yielding a supersymmetric configuration. Therefore in the following we will always work at the "supersymmetric locus" $\sigma = 0$.

In order to study the dynamics of the probe brane, we will allow the transverse coordinates $X^i = \{r, \theta, \phi\}$ to depend on the "flat" world-volume coordinates ξ^α but not on the "wrapped" ones x and y. Moreover, the gauge field $F_{\alpha\beta}$ is defined to be nonvanishing only on the "flat" part of the world-volume. Let us start from the DBI part of the action in eq. (10.1.10). By expanding the determinant, we find:

$$S_{\rm DBI} \simeq -\frac{T_4}{\kappa}\int d^3\xi d\tilde{\theta} d\tilde{\varphi} e^{-\phi}\sqrt{-\det G_{ab}} \times \left\{1 + \frac{1}{2}G^{\alpha\beta}G_{ij}\partial_\alpha X^i \partial_\beta X^j + \frac{(2\pi\alpha')^2}{4}G^{\alpha\gamma}G^{\beta\delta}F_{\alpha\beta}F_{\gamma\delta}\right\} . \quad (10.1.13)$$

Inserting the expressions (10.1.6a) and (10.1.6b) for the metric and dilaton we get:

$$\begin{aligned} S_{\rm DBI} &= -\frac{T_4}{2\kappa}\int d^3\xi d\tilde{\theta} d\tilde{\varphi}\sin\tilde{\theta}\frac{\mathcal{Z}R_0^2}{H} \quad (10.1.14) \\ &\times\left\{1 + \frac{1}{2}H\left[(\partial r)^2 + r^2\left((\partial\theta)^2 + \sin^2\theta(\partial\varphi)^2\right)\right] + \frac{(2\pi\alpha')^2}{4}HF^2\right\} , \end{aligned}$$

where we have included an additional factor of $1/2$ due to the normalization of the generators of the gauge group.

Notice that when $\mathcal{Z} = 0$, the effective tension of the brane vanishes, meaning that an enhançon mechanism is taking place [225]. Since, in order to preserve supersymmetry, we have fixed $\sigma = 0$, the enhançon radius is given by:

$$r_{\rm e} = \frac{R_A^3}{8R_0^2} = \frac{\pi g_s(\alpha')^{3/2}N}{R_0^2} . \quad (10.1.15)$$

In fact, this seems to be a general feature (albeit somewhat unnoticed in the literature) of the supergravity solutions corresponding to D-branes wrapped on cycles[6]. The fact that the solution is no longer valid inside the enhançon radius seems to prevent us from getting nonperturbative information on the gauge theory under study.

The transverse scalars have to be interpreted as Higgs fields for the gauge theory living on the brane: $X^i = 2\pi\alpha'\Phi^i$. Then, defining μ such that $r =$

[6]Notice however that the nature and location of the singularities of the metric depend on the value of the constant k appearing in eq. (10.1.2), as discussed in very similar cases in Ref.s [195,216,224]. Nonetheless, gauge theory physics as seen by the brane probe at the supersymmetric locus is independent of k, and only feels the existence of the enhançon.

$2\pi\alpha'\mu$ and integrating over the volume of the two-sphere on which the brane is wrapped, we obtain the final expression for the DBI part:

$$S_{\rm DBI} = -\frac{4\pi T_4}{2\kappa}\int d^3\xi \frac{\mathcal{Z} R_0^2}{H} \times \left\{ 1 + \frac{(2\pi\alpha')^2}{2} H\left[(\partial\mu)^2 + \mu^2\left((\partial\theta)^2 + \sin^2\theta(\partial\varphi)^2 \right) \right] + \frac{(2\pi\alpha')^2}{4} H F^2 \right\} . \tag{10.1.16}$$

Turning now to the WZ part, the pullback of C_3 is given by:

$$C_3 = \frac{1}{8} R_A^3 \cos\theta \sin\tilde{\theta} \partial_\alpha \varphi d\xi^\alpha \wedge d\tilde{\theta} \wedge d\tilde{\varphi} \, . \tag{10.1.17}$$

Then from eq. (10.1.10) we get:

$$\begin{aligned} S_{\rm WZ} &= \frac{T_4}{\kappa}\int d^3\xi d\tilde{\theta} d\tilde{\varphi} \sin\tilde{\theta} \left\{ \frac{\mathcal{Z} R_0^2}{H} + \frac{2\pi\alpha' R_A^3}{16} \cos\theta \varepsilon^{\alpha\beta\gamma} \partial_\alpha\varphi F_{\beta\gamma} \right\} \\ &= \frac{4\pi T_4}{\kappa}\int d^3\xi \left\{ \frac{\mathcal{Z} R_0^2}{H} + \frac{2\pi\alpha' R_A^3}{16} \cos\theta \varepsilon^{\alpha\beta\gamma} \partial_\alpha\varphi F_{\beta\gamma} \right\} \end{aligned} \tag{10.1.18}$$

Putting eq.s (10.1.16) and (10.1.18) together and substituting the expressions for T_4 , κ , R_A and for the function $\mathcal{Z}$, the probe action finally becomes:

$$\begin{aligned} S_{\rm probe} \;\; = \;\; & -\frac{R_0^2}{2\pi g_s\sqrt{\alpha'}}\int d^3\xi \left(1 - \frac{g_s\sqrt{\alpha'} N}{2R_0^2\mu} \right) \times \\ & \left\{ \frac{1}{2}\left[(\partial\mu)^2 + \mu^2\left((\partial\theta)^2 + \sin^2\theta(\partial\varphi)^2 \right) \right] + \frac{1}{4} F^2 \right\} \\ & + \frac{N}{8\pi}\int d^3\xi \cos\theta \varepsilon^{\alpha\beta\gamma} \partial_\alpha\varphi F_{\beta\gamma} \, . \end{aligned} \tag{10.1.19}$$

From the coefficient of F^2 in eq. (10.1.19) we can read the running gauge coupling constant of the three-dimensional gauge theory as a function of the scale μ . Defining the bare coupling as:

$$g_{\rm YM}^2 = \frac{2\pi g_s \sqrt{\alpha'}}{R_0^2} \, , \tag{10.1.20}$$

the running coupling constant is given by:

$$\frac{1}{g_{\rm YM}^2(\mu)} = \frac{1}{g_{\rm YM}^2}\left(1 - \frac{g_{\rm YM}^2 N}{4\pi\mu}\right) , \tag{10.1.21}$$

in perfect agreement with gauge theory expectations, as shown in appendix C.3.

Eq. (10.1.19) does not give explicitly the full metric on the moduli space of $\mathcal{N}=4$, $D=2+1$ SYM theory. In fact such a metric must be hyperKähler [230] and in eq. (10.1.19) we have only three moduli and not four as it should be in a hyperKähler metric. We need an extra modulus that can be obtained by dualising the vector field. In order to do that, we regard the original action in eq. (10.1.19) as a function of $F_{\alpha\beta}$ and we add to it a term:

$$-\int \Sigma\, dF\,, \tag{10.1.22}$$

so that the equation of motion for the auxiliary field Σ enforces the Bianchi identity for F on shell. By partially integrating the additional term in eq. (10.1.22), we are left with the following action:

$$\begin{aligned} S_{\rm probe} = &-\int d^3\xi\, \frac{1}{g_{\rm YM}^2(\mu)}\left\{\frac{1}{2}\left[(\partial\mu)^2 + \mu^2\left((\partial\theta)^2 + \sin^2\theta(\partial\varphi)^2\right)\right] + \frac{1}{4}F^2\right\} \\ &+ \frac{N}{8\pi}\int d^3\xi \cos\theta\varepsilon^{\alpha\beta\gamma}\partial_\alpha\varphi F_{\beta\gamma} + \frac{1}{2}\int d^3\xi\varepsilon^{\alpha\beta\gamma}\partial_\alpha\Sigma F_{\beta\gamma}\,. \end{aligned} \tag{10.1.23}$$

We can then eliminate F by means of its equation of motion that follows from eq. (10.1.23):

$$F_{\beta\gamma} = g_{\rm YM}^2(\mu)\varepsilon^{\alpha\beta\gamma}\left[\frac{N}{4\pi}\cos\theta\partial_\alpha\varphi + \partial_\alpha\Sigma\right] , \tag{10.1.24}$$

and we arrive at an action that contains four moduli, given by:

$$\begin{aligned} S_{\rm probe} = -\frac{1}{2}\int d^3\xi\Bigg\{ &\frac{1}{g_{\rm YM}^2(\mu)}\left[(\partial\mu)^2 + \mu^2\left((\partial\theta)^2 + \sin^2\theta(\partial\varphi)^2\right)\right] \\ &+ g_{\rm YM}^2(\mu)\left(\frac{N\cos\theta}{4\pi}\partial\varphi + \partial\Sigma\right)^2\Bigg\}\,. \end{aligned} \tag{10.1.25}$$

The complete metric on the moduli space $\mathcal{M}$ of the gauge theory, in terms of the 4 scalars μ , θ , ϕ and Σ is finally given by:

$$ds^2_{\mathcal{M}} = \frac{1}{g^2_{\rm YM}(\mu)}\left(d\mu^2 + \mu^2 d\Omega^2\right) + g^2_{\rm YM}(\mu)\left(d\Sigma + \frac{N\cos\theta}{4\pi}d\varphi\right)^2 , \quad (10.1.26)$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$. The metric in eq. (10.1.26) is indeed hyperKähler since it has precisely the form of the Taub-NUT metric [237]. However, because of the form given in eq. (10.1.21) of the function $g_{\rm YM}(\mu)$ our metric has a "negative mass" and thus is singular. This is due to the fact that in our probe analysis we are only able to reproduce the perturbative behaviour of the gauge theory. As discussed in Ref.s [225, 236], the complete metric should also include the instanton contribution, becoming a completely nonsingular generalization of the Atiyah-Hitchin metric.

10.2 Fractional D2/D6-brane system

10.2.1 Setup

In this section we consider a system of fractional branes on the orbifold:

$$\mathbb{R}^{1,5} \times \mathbb{R}^4/\mathbb{Z}_2 , \quad (10.2.1)$$

where $\mathbb{Z}_2$ acts by changing sign to the last four coordinates:

$$\left\{x^6, x^7, x^8, x^9\right\} \longrightarrow \left\{-x^6, -x^7, -x^8, -x^9\right\} .$$

To be precise, we are going to study a configuration of type IIA string theory[7], made of N fractional D2-branes extended along x^0, x^1, x^2 and M D6-branes extended along $x^0, x^1, x^2, x^6, \ldots, x^9$, as shown schematically in the following table, where the symbols – and · denote respectively coordinates which are longitudinal and transverse to the branes:

	0	1	2	3	4	5	6	7	8	9
							$\mathbb{R}^4/\mathbb{Z}_2$			
D2	–	–	–	·	·	·	·	·	·	·
D6	–	–	–	·	·	·	–	–	–	–

[7]Classical solutions describing fractional D-branes in type IIA orbifolds were constructed in Ref. [203].

A peculiar feature of the fractional branes transverse to the orbifold space (as the D2-branes that we are considering) is that they are stuck at the orbifold fixed point $x^6 = x^7 = x^8 = x^9 = 0$.

The orbifold projection breaks half of the supersymmetry of type IIA theory, and so does the considered D2/D6-brane system. We are then left with 8 supercharges. Thus, at low energy the theory living on N fractional D2-branes is $\mathcal{N} = 4$, $D = 2+1$ SYM theory with gauge group $SU(N)$. Moreover, from the point of view of this gauge theory the strings stretching from the D2 to the D6-branes and vice-versa make up M hypermultiplets which transform in the fundamental representation of the gauge group.

In order to describe the above system by means of a supergravity solution, we have to study how the low-energy fields which appear in the effective action behave in the background (10.2.1). Our background is characterized by the presence of a 2-form ω_2, Poincaré dual to the exceptional 2-cycle Σ_2 of the ALE space which is obtained by the resolution of the orbifold. In the orbifold limit, the volume of Σ_2 vanishes, but the background value of the integral of B_2 on it has to remain finite in order to define a sensible CFT [238, 239]:

$$\int_{\Sigma_2} B_2 = \frac{\left(2\pi\sqrt{\alpha'}\right)^2}{2} . \tag{10.2.2}$$

The 2-form ω_2 satisfies the following properties:

$$\omega_2 = -{}^{\star 4}\omega_2 , \qquad \int_{\Sigma_2} \omega_2 = 1 , \qquad \int_{\mathrm{ALE}} \omega_2 \wedge \omega_2 = -\frac{1}{2} . \tag{10.2.3}$$

The supergravity fields can have components along the vanishing cycle, so the following decomposition holds for the NS-NS two-form and the R-R three-form:

$$B_2 = \bar{B}_2 + b\,\omega_2 , \qquad C_3 = \bar{C}_3 + A_1 \wedge \omega_2 . \tag{10.2.4}$$

Since we will be looking for supergravity solutions which represent branes without a B_2 field in their world-volume, in the following we will put $\bar{B}_2 = 0$, so we simply have:

$$B_2 = b\,\omega_2 , \qquad C_3 = \bar{C}_3 + A_1 \wedge \omega_2 , \tag{10.2.5}$$

where, because of eq. (10.2.2):

$$b = \frac{\left(2\pi\sqrt{\alpha'}\right)^2}{2} + \tilde{b} , \tag{10.2.6}$$

and $\tilde{b}$ represents the fluctuation around the background value of b. We will sometimes refer to the fields b and A_1 in eq. (10.2.5) as "twisted" fields because they correspond to the massless states of the twisted sector of type IIA string theory on the orbifold.

Having given the main features of the system that we are going to study, we now turn to the supergravity solution.

10.2.2 The supergravity solution

In this subsection we will discuss the supergravity solution describing the fractional D2/D6 system that we have introduced in the previous subsection. The solution is derived in detail in appendix C.1.2. Here we will only summarize the procedure followed to find it.

The first step is to substitute the decompositions for B_2 and C_3 given in eq. (10.2.5) into the type IIA supergravity action, and to derive the equations of motion for the "untwisted" fields $G_{\mu\nu}$, ϕ , $\bar{C}_3$ and C_1 and for the "twisted" ones b and A_1 .

Then, we impose the standard Ansatz for the "untwisted" fields corresponding to a D2/D6 system[8]:

$$ds^2 = H_2^{-5/8} H_6^{-1/8} \eta_{\alpha\beta} dx^\alpha dx^\beta + H_2^{3/8} H_6^{7/8} \delta_{ij} dx^i dx^j + H_2^{3/8} H_6^{-1/8} \delta_{pq} dx^p dx^q \,, \tag{10.2.7a}$$

$$e^\phi = H_2^{1/4} H_6^{-3/4} \,, \tag{10.2.7b}$$

$$\bar{C}_3 = \left(H_2^{-1} - 1\right) dx^0 \wedge dx^1 \wedge dx^2 \,, \tag{10.2.7c}$$

where the function H_2 depends on the radial coordinate

$$\rho = \sqrt{(x^3)^2 + \ldots + (x^9)^2}$$

of the space transverse to the D2-brane, while the function H_6 depends only on the radial coordinate of the common transverse space $r = \sqrt{\delta_{ij} x^i x^j}$.

In order to write down a sensible Ansatz for the fields A_1 and C_1 , we need to take into account the contribution coming from the boundary action describing the world-volume theory of the branes. After some calculation it is easy to get convinced that the following[9] is a sensible Ansatz for the fields

[8]The coordinates are labeled as: $\alpha, \beta = \{0, 1, 2\}$, $i, j = \{3, 4, 5\}$ and $p, q = \{6, 7, 8, 9\}$.

[9]When $M = 0$, the Ansätze for A_1 and C_1 coincide with the ones given in Ref. [240].

A_1 and C_1 :

$$dA_1 = C_1 \wedge db + \frac{1}{2}\varepsilon_{ijk} H_6 \partial_i b dx^j \wedge dx^k \,, \tag{10.2.8a}$$

$$dC_1 = \frac{1}{2}\varepsilon_{ijk} \partial_i H_6 dx^j \wedge dx^k \,, \tag{10.2.8b}$$

where $\varepsilon_{345} = \varepsilon^{345} = +1$.

Substituting our Ansätze in the equations of motion and computing all the relevant contributions coming from the boundary action S_b , the final solution for the fractional D2/D6 system can be expressed in the following form[10]:

$$ds^2 = H_2^{-5/8} H_6^{-1/8} \eta_{\alpha\beta} dx^\alpha dx^\beta + H_2^{3/8} H_6^{7/8} \delta_{ij} dx^i dx^j + H_2^{3/8} H_6^{-1/8} \delta_{pq} dx^p dx^q \,, \tag{10.2.9a}$$

$$e^\phi = H_2^{1/4} H_6^{-3/4} \,, \tag{10.2.9b}$$

$$\bar{C}_3 = \left(H_2^{-1} - 1\right) dx^0 \wedge dx^1 \wedge dx^2 \,, \tag{10.2.9c}$$

$$C_1 = \frac{g_s \sqrt{\alpha'} M}{2} \cos\theta d\varphi \,, \tag{10.2.9d}$$

$$A_1 = -\pi^2 \alpha' \frac{g_s \sqrt{\alpha'}(4N - M)}{H_6} \cos\theta d\varphi \,, \tag{10.2.9e}$$

$$b = \frac{Z}{H_6} \,, \tag{10.2.9f}$$

where:

$$\begin{aligned} H_6(r) &= 1 + \frac{g_s \sqrt{\alpha'} M}{2r} \,, \\ Z(r) &= \frac{\left(2\pi\sqrt{\alpha'}\right)^2}{2} \left(1 - \frac{g_s\sqrt{\alpha'}(2N - M)}{r}\right) \,, \end{aligned} \tag{10.2.10}$$

and where H_2 is the solution of the following equation (see eq. (C.1.23a) in appendix C.1.2):

$$\left(\delta^{ij}\partial_i\partial_j + H_6 \delta^{pq}\partial_p\partial_q\right) H_2 + \frac{1}{2} H_6 \delta^{ij} \partial_i b \partial_j b \delta(x^6) \cdots \delta(x^9) + \kappa T_2 N \delta(x^3) \cdots \delta(x^9) = 0 \,. \tag{10.2.11}$$

[10]Notice that, in order to easily express the fields A_1 and C_1 in eq.s (10.2.9), we have changed coordinates in the common transverse space into polar coordinates: $(x^3, x^4, x^5) \longrightarrow (r, \theta, \varphi)$.

From the solution given in eq. (10.2.9) we can also compute the expressions for the fields C_7 and A_3 which appear naturally in the string theory. The duality relations are[11]:

$$dC_7 = -e^{3\phi/2\star} dC_1 \,, \tag{10.2.12a}$$

$$dA_3 = e^{\phi/2} \, {}^{\star 6} G_2 - db \wedge \bar{C}_3 \,, \tag{10.2.12b}$$

and the explicit computation gives:

$$C_7 = (H_6^{-1} - 1) \; dx^0 \wedge \cdots \wedge dx^6 \,, \tag{10.2.13a}$$

$$A_3 = \tilde{b} \; dx^0 \wedge dx^1 \wedge dx^2 \,. \tag{10.2.13b}$$

Notice that the field C_7 has a quite standard expression, due to the specific form of the Ansatz in eq. (10.2.8).

10.2.3 Probing the fractional brane solution

In this section we will study the world-volume theory of a probe fractional D2-brane, which is placed in the background given in eq.s (10.2.9) at some finite distance r in the transverse space $\{x^3, x^4, x^5\}$. This will give us information about the Coulomb branch of $\mathcal{N} = 4$ three-dimensional super Yang–Mills theory with gauge group $SU(N+1)$ broken into $SU(N) \times U(1)$, coupled to M hypermultiplets in the fundamental representation of the gauge group.

Let us start from the world-volume action for a single fractional D2-brane, which is given by eq. (C.2.9) in the case of $p = 2$:

$$S_{\text{probe}} = -\frac{T_2}{2\kappa} \int d^3\xi e^{-\phi/4} \sqrt{-\det\left[G_{\alpha\beta} + e^{-\phi/2} 2\pi\alpha' F_{\alpha\beta}\right]} \left(1 + \frac{\tilde{b}}{2\pi^2\alpha'}\right) + \frac{T_2}{2\kappa} \int_{\mathcal{M}_3} \left(\mathcal{C}_3 + 2\pi\alpha' \mathcal{C}_1 \wedge F\right) , \tag{10.2.14}$$

where all bulk fields are understood to be pullbacks onto the brane world-

[11] The duality relations can be derived from the equations of motion (C.1.13a) and (C.1.13c), for which the terms coming from the boundary action vanish (see appendix C.1.2).

volume and the fields $\mathcal{C}_3$ and $\mathcal{C}_1$ are given by:

$$\mathcal{C}_3 = \bar{C}_3 \left(1 + \frac{\tilde{b}}{2\pi^2\alpha'}\right) + \frac{A_3}{2\pi^2\alpha'} = \frac{1}{2\pi^2\alpha'}\frac{Z}{H_2H_6} - 1\,, \tag{10.2.15a}$$

$$\mathcal{C}_1 = C_1 \left(1 + \frac{\tilde{b}}{2\pi^2\alpha'}\right) + \frac{A_1}{2\pi^2\alpha'} = -g_s\sqrt{\alpha'}(2N - M)\cos\theta d\varphi\,. \tag{10.2.15b}$$

The computation is analogous to the one performed in section 10.1.3. We fix the static gauge, and regard the coordinates $\{x^3, x^4, x^5\}$ transverse to the probe brane as Higgs fields of the dual gauge theory: $x^i = 2\pi\alpha'\Phi^i$. We also define polar coordinates (μ, θ, φ) in the moduli space of the Φ^i, so that $r = 2\pi\alpha'\mu$.

Expanding the DBI action for slowly varying world-volume fields and keeping only up to quadratic terms in their derivatives we get:

$$\begin{aligned} S_{\text{DBI}} \;\simeq\; & -\frac{\sqrt{\alpha'}}{2g_s}\int d^3x \frac{Z}{2\pi^2\alpha'}\left(\frac{1}{2}\eta^{\alpha\beta}\delta_{ij}\partial_\alpha\Phi^i\partial_\beta\Phi^j + \frac{1}{4}\eta^{\alpha\beta}\eta^{\gamma\delta}F_{\alpha\gamma}F_{\beta\delta}\right) \\ & -\frac{T_2}{2\kappa}\int d^3x \frac{1}{2\pi^2\alpha'}\frac{Z}{H_2H_6}\,. \end{aligned} \tag{10.2.16}$$

Turning to the WZ part and substituting the expressions (10.2.15) into eq. (10.2.14) we get:

$$S_{\text{WZ}} = \frac{T_2}{2\kappa}\left[\int d^3x \left(\frac{1}{2\pi^2\alpha'}\frac{Z}{H_2H_6} - 1\right) + \int_{\mathcal{M}_3} 2\pi\alpha'\mathcal{C}_\varphi d\varphi \wedge F\right]\,. \tag{10.2.17}$$

We easily see that the position-dependent terms cancel as expected because fractional D2-branes are BPS states and do not interact with the D2/D6 system. Ignoring the constant potential, the final result is:

$$\begin{aligned} S_{\text{probe}} = & -\frac{\sqrt{\alpha'}}{4g_s}\int d^3x \frac{Z}{2\pi^2\alpha'}\left\{\frac{1}{2}\left[(\partial\mu)^2 + \mu^2\left((\partial\theta)^2 + \sin^2\theta(\partial\varphi)^2\right)\right] + \frac{1}{4}F^2\right\} \\ & - \frac{1}{16\pi}\int d^3x(2N - M)\cos\theta\varepsilon^{\alpha\beta\gamma}\partial_\alpha\varphi F_{\beta\gamma}\,. \end{aligned} \tag{10.2.18}$$

When $Z = 0$ the effective tension of the probe vanishes and this means that also in this case, as expected, an enhançon mechanism is taking place at the radius:

$$r_{\text{e}} = \sqrt{\alpha'}g_s(2N - M)\,. \tag{10.2.19}$$

Substituting in (10.2.18) the expression of Z in terms of μ, we obtain:

$$\begin{aligned} S_{\rm probe} &= -\frac{\sqrt{\alpha'}}{4g_s}\int d^3x\left[1-\frac{g_s(2N-M)}{2\pi\sqrt{\alpha'}\mu}\right]\times \\ &\quad \left\{\frac{1}{2}\left[(\partial\mu)^2+\mu^2\left((\partial\theta)^2+\sin^2\theta(\partial\varphi)^2\right)\right]+\frac{1}{4}F^2\right\} \\ &\quad -\frac{1}{16\pi}\int d^3x(2N-M)\cos\theta\varepsilon^{\alpha\beta\gamma}\partial_\alpha\varphi F_{\beta\gamma}\,. \end{aligned} \tag{10.2.20}$$

From the coefficient of the gauge field kinetic term in the previous action we can read the running coupling constant:

$$\frac{1}{g^2_{\rm YM}(\mu)}=\frac{1}{g^2_{\rm YM}}\left(1-g^2_{\rm YM}\frac{2N-M}{8\pi\mu}\right), \tag{10.2.21}$$

where we have defined the bare coupling as:

$$g^2_{\rm YM}=\frac{4g_s}{\sqrt{\alpha'}}\,. \tag{10.2.22}$$

Eq. (10.2.21) is exactly what expected for the gauge theory under consideration, as shown in appendix C.3.

Exactly as in the case of the wrapped branes described in section 10.1, eq. (10.2.20) does not give explicitly the full hyperKähler metric on the moduli space of the gauge theory. We can obtain the needed extra modulus by dualising the vector field into a scalar, using exactly the same procedure which brought us from eq. (10.1.19) to eq. (10.1.26) in section 10.1. The final result is[12]:

$$ds^2_{\mathcal{M}}=\frac{1}{g^2_{\rm YM}(\mu)}\left(d\mu^2+\mu^2d\Omega^2\right)+g^2_{\rm YM}(\mu)\left(d\Sigma+\frac{(2N-M)\cos\theta}{8\pi}d\varphi\right)^2, \tag{10.2.23}$$

where $d\Omega^2=d\theta^2+\sin^2\theta d\varphi^2$. If we put $M=0$ this metric coincides with the one found in eq. (10.1.26) by probing the geometry of N D4-branes wrapped on S^2. Again, we have found the hyperKähler Taub-NUT metric, but with a "negative mass" which makes it singular. Also in this case we have only recovered the perturbative behaviour of the gauge theory.

[12]In this case, the dualisation procedure can also be done directly in the original three-dimensional world-volume action, as in Ref.s [225, 241]. The result that one obtains coincides with that in eq. (10.2.23).

Complete action for a D6-brane extended along the orbifold

As in the case of the D7-brane analyzed in Ref. [215], the supergravity solution corresponding to our D2/D6 system can also be used to get the form of the complete world-volume action for a D6-brane extended along the whole orbifold space. In fact, in deriving the classical solution corresponding to the D2/D6 system it was enough to consider only the linear terms in the bulk fields. However, since the D2/D6 system is BPS, we expect that when we insert the corresponding classical solution into the world-volume action of either the D2-brane or the D6-brane, we obtain a constant result. In the previous subsection we have seen that this is the case for a fractional D2-brane. If we instead insert the classical solution into the action of a D6-brane given by (as it follows from eq. (C.1.18b) for $p = 2$):

$$\begin{aligned} S_6 = \frac{T_6}{\kappa}\left\{-\int d^7\xi e^{\frac{3}{4}\phi}\sqrt{-\det G_{\rho\sigma}} + \int_{\mathcal{M}_7} C_7\right\} \\ + \frac{T_2}{2\kappa}\frac{1}{2(2\pi\sqrt{\alpha'})^2}\left\{\int d^3\xi\sqrt{-\det G_{\alpha\beta}}\,\tilde{b} - \int_{\mathcal{M}_3} A_3\right\} + \ldots\,, \end{aligned} \tag{10.2.24}$$

we get terms that are dependent on the distance r between the D6-brane and the system D2/D6 described by the classical solution. The situation here is exactly the same as the one found in Ref. [215], and as in that case we must add to the previous action higher order terms that restore the no-force condition. Including them we arrive at the following boundary action for a D6-brane extended along the whole orbifold space:

$$\begin{aligned} S_6 = \frac{T_6}{\kappa}\left\{-\int d^7x e^{\frac{3}{4}\phi}\sqrt{-\det G_{\rho\sigma}} + \int_{\mathcal{M}_7} C_7\right\} \\ + \frac{T_2}{2\kappa}\frac{1}{2(2\pi\sqrt{\alpha'})^2}\left\{\int d^3\xi e^{-\phi/4}\sqrt{-\det G_{\alpha\beta}}\,\tilde{b}\left(1 + \frac{\tilde{b}}{4\pi^2\alpha'}\right)\right. \\ \left. - \int_{\mathcal{M}_3} A_3\left(1 + \frac{\tilde{b}}{4\pi^2\alpha'}\right) - \int_{\mathcal{M}_3}\bar{C}_3\tilde{b}\left(1 + \frac{\tilde{b}}{4\pi^2\alpha'}\right)\right\}. \end{aligned} \tag{10.2.25}$$

10.3 Discussion and conclusions

In this final section we want to compare the two approaches that we have described in the previous section. Let us compare the two cases dual to the pure gauge theory, that is, in absence of D6-branes ($M = 0$) in the fractional brane case.

Let us start by summarizing some of the properties and differences of the two systems:

- Both systems are able to capture only the perturbative dynamics of $\mathcal{N} = 4\,,\ D = 2 + 1$ SYM theory.

- The role of the running coupling constant is played in the two supergravity solutions by two parameters: the "running volume" $\mathcal{Z}$ of the 2-cycle for the wrapped D4-branes and the "twisted B-field" b for the fractional D2-branes.

- The enhançon, where the gauge coupling $g_{\rm YM}$ diverges, is located at the locus where respectively $\mathcal{Z} = 0$ and $b = 0\,$.

Does it exist a closer relationship between the two setups? Both systems consist of wrapped branes. In fact, on the one hand, as we have seen in section 10.2, a fractional Dp-brane can be seen as a D$(p+2)$-brane wrapped on the vanishing two-cycle of the ALE space which corresponds to the blow-up of the orbifold. On the other hand we have also seen that the D4-branes considered in section 10.1 are wrapped on a two-sphere inside the same ALE space, as explicitly shown in eq. (10.1.9). In fact, the two systems provide exactly the same information about the gauge theory living on their world-volume. In order to see the connection between the two systems it is useful to write down a general formula that provides the perturbative running coupling constant of a general $(p+1)$-dimensional gauge theory living on the flat part of the world-volume of a D$(p+2)$-brane wrapped on a (vanishing or not) two-cycle Σ_2. It is given by the following expression:

$$\frac{1}{g^2_{\rm YM}(\mu)} = \frac{V_{\rm ST}\,(\Sigma_2)}{g^2_{{\rm D}p}}\,, \tag{10.3.1}$$

where $g^2_{{\rm D}p} = 2(2\pi)^{p-2} g_s {\alpha'}^{(p-3)/2}$ is the usual (bare) coupling constant of the gauge theory living on a Dp-brane in flat space and the dimensionless "stringy volume" $V_{\rm ST}$ is given by:

$$V_{\rm ST}\,(\Sigma_2) = \frac{1}{\left(2\pi\sqrt{\alpha'}\right)^2}\int d^2\zeta\,\sqrt{-\det\left(\mathcal{G}_{AB}+B_{AB}\right)}\,, \tag{10.3.2}$$

where $\zeta^{1,2}$ parameterizes the cycle Σ_2 , while $\mathcal{G}_{AB}$ and B_{AB} $(A, B = 1, 2\,)$ are the bulk metric *without any warp factors* and the B-field on the cycle.

We can easily see that the formula in eq. (10.3.1) holds for the systems considered in sections 10.1 and 10.2. For the three-dimensional gauge theory at hand, we have $g_{\rm D2}^2 = \frac{2g_s}{\sqrt{\alpha'}}$.

For the case of the D4-branes wrapped on S^2 , we have:

$$\mathcal{G}^{\rm w} = \mathcal{Z}R_0^2\begin{pmatrix}1 & 0\\ 0 & \sin^2\tilde{\theta}\end{pmatrix}\,, \qquad B^{\rm w} = 0\,, \tag{10.3.3}$$

so that the "stringy volume" is given by:

$$V_{\rm ST}^{\rm w}\,(\Sigma_2) = \frac{\mathcal{Z}R_0^2}{\pi^2\alpha'}\,. \tag{10.3.4}$$

Substituting eq. (10.3.4) in eq. (10.3.1) we get the correct running for $\mathcal{N}=4$, $D=2+1$ SYM theory with $SU(N)$ gauge group:

$$\frac{1}{(g_{\rm YM}^{\rm w})^2(\mu)} = \frac{1}{(g_{\rm YM}^{\rm w})^2}\left(1-\frac{(g_{\rm YM}^{\rm w})^2N}{4\pi\mu}\right)\,, \tag{10.3.5}$$

where the bare coupling is defined as in eq. (10.1.20) as follows:

$$(g_{\rm YM}^{\rm w})^2 = \frac{2\pi g_s\sqrt{\alpha'}}{R_0^2}\,.$$

If instead we consider the fractional D2-branes as D4-branes wrapped on the vanishing cycle Σ_2 , we find:

$$\mathcal{G}^{\rm f} = 0\,, \qquad B^{\rm f} = Z\omega_2\,. \tag{10.3.6}$$

Now the "stringy volume" is:

$$V_{\rm ST}^{\rm f}\,(\Sigma_2) = \frac{Z}{4\pi^2\alpha'}\,. \tag{10.3.7}$$

and substituting eq. (10.3.7) in eq. (10.3.1) we get again the correct running coupling constant:

$$\frac{1}{(g^{\mathrm{f}}_{\mathrm{YM}})^2(\mu)} = \frac{1}{(g^{\mathrm{f}}_{\mathrm{YM}})^2}\left(1 - \frac{(g^{\mathrm{f}}_{\mathrm{YM}})^2 N}{4\pi\mu}\right) . \tag{10.3.8}$$

where now the bare coupling constant is defined as $(g^{\mathrm{f}}_{\mathrm{YM}})^2 = \frac{4g_s}{\sqrt{\alpha'}}$ as in eq. (10.2.22).

Notice also that if we choose the "background value" V_0 of the geometrical volume on which the D4-branes are wrapped (that is, the volume of the two-sphere inside the Eguchi–Hanson space in eq. (10.1.9), once we remove the branes setting $H = \widetilde{\mathcal{Z}} = 1$) in such a way that it coincides with the background value of the B-field of the fractional brane case, $V_0 = 4\pi R_0^2 = \frac{(2\pi\sqrt{\alpha'})^2}{2}$, the bare coupling constants (and enhançon radii) computed in the two cases become exactly the same in terms of string parameters.

One can see that the formula (10.3.1) works perfectly also, for instance, for the case of $\mathcal{N} = 2$ SYM in four dimensions, applying it to the fractional D3-brane solution of Ref.s [200, 212] and to the wrapped D5-brane solution of Ref. [216].

Although the two systems give the same perturbative gauge coupling constant, there seems not to be a "physical" limit in which one can obtain the fractional brane solution from the wrapped one or vice-versa, playing with the volume of the cycle. This is due to the fact that the two Ansätze are radically different in the warp factors, which are respectively the ones of a D4 and of a D2-brane, and is also due to the absence of a B-field on the world-volume of the D4-branes wrapped on S^2. On the other hand, if we look at the whole moduli space of the four-dimensional ALE space, we see that it is characterized by the volume of the exceptional cycle and by the flux of the B-field on it. These are the two moduli that are combined into the "stringy volume" in eq. (10.3.2) which, as we have seen, provides the running coupling constant. The situation in the two cases can then be summarized by the following diagram:

$B = 0, \quad V \neq 0$ **Wrapped branes**	$\longrightarrow$	$B = 0, \quad V = 0$ ***Enhanced gauge symmetry***	$\longleftarrow$	$B \neq 0, \quad V = 0$ **Fractional branes**

where, in the case of the wrapped D4-branes, we are keeping the size of the cycle finite and the B-flux vanishing, while in the case of the fractional

D2-branes the geometrical volume shrinks to zero size and in order to have a *conformal* orbifold we must give a definite fixed background value to the B-flux, which in the case of the $\mathbb{Z}_2$ orbifold is $\frac{(2\pi\sqrt{\alpha'})^2}{2}$.

The limiting case in which both the geometrical volume of the cycle and the B-flux are taken to vanish is the point where the theory on the Calabi–Yau space manifests an enhanced gauge symmetry, which is at the origin of the enchançon mechanism. This is the point where the "stringy volume" vanishes and the supergravity solutions break down.

Summary and Conclusions

Let us summarize the main results that we have obtained and extract some conclusions and ideas for future work.

Part I

In Chapter 3 we presented the most general family of pointlike solutions of $N = 4$ Supergravity in four dimensions. This theory is a consistent truncation of the heterotic string compactified on a six-torus, therefore all these solutions could be uplifted to ten dimensions. They are, in general, non-supersymmetric: they include all possible pointlike configurations and all previously known solutions of this theory. Their embedding in ten dimensions could be useful if one is interested in studying the black hole physics of these four dimensional solutions from a String Theory perspective. Since in general they describe non-extremal configurations of finite Hawking temperature, they are suitable to the study of thermal emission of black holes, at least in the near-extremal cases.

The whole set of configurations is invariant, as a family, with respect to the duality symmetries of the theory. $N = 4, d = 4$ Supergravity is a rather simple theory which has the interesting property that it exhibits both S- and T-dualities. In this sense, it provides us with a simple framework to check which are the physical configurations of a given theory that are contained within the "representations" of its duality group. We found the most general duality-invariant family of solutions, and our results prove that, for this theory, all possible physical solutions (supersymmetric or not) with well defined physical charges are included in the orbits of the duality group of the theory. At least in the case of black hole spacetimes, we know that all possible physical configurations are included within the general solution because all charges allowed by the "no-hair theorem" are included and can take completely independent values. The duality-invariant properties of the thermodynamic quantities in all cases have also been checked.

In Chapter 4 we tried to see the implications of this idea in a context directly related to String Theory. The goal there was to investigate if all known solitons of Type II Supergravities provided a "complete representation" of both S- and T-duality or not. We found that in fact this is not the case if one just considers the standard solitons. By finding the successive T-duals of the S-dual of the Supergravity solution describing the D7-brane, we found a whole chain of previously unknown solutions of both Type II theories. Furthermore, acting on the Type IIB solutions that we found with S-duality, and uplifting the Type IIA ones to eleven dimensions, we ended up with a whole family of (again, previously unknown) Type II and eleven dimensional solitons. The most interesting property shared by all of them is that they have the characteristics of being describing nonperturbative, elementary (as opposed to composite) string states: all of them preserve 1/2 of the supersymmetries and are charged under a single gauge field. In addition, they turn out to be highly nonperturbative, as their masses scale with higher inverse powers of the string coupling. In fact, our computation of these masses shows that they exactly match with the predictions made some years ago by Hull, Obers, Rabinovici and others: based on U-duality arguments, they conjectured the existence of some ten dimensional nonperturbative states which would be missing from the known ten-dimensional spectrum. They were able to compute their masses, and their results match with ours.

A common property of all these solutions is that they behave as the Kaluza-Klein monopole in the sense that, in their presence, a number of transverse dimensions have always to be considered as compact. We explained this property by showing that all of them can be obtained from a lower dimensional theory that originates from a dimensional reduction of the corresponding ten dimensional action. They can be generated in lower dimensions by means of symmetries that exclusively exist in the lower-dimensional theory.

The main objection found in the literature to consider the S-dual of the D7-brane is that the Type IIB super-Poincaré algebra has no central charge accounting for this state. We have also shown that this is not the case: we provided a generic method to find all the central charges associated to all these solitonic states. As explained in Section 2.3, the structure of a given superalgebra is given by the superisometries of the vacuum. If we take into account the isometric directions of these backgrounds (which are not asymptotically flat), one can formulate consistent superalgebras that include the central charges associated to these states.

It would be very interesting to find the String Theory description of all these states. Also their near-horizon geometries, their worldvolume theories and possible applications of these solutions in the context of AdS/CFT could be of interest. We would like to address these issues in the future.

Part II

In Part II we have presented some results concerning spacetimes that no longer describe massive or charged states of String Theory, but which instead can be considered as vacua. In Chapter 6 we showed how all maximally supersymmetric vacua (other than Minkowski, i.e. basically $AdS \times S$ and Hpp-wave spacetimes) of theories with eight supercharges in 4,5 and 6 dimensions are related by uplifting and compactification. This shows that a standard belief, namely that compactifications of maximally supersymmetric solutions break supersymmetry, is indeed not correct: at least for these cases, it is always possible to find a compactification that preserves all supersymmetries (we saw that such a compactification always involve the consistent truncation of some lower dimensional supermultiplet). In fact, it so happens that it is always possible to find a frame with a specific direction with respect to which not only the solutions describing these spacetimes, but also their Killing spinors, become isometric. This also allows to study T-duality of these solutions (although, in general, supersymmetry will not be preserved after T-dualization). This provides us with a unified picture of all these vacuum spacetimes. A consequence that we found is that the maximally supersymmetric solution given by the near-horizon limit of the 5-dimensional extreme rotating black hole actually interpolates between $AdS_2 \times S^3$ and $AdS_3 \times S^2$. This interpolation turns out to be smooth in parameter space, and has a simple explanation in terms of its four dimensional compactified partner: this partner is just the near-horizon limit of the dyonic Reissner-Nordström black hole. The purely electric or magnetic cases correspond to the $AdS_2 \times S^3$ or $AdS_3 \times S^2$ "corners", while a generic four dimensional dyonic solution represents a point in between in the moduli space of five dimensional vacua.

It is a well known fact that five and eleven dimensional supergravities exhibit remarkable similarities. It would be interesting to investigate the possibility of generalizing this unified picture of five dimensional vacua to eleven dimensions. The $AdS_2 \times S^3$ and $AdS_3 \times S^2$ five dimensional vacua are very reminiscent of the $AdS_4 \times S^7$ or $AdS_7 \times S^4$ maximally supersymmetric solutions of eleven dimensional supergravity (which correspond, respectively, to the near-horizon limits of the M2- and M5-brane). In fact, one can see that the interpolating five dimensional solution corresponds to a continuous rotation of an S^1 fiber to get, in the limiting cases, a Hopf-fibration of S^3 or AdS_3 with respective base spaces given by S^2 or AdS_2. The next possible Hopf-fibration is that of S^7 with S^3 and S^4 as fiber and base space, and it should be certainly possible to find an analogue for AdS_7. The main problem to parallel in eleven dimensions the whole five dimensional construction just presented is that, now, it would involve an S^3 compactification of eleven-dimensional Supergravity down to eight dimensions. This is something which is technically quite complicated.

In Chapter 7 a formal mathematical analysis concerning the supersymmetric properties of maximally supersymmetric vacua was put forward. As explained in Chapter 5, these spacetimes admit a coset space description. For a given coset spacetime G/H, there is a well-known procedure to find a G-invariant metric for it, as well as to construct the corresponding Killing vectors. What we showed here is that not only the metric properties, but also the supersymmetric ones, are given by the group-theoretical structure of these coset manifolds. At least in the case of maximally supersymmetric spacetimes, the essential point is to notice that the spin connection part of the supercovariant derivative provides a spinorial representation of the isotropy subgroup H. In addition, the field-strength part always turns out to provide a spinorial representation for the complement of H, and hence the full supercovariant connection becomes the well-known Maurer-Cartan 1-form of the considered coset space, conveniently evaluated in some spinorial representation "chosen" by the specific background. This means that the integrability condition of the Killing spinor equations will be automatically satisfied, and in this way (prior to the computation of any Killing spinor) the solution is shown to be maximally supersymmetric. Also, the Killing spinors are immediately found: they are simply given by (the inverse of) any coset representative evaluated in the spinorial representation that one can read from the gravitino supersymmetry transformation rule.

This simple geometric interpretation of the Killing spinors has an important consequence. As explained in Section 2.3, one of the most important things associated to a given vacuum is its superisometry algebra. From it one can read off what are their possible elementary excitations, even the nonperturbative ones: the allowed solitons always have associated a corresponding central charge. Computing the superalgebra associated to a given background can be a complicated task. In our framework, obtaining the translation part in the anticommutator of two supercharges becomes a rather simple calculation. It is true, however, that we do not know a systematic method to find all possible central charges. Such a procedure it is not known in general, but the observation that the supercharges actually encode the group theory properties of homogeneous spacetime backgrounds looks promising. We expect that the direction put forward here will help us to solve this interesting problem in the future. The extension of these results to homogeneous spacetimes which less supersymmetries is also work in progress.

On the other hand, in Chapter 8 we found the coset formulation of the only maximally supersymmetric background whose description as a homogeneous spacetime was not previously known. This spacetime is the near horizon limit of the five dimensional extreme rotating black hole. We saw in chapter 6 that this spacetime plays an important role among five dimensional Supergravity vacua. Its coset description has been found to be $(SO(1,2) \times SO(3))/U(1)$, where the $U(1)$ acts on both factors with an arbitrary relative weight. The result is not so strange if one takes into account the above mentioned result that we also found in Chapter 6: namely, that this space smoothly interpolates between $AdS_2 \times S^3$ and $AdS_3 \times S^2$. This, together with the fact that AdS_3 is the group manifold $SO(1,2)$ and that S^3 is the group manifold $SO(3)$, and also that in one dimension less each of them is the coset of these groups modded out by $U(1)$ makes our result a very natural one. We applied the techniques discovered in Chapter 7 to this case, too, although there is a slightly difference here: contrary to all the cases previously studied, this spacetime is homogeneous but not symmetric. We found that the general techniques apply and work equally well here.

An interesting observation is that, with the found coset description at hand, we see that this spacetime is the analytic continuation (Lorentzian version) of the compact space which is known under the name of $T^{(1,1)}$. This is the base of the conifold, a space which has been extensively studied in the context of AdS/CFT because it provides a geometry dual to four dimensional

$N=1$ gauge theory. In $T^{(1,1)}$ the $U(1)$ acts with equal weights on both $SU(2)$ factors. It would be interesting to investigate what happens if one allows, as in the case studied here, for an arbitrary weight, and which could be its potential implications in an gauge/gravity context. We find that this a very interesting question that we also hope to address shortly.

Part III

Finally, in the last Part we studied some properties of supergravity backgrounds dual to a certain three dimensional gauge theory: SYM with eight supercharges. The main motivation to focus on this theory and their gravity duals is twofold: on the one hand, the gauge theory is non conformal. On the other hand, we wanted to see form the Supergravity side the similarities and differences between two very different geometries which are supposed to be dual to the same gauge theory. This fact is already pointing out to us that both backgrounds should be somehow related. One geometry reproduces the behaviour of D4-branes wrapped on a supersymmetric cycle, and the other one corresponds to fractional D2-branes at the fixed point of the $\mathbb{C}^2/\mathbb{Z}_2$ orbifold. It has been noted in the literature that a fractional Dp-brane can be seen as a regular D$(p+2)$-brane in a certain limit. This is the limit in which two worldvolume directions of the D$(p+2)$-brane are wrapped on a two cycle and the cycle shrinks to zero size. In this singular limit the space becomes an orbifold, and the regular brane becomes a fractional one with two less worldvolume dimensions. Both geometries considered here exactly match with this pattern, since moreover we showed that the geometry in which our D4-branes are wrapped is a warped version of the Eguchi-Hanson space, which is precisely the blown-up version of the orbifold we were considering in the fractional brane supergravity solution. We saw that, form the point of view of the respective geometries, there is no smooth limit connecting them. This is because both solutions exhibit an enhançon locus at a certain point in transverse space, and at this point new massless degrees of freedom appear and the supergravity solutions cannot be trusted anymore.

From the point of view of the dual gauge theory we computed, from both supergravity duals, the one-loop perturbative beta function of the Coulomb branch of the gauge theory. We find exact agreement between both results and, further, also with the perturbative gauge theory computation (we explicitely performed this calculation by using the background field method technique). The main question is, of course, about the non perturbative contributions. This is somewhat puzzling since, in the case of $N = 4$ four dimensional SYM, the gravity side reproduces precisely the nonperturbative behaviour of the gauge theory, not the perturbative one. The reason for the opposite to happen in the cases considered here can be traced, again, to the existence of an enhançon locus. This locus has the dual gauge theory interpretation of being the analogous of Λ_{QCD}, i.e. the scale at which the coupling becomes large and nonperturbative effects become relevant. The fact that both dual geometries fail beyond this point is what seems to be the reason due to which we cannot recover the gauge theory nonperturbative phenomena from Supergravity. It would be extremely interesting to fully understand the Supergravity physics inside the enhançon. This is an open question, although some attempts can be found in recent literature. Understanding this should provide us (at least in cases analogous to the ones studied here) with answers to the questions concerning the nonperturbative behaviour of the dual gauge theory. Also, the smooth limit connecting fractional and wrapped branes would be presumably found if we were able to fully resolve the geometry. All these are interesting questions which deserve further study.

Resumen y Conclusiones

Resumamos los resultados más importantes que hemos obtenido y extraigamos algunas conclusiones e ideas para futuros trabajos.

Parte I

En el Capítulo 3 presentamos la familia más general de soluciones puntuales de Supergravedad $N = 4$ en cuatro dimensiones. Esta teoría es una truncación consistente de la cuerda heterótica compactificada en un seis-toro, por lo tanto, todas estas soluciones podrían ser subidas a diez dimensiones. Estas son, en general, no-supersimétricas: incluyen todas las posibles configuraciones puntuales y todas las soluciones de esta teoría conocidas previamente. Su interpretación en diez dimensiones podría ser útil si uno está interesado en estudiar la física de agujeros negros de estas soluciones cuatro-dimensionales desde una perspectiva de Teoría de Cuerdas. Debido a que, en general, describen configuraciones no-extremas con la temperatura de Hawking finita, las mismas son apropiadas para el estudio de emisión térmica de agujeros negros, como mínimo en los casos cercanos al límite extremo.

El conjunto completo de configuraciones es invariante, como familia, con respecto a las dualeidades de la teoría. Supergravedad $N = 4, d = 4$ es una teoría simple que tiene la interesante propiedad de mostrar tanto dualidad S como dualidades T. En este sentido, nos proporciona un marco sencillo donde comprobar cuáles son las configuraciones físicas que están contenidas dentro de las "representaciones" de grupos de dualidad. Nosotros hemos encontrado la familia más general posible invariante bajo dualidad, y nuestros resultados demuestran que, para esta teoría, todas las posibles soluciones físicas (supersimétricas o no) con cargas bien definidas se encuentran incluidas en las órbitas del grupo de dualidad de la teoría. Al menos en los casos que se corresponden con agujeros negros, sabemos que todas las configuraciones físicas posibles están incluidas en la solución general porque todas las cargas físicas permitidas por el "teorema de no-pelos" están incluidas y éstas pueden tomar valores completamente independientes. Tambien hemos comprobado, en todos los casos, las propiedades de invariancia bajo dualidad de las cantidades termodinámicas.

En el Capítulo 4 intentamos ver las implicaciones de esta idea en un contexto directamente relacionado con la Teoría de Cuerdas. El objetivo aquí ha sido investigar si todos los solitones conocidos en las Supergravedades Tipo II constituían o no una "representación completa" de las dualidades S y T. De hecho, hemos encontrado que este no es el caso si uno considera solamente los solitones estándar. Estudiando los sucesivas soluciones T-duales de la solución S-dual a la D7-brana de la teoría Tipo IIB, encontramos una cadena de soluciones, previamente desconocidas, de ambas teorías Tipo-II. Además, S-dualizando las soluciones Tipo IIB y subiendo las de Tipo IIA a once dimensiones, obtenemos una familia completa (de nuevo, previamente desconocida) de solitones en once dimensiones y ambos en teorías Tipo II. La propiedad más interesante que comparten todos ellos es que poseen las características asociadas a los estados del espectro no pereturbativo de la teoría de cuerdas que normalmente se consideran como elementales (en oposición a estados compuestos): todos preservan 1/2 de las supersimetrías y esán cargados bajo un único campo gauge. Además, resultan ser altamente no-perturbativos, ya que sus masas don proporcionales a un potencia inversa de la constante de acoplo mayor a la usual. De hecho, nuestro cálculo de estas masas muestra que las mismas coinciden con las predicciones hechas hace algunos años por Hull, Obers, Rabinovici y otros. Estos autores, basándose en argumentos de dualidad-U, propusieron la existencia de algunos estados no-perturbativos en diez dimensiones que estarían ausentes del espectro conocido. Fueron capaces de calcular sus masas, y sus resultados coinciden con los nuestros.

Una propiedad común de todas estas soluciones es que se comportan como el monopolo de Kaluza-Klein, en el sentido de que, en su presencia, un número de dimensiones transversas tienen siempre que ser consideradas compactas. Nosotros hemos explicado esta propiedad mostrando que todas ellas pueden obtenerse a partir de una teoría en dimensiones bajas que proviene de una reducción dimensional de la correpondiente acción en diez dimensiones. Estas nuevas soluciones pueden generarse en dimensiones bajas por medio de simetrías que exclusivamente existen en la teoría compactificada.

El principal obstáculo esgrimido en la literatura para considerar la solución S-dual de la D7-brana es que el superálgebra de la teorí Tipo IIB no tiene carga una central que se corresponda con este estado. Hemos demostrado que este no es el caso: damos un método genérico para encontrar todas las cargas centrales asociadas a todos estos estados solitónicos nuevos. Como explicamos en la Sección 2.3, la estructura de un determinado superálgebra viene dada por las superisometrías del vacío. Si tenemos en cuenta las direcciones isométricas de estos “backgrounds” (que no son asitóticamente planos), uno puede formular superálgebras consistentes que incluyan las cargas centrales necesarias asociadas a estos estados.

Sería muy interesante encontrar la descripción en Teoría de Cuerdas de todos estas soluciones. Además, sus geometrías de la región cercana al horizonte, sus teorías de “worldvolume” y las posibles aplicaciones de todo ello en el contexto de AdS/CFT, podrían ser de interés. Querríamos estudiar todos estos aspectos en el futuro.

Parte II

En la Parte II hemos presentado algunos resultados relativos a espaciotiempos que no describen estados masivos o cargados de Teoría de Cuerdas sino que, en su lugar, pueden considerarse como vacíos. En el Capítulo 6 demostramos cómo todos los vacíos máximamente supersimétricos (a parte de Minkowski, i.e. basicamente $AdS \times S$ y espaciotiempos tipo onda) de teorías con ocho supercargas en 4,5 y 6 dimensiones están relacionados mediante compactificaciones. Esto demuestra que la creencia habitual de que las compactificaciones de soluciones maximamente supersimétricas siempre rompen al menos algo de supersimetría no es correcta. Al menos en estos casos, siempre es posible encontrar una compactificación que preserve todas las supersimetrías (hemos visto que una compactificación de este tipo siempre involucra una truncación consistente de algún supermultiplete de la teoría en una dimensión menos). De hecho, es siempre posible encontrar un sistema de referencia en el que hay una dirección a lo largo de la cual no sólo las soluciones que describen estos espaciotiempos, sino también sus espinores de Killing, presentan isometrías. Esto también permite estudiar T-dualidad en estas soluciones (aunque, en general, supersimetría no ha de preservarse ante una T-dualización). Esto nos proporciona un espectro unificado de todos estos espaciotiempos de vacío. Una consecuencia que hemos encontrado es que la solución maximalmente supersimétrica dada por el límite de cerca del horizonte del agujero negro extremo con rotación en 5 dimensiones, realmente interpola entre $AdS_2 \times S^3$ y $AdS_3 \times S^2$. Esta interpolación resulta ser suave en el espacio de parámetros, y tiene una explicación sencilla en términos de su version compactificada en cuatro dimensiones: dicha versión compactificada es simplemente el límite de cerca del horizonte del agujero negro diónico Reissner-Nordström. Los casos estrictamente eléctrico o magnético corresponden a las "esquinas" $AdS_2 \times S^3$ o $AdS_3 \times S^2$, mientras que una solución diónica general en cuatro dimensiones representa un punto intermedio en el espacio de "moduli" de vacíos en cinco dimensiones.

Es conocido el hecho de que las teorías de Supergraved en cinco y once dimensiones muestran un considerable parecido. Sería interesante investigar la posibilidad de generalizar a once dimensiones este esquema unificado de vacíos en cinco dimensiones. Los vacíos en cinco-dimensionales $AdS_2 \times S^3$ y $AdS_3 \times S^2$ recuerdan a las soluciones máximamente supersimetricas $AdS_4 \times S^7$ or $AdS_7 \times S^4$ de Supergravedad en once dimensiones (las cuales se corresponden, respectivamente, con los límites cercanos al horizonte de la M2 y la M5-brana). De hecho, se puede ver que la solución interpolante en cinco dimensiones corresponde a una rotación continua de una fibra S^1 para dar, en los casos límite, un fibrado de Hopf de S^3 o AdS_3, con respectivos espacios base dados por S^2 o AdS_2. El siguiente fibrado de Hopf posible es el de S^7 con S^3 y S^4 como fibra y espacio base, y debería poderse encontrar un análogo para AdS_7. El problema principal para llevar a cabo en once dimensiones una construcción paralela a la realizada en cinco es que esto involucraría una compactificación en S^3 de Supergravedad de once a ocho dimensiones. Esto es algo técnicamente bastante complicado.

En el Capítulo 7 propusimos un análisis matemático en relación con las propiedades supersimétricas de los vacíos máximamente supersimétricos. Como se explicó en el Capítulo 5, estos espaciotiempos admiten una descripción como coset. Para un variedad de coset G/H dada, existe un procedimiento bien conocido para poner en él una métrica G-invariante, así como para construir los vectores de Killing correspondientes. Lo que demostramos aquí es que no sólo las propiedades métricas, sino también las supersimétricas, vienen dadas por la estructura de teoría de grupos de estas variedades tipo coset. Como mínimo en el caso de espaciotiempos maximamente supersimétricos, el punto esencial es notar que la parte de conexión de espín de la derivada supercovariante proporciona una representación espinorial del subgrupo de isotropía H. Además , la parte del tensor de campo siempre proporciona una representación espinorial para el complemento de H, y como resultado, toda la conexión supercovariante no pasa a ser sino la uno-forma de Maurer-Cartan del coset bajo consideración, convenientemente evaluada en una representación espinorial "elegida" por el "background" específico en el que estamos trabajando. Esto significa que las condiciones de integrabilidad de las ecuaciones del espinor de Killing se satisfacen automáticamente y, de esta forma, (y antes de haber caalculado ningún espinor de Killing) se demuestra que la solución es máximamente supersimétrica. Además, los espinores de Killing se encuentran inmediatamente: vienen dados simplemenete por (el inverso de) cualquier un representante del coset, evaluado en la representación espinorial que se puede leer a partir de la regla de transformación bajo supersimetría del gravitino.

Esta sencilla interpretación geométrica de los espinores de Killing tiene una importante consecuencia. Como se explicó en la Sección 2.3, una de las cosas más importantes asociadas a un determinado vacío es su álgebra de supersimetría. A partir de ella se puede saber cuáles son las excitaciones elementales posibles, incluso aquellas no-perturbativas: los solitones permitidos siempre tienen una carga central asociada correspondiente. Calcular el superálgebra asociada a un "background" dado puede ser algo complicado. Con el procedimiento indicado aqí, obtener la parte de translaciones en el anticonmutador de dos supercargas se convierte en un cálculo muy sencillo. No obstante, no conocemos un método sistemático para encontrar todas las posibles cargas centrales. Dicho procedimiento tampoco es conocido en general. Pero el hecho de que las supercargas realmente contienen información acerca de las propiedades geométricas de "backgrounds" homogéneos puede prooporcionar un camino prometedor en este sentido. Esperamos que la dirección iniciada aquí nos ayude a resolver este interesante problema en el futuro. La extensión de estos resultados a espacio-tiempos homogéneos con menos supersimetrías es un asunto que actualmente también estamos considerando.

Por otro lado, en el Capítulo 8 encontramos la formulación de como coset del único "background" máximamente supersimétrico cuya descripción como espaciotiempo homogéneo no se conocía previamente. Este espacio-tiempo es el límite cercano al horizonte del agujero negro extremo con rotación en 5 dimensiones. En el Capítulo 6 hemos visto que este espaciotiempo juega un papel importante dentro de los vacíos de Supergravedad en cinco dimensiones. Su descripción como coset resulta ser $(SO(1,2)\times SO(3))/U(1)$, donde el grupo $U(1)$ actúa sobre ambos factores con un peso relativo arbitrario. El resultado no es demasiado extraño si tenemos en cuenta el resultado mencionado arriba y que también encontramos en el Capítulo 6: el hecho de que este espacio interpola de forma suave entre $AdS_2 \times S^3$ y $AdS_3 \times S^2$. El hecho de que AdS_3 sea la variedad de grupo $SO(1,2)$ y que S^3 sea la variedad de grupo $SO(3)$, unido al que, en una dimensión menos, cada uno de ellos sea el coset respectivo de estos grupos cocientado por $U(1)$, hace que nuestro resultado aparezca como algo natural. Hemos aplicado las técnicas propuestas en el Capítulo 7 también a este caso, aunque hay una pequeña diferencia aquí: al contrario que en todos los casos estudiados previamente, este espaciotiempo es homogéneo pero no es simétrico. Hemos encontrado que, no obstante, las técnicas generales se pueden aplicar igualmente bien.

Una observación interesante es que, con la descripción tipo coset encontrada, vemos que este espaciotiempo es la continuación analítica (versión Lorentziana) del espacio compacto conocido como $T^{(1,1)}$. Este es la base del "conifold", un espacio que ha sido ampliamente estudiado en el contexto de AdS/CFT dado que proporciona una geometría dual a una teoría gauge $N=1$ en cuatro dimensiones. En $T^{(1,1)}$ el grupo $U(1)$ actúa con igual peso en ambos factores $SU(2)$. Sería interesante investigar que ocurre si tenemos en cuenta, tal y como hacemos en el caso estudiado aquí, un peso arbitrario, y cuáles podrían ser sus implicaciones en el contexto AdS/CFT. Pensamos que esta es una interesante pregunta, la cual nos gustaría estudiar pronto.

Parte III

Finalmente, en la última Parte hemos estudiado algunas propiedades de "backgrounds" de Supergravedad duales a una teoría gauge en tres dimensiones: SYM con ocho supercargas. La motivación principal para centrarnos en esta teoría y sus duales gravitatorios es doble: por un lado, la teoría gauge es no conforme. Por otro lado, pretendíamos estudiar, desde el punto de vista de las soluciones de supergravedad, los parecidos y diferencias entre dos geometrías muy diferentes que se supone que son duales a la misma teoría gauge. Este hecho ya indica que ambos "backgrounds" deberían estar relacionados de alguna manera. Una geometría reproduce el comportamiento de D4-branas enrolladas en un ciclo supersimétrico. La otra corresponde a D2-branas fraccionarias en el punto fijo del orbifold $\mathbb{C}^2/\mathbb{Z}_2$. En la literatura ha sido señalado que una Dp-brana fraccionaria puede verse como una D$(p+2)$-brana regular en un determinado límite, el límite en el cual dos direcciones del " worldvolume" de la D$(p+2)$-brana están enrolladas en un dos ciclo y el ciclo se "encoge" hasta alcanzar tamaño cero. En este límite singular el espacio pasa a ser un orbifold y la brana regular pasa a ser una brana fraccionaria con dos dimensiones menos en el "worldvolume". Hemos cisto que ambas geometrías consideradas aquí coinciden exactamente con este esquema, dado que hemos demostrado que la geometría en la cual las D4-branas están enrolladas es un una versión "warped" del espacio de Eguchi-Hanson, y est espacio es justamente blow-up del orbifold que estamos considerando. Vimos que, desde el punto de vista de las geometrías respectivas, no hay un límite suave que las conecte. Esto es debido a que ambas soluciones muestran un "enhançon locus" en un determinado punto en el espacio transverso: en este punto aparecen nuevos grados de libertad no masivos la teoría y la solución de supergravedad no es por tanto fiable a partir de este punto.

Desde el punto de vista de la teoría gauge, hemos calculado, usando los dos soluciones de supergravedad, la función beta a 1-loop en la "rama de Coulomb" de la teoría gauge. Hemos encontrado un perfecto acuerdo entre ambos resultados, y también entre éstos y el cálculo perturbativo en la teoría gauge (damos este cálculo usando el método de campo "background"). La principal pregunta es, desde luego, acerca contribuciones no perturbativas. Esto es algo extraño ya que, en el caso de $N = 4$ SYM en cuatro dimensiones, el dual gravitatorio reproduce precisamente el comportamiento no perturbativo de la teoría gauge. La razón para que ocurra esto en los casos considerados aquí es, de nuevo, es la existencia de un "enhançon locus". Este punto en el espacio transverso es el dual al análogo de Λ_{QCD} de la teoría gauge. Ésta es la escala a la cual el acoplo se hace grande y los efectos no perturbativos pasan a ser relevantes. El hecho de que cualquiera de las geometrías falle en este punto, parece ser la razón por la cual no somos capaces de recuperar efectos no perturbativos de teoría gauge a partir de Supergravedad. Sería de gran interés poder entender completamente la física de Supergravedad dentro del "enhançon". Esta es una pregunta aún por resolver, a pesar de que algunos intentos de responderla se pueden encontrar en la literatura reciente. Entender bien este asunto daría lugar (como mínimo en casos análogos a los estudiados aquí) a respuestas a las preguntas que tienen que ver con al comportamiento no perturbativo de la teoría gauge dual. Por otro lado, el límite en el que ambas soluciones de supergravedad deberían de coincidir parece tener que ver con la resolución de la geometría. Todas estas son preguntas interesantes que merecen un estudio posterior.

Appendix A

Chapter 3

A.1 Conserved Charges and Duality Invariants

The non-geometrical conserved charges of this theory are associated to the $U(1)$ vector fields. There are electric charges $\tilde{q}^{(n)}$ whose conservation law is associated to the Maxwell equation and are defined, for point-like objects by the asymptotic behavior of the $t-r$ components of the $SL(2,\mathbb{R})$-dual field strengths

$$ {}^{\star}\tilde{F}^{(n)}{}_{tr} \sim \frac{\tilde{q}^{(n)}}{\rho^2}\,, \tag{A.1.1} $$

and magnetic charges $\tilde{p}^{(n)}$ whose conservation law is associated to the Bianchi identity and are defined, for point-like objects by the asymptotic behavior of the $t-r$ components of the field strengths

$$ {}^{\star}F^{(n)}{}_{tr} \sim \frac{\tilde{p}^{(n)}}{\rho^2}\,. \tag{A.1.2} $$

These charges can be arranged in $SO(N)$ vectors $\vec{\tilde{q}}, \vec{\tilde{p}}$ and these can be arranged in $SL(2,\mathbb{R})$ doublets

$$ \begin{pmatrix} \vec{\tilde{q}} \\ \vec{\tilde{p}} \end{pmatrix}. \tag{A.1.3} $$

Under S and T duality transformations Λ, R, this charge vector transforms according to

$$\begin{pmatrix} \vec{\tilde{q}}' \\ \\ \vec{\tilde{p}}' \end{pmatrix} = \Lambda \otimes R \begin{pmatrix} \vec{\tilde{q}} \\ \\ \vec{\tilde{p}} \end{pmatrix} . \tag{A.1.4}$$

It is also useful to introduce the $SL(2,\mathbb{R})$ matrix of scalar fields

$$\mathcal{M} = e^{2\phi} \begin{pmatrix} |\lambda|^2 & a \\ \\ a & 1 \end{pmatrix} , \qquad \mathcal{M}^{-1} = e^{2\phi} \begin{pmatrix} 1 & -a \\ \\ -a & |\lambda|^2 \end{pmatrix} , \tag{A.1.5}$$

which transforms under $SL(2,\mathbb{R})$ according to

$$\mathcal{M}' = \Lambda \mathcal{M} \Lambda^T , \tag{A.1.6}$$

and it is an $SO(N)$ singlet.

We can now construct two expressions which are manifestly $SL(2,\mathbb{R}) \otimes SO(N)$-invariant and that will be useful later to express physical results in a manifestly duality-invariant way. The first one is quadratic in the charges:

$$I_2 \equiv \begin{pmatrix} \vec{\tilde{q}} & \vec{\tilde{p}} \end{pmatrix} \mathcal{M}_0^{-1} \begin{pmatrix} \vec{\tilde{q}} \\ \\ \vec{\tilde{p}} \end{pmatrix} \tag{A.1.7}$$

where $\mathcal{M}_0$ is the constant asymptotic value of $\mathcal{M}$.

The second invariant we will use is :

$$I_4 \equiv \det \left[\begin{pmatrix} \vec{\tilde{q}} \\ \\ \vec{\tilde{p}} \end{pmatrix} \begin{pmatrix} \vec{\tilde{q}} & \vec{\tilde{p}} \end{pmatrix} \right] = \left(\vec{\tilde{q}} \cdot \vec{\tilde{q}}\right)^2 \left(\vec{\tilde{p}} \cdot \vec{\tilde{p}}\right)^2 - \left(\vec{\tilde{q}} \cdot \vec{\tilde{p}}\right)^2 , \tag{A.1.8}$$

which is quartic in the charges. Observe that I_2 is moduli-dependent and I_4 is moduli-independent.

A.2 Physical Parameters

Here we explain our notation for charges and moduli used in the solutions. m stands for the ADM mass, and n for the NUT charge. They appear combined into the complex constant $\mathfrak{M}$ defined by

$$\mathfrak{M} = m + in . \tag{A.2.1}$$

The rotation parameter α is $\alpha = J/m$, where J is the angular momentum. These charges are singlets under all duality transformations.

The asymptotic behavior of the axidilaton is characterized by the constant asymptotic value $\lambda_0 = a_0 + ie^{-2\phi_0}$, where a_0 is the constant asymptotic value of the axion and ϕ_0 that of the dilaton. The axidilaton "charge" is denoted by Υ and, thus

$$\lambda \sim \lambda_0 - ie^{-2\phi_0}\frac{2\Upsilon}{\rho}\,. \tag{A.2.2}$$

(where ρ is a radial coordinate).

λ_0 transforms as λ under duality transformations and Υ is as $SO(N)$ singlet and, under $SL(2,\mathbb{R})$

$$\Upsilon' = e^{-2i\arg(c\lambda_0+d)}\Upsilon\,. \tag{A.2.3}$$

We find it convenient to use, instead of the conserved charges $\tilde{q}^{(n)}$ and $\tilde{p}^{(n)}$ defined in Appendix A.1, the constants $Q^{(n)}$ and $P^{(n)}$ defined by

$$F_{tr}{}^{(n)} \sim \frac{e^{\phi_0}Q^{(n)}}{\rho^2}\,, \qquad {}^\star F_{tr}{}^{(n)} \sim -\frac{e^{\phi_0}P^{(n)}}{\rho^2}\,, \tag{A.2.4}$$

and combined into the complex constants $\Gamma^{(n)}$ which can be arranged into an $SO(N)$ vector

$$\Gamma^{(n)} = Q^{(n)} + iP^{(n)}\,, \qquad \vec{\Gamma} = \vec{Q} + i\vec{P}\,. \tag{A.2.5}$$

In our solutions these charges are simple combinations of the conserved charges and moduli:

$$\begin{pmatrix} \vec{Q} \\ \vec{P} \end{pmatrix} = \mathcal{V}_0 \begin{pmatrix} \vec{\tilde{q}} \\ \vec{\tilde{p}} \end{pmatrix}, \tag{A.2.6}$$

where

$$\mathcal{V}_0 = e^{\phi_0}\begin{pmatrix} -1 & a_0 \\ 0 & -e^{-2\phi_0} \end{pmatrix}, \qquad \mathcal{V}_0^T\mathcal{V}_0 = \mathcal{M}_0^{-1}\,. \tag{A.2.7}$$

$\vec{\Gamma}$ is an $SO(N)$ vector and transforms under $SL(2,\mathbb{R})$ according to

$$\vec{\Gamma}\,' = e^{i\arg(c\lambda_0+d)}\vec{\Gamma}\,, \tag{A.2.8}$$

so the duality invariants can be written

$$\begin{array}{rcl} I_2 & = & |\vec{\Gamma}\cdot\vec{\Gamma}|^2 = \sum_{n=1}^{N} |\Gamma^{(n)}|^2\,, \\ I_4 & = & -\frac{1}{4}\det\left[\begin{pmatrix}\vec{\Gamma}\\ \bar{\vec{\Gamma}}\end{pmatrix}\begin{pmatrix}\vec{\Gamma} & \bar{\vec{\Gamma}}\end{pmatrix}\right]\,. \end{array} \tag{A.2.9}$$

Our solutions do not have any primary scalar hair and the axidilaton charge is always completely determined by the electric and magnetic charges, and mass and NUT charge through

$$\Upsilon = -\frac{1}{2}\sum_{n=1}^{N}\frac{(\bar{\Gamma}^{(n)})^2}{\mathfrak{M}}\,. \tag{A.2.10}$$

The absolute value of this expression is duality invariant and can be rewritten in terms of the basic invariants (A.1.7,A.1.8) as follows:

$$|\Upsilon|^2 = \frac{1}{4|\mathfrak{M}|^2}(I_2^2 - 4I_4)\,. \tag{A.2.11}$$

A.3 Central Charge Matrix Eigenvalues

The supersymmetry parameter r_0 can be expressed in terms of the two different skew eigenvalues of the central charge matrix of $N = 4, d = 4$ Supergravity [52, 103] $\mathcal{Z}_{1,2}$. Their absolute values can be expressed in terms of the electric and magnetic charges in the following way:

$$|\mathcal{Z}_{1,2}|^2 = \frac{1}{2}\sum_{n=1}^{N}|\Gamma^{(n)}|^2 \pm \frac{1}{2}\left[\left(\sum_{n=1}^{N}|\Gamma^{(n)}|^2\right)^2 - \left|\sum_{n=1}^{N}\left(\Gamma^{(n)}\right)^2\right|^2\right]^{1/2}\,, \tag{A.3.1}$$

and in terms of the invariants I_2, I_4 defined in Eqs. (A.1.7,A.1.8) as follows

$$|\mathcal{Z}_{1,2}|^2 = \frac{1}{2}I_2 \pm I_4^{1/2}\,. \tag{A.3.2}$$

With the help of these expressions and those of the previous Appendices we can write the supersymmetry parameter r_0 as follows:

$$r_0^2 = \frac{1}{|\mathfrak{M}|^2}\left(|\mathfrak{M}|^2 - |\mathcal{Z}_1|^2\right)\left(|\mathfrak{M}|^2 - |\mathcal{Z}_2|^2\right)\,. \tag{A.3.3}$$

This last equation makes explicit the fact that, if and only if r_0^2=0, one of the two possible supersymmetry Bogomol'nyi bounds

$$|\mathfrak{M}|^2 \geq |\mathcal{Z}_{1,2}|^2\,, \tag{A.3.4}$$

is saturated.

The following expression is also useful

$$|\mathcal{Z}_1 \mathcal{Z}_2|^2 = |\mathfrak{M}|^2 |\Upsilon|^2\,. \tag{A.3.5}$$

Appendix B

Chapter 4

B.1 Holomorphic $(d-3)$-Branes

In this Appendix we briefly discuss holomorphic $(d-3)$-brane solutions of the d-dimensional $SL(2,\mathbb{R})/SO(2)$ sigma model

$$S = \int d^d x \sqrt{|g|} \left\{ R + \tfrac{1}{2} \frac{\partial\tau\partial\bar{\tau}}{(\Im\mathrm{m}\tau)^2} \right\} , \tag{B.1.1}$$

where τ lives in the complex upper half plane and is defined up to modular $PSL(2,\mathbb{Z})$ transformations, so multivalued solutions are allowed if the value of τ changes by a modular transformation.

$(d-3)$-brane-type solutions of this model were first considered in Ref. [118] in $d=4$. In these dimensions $(d-3)$-branes are strings. In that reference, the following general solution of the above model was found[1]

$$\left\{ \begin{array}{rcl} ds^2 & = & dt^2 - d\vec{y}^{\,2}_{(d-3)} - H d\omega d\bar{\omega} \,, \\ \\ \tau & = & \mathcal{H} \,, \end{array} \right. \tag{B.1.2}$$

where $\mathcal{H}$ is, in principle, any complex holomorphic or antiholomorphic function of the complex variable ω (i.e. either $\partial_{\bar{\omega}}\mathcal{H} = 0$ or $\partial_{\omega}\mathcal{H} = 0$) and $H = \Im\mathrm{m}(\mathcal{H})$. H is, therefore, a real harmonic function of the 2-dimensional Euclidean spacetime transverse to the $(d-2)$-dimensional worldvolume directions. Only functions with $H \geq 0$ are admissible.

[1] Here we write the obvious generalization to any dimension d (see also Refs. [94, 119]).

A few remarks are in order here: although $g_{\omega\bar{\omega}} = H$ is in this solution equal to the imaginary part of τ, it does not transform under $PSL(2,\mathbb{Z})$. Modular invariance of the metric is, therefore, not an issue. We could have wrongly concluded that in this solution, the metric is not modular invariant because $g_{\omega\bar{\omega}} = \Im\mathrm{m}(\tau)$ but, by definition, it is, since the metric does not transform under $PSL(2,\mathbb{Z})$. Then, the l.h.s. if that equation does not transform, and the r.h.s. does, and we get a new solution (denoted by primes) with

$$
\begin{aligned}
\tau'(\omega) &= \frac{a\tau(\omega)+b}{c\tau(\omega)+d} = \frac{a\mathcal{H}+b}{c\mathcal{H}+d} \equiv \mathcal{H}' \,, \\
g'_{\omega\bar{\omega}} &= g_{\omega\bar{\omega}} = \Im\mathrm{m}(\tau) = \frac{\Im\mathrm{m}(\tau')}{|-c\tau'+a|^2} = \frac{\Im\mathrm{m}(\mathcal{H}')}{|-c\mathcal{H}'+a|^2} \,.
\end{aligned}
\tag{B.1.3}
$$

We could remove if we wished the extra factor by a conformal reparametrization:

$$
d\omega' = \frac{d\omega}{-c\mathcal{H}'(\omega)+a} \,, \tag{B.1.4}
$$

and we then could write again the new solution in a form similar to that of the original one Eq. (B.1.2) but with a new holomorphic function $\mathcal{H}'[\omega(\omega')]$. Thus, as in Ref. [118] we could have written from the beginning the general solution in the form

$$
\left\{
\begin{aligned}
ds^2 &= dt^2 - d\vec{y}^{\,2}_{(d-3)} - H|f(\omega)|^2 d\omega d\bar{\omega} \,, \\
\tau &= \mathcal{H} \,,
\end{aligned}
\right.
\tag{B.1.5}
$$

where $f(\omega)$ is any holomorphic function of ω, but this function can always be reabsorbed into a holomorphic coordinate change $\omega' = F(\omega)\,,\ \ dF/d\omega = f$ and $\tau(\omega') = \tau[F^{-1}(\omega')]$.

All this said, it must be acknowledged that, even though modular invariance of the metric is not an issue, its single-valuedness is. Since $\mathcal{H}$ will in general be a multivalued function with monodromies in G, its imaginary part will also be multivalued and it might be necessary to multiply it by $|f(\omega)|^2$, with $f(\omega)$ multivalued to make $g_{\omega\bar{\omega}}$ single valued.

A second remark we can make here is that there exists another form of the general solution which is manifestly $SL(2,\mathbb{R})$ invariant without having to invoke coordinate changes to show it:

$$\left\{\begin{array}{rcl} ds^2 & = & dt^2 - d\vec{y}_p^{\,2} - e^{-2U} d\omega d\bar{\omega}\,, \\ \tau & = & \mathcal{H}_1/\mathcal{H}_2\,, \\ e^{-2U} & = & \Im\mathrm{m}\left(\mathcal{H}_1\bar{\mathcal{H}}_2\right)\,, \end{array}\right. \tag{B.1.6}$$

where $\mathcal{H}_{1,2}$ are two arbitrary complex functions of the complex variable ω transforming as a doublet under $SL(2,\mathbb{R})$, i.e.

$$\begin{pmatrix} \mathcal{H}_1' \\ \mathcal{H}_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{pmatrix}, \tag{B.1.7}$$

both in τ and in the metric (but e^{-2U} is invariant, as it must). The structure of this family is similar to that of the duality-invariant families of black-hole solutions of pure $N = 4, d = 4$ supergravity presented in Refs. [30, 69, 87], closely related to special geometry objects as discovered in [85]. We can relate this general solution either to the solution Eq. (B.1.2) as the particular case $\mathcal{H}_1 = \mathcal{H}\,,\ \mathcal{H}_2 = 1$ or to the solution Eq. (B.1.5) as the particular case $\mathcal{H}_1/\mathcal{H}_2 = \mathcal{H}\,,\ f = \mathcal{H}_2$ since $\Im\mathrm{m}(\mathcal{H}_1\bar{\mathcal{H}}_2) = |\mathcal{H}_2|^2\Im\mathrm{m}(\mathcal{H}_1/\mathcal{H}_2)$.

All this means that we cannot generate new solutions not in this classes via $SL(2,\mathbb{R})$ transformations.

Since all these solutions are equivalent, up to coordinate transformations, we take now Eq. (B.1.2) and now consider the choice of function $\mathcal{H}$. First, we have to choose between holomorphic and anti-holomorphic $\mathcal{H}$. This choice is related to the choice between $(d-3)$-branes and anti-$(d-3)$-branes with opposite charge with respect to the $(d-2)$-form potential dual to a. The impossibility of having $\mathcal{H}$ depending on both ω and $\bar{\omega}$ is due to the impossibility of having objects with opposite charges in equilibrium. We opt for holomorphy.

Which holomorphic function should one choose? As usual, the choice has to be based on local and global conditions. Local conditions are essentially related to the existence of extended sources (with $(d-3)$ spatial dimensions) at given points in transverse (ω) space manifold. Global conditions are essentially related to the choice of global transverse space. Not all local conditions are possible for a given choice of transverse space. For instance, there is no holomorphic function for a single $(d-3)$-brane in the Riemann sphere[2].

To clarify these issues, let us consider the simplest solution in this class: let us couple the action Eq. (B.1.1) to a charged $(d-3)$-brane source. We first have to dualize the pseudoscalar a into a $(d-2)$-form potential $A_{(d-2)}$ with field strength $F_{(d-1)} = (d-1)\partial A_{(d-2)}$: $\partial a = e^{-2\varphi}\,{}^\star F_{(d-1)}$. The bulk plus brane action is

$$
\begin{aligned}
S &= \tfrac{1}{16\pi G_N^{(d)}}\int d^dx\sqrt{|g|}\,\left\{R + \tfrac{1}{2}(\partial\varphi)^2 + \tfrac{1}{2\cdot(d-1)!}F^2_{(d-1)}\right\} \\
&\quad -\tfrac{T}{2}\int d^{d-2}\xi\sqrt{|\gamma|}\,\left\{e^{\frac{2}{(d-2)}\varphi}\gamma^{ij}g_{ij} - (d-4)\right\} \qquad \text{(B.1.8)} \\
&\quad -\alpha\tfrac{T}{(d-2)!}\int d^{d-2}\xi A_{(d-2)\ i_1\cdots i_{(d-2)}}\epsilon^{i_1\cdots i_{(d-2)}}\,,
\end{aligned}
$$

where g_{ij} and $A_{(d-2)\ i_1\cdots i_{(d-2)}}$ are the pullbacks through the embedding coordinates $X^\mu(\xi)$ of the metric and $(d-2)$-form potential. T is the tension (in principle, a positive number) and $\alpha = \pm 1$ gives the sign of the charge (which is evidently proportional to the tension). The coupling to φ is the only one that allows for solutions of the form we want.

A solution is provided by

$$
\left\{
\begin{aligned}
ds^2 &= dt^2 - d\vec{y}^{\,2}_{(d-3)} - Hd\vec{x}_2^{\,2}\,, \\
e^{-\varphi} &= H\,, \\
A_{(d-2)\ ty^1\cdots y^{(d-3)}} &= \alpha H^{-1}\,, \\
Y^i &= \xi^i\,, \qquad \vec{X}_2 = 0\,,
\end{aligned}
\right. \qquad \text{(B.1.9)}
$$

where H satisfies the equation

$$
\partial^2 H = -16\pi G_N^{(d)} T\delta^{(2)}(\vec{x}_2)\,, \qquad \text{(B.1.10)}
$$

[2]of course, one meets the same situation for other branes. However, for smaller branes one can always find harmonic functions with a single pole (describing a single brane) that lead to spaces asymptotically flat in transverse directions. This is not true for higher ($(d-3)$- and $(d-2)$-) branes).

i.e. it is a harmonic function with a pole at $\vec{x}_2 = 0$, where the brane is placed. The above equation is solved by a function H that behaves near $\vec{x}_2 = 0$

$$H \sim -8G_N^{(d)} T \log|\vec{x}_2| \,. \tag{B.1.11}$$

It is clear that this solution cannot be globally correct as H becomes negative for $|\vec{x}_2| > 1$, but the local behavior of the global solution has to be the same. Any solution behaving in this way at any given point will describe a $(d-3)$-brane placed there.

Let us now compute the charge. This is defined by

$$p = \oint_\gamma e^{-2\varphi} \, {}^\star F_{(d-1)} = \oint_\gamma da \,, \tag{B.1.12}$$

where γ is a closed loop around the origin. a is given by

$$\partial_{\underline{n}} a = \alpha \epsilon_{nm} \partial_{\underline{m}} H \,, \tag{B.1.13}$$

i.e. combining $x^1 + ix^2 \equiv \omega$

$$\begin{cases} \partial_{\bar{\omega}} \tau = 0 \,, & \alpha = +1 \,, \\ \partial_{\bar{\omega}} \bar{\tau} = 0 \,, & \alpha = -1 \,, \end{cases} \tag{B.1.14}$$

that is: a is the real part of a holomorphic or antiholomorphic function of ω, whose imaginary part is the above function H. We find $a = \alpha 8 G_N^{(d)} T \,\mathcal{A}\text{rg}(\omega)$ and $p = \alpha \frac{1}{16\pi G_N^{(d)}} T$. The choice $\alpha = +1$ then, corresponds to a single $(d-3)$-brane with charge $p = +\frac{1}{16\pi G_N^{(d)}} T$ placed at the origin and corresponds to a holomorphic function $\tau = \mathcal{H}(\omega)$ that close to the origin is given by

$$\mathcal{H} \sim -8G_N^{(d)} T i \log \omega \,. \tag{B.1.15}$$

Observe that the charge is given by the multivaluedness of τ around the source, which goes from τ to $\tau + 16\pi G_N^{(d)} T$ which should be identified with τ. The charge is usually quantized due to quantum-mechanical reasons in multiples of the unit of charge (e, say) which implies the identification $\tau \equiv \tau + ne$ and the breaking of $SL(2, \mathbb{R})$. If $e = 1$ (i.e. $16\pi G_N^{(d)} T = 1$ which we can always get by rescaling τ) then $SL(2, \mathbb{Z})$ is the unbroken symmetry of the theory and the above $(d-3)$-branes are associated to the modular group element T[3].

We see that in this context solutions (and charges) can be characterized by the non-trivial monodromies around singular points which, by hypothesis, are elements of the modular group.

We can clearly generate via modular (duality) transformations of this solution with T monodromy other solutions with different monodromies. it is easy to see that if we perform a transformation $\tau \rightarrow M(\tau)$ $M \in PSL(2\mathbb{Z})$ on the above solution, the monodromy of the new solution around the origin will be MTM^{-1}. The most interesting modular transformation is $S(\tau) = -1/\tau$ which in other contexts relates electric and magnetic ("S dual") objects. Then, the S dual of the above solution will have monodromy STS around the origin and will be given either by $\mathcal{H} = -\frac{2\pi i}{\log \omega}$ using the general solution in the form of Eq. (B.1.2) or with $\mathcal{H} = \frac{1}{2\pi i} \log \omega$ and the form (4.2.6) of the solution. This is the form we have used in the main text to stress that we are dealing with a solution different from the one with monodromy T, the difference being in the choice of holomorphic function since, as we have stressed at the beginning of this Appendix all homomorphic solutions can always be written in the form (B.1.2), no matter if the monodromy is T or STS.

B.2 The KK Origin of the $SL(2,\mathbb{R})/SO(2)$ Model

We are going to see how the modular group $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\{\pm \mathbb{I}_{2\times 2}\}$ and the $SL(2,\mathbb{Z})/SO(2)$ sigma model arise in standard Kaluza-Klein compactification on a 2-torus T^2.

B.2.1 The Modular Group

As usual in KK compactifications, we use two periodic coordinates x^m $m = 1,2$ whose periodicity is fixed to $2\pi\ell$ where ℓ is some fundamental length. This means that we make the identifications

$$\vec{x} \sim \vec{x} + 2\pi\ell\vec{n}\,, \qquad \vec{x} = \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}, \qquad \vec{n} \in \mathbb{Z}^2\,. \tag{B.2.1}$$

[3]For 10-dimensional type IIB D7-branes $16\pi G_N^{(10)} = (2\pi)^7 \ell_s^8 g^2$ and $T = (2\pi)^{-7}\ell_s^{-8} g$, and, thus, $\mathcal{H} \sim -\frac{gi}{2\pi}\log\omega$. On the other hand, $C^{(8)}_{ty^1\cdots y^7} = g^{-1}H^{-1}$ $(\alpha = +1)$ and we get $p = 1$ in a most natural way.

The information on relative sizes and angles of the periods and the size of the torus is codified in the internal metric G_{mn},

$$ds^2_{\rm Int} = d\vec{x}^T G d\vec{x}\,, \tag{B.2.2}$$

which is, by hypothesis, independent on the torus coordinates $\vec{x}$, (but may depend on the remaining coordinates).

The KK Ansatz is invariant under global diffeomorphisms in the internal manifold. These are, generically, of the form

$$\vec{x}' = R^{-1}\vec{x} + \vec{a}\,, \qquad R \in GL(2,\mathbb{R})\vec{a} \in \mathbb{R}^2\,. \tag{B.2.3}$$

$\vec{a}$ simply shifts the coordinate origin and does not affect the metric. R acts on the internal metric according to

$$G' = R^T G R\,, \qquad (G_{mn} = R^p{}_m G_{pq} R^q{}_n)\,. \tag{B.2.4}$$

We want to separate the volume part of the metric from the rest[4]. Thus, we define[5]

$$K \equiv |\text{det}G_{mn}|\,, \qquad G_{mn} \equiv -K^{1/2}\mathcal{M}_{mn}\,. \tag{B.2.5}$$

$\mathcal{M}$ has determinant $+1$ and, therefore, it is a symmetric $SL(2,\mathbb{R})$ matrix and, in fact, it can be understood as an element of the coset $SL(2,\mathbb{R})/SO(2)$ with only two independent entries. If we factor out the determinant of the $GL(2,\mathbb{R})$ transformations too,

$$R \equiv |\text{det}R^m{}_n|\,, \qquad s = \text{sign}(\text{det}R^m{}_n)\,, \qquad R^m{}_n \equiv sR^{1/2}\mathcal{S}^m{}_n\,, \tag{B.2.6}$$

then the volume element K and the matrix $\mathcal{M}$ transform according to

$$\begin{aligned} \mathcal{M}' &= S^T \mathcal{M} S\,, \\ K' &= RK\,. \end{aligned} \tag{B.2.7}$$

$|K|$ is an element of the multiplicative group $\mathbb{R}^+$ and S is an element of $SL(2,\mathbb{R})$. This decomposition reflects the decomposition $GL(2,\mathbb{R}) = SL(2,\mathbb{R}) \times \mathbb{R}^+ \times \mathbb{Z}_2$. s does not act neither on K nor on $\mathcal{M}$.

[4]This is necessary, for instance, when we are interested in conformal classes of equivalence of metrics, as in string path integrals, but convenient in general.

[5]Remember that G has signature $(--)$.

We have not yet taken into account the periodic boundary conditions of the coordinates, that have to be preserved by the diffeomorphisms in the KK setting. Clearly the rescalings R do not respect the torus boundary conditions, but they rescale ℓ. The rotations S respect the boundary conditions only if $S^{-1}\vec{n} \in \mathbb{Z}^2$ the matrix entries are integer, i.e. $S \in SL(2,\mathbb{Z})$. Up to a reflection $S = -\mathbb{I}_{2\times 2}$, these diffeomorphisms are known as *Dehn twists* and are not connected with the identity (in fact, they constitute the mapping class group of torus diffeomorphisms) and they constitute the *modular group* $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\{\pm\mathbb{I}_{2\times 2}\}$. This is the group that acts on $\mathcal{M}$.

We are going to write the modular group matrices in the slightly unconventional form

$$S = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}, \tag{B.2.8}$$

to get the conventional form of the transformation of the modular parameter Eq. (B.2.15).

B.2.2 The Modular Parameter τ

We can define a complex modular-invariant coordinate ω on T^2 by

$$\omega = \tfrac{1}{2\pi\ell}\,\vec{\omega}^T \cdot \vec{x}\,, \qquad \vec{\omega} = \mathbb{C}^2\,, \tag{B.2.9}$$

where, under modular transformations, we assume that the complex vector $\vec{\omega}$ transforms according to

$$\vec{\omega}' = S^T\vec{\omega}\,. \tag{B.2.10}$$

The periodicity of ω is

$$\omega \sim \omega + \vec{\omega}^T \cdot \vec{n}\,, \qquad \vec{n} \in \mathbb{Z}^2\,. \tag{B.2.11}$$

What we have done is to transfer the information contained in the metric (more precisely, in $\mathcal{M}$) into the complex periods $\vec{\omega}$. The relation between these two is

$$\mathcal{M} = \frac{1}{\Im\mathrm{m}(\omega_1\bar{\omega_2})} \begin{pmatrix} |\omega_1|^2 & \Re\mathrm{e}(\omega_1\bar{\omega_2}) \\ \Re\mathrm{e}(\omega_1\bar{\omega_2}) & |\omega_2|^2 \end{pmatrix}. \tag{B.2.12}$$

We can check that the transformation rules for the complex periods Eq. (B.2.10) and for the matrix $\mathcal{M}$ Eq. (B.2.7) are perfectly compatible.

In terms of the modular-invariant complex coordinate, the torus metric element takes the form

$$ds^2_{\mathrm{Int}} = K^{1/2} \frac{1}{\Im\mathrm{m}\omega_1 \bar{\omega_2}} d\omega d\bar{\omega}\,. \tag{B.2.13}$$

Observe that $\Im\mathrm{m}(\omega_1\bar{\omega_2})$ is modular-invariant (and a quite important one).

It should be clear that not all pairs of complex periods characterize different tori. Recall that $\mathcal{M}$ only has 2 independent entries while $\vec{\omega}$ contains 4 real independent quantities. In particular, we can see that multiplying $\vec{\omega}$ by any complex number leaves the matrix $\mathcal{M}$ invariant. It is customary to multiply by ω_2^{-1} both the coordinate ω and define

$$\xi = \omega/\omega_2\,, \qquad \tau = \omega_1/\omega_2\,, \tag{B.2.14}$$

that can always be chosen to belong to the upper half complex plane $\mathbb{H}$ $\Im\mathrm{m}(\tau) \geq 0$ ($-\omega_1$ defines the same torus as ω_1).

Under a modular transformation with S given by Eq. (B.2.8), the modular parameter undergoes a fractional-linear transformation

$$\tau' = \frac{\alpha\tau + \beta}{\gamma\tau + \delta}\,. \tag{B.2.15}$$

and the torus coordinate ξ transforms

$$\xi' = \frac{\xi}{(c\tau + d)}\,. \tag{B.2.16}$$

Finally, in terms of τ, the matrix $\mathcal{M}$ reads

$$\mathcal{M} = \frac{1}{\Im\mathrm{m}(\tau)} \begin{pmatrix} |\tau|^2 & \Re\mathrm{e}(\tau) \\ \Re\mathrm{e}(\tau) & 1 \end{pmatrix}\,. \tag{B.2.17}$$

B.2.3 The $SL(2,\mathbb{R})/SO(2)$ Sigma-Model

In pure KK theory (with no higher-dimensional fields apart from the metric), the toroidal compactification of the Einstein-Hilbert action from $\hat{d}$ to d dimensions with the KK Ansatz

$$\left(\hat{e}_{\hat{\mu}}{}^{\hat{a}}\right) = \begin{pmatrix} e_\mu{}^a & e_m{}^i A^m{}_\mu \\ 0 & e_m{}^i \end{pmatrix}\,, \tag{B.2.18}$$

where the internal metric

$$G_{mn} = e_m{}^i e_{n\,j} = -e_m{}^i e_n{}^j \delta_{ij}\,. \tag{B.2.19}$$

gives, upon the rescaling

$$g_{E\,\mu\nu} = K^{\frac{2}{(d-2)}} g_{\mu\nu}\,, \tag{B.2.20}$$

$$\begin{aligned} S &= \int d^{\hat{d}}\hat{x}\sqrt{|\hat{g}|}\,\hat{R} \\ &= \int d^d x \sqrt{|g_E|}\,\Big[R_E + \tfrac{(\hat{d}-2)(\hat{d}-d)}{4(d-2)}(\partial \log K)^2 + \tfrac{1}{4}\mathrm{Tr}\,(\partial\mathcal{M}\mathcal{M}^{-1})^2 \\ &\qquad -\tfrac{1}{4}K^{\frac{(\hat{d}-2)}{(d-2)}}\mathcal{M}_{mn}F^{m\,\mu\nu}F^{n}{}_{\mu\nu}\Big]\,. \end{aligned} \tag{B.2.21}$$

The kinetic term for the scalar matrix $\mathcal{M}$ is manifestly invariant under $SL(2,\mathbb{R})$ transformations (the action we started from is diffeomorphism-invariant). Using the parametrization Eq. (B.2.17), it takes the standard form

$$\tfrac{1}{2}\frac{\partial\tau\partial\bar{\tau}}{(\Im\mathrm{m}(\tau))^2}\,. \tag{B.2.22}$$

Appendix C

Chapter 10

C.1 Finding the supergravity solutions

In the two sections of this Appendix we will describe the way in which we have obtained the supergravity solutions describing, respectively, D4-branes wrapped on S^2 and fractional D2/D6-branes on the orbifold $\mathbb{R}^4/\mathbb{Z}_2$.

We will be using the following conventions:

- A metric in D dimensions has signature $(-,+^{D-1})$ [1].
- ε-symbols in D dimensions are defined in such a way that $\varepsilon^{012\cdots(D-1)} = -\varepsilon_{012\cdots(D-1)} = +1$.
- A p form is defined as $\omega_p = \frac{1}{p!}\omega_{\mu_1\cdots\mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$.
- The Hodge dual $\star_D$ in D dimensions is defined as $\star_D \omega_p = \frac{\sqrt{-\det G_D}}{p!(D-p)!}\varepsilon_{\nu_1\cdots\nu_{D-p}\mu_1\cdots\mu_p}\omega^{\mu_1\cdots\mu_p} dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_{D-p}}$. Moreover, $\star$ denotes $\star_{10}$ and $\hat{\star}$ denotes $\star_{11}$.

C.1.1 D4-branes wrapped on S^2

In this appendix we explain how we have obtained the type IIA supergravity solution describing N D4-branes wrapped on S^2, using the techniques and the solutions given in Ref. [195].

Our procedure will be the following. We want to obtain the solution for the D4-branes by compactifying the solution for the M5-branes wrapped

[1] Note that this convention is opposed to the one taken in the remaining Chapters.

on S^2. The latter is found by uplifting to eleven dimensions a solution of 7-dimensional gauged supergravity with the correct identification between spin connection and gauge connection, by means of the formulas given in Ref. [232].

The seven dimensional solution

The starting point is the seven dimensional gauged supergravity considered in Ref. [195]. Following that paper, we consider a $U(1) \times U(1)$ consistent truncation of the $SO(5)$ gauged supergravity arising when one compactifies eleven dimensional supergravity on S^4. The bosonic field content of the truncated theory consists of two $U(1)$ gauge fields ($A^{(1,2)}$), two scalar fields ($\lambda_{1,2}$) and a metric.

The full solution of seven dimensional gauged supergravity is:

$$ds^2_{(7)} = \left(\frac{R_A}{R_0}\right)^2 e^{2\rho} e^{\lambda}\, \eta_{ij} d\xi^i d\xi^j + R_A^2 \left(e^{\lambda}(e^{2\rho} - \tfrac{1}{4})(d\tilde{\theta}^2 + \sin^2\tilde{\theta} d\tilde{\varphi}^2) + e^{-4\lambda} d\rho^2\right) , \tag{C.1.1a}$$

$$A^{(1)} = \frac{R_A}{4}\cos\tilde{\theta} d\tilde{\varphi} \,, \quad A^{(2)} = 0 \,, \tag{C.1.1b}$$

$$\lambda \equiv \lambda_2 \,, \quad 2\lambda_1 + 3\lambda_2 = 0 \,, \tag{C.1.1c}$$

$$e^{5\lambda} = \frac{e^{2\rho} + k e^{-2\rho} - \frac{1}{2}}{e^{2\rho} - \frac{1}{4}} \,, \tag{C.1.1d}$$

This solution is exactly the one given in Ref. [195], although we have kept track of units and we have used standard spherical coordinates $\tilde{\theta}$ and $\tilde{\varphi}$ for the two-sphere on which the branes are wrapped. $R_A = 2(\pi N)^{1/3} l_p$ is the radius of the AdS_7 space appearing in the near horizon limit of the usual flat M5-brane solution, and R_0 is an arbitrary integration constant with dimension of a length (which is $(C_2)^{-1/2}$ of eq. (24) in Ref. [195]). Finally, k is a (dimensionless) integration constant, which was called C_1 in Ref. [195]. All the coordinates entering in the above solution are dimensionless, except those spanning the unwrapped part of the world-volume of the brane, ξ^i, $i = 0, \ldots, 3$, which have dimensions of a length.

Uplift formulas and eleven dimensional solution

The seven-dimensional solution can be lifted to eleven dimensions with the help of eq.s (4.1) and (4.2) of Ref. [232], which we rewrite here:

$$d\hat{s}^2 = \tilde{\Delta}^{1/3} ds^2_{(7)} + g^{-2}\tilde{\Delta}^{-2/3}\left(X_0^{-1} d\mu_0^2 + \sum_{i=1}^{2} X_i^{-1}\left(d\mu_i^2 + \mu_i^2\left(d\phi_i + g\mathcal{A}^{(i)}\right)^2\right)\right), \tag{C.1.2a}$$

$$\begin{aligned} \hat{\star} d\hat{C}_3 = {} & 2g \sum_{\alpha=0}^{2}\left(X_\alpha^2 \mu_\alpha^2 - \tilde{\Delta} X_\alpha\right)\varepsilon_{(7)} + g\tilde{\Delta} X_0\, \varepsilon_{(7)} + \frac{1}{2g}\sum_{\alpha=0}^{2} X_\alpha^{-1}\, {}^{\star_7} dX_\alpha \wedge d(\mu_\alpha^2) \\ & + \frac{1}{2g^2}\sum_{i=1}^{2} X_i^{-2} d(\mu_i^2) \wedge \left(d\phi_i + g\mathcal{A}^{(i)}\right) \wedge {}^{\star_7}\mathcal{F}^{(i)}. \end{aligned} \tag{C.1.2b}$$

Here and below, hats will always refer to eleven-dimensional quantities. The above formulas are written in the notation of Ref. [232]: g is the seven dimensional gauged supergravity coupling constant, $\varepsilon_{(7)}$ is the seven dimensional volume form, $\mathcal{A}^{(1,2)}$ are the two $U(1)$ gauge fields, the X_α are a suitable parameterization of the 2 scalars present in the theory and $\tilde{\Delta}$ is given by:

$$\tilde{\Delta} \equiv \sum_{\alpha=0}^{2} X_\alpha \mu_\alpha^2 \,,$$

where μ_α parameterize a two-sphere: $\mu_0^2 + \mu_1^2 + \mu_2^2 = 1$. The quantities appearing in the uplift formulas are given in terms of those appearing in eq. (C.1.1) by the following expressions:

$$\begin{aligned} \frac{1}{g^2} &= \left(\frac{R_A}{2}\right)^2, \\ X_0 &= X_1 = e^{2\lambda}, \\ X_2 &= e^{-3\lambda}, \\ \mathcal{A}^{(1,2)} &= 2A^{(2,1)}, \\ \Delta &\equiv e^{3\lambda}\tilde{\Delta} = e^{5\lambda}\cos^2\chi + \sin^2\chi, \\ \varepsilon_{(7)} &= -\sqrt{-\det G_{(7)}}\, d\xi_0 \wedge \cdots d\xi_3 \wedge d\tilde{\theta} \wedge d\tilde{\varphi} \wedge d\rho, \end{aligned} \tag{C.1.3}$$

where we have chosen the following parameterization for μ_i:

$$\begin{aligned}\mu_0 &= \cos\chi\cos\theta\,,\\ \mu_1 &= \cos\chi\sin\theta\,,\\ \mu_2 &= \sin\chi\,.\end{aligned} \tag{C.1.4}$$

By using the above expressions we are now ready to write the full solution in eleven dimensions:

$$\begin{aligned} d\hat{s}^2 = \Delta^{1/3} & \left(\left(\frac{R_A}{R_0}\right)^2 e^{2\rho}\,\eta_{ij}d\xi^i d\xi^j + R_A^2(e^{2\rho}-\tfrac{1}{4})(d\tilde{\theta}^2+\sin^2\tilde{\theta}d\tilde{\varphi}^2)\right) \\ & + \Delta^{-2/3}\left(\frac{R_A}{2}\right)^2\left(\frac{4\Delta}{e^{5\lambda}}d\rho^2 + \Delta d\chi^2 + \cos^2\chi(d\theta^2+\sin^2\theta d\varphi^2)\right. \\ & \left. + e^{5\lambda}\sin^2\chi\left(d\psi+\cos\tilde{\theta}d\tilde{\varphi}\right)^2\right)\,, \end{aligned} \tag{C.1.5a}$$

$$\begin{aligned} {}^{\hat{\star}}d\hat{C}_3 = \frac{R_A^6}{R_0^4}\, & e^{4\rho}(e^{2\rho}-\tfrac{1}{4})\sin\tilde{\theta}\Bigg(2(\Delta+2)\,d\xi^0\wedge\cdots\wedge d\xi^3\wedge d\tilde{\theta}\wedge d\tilde{\varphi}\wedge d\rho \\ & + \frac{1}{4}\partial_\rho(e^{5\lambda})\sin(2\chi)\,d\xi^0\wedge\cdots\wedge d\xi^3\wedge d\tilde{\theta}\wedge d\tilde{\varphi}\wedge d\chi \\ & + \frac{1}{16}\frac{e^{5\lambda}\sin(2\chi)}{(e^{2\rho}-\frac{1}{4})^2\sin\tilde{\theta}}\,d\xi^0\wedge\cdots\wedge d\xi^3\wedge d\rho\wedge d\chi\wedge\left(d\psi+\cos\tilde{\theta}d\tilde{\varphi}\right)\Bigg) \end{aligned} \tag{C.1.5b}$$

where we have also relabeled the angles appearing in (C.1.2): $\phi_1 = \varphi\,, \phi_2 = \psi\,$. This solution describes the near horizon geometry of an M5-brane wrapped on a two-sphere. The unwrapped world-volume coordinates are $\xi^0, ..., \xi^3$, the wrapped ones are $\tilde{\theta}$ and $\tilde{\varphi}$, and the remaining coordinates are transverse to the M5. It can be easily seen that the metric in eq. (C.1.5) reduces to the one given in eq. (26) of Ref. [195] if we restrict ourselves to work at the IR fixed point analysed there.

From eq. (C.1.5b) we can compute the three-form potential. It is equal to:

$$\begin{aligned}\hat{C}_3 =& \frac{R_A^3}{8}\,\frac{e^{5\lambda}\cos^3\chi\cos\theta\sin\tilde{\theta}}{\Delta}\,d\tilde{\theta}\wedge d\tilde{\varphi}\wedge d\varphi \\ &+ \frac{R_A^3}{8}\,\frac{e^{5\lambda}(\Delta+2)\cos^2\chi\sin\chi\cos\theta}{\Delta^2}\,d\chi\wedge d\varphi\wedge\left(d\psi+\cos\tilde{\theta}d\tilde{\varphi}\right) \\ &+ \frac{R_A^3}{8}\,\frac{\partial_\rho(e^{5\lambda})\cos^3\chi\sin^2\chi\cos\theta}{\Delta^2}\,d\rho\wedge d\varphi\wedge\left(d\psi+\cos\tilde{\theta}d\tilde{\varphi}\right)\,.\end{aligned} \tag{C.1.6}$$

The last step consists in compactifying to ten dimensions the M5-brane solution just obtained along one of its non-wrapped world-volume coordinates (that we choose to be ξ^3) to get the solution describing the geometry of N wrapped D4-branes. The compactification is obtained by means of the standard expressions in the ten dimensional string frame:

$$(G_{\rm st})_{\mu\nu} = (\hat{G}_{33})^{1/2}\hat{G}_{\mu\nu}\,, \tag{C.1.7a}$$

$$e^{2\phi} = (\hat{G}_{33})^{3/2}\,, \tag{C.1.7b}$$

$$(C_3)_{\mu\nu\rho} = (\hat{C}_3)_{\mu\nu\rho}\,, \tag{C.1.7c}$$

where we have split eleven dimensional indices in $\hat{\mu} = \{\mu, \xi^3\}$. This is all we need to get the final expression for the wrapped D4-brane solution presented in section 10.1.2.

C.1.2 Fractional D2/D6-brane system

In this section we will describe in some detail how to find the supergravity solution describing a fractional D2/D6-brane system. We will always work in the Einstein frame. The Type IIA effective action in the orbifold background (10.2.1) is given, in our conventions, by:

$$\begin{aligned}S_{\rm IIA} = \frac{1}{2\kappa^2}\Bigg\{\int d^{10}x\sqrt{-G}\,R - \frac{1}{2}\int\Big(& d\phi\wedge{}^\star d\phi + e^{-\phi}H_3\wedge{}^\star H_3 \\ &- e^{3\phi/2}F_2\wedge{}^\star F_2 - e^{\phi/2}\tilde{F}_4\wedge{}^\star\tilde{F}_4 + B_2\wedge F_4\wedge F_4\Big)\Bigg\}\,,\end{aligned} \tag{C.1.8}$$

where the field strengths are given by:

$$H_3 = dB_2\,,\qquad F_2 = dC_1\,,\qquad F_4 = dC_3\,,\qquad \tilde{F}_4 = F_4 - C_1\wedge H_3\,, \tag{C.1.9}$$

and $\kappa = 8\pi^{7/2} g_s \alpha'^2$. In order to find a D-brane solution we must add to the previous bulk action a boundary action whose corresponding Lagrangian we call $\mathcal{L}_b$. The equations of motion are then obtained by varying the total action $S_{\rm IIA} + S_b$.

The first step in order to find a supergravity solution in the orbifold background (see for instance Ref. [200]) is substituting in the action (C.1.8) the form (10.2.5) of the fields:

$$B_2 = b\,\omega_2\,, \qquad C_3 = \bar{C}_3 + A_1 \wedge \omega_2\,. \tag{C.1.10}$$

Recalling that the 2-form ω_2 is normalized as in eq. (10.2.3), one obtains:

$$\begin{aligned} S'_{\rm IIA} = \frac{1}{2\kappa^2}\Bigg\{ \int d^{10}x\sqrt{-G}\,R - \frac{1}{2}\int \Big(d\phi\wedge{}^\star d\phi - e^{3\phi/2} dC_1\wedge{}^\star dC_1 - e^{\phi/2} d\bar{C}_3\wedge{}^\star d\bar{C}_3\Big) \\ - \frac{1}{4}\int_{\mathbb{R}^{1,5}} \Big(e^{-\phi} db\wedge{}^{\star 6} db - e^{\phi/2} G_2 \wedge{}^{\star 6} G_2 - 2b\wedge d\bar{C}_3 \wedge dA_1\Big)\Bigg\}\,, \end{aligned} \tag{C.1.11}$$

where we have introduced the quantity:

$$G_2 \equiv dA_1 - C_1 \wedge db\,. \tag{C.1.12}$$

By varying the previous action one finds the equations of motion for the fields C_1, $\bar{C}_3$, A_1, b and ϕ respectively:

$$d\left(e^{3\phi/2}\,{}^\star dC_1\right) - \frac{1}{2} e^{\phi/2} db\wedge{}^{\star 6} G_2 \wedge \Omega_4 + 2\kappa^2 \frac{\delta \mathcal{L}_b}{\delta C_1} = 0\,, \tag{C.1.13a}$$

$$d\left(e^{\phi/2}\,{}^\star d\bar{C}_3\right) + \frac{1}{2} db\wedge dA_1 \wedge \Omega_4 + 2\kappa^2 \frac{\delta \mathcal{L}_b}{\delta \bar{C}_3} = 0\,, \tag{C.1.13b}$$

$$d\left(e^{\phi/2}\,{}^{\star 6} G_2\right) + db \wedge d\bar{C}_3 + 4\kappa^2 \frac{\delta \mathcal{L}_b}{\delta A_1} = 0\,, \tag{C.1.13c}$$

$$d\left(e^{-\phi}\,{}^{\star 6} db - e^{\phi/2} C_1 \wedge{}^{\star 6} G_2\right) + d\bar{C}_3 \wedge dA_1 + 4\kappa^2 \frac{\delta \mathcal{L}_b}{\delta b} = 0\,, \tag{C.1.13d}$$

$$\begin{aligned} d^\star d\phi + \frac{3}{4} e^{3\phi/2} dC_1 \wedge{}^\star dC_1 + \frac{1}{4} e^{\phi/2} d\bar{C}_3 \wedge{}^\star d\bar{C}_3 \\ + \frac{1}{4}\Big[e^{-\phi}\; db\wedge{}^{\star 6} db + \frac{1}{2} e^{\phi/2}\; G_2 \wedge{}^{\star 6} G_2\Big] \wedge \Omega_4 + 2\kappa^2 \frac{\delta \mathcal{L}_b}{\delta \phi} = 0\,, \end{aligned} \tag{C.1.13e}$$

where we have defined $\Omega_4 = \delta(x^6)\cdots\delta(x^9)\; dx^6 \wedge \cdots \wedge dx^9$. By varying the action with respect to the metric one gets also the Einstein equations that it is convenient to split into three separate equations (according to which components of the metric are involved in each case). By denoting with $x^{\rho,\sigma,\dots} = \{x^0, ..., x^5\}$ the coordinates on $\mathbb{R}^{1,5}$ and by $x^{p,q,\dots} = \{x^6, ..., x^9\}$ the orbifolded coordinates we find the following equations:

$$R_{\rho\sigma} - \frac{1}{2} R G_{\rho\sigma} + 2\kappa^2 \frac{\delta \mathcal{L}_b}{\delta G^{\rho\sigma}} = T^{\rm u}_{\rho\sigma} + \Omega_4 T^{\rm t}_{\rho\sigma} \,, \tag{C.1.14a}$$

$$R_{pq} - \frac{1}{2} R G_{pq} + 2\kappa^2 \frac{\delta \mathcal{L}_b}{\delta G^{pq}} = T^{\rm u}_{pq} \,, \tag{C.1.14b}$$

$$R_{q\sigma} - \frac{1}{2} R G_{q\sigma} + 2\kappa^2 \frac{\delta \mathcal{L}_b}{\delta G^{q\sigma}} = T^{\rm u}_{q\sigma} \,. \tag{C.1.14c}$$

The energy-momentum tensors above refer separately to those of the "twisted" and "untwisted" fields, and are given by:

$$\begin{aligned} T^{\rm u}_{\mu\nu} = {} & \frac{1}{2}(\partial_\mu \phi \, \partial_\nu \phi - \frac{1}{2}(\partial\phi)^2 G_{\mu\nu}) + \frac{1}{2} e^{3\phi/2} (F_{2\,\mu A} F_{2\,\nu}{}^{A} - \frac{1}{4}(F_2)^2 G_{\mu\nu}) \\ & + \frac{1}{2\cdot 3!} e^{\phi/2} (F_{4\,\mu ABC} F_{4\,\nu}{}^{ABC} - \frac{1}{8}(F_4)^2 G_{\mu\nu}) \,, \end{aligned} \tag{C.1.15a}$$

$$T^{\rm t}_{\rho\sigma} = \frac{1}{2} \frac{\sqrt{-G_{(6)}}}{\sqrt{-G}} \left(\frac{1}{2} e^{-\phi} (\partial_\rho b \, \partial_\sigma b - \frac{1}{2}(\partial b)^2 G_{\rho\sigma}) + \frac{1}{2} e^{\phi/2} (G_{2\,\rho A} G_{2\,\sigma}{}^{A} - \frac{1}{4}(G_2)^2 G_{\rho\sigma}) \right) \,, \tag{C.1.15b}$$

where, in the expression for $T^{\rm u}_{\mu\nu}$, indices μ, ν run over the appropriate coordinates (according to the equation in which they are used) and, in all cases, summed indices $(A, B, \dots)$ run over *all* ten dimensional coordinates. In the expression for $T^{\rm t}_{\rho\sigma}$, $G_{(6)}$ refers to the determinant of the restriction of the ten dimensional metric to the 6-dimensional subspace $\mathbb{R}^{1,5}$.

As explained in section 10.2.1, we are interested in a bound state of N fractional D2-branes and M D6-branes. The world-volume of the D2-branes extends in the directions x^0, x^1, x^2, while these branes are stuck at the orbifold fixed point $x^6 = x^7 = x^8 = x^9 = 0$. The D6-branes extend in the directions x^0, x^1, x^2 as well as along the orbifolded directions x^6, x^7, x^8, x^9.

For the "untwisted" fields we consider the following standard Ansatz for a D2/D6 system:

$$ds^2 = H_2^{-5/8} H_6^{-1/8} \eta_{\alpha\beta} dx^\alpha dx^\beta + H_2^{3/8} H_6^{7/8} \delta_{ij} dx^i dx^j + H_2^{3/8} H_6^{-1/8} \delta_{pq} dx^p dx^q\,, \tag{C.1.16a}$$

$$e^\phi = H_2^{1/4} H_6^{-3/4}\,, \tag{C.1.16b}$$

$$\bar{C}_3 = \left(H_2^{-1} - 1\right) dx^0 \wedge dx^1 \wedge dx^2\,, \tag{C.1.16c}$$

where we have divided the coordinates in three groups: $x^{\alpha,\beta,\cdots} = \{x^0, x^1, x^2\}$ denote the coordinates along the world-volume of both branes, $x^{i,j,\cdots} = \{x^3, x^4, x^5\}$ denote the ones transverse to both, while $x^{p,q,\cdots} = \{x^6, x^7, x^8, x^9\}$ denote the (orbifolded) coordinates along the world-volume of the D6-branes and transverse to the D2-branes. The function H_2 depends on the radial coordinate $\rho = \sqrt{(x^3)^2 + \ldots + (x^9)^2}$ of the space transverse to the D2-brane, while the function H_6 depends only on the radial coordinate of the common transverse space $r = \sqrt{\delta_{ij} x^i x^j}$.

In order to find a sensible Ansatz for the fields A_1 and C_1 we need to take a more careful look at the contributions coming from the boundary action describing the world-volume theory of the branes. For our system, such an action is the sum of a term describing the D2-branes and a term describing the D6-branes:

$$S_b = N S_2 + M S_6\,, \tag{C.1.17}$$

The relevant parts of the action (as explained in Ref. [242], only the linear terms contribute) are given by (see appendix C.2):

$$S_2 = \frac{T_2}{2\kappa} \Bigg\{ - \int d^3x\; e^{-\phi/4} \sqrt{-\det G_{\alpha\beta}} \left(1 + \frac{\tilde{b}}{2\pi^2\alpha'} \right) + \int_{\mathcal{M}_3} \left[\bar{C}_3 \left(1 + \frac{\tilde{b}}{2\pi^2\alpha'} \right) + \frac{A_3}{2\pi^2\alpha'} \right] \Bigg\}\,, \tag{C.1.18a}$$

$$S_6 = \frac{T_6}{\kappa} \left\{ - \int d^7x\; e^{\frac{3}{4}\phi} \sqrt{-\det G_{\rho\sigma}} + \int_{\mathcal{M}_7} C_7 \right\} + \frac{T_2}{2\kappa} \frac{1}{2(2\pi\sqrt{\alpha'})^2} \left\{ \int d^3\xi \sqrt{-\det G_{\alpha\beta}}\; \tilde{b} - \int_{\mathcal{M}_3} A_3 \right\} + \ldots\,, \tag{C.1.18b}$$

where the indices $\alpha, \beta, \ldots$ run along the common world-volume $\mathcal{M}_3$ and $\rho, \sigma, \ldots$ along the whole D6-brane world-volume $\mathcal{M}_7$.

We notice that the previous boundary actions do not depend on the fields C_1 and A_1. This means that eq.s (C.1.13a) and (C.1.13c) will not contain the contribution coming from the boundary action:

$$d\left(e^{3\phi/2}\,{}^\star dC_1\right) - \frac{1}{2} e^{\phi/2} db \wedge {}^{\star_6}G_2 \wedge \Omega_4 = 0\,, \tag{C.1.19a}$$

$$d\left(e^{\phi/2}\,{}^{\star_6}G_2\right) + db \wedge d\bar{C}_3 = 0\,. \tag{C.1.19b}$$

Taking into account the expression in eq. (C.1.16c) for $\bar{C}_3$, we see that the second equation is easily satisfied by imposing:

$$e^{\phi/2}\,{}^{\star_6}G_2 = H_2^{-1} db \wedge dx^0 \wedge dx^1 \wedge dx^2\,. \tag{C.1.20}$$

Eq. (C.1.20) implies that the second term of eq. (C.1.19a) vanishes. Then, eq. (C.1.19a) can be satisfied by imposing:

$$e^{3\phi/2}\,{}^\star dC_1 = -d\left(H_6^{-1}\right) dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^6 \wedge \cdots \wedge dx^9\,. \tag{C.1.21}$$

Eq.s (C.1.20) and (C.1.21) imply after some manipulations the following expressions for A_1 and C_1 :

$$dA_1 = C_1 \wedge db + \frac{1}{2}\varepsilon_{ijk} H_6 \partial_i b dx^j \wedge dx^k\,, \tag{C.1.22a}$$

$$dC_1 = \frac{1}{2}\varepsilon_{ijk} \partial_i H_6 dx^j \wedge dx^k\,, \tag{C.1.22b}$$

where ε_{ijk} is such that $\varepsilon_{345} = \varepsilon^{345} = +1$.

We are now ready to find the complete solution. Inserting the Ansätze (C.1.16)-(C.1.22) into the equations of motion (C.1.13) and computing all the relevant contributions coming from the boundary action S_b , after some algebra we get:

$$\begin{aligned}\left(\delta^{ij}\partial_i\partial_j + H_6\delta^{pq}\partial_p\partial_q\right) H_2 + \tfrac{1}{2} H_6 \delta^{ij} \partial_i b \partial_j b\, \delta(x^6) \cdots \delta(x^9)& \\ + \kappa T_2 N \delta(x^3) \cdots \delta(x^9) = 0&\end{aligned} \tag{C.1.23a}$$

from the eq. (C.1.13b) for $\bar{C}_3$,

$$H_2^{-1}\left(H_6 \delta^{ij}\partial_i\partial_j b + 2\delta^{ij}\partial_i H_6 \partial_j b\right) - \frac{\kappa T_2}{4\pi^2\alpha'}(4N - M)\delta(x^3)\cdots\delta(x^5) = 0\,, \tag{C.1.23b}$$

from the eq. (C.1.13d) for b and

$$\frac{1}{4}H_2^{-1}\Big(\left(\delta^{ij}\partial_i\partial_j + H_6\delta^{pq}\partial_p\partial_q\right) H_2 + \frac{1}{2}H_6\delta^{ij}\partial_i b\partial_j b\delta(x^6)\cdots\delta(x^9)\Big)$$
$$- \frac{3}{4}H_6^{-1}\delta^{ij}\partial_i\partial_j H_6 + \frac{\kappa T_2}{4}\left(N - \frac{6T_6}{T_2}M\right)\delta(x^3)\cdots\delta(x^9) = 0\,, \quad \text{(C.1.23c)}$$

from the eq. (C.1.13e) for ϕ. Plugging eq. (C.1.23a) into eq. (C.1.23c) we get the following equation for the function H_6:

$$\delta^{ij}\partial_i\partial_j H_6 = -2\kappa T_6 M\delta(x^3)\cdots\delta(x^5)\,, \quad \text{(C.1.24)}$$

whose solution is given by the following harmonic function:

$$H_6(r) = 1 + \frac{g_s\sqrt{\alpha'}M}{2r}\,. \quad \text{(C.1.25)}$$

Analogously, using eq. (C.1.24) into eq. (C.1.23b) we obtain the following equation for the function $Z \equiv H_6 b$:

$$\delta^{ij}\partial_i\partial_j Z = \frac{\kappa T_2}{2\pi^2\alpha'}(2N - M)\delta(x^3)\cdots\delta(x^5)\,, \quad \text{(C.1.26)}$$

which is solved by:

$$Z(r) = \frac{\left(2\pi\sqrt{\alpha'}\right)^2}{2}\left(1 - \frac{g_s\sqrt{\alpha'}(2N-M)}{r}\right)\,, \quad \text{(C.1.27)}$$

where we have chosen the constant term in order to satisfy the condition (10.2.6) for the background value of the field b. We are now left with eq. (C.1.23a), which is in general difficult to solve. Finally, after some computation one can show that with our Ansatz the equations (C.1.14) for the metric are also satisfied provided that eq. (C.1.23a) holds.

Finally, the fields C_1 and A_1 are obtained by integrating eq.s (C.1.22). In order to do so, we change the coordinate system into polar coordinates in the common transverse space: $(x^3, x^4, x^5) \longrightarrow (r, \theta, \varphi)$. Then we obtain:

$$C_1 = \frac{g_s\sqrt{\alpha'}M}{2}\cos\theta d\varphi\,, \quad \text{(C.1.28a)}$$
$$A_1 = -\pi^2\alpha'\frac{g_s\sqrt{\alpha'}(4N-M)}{1 + \frac{g_s\sqrt{\alpha'}M}{2r}}\cos\theta d\varphi\,. \quad \text{(C.1.28b)}$$

The supergravity solution that we have found is summarized in eq. (10.2.9).

C.2 World-volume actions for fractional branes

The world-volume action for a fractional Dp-brane ($p \leq 5$) transverse to the orbifold space $\mathbb{R}^4/\mathbb{Z}_2$ can be obtained in several ways. Recalling that a fractional Dp-brane is a D$(p+2)$-brane wrapped[2] on the vanishing cycle Σ_2 defined in section 10.2.1, one can get the action for a fractional Dp-brane starting from the one of a D$(p+2)$-brane, which in the Einstein frame is:

$$S_{p+2} = S_{\text{DBI}} + S_{\text{WZ}} \,, \tag{C.2.1}$$

with:

$$S_{\text{DBI}} = -\frac{T_{p+2}}{\kappa} \int d^{p+3}\xi e^{\frac{p-1}{4}\phi} \sqrt{-\det\left[G_{ab} + e^{-\frac{\phi}{2}}\left(B_{ab} + 2\pi\alpha' F_{ab}\right)\right]} \,, \tag{C.2.2a}$$

$$S_{\text{WZ}} = \frac{T_{p+2}}{\kappa} \int_{\mathcal{M}_{p+3}} \sum_q C_q \wedge e^{B+2\pi\alpha' F} \,, \tag{C.2.2b}$$

where $\xi^{a,b,\cdots} = \{\xi^0, \ldots, \xi^{p+2}\}$ are the coordinates of the brane world-volume and $T_p = \sqrt{\pi}(2\pi\sqrt{\alpha'})^{3-p}$. All bulk fields in eq.s (C.2.2) are pullbacks onto the world-volume $\mathcal{M}_{p+3}$ of the brane.

Let us start by considering the DBI part of the action. In order to wrap the brane on the cycle Σ_2 we have to impose the decomposition in eq. (10.2.4) for the field B_2 (we suppose that it has no components outside the cycle). The metric has no support on Σ_2, so eq. (C.2.2a) becomes:

$$\begin{aligned} S_{\text{DBI}} &= -\frac{T_{p+2}}{\kappa} \int d^{p+1}\xi e^{\frac{p-1}{4}\phi} \sqrt{-\det\left[G_{\alpha\beta} + e^{-\frac{\phi}{2}} 2\pi\alpha' F_{\alpha\beta}\right]}\; e^{-\frac{\phi}{2}} \int_{\Sigma_2} b\omega_2 \,, \\ &= -\frac{T_p}{2\kappa} \int d^{p+1}\xi e^{\frac{p-3}{4}\phi} \sqrt{-\det\left[G_{\alpha\beta} + e^{-\frac{\phi}{2}} 2\pi\alpha' F_{\alpha\beta}\right]} \left(1 + \frac{\tilde{b}}{2\pi^2\alpha'}\right) \,, \end{aligned} \tag{C.2.3}$$

where we have used eq.s (10.2.2-10.2.6) and the relation $T_p = (2\pi\sqrt{\alpha'})^2 T_{p+2}$. The coordinates $\xi^{\alpha,\beta,\cdots} = \{\xi^0, \ldots, \xi^p\}$ are the coordinates of the world-volume

[2] To be precise, we are considering fractional branes of "type 1", which have a B-flux on the shrinking cycle but not an F-flux.

$\mathcal{M}_p$ of the fractional Dp-brane. Turning to the WZ part, we have to decompose the R-R potentials in a similar fashion as in eq. (10.2.4):

$$C_q = \bar{C}_q + A_{q-2} \wedge \omega_2 \tag{C.2.4}$$

(notice that $\bar{C}_{p+3}$ vanishes). To discuss what is the result of wrapping, let us first consider the highest rank field C_{p+3}, whose contribution to the WZ action is given by:

$$\frac{T_{p+2}}{\kappa} \int_{\mathcal{M}_{p+3}} C_{p+3} = \frac{T_{p+2}}{\kappa} \int_{\mathcal{M}_{p+3}} A_{p+1} \wedge \omega_2 = \frac{T_p}{2\kappa} \int_{\mathcal{M}_{p+1}} \frac{A_{p+1}}{2\pi^2\alpha'}, \tag{C.2.5}$$

where we have used eq.s (10.2.3). Considering now C_{p+1}, one gets:

$$\frac{T_{p+2}}{\kappa} \int_{\mathcal{M}_{p+3}} C_{p+1} \wedge B \longrightarrow \frac{T_{p+2}}{\kappa} \int_{\mathcal{M}_{p+3}} \bar{C}_{p+1} b \wedge \omega_2 = \frac{T_p}{2\kappa} \int_{\mathcal{M}_{p+1}} \bar{C}_{p+1} \left(1 + \frac{\tilde{b}}{2\pi^2\alpha'}\right) . \tag{C.2.6}$$

Therefore the first term of the WZ action is:

$$\frac{T_p}{2\kappa} \int_{\mathcal{M}_{p+1}} \left[\bar{C}_{p+1} \left(1 + \frac{\tilde{b}}{2\pi^2\alpha'}\right) + \frac{A_{p+1}}{2\pi^2\alpha'}\right] . \tag{C.2.7}$$

If we consider any other lower rank potential, one can see that the relevant contributions to the WZ action always involve the following combinations of fields:

$$\mathcal{C}_q = \bar{C}_q \left(1 + \frac{\tilde{b}}{2\pi^2\alpha'}\right) + \frac{A_q}{2\pi^2\alpha'}, \tag{C.2.8}$$

This means that the world-volume action for a fractional Dp-brane can always be put in the form:

$$S_p = S_{\mathrm{DBI}} + S_{\mathrm{WZ}}, \tag{C.2.9}$$

where:

$$S_{\mathrm{DBI}} = -\frac{T_p}{2\kappa} \int d^{p+1}\xi e^{\frac{p-3}{4}\phi} \sqrt{-\det\left[G_{\alpha\beta} + e^{-\frac{\phi}{2}} 2\pi\alpha' F_{\alpha\beta}\right]} \left(1 + \frac{\tilde{b}}{2\pi^2\alpha'}\right), \tag{C.2.10a}$$

$$S_{\mathrm{WZ}} = \frac{T_p}{2\kappa} \int_{\mathcal{M}_{p+1}} \sum_q \mathcal{C}_q \wedge e^{2\pi\alpha' F} . \tag{C.2.10b}$$

The precise expression of the action for a fractional Dp-brane is confirmed by the couplings of the brane to the bulk fields, computed with the boundary state formalism [200] and with explicit computation of string scattering amplitudes on a disk [243].

In this Chapter we also consider D-branes whose world-volume directions extend along the whole orbifold space, namely D$(p+4)$-branes with four longitudinal directions along $x^6, \ldots, x^9$ and $p+1$ along $x^0, \ldots, x^p$. In this case the terms linear in the bulk fields of the boundary action can be inferred from the couplings computed with the boundary state[3] [215], and one gets:

$$S_{p+4} = \frac{T_{p+4}}{\kappa} \left\{ - \int d^{p+5}\xi e^{\frac{p+1}{4}\phi} \sqrt{-\det G_{\rho\sigma}} + \int_{\mathcal{M}_{p+5}} C_{p+5} \right\}$$
$$+ \frac{T_p}{2\kappa} \frac{1}{2(2\pi\sqrt{\alpha'})^2} \left\{ \int d^{p+1}\xi \sqrt{-\det G_{\alpha\beta}}\, \tilde{b} - \int_{\mathcal{M}_{p+1}} A_{p+1} \right\} + \ldots , \quad \text{(C.2.11)}$$

where $x^{\rho,\sigma,\ldots}$ are the coordinates of the brane world-volume $\mathcal{M}_{p+5}$, while $x^{\alpha,\beta,\ldots}$ are the coordinates along the part $\mathcal{M}_{p+1}$ of the world-volume which lie outside the orbifold directions. The ellipses in the action (C.2.11) stand for terms of higher order in the fields, not accounted by the boundary state approach.

C.3 The running coupling constant of $\mathcal{N} = 4$, $D = 2 + 1$ SYM theory

In this appendix, we briefly compute the expression of the running gauge coupling constant of $\mathcal{N} = 4$, $D = 2 + 1$ super Yang–Mills theory, in order to compare the perturbative gauge theory result [244] (see Ref. [245] for a review) with what we obtain from the supergravity solutions for the wrapped and the fractional brane systems.

The one-loop effective action for a D-dimensional field theory expanded around a background which is a solution of the classical field equations can

[3]One has also to take into account the fact that the boundary state sees the fields correctly normalized on the covering space, while we are using fields that are correctly normalized on the orbifold [213].

be expressed as[4]:

$$S_{\text{eff}} = \frac{1}{4g_{\text{YM}}^2} \int d^D x \left\{ \bar{F}^a_{\mu\nu} \bar{F}^a_{\mu\nu} + \frac{1}{2} \operatorname{Tr} \log \Delta_1 + \left(\frac{N_s}{2} - 1 \right) \operatorname{Tr} \log \Delta_0 \right.$$
$$\left. - N_f \operatorname{Tr} \log \Delta_{1/2} \right\} , \qquad \text{(C.3.1)}$$

where N_s and N_f are respectively the number of scalars and Dirac fermions and where:

$$(\Delta_1)^{ab}_{\mu\nu} = - \left(\bar{D}^2 \right)^{ab} \delta_{\mu\nu} + 2 f^{acb} \bar{F}^c_{\mu\nu} , \quad (\Delta_0)^{ab} = - \left(\bar{D}^2 \right)^{ab} , \quad \Delta_{1/2} = i \bar{\not{D}} , \qquad \text{(C.3.2)}$$

D_μ being the covariant derivative and f^{abc} the gauge group structure constants. The part of the determinants in eq. (C.3.1) quadratic in the gauge fields can be extracted obtaining:

$$S_{\text{eff}} = \frac{1}{4} \int d^D x F^2 \left\{ \frac{1}{g_{\text{YM}}^2} + I \right\} , \qquad \text{(C.3.3)}$$

where:

$$I = \frac{1}{(4\pi)^{D/2}} \int_0^\infty \frac{ds}{s^{D/2-1}} e^{-\mu^2 s} R , \qquad \text{(C.3.4)}$$

where μ is the mass of the fields, and

$$R = 2 \left[\frac{N_s}{12} c_s + \frac{D-26}{12} c_v + \frac{2^{[D/2]} N_f}{6} c_f \right] , \qquad \text{(C.3.5)}$$

where $[D/2] = D/2$ if D is even and $[D/2] = \frac{D-1}{2}$ if D is odd, and where the constants c set the normalization of the generators of the gauge group ($Tr(\lambda^a \lambda^b) = c\delta^{ab}$) in the representations under which the scalars, the vector and the fermions respectively transform. Concentrating on the case $D = 3$, we get:

$$I = \frac{1}{(4\pi)^{3/2}} \int_0^\infty \frac{ds}{s^{1/2}} e^{-\mu^2 s} R = \frac{1}{8\pi\mu} R , \qquad \text{(C.3.6)}$$

with

$$R = 2 \left[\frac{N_s}{12} c_s - \frac{23}{12} c_v + \frac{N_f}{3} c_f \right] . \qquad \text{(C.3.7)}$$

[4] A bar on an operator in eq.s (C.3.1) and (C.3.2) indicates that the operator is evaluated at the background value $\bar{A}^a_\mu$ of the gauge field.

In our case we have a theory with gauge group $SU(N)$ and 8 supercharges, coupled to M hypermultiplets in its fundamental representation. The vector multiplet contains 2 Dirac fermions and 3 scalars, while each hypermultiplet is made up of 2 Dirac fermions and 4 scalars. Recalling that $c = \frac{1}{2}$ for the fundamental representation and $c = N$ for the adjoint representation, eq. (C.3.7) gives:

$$R = -2N + M\,, \tag{C.3.8}$$

and the running effective coupling given by eq.s (C.3.3)-(C.3.6) is equal to:

$$\frac{1}{g^2_{\rm YM}(\mu)} = \frac{1}{g^2_{\rm YM}}\left(1 - g^2_{\rm YM}\frac{2N - M}{8\pi\mu}\right)\,, \tag{C.3.9}$$

This expression of the one-loop running coupling constant is in complete agreement with both our results in eq.s (10.1.21) and (10.2.21).

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Bibliography

[1] T. Goto, *Relativistic Quantum Mechanics Of One-Dimensional Mechanical Continuum And Subsidiary Condition Of Dual Resonance Model*, Prog. Theor. Phys. **46** (1971) 1560.

[2] J. Scherk and J. H. Schwarz, *Dual Models For Nonhadrons*, Nucl. Phys. **B81** (1974) 118.

[3] P. Goddard, J. Goldstone, C. Rebbi and C. B. Thorn, *Quantum Dynamics Of A Massless Relativistic String*, Nucl. Phys. **B56** (1973) 109.

[4] P. Goddard and C. B. Thorn, *Compatibility Of The Dual Pomeron With Unitarity And The Absence Of Ghosts In The Dual Resonance Model*, Phys. Lett. **B40** (1972) 235.

[5] A. M. Polyakov, *Quantum Geometry Of Bosonic Strings*, Phys. Lett. **B103** (1981) 207.

[6] A. M. Polyakov, *Quantum Geometry Of Fermionic Strings*, Phys. Lett. **B103** (1981) 211.

[7] D. J. Gross, J. A. Harvey, E. J. Martinec and R. Rohm, *The Heterotic String*, Phys. Rev. Lett. **54** (1985) 502.

[8] D. J. Gross, J. A. Harvey, E. J. Martinec and R. Rohm, *Heterotic String Theory. 1. The Free Heterotic String*, Nucl. Phys. **B256** (1985) 253.

[9] D. J. Gross, J. A. Harvey, E. J. Martinec and R. Rohm, *Heterotic String Theory. 2. The Interacting Heterotic String*, Nucl. Phys. **B267** (1986) 75.

[10] M. B. Green and J. H. Schwarz, *Anomaly Cancellation In Supersymmetric D=10 Gauge Theory And Superstring Theory*, Phys. Lett. **B149** (1984) 117.

[11] T. Kaluza, *On The Problem Of Unity In Physics*, Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) **K1** (1921) 966.

[12] A. Salam and J. Strathdee, *On Kaluza-Klein Theory*, Annals Phys. **141** (1982) 316.

[13] G. Aldazabal, A. Font, L. E. Ibáñez and A. M. Uranga, *String GUTs*, Nucl. Phys. **B452** (1995) 3 [arXiv:hep-th/9410206].

[14] L. E. Ibáñez, F. Marchesano and R. Rabadan, *Getting just the standard model at intersecting branes*, JHEP **0111** (2001) 002 [arXiv:hep-th/0105155].

[15] D. Cremades, L. E. Ibáñez and F. Marchesano, *Towards a theory of quark masses, mixings and CP-violation*, arXiv:hep-ph/0212064.

[16] T. H. Buscher, *Quantum Corrections And Extended Supersymmetry In New Sigma Models*, Phys. Lett. **B159** (1985) 127.

[17] A. Font, L. E. Ibáñez, D. Lust and F. Quevedo, *Strong-Weak Coupling Duality And Nonperturbative Effects In String Theory*, Phys. Lett. **B249** (1990) 35.

[18] C.M. Hull and P.K. Townsend, *Unity of Superstring Dualities*, Nucl. Phys. **B438** (1995) 109-137. [arXiv:hep-th/9410167].

[19] P. K. Townsend, *The eleven-dimensional supermembrane revisited*, Phys. Lett. **B350** (1995) 184 [arXiv:hep-th/9501068].

[20] E. Witten, *String theory dynamics in various dimensions*, Nucl. Phys. **B443**, 85 (1995) [arXiv:hep-th/9503124].

[21] M. J. Duff, R. R. Khuri and J. X. Lu, *String solitons*, Phys. Rept. **259** (1995) 213 [arXiv:hep-th/9412184].

[22] J. Polchinski, *Dirichlet-Branes and Ramond-Ramond Charges*, Phys. Rev. Lett. **75** (1995) 4724 [arXiv:hep-th/9510017].

[23] J. Maldacena, *The large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. **2** (1998) 231 [arXiv:hep-th/9711200].

[24] J. Scherk, *Zero-Slope Limit Of The Dual Resonance Model*, Nucl. Phys. **B31** (1971) 222.

[25] M. B. Green, J. H. Schwarz and L. Brink, *N=4 Yang-Mills And N=8 Supergravity As Limits Of String Theories*, Nucl. Phys. **B198** (1982) 474.

[26] E. Bergshoeff, C.M. Hull and T. Ortín, *Duality in the Type II Superstring Effective Action*, Nucl. Phys. **B451** (1995) 547-578. [arXiv:hep-th/9504081].

[27] E. Cremmer, B. Julia and J. Scherk, *Supergravity Theory In 11 Dimensions*, Phys. Lett. **B76** (1978) 409.

[28] M. Huq and M. A. Namazie, *Kaluza-Klein Supergravity In Ten-Dimensions*, Class. Quant. Grav. **2** (1985) 293 [Erratum-ibid. **2** (1985) 597].

[29] P. Horava and E. Witten, *Heterotic and type I string dynamics from eleven dimensions*, Nucl. Phys. **B460** (1996) 506 [arXiv:hep-th/9510209].

[30] E. Lozano-Tellechea and T. Ortín, *The General, Duality-Invariant Family of Non-BPS Black-Hole Solutions of* $N = 4$, $d = 4$ *Supergravity*, Nucl. Phys. **B569** (2000) 435-450. [arXiv:hep-th/9910020].

[31] E. Lozano-Tellechea and T. Ortin, *7-branes and higher Kaluza-Klein branes*, Nucl. Phys. **B607** (2001) 213 [arXiv:hep-th/0012051].

[32] E. Lozano-Tellechea, P. Meessen and T. Ortin, *On* $d = 4, 5, 6$ *vacua with 8 supercharges*, Class. Quant. Grav. **19** (2002) 5921 [arXiv:hep-th/0206200].

[33] N. Alonso-Alberca, E. Lozano-Tellechea and T. Ortín, *Geometric construction of Killing spinors and supersymmetry algebras in homogeneous space times*, Class. Quant. Grav. **19** (2002) 6009 [arXiv:hep-th/0208158].

[34] N. Alonso-Alberca, E. Lozano-Tellechea and T. Ortin, *The near-horizon limit of the extreme rotating $d = 5$ black hole as a homogeneous spacetime*, to appear in Class. Quant. Grav. arXiv:hep-th/0209069.

[35] P. Di Vecchia, H. Enger, E. Imeroni and E. Lozano-Tellechea, *Gauge theories from wrapped and fractional branes*, Nucl. Phys. **B631** (2002) 95 [arXiv:hep-th/0112126].

[36] D. J. Gross and E. Witten, *Superstring Modifications Of Einstein's Equations*, Nucl. Phys. **B277** (1986) 1.

[37] R. I. Nepomechie, *On The Low-Energy Limit Of Strings*, Phys. Rev. **D32** (1985) 3201.

[38] R. R. Metsaev and A. A. Tseytlin, *On Loop Corrections To String Theory Effective Actions*, Nucl. Phys. **B298** (1988) 109.

[39] E. S. Fradkin and A. A. Tseytlin, *Effective Field Theory From Quantized Strings*, Phys. Lett. **B158** (1985) 316.

[40] C. G. Callan, E. J. Martinec, M. J. Perry and D. Friedan, *Strings In Background Fields*, Nucl. Phys. **B262** (1985) 593.

[41] D. Friedan, *Nonlinear Models In Two+ϵ Dimensions*, Phys. Rev. Lett. **45** (1980) 1057;

[0.2cm] E. Witten, *Nonabelian Bosonization In Two Dimensions*, Commun. Math. Phys. **92** (1984) 455;

T. L. Curtright and C. K. Zachos, *Geometry, Topology And Supersymmetry In Nonlinear Models*, Phys. Rev. Lett. **53** (1984) 1799.

[42] M. T. Grisaru, P. S. Howe, L. Mezincescu, B. Nilsson and P. K. Townsend, *N=2 Superstrings In A Supergravity Background*, Phys. Lett. **B162** (1985) 116.

[43] C. G. Callan, I. R. Klebanov and M. J. Perry, *String Theory Effective Actions*, Nucl. Phys. **B278** (1986) 78.

[44] C. Lovelace, *Stability Of String Vacua. 1. A New Picture Of The Renormalization Group*, Nucl. Phys. **B273** (1986) 413.

[45] W. Fischler and L. Susskind, *Dilaton Tadpoles, String Condensates And Scale Invariance*, Phys. Lett. **B171** (1986) 383;
Dilaton Tadpoles, String Condensates And Scale Invariance. 2, Phys. Lett. **B173** (1986) 262.

[46] G. T. Horowitz and A. Strominger, *Black Strings and p-Branes*, Nucl. Phys. **B360** (1991) 197-209.

[47] J. Dai, R. G. Leigh and J. Polchinski, *New Connection Between String Theories*, Mod. Phys. Lett. **A4** (1989) 2073-2083.

[48] M. J. Duff and J. X. Lu, *Elementary Fivebrane Solutions of $D = 10$ Supergravity*, Nucl. Phys. **B354** (1991) 141-153.

[49] C. Callan, J. Harvey, A. Strominger, *World Sheet Approach to Heterotic Instantons and Solitons*, Nucl. Phys. **B359** (1991) 611.

[50] R. Arnowitt, S. Deser and C. Misner, in *Gravitation: an Introduction to Current Research*, Ed. L. Witten, Wiley, New York (1962).

[51] L. F. Abbott and S. Deser, *Stability Of Gravity With A Cosmological Constant*, Nucl. Phys. **B195** (1982) 76.

[52] E. Witten and D. Olive, *Supersymmetry Algebras that Include Topological Charges*, Phys. Lett. **B78** (1978) 97.

[53] R. Haag, J. T. Lopuszanski and M. Sohnius, *All Possible Generators Of Supersymmetries Of The S Matrix*, Nucl. Phys. **B88** (1975) 257.

[54] J. A. de Azcarraga, J. P. Gauntlett, J. M. Izquierdo and P. K. Townsend, *Topological Extensions Of The Supersymmetry Algebra For Extended Objects*, Phys. Rev. Lett. **63** (1989) 2443.

[55] R. Kallosh, A. D. Linde, T. Ortín, A. Peet and A. Van Proeyen, *Supersymmetry as a cosmic censor*, Phys. Rev. **D46** (1992) 5278 [arXiv:hep-th/9205027].

[56] W. Israel and J.M. Nester, *Positivity of the Bondi Gravitational Mass*, Phys. Lett. **A85**, (1981) 259.

G. W. Gibbons and C. M. Hull, *A Bogomolny Bound For General Relativity And Solitons In N=2 Supergravity*, Phys. Lett. **B109** (1982) 190.

[57] P. Di Vecchia, M. Frau, I. Pesando, S. Sciuto, A. Lerda and R. Russo, *Classical p-branes from boundary state*, Nucl. Phys. **B507** (1997) 259 [arXiv:hep-th/9707068].

[58] A. Strominger and C. Vafa, *Microscopic Origin of the Bekenstein-Hawking Entropy*, Phys. Lett. **B379** (1996) 99 [arXiv:hep-th/9601029].

[59] G. W. Gibbons and P. K. Townsend, *Vacuum Interpolation In Supergravity Via Super P-Branes*, Phys. Rev. Lett. **71** (1993) 3754 [arXiv:hep-th/9307049].

[60] R. Dijkgraaf, *Fields, strings and duality*, arXiv:hep-th/9703136.

[61] A. Dabholkar, *Lectures on orientifolds and duality*, arXiv:hep-th/9804208.

[62] A. Giveon, M. Porrati and E. Rabinovici, *Target space duality in string theory*, Phys. Rept. **244** (1994) 77 [arXiv:hep-th/9401139].

[63] E. Álvarez, L. Álvarez-Gaumé and Y. Lozano, *An introduction to T duality in string theory*, Nucl. Phys. Proc. Suppl. **41** (1995) 1 [arXiv:hep-th/9410237].

[64] C. Montonen and D. I. Olive, *Magnetic Monopoles As Gauge Particles?*, Phys. Lett. **B72** (1977) 117.

[65] J. Maharana and J. H. Schwarz, *Noncompact Symmetries In String Theory*, Nucl. Phys. **B390** (1993) 3 [arXiv:hep-th/9207016].

[66] A. Sen, *Stable non-BPS states in string theory*, JHEP **9806** (1998) 007 [arXiv:hep-th/9803194].

[67] M. Cvetič, *Properties of Black Holes in Toroidally Compactified String Theory*, Nucl. Phys. Proc. Suppl. **B56** (1997) 1. [arXiv:hep-th/9701152].

[68] D. Youm, *Black Holes and Solitons in String Theory*, Phys. Rept. **316** (1999) 1-232. [arXiv:hep-th/9710046].

[69] R. Kallosh and T. Ortín, *Charge Quantization of Axion–Dilaton Black Holes* Phys. Rev. **D48** (1993) 742-747. [arXiv hep-th/9302109].

[70] Z. Perjés, *Solutions of the Coupled Einstein-Maxwell Equations Representing the Fields of Spinning Sources*, Phys. Rev. Lett. **27** (1971) 1668.

[71] W. Israel and G.A. Wilson, *A Class of Stationary Electromagnetic Vacuum Fields*, J. Math. Phys. **13**, (1972) 865.

[72] K.P. Tod, *All Metrics Admitting Supercovariantly Constant Spinors*, Phys. Lett. **121B**, (1981) 241.

[73] R. Kallosh, D. Kastor, T. Ortín and T. Torma, *Supersymmetry and Stationary Solutions in Dilaton-Axion Gravity*, Phys. Rev. **D50** (1994) 6374. [arXiv:hep-th/9406059].

[74] D.V. Gal'tsov, A.A. Garcia, O.V. Kechkin. Mar 1995. *Symmetries of the Stationary Einstein-Maxwell Dilaton-Axion Theory*, J. Math. Phys. **36** (1995) 5023-5041.

[75] D.V. Gal'tsov and O.V. Kechkin, *Ehlers-Harrison-Type Transformations in Dilaton-Axion Gravity*, Phys. Rev. **D50** (1994) 7394. [arXiv:hep-th/9407155].

[76] M. Rogatko, *Stationary Axisymmetric Axion-Dilaton Black Holes: Mass Formulae*, Class. Quant. Grav. **11** (1994) 689-693.

[77] A. Garcia, D.V. Gal'tsov and O.V. Kechkin, *Class of Stationary Axisymmetric Solutions of the Einstein-Maxwell Dilaton-Axion Field Equations*, Phys. Rev. Lett. **74** (1995) 1276-1279.

[78] M. Rogatko, *The Bogomolnyi-Type Bound in Axion - Dilaton Gravity*, Class. Quant. Grav. **12** (1995) 3115-3118.

[79] D.V. Gal'tsov and O.V. Kechkin, *U Duality and Symplectic Formulation of Dilaton-Axion Gravity*, Phys. Rev. **D54** (1996) 1656-1666. [arXiv:hep-th/9507005].

[80] G. Clément and D.V. Gal'tsov, *Stationary BPS Solutions to Dilaton-Axion Gravity*, Phys. Rev. **D54** (1996) 6136. [arXiv:hep-th/9607043].

[81] I. Bakas, *Solitons of Axion - Dilaton Gravity*, Phys. Rev. **D54** (1996) 6424-6434. [arXiv:hep-th/9605043].

[82] D.V. Gal'tsov and P.S. Letelier, *Ehlers-Harrison Transformations and Black Holes in Dilaton-Axion Gravity with Multiple Vector Fields*, Phys. Rev. **D55** (1997) 3580. [arXiv:gr-qc/9612007].

[83] D.V. Gal'tsov and S.A. Sharakin, *Matrix Ernst Potentials for EMDA with Multiple Vector Fields*, Phys. Lett. **B399** (1997) 250-257. [arXiv:hep-th/9702039].

[84] M. Rogatko, *Extrema of Mass, First Law of Black Hole Mechanics and Staticity Theorem in Einstein-Maxwell Axion Dilaton Gravity*, Phys. Rev. **D58** (1998) 044011. [arXiv:hep-th/9807012].

[85] S. Ferrara, R. Kallosh and A. Strominger, $N = 2$ *Extremal Black Holes*, Phys. Rev. **D52** (1995) 5412. [arXiv:hep-th/9508072].

[86] K.P. Tod, *More on Supercovariantly Constant Spinors*, Class. Quantum Grav. **12** (1995) 1801-1820.

[87] E. Bergshoeff, R. Kallosh and T. Ortín, *Stationary Axion/Dilaton Solutions and Supersymmetry*, Nucl. Phys. **B478** (1996) 156-180. [arXiv:hep-th/9605059].

[88] M. Bertolini, P. Fré and M. Trigiante, *The Generating Solution of Regular* $N = 8$ *BPS Black Holes*, Clas. Quant. Grav. **16** (1999) 2987-3004. [arXiv:hep-th/9905143].

[89] K. Behrndt, D. Lüst and W.A. Sabra, *Stationary Solutions of* $N = 2$ *Supergravity*, Nucl. Phys. **B510** (1998) 264. [arXiv:hep-th/9705169].

[90] R. D'Auria and P. Fré, *BPS black holes in supergravity: Duality groups, p-branes, central charges and the entropy*, arXiv:hep-th/9812160.

[91] D. Kastor and K.Z. Win, *Non-Extreme Calabi-Yau black Holes*, Phys. Lett. **B411** (1997) 33. [arXiv:hep-th/9705090].

[92] A.G. Agnese, M. La Camera, *General Spherically Symmetric Solutions in Charged Dilaton Gravity*, Phys. Rev. **D49** (1994) 2126-2128.

[93] E. Álvarez, P. Meessen and T. Ortín, *Transformation of Black Hole Hair Under Duality and Supersymmetry*, Nucl. Phys. **B508** (1997) 181-218. [arXiv:hep-th/9705094].

[94] P. Meessen and T. Ortín, *An $Sl(2,Z)$ Multiplet of Nine-Dimensional Type II Supergravity Theories*, Nucl. Phys. **B541** (1999) 195-245. [arXiv:hep-th/9806120].

[95] J.H. Schwarz, *Dilaton-Axion Symmetry*, arXiv:hep-th/9209125.

[96] T. Ortín, *Electric-Magnetic Duality and Supersymmetry in Stringy Black Holes*, Phys. Rev. **D47** (1993) 3136-3143. [arXiv:hep-th/9208078].

[97] G.W. Gibbons, *Antigravitating Black Hole Solitons with Scalar Hair in $N=4$ Supergravity*, Nucl. Phys. **B207**, (1982) 337.

[98] D. Garfinkle, G. Horowitz and A. Strominger, *Charged Black Holes in String Theory*, Phys. Rev. **D43** (1991) 3140, *Erratum ibid.* **D45** (1992) 3888.

[99] A. Shapere, S. Trivedi and F. Wilczek, *Dual Dilaton Dyons*, Mod. Phys. Lett. **A6** (1991) 2677-2686.

[100] P.K. Townsend, *Black Holes*, Lecture notes, arXiv:gr-qc/9707012.

[101] R. Wald, *General Relativity*, The University of Chicago Press, Chicago, 1984.

[102] G.W. Gibbons and S.W. Hawking, *Action Integrals and Partition Functions in Quantum Gravity*, Phys. Rev. **D15** (1977) 2752.

[103] S. Ferrara, C.A. Savoy and B. Zumino, *General Massive Multiplets in Extended Supersymmetry*, Phys. Lett. **100B** (1981) 393.

[104] C.M. Hull, *Gravitational Duality, Branes and Charges*, Nucl. Phys. **B509** (1998) 216-251. [arXiv:hep-th/9705162].

[105] C.M. Hull, *U-Duality and BPS Spectrum of Super Yang-Mills Theory and M-Theory*, JHEP **9807** (1998) 018. [arXiv:hep-th/9712075].

[106] M. Blau and M. O'Loughlin, *Aspects of U-Duality in Matrix Theory*, Nucl. Phys. **B525** (1998) 182-214. [arXiv:hep-th/9712047].

[107] N.A. Obers, B. Pioline and E. Rabinovici, *M-Theory and U-duality on T^d with Gauge Backgrounds*, Nucl. Phys. **B525** (1998) 163-181. [arXiv:hep-th/9712084].

[108] R.D. Sorkin, *Kaluza-Klein Monopole*, Phys. Rev. Lett. **51** (1983) 87.

[109] D.J. Gross and M.J. Perry, *Magnetic Monopoles in Kaluza-Klein Theories*, Nucl. Phys. **B226** (1983) 29.

[110] E. Bergshoeff, H.-J. Boonstra and T. Ortín, *S Duality and Dyonic p-Brane Solutions in Type II String Theory*, Phys. Rev. **D53** 7206-7212. [arXiv:hep-th/9508091].

[111] J.M. Izquierdo, N.D. Lambert, G. Papadopoulos and P.K. Townsend, *Dyonic Membranes*, Nucl. Phys. **B460** (1996) 560-578. [arXiv:hep-th/9508177].

[112] N. Alonso-Alberca, P. Meessen and T. Ortín, *An SL(3,Z) multiplet of 8-dimensional type II supergravity theories and the gauged supergravity inside*, Nucl. Phys. **B602** (2001) 329 [arXiv:hep-th/0012032].

[113] P.K. Townsend, *M Theory from its Superalgebra*, Lectures delivered at Cargèse 1997. arXiv:hep-th/9712004.

[114] E. Bergshoeff, B. Janssen and T. Ortín, *Kaluza-Klein Monopoles and Gauged Sigma-Models*, Phys. Lett. **B410** (1997) 131-141. [arXiv:hep-th/9706117].

[115] E. Bergshoeff and J.P. van der Schaar, *On M-9-branes*, Class. Quant. Grav. **16**(1999), 23. [arXiv:hep-th/9806069].

[116] E. Eyras and Y. Lozano, *Exotic Branes and Nonperturbative Seven-Branes*, Nucl. Phys. **B573** (2000) 735-767. [arXiv:hep-th/9908094].

[117] E. Bergshoeff, E. Eyras, R. Halbersma, J.P. van der Schaar, C.M. Hull and Y. Lozano, *Space-Time Filling Branes and Strings with Sixteen Supercharges*, Nucl. Phys. **B564** (2000) 29-59. [arXiv:hep-th/9812224].

[118] B.R. Greene, A. Shapere, C. Vafa and S.-T. Yau, *Stringy Cosmic Strings and Noncompact Calabi-Yau Manifolds*, Nucl. Phys. **B337** (1990) 1.

[119] G.W. Gibbons, M.B. Green and M.J. Perry, *Instantons and Seven-Branes in Type IIB Superstring Theory*, Phys. Lett. **B370** (1996) 37-44. [arXiv:hep-th/9511080].

[120] S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press Inc., London 1978.

F. W. Warner, *Foundations of Differentiable Manifolds and Lie Groups*, Springer-Verlag, New York 1983.

[121] E. Witten, *Instability Of The Kaluza-Klein Vacuum*, Nucl. Phys. **B195** (1982) 481.

[122] M. J. Duff, B. E. Nilsson and C. N. Pope, *The Criterion For Vacuum Stability In Kaluza-Klein Supergravity*, Phys. Lett. **B139** (1984) 154.

[123] G. T. Horowitz and A. R. Steif, *Space-Time Singularities In String Theory*, Phys. Rev. Lett. **64** (1990) 260.

[124] D. Amati and C. Klimcik, *Nonperturbative Computation Of The Weyl Anomaly For A Class Of Nontrivial Backgrounds*, Phys. Lett. **B219** (1989) 443;

[125] R. Gueven, *Plane Waves In Effective Field Theories Of Superstrings*, Phys. Lett. **B191** (1987) 275;

H. J. de Vega and N. Sanchez, *Quantum String Propagation Through Gravitational Shock Waves*, Phys. Lett. **B244** (1990) 215;

G. T. Horowitz and A. R. Steif, *Space-Time Singularities In String Theory*, Phys. Rev. Lett. **64** (1990) 260.
Strings In Strong Gravitational Fields, Phys. Rev. **D42** (1990) 1950;

E. A. Bergshoeff, R. Kallosh and T. Ortín, *Supersymmetric string waves*, Phys. Rev. **D47** (1993) 5444 [arXiv:hep-th/9212030];

C. R. Nappi and E. Witten, *A WZW model based on a nonsemisimple group*, Phys. Rev. Lett. **71** (1993) 3751 [arXiv:hep-th/9310112];

O. Jofre and C. Núñez, *Strings In Plane Wave Backgrounds Revisited*, Phys. Rev. **D50** (1994) 5232 [arXiv:hep-th/9311187];

A. A. Kehagias and P. Meessen, *Exact string background from a WZW model based on the Heisenberg group*, Phys. Lett. **B331** (1994) 77 [arXiv:hep-th/9403041];

K. Sfetsos and A. A. Tseytlin, *Four-dimensional plane wave string solutions with coset CFT description*, Nucl. Phys. **B427** (1994) 245 [arXiv:hep-th/9404063];

E. Kiritsis, C. Kounnas and D. Lust, *Superstring gravitational wave backgrounds with space-time supersymmetry*, Phys. Lett. **B331** (1994) 321 [arXiv:hep-th/9404114];

A. A. Tseytlin, *Exact solutions of closed string theory*, Class. Quant. Grav. **12** (1995) 2365 [arXiv:hep-th/9505052].

[126] M.B. Green, J.H. Schwarz, E. Witten, *Superstring Theory*, Cambridge Monographs in Mathematical Physics, Cambridge University Press, 1987; p. 172 ff.

[127] R. Rohm, *Spontaneous Supersymmetry Breaking In Supersymmetric String Theories*, Nucl. Phys. **B237** (1984) 553.

[128] J. Polchinski, *Evaluation Of The One Loop String Path Integral*, Commun. Math. Phys. **104** (1986) 37.

[129] C. G. Callan, C. Lovelace, C. R. Nappi and S. A. Yost, *String Loop Corrections To Beta Functions*, Nucl. Phys. **B288** (1987) 525.

[130] E. J. Martinec, *Nonrenormalization Theorems And Fermionic String Finiteness*, Phys. Lett. **B171** (1986) 189.

[131] M. Dine and N. Seiberg, *Nonrenormalization Theorems In Superstring Theory*, Phys. Rev. Lett. **57** (1986) 2625.

[132] C. G. Callan, C. Lovelace, C. R. Nappi and S. A. Yost, *Loop Corrections To Conformal Invariance For Type 1 Superstrings*, Phys. Lett. **B206** (1988) 41.

[133] C. G. Callan, C. Lovelace, C. R. Nappi and S. A. Yost, *Loop Corrections To Superstring Equations Of Motion*, Nucl. Phys. **B308** (1988) 221.

[134] J. Kowalski-Glikman, *Vacuum States In Supersymmetric Kaluza-Klein Theory*, Phys. Lett. **B134** (1984) 194.

[135] J. Figueroa-O'Farrill and G. Papadopoulos, *Homogeneous fluxes, branes and a maximally supersymmetric solution of M-theory*, JHEP **0108** (2001) 036 [arXiv:hep-th/0105308].

[136] M. Blau, J. Figueroa-O'Farrill, C. Hull and G. Papadopoulos, *A new maximally supersymmetric background of IIB superstring theory*, JHEP **0201** (2002) 047 [arXiv:hep-th/0110242].

[137] R.R. Metsaev, *Type IIB Green-Schwarz superstring in plane wave Ramond-Ramond background*, Nucl. Phys. **B625** (2002) 70 [arXiv:hep-th/0112044];

R.R. Metsaev and A.A. Tseytlin, *Exactly solvable model of superstring in plane wave Ramond-Ramond background*, Phys. Rev. **D65** (2002) 126004 [arXiv:hep-th/0202109];

J.G. Russo and A.A. Tseytlin, *On solvable models of type IIB superstring in NS-NS and R-R plane wave backgrounds*, JHEP **0204** (2002) 021 [arXiv:hep-th/0202179].

[138] M. Billo and I. Pesando, *Boundary states for GS superstrings in an Hpp wave background*, Phys. Lett. **B536** (2002) 121 [arXiv:hep-th/0203028];

A. Dabholkar and S. Parvizi, *Dp branes in pp-wave background*, Nucl. Phys. **B641** (2002) 223 [arXiv:hep-th/0203231].

A. Kumar, R. R. Nayak and Sanjay, *D-brane solutions in pp-wave background*, Phys. Lett. **B541** (2002) 183 [arXiv:hep-th/0204025].

P. Bain, P. Meessen and M. Zamaklar, *Supergravity solutions for D-branes in Hpp-wave backgrounds*, arXiv:hep-th/0205106.

[139] K. Skenderis and M. Taylor, *Branes in AdS and pp-wave spacetimes*, JHEP **0206** (2002) 025 [arXiv:hep-th/0204054].

[140] R. Penrose, *Any Spacetime has a Plane Wave as a Limit*, in *Differential Geometry and Relativity*, Reidel, Dordrecht (1976) 271-275.

[141] R. Gueven, *Plane wave limits and T-duality*, Phys. Lett. **B482** (2000) 255 [arXiv:hep-th/0005061].

[142] M. Blau, J. Figueroa-O'Farrill, C. Hull and G. Papadopoulos, *Penrose limits and maximal supersymmetry*, Class. Quant. Grav. **19** (2002) L87 [arXiv:hep-th/0201081].

[143] M. Blau, J. Figueroa-O'Farrill and G. Papadopoulos, *Penrose limits, supergravity and brane dynamics*, Class. Quant. Grav. **19** (2002) 4753 [arXiv:hep-th/0202111].

[144] D. Berenstein, J.M. Maldacena and H. Nastase, *Strings in flat space and pp waves from N = 4 super Yang Mills*, JHEP **0204** (2002) 013 [arXiv:hep-th/0202021];

N. Itzhaki, I.R. Klebanov and S. Mukhi, *PP wave limit and enhanced supersymmetry in gauge theories*, JHEP **0203** (2002) 048 [arXiv:hep-th/0202153];

J. Gomis and H. Ooguri, Nucl. Phys. **B635** (2002) 106 [arXiv:hep-th/0202157];

U. Gursoy, C. Núñez and M. Schvellinger, *RG flows from Spin(7), CY 4-fold and HK manifolds to AdS, Penrose limits and pp waves*, JHEP **0206** (2002) 015 [arXiv:hep-th/0203124];

S. R. Das, C. Gómez and S. J. Rey, *Penrose limit, spontaneous symmetry breaking and holography in pp-wave background*, Phys. Rev. **D66** (2002) 046002 [arXiv:hep-th/0203164];

D. Brecher, C. V. Johnson, K. J. Lovis and R. C. Myers, *Penrose limits, deformed pp-waves and the string duals of N = 1 large N gauge theory* JHEP **0210** (2002) 008 [arXiv:hep-th/0206045].

[145] M.J. Duff, H. Lu and C. N. Pope, *AdS(5) x S(5) untwisted*, Nucl. Phys. **B532** (1998) 181 [arXiv:hep-th/9803061].

[146] J. Michelson, J. Michelson, *(Twisted) toroidal compactification of pp-waves*, Phys. Rev. **D66** (2002) 066002 [arXiv:hep-th/0203140].

[147] J. Gheerardyn and P. Meessen, *Supersymmetry of massive D = 9 supergravity*, Phys. Lett. **B525** (2002) 322 [arXiv:hep-th/0111130].

[148] E. Bergshoeff, R. Kallosh and T. Ortín, *Duality versus supersymmetry and compactification*, Phys. Rev. **D51** (1995) 3009 [arXiv:hep-th/9410230].

[149] I. Bakas, *Space-time interpretation of S duality and supersymmetry violations of T duality*, Phys. Lett. **B343** (1995) 103 [arXiv:hep-th/9410104].

[150] M. J. Duff, H. Lu and C. N. Pope, *Supersymmetry without supersymmetry*, Phys. Lett. **B409** (1997) 136 [arXiv:hep-th/9704186].

[151] J. P. Gauntlett, J. B. Gutowski, C. M. Hull, S. Pakis and H. S. Reall, *All supersymmetric solutions of minimal supergravity in five dimensions*, arXiv:hep-th/0209114.

[152] P. Meessen, *A small note on PP-wave vacua in 6 and 5 dimensions*, Phys. Rev. **D65** (2002) 087501 [arXiv:hep-th/0111031].

[153] G. W. Gibbons, G. T. Horowitz and P. K. Townsend, *Higher dimensional resolution of dilatonic black hole singularities*, Class. Quant. Grav. **12** (1995) 297 [arXiv:hep-th/9410073].

[154] R. Kallosh, A. Rajaraman and W. K. Wong, *Supersymmetric rotating black holes and attractors*, Phys. Rev. **D55** (1997) 3246 [arXiv:hep-th/9611094].

[155] J. P. Gauntlett, R. C. Myers and P. K. Townsend, *Black holes of $D = 5$ supergravity*, Class. Quant. Grav. **16** (1999) 1 [arXiv:hep-th/9810204].

[156] J. Kowalski-Glikman, *Positive Energy Theorem And Vacuum States For The Einstein-Maxwell System*, Phys. Lett. **B150** (1985) 125.

[157] I. Robinson, *A Solution of the Maxwell-Einstein Equations*, Bull. Acad. Polon. Sci. **7** (1959), 351;

B. Bertotti, *Uniform Electromagnetic Field in the Theory of GR*, Phys. Rev. **116** (1959) 1331.

[158] H. J. Boonstra, B. Peeters and K. Skenderis, *Brane intersections, anti-de Sitter spacetimes and dual superconformal theories*, Nucl. Phys. **B533** (1998) 127 [arXiv:hep-th/9803231].

[159] G. W. Gibbons and C. A. Herdeiro, *Supersymmetric rotating black holes and causality violation*, Class. Quant. Grav. **16** (1999) 3619 [arXiv:hep-th/9906098].

[160] M. Cahen and N. Wallach, *Lorentzian Symmetric Spaces*, Bull. Am. Math. Soc. **76** (1970) 585-591.

[161] G. Papadopoulos, *Rotating rotated branes*, JHEP **9904** (1999) 014 [arXiv:hep-th/9902166];

M. Cvetic, H. Lu and C. N. Pope, *M-theory pp-waves, Penrose limits and supernumerary supersymmetries*, Nucl. Phys. **B644** (2002) 65

[arXiv:hep-th/0203229];
Penrose limits, pp-waves and deformed M2-branes, arXiv:hep-th/0203082;

J. P. Gauntlett and C. M. Hull, *pp-waves in 11-dimensions with extra supersymmetry*, JHEP **0206** (2002) 013 [arXiv:hep-th/0203255];

I. Bena and R. Roiban, *Supergravity pp-wave solutions with 28 and 24 supercharges*, arXiv:hep-th/0206195.

[162] H. Nishino and E. Sezgin, *The Complete N=2, D = 6 Supergravity With Matter And Yang-Mills Couplings*, Nucl. Phys. **B278** (1986) 353.

[163] E. Cremmer, *Supergravities In 5 Dimensions*, in *C80-06-22.1.1* LPTENS 80/17 *Invited paper at the Nuffield Gravity Workshop, Cambridge, Eng., Jun 22 - Jul 12, 1980.*

[164] A. H. Chamseddine and H. Nicolai, *Coupling The SO(2) Supergravity Through Dimensional Reduction*, Phys. Lett. **B96** (1980) 89.

[165] S.D. Majumdar, *A Class of Exact Solutions of Einstein's Field Equations*, Phys. Rev. **72** (1947) 390-398;

A. Papapetrou, *A Static Solution of the Equations of the Gravitational Field for an Arbitrary Charge-Distribution*, Proc. Roy. Irish. Acad. **A51** (1947) 191.

[166] G. W. Gibbons, *Aspects Of Supergravity Theories*, Print-85-0061 (CAMBRIDGE) *Three lectures given at GIFT Seminar on Theoretical Physics, San Feliu de Guixols, Spain, Jun 4-11, 1984.*

[167] R. Kallosh, *Supersymmetric black holes*, Phys. Lett. **B282** (1992) 80 [arXiv:hep-th/9201029].

[168] A. H. Chamseddine, S. Ferrara, G. W. Gibbons and R. Kallosh, *Enhancement of supersymmetry near 5d black hole horizon*, Phys. Rev. **D55** (1997) 3647 [arXiv:hep-th/9610155].

[169] P. K. Townsend, *Killing spinors, supersymmetries and rotating intersecting branes*, arXiv:hep-th/9901102.

[170] A. Fujii, R. Kemmoku and S. Mizoguchi, *D = 5 simple supergravity on AdS(3) x S(2) and N = 4 superconformal field theory*, Nucl. Phys. **B574** (2000) 691 [arXiv:hep-th/9811147].

[171] M. Cvetic and F. Larsen, *Near horizon geometry of rotating black holes in five dimensions*, Nucl. Phys. **B531** (1998) 239 [arXiv:hep-th/9805097].

[172] J.P. Gauntlett, R.C. Myers and P.K. Townsend, *Supersymmetry of rotating branes*, Phys. Rev. **D59** (1999) 025001 [arXiv:hep-th/9809065].

[173] M. Hatsuda, K. Kamimura and M. Sakaguchi, *Super-PP-wave algebra from super-AdS x S algebras in eleven-dimensions*, Nucl. Phys. **B637** (2002) 168 [arXiv:hep-th/0204002].

[174] J. M. Figueroa-O'Farrill, *On the supersymmetries of anti de Sitter vacua*, Class. Quant. Grav. **16** (1999) 2043 [arXiv:hep-th/9902066].

[175] L. Castellani, R. D'Auria and P. Fré, *Supergravity And Superstrings: A Geometric Perspective. Vol. 1: Mathematical Foundations*, World Scientific, Singapore (1991).

[176] R. Coquereaux and A. Jadczyk, *Riemannian Geometry, Fiber Bundles, Kaluza-Klein Theories And All That*, World Sci. Lect. Notes Phys. **16** (1988) 1.

[177] A. Lichnerowicz, *Spineurs harmoniques*, C. R. Acad. Sci. Paris **257** (1963) 7-9.

[178] Y. Kosmann, *Dérivées de Lie des spineurs*, Annali di Mat. Pura Appl. **(IV) 91** (1972) 317-395.

[179] Y. Kosmann, *Dérivées de Lie des spineurs*, C. R. Acad. Sci. Paris, Sér. A-B, **262** (1966) A289-92.

[180] T. Ortín, *A note on Lie-Lorentz derivatives*, Class. Quant. Grav. **19** (2002) L143 [arXiv:hep-th/0206159].

[181] M. Godina and P. Matteucci *Reductive G-structures and Lie derivatives* [arXiv:math.DG/0201235].

[182] H. Lu, C. N. Pope and J. Rahmfeld, *A construction of Killing spinors on S^n*, J. Math. Phys. **40** (1999) 4518 [arXiv:hep-th/9805151].

[183] P. G. Freund and M. A. Rubin, *Dynamics Of Dimensional Reduction*, Phys. Lett. **B97** (1980) 233.

[184] L. Castellani, L. J. Romans and N. P. Warner, *A Classification Of Compactifying Solutions For D = 11 Supergravity*, Nucl. Phys. **B241** (1984) 429.

[185] M. J. Duff, H. Lu, C. N. Pope and E. Sezgin, *Supermembranes with fewer supersymmetries*, Phys. Lett. **B371** (1996) 206 [arXiv:hep-th/9511162].

[186] L. Castellani, A. Ceresole, R. D'Auria, S. Ferrara, P. Fré and M. Trigiante, *G/H M-branes and AdS(p+2) geometries*, Nucl. Phys. **B527** (1998) 142 [arXiv:hep-th/9803039].

[187] C. A. Herdeiro, *Special properties of five dimensional BPS rotating black holes*, Nucl. Phys. **B582** (2000) 363 [arXiv:hep-th/0003063].

[188] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, *Gauge theory correlators from non-critical string theory*, Phys. Lett. **B428** (1998) 105 [arXiv:hep-th/9802109].

[189] E. Witten, *Anti-de Sitter space and holography*, Adv. Theor. Math. Phys. **2** (1998) 253 [arXiv:hep-th/9802150].

[190] G. 't Hooft, *A Planar Diagram Theory For Strong Interactions*, Nucl. Phys. **B72** (1974) 461.

[191] G. 't Hooft, *Dimensional Reduction In Quantum Gravity*, arXiv:gr-qc/9310026.

[192] R. Bousso, *The holographic principle*, Rev. Mod. Phys. **74** (2002) 825 [arXiv:hep-th/0203101].

[193] J. D. Bekenstein, *Black Holes And Entropy*, Phys. Rev. **D7** (1973) 2333;

J. D. Bekenstein, *Generalized Second Law Of Thermodynamics In Black Hole Physics*, Phys. Rev. **D9** (1974) 3292.

[194] S. W. Hawking, *Particle Creation By Black Holes*, Commun. Math. Phys. **43** (1975) 199.

[195] J. Maldacena and C. Núñez, *Supergravity description of field theories on curved manifolds and a no-go theorem*, Int. J. Mod. Phys. **A16** (2001) 82, [arXiv:hep-th/0007018].

[196] M. Bershadsky, C. Vafa and V. Sadov, *D-Branes and Topological Field Theories*, Nucl. Phys. **B463** (1996) 420, [arXiv:hep-th/9511222].

[197] E. Witten, *Topological Quantum Field Theory*, Commun. Math. Phys. **117** (1988) 353.

[198] J. M. Labastida and M. Marino, *Twisted baryon number in N=2 supersymmetric QCD*, Phys. Lett. **B400** (1997) 323 [arXiv:hep-th/9702054].

[199] M. Cvetič, H. Lü and C. N. Pope, *Consistent Kaluza-Klein Sphere Reductions*, Phys. Rev. **D62** (2000) 064028, [arXiv:hep-th/0003286].

[200] M. Bertolini, P. Di Vecchia, M. Frau, A. Lerda, R. Marotta and I. Pesando, *Fractional D-branes and their gauge duals*, JHEP **0102** (2001) 014, [arXiv:hep-th/0011077].

[201] M. Bertolini, P. Di Vecchia and R. Marotta, *N=2 four-dimensional gauge theories from fractional branes*, arXiv:hep-th/0112195.

[202] A. W. Peet and J. Polchinski, *UV/IR relations in AdS dynamics*, Phys. Rev. **D59** (1999) 065011 [arXiv:hep-th/9809022].

[203] M. Frau, A. Liccardo and R. Musto, *The geometry of fractional branes*, Nucl. Phys. **B602** (2001) 39, [arXiv:hep-th/0012035].

[204] M. R. Douglas and G. W. Moore, *D-branes, Quivers, and ALE Instantons*, hep-th/9603167.

[205] C. V. Johnson and R. C. Myers, *Aspects of type IIB theory on ALE spaces*, Phys. Rev. **D55** (1997) 6382 [arXiv:hep-th/9610140].

[206] D. Diaconescu, M. R. Douglas and J. Gomis, *Fractional branes and wrapped branes*, JHEP **9802** (1998) 013, [arXiv:hep-th/9712230].

[207] I. R. Klebanov and N. A. Nekrasov, *Gravity duals of fractional branes and logarithmic RG flow*, Nucl. Phys. **B574** (2000) 263, [arXiv:hep-th/9911096].

[208] C. P. Herzog, I. R. Klebanov and P. Ouyang, *Remarks on the warped deformed conifold*, hep-th/0108101.

[209] A. Fayyazuddin and D. J. Smith, *Localized intersections of M5-branes and four-dimensional superconformal field theories*, JHEP **9904** (1999) 030, [arXiv:hep-th/9902210].

[210] A. Fayyazuddin and D. J. Smith, *Warped AdS near-horizon geometry of completely localized intersections of M5-branes*, JHEP **0010** (2000) 023, [arXiv:hep-th/0006060].

[211] B. Brinne, A. Fayyazuddin, S. Mukhopadhyay and D. J. Smith, *Supergravity M5-branes wrapped on Riemann surfaces and their QFT duals*, JHEP **0012** (2000) 013, [arXiv:hep-th/0009047].

[212] J. Polchinski, *$N = 2$ gauge-gravity duals*, Int. J. Mod. Phys. **A16** (2001) 707, [arXiv:hep-th/0011193].

[213] M. Billo, L. Gallot and A. Liccardo, *Classical geometry and gauge duals for fractional branes on ALE orbifolds*, Nucl. Phys. **B614** (2001) 254 [arXiv:hep-th/0105258].

[214] M. Grana and J. Polchinski, *Gauge/gravity duals with holomorphic dilaton*, Phys. Rev. **D65** (2002) 126005 [arXiv:hep-th/0106014].

[215] M. Bertolini, P. Di Vecchia, M. Frau, A. Lerda and R. Marotta, *N=2 gauge theories on systems of fractional D3/D7 branes*, Nucl. Phys. **B621** (2002) 157 [arXiv:hep-th/0107057].

[216] J. P. Gauntlett, N. Kim, D. Martelli and D. Waldram, *Wrapped five-branes and $N = 2$ super Yang–Mills theory*, Phys. Rev. **D64** (2001) 106008, [arXiv:hep-th/0106117].

[217] F. Bigazzi, A. L. Cotrone and A. Zaffaroni, *N=2 gauge theories from wrapped five-branes*, Phys. Lett. **B519** (2001) 269, [arXiv:hep-th/0106160].

[218] J. M. Maldacena and C. Núñez, *Towards the large N limit of pure N = 1 super Yang Mills*, Phys. Rev. Lett. **86** (2001) 588, [arXiv:hep-th/0008001].

[219] C. Núñez, I. Y. Park, M. Schvellinger and T. A. Tran, *Supergravity duals of gauge theories from F(4) gauged supergravity in six dimensions*, JHEP **0104** (2001) 025, [arXiv:hep-th/0103080].

[220] J. Gomis, *D-branes, holonomy and M-theory*, Nucl. Phys. **B606** (2001) 3, [arXiv:hep-th/0103115].

[221] J. D. Edelstein and C. Núñez, *D6 branes and M-theory geometrical transitions from gauged supergravity*, JHEP **0104** (2001) 028, [arXiv:hep-th/0103167].

[222] J. Gomis and T. Mateos, *D6 branes wrapping Kaehler four-cycles*, Phys. Lett. **B524** (2002) 170 [arXiv:hep-th/0108080].

[223] J. Gomis and J. G. Russo, *D=2+1 N=2 Yang-Mills theory from wrapped branes*, JHEP **0110** (2001) 028 [arXiv:hep-th/0109177].

[224] J. P. Gauntlett, N. w. Kim, D. Martelli and D. Waldram, *Fivebranes wrapped on SLAG three-cycles and related geometry*, JHEP **0111** (2001) 018 [arXiv:hep-th/0110034].

[225] C. V. Johnson, A. W. Peet and J. Polchinski, *Gauge theory and the excision of repulson singularities*, Phys. Rev. **D61** (2000) 086001, [arXiv:hep-th/9911161].

[226] M. Wijnholt and S. Zhukov, *Inside an enhancon: Monopoles and dual Yang-Mills theory*, Nucl. Phys. **B639** (2002) 343 [arXiv:hep-th/0110109].

[227] M. Petrini, R. Russo and A. Zaffaroni, *N=2 gauge theories and systems with fractional branes*, Nucl. Phys. **B608** (2001) 145, [arXiv:hep-th/0104026].

[228] C. V. Johnson, R. C. Myers, A. W. Peet and S. F. Ross, *The enhançon and the consistency of excision*, Phys. Rev. **D64** (2001) 106001, [arXiv:hep-th/0105077].

[229] P. Merlatti, *The enhancon mechanism for fractional branes*, Nucl. Phys. **B624** (2002) 200 [arXiv:hep-th/0108016].

[230] L. Álvarez-Gaumé and D. Z. Freedman, *Geometrical Structure And Ultraviolet Finiteness In The Supersymmetric Sigma Model*, Commun. Math. Phys. **80** (1981) 443.

[231] N. Seiberg and E. Witten, *Gauge dynamics and compactification to three dimensions*, hep-th/9607163.

[232] M. Cvetič, M. J. Duff, P. Hoxha, J. T. Liu, H. Lü, J. X. Lu, R. Martínez-Acosta, C. N. Pope, H. Sati and T. A. Tran, *Embedding of AdS Black Holes in Ten and Eleven Dimensions*, Nucl. Phys. **B558** (1999) 96, [arXiv:hep-th/9903214].

[233] J. P. Gauntlett, N. Kim, S. Pakis and D. Waldram, *Membranes wrapped on holomorphic curves*, Phys. Rev. **D65** (2002) 026003 [arXiv:hep-th/0105250].

[234] J. Babington and N. Evans, *Field theory operator encoding in $N = 2$ geometries*, JHEP **0201** (2002) 016 [arXiv:hep-th/0111082].

[235] T. Eguchi and A. J. Hanson, *Selfdual Solutions To Euclidean Gravity*, Annals Phys. **120** (1979) 82.

[236] C. V. Johnson, *D-brane primer*, hep-th/0007170.

[237] S. W. Hawking, *Gravitational Instantons*, Phys. Lett. **A60** (1977) 81.

[238] P. S. Aspinwall, *Enhanced gauge symmetries and K3 surfaces*, Phys. Lett. **B357** (1995) 329, [arXiv:hep-th/9507012].

[239] M. R. Douglas, *Enhanced gauge symmetry in M(atrix) theory*, JHEP **9707** (1997) 004, [arXiv:hep-th/9612126].

[240] M. Cvetič, H. Lü and C. N. Pope, *Brane resolution through transgression*, Nucl. Phys. **B600** (2001) 103, [arXiv:hep-th/0011023].

[241] C. Schmidhuber, *D-brane actions*, Nucl. Phys. **B467** (1996) 146, [arXiv:hep-th/9601003].

[242] M. Bertolini, P. Di Vecchia, M. Frau, A. Lerda, R. Marotta and R. Russo, *Is a classical description of stable non-BPS D-branes possible?*, Nucl. Phys. **B590** (2000) 471, [arXiv:hep-th/0007097].

[243] P. Merlatti and G. Sabella, *World volume action for fractional branes*, Nucl. Phys. **B602** (2001) 453, [arXiv:hep-th/0012193].

[244] E. S. Fradkin and A. A. Tseytlin, *Quantum Properties Of Higher Dimensional And Dimensionally Reduced Supersymmetric Theories*, Nucl. Phys. **B227** (1983) 252.

[245] P. Di Vecchia, *Duality in N=2,4 supersymmetric gauge theories*, hep-th/9803026.