Black Holes in Supergravity with Applications to String Theory

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Contents

Chapter 1

Introduction

The aim of this thesis is to study black holes in String Theory (ST) through their classical description as Supergravity solutions. ST [1, 2] is a framework that attempts to offer a unified quantum description of all the known fundamental interactions and, in particular, of gravity. It would solve, if correct, the long-standing problem of Quantum Gravity. In the present day there is no experimental evidence of ST, and there is no hope in the ST community that such an evidence will soon be found. Despite the lack of experimental results, ST has passed several self-consistency checks that are expected to hold in the *right* unifying theory of nature [3–8], if it exists. As a consequence, an enormous effort has been devoted over the last decades to develop ST, leading to beautiful and important advances and insights in modern Theoretical Physics [6, 9, 10] and Mathematics [11–19]. The ramifications of ST-inspired results are nowadays virtually everywhere in Theoretical Physics, and thus even if the *right* theory of nature was not ST, we can expect that they will have some ingredients in common. Hundreds of thesis, books and reviews have been written over the years dealing with the principal aspects of ST, such as supersymmetry, perturbative ST, conformal field theory, D-branes, ST dualities, aDS/CFT correspondence, M-Theory... and hence we refer the interested reader to the existing literature, for instance [7, 8, 20–22] and references therein.

I will focus instead (see chapters 2 and 3), for reasons that will become apparent later, on a different area, sometimes forgotten in ST applications: the precise mathematical structure of the ST *effective actions*, *i.e.* field theories that describe the dynamics of the massless modes of the ST spectrum and are used to make contact with four-dimensional low-energy physics. The mathematical structure of these effective actions is crucial in order to ensure the consistency of the theory and it is the necessary background that we will need to pursue our goal, namely the study of black holes in ST. Let us see first how to go from the full-fledged ST to four-dimensional Supergravity.

1.1 From String Theory to Supergravity

ST *lives* in ten dimensions, that is, the mathematical object that represents the space-time in ST is a ten-dimensional manifold. Since experimentally, that is, at low energies, we observe four dimensions, some mechanism must be used to reconcile theory and experiment. The standard way to proceed is to assume that the space-time manifold $\mathcal{M}$[1] has the following fibre bundle structure

$$\mathcal{M} \xrightarrow{\pi} \mathcal{M}_4 \,, \tag{1.1}$$

where $\mathcal{M}_4$ is the base space manifold (which represents the space-time that we observe at low energies) and the fibre $\mathcal{M}_6\,(p) = \pi^{-1}(p)$ at each $p \in \mathcal{M}_4$ is a compact manifold, small enough to not be accessible in current high-energy experiments. So to say, $\mathcal{M}_6\,(p)$ is so small that we cannot *see it* with the available technology. Notice that $\mathcal{M}_6\,(p) \cong \mathcal{M}_6\,(q)\ , \forall\ p, q \in \mathcal{M}_4$, so we can denote the typical fibre simply by $\mathcal{M}_6$.

The general mechanism described above can be put in practice in several different ways, on which the precise size limits of $\mathcal{M}_6$ depend [23–26]. Unfortunately, it is currently not known how to compactify ST on a non-trivial compact manifold $\mathcal{M}_6$, except for some particular cases. By non-trivial manifold here we mean a Riemannian manifold with a curved metric. The usual procedure to deal with this situation is to cook up an effective field theory action that encodes the dynamics of the massless modes present in the ST spectrum, which are the ones

[1] For more details about the precise properties of $\mathcal{M}$, see chapter 2.

relevant to low energy physics. The reason is that the first massive states in the spectrum have masses of the order of the Planck mass, which is way out of reach for current particle accelerators.

As a consequence of ST having local space-time supersymmetry, the ST massless states action is a very special one: a Supergravity [27–31]. The key point is that, even though it is not know how to compactify ST, it is known how to compactify a field theory, and in particular a Supergravity. Therefore, although the procedure described above is sometimes called "ST compactification" what is actually compactified is a ten-dimensional Supergravity. We can add to it ST corrections, that is, modifications to the *tree level* result as prescribed by ST, which is not a field theory but a theory of extended objects. The resulting corrected action is, again, a Supergravity but, tipically, it has higher order terms in the Lagrangian and modified couplings.

If we compactify now low energy ST, that is, ten-dimensional Supergravity, on a particular compact six-dimensional manifold $\mathcal{M}_6$[2], we obtain a four-dimensional effective action which will also be a Supergravity, four-dimensional in this case, if some technical requirements are obeyed by the compact manifold $\mathcal{M}_6$. I will come back to this point later, but for the moment I refer the reader to [26] and references therein.

To summarize, we have started with ten-dimensional ST and we have ended up with four-dimensional Supergravity, which is a theory of gravity and matter that can be embedded in ST, and therefore seems to be the perfect starting point to study four-dimensional black holes in ST.

1.2 Supergravity

Supergravity is a locally supersymmetric theory of gravity, that is, a field theory invariant under local supersymmetry transformations. Supersymmetry is a not so new, and yet to be observed, hypothetical symmetry between bosons and fermions [29, 32–34].

If we consider a field theory content with spin less or equal than two, there are seven different types of four-dimensional Supergravity $\mathcal{N} = 1, \cdots, 6, 8$, depending on the amount $\mathcal{N}$ of supersymmetry of the theory. Supersymmetry transformations are generated by a set of spinors $\epsilon_I(p)\,, I = 1, \cdots, \mathcal{N}$, where $\mathcal{N}$ is the *number of supersymmetries* of the given Supergravity. In four dimensions, the minimal spinors $\epsilon_I(p)$ can be taken to be Weyl or Majorana, and therefore we have respectively $2\mathcal{N}$ complex or $4\mathcal{N}$ real associated *charges*. Supersymmetry transformations can be schematically written as

$$\delta_\epsilon \phi_b \sim \bar{\epsilon}(p)\left(\phi_b + \bar{\phi}_f \phi_f\right)\phi_f\,, \qquad \delta_\epsilon \phi_f \sim \partial\epsilon(p) + \left(\phi_b + \bar{\phi}_f\phi_f\right)\epsilon(p)\phi_f\,, \tag{1.2}$$

where ϕ_b denotes the bosonic fields and ϕ_f denotes the fermionic fields.

Since Supergravity is a supersymmetric theory, and supersymmetry relates bosonic a fermionic fields, every Supergravity contains bosons and fermions. Truncating the fermions is always consistent, thanks to the following $\mathbb{Z}_2$ symmetry, present in every Supergravity Lagrangian

$$\phi_b \to \phi_b\,, \qquad \phi_f \to -\phi_f\,. \tag{1.3}$$

The bosonic sector of four-dimensional Supergravity is a particular instance of General Relativity, as formulated by Albert Einstein in 1915 [35]. That is, it is a *metric theory* of gravity coupled to a particular matter content, which includes scalars and vector fields, and where the equation of motion for the metric g is given by

$$\mathbf{R}\left(\nabla\right) = \mathbf{T}\,. \tag{1.4}$$

Here $\mathbf{R}\left(\nabla\right)$ is the Ricci tensor of the Levi-Civita connection ∇ associated to g on the space-time tangent bundle [36], and $\mathbf{T}$ is the *geometrized* energy-momentum tensor corresponding to the matter content of the theory.

General Relativity cannot be, in principle, coupled to fermions, since it is formulated in a way on which only the diffeomorphisms group $\mathrm{Diff}\left(\mathcal{M}\right)$ acts naturally on the matter content. We need to make manifest the local action of the Lorentz group $\mathrm{SO}(1,3)$[3] on the matter content of the theory, since fermions are associated to spinorial representations of $\mathrm{SO}(1,3)$. Therefore, if we want to consider the complete Supergravity action, we have to change the set-up and use a more general formalism, which turns out to be the Cartan-Sciama-Kibble theory [7, 37–39], a generalization of Einstein's General Relativity. Just as the bosonic sector of Supergravity is a particular case

[2]Notice that it is possible to compactify in manifolds with dimension other than six, obtaining as effective actions Supergravities in dimensions other than four.

[3]Rather, the action of its double-cover, the spin group $\mathrm{Spin}(1,3)$, see below.

of General Relativity, the complete Supergravity theory is a particular case of the Cartan-Sciama-Kibble theory, which is an extension of General Relativity that can accommodate fermions.

Before introducing the Cartan-Sciama-Kibble theory it is necessary to modify a bit the *geometric* set-up, in order to geometrically introduce fermions. General Relativity coupled to matter can be described in terms of objects that transform as tensors under space-time diffeomorphisms (such as sections over the tensor products of $T\mathcal{M}$ and $T^*\mathcal{M}$ or connections on principal bundles over $\mathcal{M}$). For instance the metric, which *describes* the gravitational interaction, is a non-degenerate section of $\mathrm{S}^2T^*\mathcal{M}$. The electromagnetic interaction, on the other hand, is described by a connection on a principal $\mathrm{U}(1)$ bundle over $\mathcal{M}$.

However, fermions are described in classical field theory as spinors, that is, representations of the spin group $\mathrm{Spin}(1,3)$, and they do not correspond to any section of the tangent or cotangent bundle. The spin group $\mathrm{Spin}(1,3)$ is the double cover of the Lorentz group $\mathrm{SO}(1,3)$ such that the following short exact sequence holds

$$\mathbb{Z}_2 \to \mathrm{Spin}(1,3) \xrightarrow{\rho} \mathrm{SO}(1,3)\,. \tag{1.5}$$

In order to properly include fermions into the game, we have to consider first the bundle of frames $\mathrm{F}(T\mathcal{M})$ instead of the tangent $T\mathcal{M}$ and the cotangent $T^*\mathcal{M}$ bundles. The frame bundle $\mathrm{F}(T\mathcal{M})$ is the principal vector bundle associated to $T\mathcal{M}$, defined as follows

Definition 1.2.1. Let $\mathcal{M}$ be a differentiable manifold, in our case the space-time minfold. Define a manifold $\mathrm{F}(T\mathcal{M})$ as

$$\mathrm{F}(T\mathcal{M}) = \{(p, \mathrm{e}_1, ..., \mathrm{e}_4) : p \in \mathcal{M}\,, (\mathrm{e}_1, ..., \mathrm{e}_4) \text{ ordered basis of } T_p\mathcal{M}\}\,. \tag{1.6}$$

Define now a projection $\pi_{\mathrm{F}} : \mathrm{F}(T\mathcal{M}) \to \mathcal{M}$ by $\pi(p, \mathrm{e}_1, \cdots, \mathrm{e}_4) = p$ and define an action of $\mathrm{GL}(4,\mathbb{R})$ by $\mathrm{e}'_a = A_a^{\ b}\mathrm{e}_b\,, a = 1, \cdots, 4$, where $A \in \mathrm{GL}(4,\mathbb{R})$. This makes $\mathrm{F}(T\mathcal{M})$ into a principal bundle with fibre $\mathrm{GL}(4,\mathbb{R})$.

Since the space-time manifold $\mathcal{M}$ is equipped with a Lorentzian metric g, we can actually consider the oriented orthonormal frame bundle $\mathrm{F}_{\mathrm{SO}}(T\mathcal{M})$, built only from the bases orthonormal respect to the metric g and positively oriented. Notice that this is possible because we are assuming from the onset that the space-time manifold is oriented (otherwise we could not write actions as integrals over the space-time). The structure group of $\mathrm{F}_{\mathrm{SO}}(T\mathcal{M})$ is then reduced from $\mathrm{GL}(4,\mathbb{R})$ to $\mathrm{SO}(1,3)$.

Definition 1.2.2. Let $\mathcal{M}$ be a differentiable manifold. Define a manifold $\mathrm{F}_{\mathrm{SO}}(T\mathcal{M})$ as

$$\mathrm{F}_{\mathrm{SO}}(T\mathcal{M}) = \{(p, \mathrm{e}_1, ..., \mathrm{e}_4) : p \in \mathcal{M}\,, (\mathrm{e}_1, ..., \mathrm{e}_4)\ T_p\mathcal{M} \text{ ordered basis} \mid \mathrm{g}(\mathrm{e}_a, \mathrm{e}_b) = \eta_{ab}\,, \det_\partial(\mathrm{e}) > 0\}\,. \tag{1.7}$$

Define now a projection $\pi_{\mathrm{SO}} : \mathrm{F}_{\mathrm{SO}}(T\mathcal{M}) \to \mathcal{M}$ by $\pi(p, \mathrm{e}_1, \cdots, \mathrm{e}_4) = p$ and define an action of $\mathrm{SO}(1,3)$ by $\mathrm{e}'_a = O_a^{\ b}\mathrm{e}_b\,, a = 1, \cdots, 4$, where $O \in \mathrm{SO}(1,3)$. This makes $\mathrm{F}_{\mathrm{SO}}(T\mathcal{M})$ into a principal bundle with fibre $\mathrm{SO}(1,3)$.

Now, in order for the space-time manifold $\mathcal{M}$ to admit fermions, a technical requirement must be fulfilled: the bundle $\mathrm{F}_{\mathrm{SO}}(T\mathcal{M})$ must admit an *equivariant lift* with respect to $\mathrm{Spin}(1,3)$. In that case we can construct the spin bundle $\mathrm{P} \to \mathcal{M}$, which is a principal bundle with fibre $\mathrm{Spin}(1,3)$. The precise definition goes as follows

Definition 1.2.3. A pair (P, Ω_P) is a spin structure on the principal bundle $\mathrm{F}_{\mathrm{SO}} \to \mathcal{M}$ when

1. $\mathrm{P} \xrightarrow{\pi_{\mathrm{P}}} \mathcal{M}$ is a principal bundle with fibre $\mathrm{Spin}(1,3)$.

2. $\Omega_{\mathrm{P}} : \mathrm{P} \to \mathrm{F}_{\mathrm{SO}}$ is an equivariant two-fold covering map such that $\pi_{\mathrm{SO}} \circ \Omega_{\mathrm{P}} = \pi_{\mathrm{P}}$ and $\Omega_{\mathrm{P}}(p, O) = \Omega_{\mathrm{P}}(p)\rho(O)\,, \forall\, p \in \mathrm{P}\,, \forall\, O \in \mathrm{Spin}(1.3)$.

The vector bundle $\mathrm{S} \to \mathcal{M}$ associated to P and the spin representation of $\mathrm{Spin}(1,3)$ is then the spin bundle, and spinors are thus section of S. The vector bundle associated to a principal bundle can be defined in general. In our case the construction is given by

Definition 1.2.4. Let $\kappa : \mathrm{Spin}(1,3) \to \mathrm{U}(V)$ a unitary representation of $\mathrm{Spin}(1,3)$ on a complex vector space V. Then $\mathrm{Spin}(1,3)$ acts on the product space $\mathrm{P} \times V$ by the principal bundle action on the first factor and κ on the second. We define

$$\mathrm{S} = \left(\mathrm{P} \times V\right) / \mathrm{Spin}(1,3)\,. \tag{1.8}$$

Since $\mathrm{P}/\mathrm{Spin}(1,3) = \mathcal{M}$, the obvious map $\pi_{\mathrm{S}} : \left(\mathrm{P} \times V\right) / \mathrm{Spin}(1,3) \rightarrow \mathrm{P}/\mathrm{Spin}(1,3)$ gives the projection on the base space $\mathcal{M}$. Since $\mathrm{Spin}(1,3)$ acts freely on P, this projection has fibre V. Therefore S is a vector space with base $\mathcal{M}$ and fibre V.

Sections of S are spinors. If we want fermions to exist in the space-time, we must require $\mathcal{M}$ to admit a spin bundle S, since, as we have seen, spinor fields are sections of such bundle. The obstruction to construct a spin bundle over $\mathcal{M}$ is given by the second Stiefel-Whitney class w_2, which should be zero. For more details, the reader is referred to [40].

Local sections $\mathbf{e}$ of P are called local spin frames or *vierbiens*. In terms of them, the metric is given by

$$\mathbf{g} = \eta\left(\mathbf{e}, \mathbf{e}\right)\,, \tag{1.9}$$

where $\eta = \mathrm{Diag}(+,-,-,-)$ is the Minkowski metric. The principal bundle P can now be equipped with the so-called *spin connection* ω, which makes it possible to construct derivatives covariant under local $\mathrm{Spin}(1,3)$ transformations. Using (1.9) it is possible to rewrite General Relativity in terms of the vierbeins $\mathbf{e}$ instead of the metric. This is important because the choice of a vierbein makes each $(T_p\mathcal{M}\,, p \in \mathcal{M})$ into Minkowski space, where we know how to write actions for spinor fields. In addition, the use of vierbeins makes explicit the invariance of the theory under local $\mathrm{Spin}(1,3)$ transformations. This is precisely what we need to include fermions in the theory, using the so-called *first-order formalism* of the Cartan-Sciama-Kibble theory. It can be summarized as follows

1. The dynamical field associated to gravity is taken to be $\mathbf{e}$ instead of the metric $\mathbf{g}$.
2. The Einstein-Hilbert term is now written in terms of the curvature $R_{a_1 a_2}(\omega)$ of the spin connection ω as follows

   $$S\left[\mathbf{e}, \omega\right] = \frac{1}{2}\int R_{a_1 a_2}(\omega) \wedge \mathbf{e}_{a3} \wedge \mathbf{e}_{a4}\epsilon^{a_1 a_2 a_3 a_4}\,, \tag{1.10}$$

   where $\epsilon^{a_1 a_2 a_3 a_4}$ is the flat Levi-Civita tensor. The action (1.10) is equivalent to the first-order Einstein-Hilbert action for the metric and the affine connection Γ. See [7] for a detailed explanation.
3. The covariant derivatives are constructed using the connection ω which is considered an independent field of the theory.
4. Kinetic terms for the spinors are introduced using the covariant derivative constructed from ω. For instance, if ψ is a Dirac spinor, then a kinetic term of the action would be of the form $\bar{\psi}\not{\mathcal{D}}\psi$, where $\mathcal{D}\psi \sim \partial\psi + \omega\psi$.
5. The equation of motion for the spin connection ω is an algebraic constraint which relates it with the other fields of the theory. In particular, it may have torsion. A metric compatible connection ω can be written as follows

   $$\omega = \omega_{\mathrm{LC}} + \mathrm{K}\,, \tag{1.11}$$

   where ω_{LC} is the Levi-Civita connection and K is the *contorsion* tensor [7]. For Supergravity theories, K depends only on the fermionic fields of the theory, and therefore vanishes in a purely bosonic background.

Therefore, using the Cartan-Sciama-Kibble theory we can extend the Lagrangian density $\mathcal{L}(\mathbf{g}, \phi_b)$ of General Relativity to the Lagrangian density $\mathcal{L}(\mathbf{e}, \omega, \phi_b, \phi_f)$, which includes fermions and instead of the metric $\mathbf{g}$ has $\mathbf{e}$ as the dynamical field associated to gravity. Notice that, although every Lagrangian density of the form $\mathcal{L}(\mathbf{g}, \phi_b)$ can be written as $\mathcal{L}(\mathbf{e}, \omega, \phi_b, \phi_f = 0)$, the converse is not true and, therefore, the Cartan-Sciama-Kibble theory is in general not equivalent to General Relativity.
As it happens in General Relativity, four-dimensional Supergravity assumes that the space-time can be described by a differentiable Pseudo-Riemannian manifold $\mathcal{M}$ modulo isometries of the Pseudo-Riemannian metric, that is, by an equivalence class of Pseudo-Riemannian isometric manifolds $[\mathcal{M}]$. Physicists are used to consider diffeomorphic space-times as equivalent. This can be easily accommodated in the definition of equivalence classes by isometries by equipping the image manifold with the induced metric. That is, if f is the corresponding diffeomorphism, we

would have $f : (\mathcal{M}, \mathrm{g}) \to \left(\mathcal{M}, \tilde{\mathrm{g}} = \left(f^{-1}\right)^* \mathrm{g}\right)$, which are isometric manifolds, and hence equivalent from a physical point of view.

Therefore Supergravity can be constructed explicitly in terms of geometric objects, that is, suitable sections of appropriate fibre bundle constructed over the space-time manifold, and thus it is a *geometric theory*. In fact, as we will see in chapter 3, the structure of the Lagrangian itself can be determined in a geometric way by using appropriate manifolds and sections therein.

Supersymmetry imposes severe constraints on the field content and structure of the Lagrangian, making possible the study and classification of all the possible Supergravities. Initially developed independently from ST, it was realized in the eighties that the three different ten-dimensional Supergravities are the low-energy limit of the five, duality-related, ten-dimensional STs. Not only that: Supergravity includes crucial non-perturbative information about ST, through its BPS spectrum. That is, although Supergravity encodes the dynamics of only the massless states of the ST spectrum, it contains solitonic supersymmetric solutions with physical properties that are protected by supersymmetry from the ST corrections to the Supergravity action. Some of these supersymmetric solutions, like particular instances of supersymmetric black holes, correspond to the long range fields created by bound states of non-perturbative ST objects, like D-branes.

Its crucial relation with ST, together with the fact that Supergravity may be the right effective theory of nature (up to some scale) even if it is not embedded in ST, and the beautiful mathematical formulation that underlies the theory, has made Supergravity an extremely important research topic, which is still extensively studied, with never decreasing interest.

In addition, a remarkable and fascinating discovery that focused a lot of attention, and increased even more the interest in Supergravity, was made in 2007: explicit computations showed that $\mathcal{N} = 8$ Supergravity was perturbatively UV finite up to three loops [10]! (Divergences were expected at one loop). Soon enough the computation was extended, with finite results, to four loops [41] and to three loops for $\mathcal{N} = 4$ pure Supergravity [42]. Several powerful arguments have appeared in favor of $\mathcal{N} = 8$ Supergravity and $\mathcal{N} = 4$ pure Supergravity being perturbatively finite to all loops [43–46][4]. Intense research is being performed to extend the explicit calculations to higher loops and to provide a final proof of the conjectured UV perturbative finitness of $\mathcal{N} = 8$ and pure $\mathcal{N} = 4$ Supergravities.

On a more mathematical side, Supergravity is suffering a quiet and elegant revolution: a new mathematical tool, called Generalized Complex Geometry [18, 19] and its *extension*, Exceptional Generalized Geometry [48–51], allow for a new and more geometric formulation of Supergravity, considering sections on a generalized bundle instead of sections of the tangent bundle (and its tensorial products) of the spacetime manifold to represent the bosonic fields of the theory. This new formulation may be relevant for the role that Supergravity plays in ST, since it allows a covariant and natural action of the ST duality groups on its massless content, through the *extended structure group* that acts on the generalized tangent bundle.

Generalized Complex Geometry considers the vector bundle $\mathbb{TM} = T\mathcal{M} \oplus T^*\mathcal{M}$ over $\mathcal{M}$ instead of the tangent or contangent bundles. Here $\oplus$ stands for the *Whitney sum* of vector bundles over the same base space $\mathcal{M}$. Elements of $\mathbb{TM}$ are of the form $\mathbb{X} = X + \xi$, where $X \in T\mathcal{M}$ and $\xi \in T^*\mathcal{M}$. $\mathbb{TM}$, in contrast to the tangent bundle $T\mathcal{M}$, is naturally equipped with a canonical metric G, defined as follows

$$\mathrm{G}\left(\mathbb{X}, \mathbb{Y}\right) = \frac{1}{2}\left(\xi(Y) + \eta(X)\right) , \qquad \forall\, \mathbb{X} = X + \xi\,, \mathbb{Y} = Y + \eta \in \mathbb{TM}\,. \tag{1.12}$$

G has signature (d, d), where d is the dimension of $\mathcal{M}$. Interestingly enough, at every point $p \in \mathcal{M}$, G is invariant under the action of $\mathrm{SO}(d, d)$, the T-duality symmetry group of ST compactified on a d-torus. A generalized complex structure J on $\mathbb{TM}$ is a bundle map

$$\mathrm{J} : T\mathcal{M} \oplus T^*\mathcal{M} \to T\mathcal{M} \oplus T^*\mathcal{M}\,, \tag{1.13}$$

such that $\mathrm{J}^2 = -1$ and $\mathrm{G}(\cdot, \cdot) = \mathrm{G}(\mathrm{J}\cdot, \mathrm{J}\cdot)$, where G is the canonical metric on $T\mathcal{M} \oplus T^*\mathcal{M}$. Remarkably enough, a complex structure $\mathcal{J}$ on $T\mathcal{M}$ and a symplectic structure ω on $\Lambda^2 T^*\mathcal{M}$ are special instances of J, when their actions are suitable extended to $T\mathcal{M} \oplus T^*\mathcal{M}$.

This is extremely important, and allows for Generalized Complex Geometry to give a unified description of the ten-dimensional supersymmetric Supergravity backgrounds. If we look for a supersymmetric solution of the equations of motion of ten-dimensional Supergravity of the form

$$\mathcal{M} = \mathcal{M}_4 \times \mathcal{M}_6\,, \qquad \mathrm{g}(x, y) = \mathrm{g}_4(x) + \mathrm{g}_6(y)\,, \tag{1.14}$$

[4]For a pessimistic opinion, see [47].

where the metric is written in a patch $\mathcal{U}$ with coordinates $(x^1, \dots, x^4, y^1, \dots, y^6)$, then a possible solution is

$$\mathrm{g}(y) = \eta + \mathrm{g}_{CY}(y) \tag{1.15}$$

where all the fluxes are taken to be zero and the dilaton is constant. This solution is phenomenologically relevant because, for Heterotic ST, it preserves four-dimensional $\mathcal{N} = 1$ supersymmetry [24]. Here $\mathrm{g}_{CY}(y)$ stands for the metric of the Calabi-Yau internal space $\mathcal{M}_6$. However, supersymmetric backgrounds with non-trivial fluxes are far from being Calabi-Yau[5]. A Calabi-Yau manifold is in particular a symplectic manifold and a complex manifold in a compatible way, but more general supersymmetric backgrounds are in general not simultaneously complex and symplectic. It turns out that Generalized Complex Geometry gives a unified description of all internal spaces of supersymmetric flux backgrounds [26].

Generalized Complex Geometry can be also used to naturally describe the supersymmetric embedding of D-branes with world-volume fluxes into these backgrounds [52–55]. For more details and other applications of Generalized Complex Geometry, the reader is referred to the excellent reviews [26, 56].

Generalized Complex geometry and Exceptional Generalized Geometry could be the first step towards the correct mathematical formalism to describe ST.

1.3 Remarks about Black Holes

A black hole space-time [57–62] is a particular kind of space-time that contains a region, called black hole, from which gravity prevents anything, including light, from escaping (we will give a precise definition in a moment). In this extreme situation, the classical laws of physics break down at the singularity and a quantum description of gravity seems to be needed. Since ST is one of the candidates to a theory of Quantum Gravity, it makes sense to use it to study situations where Quantum Gravity effects are expected to be important, as it happens for black holes, and see what we can learn from it, even if we don't know if ST actually describes the universe.

All the black holes that we will obtain and study in this thesis are asymptotically flat, static and spherically symmetric[6]. Notice that we truncate the fermions, which is always a consistent truncation, because black-hole solutions describe classical macroscopic objects. Therefore, by an asymptotically flat, static and spherically symmetric black-hole solution we mean a solution of the equations of motion of the corresponding theory for all the bosonic fields in the Lagrangian, such that the space-time manifold $\mathcal{M}$ is asymptotically flat, static, spherically symmetric and contains a black hole in the following sense [63]

- A space-time $\mathcal{M}$ is asymptotically simple if and only if it admits a conformal compactification and all the null geodesics of $\mathcal{M}$ start and end on $\partial\mathcal{M}$.
- A space-time $\mathcal{M}$ is asymptotically flat if and only if it has an open neighbourhood $\mathcal{U}$ isometric to an open neighbourhood of the boundary of the conformal compactification of an asymptotically simple space-time, and the Ricci tensor vanishes on $\mathcal{U}$.
- A asymptotically flat space-time is stationary if and only if there exists a time-like Killing vector ξ in a neighbourhood of the spatial infinity.
- A stationary space-time $\mathcal{M}$ is static if and only if there exists an space-like hypersurface orthogonal to the Killing vector ξ.
- In a strongly asymptotically predictable space-time $\mathcal{M}$, we will call black hole, if it exists, the region $\mathcal{B} \subset \mathcal{M}$ which is not contained in the causal past of the infinite null future.

Although the definition given above is completely precise, it is not very useful for practical purposes. A clearer characterization can be given using local coordinates (t, r, θ, ϕ) adapted to an spherically symmetric and static space-time $\mathcal{M}$. Here t is a time coordinate, r is a radial coordinate, and (θ, ϕ) are angular coordinates. The metric can thus be written as

$$\mathrm{g} = g_{tt}(r) dt \otimes dt - g_{rr}(r) dr \otimes dr - r^2 \mathrm{h}_{S^2} \,, \tag{1.16}$$

[5]Notice that fluxes cannot be turned on in compact spaces, unless negative tension sources, the so-called orientifold planes, are included in the vacuum structure.

[6]With one exception: in chapter 5 we will consider momentarily stationary space-times, which are anyway *composed* of several *static* black holes.

where $h_{S^2} = d\theta \otimes d\theta + \sin^2\theta d\phi \otimes d\phi$ is the round metric on the unit two-sphere. The event horizon typically lies at one of the the solutions of the equation $g_{tt}(r) = 0$. In general, the equation $g_{tt}(r) = 0$ has several solutions $r_1 <, \cdots, < r_n$, each one typically corresponding to the position of a particular *horizon*.

In the cases considered in this thesis, the maximum number of solutions of the equation $g_{tt}(r) = 0$ is going to be two. Let us denote them by $r_- < r_+$. Then, at $r = r_+$ we have the event horizon and at $r = r_-$ we have the Cauchy horizon. When the physical parameters of the solution (such as the mass, the charge or the *moduli*[7]) are adjusted so $r_- = r_+$ we have an extremal black hole. Let's see how this works for the Reissner-Nordström black hole [58, 59] of mass m and charge q, which is the simplest example of the kind of black holes that are considered in this thesis. By Birkhoff's theorem, the Reissner-Nordström solution is the only spherically symmetric solution of the Einstein-Maxwell theory, which in addition is also static and asymptotically flat. The metric is therefore of the form (1.16), with

$$g_{tt}(r) = 1 - \frac{2m}{r} + \frac{q^2}{r^2}\,, \qquad g_{rr}(r) = g_{tt}^{-1}(r)\,. \tag{1.17}$$

$g_{tt}(r)$ can be written as

$$g_{tt}(r) = \frac{(r - r_+)(r - r_-)}{r^2}\,, \qquad r_\pm = m \pm \sqrt{m^2 - q^2}\,. \tag{1.18}$$

Hence, the equation $g_{tt}(r) = 0$ has two solutions, given by $r_\pm$. The event horizon lies at $r = r_+$ and the Cauchy horizon lies at $r = r_-$. Therefore, the entropy and temperatures of the black hole are given by [64, 65]

$$S = \frac{A}{4} = \pi r_+^2\,, \qquad T = \frac{1}{2\pi}\left(\frac{r_0}{r_+}\right)^2\,, \tag{1.19}$$

where A is the area of the event horizon and $r_0 = \mp M \pm r_\pm$ is the extremality parameter (see below). The black hole is extremal when the two horizons coincide, that is, when $m = q$ (and hence $r_0 = 0$). Notice that in that case, the temperature T of the black hole is always zero but the entropy S is not, if the extremal black hole is regular. Extremal black holes cannot radiate [66] and are thus stable. For more details the reader is referred to [7, 63, 67, 68].

1.4 Supergravity solutions and the attractor mechanism

Going back to the topic of this thesis, black holes in ST, we now have all the elements necessary to precisely define our goal: in order to study black holes in ST we are going to develop a mathematical procedure to obtain black-hole solutions of four-dimensional Supergravity theories. These classical solutions (they are classical simply because the theory is not quantized) play a relevant role in ST, since they are necessary, for instance, in order to check the match between the microscopic entropy, computed as an ensemble of D-branes and other ST objects, and the macroscopic entropy, given by the area of the classical solution. This match is mandatory if ST is to be the correct theory of nature and contains a consistent theory of Quantum Gravity. In fact, the match has been checked to some extent in several specific cases of extremal and near-extremal black holes [6, 69–75]. However, there are supersymmetric (and hence extremal and stable) simple black holes for which the microscopic interpretation of the entropy is not even known at the leading order [76]. This means that even for the simplest kind of black holes the complete microscopic description of the entropy in ST has not always been achieved.

With the currently available tools, a necessary condition to obtain a match between the microscopic and the macroscopic computation is that the Supergravity solution depends only on quantized quantities, such as the charges of the black hole, but not, for example on the moduli, which are free parameters. Remarkably enough, for extremal black holes this is what generically happens: all the information about infinity is lost at the horizon[8] and the entropy only depends on the quantized charges, *e.i.*, the dependence on the moduli drops out. This is a consequence of the so-called *attractor mechanism* [77–89] for Supergravity black holes, which roughly speaking states that the scalar fields of a Supergravity extremal black-hole solution flow from a completely arbitrary value at spatial infinity to a completely fixed (in terms of the quantized charges of the black hole) value on the horizon, independent of the asymptotic value of the scalar at spatial infinity.

[7]In this context, the moduli is the arbitrary value at spatial infinity of the scalar fields present in the theory, which is not fixed by the equations of motion.

[8]In the absence of flat directions.

In fact, the attractor mechanism holds in a class of theories larger than the Supergravities. In particular, it holds in any theory of the form (4.1). As we will see in chapter 4, the most general spherically symmetric and static metric solution of (3.28) is given by

$$\mathrm{g} = e^{2U} dt \otimes dt - e^{-2U} \left[\frac{r_0^4}{\sinh^4 r_0 \tau} d\tau \otimes d\tau + \frac{r_0^2}{\sinh^2 r_0 \tau} \mathrm{h}_{S^2} \right] , \tag{1.20}$$

where τ is the radial coordinate and r_0 is the non-extremality parameter when (1.20) represents a black hole. In that case, the exterior of the event horizon is covered by $\tau \in (-\infty, 0)$, the event horizon being located at $\tau \to -\infty$ and the spatial infinity at $\tau \to 0^-$. The interior of the Cauchy horizon (if any) is covered by $\tau \in (\tau_S, \infty)$, the inner horizon being located at $\tau \to +\infty$ while the singularity is located at some finite, positive, value τ_S of the radial coordinate τ [90]. Since we are assuming that the space-time is spherically symmetric, all the fields of the theory, that is, the scalars and the vector fields, depend only on the radial coordinate τ. In the background given by (1.20) the Maxwell equations can be explicitly integrated, giving the vector fields as functions of τ and the electric q_Λ and magnetic p^Λ charges. The other equations of motion form a system of second order ordinary differential equations for $(U(\tau), \phi(\tau))$, namely (4.18), (4.19) and (4.20).

The extremal limit $r_0 \to 0$ of (1.20) is given by

$$\mathrm{g} = e^{2U} dt \otimes dt - e^{-2U} \left[\frac{1}{\tau^4} d\tau \otimes d\tau + \frac{1}{\tau^2} \mathrm{h}_{S^2} \right] = e^{2U} dt \otimes dt - e^{-2U} \left[\delta_{ij} dx^i \otimes dx^j \right] , \tag{1.21}$$

where x^i , $i = 1, \dots, 3$ are three-dimensional *cartesian coordinates* and δ_{ij} is the Kronecker delta. For an extremal regular black-hole solution of a theory with non-degenerate scalar metric $\mathcal{G}_{ij}$ and non-divergent scalars at the horizon, it can be proven that in the near horizon limit we have

$$\lim_{\tau \to -\infty} e^{-2U} = \frac{A}{4\pi} \lim_{\tau \to -\infty} \tau^2 , \qquad \lim_{\tau \to -\infty} \tau \frac{d\phi^i}{d\tau} = 0 \, , i = 1, \cdots, n_v \, , \tag{1.22}$$

where A is the area of the event horizon and n_v is the number of scalars. Using Eq. (1.22) we can obtain the near-horizon limit of the Eq. (4.20), which reads

$$\lim_{\tau \to -\infty} \left[\frac{d^2 \phi^i}{d\tau^2} + \frac{4\pi}{A} \mathcal{G}^{ij}(\phi_{\mathrm{h}}) \partial_j V_{\mathrm{bh}}(\phi_{\mathrm{h}}, \mathcal{Q}) \frac{1}{\tau^2} \right] = 0 \, , \tag{1.23}$$

where $V_{\mathrm{bh}}(\phi, \mathcal{Q})$ is the *black-hole potential* and $\mathcal{Q} = \left(p^\Lambda, q_\Lambda\right)^T$ denotes the electric and magnetic charges of the black hole. The solution to the above differential equation is given by

$$\lim_{\tau \to -\infty} \phi^i = \lim_{\tau \to -\infty} \left[-\frac{4\pi}{A} \mathcal{G}^{ij}(\phi_{\mathrm{h}}) \partial_j V_{\mathrm{bh}}(\phi_{\mathrm{h}}, \mathcal{Q}) \log(-\tau) + c_1^i \tau + c_2^i \right] , \tag{1.24}$$

where c_1^i and c_2^i are arbitrary constants. Therefore, taking the τ-derivative we obtain

$$\lim_{\tau \to -\infty} \frac{d\phi^i}{d\tau} = \lim_{\tau \to -\infty} \left[-\frac{4\pi}{A} \mathcal{G}^{ij}(\phi_{\mathrm{h}}) \partial_j V_{\mathrm{bh}}(\phi_{\mathrm{h}}, \mathcal{Q}) \frac{1}{\tau} + c_1^i \right] . \tag{1.25}$$

Since Eq. (1.22) must hold and, by assumption, the scalars do not diverge at the horizon, we conclude that

$$c_1^i = 0 \, , \qquad c_2^i = \phi_{\mathrm{h}}^i \, , \qquad \mathcal{G}^{ij}(\phi_{\mathrm{h}}) \partial_j V_{\mathrm{bh}}(\phi_{\mathrm{h}}, \mathcal{Q}) = 0 \, . \tag{1.26}$$

Finally, since the scalar metric is non-degenerate, the condition involving the black-hole potential can be rewritten as

$$\partial_i V_{\mathrm{bh}}(\phi_{\mathrm{h}}, \mathcal{Q}) = 0 \, , \tag{1.27}$$

which is the essence of the attractor mechanism [91]. The value of the scalars at the horizon ϕ_{h}^i must be a critical point of the black hole potential $V_{\mathrm{bh}}(\phi, \mathcal{Q})$. If $V_{\mathrm{bh}}(\phi, \mathcal{Q})$ has no flat directions, that is, if (1.27) is a compatible system of n_v independent equations, all the scalars are fixed at the horizon in terms of the charges of the black hole. From the near-horizon limit of Eq. (4.19) it can be easily obtained that, for extremal black holes, the entropy S is given by

$$S = \pi V_{\rm bh}(\phi_{\rm h}, \mathcal{Q})\,. \tag{1.28}$$

Therefore, we obtain a remarkable result: in the absence of flat directions the entropy S and the value of the scalars at the horizon only depend on the charges of the black hole, that is

$$\phi_{\rm h} = \phi_{\rm h}(\mathcal{Q})\,, \qquad S = \pi V_{\rm bh}(\phi_{\rm h}(\mathcal{Q}), \mathcal{Q})\,. \tag{1.29}$$

When flat directions are present, the scalars are only partially fixed in terms of the charges, but the dependence of the entropy S on the moduli still drops out. Supersymmetric attractors are always minima of the black hole potential. For non-supersymmetric attractors this issue must be studied on a case by case basis, although for homogeneous scalar manifolds it has been proven that the critical points of the black hole potential are always stable, but only up to possible flat directions [84, 92]. The attractor mechanism is only one example of the interesting features that Supergravity black holes, and other black objects such a strings or black p-branes display [93–97].

The search of solutions of Supergravity theories started with the class of supersymmetric solutions, since they are easier to obtain and classify, thanks to the first order differential equations (the so-called *Killing spinor equations*) that they obey.

A configuration of fields (ϕ_b, ϕ_f) is supersymmetric invariant (also called B.P.S.) if and only if

$$\delta_\epsilon \phi_b = 0\,, \qquad \delta_\epsilon \phi_f = 0\,, \tag{1.30}$$

for at least one spinor ϵ. In that case, ϵ is called a *Killing spinor*. If a given configuration is invariant under the maximum number of independent Killing spinors, given by $\mathcal{N}$, then it is said to be *maximally supersymmetric*. Notice that a supersymmetric configuration in principle does not solve the equations of motion of the corresponding theory. However, a careful analysis of the Killing spinor equations and their integrability conditions, the so called *Killing spinor identities*, shows that a supersymmetric configuration usually solves almost all the equations of motion of the theory, if not all [98, 99].

We are interested in black-hole solutions where the fermions are set to zero. For bosonic configurations, the equation $\delta_\epsilon \phi_b = 0$ is automatically solved and we have to solve only $\delta_\epsilon \phi_f = 0$ to obtain a supersymmetric configuration. These are the Killing spinor equations that must be solved in any supersymmetric theory to obtain a bosonic supersymmetric configuration. When a supersymmetric configuration also obeys the equations of motion of the theory it becomes a supersymmetric solution. Examples of supersymmetric solutions of Supergravity are the Reissner-Nordström extreme black hole, Minkowski space-time or the aDS space-time.

The supersymmetric solutions of Supergravity theories that describe vacua, black holes or topological defects, play a fundamental role in the progress of ST, since they represent non-perturbative stable states that can be trusted beyond the Supergravity approximation, and that, if supersymmetric enough, can be taken as exact states of ST [100, 101]. Supersymmetric solutions have also played an important role in mathematics, and, in particular in differential geometry, since the classification of supersymmetric solutions is closely related to the classification of manifolds with special holonomy, given that a supersymmetric solution has at least a globally defined section of the corresponding spin bundle, and therefore has reduced *generalized holonomy*. Therefore, great effort has been devoted in the last decades to study and classify as many supersymmetric solutions as possible.

In his seminal work [102], Tod showed that the Killing spinor equations, together with the corresponding integrability conditions, could be used to systematically classify all the supersymmetric solutions of a given Supergravity theory. He used the Newman-Penrose formalism and focused on pure $\mathcal{N} = 2$ Supergravity, following [103]. Since the Newman-Penrose formalism can only be applied in four dimensions, new techniques had to be developed in order to deal with higher dimensional cases. One of the techniques developed was the spinor-bilinear method [104], which was used to classify all the supersymmetric solutions of minimal five-dimensiona Supergravity. This result was soon extended to the Abelian gauged case [105], to general matter contents and couplings [106] and to other Supergravities [107–120].

Another approach, closer to the geometrical properties of having a spin manifold with global sections (it exploits the fact that a global spinor defines a *G structure*) was developed in [104, 110, 121]. Finally, we can mention yet another approach, that can be used to find black-hole solutions of four-dimensional theories, which exploits the symmetries of the time-like dimensionally-reduced theories which become a non-linear σ-model coupled to 3-dimensional gravity [122–128].

1.5 Outline of the thesis

The outline of this thesis goes as follows: on chapter 2 we introduce the relevant mathematical background to formulate extended ungauged Supergravity[9] in four-dimensions, that is, Special Kähler Geometry and homogeneous spaces. On chapter 3 we briefly comment on the structure of four-dimensional extended ungauged Supergravity relevant for black-hole solutions. On chapter 4, based on [129], we characterize the most general spherically symmetric and static black-hole solution of ungauged Supergravity, and use the result to study the hidden conformal symmetries of Supergravity black holes, obtaining the full Virasoro algebra of the dual conformal field theory. On chapter 5, based on [130], we obtain all the supersymmetric black-hole solutions of extended Supergravity by means of the algorithm provided in [131]. On chapter 6, based on [132, 133], we introduce the *H-F.G.K. formalism*, which simplifies the construction of non-supersymmetric black-hole solutions in $\mathcal{N} = 2$ Supergravity. On chapter 7, based on [76, 134], we apply the H-F.G.K. formalism to a class of theories corresponding to Type-IIA String Theory compactified on a Calabi-Yau (C.Y.) threefold, obtaining the so-called *quantum* black holes, which only exist when certain quantum corrections (perturbative or non-perturbative, depending on the solution) are included in the prepotential. For the case of non-perturbative black holes we elaborate on the potential consequences of the appearance on the solution of multi-valued functions in relation to the no-hair theorem for four-dimensional black holes. This thesis is based on [76, 129–134]. Other works finished during my doctoral studies are [90, 97, 135–139].

[9] For the case of $\mathcal{N} = 2$ Supergravity, in the absence of hypermultiplets.

Chapter 2

Mathematical preliminaries

Extended four-dimensional Supergravity, as a classical field theory, admits an elegant *geometric* formulation in the following sense: the lagrangian can be constructed from geometrical structures (such as sections) of particular manifolds that are naturally associated to the theory. Therefore, extended four-dimensional Supergravity can be completely specified by some *geometrical data*, that is, particular fibre bundles and global sections thereof.

In fact, such geometric formulation is not just an elegant mathematical tool to describe Supergravity, it is also useful in a wide range of applications of Supergravity, for instance to relate Supergravity to ST compactifications or to provide the general formalism to deal with gaugings. In this thesis we will use it to simplify the task of obtaining black hole solutions: using the mathematical structure of the theory we can introduce a new set of variables which considerably eases the construction of non-supersymmetric black hole solutions.

As far as Quantum maximally and half-maximally extended Supergravity [10, 41, 42] is concerned, the mathematical structure of the theory at the classical level is also of outermost importance: for example, in Refs. [44, 45] the symplectic action of the $E_{7(7)}$ group was key in order to explain the conjectured finiteness of $\mathcal{N} = 8$ ungauged Supergravity at all orders in perturbation theory.

Before dealing with the geometric formulation of four-dimensional extended Supergravity[1], it is therefore necessary to introduce the mathematical background that will play a relevant role in the construction of the theories, namely, Special Kähler geometry for $\mathcal{N} = 2$ Supergravity (in the absence of Hypermultiplets) and *Irreducible Riemannian Globally Symmetric* (I.R.G.S.) spaces for $\mathcal{N} > 2$ Supergravity. That is the goal of this chapter.

2.1 Special Kähler Geometry

Some basic references for this section are [140–146]. See the appendices of [113, 147, 148], for an extremely well written short review of Special Kähler Geometry and its relation to $\mathcal{N} = 2$ Supergravity coupled to vector multiplets and its gaugings. The definition of Special Kähler manifold was made in [13], formalizing the original results of [11]. We will follow [113, 144, 146].

A Special Kähler manifold is a particular instance of real differentiable manifold $\mathcal{M}$, that is, it is a differentiable manifold with some extra-structure defined on it. Therefore, we will proceed defining step-by-step all the necessary ingredients until we arrive to a Special Kähler manifold, By a real differentiable manifold $\mathcal{M}$ we mean a Hausdorff and second countable topological space equipped with a differentiable structure (therefore it is paracompact and metrizable).

Definition 2.1.1. A differentiable Riemannian manifold $(\mathcal{M}, \mathcal{G})$ is a differentiable real manifold equipped with a smooth,non-degenerate, point-wise positive definite, global section g of $S^2T^*\mathcal{M}$.

Definition 2.1.2. A Symplectic manifold $(\mathcal{M}, \omega)$ is a differentiable manifold real equipped with a smooth, non-degenerate , global section ω of $\Lambda^2T^*\mathcal{M}$.

Definition 2.1.3. An almost-complex structure $\mathcal{J} : T\mathcal{M} \to T\mathcal{M}$ on a tangent bundle $T\mathcal{M}$ is a bundle endomorphism such that $\mathcal{J}^2 = -1$.

[1] See chapter 3.

Definition 2.1.4. A differentiable almost-complex manifold $(\mathcal{M}, \mathcal{J})$ is a differentiable $2n$-dimensional real manifold equipped with an almost-complex structure.

Let $(\mathcal{M}, \mathcal{J})$ be an almost-complex manifold. The complexified tangent bundle of $\mathcal{M}$ is the bundle $T\mathcal{M} \otimes \mathbb{C} \to \mathcal{M}$ with fibre $(T\mathcal{M} \otimes \mathbb{C})_p = T\mathcal{M}_p \otimes \mathbb{C}$ at each $p \in \mathcal{M}$. If $T_p\mathcal{M}$ is a real $2n$-dimensional vector space, then $T_p\mathcal{M} \otimes \mathbb{C}$ is a complex $2n$-dimensional vector space.

We can extend linearly $\mathcal{J}$ on $T\mathcal{M} \otimes \mathbb{C}$ as follows

$$\mathcal{J}(v \otimes c) = \mathcal{J}v \otimes c \,, \qquad v \in T\mathcal{M},\ c \in \mathbb{C}\,. \tag{2.1}$$

Since $\mathcal{J}^2 = -1$, $\mathcal{J}_p$ acting on $T_p\mathcal{M} \otimes \mathbb{C}$ has eigenvalues $\pm i$. We define

$$\begin{aligned} T_{(1,0)}\mathcal{M} &= \{v \in T\mathcal{M} \otimes \mathbb{C} \mid \mathcal{J}v = iv \,, \forall\, v \in T\mathcal{M} \otimes \mathbb{C}\}\,, \\ T_{(0,1)}\mathcal{M} &= \{v \in T\mathcal{M} \otimes \mathbb{C} \mid \mathcal{J}v = -iv \,, \forall\, v \in T\mathcal{M} \otimes \mathbb{C}\}\,. \end{aligned} \tag{2.2}$$

Since

$$\begin{aligned} \pi_{(1,0)} : T\mathcal{M} \otimes \mathbb{C} &\to T_{(1,0)}\mathcal{M} \\ v &\to \frac{1}{2}(v \otimes 1 - \mathcal{J}v \otimes i)\,, \end{aligned} \tag{2.3}$$

$$\begin{aligned} \pi_{(0,1)} : T\mathcal{M} \otimes \mathbb{C} &\to T_{(0,1)}\mathcal{M} \\ v &\to \frac{1}{2}(v \otimes 1 + \mathcal{J}v \otimes i)\,, \end{aligned} \tag{2.4}$$

are a real bundle isomorphisms such that $\pi_{(1,0)} \circ \mathcal{J} = -i\pi_{(0,1)}$, we have

$$T\mathcal{M} \cong T_{(1,0)}\mathcal{M} \cong \overline{T_{(0,1)}\mathcal{M}}\,, \tag{2.5}$$

$$\left(\pi_{(1,0)}, \pi_{(0,1)}\right) \;:\; T\mathcal{M} \otimes \mathbb{C} \xrightarrow{\cong} T_{(1,0)}\mathcal{M} \oplus T_{(0,1)}\mathcal{M}\,. \tag{2.6}$$

Analogously, for the complexified cotangent bundle $T^*\mathcal{M} \otimes \mathbb{C} \to \mathcal{M}$ we can conclude

$$T^*\mathcal{M} \cong T^{(1,0)}\mathcal{M} \cong T^{(0,1)}\mathcal{M}\,, \tag{2.7}$$

$$\left(\pi^{(1,0)}, \pi^{(0,1)}\right) \;:\; T^*\mathcal{M} \otimes \mathbb{C} \xrightarrow{\cong} T^{(1,0)}\mathcal{M} \oplus T^{(0,1)}\mathcal{M}\,. \tag{2.8}$$

where

$$\begin{aligned} T^{(1,0)}\mathcal{M} &= \{\alpha \in T^*\mathcal{M} \otimes \mathbb{C} \mid \alpha(\mathcal{J}v) = i\alpha(v)\,, \forall\, v \in T\mathcal{M} \otimes \mathbb{C}\}\,, \\ T^{(0,1)}\mathcal{M} &= \{v\alpha \in T^*\mathcal{M} \otimes \mathbb{C} \mid \alpha(\mathcal{J}v) = -i\alpha(v)\,, \forall\, v \in T\mathcal{M} \otimes \mathbb{C}\}\,, \end{aligned} \tag{2.9}$$

and we have defined the natural projections $\pi^{(1,0)}$ and $\pi^{(0.1)}$ of the complexified cotangent bundle as follows

$$\begin{aligned} \pi^{(1,0)} : T^*\mathcal{M} \otimes \mathbb{C} &\to T^{(1,0)}\mathcal{M} \\ \alpha &\to \frac{1}{2}(\alpha \otimes 1 - \alpha \otimes i \circ \mathcal{J})\,, \end{aligned} \tag{2.10}$$

$$\begin{aligned} \pi^{(0,1)} : T^*\mathcal{M}\otimes\mathbb{C} &\rightarrow T^{(0,1)}\mathcal{M} \\ \alpha &\rightarrow \frac{1}{2}\left(\alpha\otimes 1+\alpha\otimes i\circ\mathcal{J}\right)\,, \end{aligned} \tag{2.11}$$

We are going to elucidate the structure of the space of forms on $(\mathcal{M},\mathcal{J})$, which, since $(\mathcal{M},\mathcal{J})$ is almost-complex, is going to be constructed from sections of $T^*\mathcal{M}\otimes\mathbb{C}$ and its exterior powers, and not from $T^*\mathcal{M}$. The main reason is that $\mathcal{J}$ can be diagonalized on $T^*\mathcal{M}\otimes\mathbb{C}$ but not on $T^*\mathcal{M}$. For an almost-complex manifold $(\mathcal{M},\mathcal{J})$ let

$$\Omega^k\left(\mathcal{M},\mathbb{C}\right)\equiv\Gamma\left(\Lambda^k\left(T^*\mathcal{M}\otimes\mathbb{C}\right)\right)\,, \tag{2.12}$$

where $\Gamma(\cdot)$ stands for the space of sections of the corresponding fibre bundle $\cdot$ and

$$\Lambda^k\left(T^*\mathcal{M}\otimes\mathbb{C}\right)=\Lambda^k\left(T^{(0,1)}\mathcal{M}\oplus T^{(1,0)}\mathcal{M}\right)=\bigoplus_{l+m=k}\Lambda^l\left(T^{(0,1)}\mathcal{M}\right)\wedge\Lambda^m\left(T^{(1,0)}\mathcal{M}\right)\,. \tag{2.13}$$

Definition 2.1.5. The differential forms of type (l,m) on an almost-complex manifold $(\mathcal{M},\mathcal{J})$ are the sections of $\bigoplus_{l+m=k}\Lambda^l\left(T^{(0,1)}\mathcal{M}\right)\wedge\Lambda^m\left(T^{(1,0)}\mathcal{M}\right)$.

It is convenient to define

$$\Omega^{(l,m)}\left(\mathcal{M},\mathbb{C}\right)=\Gamma\left(\Lambda^l\left(T^{(0,1)}\mathcal{M}\right)\wedge\Lambda^m\left(T^{(1,0)}\mathcal{M}\right)\right)\,, \tag{2.14}$$

and hence

$$\Omega^k\left(\mathcal{M},\mathbb{C}\right)=\bigoplus_{l+m=k}\Omega^{(l,m)}\left(\mathcal{M},\mathbb{C}\right)\,. \tag{2.15}$$

Therefore, when speaking of tensors on a complex manifold, it is generally referred to sections of the complexified tangent or cotangent bundle and its tensorial products. Let $\pi^{(l,m)}$ be the natural projection $\pi^{(l,m)}:\Lambda^k\left(T^*\mathcal{M}\otimes\mathbb{C}\right)\rightarrow\Lambda^l\left(T^{(0,1)}\mathcal{M}\right)\wedge\Lambda^m\left(T^{(1,0)}\mathcal{M}\right)$. We define then

$$\begin{aligned} \partial &\equiv\pi^{(l+1,m)}\circ d:\Omega^{(l,m)}(\mathcal{M},\mathbb{C})\rightarrow\Omega^{(l+1,m)}(\mathcal{M},\mathbb{C})\,, \\ \bar\partial &\equiv\pi^{(l,m+1)}\circ d:\Omega^{(l,m)}(\mathcal{M},\mathbb{C})\rightarrow\Omega^{(l,m+1)}(\mathcal{M},\mathbb{C})\,, \end{aligned} \tag{2.16}$$

which are differential operators that act on forms of type (m,l). When considering complex manifolds, ∂ and $\bar\partial$ will become natural differential operators in terms of the complex coordinates of a given chart.

It is possible to define in a compatible, natural, way, an almost-complex structure $\mathcal{J}$ on a Symplectic manifold, which makes it also Riemannian.

Definition 2.1.6. Let $(\mathcal{M},\omega)$ be a Symplectic manifold. An almost-complex structure $\mathcal{J}$ is called compatible if $\mathcal{G}\left(\cdot,\cdot\right)=\omega\left(\cdot,\mathcal{J}\cdot\right)$ is a Riemannian metric on $\mathcal{M}$. The triple $(\omega,\mathcal{G},\mathcal{J})$ is then called a compatible triple.

For a compatible triple $(\omega,\mathcal{G},\mathcal{J})$ we have that

$$\mathcal{G}(\mathcal{J}\cdot,\mathcal{J}\cdot)=\omega(\mathcal{J}\cdot,\mathcal{J}^2\cdot)=\omega(\cdot,\mathcal{J}\cdot)=\mathcal{G}(\cdot,\cdot)\,, \tag{2.17}$$

that is, $\mathcal{J}$ preserves the Riemannian metric G.

Definition 2.1.7. An almost complex structure $\mathcal{J}$ is integrable if and only if $N(u,v)\equiv[\mathcal{J}u,\mathcal{J}v]-\mathcal{J}[u,\mathcal{J}v]-\mathcal{J}[\mathcal{J}u,v]-[u,v]=0\,\forall u,v\in T\mathcal{M}$.

$N(\cdot,\cdot)$ is the Nijenhuis tensor, and intuitively it parametrizes the obstruction to the possibility of defining holomorphic changes of coordinates in $\mathcal{M}$. When it vanishes, it is possible to construct an holomorphic atlas and therefore $\mathcal{M}$ is a complex manifold.

Definition 2.1.8. A differentiable complex manifold $(\mathcal{M}, \mathcal{J})$ is a differentiable $2n$-dimensional real manifold equipped with an integrable complex structure.

Let $\mathcal{U} \subset \mathcal{M}$ a chart on a complex manifold $(\mathcal{M}, \mathcal{J})$ with coordinates $z^i = x^i + iy^i$ and real coordinates (x^i, y^i). At any $p \in \mathcal{M}$ we have

$$\begin{aligned} T^*_p\mathcal{M} &= \mathbb{R}\text{−Span}\left[dx^i, dy^i\right]_p \,, \\ T^*_p\mathcal{M} \otimes \mathbb{C} &= \mathbb{C}\text{−Span}\left[dx^i, dy^i\right]_p \,. \end{aligned} \tag{2.18}$$

Then, defining $dz^i = dx^i + idy^i$ and $d\bar{z}^{\bar{i}} = dx^i - idy^i$ we can write

$$T^*_p\mathcal{M} \otimes \mathbb{C} = \mathbb{C}\text{−Span}\left[dz^i\right]_p \oplus \mathbb{C}\text{−Span}\left[d\bar{z}^{\bar{i}}\right]_p = T^{(1,0)}\mathcal{M} \oplus T^{(0,1)}\mathcal{M}\,, \tag{2.19}$$

since $dz^i \circ \mathcal{J} = idz^i$ and $d\bar{z}^{\bar{i}} \circ \mathcal{J} = -id\bar{z}^{\bar{i}}$. Therefore, a $(1,0)$-form α is written as $\alpha = \alpha_i dz^i$ and $(0,1)$-form β is written as $\beta = \beta_{\bar{i}} d\bar{z}^{\bar{i}}$. Since the manifold $(\mathcal{M}, \mathcal{J})$ is complex, we can consistently define complex coordinates coordinates $\tilde{z}^i$ (this may fail in an almost-complex manifold) such that $d\tilde{z}^i = dz^i$. Of course, we will drop the tilde and call them z^i. An similar derivation allows to conclude that

$$\frac{\partial}{\partial z^i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} - i\frac{\partial}{\partial y^i}\right)\,, \qquad \frac{\partial}{\partial \bar{z}^{\bar{i}}} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} + i\frac{\partial}{\partial y^i}\right)\,, \tag{2.20}$$

where $\frac{\partial}{\partial z^i}$ spans $T_{(1,0)}\mathcal{M}$ and $\frac{\partial}{\partial \bar{z}^{\bar{i}}}$ spans $T_{(0,1)}\mathcal{M}$.

It can be also proven that the exterior derivative d can be written in terms of ∂ and $\bar{\partial}$ as $d = \partial + \bar{\partial}$, where, given a function $f \in C^\infty(\mathcal{M}, \mathbb{C})$ ∂ and $\bar{\partial}$ act as follows

$$\partial f = \frac{\partial f}{\partial z^i} dz^i\,, \qquad \bar{\partial} f = \frac{\partial f}{\partial \bar{z}^{\bar{i}}} d\bar{z}^{\bar{i}}\,. \tag{2.21}$$

We are now ready to define Kähler manifolds:

Definition 2.1.9. A Kähler manifold is a Symplectic manifold $(\mathcal{M}, \omega)$, equipped with an integrable compatible almost complex structure $\mathcal{J}$. The symplectic form ω is then called the Kähler form.

Since ω and $\mathcal{J}$ are compatible, they define a Riemannian metric: $\mathcal{G}(\cdot,\cdot) = \omega(\cdot, \mathcal{J}\cdot)$. Therefore, a Kähler manifold is a differentiable manifold which is Symplectic, Riemannian, and Complex in a compatible way, and therefore incorporates together the three basics types of geometry: Symplectic, Riemannian and Complex [2].

Since a Kähler manifold has an integrable complex structure, it immediately follows that a Kähler manifold is a complex manifold. As a consequence, tensors on a Kähler manifold are sections of the complexified tangent and cotangent bundles and their tensorial products. In addition we have the following propostion

Proposition 2.1.10. *Let $(\mathcal{M}, \omega, \mathcal{J})$ a Kähler manifold, then $\mathcal{J}$ is a symplectomorphism, that is, $\mathcal{J}^* = \mathcal{J}$.*

Proof.

$$\mathcal{J}^*_p\omega_p(u,v) = \omega_p(\mathcal{J}_p u, \mathcal{J}_p v) = \mathcal{G}_p(v, \mathcal{J}_p u) = \omega_p(u,v)\,, \qquad \forall p \in \mathcal{M} \text{ and } u, v \in T_p\mathcal{M}\,. \tag{2.22}$$

In fact, propostion 2.1.10 holds also when ω is not closed, as long as $(\omega, \mathcal{G}, \mathcal{J})$is a compatible triple. Since ω is a two-form, in principle $\omega \in \Omega^2(\mathcal{M}, \mathbb{C})$, that is, it is a section of the exterior product of two copies of the complexified cotangent bundle $T\mathcal{M} \otimes \mathbb{C}$. Therefore

$$\omega \in \Omega^2(\mathcal{M}, \mathbb{C}) = \Omega^{(2,0)}(\mathcal{M}, \mathbb{C}) \oplus \Omega^{(1,1)}(\mathcal{M}, \mathbb{C}) \oplus \Omega^{(0,2)}(\mathcal{M}, \mathbb{C})\,. \tag{2.23}$$

However, imposing proposition 2.1.10 in a local chart $(\mathcal{U}, z^i)$, one can see that

[2]Kähler manifolds are of outermost importance in Theoretical Physics: they appear for example in every supersymmetric non-linear σ model, String Theory compactifications...

$$\omega \in \Omega^{(1,1)}(\mathcal{M}, \mathbb{C}) \,, \tag{2.24}$$

i.e. ω defines a Dolbeault $(1,1)$-cohomology class $[\omega] \in H^{(1,1)}_{\text{Dolbeault}}(\mathcal{M})$. In other words, the local expression for ω does not contain terms of the form $dz^i \wedge dz^j$ or $d\bar{z}^{\bar{i}} \wedge d\bar{z}^{\bar{j}}$. Therefore, in a local chart $\left(\mathcal{U}, z^i\right)$, we can write ω as follows

$$\omega = i h_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}} \,, \tag{2.25}$$

where $h_{i\bar{j}} \in C^\infty(\mathcal{U}, \mathbb{C})$ and the i has been introduced by convenience. Using that $\mathcal{G}(\cdot,\cdot) = \omega(\cdot, \mathcal{J}\cdot)$, we can write $\mathcal{G}$ in the same local chart as

$$\mathcal{G} = h_{i\bar{j}} \left(dz^i \otimes d\bar{z}^{\bar{j}} + dz^{\bar{j}} \otimes d\bar{z}^i\right) . \tag{2.26}$$

Since a Kähler manifold is in particular Symplectic, the symplectic form ω must be real, that is $\omega = \bar{\omega}$, which is equivalent to $h_{ij} = \bar{h_{ji}}$. For the same reason, ω is non-degenerate and its n-th exterior power defines a volume form on $(\mathcal{M}, \omega, \mathcal{J})$, making $(\mathcal{M}, \omega, \mathcal{J})$ orientable. To summarize, the Kähler form ω is a two-form compatible with the complex structure, closed, real valued and non-degenerate.

Theorem 2.1.11. *Let ω be a closed real-valued $(1,1)$-form on a complex manifold $\mathcal{M}_{\text{Complex}}$ and let $p \in \mathcal{M}_{\text{Complex}}$. Then, $\exists\, \mathcal{U}(p)$* and *$\mathcal{K} \in C^\infty(\mathcal{U}, \mathbb{R})$ such that, on $\mathcal{U}(p)$, $\omega = i\partial\bar{\partial}\mathcal{K}$.*

Proof. See proposition 8.8 in [146].

If we apply theorem 2.1.11 to a Kähler manifold, which is in particular complex, we obtain

$$\mathcal{G} = \partial_i \partial_{\bar{j}} \mathcal{K} \left(dz^i \otimes d\bar{z}^{\bar{j}} + dz^{\bar{j}} \otimes d\bar{z}^i\right) \tag{2.27}$$

and $\mathcal{K}$ is called the *Kähler potential.* The metric therefore can be obtained from a single function, which allows to write simple expressions for the Levi-Cività connection and the curvature tensors. Remarkably enough, the existence of normal holomorphic coordinates on a complex manifold around each point is equivalent to the metric being Kähler. The Levi-Cività connection on a Kähler manifold is given by

$$\Gamma_{jk}{}^i = \mathcal{G}^{i\bar{i}} \partial_j \mathcal{G}_{\bar{i}k} \,, \qquad \Gamma_{\bar{j}\bar{k}}{}^{\bar{i}} = \mathcal{G}^{\bar{i}i} \partial_{\bar{j}} \mathcal{G}_{\bar{k}i} \,. \tag{2.28}$$

The Riemann curvature tensor has as only non-vanishing components $R_{i\bar{j}k\bar{l}}$, but we will not need their explicit expression. The Ricci tensor is given by

$$R_{i\bar{i}} = \partial_i \partial_{\bar{i}} \left(\tfrac{1}{2} \log \det \mathcal{G}\right) . \tag{2.29}$$

The Kähler potential is not unique: it is defined only up to *Kähler transformations* of the form

$$\mathcal{K}' = \mathcal{K} + f + \bar{f} \,, \tag{2.30}$$

where f is any holomorphic function on $(\mathcal{M}, \omega, \mathcal{J})$. We need to introduce now the concepts of holomorphic vector bundle, line-bundel and *Kähler-Hodge* manifold

Definition 2.1.12. An holomorphic vector bundle is a complex vector bundle over a complex manifold $\mathcal{M}$ such that the total space E is a complex manifold and the projection map $\pi : E \to \mathcal{M}$ is holomorphic.

In particular, the transition functions of an holomorphic vector bundle are holomorphic.

Definition 2.1.13. A line bundle $\mathcal{L} \xrightarrow{\pi} \mathcal{M}$ is a holomorphic vector bundle of rank $r = 1$.

Definition 2.1.14. A Kähler manifold $(\mathcal{M}, \omega, \mathcal{J})$ is Hodge if and only if there exists a line bundle $\mathcal{L} \xrightarrow{\pi} \mathcal{M}$ such that $c_1(\mathcal{L}) = [\omega]$, where $c_1(\mathcal{L})$ denotes the first Chern class of $\mathcal{L}$.

The condition on the Chern class is rather abstract, and we will not discuss it here. Intuitively, what is required in definition 2.1.14 is that the curvature of the line-bundle bundle is equal, up to an exact form, to the Kähler form ω.

Let us define now, motivated by the structure of $\mathcal{N}=1$ Supergravity, a family of (complex) rank one bundles $\{\mathcal{L}^{(q,\bar{q})}, q\in\mathbb{R}, \bar{q}\in\mathbb{R}\}$ over a Kähler manifold $(\mathcal{M},\omega,\mathcal{J})$ such that, given a section $W\in\Gamma\left(\mathcal{L}^{(q,\bar{q})}\right)$, on the overlap of two patches $\mathcal{U}_{(\alpha)}$ and $\mathcal{U}_{(\beta)}$, W and $\mathcal{K}$ are related as follows

$$W_{(\alpha)} = e^{-(qf+\bar{q}\bar{f})/2}W_{(\beta)}\,, \qquad \mathcal{K}_{(\alpha)} = \mathcal{K}_{(\alpha)} + f + \bar{f}\,, \tag{2.31}$$

and define the *Kähler (connection) 1-form* $\mathcal{Q}$ as

$$\mathcal{Q} \equiv (2i)^{-1}(dz^i\partial_i\mathcal{K} - d\bar{z}^{\bar{i}}\partial_{\bar{i}}\mathcal{K})\,, \tag{2.32}$$

such that the covariant derivative on sections of $\mathcal{L}^{(q,\bar{q})}$ is given by

$$\mathfrak{D}_i \equiv \nabla_i + iq\mathcal{Q}_i\,, \qquad \mathfrak{D}_{\bar{i}} \equiv \nabla_{\bar{i}} - i\bar{q}\mathcal{Q}_{\bar{i}}\,, \tag{2.33}$$

where ∇ is the standard covariant derivative associated to the Levi-Cività connection on $\mathcal{M}$. The Kähler connection one-form, on the overlap of two patches $\mathcal{U}_{(\alpha)}$ and $\mathcal{U}_{(\beta)}$, is related as follows

$$\mathcal{Q}_{(\alpha)} = \mathcal{Q}_{(\beta)} - \tfrac{i}{2}\partial f\,. \tag{2.34}$$

Then, for $q=1$, $\bar{q}=0$, $W\in\Gamma\left(\mathcal{L}^{(1,0)}\right)$ is the superpotential of $\mathcal{N}=1$ Supergravity and $\mathcal{L}^{(1,0)}$ is a line-bundle which is also Kähler-Hodge in the sense of definiton 2.1.14. That is, the curvature of the line-bundle, obtaining by computing $[\mathfrak{D}_i,\mathfrak{D}_j]$ is equal to the ω up to an exact form, which in this case is zero. Other objects of $\mathcal{N}=1$ Supergravity correspond to sections of $\mathcal{L}^{(q,\bar{q})}$ for other values of q and $\bar{q}$.

A Kähler-Hodge manifold provides the formal starting point for the definition of a Special Kähler manifold: Special Kähler Geometry appears in the scalar manifold corresponding to the vector multiplets of $\mathcal{N}=2$ Supergravity. Since $\mathcal{N}=2$ supersymmetry includes $\mathcal{N}=1$ supersymmetry, we expect Special Kähler manifolds to be also Kähler-Hodge, equipped with some extra structure, which is indeed the case. Before giving the definition of Special Kähler manifold, let us state that there are two kinds of Special Kähler geometry:

1. Rigid Special Kähler Geometry $\to$ vector-multiplet scalar field sector of $\mathcal{N}=2$ Yang-Mills theories.

2. Local Special Kähler Geometry $\to$ vector-multiplet scalar field sector of $\mathcal{N}=2$ Supergravity theories.

Here we will deal with the local case, since we are interested in Supergravity. In any case, the definition for the global case is similar to the local case, see [149].

Definition 2.1.15. A Kähler-Hodge manifold $\mathcal{L}\xrightarrow{\pi}\mathcal{M}$ is Special Kähler of the local type if there exists a bundle $\mathcal{SV}=\mathcal{SM}\otimes\mathcal{L}\xrightarrow{\pi}\mathcal{M}$ such that for some holomorphic section $\Omega\in\Gamma(\mathcal{SV})$ the Kähler 2-form is given by $\omega=i\partial\bar{\partial}\log\left(i\Omega_M\bar{\Omega}^M\right)$, where $\mathcal{SM}\xrightarrow{\pi}\mathcal{M}$ is a flat $(2n_v+2)$-dimensional vector bundle with structure group $\mathrm{Sp}(2n_v+2,\mathbb{R})$.

$\Omega_M\bar{\Omega}^M=\langle\Omega\mid\bar{\Omega}\rangle$ denotes the hermitian and symplectic inner product of fibres. $M,N,\dots=1\cdots 2n_v+2$ are called *symplectic indices*, and can be *decomposed* in two sets of indices $\Lambda,\Sigma\cdots=1,\cdots,n_v+1$ such that, for example, the section Ω would be written as $\Omega^M=(\mathcal{X}^\Lambda,\mathcal{F}_\Lambda)^T$, and therefore we have

$$\Omega = \begin{pmatrix}\mathcal{X}^\Lambda\\ \mathcal{F}_\Sigma\end{pmatrix} \;\to\; \begin{cases}\langle\Omega\mid\bar{\Omega}\rangle & \equiv\; \bar{\mathcal{X}}^\Lambda\mathcal{F}_\Lambda - \mathcal{X}^\Lambda\bar{\mathcal{F}}_\Lambda = -i\,e^{-\mathcal{K}}\,,\\ \partial_{\bar{i}}\Omega & =\; 0\,,\\ \langle\partial_i\Omega\mid\Omega\rangle & =\; 0\,.\end{cases} \tag{2.35}$$

Ω is everything we need to know in order to define $\mathcal{N}=2$ ungauged supergravity in the absence of hyper-multiplets: if we know Ω^M we can write the complete Lagrangian.

It is convenient to define a covariantly holomorphic symplectic section $\mathcal{V}=e^{\frac{\mathcal{K}}{2}}\Omega$[3], which therefore obeys

[3]This is the section of a different vector bundle, which cannot be holomorphic since $\mathcal{V}$ is related in different patches through non-holomorphic transition functions.

$$\mathcal{V} = \begin{pmatrix} \mathcal{L}^\Lambda \\ \mathcal{M}_\Sigma \end{pmatrix} \rightarrow \begin{cases} \langle \mathcal{V} \mid \bar{\mathcal{V}} \rangle & \equiv \bar{\mathcal{L}}^\Lambda \mathcal{M}_\Lambda - \mathcal{L}^\Lambda \bar{\mathcal{M}}_\Lambda = -i\,, \\ \mathfrak{D}_{\bar{i}} \mathcal{V} & = (\partial_{\bar{i}} + \tfrac{1}{2}\partial_{\bar{i}}\mathcal{K})\mathcal{V} = 0\,, \\ \langle \mathfrak{D}_i \mathcal{V} \mid \mathcal{V} \rangle & = 0\,. \end{cases} \tag{2.36}$$

Notice if we define

$$\mathcal{U}_i \equiv \mathfrak{D}_i \mathcal{V} = \begin{pmatrix} f^\Lambda{}_i \\ h_{\Sigma\, i} \end{pmatrix}\,, \quad \bar{\mathcal{U}}_{\bar{i}} = \overline{\mathcal{U}_i}\,, \tag{2.37}$$

then it follows from the basic definitions that

$$\begin{aligned} \mathfrak{D}_{\bar{i}}\, \mathcal{U}_i &= \mathcal{G}_{i\bar{i}}\, \mathcal{V} & \langle \mathcal{U}_i \mid \bar{\mathcal{U}}_{\bar{i}} \rangle &= i\mathcal{G}_{i\bar{i}}\,, \\ \langle \mathcal{U}_i \mid \bar{\mathcal{V}} \rangle &= 0\,, & \langle \mathcal{U}_i \mid \mathcal{V} \rangle &= 0\,. \end{aligned} \tag{2.38}$$

Taking the covariant derivative of the last identity $\langle \mathcal{U}_i \mid \mathcal{V} \rangle = 0$ we find immediately that $\langle \mathfrak{D}_i \mathcal{U}_j \mid \mathcal{V} \rangle = -\langle\, \mathcal{U}_j \mid \mathcal{U}_i \rangle$. It can be shown that the r.h.s. of this equation is antisymmetric while the l.h.s. is symmetric, so that

$$\langle \mathfrak{D}_i \mathcal{U}_j \mid \mathcal{V} \rangle = \langle \mathcal{U}_j \mid \mathcal{U}_i \rangle = 0\,. \tag{2.39}$$

The importance of this last equation is that if we group together $\mathcal{E}_\Lambda = (\mathcal{V}, \mathcal{U}_i)$, we can see that $\langle \mathcal{E}_\Sigma \mid \bar{\mathcal{E}}_\Lambda \rangle$ is a non-degenerate matrix. This then allows us to construct an identity operator for the symplectic indices, such that for a given section of $\mathcal{A} \ni \Gamma(E, \mathcal{M})$ we have

$$\mathcal{A} = i\langle \mathcal{A} \mid \bar{\mathcal{V}} \rangle \mathcal{V} - i\langle \mathcal{A} \mid \mathcal{V} \rangle\, \bar{\mathcal{V}} + i\langle \mathcal{A} \mid \mathcal{U}_i \rangle \mathcal{G}^{i\bar{i}}\, \bar{\mathcal{U}}_{\bar{i}} - i\langle \mathcal{A} \mid \bar{\mathcal{U}}_{\bar{i}} \rangle \mathcal{G}^{i\bar{i}} \mathcal{U}_i\,. \tag{2.40}$$

As we have seen $\mathfrak{D}_i \mathcal{U}_j$ is symmetric in i and j, but what more can be said about it: as one can easily see, the inner product with $\bar{\mathcal{V}}$ and $\bar{\mathcal{U}}_{\bar{i}}$ vanishes due to the basic properties. Let us then define the Kähler-weight 2 object

$$\mathcal{C}_{ijk} \equiv \langle \mathfrak{D}_i\, \mathcal{U}_j \mid \mathcal{U}_k \rangle \;\rightarrow\; \mathfrak{D}_i\, \mathcal{U}_j = i\mathcal{C}_{ijk} \mathcal{G}^{k\bar{l}} \bar{\mathcal{U}}_{\bar{l}}\,, \tag{2.41}$$

where the last equation is a consequence of Eq. (2.40). Since the $\mathcal{U}$'s are orthogonal, however, one can see that $\mathcal{C}$ is completely symmetric in its 3 indices. Furthermore one can show that

$$\mathfrak{D}_{\bar{i}}\, \mathcal{C}_{jkl} = 0\,, \qquad \mathfrak{D}_{[i}\, \mathcal{C}_{j]kl} = 0\,. \tag{2.42}$$

The period or monodromy matrix $\mathcal{N}$ is defined by the relations

$$\mathcal{M}_\Lambda = \mathcal{N}_{\Lambda\Sigma} \mathcal{L}^\Sigma\,, \qquad h_{\Lambda\, i} = \bar{\mathcal{N}}_{\Lambda\Sigma} f^\Sigma{}_i\,. \tag{2.43}$$

The relation $\langle \mathcal{U}_i \mid \overline{\mathcal{V}} \rangle = 0$ then implies that $\mathcal{N}$ is symmetric, which then also trivializes $\langle \mathcal{U}_i \mid \mathcal{U}_j \rangle = 0$.

From the other basic properties in (2.38) we find

$$\mathcal{L}^\Lambda \Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma} \bar{\mathcal{L}}^\Sigma = -\tfrac{1}{2}\,, \tag{2.44}$$

$$\mathcal{L}^\Lambda \Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma} f^\Sigma{}_i = \mathcal{L}^\Lambda \Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma} \bar{f}^\Sigma{}_{\bar{i}} = 0\,, \tag{2.45}$$

$$f^\Lambda{}_i\, \Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma} \bar{f}^\Sigma{}_{\bar{i}} = -\tfrac{1}{2}\mathcal{G}_{i\bar{i}}\,. \tag{2.46}$$

Further identities that can be derived are

$$(\partial_i \mathcal{N}_{\Lambda\Sigma})\mathcal{L}^\Sigma = -2i\Im\mathrm{m}(\mathcal{N})_{\Lambda\Sigma}\ f^\Sigma{}_i\,, \tag{2.47}$$

$$\partial_i \bar{\mathcal{N}}_{\Lambda\Sigma}\ f^\Sigma{}_j = -2\mathcal{C}_{ijk}\mathcal{G}^{k\bar{k}}\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}\bar{f}^\Sigma{}_{\bar{k}}\,, \tag{2.48}$$

$$\mathcal{C}_{ijk} = f^\Lambda{}_i f^\Sigma{}_j \partial_k \bar{\mathcal{N}}_{\Lambda\Sigma}\,, \tag{2.49}$$

$$\mathcal{L}^\Sigma \partial_{\bar{i}} \mathcal{N}_{\Lambda\Sigma} = 0\,, \tag{2.50}$$

$$\partial_{\bar{i}} \bar{\mathcal{N}}_{\Lambda\Sigma}\ f^\Sigma{}_i = 2i\mathcal{G}_{i\bar{i}}\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}\mathcal{L}^\Sigma\,. \tag{2.51}$$

An important identity one can derive, and that will be used various times in the main text, is given by

$$U^{\Lambda\Sigma} \equiv f^\Lambda{}_i \mathcal{G}^{i\bar{i}} \bar{f}^\Sigma{}_{\bar{i}} = -\tfrac{1}{2}\Im\mathrm{m}(\mathcal{N})^{-1|\Lambda\Sigma} - \bar{\mathcal{L}}^\Lambda \mathcal{L}^\Sigma\,, \tag{2.52}$$

whence $\bar{U}^{\Lambda\Sigma} = U^{\Sigma\Lambda}$.

We can define the graviphoton and matter vector projectors

$$\mathcal{T}_\Lambda \equiv 2i\mathcal{L}_\Lambda = 2i\mathcal{L}^\Sigma \Im\mathrm{m}\,\mathcal{N}_{\Sigma\Lambda}\,, \tag{2.53}$$

$$\mathcal{T}^i{}_\Lambda \equiv -\bar{f}_\Lambda{}^i = -\mathcal{G}^{i\bar{j}} \bar{f}^\Sigma{}_{\bar{j}} \Im\mathrm{m}\,\mathcal{N}_{\Sigma\Lambda}\,. \tag{2.54}$$

Using these definitions and the above properties one can show the following identities for the derivatives of the period matrix:

$$\begin{aligned} \partial_i \mathcal{N}_{\Lambda\Sigma} &= 4\mathcal{T}_{i(\Lambda}\mathcal{T}_{\Sigma)}\,, \\ \partial_{\bar{i}} \mathcal{N}_{\Lambda\Sigma} &= 4\bar{\mathcal{C}}_{\bar{i}\bar{j}\bar{k}}\mathcal{T}^{\bar{i}}{}_{(\Lambda}\mathcal{T}^{\bar{j}}{}_{\Sigma)}\,. \end{aligned} \tag{2.55}$$

Observe that the first of Eqs. (2.35) together with the definition of the period matrix $\mathcal{N}$ imply the following expression for the Kähler potential:

$$e^{-\mathcal{K}} = -2\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}\mathcal{X}^\Lambda \bar{\mathcal{X}}^\Sigma\,. \tag{2.56}$$

If we now assume that $\mathcal{F}_\Lambda$ depends on z^i through the $\mathcal{X}$'s, then from the last equation we can derive that

$$\partial_i \mathcal{X}^\Lambda \left[2\mathcal{F}_\Lambda - \partial_\Lambda \left(\mathcal{X}^\Sigma \mathcal{F}_\Sigma\right)\right] = 0\,. \tag{2.57}$$

If $\partial_i \mathcal{X}^\Lambda$ is invertible as an $n \times \bar{n}$ matrix, then we must conclude that

$$\mathcal{F}_\Lambda = \partial_\Lambda \mathcal{F}(\mathcal{X})\,, \tag{2.58}$$

where $\mathcal{F}$ is a homogeneous function of degree 2, called the *prepotential*.

Making use of the prepotential and the definitions (3.24), we can calculate

$$\mathcal{N}_{\Lambda\Sigma} = \bar{\mathcal{F}}_{\Lambda\Sigma} + 2i\frac{\Im\mathrm{m}\mathcal{F}_{\Lambda\Lambda'}\mathcal{X}^{\Lambda'}\Im\mathrm{m}\mathcal{F}_{\Sigma\Sigma'}\mathcal{X}^{\Sigma'}}{\mathcal{X}^\Omega \Im\mathrm{m}\mathcal{F}_{\Omega\Omega'}\mathcal{X}^{\Omega'}}\,. \tag{2.59}$$

Having the explicit form of $\mathcal{N}$, we can also derive an explicit representation for $\mathcal{C}$ by applying Eq. (2.50). One finds

$$\mathcal{C}_{ijk} = e^{\mathcal{K}}\partial_i \mathcal{X}^\Lambda \partial_j \mathcal{X}^\Sigma \partial_k \mathcal{X}^\Omega \mathcal{F}_{\Lambda\Sigma\Omega}\,, \tag{2.60}$$

so that the prepotential really determines all structures in special geometry.

A last remark has to be made about the existence of a prepotential: clearly, given a holomorphic section Ω a prepotential need not exist. It was shown in Ref. [142], however, that one can always apply an $Sp(2n_v+2,\mathbb{R})$ transformation such that a prepotential exists.

2.2 Homogeneous spaces

The relevant spaces in $\mathcal{N} > 2$ Supergravity are *Irreducible Riemannian Globally Symmetric* (I.R.G.S.) spaces, which are particular instances of homogeneous spaces (see [150–153] and references therein). We will also closely follow the appendix of [131]. We will start by defining the concept of homogeneous space and then we will move to the cases of interest in Supergravity, the symmetric spaces.

Theorem 2.2.1. *Let H be a closed subgroup of a Lie group G, and let G/H be the set $\{\sigma H : \sigma \in G\}$ of left cosets modulo H. Let $\pi : G \to G/H$ denote the natural projection $\pi(\sigma) = \sigma H$. Then G/H has a unique manifold structure such that*

1. *π is C^∞.*
2. *There exist local smooth sections of G/H in G; that is, if $\sigma H \in G/H$, there is a neighbourhood $\mathcal{U}(\sigma H)$ and a C^∞ map $\tau : \mathcal{U}(\sigma H) \to G$ such that $\pi \circ \tau = \mathrm{I}$.*

Definition 2.2.2. Manifolds of the form G/H where G is a Lie group, H is a closed subgroup of G, and the manifold structure is the unique satisfying theorem 2.2.1, are called *homogeneous manifolds.*

Definition 2.2.3. Let $\eta : G \times \mathcal{M} \to \mathcal{M}$ be an action of G on $\mathcal{M}$ on the left. We denote $\eta_\sigma(m) = \eta(\sigma, m)$.

1. The action is called *effective* if the identity e of G is the only element of G for which η_e is the identity map on $\mathcal{M}$.
2. The action is called *transitive* if $\forall\ p, q \in \mathcal{M}\ \ \exists\ \sigma \in G\ /\ \eta_\sigma(p) = q$.
3. Let $p_0 \in \mathcal{M}$ and let $H = \{\sigma \in G : \eta_\sigma(p_0) = p_0\}$. Then H is a closed subgroup of G, the isotropy group at m_0.
4. The action η restricted to H gives an action of H on $\mathcal{M}$ on the left with fixed point m_0. It can then be proven that $\alpha : H \to \mathrm{Aut}(T_{p_0}\mathcal{M})$, where $\alpha(\sigma) = d\eta_\sigma\,(T_{p_0}\mathcal{M})$, is a representation of H, the *linear isotropy group* at p_0.

Theorem 2.2.4. *Let $\eta \times \mathcal{M} \to \mathcal{M}$ be a transitive action of the Lie group G on the manifold $\mathcal{M}$ on the left. Let $p_0 \in \mathcal{M}$, and let H be the isotropy group at p_0. Then the map $\beta : G/H \to \mathcal{M}$ given by $\beta(\sigma H) = \eta_\sigma(p_0)$ is a diffeomorphism.*

Proof. See [152], Theorem 3.62.

If G is a Lie group and H a closed subgroup of G, then there is a natural transitive action l of G on the homogeneous manifold G/H on the left, namely $l : G \times G/H \to G/H$ given by $l(\sigma, \tau H) = \sigma\tau H$.

A manifold G/H of the kind defined in 2.2.2 is called homogeneous because it admits a transitive action of a group, in particular of G. Thanks to theorem 2.2.4 we know that the converse is also true, if a manifold $\mathcal{M}$ admits a transitive action of a group G, then it is diffeomorphic to G/H, where H is the isotropy group of the action.

Theorem 2.2.5. *If the isotropy group H of a homogeneous space G/H is compact, then it can be equipped with a G-invariant Riemannian metric.*

Proof. See [154], theorem 1, chapter 4.

Notice that, under the assumptions of 2.2.1, G/H is a differentiable manifold, but it may not be a Lie group itself. The following theorem gives a sufficient condition for that to happen.

Theorem 2.2.6. *Let G be a Lie group and H a closed normal subgroup of G. Then the homogeneous manifold G/H with its natural group structure is a Lie group.*

Proof. See [152], theorem 3.64.

The non-linear σ-models appearing in extended Supergravity are homogeneous manifolds G/H, but H is in general not a normal subgroup of G, and therefore we cannot use the tools available for group manifolds in order to study them. They are, however, of a particular kind: they are symmetric spaces, a specific type of homogeneous manifold, and in addition, its isotropy group H is compact. Therefore, they can be endowed with a G-invariant

Riemannian metric. We will thus consider Riemannian symmetric spaces, since they are the ones appearing in the non-linear σ-models of extended Supergravity.

A Riemannian symmetric space is a Riemannian manifold $(\mathcal{S}, \mathcal{G})$ with the property that the geodesic reflection at any point is an isometry of $\mathcal{S}$. That is, $\forall p \in \mathcal{M} \ \exists \ s_p \in \ I_{\text{Isometries}}(\mathcal{M})$ with the properties

$$s_p(p) = p\,, \qquad (ds_p)_p = -\mathrm{I}\,. \tag{2.61}$$

As a consequence of this definition, every symmetric space $\mathcal{S}$ is homogeneous: $\mathcal{S}$ can be shown to admit the transitive action of a Lie group G, which is indeed the isometry group of the Riemannian metric $\mathcal{G}$.

Definition 2.2.7. A symmetric space $\mathcal{S}$ is precisely a homogeneous space with a symmetry s_p at some point $p \in \mathcal{S}$.

For our purposes it is therefore better to characterize a symmetric space in the form G/H (which is possible since they are homogeneous), where G is its isometry group and H a closed subgroup of G, through the following result

Theorem 2.2.8. *Let G be a connected Lie group with an involution $\sigma : G \to G$ and a left invariant metric $\mathcal{G}$, which is also right invariant under the closed subgroup $\tilde{K} = \{g \in G\,, g^\sigma = g\}$. Let K be a closed subgroup of G with $\tilde{K}^0 \subset K \subset \tilde{K}$ where $\tilde{K}^0$ denotes the connected component (identity component) of $\tilde{K}$. Then $S = G/H$ is a symmetric space where the metric is induced from the given metric on G. Every symmetric space S arises this way.*

Proof. See [153], theorem 4.1.

Lemma 2.2.9. *A vector space decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is the eigenspace decomposition of a order-two automorphism σ of $\mathfrak{g}$ if and only if*

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}\,, [\mathfrak{h}, \mathfrak{t}] \subset \mathfrak{t}\,, [\mathfrak{t}, \mathfrak{t}] \subset \mathfrak{h}\,, \tag{2.62}$$

A decomposition of Lie algebra $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{h}$ obeying lemma 2.2.9 and such that $\mathrm{ad}(\mathfrak{h})_{\mathfrak{t}}$ is the Lie algebra of a compact subgroup of $GL(\mathfrak{t})$ is called the Cartan decomposition*, and the corresponding involution σ the* Cartan involution. *We have the following result*

Theorem 2.2.10. *Any symmetric space $\mathcal{S}$ determines a Cartan decomposition on the Lie algebra of Killing fields. Vice versa, to any Lie algebra $\mathfrak{g}$ with Cartan decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{h}$ there exists a unique simply connected symmetric space $\mathcal{S} = G/H$ where G is the simply connected Lie group with Lie algebra $\mathfrak{g}$ and H is the connected subgroup with Lie algebra $\mathfrak{h}$.*

Proof. See [153], theorem 4.2.

The extended Supergravity scalar manifolds are symmetric spaces: they are of the form G/H, where G is the non-compact real form of a simple, finite-dimensional, Lie group and a H is its maximal compact subgroup, which is the isotropy group of the manifold. It is equipped with a $G_L \times H_R$ invariant Riemannian metric, which has strictly negative definite signature.

Definition 2.2.11. An Irreducible Riemannian Globally Symmetric space is a symmetric space with strictly negative definite metric signature.[4]

We have arrived therefore to the specific kind of scalar manifolds appearing in the non-linear σ-models of extended four-dimensional Supergravity. In order to implement electromagnetic duality rotations in the theory, as it will be explained in section 3.1, it is needed to embed the group G appearing in the Supergravity scalar manifold G/H into the symplectic group $\mathrm{Sp}(2\bar{n}, \mathbb{R})$, or, going to a complex basis, into $\mathrm{Usp}(\bar{n}, \bar{n})$, a procedure that can be always performed in Supergravity. Therefore, all the scalar manifolds can be described by a $\mathrm{Usp}(\bar{n}, \bar{n})$ matrix U which is constructed in terms of the matrices[5]

$$f \equiv (f^{\Lambda}{}_{IJ}, f^{\Lambda}{}_{i})\,, \qquad h \equiv (h_{\Lambda\, IJ}, h_{\Lambda\, i})\,, \tag{2.63}$$

[4] In our conventions, the σ-model metric is positive definite. Therefore it is minus the metric of the corresponding I.R.G.S. space.

[5] When we multiply these matrices we must include a factor $1/2$ for each contraction of pairs of antisymmetric indices IJ.

which formally are sections of the following trivial flat, symplectic bundle

$$G \times_H \mathbb{R}^{2n} \to G/H \tag{2.64}$$

$I, J = 1, \ldots \mathcal{N}$ are the graviton-supermultiplet, or equivalently $\mathrm{U}(\mathcal{N})$, indices and $i (= 1, \ldots n_v)$ are indices labeling the vector multiplets, and the embedding then imposes that

$$\bar{n} = n + \frac{\mathcal{N}(\mathcal{N}-1)}{2} . \tag{2.65}$$

This information is represented in the following table:[6]

N	3	4	5	6	8
n	n	n	0	1	0
$\bar{n}$	$n+3$	$n+6$	10	16	28

Using the above matrices one can then embed the generic scalar manifolds as

$$U \equiv \tfrac{1}{\sqrt{2}} \begin{pmatrix} f + ih & \bar{f} + i\bar{h} \\ f - ih & \bar{f} - i\bar{h} \end{pmatrix} . \tag{2.66}$$

The condition that $U \in \mathrm{Usp}(\bar{n}, \bar{n})$

$$\begin{aligned} U^{-1} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} U^\dagger \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} U^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ &= \tfrac{1}{\sqrt{2}} \begin{pmatrix} f^\dagger - ih^\dagger & -(f^\dagger + ih^\dagger) \\ -(f - ih) & f + ih \end{pmatrix} , \end{aligned} \tag{2.67}$$

leads to the following conditions for f and h:

$$i(f^\dagger h - h^\dagger f) = 1 , \qquad f^T h - h^T f = 0 . \tag{2.68}$$

In terms of the symplectic sections

$$\mathcal{V}_{IJ} = \begin{pmatrix} f^\Lambda{}_{IJ} \\ h_{\Lambda IJ} \end{pmatrix} , \qquad \mathcal{V}_i = \begin{pmatrix} f^\Lambda{}_i \\ h_{\Lambda i} \end{pmatrix} , \tag{2.69}$$

these constraints take the form[7]

$$\begin{aligned} \langle \mathcal{V}_{IJ} \mid \bar{\mathcal{V}}^{KL} \rangle &= -2i\delta^{KL}{}_{IJ} , \\ \langle \mathcal{V}_i \mid \bar{\mathcal{V}}^j \rangle &= -i\delta_i{}^j , \end{aligned} \tag{2.71}$$

with the rest of the symplectic products vanishing.

The left-invariant Maurer-Cartan 1-form can be split into the Vielbeine P and the connection Ω as follows:

$$\Gamma \equiv U^{-1} dU = \begin{pmatrix} \Omega & \bar{P} \\ P & \bar{\Omega} \end{pmatrix} . \tag{2.72}$$

Thus, the different components of the connection are

$$\Omega = \begin{pmatrix} \Omega^{KL}{}_{IJ} & \Omega^j{}_{IJ} \\ \Omega^{KL}{}_i & \Omega^j{}_i \end{pmatrix} = \begin{pmatrix} i\langle d\mathcal{V}_{IJ} \mid \bar{\mathcal{V}}^{KL} \rangle & i\langle d\mathcal{V}_{IJ} \mid \bar{\mathcal{V}}^j \rangle \\ i\langle d\mathcal{V}_i \mid \bar{\mathcal{V}}^{KL} \rangle & i\langle d\mathcal{V}_i \mid \bar{\mathcal{V}}^j \rangle \end{pmatrix} , \tag{2.73}$$

[6] Observe that $\mathcal{N} = 6$ has $n = 1$, even though there are no vector supermultiplets in this case.

[7] We use the convention

$$\langle \mathcal{A} \mid \mathcal{B} \rangle \equiv \mathcal{B}^\Lambda \mathcal{A}_\Lambda - \mathcal{B}_\Lambda \mathcal{A}^\Lambda . \tag{2.70}$$

and those of the Vielbeine are

$$P = \begin{pmatrix} P_{KLIJ} & P_{jIJ} \\ P_{KLi} & P_{ij} \end{pmatrix} = \begin{pmatrix} -i\langle d\mathcal{V}_{IJ} \mid \mathcal{V}_{KL}\rangle & -i\langle d\mathcal{V}_{IJ} \mid \mathcal{V}_j\rangle \\ -i\langle d\mathcal{V}_i \mid \mathcal{V}_{KL}\rangle & -i\langle d\mathcal{V}_i \mid \mathcal{V}_j\rangle \end{pmatrix} . \tag{2.74}$$

The period matrix $\mathcal{N}_{\Lambda\Sigma}$ is defined by

$$\mathcal{N} = hf^{-1} = \mathcal{N}^T , \tag{2.75}$$

which implies properties which should be familiar from the $N = 2$ case: for instance

$$\mathfrak{D}h_\Lambda = \bar{\mathcal{N}}_{\Lambda\Sigma}\mathfrak{D}f^\Lambda , \qquad h_\Lambda = \mathcal{N}_{\Lambda\Sigma}f^\Sigma , \tag{2.76}$$

and

$$-\tfrac{1}{2}(\Im\mathrm{m}\mathcal{N})^{-1|\Lambda\Sigma} = \tfrac{1}{2}f^\Lambda{}_{IJ}\bar{f}^{\Sigma IJ} + f^\Lambda{}_i\bar{f}^{\Sigma i} , \tag{2.77}$$

which can be derived from the definition of $\mathcal{N}$ and Eq. (2.68).

We also quote the completeness relation

$$\tfrac{1}{2} \mid \mathcal{V}_{IJ}\rangle\langle\bar{\mathcal{V}}^{IJ} \mid \ -\tfrac{1}{2} \mid \bar{\mathcal{V}}^{IJ}\rangle\langle\mathcal{V}_{IJ} \mid \ + \mid \mathcal{V}_i\rangle\langle\bar{\mathcal{V}}^i \mid \ - \mid \bar{\mathcal{V}}^i\rangle\langle\mathcal{V}_i \mid = i . \tag{2.78}$$

Defining the $H_{Aut} \times H_{Matter}$ covariant derivative according to

$$\mathfrak{D}\mathcal{V} = d\mathcal{V} - \mathcal{V}\Omega , \tag{2.79}$$

and using Eq. (2.76) we obtain from (2.73)

$$\Omega^{KL}{}_i = \Omega^j{}_{IJ} = 0 , \tag{2.80}$$

and from (2.74)

$$P_{IJKL} = -2f^\Lambda{}_{IJ}\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}\ \mathfrak{D}f^\Sigma{}_{KL} , \tag{2.81}$$

$$P_{iIJ} = -2f^\Lambda{}_i\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}\ \mathfrak{D}f^\Sigma{}_{IJ} , \tag{2.82}$$

$$P_{ij} = -2f^\Lambda{}_i\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}\ \mathfrak{D}f^\Sigma{}_j . \tag{2.83}$$

The above equation can be inverted to give

$$\mathfrak{D}f^\Lambda{}_{IJ} = \bar{f}^{\Lambda i}P_{iIJ} + \tfrac{1}{2}\bar{f}^{\Lambda KL}P_{IJKL} , \tag{2.84}$$

$$\mathfrak{D}f^\Lambda{}_i = \bar{f}^{\Lambda j}P_{ij} + \tfrac{1}{2}\bar{f}^{\Lambda IJ}P_{iIJ} , \tag{2.85}$$

using Eq. (2.77).

The definition of the covariant derivative leads to the identities

$$\langle\, \mathfrak{D}\mathcal{V} \mid \bar{\mathcal{V}}\,\rangle = 0 , \qquad \langle\, \mathfrak{D}\mathcal{V} \mid \mathcal{V}\,\rangle = \langle\, d\mathcal{V} \mid \mathcal{V}\,\rangle = iP . \tag{2.86}$$

The inverse Vielbeine $\bar{P}^{IJKL}$, $\bar{P}^{iIJ}$, $\bar{P}^{ij}$, satisfy (here A labels the physical fields)

$$\bar{P}^{IJKL\,A}P_{MNOP\,A} = 4!\delta^{IJKL}{}_{MNOP} , \qquad \bar{P}^{iIJ\,A}P_{jKL\,A} = 2\delta^i{}_j\delta^{IJ}{}_{KL} . \tag{2.87}$$

Their crossed products vanish but their products with $P_{ij\,A}$ do not.

We find

$$\langle \mathfrak{D}_A \mathcal{V}_{IJ} \mid \mathfrak{D}_B \bar{\mathcal{V}}^{KL} \rangle = \tfrac{i}{2} P_{IJMNA} \bar{P}^{KLMN}{}_B + i P_{iIJA} \bar{P}^{iKL}{}_B \,, \tag{2.88}$$

$$\langle \mathfrak{D}_A \mathcal{V}_{IJ} \mid \mathfrak{D}_B \bar{\mathcal{V}}^{i} \rangle = \tfrac{i}{2} P_{IJKLA} \bar{P}^{iKL}{}_B + i P_{jIJA} \bar{P}^{ij}{}_B \,, \tag{2.89}$$

$$\langle \mathfrak{D}_A \mathcal{V}_{i} \mid \mathfrak{D}_B \bar{\mathcal{V}}^{j} \rangle = \tfrac{i}{2} P_{iIJA} \bar{P}^{iIJ}{}_B + i P_{ikA} \bar{P}^{jk}{}_B \,, \tag{2.90}$$

while $\langle \mathfrak{D}_A \mathcal{V}_{IJ} \mid \mathfrak{D}_B \mathcal{V}_{KL} \rangle = \langle \mathfrak{D}_A \mathcal{V}_{IJ} \mid \mathfrak{D}_B \mathcal{V}_{i} \rangle = \langle \mathfrak{D}_A \mathcal{V}_{i} \mid \mathfrak{D}_B \mathcal{V}_{j} \rangle = 0$.
Using the definition of the period matrix Eq. (3.39), equation (2.76) and the first of Eqs. (2.68) we get

$$d\mathcal{N} = 4i \Im\mathrm{m}\mathcal{N}\, \mathfrak{D} f f^\dagger\, \Im\mathrm{m}\mathcal{N} \,. \tag{2.91}$$

This expression can be expanded in terms of the Vielbeine, using Eqs. (2.84) and (2.85)

$$d\mathcal{N}_{\Lambda\Sigma} = i \Im\mathrm{m}\mathcal{N}_{\Gamma(\Lambda} \Im\mathrm{m}\mathcal{N}_{\Sigma)\Omega} \left[P_{IJKL} \bar{f}^{\Gamma IJ} \bar{f}^{\Omega KL} + 4 P_{iIJ} \bar{f}^{\Gamma i} \bar{f}^{\Omega IJ} + 4 P_{ij} \bar{f}^{\Gamma i} \bar{f}^{\Omega j} \right] \,. \tag{2.92}$$

and, using Eqs. (2.87) and taking into account that their contraction with P_{ij} does not necessarily vanish, implies

$$\bar{P}^{IJKL\,A} \frac{\partial}{\partial \phi^A} \mathcal{N}_{\Lambda\Sigma} = 4! i \Im\mathrm{m}\mathcal{N}_{\Omega(\Lambda} \Im\mathrm{m}\mathcal{N}_{\Sigma)\Delta} \bar{f}^{\Omega[IJ|} \bar{f}^{\Delta|KL]} \,, \tag{2.93}$$

$$\bar{P}^{iIJ\,A} \frac{\partial}{\partial \phi^A} \mathcal{N}_{\Lambda\Sigma} = 8 i \Im\mathrm{m}\mathcal{N}_{\Omega(\Lambda} \Im\mathrm{m}\mathcal{N}_{\Sigma)\Delta} \bar{f}^{\Omega i} \bar{f}^{\Delta IJ} \,. \tag{2.94}$$

$$\bar{P}^{IJKL\,A} \frac{\partial}{\partial \phi^A} \bar{\mathcal{N}}_{\Lambda\Sigma} = -4 i \Im\mathrm{m}\mathcal{N}_{\Omega(\Lambda} \Im\mathrm{m}\mathcal{N}_{\Sigma)\Delta} \bar{P}^{IJKL\,A} \bar{P}^{ij}{}_A f^{\Omega}{}_i f^{\Delta}{}_j \,, \tag{2.95}$$

$$\bar{P}^{iIJ\,A} \frac{\partial}{\partial \phi^A} \bar{\mathcal{N}}_{\Lambda\Sigma} = -4 i \Im\mathrm{m}\mathcal{N}_{\Omega(\Lambda} \Im\mathrm{m}\mathcal{N}_{\Sigma)\Delta} \bar{P}^{iIJ\,A} \bar{P}^{jk}{}_A f^{\Omega}{}_i f^{\Delta}{}_j \,. \tag{2.96}$$

Using the Maurer-Cartan equations $d\Gamma + \Gamma \wedge \Gamma = 0$ and direct calculations we find that the curvatures of $\Omega^{KL}{}_{IJ}$ and $\Omega^j{}_i$ are

$$\begin{aligned} R^{KL}{}_{IJ} &= d\Omega^{KL}{}_{IJ} + \tfrac{1}{2} \Omega^{KL}{}_{MN} \wedge \Omega^{MN}{}_{IJ} \\ &= -\tfrac{1}{2} \bar{P}^{KLMN} \wedge P_{MNIJ} - \bar{P}^{iKL} \wedge P_{iIJ} && (2.97) \\ &= -i \langle \mathfrak{D} \mathcal{V}_{IJ} \mid \mathfrak{D} \bar{\mathcal{V}}^{KL} \rangle \,, && (2.98) \\ R^j{}_i &= d\Omega^j{}_i + \Omega^j{}_k \wedge \Omega^k{}_i = -\tfrac{1}{2} \bar{P}^{jIJ} \wedge P_{iIJ} - \bar{P}^{ik} \wedge P_{ik} && (2.99) \\ &= -i \langle \mathfrak{D} \mathcal{V}_i \mid \mathfrak{D} \bar{\mathcal{V}}^j \rangle \,. && (2.100) \end{aligned}$$

The vanishing of the curvature of $\Omega^i{}_{IJ}$ leads to

$$\tfrac{1}{2} P_{IJKL} \wedge \bar{P}^{iKL} + P_{jIJ} \wedge \bar{P}^{ij} = -i \langle \mathfrak{D} \mathcal{V}_{IJ} \mid \mathfrak{D} \bar{\mathcal{V}}^i \rangle = 0 \,. \tag{2.101}$$

Chapter 3

Extended ungauged Supergravity in four dimensions

After the mathematical background provided in chapter 2, we are ready to summarize the *geometric* formulation of extended four-dimensional ungauged Supergravity, following [149, 155], where the interested reader will find a more detailed exposition. In the case of $\mathcal{N}=2$ Supergravity, we will focus only on the vector multiplet sector, given that black hole solutions with non-trivial hyper-scalars are believed to be singular since they would have *scalar hair* and it is always consistent to set the hyper-scalars to a constant value. We start by reviewing electromagnetic duality in a class of gravity theories coupled to scalars and vector fields that includes the bosonic sector of any ungauged Supergravity.

3.1 Extended electromagnetic duality

For this section, the basic references are [141, 156, 157]. An excellent review and extension of these works can be found in [158]. We are going to consider four-dimensional theories of the general form

$$I = \int d^4x\sqrt{|g|}\,\{R + \mathcal{G}_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j + 2\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda}{}_{\mu\nu}F^{\Sigma\,\mu\nu} - 2\Re\mathrm{e}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda}{}_{\mu\nu}\star F^{\Sigma\,\mu\nu}\}\ , \tag{3.1}$$

which includes the bosonic sectors of all four-dimensional ungauged supergravities for appropriate σ-model metrics $\mathcal{G}_{ij}(\phi)$ and (complex) kinetic matrix $\mathcal{N}_{\Lambda\Sigma}(\phi)$, with negative-definite imaginary part (see sections 3.2 and 3.3). The indices $i, j, \ldots = 1, \ldots, n_s$ run over the scalar fields and the indices $\Lambda, \Sigma, \ldots = 0, \ldots, n_v$ over the 1-form fields. Their numbers are related only for $\mathcal{N} \geq 2$ supergravity theories.

We denote to the equations of motion corresponding to the action (3.28) by

$$\mathcal{E}_a{}^{\mu} \equiv -\frac{1}{2\sqrt{|g|}}\frac{\delta S}{\delta e^a{}_\mu}\,, \qquad \mathcal{E}_i \equiv -\frac{1}{2\sqrt{|g|}}\frac{\delta S}{\delta\phi^i}\,, \qquad \mathcal{E}_\Lambda{}^{\mu} \equiv \frac{1}{8\sqrt{|g|}}\frac{\delta S}{\delta A^{\Lambda}{}_\mu}\,, \tag{3.2}$$

and denote the Bianchi identities for the vector field strengths by

$$\mathcal{B}^{\Lambda\,\mu} \equiv \nabla_\nu{}^\star F^{\Lambda\,\nu\mu}\,. \tag{3.3}$$

The explicit form of the equations of motion can be found to be

$$\begin{aligned}
\mathcal{E}_{\mu\nu} &= G_{\mu\nu} + \mathcal{G}_{ij}[\partial_\mu\phi^i\partial_\nu\phi^j - \tfrac{1}{2}g_{\mu\nu}\partial_\rho\phi^i\partial^\rho\phi^j] \\
&\quad + 8\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,+}{}_{\mu}{}^{\rho}F^{\Sigma\,-}{}_{\nu\rho}\,, && (3.4)\\
\mathcal{E}_i &= \nabla_\mu(\mathcal{G}_{ij}\partial^\mu\phi^i) - \frac{1}{2}\partial_i\mathcal{G}_{jk}\partial_\rho\phi^j\partial^\rho\phi^k + \partial_i[\tilde{F}_\Lambda{}^{\mu\nu\star}F^{\Lambda}{}_{\mu\nu}]\,, && (3.5)\\
\mathcal{E}_\Lambda{}^{\mu} &= \nabla_\nu{}^\star\tilde{F}_\Lambda{}^{\nu\mu}\,, && (3.6)
\end{aligned}$$

where we have defined the dual vector field strength $\tilde{F}_\Lambda$ by

$$\tilde{F}_{\Lambda\,\mu\nu} \equiv -\frac{1}{4\sqrt{|g|}}\frac{\delta S}{\delta^\star F^\Lambda{}_{\mu\nu}} = \Re\mathrm{e}\mathcal{N}_{\Lambda\Sigma}F^\Sigma{}_{\mu\nu} + \Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}{}^\star F^\Sigma{}_{\mu\nu}\,. \tag{3.7}$$

Let's focus our attention on the equations of motion for the vector fields (that is, the Maxwell identities) $\mathcal{E}_\Lambda{}^\mu$ together with the Bianchi identities $\mathcal{B}^{\Lambda\,\mu}$ and define the *doublet*

$$\mathcal{E}^M_\mu \equiv \begin{pmatrix} \mathcal{B}^\Lambda{}_\mu \\ \\ \mathcal{E}_{\Lambda\ \mu} \end{pmatrix}, \tag{3.8}$$

where $M = (\Lambda, \Lambda)$. The Maxwell and Bianchi identities can be now succinctly written as

$$\mathcal{E}^M_\mu = 0\,, \tag{3.9}$$

and therefore they admit as a symmetry an arbitrary $\mathrm{GL}(2n_v + 2, \mathbb{R})$ rotation acting on M. That is

$$\mathcal{E}^M_\mu = 0 \Rightarrow \mathfrak{A}^M{}_N\mathcal{E}^N_\mu = 0\,, \qquad \mathfrak{A} \in \mathrm{G}L(2n_v + 2, \mathbb{R})\,. \tag{3.10}$$

It is convenient to write $\mathfrak{A}$ in terms of $(n_v + 1) \times (n_v + 1)$ blocks

$$\mathfrak{A} = \begin{pmatrix} D & C \\ \\ B & A \end{pmatrix}, \tag{3.11}$$

These transformations act in the same form on the vector of $2n_v + 2$ two-forms

$$F^M \equiv \begin{pmatrix} F^\Lambda \\ \\ \tilde{F}_\Lambda \end{pmatrix}, \qquad F'^{\,M} = \mathfrak{A}^M{}_N F^N\,. \tag{3.12}$$

However, $\tilde{F}_\Lambda$ is not an independent set of fields, it is related to F^Λ by Eq. (3.7), and therefore we must require the same definition to hold for the transformed $\tilde{F}'_\Lambda$ in terms of the transformed action S' and the transformed $F'^{\,\Lambda}$, that is, we require that

$$\tilde{F}'_{\Lambda\,\mu\nu} \equiv -\frac{1}{4\sqrt{|g|}}\frac{\delta S'}{\delta^\star F'^{\,\Lambda}{}_{\mu\nu}}\,. \tag{3.13}$$

In order to implement (3.13) consistently, we have to consider simultaneously a transformation $\xi \in \mathrm{Diff}\,(\mathcal{M}_{\text{scalar}})$ on the scalar manifold $\mathcal{M}_{\text{scalar}}$, since imposing (3.13) requires the scalar matrix $\mathcal{N}_{\Lambda\Sigma}$ to transform in a prescribed way, which in turn has to be implemented through a transformation of the scalars. Therefore, we assume the existence of a group homomorphism

$$\mathrm{i}:\ \mathrm{Diff}\,(\mathcal{M}_{\text{scalar}}) \to \mathrm{GL}\,(2n_v + 2, \mathbb{R})\,, \tag{3.14}$$

which maps every diffeomorphism $\xi \in \mathrm{Diff}\,(\mathcal{M}_{\text{scalar}})$ to a general linear transformation $\mathrm{i}\,(\xi) \in \mathrm{GL}(2n_v + 2, \mathbb{R})$.

Using the homomorphism i we can define now a simultaneous action of ξ in all of the fields of the theory. Writing schematically $\phi' = \xi\,(\phi)$, we define the action of an arbitrary diffeomorphism $\xi \in \mathrm{Diff}\,(\mathcal{M}_{\text{scalar}})$ on the theory (3.28) to be given by [149]

$$\left\{\phi, F^M, \mathcal{N}_{\Lambda\Sigma}\right\} \xrightarrow{\xi} \left\{\phi', (\mathrm{i}\,(\xi))^M{}_N F^N, \mathcal{N}'_{\Lambda\Sigma}\,(\phi')\right\}\,, \tag{3.15}$$

and consistency with Eq. (3.13) requires that

$$\mathcal{N}'\,(\phi') = (A\mathcal{N}\,(\phi) + B)(C\mathcal{N}\,(\phi) + D)^{-1}\,. \tag{3.16}$$

Furthermore, the transformations must preserve the symmetry of the period matrix, which requires

$$A^T C = C^T A\,, \qquad D^T B = B^T D\,, \qquad A^T D - C^T B = 1\,, \tag{3.17}$$

i.e. the transformations must belong to $\mathrm{Sp}(2n_v+2,\mathbb{R})$. Therefore, the homomorphism $\mathfrak{i}$ must be reduced to

$$\mathfrak{i}:\ \mathrm{Diff}\,(\mathcal{M}_{\mathrm{scalar}}) \to \mathrm{Sp}\,(2n_v+2,\mathbb{R})\ . \tag{3.18}$$

Notice that (3.18) can never be an isomorphism, since $\mathrm{Sp}\,(2n_v+2,\mathbb{R})$ is a finite-dimensional Lie group and $\mathrm{Diff}\,(\mathcal{M}_{\mathrm{scalar}})$ is infinite-dimensional. The above transformation rules for the vector field strength and period matrix imply

$$\Im\mathrm{m}\mathcal{N}' = (C\mathcal{N}^* + D)^{-1\,T}\Im\mathrm{m}\mathcal{N}(C\mathcal{N}+D)^{-1}\,, \qquad F'^{\Lambda\,+} = (C\mathcal{N}^*+D)_{\Lambda\Sigma}F^{\Sigma\,+}\,, \tag{3.19}$$

so the combination $\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,+}{}_{\mu}{}^{\rho}F^{\Lambda\,-}{}_{\nu\rho}$ that appears in the energy-momentum tensor is automatically invariant.

So far, the situation is the following: we have noticed that the set of Maxwell and Bianchi identities admit a global group of linear symmetries given by $\mathrm{GL}\,(2n_v+2,\mathbb{R})$. In order to define an action consistent with (3.13) we have to impose three conditions

1. The transformation group mas be reduced from $\mathrm{GL}\,(2n_v+2,\mathbb{R})$ to $\mathrm{Sp}\,(2n_v+2,\mathbb{R})$
2. The $\mathrm{Sp}\,(2n_v+2,\mathbb{R})$ rotation must be performed together with a transformation on the scalars, given by a diffeomorphism $\xi\in\mathrm{Diff}\,(\mathcal{M}_{\mathrm{scalar}})$.
3. The couplings of scalars and vector fields $\mathcal{N}_{\Lambda\Sigma}$ must transform as indicated in (3.16).

So, as long as our *kinetic matrix* $\mathcal{N}_{\Lambda\Sigma}$ obeys (3.16) we can consistently define *symplectic/duality rotations* on the theory in such a way that they act as symmetries of the Maxwell and the Bianchi identities. Notice that this does not mean at all that the duality rotations are symmetries of the action: duality rotations are not even a symmetry of

$$I_{\mathrm{Maxwell}} = \int d^4x\sqrt{|g|}\left\{+2\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda}{}_{\mu\nu}F^{\Sigma\,\mu\nu} - 2\Re\mathrm{e}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda}{}_{\mu\nu}\star F^{\Sigma\,\mu\nu}\right\}\,, \tag{3.20}$$

i.e., the part of the Lagrangian corresponding to the vector fields. But on top of that, we are considering arbitrary scalar diffeomorphisms, which will not preserve the non-linear σ-model nor the equations of motion for the scalars. Therefore, if we require at least the duality rotations to be symmetries of the equations of motion, we have to consider not arbitrary diffeomorphisms but only isometries of the scalar metric $\mathcal{G}_{ij}(\phi)$, which are exact symmetries of

$$I_{\mathrm{Scalars}} = \int d^4x\sqrt{|g|}\left\{\mathcal{G}_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j\right\}\ . \tag{3.21}$$

Therefore, the homomorphism $\mathfrak{i}$ must be again reduced to

$$\mathfrak{i}:\ \mathrm{Isometries}\,(\mathcal{M}_{\mathrm{scalar}},\mathcal{G}_{ij}) \to \mathrm{Sp}\,(2n_v+2,\mathbb{R})\ . \tag{3.22}$$

Thus, the duality/symplectic transformations, *i.e.* global symmetries of the equations of motion, are the isometries of the scalar manifold. considered as a Riemannian manifold equipped with the metric $\mathcal{G}_{ij}$, which act on the scalars as usual diffeomorphisms and on the vector fields linearly through the homomorphism (3.22), as long as the matrix $\mathcal{N}_{\Lambda|sigma}$ transforms as prescribed by (3.16).

It can checked that the strict symmetries of the Lagrangian are the isometries of the scalar manifold with a block diagonal embedding on the symplectic group $\mathrm{Sp}\,(2n_v+2,\mathbb{R})$, and the symmetries of the Lagrangian up to a total derivative are those whose embedding obeys $C=0$.

3.2 $\mathcal{N}=2,\ d=4$ ungauged Supergravity

$\mathcal{N}=2$ four-dimensional Supergravity makes reference generically to *any four-dimensional theory* of gravity invariant under two supersymmetries. Here we will consider such theory up to two derivatives in the Lagrangian and in the absence of gauings, and in consequence, we call it ungauged four-dimensional classical Supergravity. The matter content of the theory is the following

1. *Gravitational multiplet:* $\left(e^a_\mu, \bar{\psi}^I, \psi_I, A^0\right)$, where e^a_μ is the Vielbein (together with the spin connection one-form ω^{ab}), ψ_I is an $SU(2)$ doublet of gravitino one-forms, and A^0 is the graviphoton one-form.

2. n_v *vector supermultiples:* $\left(A^i, \bar{\lambda}^i_I, \lambda^{i\,I}, z^i\right)$, where A^i $i = 1, \cdots, n_v$ is a one-form, λ^i_I is a zero-form spinor, and z^i is a complex scalar. The scalars z^i parametrize the n_v-dimensional base of a Special Kähler bundle $\mathcal{SV}$.

3. n_h *hypermultiplets:* $(\chi_\alpha, \chi^\alpha, q^u)$, where χ_α is zero-form spinor $(\alpha = 1, \cdots, 2n_h)$, and four real scalars $(u = 1, \cdots, 4n_h)$ which parametrize a $4n_h$-dimensional Quaternionic manifold $\mathcal{HM}$.

It is convenient tu define a new index $\Lambda = (0, i)$, which allows to write all the vector fields of the theory as $\left\{A^\Lambda_\mu\,, \Lambda = 0, \cdots, n_v\right\}$. Since we are interested in bosonic solutions, we will set all the fermions to zero, which is always a consisten truncation. The general bosonic Lagrangian is given by

$$\begin{aligned} S &= \int d^4x\sqrt{|g|}\,\Big(R + \mathcal{G}_{i\bar{j}}\,(z,\bar{z})\,\partial_\mu z^i\partial^\mu\bar{z}^{\bar{j}} + h_{uv}\,(q)\,\partial_\mu q^u\partial^\mu q^v \\ &+ 2\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}\,(z,\bar{z})\,F^\Lambda{}_{\mu\nu}F^{\Sigma\,\mu\nu} - 2\Re\mathrm{e}\mathcal{N}_{\Lambda\Sigma}\,(z,\bar{z})\,F^\Lambda{}_{\mu\nu}\star F^{\Sigma\,\mu\nu}\Big)\ , \end{aligned}$$

Observe that the canonical normalization of the vector fields kinetic terms implies that $\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}$ is negative definite, as is guaranteed by special geometry [142]. The equations of motion for the hyper-scalars corresponding to (3.23) are given by

$$\mathfrak{D}_\mu\partial^\mu q^u = \nabla_\mu\partial^\mu q^u + \Gamma_{vw}{}^u\partial^\mu q^v\partial_\mu q^w = 0\,, \tag{3.23}$$

where $\Gamma_{vw}{}^u$ are the Christoffel symbols of the 2^{nd} kind for the metric h_{uv}. Therefore, it is always consistent to truncate the hyper-scalars to a constant value $q^u = q^u_0$, and we will do so in the sequel, since black hole solutions with non-trivial hyper-scalars are believed to be singular, since the would develop *scalar hair*. Let's state for completeness that the scalars q^u parametrize a Quaternionic manifold [159], *i.e.* a Riemannian manifold of special holonomy, which will not be discussed here.

As a consequence of supersymmetry, the metric $\mathcal{G}_{i\bar{j}}\,(z,\bar{z})$ of the non-linear σ model and *period matrix* $\mathcal{N}_{\Lambda\Sigma}\,(z,\bar{z})$ are constrained in a very precise way. We have the following structure

1. The scalars z^i parametrize a Special Kähler manifold[1]: a holomorphic non-trivial flat tensor bundle $\mathcal{SV} = \mathcal{SM}\otimes\mathcal{L}$ with structural group $Sp\,(2n_v+2,\mathbb{R})\otimes U(1)$.

2. All the couplings of the theory (in the absence of hyper-multiplets) can be constructed in terms of the holomorphic section $\Omega\in\Gamma\,(\mathcal{SM})$ or the covariantly holomorphic symplectic section $\mathcal{V}$.

Since we are interested in bosonic configurations, the only couplings that we need to consider are those of vector fields and scalars, given by the period matrix $\mathcal{N}_{\Lambda\Sigma}\,(z,\bar{z})$. It can be shown that, in terms of the covariantly holomorphic symplectic section $\mathcal{V}$, $\mathcal{N}_{\Lambda\Sigma}\,(z,\bar{z})$ is given by [160]

$$\mathcal{M}_\Lambda = \mathcal{N}_{\Lambda\Sigma}\mathcal{L}^\Sigma\,, \qquad h_{\Lambda\,i} = \mathcal{N}^*{}_{\Lambda\Sigma}f^\Sigma{}_i\,, \tag{3.24}$$

where $\mathcal{M}_\Lambda, \mathcal{L}^\Sigma, h_{\Lambda\,i}$ and $f^\Sigma{}_i$ have been defined in (2.36) and (2.37).

Notice that Eq. (3.24) implies that $\mathcal{N}_{\Lambda\Sigma}\,(z,\bar{z})$ transforms under diffeomorphisms of the base space as required by (3.16). Therefore, we can apply the formalism of section 3.1[2] and conclude that the equations of motion of ungauged $\mathcal{N}=2$ Supergravity enjoy duality invariance. The same conclusion holds in the presence of hyper-multiplets.

To summarize, four-dimensional ungauged $\mathcal{N}=2$ Supergravity in the absence of hyper-multiplets is completely specified once the Special Kähler bundle $\mathcal{SM}$ describing the self-interactions of the vector multiplets is given. As explained in chapter 2, specifying such bundle is equivalent to specify the holomorphic symplectic section, which is equivalente, when it exists, to specify the second-order homogeneous prepotential $F(\mathcal{X})$. Therefore, four-dimensional ungauged $\mathcal{N}=2$ Supergravity in the absence of hyper-multiplets can be completely specified in terms of just a function, the prepotential.

[1] For more details about Special Kähler geometry, see chapter 2 and references therein.
[2] Obviously the action (3.23) is a particular case of (3.28).

3.2.1 Type-IIA String Theory on a Calabi Yau manifold

Let's consider a particular example of ungauged $\mathcal{N}=2$ Supergravity embeddable in String Theory. In particular, we are going to consider the class of Supergravities that are obtained by compactifying Type-IIA String Theory on a Calabi Yau manifold in the absence of fluxes. We will come back to this example in chapter 7, obtaining some of its black hole solutions. As in the previous section, we will omit the hyper-scalar sector.

Type-IIA String Theory on a Calabi-Yau threefold yields four-dimensional, ungauged, $\mathcal{N}=2$ Supergravity, given by [14, 161–163]

$$F(\mathcal{X}) = -\frac{1}{3!}\kappa^0_{ijk}\frac{\mathcal{X}^i\mathcal{X}^j\mathcal{X}^k}{\mathcal{X}^0} + i\frac{\chi\zeta(3)}{2(2\pi)^3} + \frac{i(\mathcal{X}^0)^2}{(2\pi)^3}\sum_{\{d_i\}} n_{\{d_i\}} Li_3\left(e^{2\pi i d_i \frac{\mathcal{X}^i}{\mathcal{X}^0}}\right) \ ,$$

where $Li_3(x)=\sum_{k=1}^{\infty}\frac{x^k}{k^3}$ is the third polylogarithmic function, χ is the Euler characteristic of the Calabi-Yau, $\zeta(3)$ is the Riemann zeta function of 3, $n_{\{d_i\}}$ is the number of genus 0 instantons[3] and κ^0_{ijk} are the classical intersection numbers. Using special coordinates $z^i=\frac{\mathcal{X}^i}{\mathcal{X}^0}$ we can write the prepotential as follows (gauge $\mathcal{X}^0=1$)

$$\mathcal{F} = -\frac{1}{3!}\kappa^0_{ijk}z^iz^jz^k + i\frac{\chi\zeta(3)}{2(2\pi)^3} + \frac{i}{(2\pi)^3}\sum_{\{d_i\}} n_{\{d_i\}} Li_3\left(e^{2\pi i d_i z^i}\right) \ ,$$

The theory defined by (3.25) is extremely involved due to the infinite sum of polylogarithms. We can simplify it by considering the *large-volume* compactification limit $\Im\mathrm{m} z^i \to \infty$, where the prepotential is given by

$$\mathcal{F}_0 = -\frac{1}{3!}\kappa^0_{ijk}z^iz^jz^k \ .$$

Let's consider now the compactification on a specific Calabi-Yau threefold, the Quintic manifold. The effective theory of Type-IIA String Theory compactified on the Quintic C.Y. three-fold, in the large-volume compactification limit, is given by[4] [14, 161]

$$\mathcal{F}_0 = -t^3 \ .$$

Let's construct the bosonic Lagrangian. First, we need the geometric data of the Special Kähler manifold relevant to construct the Lagrangian, that is, the scalar metric and the covariantly holomorphic symplectic section. Using the formulae of chapter (2) we obtain

$$\mathcal{G}_{t\bar{t}} = \frac{-3}{(t-\bar{t})^2}\,, \qquad \mathcal{V}^T = \left(1, t, t^3, -3t^2\right) \ .$$

We can identify the Special Kähler manifold corresponding to (3.25), which is an homogeneous, symmetric space:

$$\mathcal{M} = \frac{SU(1,1)}{U(1)} \tag{3.25}$$

Such geometry would be completely modified if we were to introduce the constant and the polilogarithmic corrections on the prepotential: the scalar manifold would not be an homogeneous space anymore!

With (3.25) we can now compute $\mathcal{N}_{\Lambda\Omega}$, using (3.24). The result is

$$\mathrm{Re}\,\mathcal{N}_{IJ} = \begin{pmatrix} -2\Re^3 & 3\Re^2 \\ 3\Re^2 & -6\Re \end{pmatrix}, \tag{3.26}$$

$$\mathrm{Im}\,\mathcal{N}_{IJ} = \begin{pmatrix} -(\Im^3+3\Re^2\Im) & 3\Re\Im \\ 3\Re\Im & -3\Im \end{pmatrix}, \tag{3.27}$$

where $\Re=\mathrm{Re}(t)$ and $\Im=\mathrm{Im}(t)$. Therefore, the bosonic Lagrangian, in the absence of hyper-multiplets, of Type-IIA String Theory compactified on the mirror Quintic manifold, is finally given by

[3] That is, the number of distinct holomorphic mappings of the genus 0 world-sheet onto holomorphic two-cycles with degrees $\{d_i\}$
[4] Up to an unimportant constat for our purposes,

$$S = \int d^4x\sqrt{|g|}\left(R - \frac{3\partial_\mu t\partial^\mu \bar{t}}{(t-\bar{t})^2} + 2\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}\left(z,\bar{z}\right)F^{\Lambda}{}_{\mu\nu}F^{\Sigma\,\mu\nu} - 2\Re\mathrm{e}\mathcal{N}_{\Lambda\Sigma}\left(z,\bar{z}\right)F^{\Lambda}{}_{\mu\nu}\star F^{\Sigma\,\mu\nu}\right).$$

3.3 $\mathcal{N} > 2,\ d = 4$ ungauged Supergravity

The structure of four-dimensional ungauged $\mathcal{N} > 2$ Supergravity is very similar to the structure of four-dimensional ungauged $\mathcal{N} = 2$ Supergravity coupled to vector multiplets. A similar *symplectic formulation*, that is, based on the construction of a vector bundle with a symplectic structure group over the scalar manifold can be realized, and all the couplings of the theory written in terms of a section of such bundle. The basic reference for this section is [155], where a much more detailed exposition can be found. We will use the notation and conventions of [112, 131].

If we restrict ourselves to theories with maximum spin two, we can conclude that the amount of supersymmetry is constrained to be less or equal than eight $\mathcal{N} \leq 8$[5]. In addition, we will consider only terms up to two derivatives on the Lagrangian. As in the $\mathcal{N} = 2$ case, all the $\mathcal{N} > 2$ Supergravities contain in the lagrangian a non-linear σ model of the form

$$I_{\text{Scalars}} = \int d^4x\sqrt{|g|}\left\{\mathcal{G}_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j\right\}. \tag{3.28}$$

As we saw, for $\mathcal{N} = 2$ ungauged Supergravity coupled to vector multiplets, the scalars parametrize the base space of a Special Kähler bundle, whose defining section $\mathcal{V}$ specifies the $\mathcal{N} = 2$ model, since the Lagrangian, couplings and scalar metric, can be completely specified in terms of $\mathcal{V}$. Here the situation is similar. However the *extra* amount supersymmetry imposes stronger constraints on the theory, and now the Riemannian scalar manifold $(\mathcal{M}_{\text{Scalar}}, \mathcal{G}_{ij})$ has to be an Irreducible Riemannian Globally Symmetric space[6][7] [150]. Therefore, all the σ-models for $\mathcal{N} > 2$ Supergravity are of the form

$$\mathcal{M}_{scalar} = \frac{G}{H} \tag{3.29}$$

where G is the group of isometries of the scalar manifold $(\mathcal{M}_{\text{Scalar}}, \mathcal{G}_{ij})$ and H is its isotropy group. In particular, for $\mathcal{N} > 2$ Supergravity G is the maximally non-compact real form of a simple Lie group, which depends on the particular $\mathcal{N} > 2$ Supergravity. The different groups appearing in Supergravity $\mathcal{N} > 2$ are summarized in table 3.3. The isotropy group H is in turn of the form

$$H = H_{\text{A}} \times H_{\text{M}}, \tag{3.30}$$

where H_{A} corresponds to the automorphisms group of the corresponding supersymmetry algebra, and H_{M} is related to the matter vector multiplets. When matter cannot be incorporated to the theory, namely for $\mathcal{N} > 4$, we have $H_{\text{M}} = \text{I}$. The geometric formulation of $\mathcal{N} > 2$ can be given in terms of an specific fibre bundle over $\frac{G}{H}$, namely, a trivial flat symplectic bundle of the form [155, 158]

$$G \times_H \mathbb{R}^{2n} \to \frac{G}{H}, \tag{3.31}$$

which justifies its construction in section 2.2. The couplings of the theory can be obtained now from the symplectic section $\mathcal{V}_{IJ}$ defined in 2.69.

All the four-dimensional ungauged $\mathcal{N} > 2$ Supergravities can be described in a unified way. In particular, all the Supergravity $\mathcal{N} > 2$ matter contents can be written in the same generic form; we only need to take into account the range of values taken by the U($\mathcal{N}$) R-symmetry indices, denoted by uppercase Latin letters I *etc.* taking on values $1, \cdots, \mathcal{N}$, in each particular case. Only fields and terms that should be considered are those whose number of antisymmetric SU($\mathcal{N}$) indices is correct, i.e. objects with more than $\mathcal{N}$ antisymmetric indices are zero and terms with Levi-Cività symbols $\epsilon^{I_1 \cdots I_M}$ should only be considered when M equals the $\mathcal{N}$ of the supergravity theory under consideration. There are also constraints on the generic fields for specific values of $\mathcal{N}$ that we are going to review.

The generic supergravity multiplet in four dimensions is

[5] That is, thirty two real charges, if we consider Majorana spinors.

[6] See section 2.2 for more details.

[7] Notice that $\mathcal{G}_{ij}$ is minus the metric of the Supergravity σ-model.

$$\left\{e^a{}_\mu, \psi_{I\,\mu}, A^{IJ}{}_\mu, \chi_{IJK}, \chi^{IJKLM},\, P_{IJKL\,\mu}\right\}\,, \quad I,J,\cdots = 1,\cdots,N\,, \tag{3.32}$$

and the generic vector multiplets (labeled by $i = 1,\cdots,n$) are

$$\left\{A_{i\,\mu}, \lambda_{iI}, \lambda_i{}^{IJK}, P_{iIJ\,\mu}\right\}\,. \tag{3.33}$$

The spinor fields $\psi_{I\,\mu}, \chi_{IJK}, \chi^{IJKLM}, \lambda_{iI}, \lambda_i{}^{IJK}$ have positive chirality with the given positions of the SU($\mathcal{N}$) indices.

The scalars of these theories are encoded into the $2\bar{n}$-dimensional ($\bar{n} \equiv n + \frac{\mathcal{N}(\mathcal{N}-1)}{2}$) symplectic sections ($\Lambda = 1,\dots\bar{n}$) $\mathcal{V}_{IJ}$ and $\mathcal{V}_i$ (see section 2.2). They appear in the bosonic sector of the theory via the pullbacks of the Vielbeine $P_{IJKL\mu}$ (Supergravity multiplet) and $P_{iIJ\,\mu}$ (matter multiplets)[8]. There are three instances of theories for which the scalar Vielbeine are constrained: first, when $\mathcal{N} = 4$ the matter scalar Vielbeine are constrained by the SU(4) complex self-duality relation[9]

$$\mathcal{N} \;=\; 4 :: \qquad P^{*\,i\,IJ} \;=\; \tfrac{1}{2}\varepsilon^{IJKL}\, P_{i\,KL}\,. \tag{3.34}$$

Secondly, in $\mathcal{N} = 6$ the scalars in the supergravity multiplet are represented by one Vielbein P_{IJ} and one Vielbein P_{IJKL} related by the SU(6) duality relation

$$\mathcal{N} \;=\; 6 :: \qquad P^{*\,IJ} \;=\; \tfrac{1}{4!}\varepsilon^{IJK_1\cdots K_4}\, P_{K_1\cdots K_4}\,, \tag{3.35}$$

and lastly the $\mathcal{N} = 8$ case, in which the Vielbeine is constrained by the SU(8) complex self-duality relation

$$\mathcal{N} \;=\; 8 :: \qquad P^{*\,I_1\cdots I_4} = \tfrac{1}{4!}\varepsilon^{I_1\cdots I_4 J_1\cdots J_4}\, P_{J_1\cdots J_4}\,. \tag{3.36}$$

These constraints must be taken into account in the action.

The graviphotons $A^{IJ}{}_\mu$ do not appear directly in the theory, rather they only appear through the "dressed" vectors, which are defined by

$$A^\Lambda{}_\mu \equiv \tfrac{1}{2}f^\Lambda{}_{IJ}A^{IJ}{}_\mu + f^\Lambda{}_i A^i{}_\mu\,. \tag{3.37}$$

For $\mathcal{N} > 4$ the larger amount of supersymmetry implies that the theory is unique: no matter can be added and all the fields belong to the gravitational multiplet. Therefore, the scalar manifold is also completely fixed, as can be seen in table 3.3.

As in the $\mathcal{N} = 2$ Supergravity case, we are only interested in the bosonic part of the Lagrangian, since we are going to consider exclusively bosonic solutions. The bosonic Lagrangian is again of the form

$$\begin{aligned} S \;=\; & \int d^4x\sqrt{|g|}\,\big[R + 2\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,\mu\nu}F^{\Sigma}{}_{\mu\nu} - 2\Re\mathrm{e}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,\mu\nu}\star F^{\Sigma}{}_{\mu\nu} \\ & + \tfrac{2}{4!}\alpha_1 P^{*\,IJKL}{}_\mu P_{IJKL}{}^\mu + \alpha_2 P^{*\,iIJ}{}_\mu P_{iIJ}{}^\mu\big]\,, \end{aligned} \tag{3.38}$$

where $\mathcal{N}_{\Lambda\Sigma}$ is the generalization of the $\mathcal{N} = 2$ period matrix, defined in Eq. (3.39), and where the parameters α_1, α_2 are equal to 1 in all cases except for $\mathcal{N} = 4, 6$ and 8 as one needs to take into account the above constraints on the Vielbeine: $\alpha_2 = 1/2$ for $\mathcal{N} = 4$, $\alpha_1 + \alpha_2 = 1$ for $\mathcal{N} = 6$ (the simplest choice being $\alpha_2 = 0$) and $\alpha_1 = 1/2$ for $\mathcal{N} = 8$. The action is good enough to compute the Einstein and Maxwell equations, but not the scalars' equations of motion in the cases in which the scalar Vielbeine are constrained: these constraints have to be properly dealt with and the resulting equations of motion are given below. The period matrix $\mathcal{N}_{\Lambda\Sigma}$ is defined by[10]

$$\mathcal{N} = hf^{-1} = \mathcal{N}^T\,, \tag{3.39}$$

[8] The Vielbeine $P_{ij\,\mu}$ either vanish identically or depend on $P_{IJKL\mu}$ and $P_{iIJ\,\mu}$, depending on the specific value of $\mathcal{N}$. Thus, they are not needed as independent variables to construct the theories.

[9] In order to highlight the fact that an equation holds for a specific $\mathcal{N}$ only, we write a numerical variation of the token "$\mathcal{N} = 4$ ::" to the left of the equation.

[10] See section 2.2 for more details.

$\mathcal{N}$	G/H	R
$\mathcal{N}=3$	$\frac{SU(3,n_v)}{SU(3)\times SU(n_v)}$	$\mathbf{(3+n_v)_c}$
$\mathcal{N}=4$	$\frac{SL(2,\mathbb{R})}{U(1)}\times\frac{SO(6,n_v)}{SO(6)\times SO(n_v)}$	$\mathbf{(2,6+n_v)}$
$\mathcal{N}=5$	$\frac{SU(1,5)}{U(5)}$	**20**
$\mathcal{N}=6$	$\frac{SO^*(12)}{U(6)}$	**32**
$\mathcal{N}=8$	$\frac{E_{7(7)}}{SU(8)/\mathbb{Z}_2}$	**56**

Table 3.1: $N\geqslant 3$ supergravity sequence of groups G of the corresponding $\frac{G}{H}$ symmetric spaces, and their symplectic representations $\mathbf{R}$

Notice that, as in the $\mathcal{N}=2$ case, Eq. (3.39) implies that $\mathcal{N}_{\Lambda\Sigma}$ transforms under diffeomorphisms of the base space as required by (3.16). Therefore, we can apply the formalism of section 3.1[11] and conclude that the equations of motion of ungauged $\mathcal{N}>2$ Supergravity enjoy duality invariance.

As corresponds to the general formalism explained in section 3.1, the isometry group, for each $\mathcal{N}=3,\dots,8$ must be embedded in the corresponding symplectic group, something which is always possible. As explained in section 2.2, the isometry group of I.G.R.S spaces is G itself. The scalar manifold for all $\mathcal{N}>2$ Supergravities, together its symplectic representation, is detailed in table 3.3.

[11]Obviously the action (3.38) is a particular case of (3.28).

Chapter 4

Supergravity black holes

In this chapter we are going to obtain the form of the most general black hole solution of four-dimensional, ungauged Supergravity[1], using the so-called *conform-static* coordinates, since they provide a *universal* way, that is, model independent, to identify the extremal limits and the horizon of the black hole. As a result, these coordinates allow a general study of the attractor behavior of several physical quantities on the event horizon of the black hole, yielding, as a plus, simplified equations of motion.

We will then use the (previously obtained) general form of a Supergravity black hole to identify its *hidden* conformal symmetries, extending them to a full Virasoro algebra, which provides the link to the dual conformal description of the microscopic degrees of freedom of the entropy.

4.1 The general form a Supergravity black hole

We are going to consider black-hole solutions of four-dimensional theories of the general form

$$I = \int d^4x\sqrt{|g|}\left\{R + \mathcal{G}_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j + 2\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda}{}_{\mu\nu}F^{\Sigma\,\mu\nu} - 2\Re\mathrm{e}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda}{}_{\mu\nu}\star F^{\Sigma\,\mu\nu}\right\}\,, \tag{4.1}$$

which, as explained in chapter 3, includes the bosonic sectors of all four-dimensional ungauged supergravities for appropriate σ-model metrics $\mathcal{G}_{ij}(\phi)$ and (complex) kinetic matrix $\mathcal{N}_{\Lambda\Sigma}(\phi)$, with negative-definite imaginary part. The indices $i, j, \ldots = 1, \ldots, n_v$ run over the scalar fields and the indices $\Lambda, \Sigma, \ldots = 0, \ldots, n_v$ over the 1-form fields. Their numbers are related only for $N \geq 2$ supergravity theories. Since we want to obtain static solutions, we consider the metric

$$ds^2 = e^{2U}dt^2 - e^{-2U}\gamma_{\underline{mn}}dx^{\underline{m}}dx^{\underline{n}}\,, \tag{4.2}$$

where $\gamma_{\underline{mn}}$ is a 3-dimensional (*transverse*) Riemannian metric, to be specified later. Using Eq. (4.2) and the assumption of staticity for all the fields, we perform a dimensional reduction over time in the equations of motion that follow from the above general action. We obtain a set of reduced equations of motion that can be written in the form[2]

$$\nabla_{\underline{m}}\left(\mathcal{G}_{AB}\partial^{\underline{m}}\tilde{\phi}^B\right) - \tfrac{1}{2}\partial_A\mathcal{G}_{BC}\partial_{\underline{m}}\tilde{\phi}^B\partial^{\underline{m}}\tilde{\phi}^C = 0\,, \tag{4.3}$$

$$R_{\underline{mn}} + \mathcal{G}_{AB}\partial_{\underline{m}}\tilde{\phi}^A\partial_{\underline{n}}\tilde{\phi}^B = 0\,, \tag{4.4}$$

$$\partial_{[\underline{m}}\psi^\Lambda\partial_{\underline{n}]}\chi_\Lambda = 0\,, \tag{4.5}$$

where all the tensor quantities refer to the 3-dimensional metric $\gamma_{\underline{mn}}$ and where we have defined the metric $\mathcal{G}_{AB}$

$$\mathcal{G}_{AB} \equiv \begin{pmatrix} 2 & & \\ & \mathcal{G}_{ij} & \\ & & 4e^{-2U}\mathcal{M}_{MN} \end{pmatrix}\,, \tag{4.6}$$

[1] In fact, we are going to obtain the form of the most general static, spherically symmetric solution of the action (4.1), which basically covers any theory of gravity coupled to scalars and vector fields, up to two derivatives.

[2] See Ref. [78] for more details.

$$(\mathcal{M}_{MN}) \equiv \begin{pmatrix} (\mathfrak{I} + \mathfrak{R}\mathfrak{I}^{-1}\mathfrak{R})_{\Lambda\Sigma} & -(\mathfrak{R}\mathfrak{I}^{-1})_{\Lambda}{}^{\Sigma} \\ -(\mathfrak{I}^{-1}\mathfrak{R})^{\Lambda}{}_{\Sigma} & (\mathfrak{I}^{-1})^{\Lambda\Sigma} \end{pmatrix}, \quad \mathfrak{R}_{\Lambda\Sigma} \equiv \Re\mathrm{e}\mathcal{N}_{\Lambda\Sigma}\,, \quad \mathfrak{I}_{\Lambda\Sigma} \equiv \Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}\,, \tag{4.7}$$

in the *extended* manifold of coordinates $\tilde{\phi}^A = (U, \phi^i, \psi^\Lambda, \chi_\Lambda)$.

Eqs. (4.3) and (4.4) can be obtained from the three-dimensional effective action

$$I = \int d^3x\sqrt{|\gamma|}\left\{R + \mathcal{G}_{AB}\partial_{\underline{m}}\tilde{\phi}^A\partial^{\underline{m}}\tilde{\phi}^B\right\}, \tag{4.8}$$

to which we have to add as a constraint the Eq. (4.5).

In order to further dimensionally reduce the theory to a mechanical, one-dimensional, problem, we introduce the following transverse metric

$$\gamma_{\underline{mn}}dx^{\underline{m}}dx^{\underline{n}} = \frac{d\tau^2}{W_\kappa^4} + \frac{d\Omega_\kappa^2}{W_\kappa^2}, \tag{4.9}$$

where W_κ is an arbitrary function of τ and $d\Omega_\kappa^2$ is the metric of the 2-dimensional symmetric space of curvature κ and unit radius:

$$d\Omega_{(1)}^2 \equiv d\theta^2 + \sin^2\theta\, d\phi^2\,, \tag{4.10}$$

$$d\Omega_{(-1)}^2 \equiv d\theta^2 + \sinh^2\theta\, d\phi^2\,, \tag{4.11}$$

$$d\Omega_{(0)}^2 \equiv d\theta^2 + d\phi^2\,. \tag{4.12}$$

Notice that for $k = 1$, Eq. (4.2), assuming Eq. (4.9), is the most general spherically symmetric, static metric of a four-dimensional space-time. In the three cases $k = 1, 0, -1$ the equation for $W_\kappa(\tau)$ can be integrated and the result is

$$W_1 = \frac{\sinh r_0\tau}{r_0}, \tag{4.13}$$

$$W_{-1} = \frac{\cosh r_0\tau}{r_0}, \tag{4.14}$$

$$W_0^{\pm} = a e^{\mp r_0\tau}\,. \tag{4.15}$$

a is an arbitrary real constant with dimensions of inverse length and r_0 is an integration constant whose interpretation depends on k. We are interested in the case $k = 1$, corresponds to asymptotically flat, spherically symmetric, static black holes. The case $k = 0$ has been recently studied in Ref. [137] and corresponds to a rich spectrum of Lifshitz-like solutions with hyper-scaling violation. The case $k = -1$ has been studied in [164] and corresponds to topological solutions with a particular singular behavior.

Remarkably enough, in the three cases (4.13), (4.14) and (4.15) we are left with the same equations for the one-dimensional fields, which are given by

$$\frac{d}{d\tau}\left(\mathcal{G}_{AB}\frac{d\tilde{\phi}^B}{d\tau}\right) - \tfrac{1}{2}\partial_A\mathcal{G}_{BC}\frac{d\tilde{\phi}^B}{d\tau}\frac{d\tilde{\phi}^C}{d\tau} = 0\,, \tag{4.16}$$

$$\mathcal{G}_{BC}\frac{d\tilde{\phi}^B}{d\tau}\frac{d\tilde{\phi}^C}{d\tau} = 2r_0^2\,. \tag{4.17}$$

The electrostatic and magnetostatic potentials $\psi^\Lambda, \chi_\Lambda$ only appear through their τ-derivatives. The associated conserved quantities are the magnetic and electric charges p^Λ, q_Λ and can be used to eliminate completely the potentials. The remaining equations of motion can be put in the convenient form

$$U'' + e^{2U}V_{\rm bh} = 0\,, \tag{4.18}$$

$$(U')^2 + \tfrac{1}{2}\mathcal{G}_{ij}\phi^{i\,\prime}\phi^{j\,\prime} + e^{2U}V_{\rm bh} = r_0^2\,, \tag{4.19}$$

$$(\mathcal{G}_{ij}\phi^{j\,\prime})' - \tfrac{1}{2}\partial_i\mathcal{G}_{jk}\phi^{j\,\prime}\phi^{k\,\prime} + e^{2U}\partial_i V_{\rm bh} = 0\,, \tag{4.20}$$

in which the primes indicate differentiation with respect to τ and the so-called *black-hole potential* $V_{\rm bh}$ is given by

$$-V_{\rm bh}(\phi,\mathcal{Q}) \equiv -\tfrac{1}{2}\mathcal{Q}^M\mathcal{Q}^N\mathcal{M}_{MN}\,, \qquad (\mathcal{Q}^M) \equiv \begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix}. \tag{4.21}$$

Eqs. (4.18) and (4.20) can be derived from the effective action

$$I_{\rm eff}[U,\phi^i] = \int d\tau\,\left\{(U')^2 + \tfrac{1}{2}\mathcal{G}_{ij}\phi^{i\,\prime}\phi^{j\,\prime} - e^{2U}V_{\rm bh}\right\}. \tag{4.22}$$

Eq. (4.19) is nothing but the conservation of the Hamiltonian (due to absence of explicit τ-dependence of the Lagrangian) with a particular value of the integration constant r_0^2.

A large number of solutions of this system, for different theories of $\mathcal{N}=2, d=4$ supergravity coupled to vector supermultiplets, have been found (see *e.g.* Refs. [76, 90, 132, 133, 165]), focusing always on the case $k=1$. With this choice of transverse metric, they describe single, charged, static, spherically-symmetric, asymptotically-flat, non-extremal black holes. However, since the equations of motion are exactly the same in the three cases $k=1,0,-1$, these solutions are still solutions if we set $\kappa=0,-1$ in the transverse metric, and therefore they can be used to describe Lifshitz-like and topological solutions. Hence, for each solution of the effective system of equations we can build three different metrics, representing three different, non-equivalent space-times, that solve the equations of motion of the original theory (this is the essence of the *Bh-hvLif-T triality* [138]).

To summarize, and focusing only in the $k=1$ case, the metrics of all spherically symmetric, static, black-hole solutions of the action (4.1) have the general form

$$\begin{aligned} ds^2 &= e^{2U}dt^2 - e^{-2U}\gamma_{\underline{mn}}dx^{\underline{m}}dx^{\underline{n}}\,, \\ \gamma_{\underline{mn}}dx^{\underline{m}}dx^{\underline{n}} &= \left(\frac{r_0}{\sinh r_0\tau}\right)^2\left[\left(\frac{r_0}{\sinh r_0\tau}\right)^2 d\tau^2 + d\Omega^2_{(2)}\right], \end{aligned} \tag{4.23}$$

where r_0 is the non-extremality parameter and $U(\tau)$ is a function of the radial coordinate τ that characterizes each particular solution. In these coordinates the exterior of the event horizon is covered by $\tau\in(-\infty,0)$, the event horizon being located at $\tau\to-\infty$ and the spatial infinity at $\tau\to 0^-$. The interior of the Cauchy horizon (if any) is covered by $\tau\in(\tau_S,\infty)$, the inner horizon being located at $\tau\to+\infty$ while the singularity is located at some finite, positive, value τ_S of the radial coordinate τ [90].

The task of obtaining black hole solutions to the action (4.1) is therefore reduced to find the solution $(U(\tau),\phi^i(\tau))$ of the corresponding effective, one-dimensional, system of ordinary differential equations. All the four-dimensional fields, solving the original, four-dimensional, equations of motion, can be constructed from $(U(\tau),\phi^i(\tau))$[3].

Using (4.23), we can compute the area of a 2-sphere at fixed radial coordinate $\tau=\tau_0$, which is given by

$$A(\tau_0) = 4\pi f^2(\tau_0)e^{-2U(\tau_0)}\,, \tag{4.24}$$

where

$$f(\tau) \equiv \frac{r_0}{\sinh r_0\tau}\,. \tag{4.25}$$

Therefore, the areas of the event and Cauchy horizons, A_+ and A_- respectively, read

$$A_\pm = \lim_{\tau_0\to\mp\infty} A(\tau_0)\,. \tag{4.26}$$

We will use Eq. (4.26) later in order to correctly interpret the near-horizon limits of the massless Klein-Gordon equation.

4.2 Hidden symmetry and the microscopic description of the entropy

In Ref. [166] it was shown that the massless Klein-Gordon equation in the background of the four-dimensional Schwarzschild black hole exhibits a $SL(2,\mathbb{R})$ invariance in the near-horizon limit which extends to spatial infinity

[3]For $\mathcal{N}=2$ Supergravity, we will use a complex notation $(U(\tau),z^i(\tau))$, where the z^i are complex scalars.

at sufficiently low frequencies. Here we will generalize these results to every charged, static, spherically symmetric black-hole solution of (4.1), whose general black hole solution is of the form Eq. (4.23).

In the space-time background given by the metric (4.23), the massless Klein-Gordon equation

$$\frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}g^{\mu\nu}\partial_\nu\Phi\right)=0\,, \tag{4.27}$$

can be written in the form

$$e^{-2U}\partial_t^2\Phi - e^{2U}f^{-4}\partial_\tau^2\Phi - e^{2U}f^{-2}\Delta_{S^2}\Phi = 0\,, \tag{4.28}$$

where $f(\tau)$ has been defined in Eq. (4.25) and

$$\Delta_{S^2}\Phi = \frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\Phi\right) + \frac{1}{\sin^2\theta}\partial_\phi^2\Phi\,, \tag{4.29}$$

is the Laplacian on the round 2-sphere of unit radius. Using the separation ansatz

$$\Phi = e^{-i\omega t}R(\tau)Y_m^l(\theta,\phi)\,, \tag{4.30}$$

and

$$\Delta_{S^2}Y_m^l(\theta,\phi) = -l(l+1)Y_m^l(\theta,\phi)\,, \tag{4.31}$$

we find

$$\omega^2 e^{-4U}f^2R(\tau) + f^{-2}\partial_\tau^2R(\tau) = l(l+1)R(\tau)\,, \tag{4.32}$$

so we can write Eq. (4.28) as

$$\mathcal{K}_4\Phi = l(l+1)\Phi\,, \tag{4.33}$$

where $\mathcal{K}_4$ is the second-order differential operator

$$\mathcal{K}_4 \equiv -e^{-4U}f^2\partial_t^2 + f^{-2}\partial_\tau^2\,. \tag{4.34}$$

In order to exhibit the hidden conformal structure of the given space-time, we want to find a representation of $SL(2,\mathbb{R})$ in terms of first-order differential operators (vector fields) in the $t-\tau$ submanifold, such as the $SL(2,\mathbb{R})$ quadratic Casimir, constructed from those vector fields is equal to the second-order differential operator $\mathcal{K}_4$. Thus, we want to find three real vector fields

$$L_m = a_{mt}\partial_t + a_{m\tau}\partial_\tau\,, \qquad m=0,\pm1\,, \tag{4.35}$$

for some functions $a_{mt}(t,\tau), a_{m\tau}(t,\tau)$, whose Lie brackets satisfy $\mathfrak{sl}(2)$ Lie algebra

$$[L_m, L_n] = (m-n)L_{m+n}\,, \qquad m=0,\pm1\,, \tag{4.36}$$

and such that

$$\mathcal{H}^2 \equiv L_0^2 - \tfrac{1}{2}\left(L_1L_{-1} + L_{-1}L_1\right) = \mathcal{K}_4\,. \tag{4.37}$$

In order to simplify the problem, following [166], we have to make some additional assumptions on the functions $a_{It}(t,\tau), a_{I\tau}(t,\tau)$, Thus, we make the following ansatz

$$L_1 = l(t)\left[-m(\tau)\partial_t + n(\tau)\partial_\tau\right]\,, \tag{4.38}$$

$$L_0 = -\frac{c}{r_0}\partial_t\,, \tag{4.39}$$

$$L_{-1} = -l^{-1}(t)\left[m(\tau)\partial_t + n(\tau)\partial_\tau\right]\,, \tag{4.40}$$

where m and n are functions of τ, l is a function of t and c is a real constant.

Plugging this ansatz into Eq. (4.36) we obtain two differential equations

$$m^2\partial_t\log l + n\partial_\tau m = \frac{c}{r_0}\,, \tag{4.41}$$

$$\frac{c}{r_0}\partial_t\log l = 1\,, \tag{4.42}$$

and plugging it into Eq. (4.37) we obtain three equations

$$m = h\partial_\tau n\,, \tag{4.43}$$

$$m^2 = e^{-4U}f^2 + (c/r_0)^2\,, \tag{4.44}$$

$$n^2 = f^{-2}\,. \tag{4.45}$$

These equations cannot be solved for arbitrary $U(\tau)$: we can find l, m, n as functions of $f(\tau)$ and the constant c

$$l(t) = c_0 e^{r_0 t/c}\,, \qquad n^2(\tau) = f^{-2}\,, \qquad m(\tau) = h\cosh(r_0\tau)\,, \tag{4.46}$$

for some real constant c_0, leaving the following equation for the constant c to be solved:

$$c^2 = \left(e^{-2U}f^2\right)^2\,. \tag{4.47}$$

This equation has only one exact solution, given by $e^U \sim f$, which does not correspond to any asymptotically flat black hole. We have to content ourselves with a range of values of the coordinate τ in which the above equation can be solved approximately. The two ranges that we have identified correspond to the two near-horizon regions (event and Cauchy horizons $\tau \to -\infty$ or $\tau \to -\infty$, respectively) in which

$$\left(e^{-2U}f^2\right)^2 \overset{\tau\to\mp\infty}{\sim} \left(\frac{A_\pm}{4\pi}\right)^2 + \mathcal{O}(e^{\pm r_0\tau}) = c^2 + \mathcal{O}(e^{\pm r_0\tau})\,, \tag{4.48}$$

according to Eq. (4.26).

We conclude that in the geometry of any four-dimensional, charged, static, black-hole solution of a theory of the form (4.1), there are two triplets of vector fields L^+_m and L^-_m, $m = 0, \pm 1$ given by

$$L^\pm_1 = -\frac{e^{r_0\pi t/S_\pm}}{r_0}\left(\frac{S_\pm}{\pi}\cosh(r_0\tau)\partial_t + \sinh(r_0\tau)\partial_\tau\right) \tag{4.49}$$

$$L^\pm_0 = -\frac{S_\pm}{r_0\pi}\partial_t\,, \tag{4.50}$$

$$L^\pm_{-1} = -\frac{e^{-r_0\pi t/S_\pm}}{r_0}\left(\frac{S_\pm}{\pi}\cosh(r_0\tau)\partial_t - \sinh(r_0\tau)\partial_\tau\right)\,, \tag{4.51}$$

where $S_\pm = \frac{A_\pm}{4}$, which generate two $\mathfrak{sl}(2)$ algebras whose quadratic Casimirs

$$\mathcal{H}^{\pm\,2} \equiv (L^\pm_0)^2 - \tfrac{1}{2}\left(L^\pm_1 L^\pm_{-1} + L^\pm_{-1}L^\pm_1\right)\,, \tag{4.52}$$

approximate the massless Klein-Gordon equation in the two near-horizon regions[4]:

$$\mathcal{K}_4\Phi = \left\{-e^{-4U}f^2\partial_t^2 + f^{-2}\partial_\tau^2\right\}\Phi \overset{\tau\to\mp\infty}{\longrightarrow} f^{-2}\left\{-\left(S_\pm/\pi\right)^2\partial_t^2 + \partial_\tau^2\right\}\Phi = \mathcal{H}^{\pm\,2}\Phi\,. \tag{4.54}$$

We can see from Eq. (4.49) that the extremal limit $r_0 \to 0$ is singular. The reason is that the operations of taking the near-horizon limit and of taking the extremal limit $r_0 \to 0$ do not commute.

The $\mathfrak{sl}(2)$ algebra that we have just found can be immediately extended to a complete Witt algebra (or a Virasoro algebra with vanishing central charge) with the commutation relations (4.36) for all $m \in \mathbb{Z}$. The generators of the Witt algebra are given by

$$L^\pm_m = -\frac{e^{mr_0\pi t/S_\pm}}{r_0}\left(\frac{S_\pm}{\pi}\cosh(mr_0\tau)\partial_t + \sinh(mr_0\tau)\partial_\tau\right)\,. \tag{4.55}$$

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[4]Observe that we only approximate some terms (i.e. we keep some sub-dominating terms):

$$e^{-4U}f^2 = f^{-2}(e^{-2U}f^2)^2 \sim f^{-2}\left[\left(\frac{A_\pm}{4\pi}\right)^2 + \mathcal{O}(e^{\pm r_0\tau})\right] \sim f^{-2}\left(\frac{A_\pm}{4\pi}\right)^2 + \mathcal{O}(e^{\pm r_0\tau})\,, \tag{4.53}$$

which is correct to that order. On the other hand, we do not need to restrict ourselves to any particular range of frequencies.

To summarize, we have constructed two Witt algebras which have a well-defined action in the space of solutions to the wave equation in the background of the exterior and interior near-horizon limits of a generic, charged, static black hole solution of (4.1). The two $\mathfrak{sl}(2)$ subalgebras are symmetries of these wave equations, since the wave operators can be seen as their Casimirs, but they are not symmetries of the background metrics which, being essentially the products of Rindler spacetime (locally Minkowski) and spheres, have abelian (in the time-radial part) isometry algebras.

This result generalizes those obtained in Refs. [166–169], and present an opportunity to put to test some conjectures and common lore of this field. To start with, is there a CFT associated to the Witt algebras and can one compute the central charge of the Virasoro algebra? A most naive computation does not seem to give meaningful results. This, of course, does not preclude the possibility that a more rigorous calculation, preceded of careful definitions of the boundary conditions of the fields at the relevant boundaries (which have to be identified first) may give a meaningful answer.

Meanwhile, it is amusing to speculate on the possible consequences of the existence of such a CFT withe left and right sectors whose entropies $S_{\rm R}, S_{\rm L}$ and temperatures would be related to the temperatures and entropies of the outer and inner horizons (T_+, T_- and S_+, S_-, respectively) by

$$S_\pm = S_{\rm R} \pm S_{\rm L}\,, \tag{4.56}$$

$$\frac{1}{T_\pm} = \tfrac{1}{2}\left(\frac{1}{T_{\rm R}} \pm \frac{1}{T_{\rm L}}\right)\,, \tag{4.57}$$

and obeying the fundamental relation

$$S_+ = \frac{\pi^2}{12}(c_{\rm R} T_R + c_{\rm L} T_{\rm L})\,, \tag{4.58}$$

where $c_{\rm L,R}$ are the central charges of the left and right sectors, which will be assumed to be equal $c_{\rm R} = c_{\rm L} = c$.

The temperatures and entropies of the outer and inner horizons are related to the non-extremality parameter r_0 by

$$2S_\pm T_\pm = r_0\,, \tag{4.59}$$

which implies for the temperatures of the left and right sectors

$$4S_{\rm L,R} T_{\rm L,R} = r_0\,. \tag{4.60}$$

In the extremal limit

$$S_{\rm L} \to 0\,, \quad T_{\rm R} \to 0\,, \quad T_\pm \to 0\,, \quad S_\pm \to S_{\rm R}\,, \tag{4.61}$$

and both $S_{\rm R}$ and $T_{\rm L}$ remain finite and are convenient quantities to work with. In particular, we can express the central charge that the CFT should have in order to reproduce the Bekenstein-Hawking entropy consistently with this picture, in terms of these two parameters:

$$c = \frac{12}{\pi^2}\frac{S_{\rm R}}{T_{\rm L}}\,. \tag{4.62}$$

Chapter 5

All the supersymmetric black holes of extended Supergravity

In this chapter we are going to explicitly construct the most general (single and multi-center) supersymmetric black hole metric of $\mathcal{N} > 2$ ungauged Supergravity in four dimensions, using the algorithm provided to that effect in [131][1], where the exhaustive classification of all the time-like supersymmetric solutions of any extended four-dimensional ungauged Supergravity was performed. Although we are going first to specialize to the case of $\mathcal{N} = 8$ Supergravity, thanks to the properties of the *groups of Type* E_7 we will see that our results also apply to all $\mathcal{N} > 2$ Supergravities and also to specifc instances of $\mathcal{N} = 2$, namely those with symmetric scalar manifolds. For the remaining $\mathcal{N} = 2$ Supergravity cases, since the theory (in the absence of hypermultiplets) is specified upon the choice of a Special Kähler manifold, we can only characterize the form of the supersymmetric solution, whose details depend on the particular $\mathcal{N} = 2$ model. Previous results on black holes and attractors in $\mathcal{N} = 8$ Supergravity can be found in [170–182].

5.1 The mathematical formalism

According to the results of [131], in order to construct a timelike black-hole-type supersymmetric solution of $\mathcal{N} = 8$ supergravity we may proceed as follows[2]:

1. Choose an x-dependent rank-2, 8×8 complex antisymmetric M_{IJ}, These matrices must satisfy a number of constraints that are difficult to solve. This implies that, in practice, we cannot construct the most general matrices that satisfy them. Nevertheless, with those matrices we can proceed to the next step.

2. The scalars are encoded into the 56-dimensional symplectic vector

$$(\mathcal{V}^M{}_{IJ}) = \begin{pmatrix} f^{ij}{}_{IJ} \\ h_{ij\,IJ} \end{pmatrix}, \tag{5.1}$$

   antisymmetric in the *local* $SU(8)$ indices $I, J = 1, \cdots 8$. It transforms in the fundamental $(\mathbf{56})$ of $E_{7(7)}$ (ij indices) and as antisymmetric $U(8)$ tensor (IJ indices), It satisfies[3]

$$\langle \mathcal{V}_{IJ} \mid \bar{\mathcal{V}}^{KL} \rangle = \tfrac{1}{2} \bar{f}^{ij\,KL} h_{ij\,IJ} - \tfrac{1}{2} \bar{h}_{ij}{}^{KL} f^{ij}{}_{IJ} = -2i\delta^{KL}{}_{IJ}\,, \qquad \langle \mathcal{V}_{IJ} \mid \mathcal{V}_{KL} \rangle = 0\,, \tag{5.4}$$

   Using the matrix M_{IJ} chosen in the previous step, we define the real symplectic vectors $\mathcal{R}^M$ and $\mathcal{I}^M$

[1] Generalizing the results of the seminal work [104] by Gauntlett *et al.*

[2] We have included in this recipe, to simplify it, the vanishing of the "hyperscalars".

[3] The symplectic product of two vectors $\langle \mathcal{A} \mid \mathcal{B} \rangle$ is defined by

$$\langle \mathcal{A} \mid \mathcal{B} \rangle \equiv \mathcal{A}_M \mathcal{B}^M \equiv \mathcal{A}^N \mathcal{B}^M \Omega_{MN}\,, \tag{5.2}$$

where

$$(\Omega_{MN}) \equiv \begin{pmatrix} 0 & \mathbb{1}_{28\times 28} \\ -\mathbb{1}_{28\times 28} & 0 \end{pmatrix}, \tag{5.3}$$

is the skew metric of $\mathrm{Sp}(56,\mathbb{R})$ that we use to lower (as above) or raise symplectic indices.

$$\mathcal{R}^M + i\mathcal{I}^M \equiv \mathcal{V}^M{}_{IJ}\frac{M^{IJ}}{|M|^2}\,, \qquad |M|^2 = M_{IJ}M^{IJ}\,. \tag{5.5}$$

These two are, by definition, $U(8)$ singlets (no $U(8)$ gauge-fixing necessary) and only transform in the fundamental of $E_{7(7)}$.

3. The components of $\mathcal{I}$ are 56 real functions $\mathcal{H}^M$ harmonic in the Euclidean $\mathbb{R}^3$ transverse space.
4. $\mathcal{R}$ is to be be found from $\mathcal{I}$ exploiting the redundancy in the description of the scalars by the sections $\mathcal{V}^M{}_{IJ}$[4]. Even with the knowledge of M_{IJ} this is a very difficult step.
5. The metric is

$$ds^2 \;=\; e^{2U}(dt+\omega)^2 - e^{-2U}d\vec{x}^{\,2}\,, \tag{5.6}$$

where

$$e^{-2U} \;=\; |M|^{-2} = \langle\,\mathcal{R}\mid\mathcal{I}\,\rangle = \tfrac{1}{2}\mathcal{I}^{ij}\mathcal{R}_{ij} - \tfrac{1}{2}\mathcal{I}_{ij}\mathcal{R}^{ij}\,, \tag{5.7}$$

$$(d\omega)_{mn} \;=\; 2\epsilon_{mnp}\langle\,\mathcal{I}\mid\partial_p\mathcal{I}\,\rangle\,, \tag{5.8}$$

and can be constructed automatically provided one has been given the harmonic functions corresponding to $\mathcal{I}$ and $\mathcal{R}(\mathcal{I})$, quite independently of the construction of these objects from M_{IJ} and $\mathcal{V}^M{}_{IJ}$. The same is true for the vector field strengths.
6. The vector field strengths are given by

$$\mathcal{F} = -\tfrac{1}{2}d(\mathcal{R}\hat{V}) - \tfrac{1}{2}\star(\hat{V}\wedge d\mathcal{I})\,, \qquad \hat{V} = \sqrt{2}e^{2U}(dt+\omega)\,. \tag{5.9}$$

7. The Vielbeins describing the scalars in the coset $E_{7(7)}/SU(8)$ $P_{IJKL,\mu}$ are split into two complementary sets:

$$P_{IJKL}\,\mathcal{J}^I{}_{[M}\mathcal{J}^J{}_N\mathcal{J}^K{}_P\tilde{\mathcal{J}}^L{}_{Q]}\,, \quad \text{and} \quad P_{IJKL}\,\mathcal{J}^I{}_{[M}\tilde{\mathcal{J}}^J{}_N\tilde{\mathcal{J}}^K{}_P\tilde{\mathcal{J}}^L{}_{Q]}\,, \tag{5.10}$$

where we have defined the projectors

$$\mathcal{J}^I{}_J \equiv \frac{2M^{IK}M_{JK}}{|M|^2}\,, \qquad \mathcal{J}^I{}_J = \delta^I{}_J - \tilde{\mathcal{J}}^I{}_J\,. \tag{5.11}$$

All those in the second set have been assumed to vanish from the start, since they would lead to a non-trivial metric in the transverse 3-dimensional space, while those in the first set can in principle be found from $\mathcal{R}$ and $\mathcal{I}$, using the definitions of these vectors and of the Vielbein and the explicit form of the chosen M_{IJ}, setting $\mathcal{I}^M = H^M(x)$ and confronting the third step: the resolution of the stabilization equations.

5.2 The supersymmetric black hole solution of $\mathcal{N}=8$ Supergravity

For the last 20 years, black holes have been intensively studied in string theory and supergravity with never-decreasing interest. A large part of effort has been focused on two subjects: the construction of the most general black-hole solutions of these theories and the understanding and computation of different physical properties, specially the entropy, of the black-hole solutions, following the seminal result of Strominger and Vafa [6].

The attractor mechanism [78, 183] has provided a bridge between these two subjects, allowing the computation of the entropy and other black-hole properties on the black-hole horizon without the knowledge of the complete black-hole solutions, at least in the extremal cases. In theories with a very high degree of (super-)symmetry, though, it is not necessary to use this mechanism and the entropy of the extremal black holes can be

[4] $\mathcal{V}^M{}_{IJ}$ uses 56^2 complex components to describe just 70 physical scalars. The constraints that it satisfies imply a large number of relations between the components. The same is true for the components projected with M_{IJ}. This step is equivalent to the resolution of the *stabilization equations* in $\mathcal{N}=2$ theories.

determined requiring duality-invariance, correct dimensionality and moduli-independence (which is a consequence of the attractor mechanism [78]). In particular, the entropy of the extremal black holes of $\mathcal{N}=8$ supergravity [184, 185] was found in [186] to be given by the unique quartic invariant of the $E_{7(7)}$ duality group. If we use the real basis

$$\mathcal{Q} \equiv \begin{pmatrix} p^{ij} \\ q_{ij} \end{pmatrix}, \tag{5.12}$$

for the charges, where the indices $i,j=1,\cdots,8$ transform homogeneously under the $SL(8,\mathbb{R}) \subset E_{7(7)}$ and each pair of indices is antisymmetrized (so there are 28 electric plus 28 magnetic independent charges), the quartic invariant is known as the Cartan invariant $J_4(\mathcal{Q})$ [187]

$$J_4(\mathcal{Q}) = p^{ij}q_{jk}p^{kl}q_{li} - \tfrac{1}{4}(p^{ij}q_{ij})^2 + \tfrac{1}{96}\,\varepsilon_{ijklmnpq}p^{ij}p^{kl}p^{mn}p^{pq} + \tfrac{1}{96}\,\varepsilon^{ijklmnpq}q_{ij}q_{kl}q_{mn}q_{pq}\,. \tag{5.13}$$

In the complex basis, the quartic invariant is known as the Julia-Cremmer invariant $\diamondsuit(\mathcal{Q})$ [184]. They are equal up to a sign [188, 189] and we will not be concerned with its explicit form.

Although it has not been proven directly[5], the entropy formula for the extremal black holes of $\mathcal{N}=8$ supergravity

$$S = \pi\sqrt{|J_4(\mathcal{Q})|}\,, \tag{5.14}$$

has passed all checks and, in particular, it has been shown to reproduce the entropies of black holes of supergravity theories with $\mathcal{N}<8$ (specially $\mathcal{N}=2$) obtained by truncation of $\mathcal{N}=8$. For supersymmetric black holes $J_4(\mathcal{Q})>0$ and one does not need to take the absolute value.

One of the main obstructions for proving this formula is our lack of knowledge of the general extremal black-hole solutions of $\mathcal{N}=8$ supergravity as opposite to our complete knowledge of those of the $\mathcal{N}=2$ theories [113, 190–193]. This, and the standard lore that all the 1/8 supersymmetric (the ones with a potentially regular horizon) black-hole solutions of $\mathcal{N}=8$ are supersymmetric black-hole solutions of some of the $\mathcal{N}=2$ truncations of that theory (which seems to have been disproven by the explicit examples of [194, 195]) explains why most of the literature on $\mathcal{N}=8$ black holes deals with such truncations.

The supersymmetric black-hole solutions of $\mathcal{N}=2$ supergravity were re-discovered in [113] among the time-like supersymmetric solutions of the theory, which were found by exploiting the integrability conditions of the Killing spinor equations following Tod [102] along the lines of [104]. The same procedure was followed in [131] for all $\mathcal{N}\geq 2, d=4$ ungauged supergravities, using the (almost) $\mathcal{N}$-independent formalism of [155], but the result, which we are going to explain in the next section, looked too complicated to be used in the explicit construction of the solutions, in spite to its similarity to the result found in the $\mathcal{N}=2$ case.

We have recently realized, though, that the results found in [131] do permit the explicit construction of the metric of the most general single and multi-black-hole solutions of ungauged $\mathcal{N}=8$ supergravity. The complications are restricted to the explicit construction of the scalar fields. Thus, we are going to show how to construct the metrics of the most general black holes ungauged $\mathcal{N}=8$ supergravity, but we will not be able to provide a simple algorithm to find the scalar fields corresponding to those solution. Nevertheless, the consistency of the formalism ensures their existence and there is much that can be learned from the metrics.

In the next section we are going to give the general form of the supersymmetric metric solution of $\mathcal{N}=8$ ungauged Supergravity, after which we will discuss the black-hole case, showing how the entropy formula (5.14) arises for supersymmetric black holes and which of $E_{7(7)}$ invariants studied in [196] actually arise in the two-center case.

5.2.1 The metrics of the supersymmetric black-hole solutions of $\mathcal{N}=8$ supergravity

If we want to construct the most general black-hole solutions of $\mathcal{N}=8$ supergravity, the recipe demands a parametrization of the space of all the matrices $M_{IJ}(x)$ that satisfy all the technical requirements, which is very difficult to find.

We have realized, however, that this is a problem that we only need to solve explicitly if we want to construct explicitly the scalar fields. If we are only interested in constructing the metric (and perhaps the vector fields) all we really need is to assume that the problem has been solved and the resulting $M_{IJ}(x)$ has been used to define $\mathcal{R}$ and $\mathcal{I}$.

[5] To the best of our knowledge, not even within the FGK formalism of [78].

One may naively think that both the explicit form of $M_{IJ}(x)$ and the explicit expression of the components $\mathcal{V}^M{}_{IJ}$ are needed to set up the stabilization equations and to solve them, finding $\mathcal{R}$ as a function of $\mathcal{I}$. Fortunately, this problem can be reformulated as follows: with a real vector in the **56** of $E_{7(7)}$, $\mathcal{I}$, we want to construct another one in the same representation which is a non-trivial function of the former, $\mathcal{R}(\mathcal{I})$. For a single $\mathcal{I}$, there is a unique way of constructing a **56** from another **56**, provided by the Jordan triple product[6]. Thus, $\mathcal{R}^M(\mathcal{I})$ must be given by

$$\mathcal{R}^M(\mathcal{I}) \sim (\mathcal{I},\mathcal{I},\mathcal{I})^M \,, \tag{5.15}$$

where

$$\begin{aligned} (\mathcal{I},\mathcal{I},\mathcal{I})^{ij} &= \tfrac{1}{2}\mathcal{I}^{ik}\mathcal{I}_{kl}\mathcal{I}^{lj} + \tfrac{1}{8}\mathcal{I}^{ij}\mathcal{I}_{kl}\mathcal{I}^{kl} - \tfrac{1}{96}\varepsilon^{ijklmnpq}\mathcal{I}_{kl}\mathcal{I}_{mn}\mathcal{I}_{pq}\,, \\ (\mathcal{I},\mathcal{I},\mathcal{I})_{ij} &= -\tfrac{1}{2}\mathcal{I}_{ik}\mathcal{I}^{kl}\mathcal{I}_{lj} - \tfrac{1}{8}\mathcal{I}_{ij}\mathcal{I}_{kl}\mathcal{I}^{kl} + \tfrac{1}{96}\varepsilon_{ijklmnpq}\mathcal{I}^{kl}\mathcal{I}^{mn}\mathcal{I}^{pq}\,. \end{aligned} \tag{5.16}$$

To determine the proportionality factor we must first take into account that we expect the $\mathcal{R}^M(\mathcal{I})$ to be homogenous of first order in $\mathcal{I}$, which requires that we divide $(\mathcal{I},\mathcal{I},\mathcal{I})$ by an $E_{7(7)}$ invariant (to preserve the symmetry properties) homogenous of second degree in $\mathcal{I}$, which can only be $\sqrt{J_4(\mathcal{I})}$.

We, thus, conclude that, up to a normalization constant β to be determined later, the solution to the stabilization equations of $\mathcal{N}=8$ supergravity defined in the previous section is

$$\mathcal{R}^M(\mathcal{I}) = \beta\frac{(\mathcal{I},\mathcal{I},\mathcal{I})^M}{\sqrt{J_4(\mathcal{I})}}\,, \tag{5.17}$$

which is our main result and allows the complete construction of the metrics of all the supersymmetric black holes of the theory.

Actually, since, as we are going to show in the next section, $\beta = 2$, $\mathcal{R}^M$ coincides exactly with the *Freudenthal dual*[7] of $\mathcal{I}^M$, which we can denote by $\tilde{I}^M$ defined in [197]. The Freudenthal dual $\tilde{Q}$ enjoys several remarkable properties. Firstly,

$$\langle\, \tilde{\mathcal{Q}} \mid \mathcal{Q}\,\rangle = 2J_4(\mathcal{Q})\,, \tag{5.18}$$

which follows from the property of the Jordan triple product

$$\langle\, (\mathcal{Q},\mathcal{Q},\mathcal{Q}) \mid \mathcal{Q}\,\rangle = J_4(\mathcal{Q})\,. \tag{5.19}$$

Secondly,

$$\tilde{\tilde{\mathcal{Q}}} = -\mathcal{Q}\,, \tag{5.20}$$

which eliminates a possible solution to the stabilization equations (namely $\mathcal{R}^M = \tilde{\tilde{\mathcal{I}}}^{\mathcal{M}}$) because $e^{-2U} = \langle\, \mathcal{R} \mid \mathcal{I}\rangle$ would vanish identically.

Thirdly,

$$J_4(\tilde{\mathcal{Q}}) = J_4(\mathcal{Q})\,. \tag{5.21}$$

Finally, in [198] (where the definition of Freudenthal dual was generalized to all $\mathcal{N} \geq 2$ theories) is has been shown to be a symmetry of the space of critical points of the black-hole potential introduced in [78].

Thus, following the recipe, and choosing some harmonic functions $H^M(x)$, the metric function e^{-2U} is always given by

$$e^{-2U} = \beta\sqrt{J_4(H)}\,, \tag{5.22}$$

and the 1-form ω is always given by the solution to

[6]The Jordan triple product of three different **56**s is defined only up to terms proportional to the symplectic products of two of the three **56**s. The ambiguity disappears when we consider them to be equal, since the symplectic products will automatically vanish.

[7]We thank M. Duff and L. Borsten for pointing out this fact to us.

$$(d\omega)_{mn} = \varepsilon_{mnp}\left(\mathcal{I}_{ij}\partial_p\mathcal{I}^{ij} - \mathcal{I}^{ij}\partial_p\mathcal{I}_{ij}\right) . \tag{5.23}$$

the vector field strengths follow from the general formula and the scalars, as mentioned before, cannot be easily recover, even if we now introduce an M_{IJ} with all the required properties. This is an evident shortcoming of this procedure, but we believe it is compensated by the possibility of studying explicitly the general black-hole metric.

Observe that, as expected, $\mathcal{R}_M$ can be obtained from the metric function as

$$2\mathcal{R}_M(\mathcal{I}) = \frac{\partial\, e^{-2U}}{\partial\mathcal{I}^M} . \tag{5.24}$$

Furthermore, observe that the expression that we have given for the metric function reduces to those found in [199] for all the magic $\mathcal{N} = 2$ truncations of $\mathcal{N} = 8$ supergravity and another simple truncation also reduces it to that of the well-known STU model. The solution to the stabilization equations of the 4-dimensional supergravities with duality groups of Type $E7$ [200–202] is also given by an analogous expression.

In the next sections we analyze what these formulae mean for 1- and 2-center solutions.

5.2.2 Single supersymmetric black-hole solutions

To study more closely these black-hole metrics it is convenient to introduce the so-called $\mathbb{K}$-tensor [196, 203], which is associated to the completely symmetric linearization of the Cartan invariant performed in [204] (see [205] for more details):

$$\begin{aligned} J_4'(\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3, \mathcal{Q}_4) \;\equiv\;& \tfrac{1}{6}\mathrm{Tr}_{SL(8,\mathbb{R})}\left\{p_1\cdot q_2\cdot p_3\cdot q_4 + p_1\cdot q_3\cdot p_4\cdot q_2 + p_1\cdot q_4\cdot p_2\cdot q_3 + (p\leftrightarrow q)\right\} \\ & -\tfrac{1}{12}\left\{[\mathcal{Q}_1\mid\mathcal{Q}_2][\mathcal{Q}_3\mid\mathcal{Q}_4] + [\mathcal{Q}_1\mid\mathcal{Q}_3][\mathcal{Q}_2\mid\mathcal{Q}_4] + [\mathcal{Q}_1\mid\mathcal{Q}_4][\mathcal{Q}_2\mid\mathcal{Q}_3]\right\} \\ & +\tfrac{1}{4}\left[\mathrm{Pf}_{SL(8,\mathbb{R})}||p_1p_2p_3p_4|| + (p\leftrightarrow q)\right] , \end{aligned} \tag{5.25}$$

where $\mathrm{Tr}_{SL(8,\mathbb{R})}$ stands for the trace of the products of p and q matrices (always one upper and one lower index), we have defined, for convenience, the symmetric product

$$[\mathcal{Q}_1\mid\mathcal{Q}_2] \equiv -\tfrac{1}{2}\mathrm{Tr}_{SL(8,\mathbb{R})}[p_1\cdot q_2 + (p\leftrightarrow q)] , \tag{5.26}$$

and

$$\begin{aligned} \mathrm{Pf}||p_1p_2p_3p_4|| \;&\equiv\; \tfrac{1}{4!}\varepsilon_{ijklmnop}p_1^{ij}p_2^{kl}p_3^{mn}p_4^{op} , \\ \mathrm{Pf}||q_1q_2q_3q_4|| \;&\equiv\; \tfrac{1}{4!}\varepsilon^{ijklmnop}q_{1\,ij}q_{2\,kl}q_{3\,mn}q_{4\,op} . \end{aligned} \tag{5.27}$$

The $\mathcal{K}$-tensor can be defined by its contraction with four different fundamentals:

$$\mathbb{K}_{MNPQ}\,\mathcal{Q}_1{}^M\mathcal{Q}_2{}^N\mathcal{Q}_3{}^P\mathcal{Q}_4{}^Q \equiv J_4'(\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3, \mathcal{Q}_4) , \tag{5.28}$$

and, since J_4' is completely symmetric in the four **56**s, the $\mathbb{K}$-tensor is also completely symmetric in the four symplectic indices

$$\mathbb{K}_{MNPQ} = \mathbb{K}_{(MNPQ)} . \tag{5.29}$$

By construction

$$J_4'(\mathcal{Q}, \mathcal{Q}, \mathcal{Q}, \mathcal{Q}) = J_4(\mathcal{Q}) = \mathbb{K}_{MNPQ}\,\mathcal{Q}^M\mathcal{Q}^N\mathcal{Q}^P\mathcal{Q}^Q , \tag{5.30}$$

and the Jordan triple product can be also written in terms of this tensor as

$$(\mathcal{Q}, \mathcal{Q}, \mathcal{Q})^M = \mathbb{K}^M{}_{NPQ}\,\mathcal{Q}^N\mathcal{Q}^P\mathcal{Q}^Q , \tag{5.31}$$

so we can write the symplectic vector $\mathcal{R}$ (5.17) and the metric function e^{-2U} (5.22) in the more useful form

$$\mathcal{R}_M = \beta \frac{\mathbb{K}_{MNPQ} H^N H^P H^Q}{\sqrt{J_4(H)}}\,, \tag{5.32}$$

$$e^{-2U} = \beta \sqrt{\mathbb{K}_{MNPQ} H^M H^N H^P H^Q}\,. \tag{5.33}$$

Single, extremal, static ($\omega = 0$) black-hole solutions are associated to harmonic functions of the form

$$H^M = A^M + \frac{\mathcal{Q}^M/\sqrt{2}}{r}\,, \qquad r \equiv |\vec{x}|\,, \tag{5.34}$$

where the A^M are constants to be determined in terms of the physical constants of the solution. This is done by requiring asymptotic flatness and absence of sources of NUT charge and using the relation between these constants and the asymptotic values of the scalars (which we do not know explicitly). This means that we will not be able to find the general form of these constants. Nevertheless, let us see how far we can go.

Asymptotic flatness implies

$$|M_\infty|^{-2} = \langle\, \mathcal{R}_\infty \mid \mathcal{I}_\infty \,\rangle = e^{-2U_\infty} = 1\,, \tag{5.35}$$

and requires the normalization

$$\mathbb{K}_{MNPQ} A^M A^N A^P A^Q = \beta^{-2}\,. \tag{5.36}$$

The absence of sources of NUT charge follows from setting $\omega = 0$ in Eq. (5.8):

$$0 = \langle\, A \mid \mathcal{Q} \,\rangle = \Im\mathrm{m}\left(\mathcal{Z}_{\infty\, IJ} M_\infty^{IJ}\right)\,, \tag{5.37}$$

where we have used the definition of $\mathcal{I}$, we have also used asymptotic flatness and the definition of the central charge matrix of $\mathcal{N} = 8$ supergravity

$$\mathcal{Z}_{IJ} \equiv \langle\, \mathcal{V}_{IJ} \mid \mathcal{Q} \,\rangle\,. \tag{5.38}$$

The projection

$$\mathcal{Z} \equiv \tfrac{1}{\sqrt{2}} \mathcal{Z}_{IJ} \frac{M^{IJ}}{|M|^2}\,, \tag{5.39}$$

plays the rôle of central charge for the solutions associated to M^{IJ}, which projects in the $U(8)$ directions in which supersymmetry is preserved. As shown in [179], it drives the flow of the metric function (but not that of the $\mathcal{N} = 8$ scalars). The condition of vanishing NUT charge can be written in the form

$$N = \Im\mathrm{m}\, \mathcal{Z}_\infty = 0\,, \tag{5.40}$$

as in an $N = 2$ theory with central charge $\mathcal{Z}$. As we are going to see the mass of the black hole is given by the real part of $\mathcal{Z}_\infty$ which coincides with the absolute value (because the imaginary part vanishes)[8]:

$$M = |\mathcal{Z}_\infty| = \Re\mathrm{e}\, \mathcal{Z}_\infty = \tfrac{1}{\sqrt{2}} \langle\, \mathcal{R}_\infty \mid \mathcal{Q} \,\rangle = \tfrac{1}{\sqrt{2}} \beta^2 \mathbb{K}_{MNPQ} A^M A^N A^P \mathcal{Q}^Q\,. \tag{5.41}$$

Taking these conditions and relations into account[9], we find that the metric function has the form

$$e^{-2U} = \sqrt{1 + \frac{4M}{r} + \frac{3\beta^2 \mathbb{K}_{MNPQ} A^M A^N \mathcal{Q}^P \mathcal{Q}^Q}{r^2} + \frac{\sqrt{2}\beta^2 \mathbb{K}_{MNPQ} A^M \mathcal{Q}^N \mathcal{Q}^P \mathcal{Q}^Q}{r^3} + \frac{\beta^2 J_4(\mathcal{Q})/4}{r^4}}\,. \tag{5.42}$$

[8] Entirely analogous expressions have been given in [199] for the masses of the black holes of the magic $\mathcal{N} = 2$ truncations of $\mathcal{N} = 8$ supergravity.

[9] We will have to impose additional conditions, like the positivity of the mass, to ensure the regularity of the metric.

The asymptotic behavior confirms the identification of the mass parameter, which, as all the other coefficients of the $1/r^n$ terms in the square root (in particular $J_4(\mathcal{Q})$), has to be positive for the metric to be regular. In the near-horizon limit $r \to 0$, the last term dominates the metric function and we recover the well-known entropy formula (5.14) setting $\beta = 2$. The coefficients of $1/r^2$ and $1/r^3$ do not have a simple expression in terms of the physical parameters.

5.2.3 Supersymmetric 2-center solutions

Multicenter solutions can be constructed by choosing harmonic functions with several poles, as in $\mathcal{N} = 2$ theories [192, 193],

$$H^M = A^M + \sum_a \frac{\mathcal{Q}_a^M/\sqrt{2}}{|\vec{x} - \vec{x_a}|}, \tag{5.43}$$

and tuning the parameters A^M, $\mathcal{Q}_a^M$, $\vec{x}_a$,so the integrability conditions of the equation for ω (5.8)

$$\langle A \mid \mathcal{Q}_a \rangle + \sum_b \frac{\langle \mathcal{Q}_b \mid \mathcal{Q}_a \rangle/\sqrt{2}}{|\vec{x}_a - \vec{x}_b|} = 0. \tag{5.44}$$

Summing the above equations over a and taking into account the antisymmetry of the symplectic product, we find that the constants A, apart from satisfying (5.36), also satisfy the condition (5.37) where $\mathcal{Q} = \sum_a \mathcal{Q}_a$.

When these equations are satisfied, ω exists and describes the total angular momentum of the multi-black-hole system, just as in the $\mathcal{N} = 2$ cases, since the equations are identical.

The (square of) the metric function will contain many terms, up to order $|\vec{x} - \vec{x}_a|^{-4}$. The term of order $|\vec{x} - \vec{x}_a|^{-1}$ has the coefficient

$$M_a \equiv 2\sqrt{2}\mathbb{K}_{MNPQ} A^M A^N A^P \mathcal{Q}_a^Q, \tag{5.45}$$

which corresponds to the mass that the a^{th} center if it was isolated. The mass of the solution is the sum of these parameters $M = \sum_a M_a$.

The coefficient of $|\vec{x} - \vec{x}_a|^{-n} |\vec{x} - \vec{x}_b|^{-m}$ with $m + n = 4$ is one the five quartic invariants listed in [196] for 2-center solutions

$$\begin{aligned}
I_{+2} &= \mathbb{K}_{MNPQ} \mathcal{Q}_a{}^M \mathcal{Q}_a{}^N \mathcal{Q}_a{}^P \mathcal{Q}_a{}^Q = J_4'(\mathcal{Q}_a, \mathcal{Q}_a, \mathcal{Q}_a, \mathcal{Q}_a) = J_4(\mathcal{Q}_a), \\
I_{+1} &= \mathbb{K}_{MNPQ} \mathcal{Q}_a{}^M \mathcal{Q}_a{}^N \mathcal{Q}_a{}^P \mathcal{Q}_b{}^Q = J_4'(\mathcal{Q}_a, \mathcal{Q}_a, \mathcal{Q}_a, \mathcal{Q}_b), \\
I_0 &= \mathbb{K}_{MNPQ} \mathcal{Q}_a{}^M \mathcal{Q}_a{}^N \mathcal{Q}_b{}^P \mathcal{Q}_b{}^Q = J_4'(\mathcal{Q}_a, \mathcal{Q}_a, \mathcal{Q}_b, \mathcal{Q}_b), \\
I_{-1} &= \mathbb{K}_{MNPQ} \mathcal{Q}_a{}^M \mathcal{Q}_b{}^N \mathcal{Q}_b{}^P \mathcal{Q}_b{}^Q = J_4'(\mathcal{Q}_a, \mathcal{Q}_b, \mathcal{Q}_b, \mathcal{Q}_b), \\
I_{-2} &= \mathbb{K}_{MNPQ} \mathcal{Q}_b{}^M \mathcal{Q}_b{}^N \mathcal{Q}_b{}^P \mathcal{Q}_b{}^Q = J_4'(\mathcal{Q}_b, \mathcal{Q}_b, \mathcal{Q}_b, \mathcal{Q}_b), = J_4(\mathcal{Q}_b).
\end{aligned} \tag{5.46}$$

The I_{+2} I_{-2} give the contributions of each center to the entropy.

With more than two centers, other combinations will appear based on the quartic invariant. The sextic invariant found in [196] does not seem to occur in these solutions.

5.3 Supersymmetric black holes and groups of Type E_7

The concept of group of Type E_7 axiomatizes the key properties of the fundamental (symplectic) representation the exceptional group E_7.

E_7 is one of the exceptional simple groups on the classification of all the simple compact groups (or, analogously, complex simple Lie algebras) made by Élie Cartan [187]. The maximally non-compact real form $G = E_{7(7)}$ of E_7 is precisely the group that appears in the coset scalar manifold G/H of $\mathcal{N} = 8$ Supergravity.

The first axiomatic characterization of groups *"of Type E_7"* through a module (irreducible representation) was given in 1967 by Brown [206] (here we will follow [202], see also [201]).

A group G of Type E_7 is a Lie group endowed with a representation $\mathbf{R}$ such that:

1. $\mathbf{R}$ is *symplectic*, *i.e.* (the subscripts "s" and "a" stand for symmetric and skew-symmetric throughout):

$$\exists!\Omega_{[MN]} \equiv \mathbf{1} \in \mathbf{R}\times_a\mathbf{R}; \tag{5.47}$$

   $\Omega_{[MN]}$ defines a non-degenerate skew-symmetric bilinear form, that is, a *symplectic product*$\langle\cdot,\cdot\rangle$: given two different vectors $\mathcal{Q}_1, \mathcal{Q}_2 \in \mathbf{R}$, $\langle\cdot\cdot\rangle$ is defined as

$$\langle \mathcal{Q}_1, \mathcal{Q}_2\rangle \equiv \mathcal{Q}_1^M \mathcal{Q}_2^N \Omega_{MN} = -\langle \mathcal{Q}_2, \mathcal{Q}_1\rangle \,. \tag{5.48}$$

2. $\mathbf{R}$ admits a unique rank-4 completely symmetric primitive G-invariant structure, the so-called K-tensor

$$\exists!\mathbb{K}_{(MNPQ)} \equiv \mathbf{1} \in [\mathbf{R}\times\mathbf{R}\times\mathbf{R}\times\mathbf{R}]_s \,; \tag{5.49}$$

   thus, by contracting the K-tensor with the same vector $\mathcal{Q} \in \mathbf{R}$, we obtain a rank-four homogeneous G-invariant polynomial (here ς is a normalization constant):

$$\mathbf{q}(\mathcal{Q}) \equiv \varsigma\mathbb{K}_{MNPO}\mathcal{Q}^M\mathcal{Q}^N\mathcal{Q}^P\mathcal{Q}^O, \tag{5.50}$$

   which corresponds to the evaluation of the rank-four symmetric invariant $\mathbf{q}$-structure induced by the K-tensor on four identical modules $\mathbf{R}$.

3. If a trilinear map t: $\mathbf{R}\times\mathbf{R}\times\mathbf{R}\to\mathbf{R}$ is defined such that

$$\langle t(\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3), \mathcal{Q}_4\rangle = \mathbf{q}(\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3, \mathcal{Q}_4)\,, \tag{5.51}$$

   then it holds that

$$\langle t(\mathcal{Q}_1, \mathcal{Q}_1, \mathcal{Q}_2), t(\mathcal{Q}_2, \mathcal{Q}_2, \mathcal{Q}_2)\rangle = -2\langle \mathcal{Q}_1, \mathcal{Q}_2\rangle\, \mathbf{q}(\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_2, \mathcal{Q}_2)\,. \tag{5.52}$$

Therefore, for any Supergravity whose scalar manifold is a coset space G/H with G of the E_7 type, we can apply the same discussion that we made in section 5.2.1 for the $\mathcal{N}=8$ case and conclude that *a solution* to the stabilization equations is given by

$$\mathcal{R}^M(\mathcal{I}) = \beta(\mathcal{I},\mathcal{I},\mathcal{I})^M\,, \tag{5.53}$$

where now $t(\cdot,\cdot,\cdot) = (\cdot,\cdot,\cdot)$ is the trilinear product of the corresponding group G of Type E_7. Once we have $t(\cdot,\cdot,\cdot)$ for a particular G corresponding to a Supergravity, the supersymmetric black hole metric can be constructed following exactly the same steps as in the $\mathcal{N}=8$ case.

Remarkably enough, all $2\leq\mathcal{N}\leq 8$-extended four-dimensional Supergravities with symmetric scalar manifolds $\frac{G}{H}$ have G of Type E_7 [197, 202]. The Supergravity duality groups G can be then classified into three different classes [202, 207, 208], simple non-degenerate, simple degenerate and semi-simple non-degenerate, all of them belonging to the class of groups of Type E_7.

Simple non-degenerate A group of Type E_7 is non-degenerate if the quartic form $q(\cdot,\cdot,\cdot,\cdot)$ is absolutely irreducible (irreducible over a separable closure of the base field). The simple non-degenerate four-dimensional Supergravity groups G of Type E_7 are given by [202]

1. $G = E_{7(7)} \to \mathcal{N}=8$ Supergravity.

2. $G = SO^*(12) \to \mathcal{N}=6$ Supergravity.

3. $G = SU(1,5) \to \mathcal{N}=5$ Supergravity.

Simple degenerate The simple degenerate four-dimensional Supergravity groups G of Type E_7 are given by [202]

1. $G = U(3, n_v) \rightarrow \mathcal{N} = 3$ Supergravity coupled to n_v vector multiplets.
2. $G = U(1, n_v) \rightarrow \mathcal{N} = 2$ Supergravity *minimally* coupled to n_v vector multiplets.

For either $G = U(3, n_v)$ or $G = U(1, n_v)$, as well as for generic reducible groups of Type E_7, one can see that the quartic form is reducible [202, 208] as follows

$$\mathbb{K}_{MNPQ} = \alpha \mathbb{S}_{M(N}\mathbb{S}_{PQ)}\,, \tag{5.54}$$

where $\alpha \in \mathbb{R}$ is a constant and $\mathbb{S}$ is the rank-two symmetric symplectic tensor defined by

$$\mathbf{Q}_1^i \overline{\mathbf{Q}}_2^{\bar{j}} \eta_{i\bar{j}} = \mathbb{S}_{MN} \mathcal{Q}_1^M \mathcal{Q}_2^N + i\mathbb{C}_{MN} \mathcal{Q}_1^M \mathcal{Q}_2^N\,, \tag{5.55}$$

where $\eta_{i\bar{j}}$ is the invariant metric of the fundamental irrep. $\mathbf{r} + \mathrm{s}$ of $U(r, s)$, and $\mathbf{Q}_x^i$ and $\mathbf{Q}_x^i$ are the charge vectors in the *complex* (manifestly $U(r, s)$-covariant) symplectic frame. It follows that

$$q(\mathcal{Q}, \mathcal{Q}, \mathcal{Q}, \mathcal{Q}) = \gamma \left(\mathbb{S}_{MN} \mathcal{Q}^M \mathcal{Q}^N\right)^2\,, \tag{5.56}$$

for an appropriate number γ. This means that for these Supergravities the metric factor e^{-2U} can be written simply as

$$e^{-2U} \sim \left|\mathbb{S}_{MN} H^M H^N\right|\,, \tag{5.57}$$

which indeed can be explicitly checked by solving the stabilization equations by algebraic methods.

Semi-simple non-degenerate The semi-simple non-degenerate four-dimensional Supergravity groups G of Type E_7 are given by [202]

1. $G = SL(2, \mathbb{R}) \times SO(6, n_v) \rightarrow \mathcal{N} = 4$ Supergravity coupled to n_v vector multiplets.
2. $G = SL(2, \mathbb{R}) \times SO(2, n_v) \rightarrow \mathcal{N} = 2$ Supergravity coupled to $n_v + 1$ vector multiplets.

5.4 $\mathcal{N} = 2$ Supergravity supersymmetric black holes

The case of $\mathcal{N} = 2$ Supergravity cannot be solved in general since the scalar manifold is not specified unambiguously. As explained in chapter 3, supersymmetry only constraint the manifold spanned by the scalar fields on the vector multiplets to be of Special Kähler type, a condition which does not fix the manifold unequivocally. What can been done, however, is to characterize and classify the most general supersymmetric solutions of the theory, a knowledge that will be of outermost importance when trying to propose a formalism to deal with general (not only supersymmetric) black hole solutions .

The supersymmetric solutions of Supergravity, and in particular of $\mathcal{N} = 2$, can be classified in two classes, the time-like and null, regarding the casual character of the Killing vector constructed from the Killing spinors and the Clifford-algebra γ-matrices [102, 104–110, 119, 120]. The black hole solutions belong to the time-like class, so that's the particular class that we are going to review here, assuming from the beginning that the hyper-scalars have been set to a constant value, since otherwise we don't expect regular solutions. Notice that this is a particular case of the exposition made in section 5.1, but we include it here since it is of capital importance for chapter 6.

The black hole supersymmetric metric is given by

$$ds^2 = e^{2U}(dt + \omega)^2 - e^{-2U}\delta_{mn} dx^m dx^n\,. \tag{5.58}$$

Notice that the only difference respect to the general Supergravity black hole metric (4.2) (assuming staticity $\omega = 0$) is that now the transverse metric is flat, as corresponds in (4.2) to the extremal limit $r_0 \rightarrow 0$. That is, supersymmetry always implies extremalily[10], but not the other way around [209, 210].

[10] Of course, this conclusion relies on the assumption that r_0 can be interpreted as the extremality parameter for all the Supergravity black holes.

We take now the covariantly holomorphic section $\mathcal{V}$ of Special Kahler geometry and a function $X(z,\bar{z})$, with Kähler weight such that $\mathcal{V}/X$ is Kähler neutral, and define

$$\mathcal{R} + i\mathcal{I} = \mathcal{V}/X\,. \tag{5.59}$$

Then, it can be shown that a for a supersymmetric black hole we have

$$e^{-2U} = \langle \mathcal{R} \mid \mathcal{I} \rangle\,, \tag{5.60}$$

$$(d\omega)_{xy} = 2\epsilon_{xyz}\langle\, \mathcal{I} \mid \partial^z \mathcal{I}\,\rangle\,. \tag{5.61}$$

$$\mathcal{I}^M = a^M - \frac{\mathcal{Q}^M}{\sqrt{2}}\tau\,. \tag{5.62}$$

The vector field strengths are given by

$$F = -\tfrac{1}{\sqrt{2}}\{d[e^{2U}\mathcal{R}(dt+\omega)] - {}^\star[e^{-2U}d\mathcal{I} \wedge (dt+\omega)]\}\,, \tag{5.63}$$

and the scalar fields z^i can be computed by taking the quotients

$$z^i = (\mathcal{V}/X)^i/(\mathcal{V}/X)^0\,. \tag{5.64}$$

Given $\mathcal{I}$, $\mathcal{R} \equiv \Re\mathrm{e}(\mathcal{V}/X)$ can in principle be found by solving the generalized stabilization, which depend on the specific model under consideration. Solving the stabilization equations completely determines the solution, since all the physical fields can be constructed in terms of $\mathcal{I}^M = a^M - \frac{\mathcal{Q}^M}{\sqrt{2}}\tau$, as it can be checked from the previous formulae.

Chapter 6

The H-F.G.K. formalism

In this chapter we are going to introduce a new formalism, the so-called H-F.G.K. formalism, based on a dimensional reduction of the original four-dimensional action and the use of a new set of duality-covariant variables inspired by supersymmetry, which eases the explicit construction of non-supersymmetric black hole solutions of $\mathcal{N}=2$ four-dimensional ungauged Supergravity.

6.1 H-FGK for $\mathcal{N}=2$, $d=4$ supergravity

In [132, 165] it was shown that the search of single, static, spherically-symmetric black-hole solutions of an $\mathcal{N}=2$, $d=4$ supergravity coupled to n_v vector multiplets (and, correspondingly, including n_v complex scalars z^i and n_v+1 Abelian vector fields $A^{\Lambda}{}_{\mu}$) with electric (q_Λ) and magnetic (p^Λ) charges described by the $2(n_v+1)$-dimensional symplectic vector $(\mathcal{Q}^M)\equiv(p^\Lambda,q_\Lambda)^T$, is remarkably simplified by going to a new set of $2(n_v+1)$ variables H^M which form a linear, symplectic, representation of the U-duality group and that become harmonic functions on euclidean $\mathbb{R}^3$ for supersymmetric black hole solutions.

We proceed now to describe the change of variables, from those defining a black-hole solution for given electric and magnetic charges $(\mathcal{Q}^M)=(p^\Lambda,q_\Lambda)^{\mathrm{T}}$, namely the metric function U and the complex scalars z^{i}[1], to the variables $(H^M)=(H^\Lambda,H_\Lambda)^{\mathrm{T}}$ that have the same transformation properties as the charges. There is an evident mismatch between these two sets of variables, because U is real. For consistency we will introduce a complex variable X of the form[2]

$$X=\tfrac{1}{\sqrt{2}}e^{U+i\alpha}\,, \tag{6.1}$$

although the phase α does not occur in the original FGK formalism. The change of variables will then be well defined, and the absence of α will lead to a constraint on the new set of variables: this constraint is related to the absence of NUT charge, a possibility which in $d=4$ is allowed for by spherical symmetry.

The theory is specified by the prepotential[3] $\mathcal{F}$, a homogeneous function of second degree in the complex coordinates $\mathcal{X}^\Lambda$. Consequently, defining

$$\mathcal{F}_\Lambda\equiv\frac{\partial\mathcal{F}}{\partial\mathcal{X}^\Lambda}\quad\text{and}\quad\mathcal{F}_{\Lambda\Sigma}\equiv\frac{\partial^2\mathcal{F}}{\partial\mathcal{X}^\Lambda\partial\mathcal{X}^\Sigma}\,,\quad\text{we have:}\quad\mathcal{F}_\Lambda=\mathcal{F}_{\Lambda\Sigma}\mathcal{X}^\Sigma\,. \tag{6.2}$$

Since the matrix $\mathcal{F}_{\Lambda\Sigma}$ is homogenous of degree zero and X has the same Kähler weight as the covariantly holomorphic section

$$(\mathcal{V}^M)=\begin{pmatrix}\mathcal{L}^\Lambda\\ \mathcal{M}_\Lambda\end{pmatrix}=e^{\mathcal{K}/2}\begin{pmatrix}\mathcal{X}^\Lambda\\ \mathcal{F}_\Lambda\end{pmatrix}, \tag{6.3}$$

where $\mathcal{K}$ is the Kähler potential, we also find

$$\frac{\mathcal{M}_\Lambda}{X}=\mathcal{F}_{\Lambda\Sigma}\frac{\mathcal{L}^\Sigma}{X}\,. \tag{6.4}$$

[1]See section 4.1 for more details.

[2] In this section we will be following the conventions of Ref. [113], where the function X appears as a scalar bilinear built out of the Killing spinors.

[3]We only use the prepotential here to determine quickly the homogeneity properties of the objects we are going to deal with. These properties are, however, valid for any $\mathcal{N}=2$ theory in any symplectic frame, whether or not a prepotential exists.

Defining the Kähler-neutral, real, symplectic vectors $\mathcal{R}^M$ and $\mathcal{I}^M$ by

$$\mathcal{R}^M = \Re\mathfrak{e}\,\mathcal{V}^M/X\,, \qquad \mathcal{I}^M = \Im\mathfrak{m}\,\mathcal{V}^M/X\,, \tag{6.5}$$

and using the symplectic metric

$$(\Omega_{MN}) \equiv \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix} \tag{6.6}$$

as well as its inverse Ω^{MN} to lower and raise the symplectic indices according to the convention

$$\mathcal{R}_M = \Omega_{MN}\mathcal{R}^N\,, \qquad \mathcal{R}^M = \mathcal{R}_N\Omega^{NM}\,, \tag{6.7}$$

one can rewrite the complex relation (6.4) in the real form

$$\mathcal{R}_M = -\mathcal{M}_{MN}(\mathcal{F})\mathcal{I}^N\,. \tag{6.8}$$

The symmetric symplectic matrix

$$\mathcal{M}(\mathcal{A}) \equiv \begin{pmatrix} \Im\mathfrak{m}\,\mathcal{A}_{\Lambda\Sigma} + \Re\mathfrak{e}\,\mathcal{A}_{\Lambda\Omega}\,\Im\mathfrak{m}\,\mathcal{A}^{-1|\,\Omega\Gamma}\,\Re\mathfrak{e}\,\mathcal{A}_{\Gamma\Sigma} & -\Re\mathfrak{e}\,\mathcal{A}_{\Lambda\Omega}\,\Im\mathfrak{m}\,\mathcal{A}^{-1|\,\Omega\Sigma} \\ & \\ -\Im\mathfrak{m}\,\mathcal{A}^{-1|\,\Lambda\Omega}\,\Re\mathfrak{e}\,\mathcal{A}_{\Omega\Sigma} & \Im\mathfrak{m}\,\mathcal{A}^{-1|\,\Lambda\Sigma} \end{pmatrix}, \tag{6.9}$$

can be associated with any symmetric complex matrix $\mathcal{A}_{\Lambda\Sigma}$ with a non-degenerate imaginary part (such as $\mathcal{F}_{\Lambda\Sigma}$ and the period matrix $\mathcal{N}_{\Lambda\Sigma}$). The inverse of $\mathcal{M}_{MN}$, denoted by $\mathcal{M}^{MN}$, is the result of raising the indices with the inverse symplectic metric.

It is also immediate to prove the relation

$$d\mathcal{R}_M = -\mathcal{M}_{MN}(\mathcal{F})\,d\mathcal{I}^N\,. \tag{6.10}$$

From this equality, its inverse and the symmetry properties of $\mathcal{M}_{MN}$ we can derive the following relation between partial derivatives (see *e.g.* [211]):

$$\frac{\partial\mathcal{I}^M}{\partial\mathcal{R}_N} = \frac{\partial\mathcal{I}^N}{\partial\mathcal{R}_M} = -\frac{\partial\mathcal{R}^M}{\partial\mathcal{I}_N} = -\frac{\partial\mathcal{R}^N}{\partial\mathcal{I}_M} = -\mathcal{M}^{MN}(\mathcal{F})\,. \tag{6.11}$$

Since we want the new variables to become harmonic functions on euclidean $\mathbb{R}^3$, we introduce two dual sets of variables H^M and $\tilde{H}_M$ and replace the original $n+1$ fields X, z^i by the $2n+2$ real variables $H^M(\tau)$

$$\mathcal{I}^M(X, Z, X^*, Z^*) = H^M\,. \tag{6.12}$$

The dual variables $\tilde{H}^M$ can be identified with $\mathcal{R}^M$, which we can express as functions of the H^M through Eq. (6.8). This gives $\mathcal{V}^M/X$ as a function of the H^M. The physical fields can then be recovered by

$$z^i = \frac{\mathcal{V}^i/X}{\mathcal{V}^0/X} \qquad \text{and} \qquad e^{-2U} = \frac{1}{2|X|^2} = \mathcal{R}_M\mathcal{I}^M\,. \tag{6.13}$$

The phase of X, α, can be found by solving the differential equation (*cf.* Eqs. (3.8), (3.28) in Ref. [212])

$$\dot{\alpha} = 2|X|^2\dot{H}^M H_M - \mathcal{Q}_\star\,, \qquad \text{where} \qquad \mathcal{Q}_\star = \tfrac{1}{2i}\dot{z}^i\partial_i\mathcal{K} + \text{c.c.} \tag{6.14}$$

is the pullback of the Kähler connection 1-form

$$\mathcal{Q}_\star = \tfrac{1}{2i}\dot{z}^i\partial_i\mathcal{K} + \text{c.c.} \tag{6.15}$$

Having detailed the change of variables, we want to rewrite the FGK action for static, spherically symmetric solutions of $\mathcal{N}=2$, $d=4$ Supergravity [78], *i.e.*

$$I_{\text{FGK}}[U, z^i] = \int d\tau\left\{(\dot{U})^2 + \mathcal{G}_{ij^*}\dot{z}^i\dot{z}^{*\,j^*} - \tfrac{1}{2}e^{2U}\mathcal{M}_{MN}(\mathcal{N})\mathcal{Q}^M\mathcal{Q}^N + r_0^2\right\}, \tag{6.16}$$

in terms of the variables H^M. We start by defining the function $\mathsf{W}(H)$, which can be identified with the *Hesse potential*

$$\mathsf{W}(H) \equiv \tilde{H}_M(H)H^M = e^{-2U} = \frac{1}{2|X|^2}\,, \tag{6.17}$$

which is homogenous of second degree in the H^M. Using the properties (6.11) one can show that

$$\partial_M \mathsf{W} \equiv \frac{\partial \mathsf{W}}{\partial H^M} = 2\tilde{H}_M\,, \tag{6.18}$$

$$\partial^M \mathsf{W} \equiv \frac{\partial \mathsf{W}}{\partial \tilde{H}_M} = 2H^M\,, \tag{6.19}$$

$$\partial_M\partial_N \mathsf{W} = -2\mathcal{M}_{MN}(\mathcal{F})\,, \tag{6.20}$$

$$\mathsf{W}\,\partial_M\partial_N \log \mathsf{W} = 2\mathcal{M}_{MN}(\mathcal{N}) + 4\mathsf{W}^{-1}H_M H_N\,, \tag{6.21}$$

where the last property is based on the following relation[4]

$$-\mathcal{M}_{MN}(\mathcal{N}) = \mathcal{M}_{MN}(\mathcal{F}) + 4\mathcal{V}_{(M}\mathcal{V}^*_{N)}\,. \tag{6.22}$$

Using the special geometry identity $\mathcal{G}_{ij^*} = -i\mathcal{D}_i\mathcal{V}_M\mathcal{D}_{j^*}\mathcal{V}^{*\,M}$, we can rewrite the effective action in the form

$$-I_{\rm eff}[H] = \int d\tau \left\{ \tfrac{1}{2}\partial_M\partial_N \log \mathsf{W}\left(\dot{H}^M\dot{H}^N + \tfrac{1}{2}\mathcal{Q}^M\mathcal{Q}^N\right) - \Lambda - r_0^2 \right\}, \tag{6.23}$$

where we have defined

$$\Lambda \equiv \left(\frac{\dot{H}^M H_M}{\mathsf{W}}\right)^2 + \left(\frac{\mathcal{Q}^M H_M}{\mathsf{W}}\right)^2. \tag{6.24}$$

The τ-independence of the Lagrangian implies the conservation of the Hamiltonian $\mathcal{H}$

$$\mathcal{H} \equiv -\tfrac{1}{2}\partial_M\partial_N \log \mathsf{W}\left(\dot{H}^M\dot{H}^N - \tfrac{1}{2}\mathcal{Q}^M\mathcal{Q}^N\right) + \left(\frac{\dot{H}^M H_M}{\mathsf{W}}\right)^2 - \left(\frac{\mathcal{Q}^M H_M}{\mathsf{W}}\right)^2 - r_0^2 = 0\,. \tag{6.25}$$

The equations of motion can be written in the form

$$\tfrac{1}{2}\partial_P\partial_M\partial_N \log \mathsf{W}\left(\dot{H}^M\dot{H}^N - \tfrac{1}{2}\mathcal{Q}^M\mathcal{Q}^N\right) + \partial_P\partial_M \log \mathsf{W}\,\ddot{H}^M - \frac{d}{d\tau}\left(\frac{\partial\Lambda}{\partial\dot{H}^P}\right) + \frac{\partial\Lambda}{\partial H^P} = 0\,. \tag{6.26}$$

Contracting them with H^P and using the homogeneity properties of the different terms as well as the Hamiltonian constraint above, we find the equation (*cf.* Eq. (3.31) of Ref. [212] for the stationary extremal case)

$$\tfrac{1}{2}\partial_M \log \mathsf{W}\left(\ddot{H}^M - r_0^2 H^M\right) + \left(\frac{\dot{H}^M H_M}{\mathsf{W}}\right)^2 = 0\,, \tag{6.27}$$

which corresponds to the equation of motion of the variable U in the standard formulation.

Note that in the extremal case ($r_0 = 0$) and in the absence of the NUT charge

$$\dot{H}^M H_M = 0\,, \tag{6.28}$$

the equations of motion are solved by harmonic functions $\dot{H}^M = \mathcal{Q}^M$ [211].

6.1.1 Extremal black holes

Extremal supersymmetric black holes are expected to be described by $H^M(\tau)$ which are harmonic in Euclidean $\mathbb{R}^3$, i.e. linear in τ[5]:

[4] This relation can be derived from the identities in Ref. [140].

[5] As mentioned above, this anstaz arises quite naturally when one imposes the constraint Eq. (6.28), but it may not be the most general one. The known extremal solutions (usually non-supersymmetric) that do not conform to this ansatz do not satisfy that constrained, either [212, 213]. On the other hand, the representation of a solution in terms of the H^Mmay not be unique and the harmonicity or the fact that the constraint Eq. (6.28) is satisfied or not, may not always be a characteristic feature of a solution. In what follows we are going to explore the (large) sector of the space of black-hole solutions which can be described by harmonc H^Ms and, therefore, satisfy the constraint Eq. (6.28). Analogous remarks apply to the non-extremal hyperbolic ansatz to be studied later.

$$H^M = A^M - \tfrac{1}{\sqrt{2}} B^M \tau \,, \tag{6.29}$$

where A^M and B^M are integration constants to be determined as functions of the independent physical constants (namely, the charges $\mathcal{Q}^M$ and the values of the scalars at spatial infinity z^i_∞) by using the equations of motion (6.26)-(6.28) and the asymptotic conditions.

The equations of motion for the above ansatz can be written in a simple and suggestive form

$$\partial_P \left[V_{\rm bh}(H,\mathcal{Q}) - V_{\rm bh}(H,B)\right] = 0\,, \tag{6.30}$$

$$V_{\rm bh}(H,\mathcal{Q}) - V_{\rm bh}(H,B) = 0\,, \tag{6.31}$$

$$A^M B_M = 0\,. \tag{6.32}$$

Observe that the first two equations are automatically solved for $B^M = \mathcal{Q}^M$, which corresponds to the supersymmetric case. The third equation (enforcing absence of NUT charge) takes the form $A^M \mathcal{Q}_M$ and still has to be solved, which can be done generically [192, 193] as we are going to show.

Furthermore, observe that the Hamiltonian constraint (6.31) is equivalent to the requirement that the black-hole potential, *evaluated on the solutions* has the same form in terms of the true or the *fake central charge*[6]

$$\tilde{\mathcal{Z}}(\phi, B) \equiv \langle\, \mathcal{V} \mid B \,\rangle\,, \tag{6.33}$$

that is

$$-\,V_{\rm bh}(\phi,\mathcal{Q}) = |\tilde{\mathcal{Z}}|^2 + \mathcal{G}^{ij^*}\mathcal{D}_i\tilde{\mathcal{Z}}\mathcal{D}_{j^*}\tilde{\mathcal{Z}}^*\,. \tag{6.34}$$

The asymptotic conditions take the form

$$\mathsf{W}(A) = 1\,, \tag{6.35}$$

$$z^i_\infty = \frac{\tilde{H}^i(A) + iA^i}{\tilde{H}^0(A) + iA^0}\,, \tag{6.36}$$

but can always be solved, together with (6.32) as follows: if we write X as

$$X = \tfrac{1}{\sqrt{2}} e^{U + i\alpha}\,, \tag{6.37}$$

then, from the definition of $\mathcal{I}^M$ (6.5) we get

$$H^M = \sqrt{2} e^{-U}\, \Im\mathfrak{m}(e^{-i\alpha}\mathcal{V}^M)\,, \tag{6.38}$$

and, at spatial infinity $\tau = 0$, using asymptotic flatness (6.35)

$$A^M = \sqrt{2}\, \Im\mathfrak{m}(e^{-i\alpha_\infty}\mathcal{V}^M_\infty)\,. \tag{6.39}$$

Now, to determine α_∞ we can use (6.32) and the definition of fake central charge (6.33). Observe that

$$A_M B^M = \langle\, H \mid B \,\rangle = \Im\mathfrak{m}\langle\, \mathcal{V}/X \mid B \,\rangle = \Im\mathfrak{m}(\tilde{\mathcal{Z}}/X) = e^{-U}\Im\mathfrak{m}(e^{-i\alpha}\tilde{\mathcal{Z}}) = 0\,, \tag{6.40}$$

from which it follows first that

[6] It is worth stressing that, even though the first equation is the derivative of the second with respect to H^P, solving the second for some functions H^M does not imply having solved the first. Only if we find a B^M such that the second equation is satisfied identically for any H^M will the first equation be satisfied as well. The number of B^Ms with this property and their value depend on the particular theory under consideration, but their existence is a quite general phenomenon.

$$e^{i\alpha} = \pm\tilde{\mathcal{Z}}/|\tilde{\mathcal{Z}}|\,, \tag{6.41}$$

and is then the general expression for the A^M as a function of the B^M and the z^i_∞:

$$A^M = \pm\sqrt{2}\,\Im\mathfrak{m}\left(\frac{\tilde{\mathcal{Z}}^*_\infty}{|\tilde{\mathcal{Z}}_\infty|}\mathcal{V}^M_\infty\right)\,. \tag{6.42}$$

In general, the sign of A^M should be chosen to make H^M finite (and, generically, the metric non-singular) in the range $\tau \in (-\infty, 0)$. The positivity of the mass is a physical condition that eliminates some singularities of the metric. As we are going to see in Eq. (6.50), this requirement singles out the upper sign in the above formula.

This reduces the problem of finding a complete solution to the determination of the constants B^M as functions of the physical parameters $\mathcal{Q}^M, z^i_\infty$, which must solve equations (6.30) and (6.31).

It is useful to analyze the near-horizon and spatial-infinity limits of these two equations. The near-horizon limit of (6.31) plus the definition of the fake central charge lead to the following chain of relations[7]

$$S/\pi = \tfrac{1}{2}\mathsf{W}(B) = -V_{\rm bh}\,(B,\mathcal{Q}) = |\tilde{\mathcal{Z}}(B,B)|^2\,, \tag{6.43}$$

where $\tilde{\mathcal{Z}}(B,B)$ is the near-horizon value of the fake central charge. The last of these relations, together with the condition (6.34) imply that, on the horizon, the fake central charge reaches an extremum

$$\partial_i|\tilde{\mathcal{Z}}(\phi_h, B)| = 0\,. \tag{6.44}$$

The near-horizon limit of (6.30) leads to

$$\partial_M V_{\rm bh}(B,\mathcal{Q}) = 0\,, \tag{6.45}$$

which says that the B^M extremize the value of the black-hole potential on the horizon. Since the black-hole potential is invariant under a global rescaling of the H^M, the solutions (that we will call generically *attractors* B^M) of these equations are determined up to a global rescaling which can be fixed by imposing Eq. (6.31).

The B^M must transform under the duality group of the theory (embedded in $Sp(2n+2,\mathbb{R})$) in the same representation as the H^M, the charges $\mathcal{Q}^M$ and the constants A^M. In certain cases this poses strong constraints on the possible solutions since, building an object that transforms in the right representation of the duality group and has dimensions of length squared from $\mathcal{Q}^M$ and z^i_∞ can be far from trivial. A possibility which is always available is the Freudenthal dual defined in Ref. [198], generalizing the definition made in Ref. [197], and further explored in [214], where it was shown that the lagrangian (6.23) has indeed a gauge symmetry which is a generalization of a Frehudental transformation on H^M: Freudenthal duality in $\mathcal{N}=2, d=4$ theories can be understood as the transformation from the H^M to the $\tilde{H}_M(H)$ variables. The same transformation can be applied to any symplectic vector, such as the charge vector. Then, in our notation and conventions, the Freudenthal dual of the charge vector, $\tilde{\mathcal{Q}}_M$, is defined by

$$\tilde{\mathcal{Q}}_M = \tfrac{1}{2}\frac{\partial\mathsf{W}(\mathcal{Q})}{\partial\mathcal{Q}^M}\,. \tag{6.46}$$

It is not hard to prove that this duality transformation is an antiinvolution

$$\tilde{\tilde{\mathcal{Q}}}_{\mathcal{M}} = -\mathcal{Q}_{\mathcal{M}}\,, \tag{6.47}$$

and using Eq. (6.17) to show that

$$\mathsf{W}(\tilde{\mathcal{Q}}) = \mathsf{W}(\mathcal{Q})\,. \tag{6.48}$$

It is harder to show that the critical points of the black-hole potential are invariant under Freudenthal duality [198]. Therefore, since $B^M = \mathcal{Q}^M$ is always an attractor (the supersymmetric one),

$$B^M = \tilde{\mathcal{Q}}^M\,, \tag{6.49}$$

[7] In this and other equations, the expression $V_{\rm bh}\,(B,\mathcal{Q})$ stands for standard the black-hole potential with the functions $H^M(\tau)$ replaced by the constants B^M.

will always be another attractor.

Let us now consider the spatial infinity limit taking into account the definition of the mass in these spacetimes and the definition of the fake central charge

$$M = \dot{U}(0) = \tfrac{1}{\sqrt{2}}\langle \mathcal{R}(A) \mid B\rangle = \pm|\tilde{\mathcal{Z}}(A,B)|\,. \tag{6.50}$$

As mentioned before, to have a positive mass we must use exclusively the upper sign in (6.41) and (6.42) and we do so from now onwards. In the supersymmetric case, when $B^M = \mathcal{Q}^M$ and the fake central charge is the true one, this is the supersymmetric BPS relation.

The asymptotic limit of (6.31) plus (6.34) and the above relation give

$$M^2 + \left[\mathcal{G}^{ij^*}\mathcal{D}_i\tilde{\mathcal{Z}}\mathcal{D}_{j^*}\tilde{\mathcal{Z}}^*\right]_\infty + V_{\mathrm{bh}\,\infty} = 0\,, \tag{6.51}$$

which, when compared with the general BPS bound [78], lead to the identification of the scalar charges Σ_i with the values of the covariant derivatives of the fake central charges at spatial infinity

$$\Sigma_i = \mathcal{D}_i\tilde{\mathcal{Z}}\Big|_\infty\,. \tag{6.52}$$

First-order formalism

First-order flow equations for extremal BPS and non-BPS black holes can be easily found following [179] but using the generic harmonic functions (6.29): let us consider the Kähler-covariant derivative of the inverse of the auxiliary function

$$\begin{aligned}
\mathcal{D}X^{-1} &= i\langle\,\mathcal{V} \mid \mathcal{V}^*\rangle\mathcal{D}X^{-1} = i\langle\,\mathcal{D}(\mathcal{V}/X) \mid \mathcal{V}^*\rangle = i\langle\,d(\mathcal{V}/X) \mid \mathcal{V}^*\rangle \\
&= i\langle\,d(\mathcal{V}/X) - d(\mathcal{V}/X)^* \mid \mathcal{V}^*\rangle = -2\langle\,dH \mid \mathcal{V}^*\rangle \\
&= +\sqrt{2}\,\tilde{\mathcal{Z}}^*(\phi,B)\,d\tau\,,
\end{aligned} \tag{6.53}$$

where we have used the normalization of the symplectic section in the first step , the property $\langle\,\mathcal{D}\mathcal{V} \mid \mathcal{V}^*\rangle = 0$ in the second, the Kähler-neutrality of $\mathcal{V}/X$ in the third, $\langle\,\mathcal{D}\mathcal{V}^* \mid \mathcal{V}^*\rangle = \langle\,\mathcal{V}^* \mid \mathcal{V}^*\rangle = 0$ in the fourth, the definition of $\mathcal{I} = H$ in the fifth, and the ansatz (6.29) and the definition of the fake central charge (6.33) in the sixth.

From this equation and (6.37) and (6.41) we find the standard first-order equation for the metric function U:

$$\frac{de^{-U}}{d\tau} = |\tilde{\mathcal{Z}}(\phi,B)|\,. \tag{6.54}$$

Let us now consider the differential of the complex scalar fields:

$$\begin{aligned}
dz^i &= i\mathcal{G}^{ij^*}\langle\,\mathcal{D}_{j^*}\mathcal{V}^* \mid \mathcal{D}_k\mathcal{V}\,\rangle dz^k = iX\mathcal{G}^{ij^*}\langle\,\mathcal{D}_{j^*}\mathcal{V}^* \mid \mathcal{D}_k(\mathcal{V}/X)\,\rangle dz^k \\
&= iX\mathcal{G}^{ij^*}\langle\,\mathcal{D}_{j^*}\mathcal{V}^* \mid \partial_k(\mathcal{V}/X)\,\rangle dz^k = iX\mathcal{G}^{ij^*}\langle\,\mathcal{D}_{j^*}\mathcal{V}^* \mid d(\mathcal{V}/X)\,\rangle \\
&= iX\mathcal{G}^{ij^*}\langle\,\mathcal{D}_{j^*}\mathcal{V}^* \mid d(\mathcal{V}/X) - d(\mathcal{V}/X)^*\,\rangle = -2X\mathcal{G}^{ij^*}\langle\,\mathcal{D}_{j^*}\mathcal{V}^* \mid dH\,\rangle \\
&= +\sqrt{2}\,X\mathcal{G}^{ij^*}\langle\,\mathcal{D}_{j^*}\mathcal{V}^* \mid B\,\rangle\,d\tau = \sqrt{2}\,X\mathcal{G}^{ij^*}\mathcal{D}_{j^*}\tilde{\mathcal{Z}}^*(\phi,B)\,d\tau\,,
\end{aligned} \tag{6.55}$$

where we have used the same properties as before. To put this expression in a more conventional form we can use the covariant holomorphicity of $\tilde{\mathcal{Z}}$ writing

$$\mathcal{D}_{j^*}\tilde{\mathcal{Z}}^* = \mathcal{D}_{j^*}\frac{|\tilde{\mathcal{Z}}|^2}{\tilde{\mathcal{Z}}} = \frac{2|\tilde{\mathcal{Z}}|\partial_{j^*}|\tilde{\mathcal{Z}}|}{\tilde{\mathcal{Z}}} = 2e^{-i\alpha}\partial_{j^*}|\tilde{\mathcal{Z}}|\,, \tag{6.56}$$

and plugging this result in the expression above:

$$\frac{dz^i}{d\tau} = 2e^U \mathcal{G}^{ij^*}\partial_{j^*}|\tilde{\mathcal{Z}}|\,. \tag{6.57}$$

It is easy to check that these first order equations imply the second-order equations of motion

$$\ddot{U} + e^{2U} V_{\rm bh}(\phi, B) \;=\; 0\,, \tag{6.58}$$

$$\ddot{Z}^i + \Gamma_{jk}{}^i \dot{Z}^j \dot{Z}^k + e^{2U}\partial^i V_{\rm bh}(\phi, B) \;=\; 0\,, \tag{6.59}$$

which coincide with the original ones if

$$V_{\rm bh}(\phi, B) = V_{\rm bh}(\phi, \mathcal{Q})\,, \tag{6.60}$$

for any ϕ (not just for ths solution; see the remark in footnote 6).

6.1.2 Non-extremal black holes: the generic Ansatz

Precious experience [90] (see also [165] and, further, [215, 216] for 5-dimensional examples) suggests that a quite general ansatz for the variables H^M for non-extremal black holes of $\mathcal{N}=2, d=4$ supergravity is[8]

$$H^M(\tau) = A^M \cosh\, r_0\tau + \frac{B^M}{r_0}\sinh\, r_0\tau\,, \tag{6.61}$$

for some integration constants A^M and B^M that, as in the extremal case, have to be determined by solving the equations of motion and by imposing the standard normalization of the physical fields at spatial infinity.

Using this ansatz, the equations of motion (6.25)-(6.27) take the form

$$\tfrac{1}{2}\partial_P\partial_M\partial_N \log \mathsf{W}\, \left[B^M B^N - r_0^2 A^M A^N\right] - \partial_P\left(V_{\rm bh}(\phi,\mathcal{Q})/\mathsf{W}\right) \;=\; 0\,, \tag{6.62}$$

$$-\tfrac{1}{2}\partial_M\partial_N \log \mathsf{W}\left[B^M B^N - r_0^2 A^M A^N\right] - V_{\rm bh}(\phi,\mathcal{Q})/\mathsf{W} \;=\; 0\,, \tag{6.63}$$

$$A^M B_M \;=\; 0\,, \tag{6.64}$$

where we have used the third equation and the homogeneity properties of the Hessian potential W in order to simplify the first two.

In the non-extremal case we can define several fake central charges:

$$\tilde{\mathcal{Z}}(\phi, B) \equiv \langle\, \mathcal{V} \mid B\,\rangle\,, \qquad \tilde{\mathcal{Z}}(\phi, B_\pm) \equiv \langle\, \mathcal{V} \mid B_\pm\,\rangle\,, \tag{6.65}$$

where we have defined the shifted coefficients

$$B^M_\pm \equiv \lim_{\tau\to\mp\infty} \frac{r_0 H^M(\tau)}{\sinh r_0\tau} = B^M \mp r_0 A^M\,. \tag{6.66}$$

Imposing now the same asymptotic conditions on the fields as in the extremal case and the condition (6.64), we arrive again to (6.42). Therefore, we only have to determine the B^Ms plus the non-extremality parameter r_0 by imposing the equations of motion.

The mass is given again by Eqs. (6.50) and the expression for the event horizon area (+) and the Cauchy horizon area (−) are given by

$$\frac{A_{\rm h\pm}}{4\pi} = \mathsf{W}(B_\pm)\,. \tag{6.67}$$

[8] See the caveats in footnote 5.

In the near-horizon limit, the equations of motion, upon use of the above formulae for the area of the event horizon, lead to the following relations

$$\frac{A_{\rm h\pm}}{4\pi} = -V_{\rm bh}(B_\pm) \pm 2r_0\mathcal{M}_{MN}[\mathcal{F}(B_\pm)]A^M B^N_\pm = \mathsf{W}(B_\pm)\,, \tag{6.68}$$

$$\partial_P V_{\rm bh}(B_\pm) = \pm 2r_0\partial_P\mathcal{M}_{MN}[\mathcal{F}(B)]A^M B^N_\pm = -2r_0^2\partial_P\mathcal{M}_{MN}[\mathcal{F}(B)]A^M A^N\,, \tag{6.69}$$

which generalize Eqs. (6.43) and (6.45) to the non-extremal case. In the last identity we have used the expression

$$H^M\partial_P\mathcal{M}_{MN}(\mathcal{F}) = 0\,. \tag{6.70}$$

The right-hand side of Eq. (6.69) would, then, vanish if $A^M \propto B^M$. This is a special case that we will study in Section 6.1.3. Another possibility is that $\mathcal{F}_{\Lambda\Sigma}$ and, henceforth, $\mathcal{M}_{MN}(\mathcal{F})$ are constant, as it happens in quadratic models but, in the general case $\partial_P V_{\rm bh}(B_\pm) \neq 0$ for non-extremal black holes and we conclude that, in general, the values of the scalars on the horizon do not extremize the black-hole potential.

First-order formalism

The derivation carried out for extremal black holes in Section 6.1.1 can be straightforwardly extended to the non-extremal case. As in the 5-dimensional case studied in Ref. [216], the trick is to define a new coordinate ρ and a function $f(\rho)$

$$\rho \equiv \frac{\sinh r_0\tau}{r_0\cosh r_0\tau} \qquad\qquad f(\rho) \equiv \frac{1}{\sqrt{1-r_0^2\rho^2}} = \cosh r_0\tau\,, \tag{6.71}$$

so that the hyperbolic ansatz (6.61) for H^M can be rewritten in the "almost extremal form":

$$H^M = f(\rho)(A^M + B^M\rho) \equiv f(\rho)\hat{H}^M\,. \tag{6.72}$$

Then, following the same steps that lead to Eqs. (6.54) and (6.74) one can obtain the first-order flow equations:

$$\frac{de^{-\hat{U}}}{d\rho} = \sqrt{2}|\tilde{\mathcal{Z}}(\phi,B)|\,, \tag{6.73}$$

$$\frac{dz^i}{d\rho} = -2\sqrt{2}\,e^{\hat{U}}\mathcal{G}^{ij^*}\partial_{j^*}|\tilde{\mathcal{Z}}(\phi,B)|\,. \tag{6.74}$$

where we ave introduced the hatted warp factor $\hat{U} = U + \log f$.

Similarly to the extremal case, it is not difficult to show that these first-order equations imply the second order ones:

$$\frac{d^2\hat{U}}{d\rho^2} + e^{2\hat{U}}V_{\rm bh}(\phi,\sqrt{2}B) = 0\,, \tag{6.75}$$

$$\frac{d^2z^i}{d\rho^2} + \Gamma_{kl}{}^i\frac{dz^k}{d\rho}\frac{dz^l}{d\rho} + e^{2\hat{U}}\mathcal{G}^{ij^*}\partial_{j^*}V_{\rm bh}(\phi,\sqrt{2}B) = 0\,, \tag{6.76}$$

plus the constraint[9]

$$\left(\frac{d\hat{U}}{d\rho}\right)^2 + \mathcal{G}_{ij^*}\frac{dz^i}{d\rho}\frac{dz^{*j^*}}{d\rho} + e^{2\hat{U}}V_{\rm bh}(\phi,\sqrt{2}B) = 0\,, \tag{6.77}$$

[9]Observe that the right-hand side of this equationis not r_0^2.

but now with respect to the new variable ρ and the new function $\hat{U}$. In order to compare these equations with the actual second-order equations of the warp factor and the scalars we have to rewrite them in terms of the variable τ and rescale $\hat{U}$ to U. For the former, by using $d/drho = f^2 d/d\tau$ and Eq. (6.73) one finds:

$$\ddot{U} - \frac{2\sqrt{2}\rho}{f} e^{U} |\mathcal{Z}(\phi, \sqrt{2}B)| + \frac{r_0^2}{f^2} + \frac{e^{2U}}{f^2} V_{\rm bh}(\phi, \sqrt{2}B)\,, \tag{6.78}$$

from which it follows the relation between the true black hole potential and the fake one that must hold for the above second-order equations to imply the true ones:

$$e^{2U} V_{\rm bh}(\phi, \mathcal{Q}) = \frac{e^{2U}}{f^2} V_{\rm bh}(\phi, \sqrt{2}B) - \frac{2\sqrt{2} r_0^2 \rho}{f} e^{U} |\mathcal{Z}(\phi, \sqrt{2}B)| + \frac{r_0^2}{f^2}\,. \tag{6.79}$$

The same condition ensures that the constraint Eq. (6.77) implies the standard Hamiltonian constraint. For the scalar equations we find the condition

$$\partial_i \left\{ e^{2U} V_{\rm bh}(\phi, \mathcal{Q}) - \frac{e^{2U}}{f^2} V_{\rm bh}(\phi, \sqrt{2}B) + \frac{4\sqrt{2} r_0^2 \rho}{f} e^{U} |\mathcal{Z}(\phi, \sqrt{2}B)| \right\} = 0\,. \tag{6.80}$$

There no other conditions to be satisfied for the firs-order equations to imply all the second order ones. Taking the derivative with respecto to ρ of Eq. (6.79) we find that, if we assume that this relation is satisfied for any ϕ (or any H^M), then the last equation is also satisfied and all the second-order equations are satisfied.

Evaluating Eq. (6.79) at spatial infinity, ($\tau = 0$, which corresponds to $\rho = 0$) we find the following relation between the charges, the fake charges, the moduli at infinity and the non-extremality parameter:

$$V_{\rm bh}(\phi_\infty, \mathcal{Q}) - V_{\rm bh}(\phi_\infty, \sqrt{2}B) = r_0^2\,. \tag{6.81}$$

6.1.3 Non-extremal generalizations of doubly-extremal black holes

In this section we are going to solve the equations of the H-FGK system for the non-extremal black holes whose scalars are constant over the whole spacetime using the hyperbolic ansatz Eq. (6.61) for any theory of $\mathcal{N} = 2, d = 4$ supergravity. Thus, we assume

$$z^i_\infty = z^i_{\rm h}\,, \tag{6.82}$$

which requires

$$B^M \propto A^M\,, \tag{6.83}$$

where the constants A^M are given by Eq. (6.42).

Using the proportionality of the B^Ms and A^Ms in the $\tau \to 0^-$ or $\tau \to \pm\infty$ limit of (6.62) we get

$$\partial_K V_{\rm bh}(\phi_\infty, \mathcal{Q}) = 0\,, \tag{6.84}$$

which proves that the scalars must assume a value $\phi_\infty = \phi_{\rm h}$ which extremizes the black hole potential just as in the extremal case (this is something that has to be taken into account when one applies Eq. (6.42) to find the A^M). We can, therefore, use Eq. (6.43) that gives the value of the black-hole potential at the horizons in terms of the fake central charge there $\tilde{\mathcal{Z}}(B, B)$ (not $\tilde{\mathcal{Z}}(\phi, B_\pm)$)

$$-V_{\rm bh}(\phi_\infty, \mathcal{Q}) = |\tilde{\mathcal{Z}}(B, B)|^2\,. \tag{6.85}$$

The proportionality constant between B^M and A^M is easily determined to be $-\mathsf{W}^{1/2}(B)$ by using the normalization at infinity $\mathsf{W}(A) = 1$ and choosing the sign so as to make the functions $H^M \neq 0$ for $\tau \in (-\infty, 0)$. Then we can write

$$H^M(\tau) = A^M \left[\cosh r_0\tau - \mathsf{W}^{1/2}(B) \frac{\sinh r_0 \tau}{r_0} \right]\,. \tag{6.86}$$

and the values of $B^M_\pm$ are

$$B^M_\pm = -[\mathsf{W}^{1/2}(B) \pm r_0]A^M\,, \tag{6.87}$$

and

$$\mathsf{W}(B_\pm) = [\mathsf{W}^{1/2}(B) \pm r_0]^2\,. \tag{6.88}$$

A relation between the value of $\mathsf{W}^{1/2}(B)$ and physical parameters and r_0 can be found by taking the $\tau \to 0^-$ of Eq. (6.63)

$$\mathsf{W}(B) = r_0^2 - V_{\rm bh}(\phi_\infty, \mathcal{Q})\,. \tag{6.89}$$

Another relation comes from the definition of mass $M = \dot{U}(0)$ which gives $M = \tilde{H}_M(A)B^M$. Using the proportionality between A^M and B^M we find that

$$M = \mathsf{W}^{1/2}(B)\,. \tag{6.90}$$

The final expression for the functions $H^M(\tau)$ and the entropies of all these solutions, for any theory, is

$$H^M(\tau) = A^M\left[\cosh r_0\tau - M\frac{\sinh r_0\tau}{r_0}\right]\,, \tag{6.91}$$

$$S_\pm = \pi[M \pm r_0]^2\,, \tag{6.92}$$

(6.93)

where the non-extremality parameter is, upon use of Eq. (6.85) given by

$$r_0 = \sqrt{M^2 - |\tilde{\mathcal{Z}}(B,B)|^2}\,. \tag{6.94}$$

Chapter 7

Quantum black holes in String Theory

In this chapter we use the H-F.G.K. formalism (see section 6) to define a new class of black hole solutions in Type-IIA String Theory compactified on a Calabi-Yau (C.Y.) three-fold, in the presence of perturbative and non-perturbative corrections. We have chosen to call them *quantum* black holes, since they only exist when the *quantum* corrections to the prepotential are present, and no classical limit can be assigned to them. We will also obtain the first explicit and complete black hole solution in the presence of non-perturbative corrections to the prepotential. Supersymmetric black hole solutions to $\mathcal{N} = 2$ four-dimensional ungauged Supergravity in the presence quantum corrections have been previously considered in [133, 217]. The entropy of supersymmetric black holes in the presence of perturbative and non-perturbative corrections has been investigated in [190].

7.1 Type-IIA String Theory on a Calabi-Yau manifold

Type-IIA String Theory compactified to four dimensions on a C.Y. three-fold, with Hodge numbers $(h^{1,1}, h^{2,1})$, is described, up to two derivatives, by a $\mathcal{N} = 2, d = 4$ ungauged Supergravity whose prepotential is given in terms of an infinite series around $\Im\mathrm{m} z^i \to \infty$[1] [14, 161, 218]

$$\mathcal{F} = -\frac{1}{3!}\kappa^0_{ijk} z^i z^j z^k + \frac{ic}{2} + \frac{i}{(2\pi)^3}\sum_{\{d_i\}} n_{\{d_i\}} Li_3\left(e^{2\pi i d_i z^i}\right) \ , \tag{7.1}$$

where $z^i, \ i = 1, ..., n_v+1 = h^{1,1}$, are the scalars in the vector multiplets,[2] $c = \frac{\chi\zeta(3)}{(2\pi)^3}$ is a model dependent number[3], κ^0_{ijk} are the classical intersection numbers, $d_i \in \mathbb{Z}^+$ is a $h^{1,1}$-dimensional summation index and $Li_3(x)$ is the third polylogarithmic function, defined in section 7.3.2 together with some of its properties. The first two addends in the prepotential correspond to tree level and perturbative contributions in the α' expansion, respectively

$$\mathcal{F}_{\mathrm{P}} = -\frac{1}{3!}\kappa^0_{ijk} z^i z^j z^k + \frac{ic}{2} \ , \tag{7.2}$$

whereas the third term accounts for non-perturbative corrections produced by world-sheet instantons.

$$\mathcal{F}_{\mathrm{NP}} = \frac{i}{(2\pi)^3}\sum_{\{d_i\}} n_{\{d_i\}} Li_3\left(e^{2\pi i d_i z^i}\right) \ . \tag{7.3}$$

These configurations get produced by (non-trivial) embeddings of the world-sheet into the C.Y. three-fold. The holomorphic mappings of the (genus 0) string world sheet onto the $h^{1,1}$ two-cycles of the C.Y. three-fold are classified by the nubers d_i, which count the number of wrappings of the world sheet around the $i-$th generator of the integer homology group $H_2(\mathrm{C.Y.}, \mathbb{Z})$. The number of different mappings for each set of $\{d_i\}$ $(\equiv \{d_1, ..., d_{h^{1,1}}\})$ or, in other words, the number of genus 0 instantons is denoted by $n_{\{d_i\}}$[4]

[1] Actually, the prepotential obtained in a Type-IIA C.Y. compactification is *symplectically equivalent* to the prepotential (7.1).
[2] There are also $h^{2,1} + 1$ hypermultiplets in the theory. However, they can be consistently set to a constant value.
[3] χ is the Euler characteristic, which for C.Y. three-folds is given by $\chi = 2(h^{1,1} - h^{2,1})$.
[4] See, e.g. [219] for more details on the stringy origin of the prepotential.

The full prepotential can be rewritten in homogeneous coordinates $\mathcal{X}^\Lambda$, $\Lambda = (0, i)$ as

$$F(\mathcal{X}) = -\frac{1}{3!}\kappa^0_{ijk}\frac{\mathcal{X}^i\mathcal{X}^j\mathcal{X}^k}{\mathcal{X}^0} + \frac{ic(\mathcal{X}^0)^2}{2} + \frac{i(\mathcal{X}^0)^2}{(2\pi)^3}\sum_{\{d_i\}} n_{\{d_i\}} Li_3\left(e^{2\pi i d_i \frac{\mathcal{X}^i}{\mathcal{X}^0}}\right) \,, \tag{7.4}$$

with the scalars z^i given by[5]

$$z^i = \frac{\mathcal{X}^i}{\mathcal{X}^0} \,. \tag{7.5}$$

We are interested in studying spherically symmetric, static, black hole solutions of the theory defined by Eq. (7.1). In order to do so we are going to use the H-F.G.K. formalism [90, 132, 216], based on the use of a new set of variables H^M, $M = (\Lambda, \Lambda)$, that transform linearly under duality and reduce to harmonic functions on the transverse space $\mathbb{R}^3$ in the supersymmetric case[6]. However, the theory defined by Eq. (7.1) is extremely complicated and therefore the task of obtaining explicit black hole solutions is almost hopeless. Therefore, we are going to consider a particular approximation, namely, the large volume limit $\Im\mathrm{m} z^i \to \infty$: in section 7.2 we will discard the non-perturbative corrections and consider only the perturbative ones, and in section 7.3 we will consider a self-mirror C.Y., and therefore the perturbative correction exactly vanishes, and the first non-trivial non-perturbative correction to the prepotential.

7.2 Perturbative quantum black holes

The non-perturbative corrections (7.3) are exponentially suppressed and therefore can be safely ignored going to the large volume limit. Therefore our starting point is going to be Eq. (7.2), which in homogeneous coordinates $\mathcal{X}^\Lambda$, $\Lambda = (0, i)$, can be written as

$$F(\mathcal{X}) = -\frac{1}{3!}\kappa^0_{ijk}\frac{\mathcal{X}^i\mathcal{X}^j\mathcal{X}^k}{\mathcal{X}^0} + \frac{ic}{2}\left(\mathcal{X}^0\right)^2 \,. \tag{7.6}$$

The scalars z^i are given by

$$z^i = \frac{\mathcal{X}^i}{\mathcal{X}^0} \,. \tag{7.7}$$

The scalar geometry defined by (7.6) is the so called *quantum corrected d*-SK geometry [220, 221].The attractor points of this class of models have been extensively studied in [222–224]. In this scenario, the classical case is modified and the scalar manifold, due to the correction encoded in c, is no longer homogeneous, and therefore, the geometry has been *corrected* by quantum effects.

7.2.1 A *quantum* class of black holes

In chapter 6, thanks to the H-F.G.K. formalism, we reduced the task of obtaining black hole solutions of four-dimensional ungauged $\mathcal{N} = 2$ Supergravity to solving the following set of equations

$$\mathcal{E}_P = \tfrac{1}{2}\partial_P\partial_M\partial_N \log \mathsf{W}\left[\dot{H}^M\dot{H}^N - \tfrac{1}{2}\mathcal{Q}^M\mathcal{Q}^N\right] + \partial_P\partial_M \log \mathsf{W}\,\ddot{H}^M - \frac{d}{d\tau}\left(\frac{\partial\Lambda}{\partial\dot{H}^P}\right) + \frac{\partial\Lambda}{\partial H^P} = 0 \,, \tag{7.8}$$

together with the *Hamiltonian constraint*

$$\mathcal{H} \equiv -\tfrac{1}{2}\partial_M\partial_N \log \mathsf{W}\left(\dot{H}^M\dot{H}^N - \tfrac{1}{2}\mathcal{Q}^M\mathcal{Q}^N\right) + \left(\frac{\dot{H}^M H_M}{\mathsf{W}}\right)^2 - \left(\frac{\mathcal{Q}^M H_M}{\mathsf{W}}\right)^2 - r_0^2 = 0 \,, \tag{7.9}$$

where

[5] This coordinate system is therefore only valid away from the locus $\mathcal{X}^0 = 0$.
[6] For more details, see section 6.

$$\Lambda \equiv \left(\frac{\dot{H}^M H_M}{\mathsf{W}}\right)^2 + \left(\frac{\mathcal{Q}^M H_M}{\mathsf{W}}\right)^2 , \tag{7.10}$$

and

$$\mathsf{W}(H) \equiv \tilde{H}_M(H) H^M = e^{-2U} . \tag{7.11}$$

The theory is now expressed in terms of $2(n_v+1)$ variables H^M and depends on $2(n_v+1)+1$ parameters: $2(n_v+1)$ charges $\mathcal{Q}^M$ and the non-extremality parameter r_0, from which one can reconstruct the solution in terms of the original fields of the theory (that is it, the space-time metric, scalars and vector fields).

For Eq. (7.2), the general $\mathsf{W}(H)$ is an extremely involved function, and one cannot expect to solve in full generality the corresponding differential equations of motion, or even the associated algebraic equations of motion obtained by making use of the hyperbolic Ansatz for the H^M. Therefore, we are going to consider a particular truncation, which will give us the desired *quantum* black holes

$$H^0 = H_0 = H_i = 0, \quad p^0 = p_0 = q_i = 0 . \tag{7.12}$$

Eq. (7.31) implies

$$\mathsf{W}(H) = \alpha \left|\kappa^0_{ijk} H^i H^j H^k\right|^{2/3} , \tag{7.13}$$

where $\alpha = \frac{(3!c)^{1/3}}{2}$ must be positive in order to have a non-singular metric. Hence $c > 0$ is a necessary condition in order to obtain a regular solution and a consistent truncation. The corresponding *black hole potential* reads

$$V_{\rm bh} = \frac{\mathsf{W}(H)}{4}\partial_{ij}\log \mathsf{W}(H)\mathcal{Q}^i\mathcal{Q}^j , \tag{7.14}$$

The scalar fields, purely imaginary, are given by

$$z^i = i\,(3!c)^{1/3}\,\frac{H^i}{\left(\kappa^0_{ijk}H^iH^jH^k\right)^{1/3}} , \tag{7.15}$$

and are subject to the following constraint, which ensures the regularity of the Kähler potential ($\mathcal{X}^0 = 1$ gauge)

$$\kappa^0_{ijk}\Im{\rm m} z^i \Im{\rm m} z^j \Im{\rm m} z^k > \frac{3c}{2} . \tag{7.16}$$

Substituting Eq. (7.15) into Eq. (7.16), we obtain

$$c > \frac{c}{4} , \tag{7.17}$$

which is an identity (assuming $c > 0$) and therefore imposes no constraints on the scalars. This phenomenon can be traced back to the fact that the the Kähler potential is constant when evaluated on the solution, and given by

$$e^{-\mathcal{K}} = 6c , \tag{7.18}$$

which is well defined, again, if $c > 0$. Since the volume of the C.Y. manifold is proportional to $e^{-\mathcal{K}}$, Eq. (7.18) implies that such volume remains constant and, in particular, that the limit $\Im{\rm m} z^i \to \infty$ does not imply a large volume limit of the compactification C.Y. manifold, a remarkable fact that can be seen as a purely quantum characteristic of our solution[7]. Notice that it is also possible to obtain the classical limit $\Im{\rm m} z^i \gg 1$ taking $c \gg 1$, that is, choosing a Calabi-Yau manifold with large enough c. In this case we would have also a truly large volume limit.

We have seen that, in order to obtain a consistent truncation, a necessary condition is $c > 0$, which implies that $\mathsf{W}(H)$ is well defined. We can go even further and argue that this is a sufficient condition by studying the equations of motion $\mathcal{E}_P$:

[7] Notice that in order to consistently discard the non-perturbative terms in Eq. (7.1) we only need to take the limit $\Im{\rm m} z^i \to \infty$. Therefore, the behavior of the C.Y. volume in such limit plays no role.

A consistent truncation requires that the equation of motion of the truncated field is identically solved for the *truncation value* of the field. First, notice that the set of solutions of Eqs. (7.8) and (7.9), taking into account (7.31), is non-empty, since there is a *model-independent* solution, given by

$$H^i = a^i - \frac{p^i}{\sqrt{2}}\tau, \qquad r_0 = 0\,, \tag{7.19}$$

which corresponds to a supersymmetric black hole. However, the equations of motion $\mathcal{E}_P$ don't know about supersymmetry: it is system of differential equations whose solution can be written as

$$H^M = H^M\left(a, b\right)\,, \tag{7.20}$$

where we have made explicit the dependence in $2n_v + 2$ integration constants. When the solution (7.20) is plugged into (7.9) is when we impose, through r_0, a particular condition about the extremality of the black hole. If $r_0 = 0$ the integration constants are fixed such as the solution is extremal. In general there is not a unique way of doing it, one of the possibilities being always the supersymmetric one. Therefore, given that for our particular truncation the supersymmetric solution always exists, we can expect the existence also of the corresponding solution (7.20) of the equations of motion, from which the supersymmetric solution may be obtained through a particular choice of the integration constants that make (7.20) fulfilling (7.9) for $r_0 = 0$.
We conclude, hence, that

$$\left\{H^P = 0, \mathcal{Q}^P = 0\right\} \quad \Rightarrow \mathcal{E}_P = 0\,, \tag{7.21}$$

and therefore the truncation of as many H's as we want, together with the correspondet $\mathcal{Q}$'s, is consistent as long as $\mathsf{W}(H)$ remains well defined, something that in our case is assured if $c > 0$. From Eq. (7.1) it can be checked that the case $c = 0$, that is $h^{1,1} = h^{2,1}$, can be cured by non-perturbative effects.

It is easy to see that the truncation is not consistent in the classical limit, and therefore, we can conclude that the corresponding solutions are *genuinely* quantum solutions, which only exist when perturbative quantum effects are incorporated into the action.

Hence, we can conclude that if we require our theory to contain regular *quantum* black holes there is a topological restriction on the Calabi-Yau manifolds that we can choose to compactify Type-IIA String Theory. The condition can be expressed as

$$c > 0 \quad \Rightarrow \quad h^{11} > h^{21}\,. \tag{7.22}$$

For the supersymmetric solution (7.48) it is possible to compute the entropy in a model independent way. The result reads

$$S_{susy} = \pi\alpha \left|\kappa^0_{ijk}p^i p^j p^k\right|^{2/3}\,. \tag{7.23}$$

The class of supersymmetric black holes described here, with entropy (7.23), have no microscopic-String-Theory description, not even at the leading order, and therefore illustrate how the microscopic description of the entropy in String Theory is not well understood even for the simplest class of black holes, namely, the supersymmetric one.

7.2.2 Quantum corrected STU model

In this section we consider a very special case, the so-called STU model, obtaining the first non-extremal solution with non-constant scalars in the presence of perturbative quantum corrections. In order to do so, we set $n_v = 3,\ \kappa^0_{123} = 1$. From (7.13) we obtain

$$\mathsf{W}(H) = \alpha \left|H^1 H^2 H^3\right|^{2/3}\,, \tag{7.24}$$

where $\alpha = 3c^{1/3}$. The scalar fields are given by

$$z^i = ic^{1/3}\frac{H^i}{\left(H^1 H^2 H^3\right)^{1/3}}\,, \tag{7.25}$$

The τ-dependence of the H^M can be found by solving Eqs. (7.8) and (7.9), and the solution is given by

$$H^i = a^i \cosh(r_0\tau) + \frac{b^i}{r_0}\sinh(r_0\tau)\,, \qquad b^i = s^i_b\sqrt{r_0^2(a^i)^2 + \frac{(p^i)^2}{2}}\quad . \tag{7.26}$$

The three constants a^i can be fixed relating them to the value of the scalars at infinity and imposing asymptotic flatness. We have, hence, four conditions for three parameters and therefore one would expect a relation among the $\Im\mathrm{m} z^i_\infty$, leaving c undetermined. However, the explicit calculation shows that the fourth relation is compatible with the others, and therefore no extra constraint is necessary. The a^i are given by

$$a^i = -s^i_b\frac{\Im\mathrm{m} z^i_\infty}{\sqrt{3c}}\quad . \tag{7.27}$$

The mass and the entropy, in turn, read

$$M = \frac{r_0}{3}\sum_i\sqrt{1+\frac{3c(p^i)^2}{2r_0^2(\Im\mathrm{m} z^i_\infty)^2}}\;, \tag{7.28}$$

$$S_\pm = r_0^2\pi\prod_i\left(\sqrt{1+\frac{3c(p^i)^2}{2r_0^2(\Im\mathrm{m} z^i_\infty)^2}}\pm 1\right)^{2/3}\quad , \tag{7.29}$$

and therefore the product of the inner and outer entropy only depends on the charges

$$S_+S_- = \frac{\pi^2\alpha^2}{4}\prod_i (p^i)^{4/3}\quad , \tag{7.30}$$

In the extremal limit we obtain the supersymmetric as well as the non-supersymmetric extremal solutions, depending on the sign chosen for the charges.

7.3 Non-perturbative quantum black holes

In this section we are going to include non-perturbative corrections into the game. As we saw in section 7.2, the perturbative quantum black holes become singular when $c = 0$, *i.e.*, when the compactification C.Y. manifold is self-mirror. However, as we will see later, it is possible to make the formal limit $c \to 0$ regular by including non-perturbative corrections in the prepotential. At the same time, we will construct the first explicit black hole solution in the presence of non-perturbative quantum corrections.

Therefore we are going to consider, for the time being, the complete (7.1), and impose the same particular truncation as in section 7.2

$$H^0 = H_0 = H_i = 0, \quad p^0 = q_0 = q_i = 0\,. \tag{7.31}$$

Under this assumption, the stabilization equations, which can be directly read off from (7.11) take the form

$$\begin{pmatrix} iH^i \\ \mathcal{R}_i\end{pmatrix} = \frac{e^{\mathcal{K}/2}}{X}\begin{pmatrix}\mathcal{X}^i \\ \frac{\partial F(\mathcal{X})}{\partial\mathcal{X}^i}\end{pmatrix}, \begin{pmatrix}\mathcal{R}^0 \\ 0\end{pmatrix} = \frac{e^{\mathcal{K}/2}}{X}\begin{pmatrix}\mathcal{X}^0 \\ \frac{\partial F(\mathcal{X})}{\partial\mathcal{X}^0}\end{pmatrix}, \tag{7.32}$$

and the physical fields can be obtained in terms of the H^i as

$$e^{-2U} = \mathcal{R}_i(H)H^i\,, \quad z^i = i\frac{H^i}{\mathcal{R}^0(H)}\,, \tag{7.33}$$

as soon as $\mathcal{R}^0$ and $\mathcal{R}^i$ are determined. In order to obtain $\mathcal{R}^0$ as a function of H^i, we need to solve the highly involved equation

$$\frac{\partial F(H)}{\partial\mathcal{R}^0} = 0\,, \tag{7.34}$$

where $F(H)$ stands for the prepotential expressed in terms of the H^i

$$F(H) = \frac{i}{3!}\kappa^0_{ijk}\frac{H^i H^j H^k}{\mathcal{R}^0} + \frac{ic(\mathcal{R}^0)^2}{2} + \frac{i(\mathcal{R}^0)^2}{(2\pi)^3}\sum_{\{d_i\}} n_{\{d_i\}} Li_3\left(e^{-2\pi d_i \frac{H^i}{\mathcal{R}^0}}\right). \tag{7.35}$$

Once this is done, it is not difficult to express $\mathcal{R}^i$ in terms of H^i. Indeed, from (7.32) we simply have

$$\mathcal{R}_i = -i\frac{\partial F(H)}{\partial H^i}, \tag{7.36}$$

where $\mathcal{R}^0 = \mathcal{R}^0(H)$ must be substituted only after we perform the derivative, and must be taken to be independent before. Let's see how involved is the equation for $\mathcal{R}^0$: if we expand (7.34), we find

$$-\frac{1}{3!}\kappa^0_{ijk}\frac{H^i H^j H^k}{(\mathcal{R}^0)^3} + c + \frac{1}{4\pi^3}\sum_{\{d_i\}} n_{\{d_i\}}\left[Li_3\left(e^{-2\pi d_i \frac{H^i}{\mathcal{R}^0}}\right) + Li_2\left(e^{-2\pi d_i \frac{H^i}{\mathcal{R}^0}}\right)\left[\frac{\pi d_i H^i}{\mathcal{R}^0}\right]\right] = 0. \tag{7.37}$$

Solving (7.37) for $\mathcal{R}^0$ in full generality seems to be an extremely difficult task. However, if we go to the large volume compactification limit ($\Im\text{m}z^i >> 1$), we can make use of the following property for polylogarithmic functions

$$\lim_{|w|\to 0} Li_s(w) = w\,, \forall s \in \mathbb{N}, \tag{7.38}$$

since, in our case, $w = e^{-2\pi d_i \Im m z^i}$, $\forall \{d_i\} \in (\mathbb{Z}^+)^{h^{1,1}}$. Eq. (7.38) enables us to rewrite (7.37) as

$$-\frac{1}{3!}\kappa^0_{ijk}\frac{H^i H^j H^k}{(\mathcal{R}^0)^3} + c + \frac{1}{4\pi^3}\sum_{\{d_i\}} n_{\{d_i\}}\left[e^{-2\pi d_i \frac{H^i}{\mathcal{R}^0}} + e^{-2\pi d_i \frac{H^i}{\mathcal{R}^0}}\left[\frac{\pi d_i H^i}{\mathcal{R}^0}\right]\right] = 0, \tag{7.39}$$

keeping in mind that we are assuming $\Im\text{m}z^i >> 1$. The dominant contribution in this regime, aside from the cubic one, is given by c. In [76], the first non-extremal black hole solutions (with constant and non-constant scalars) of (7.1) were obtained ignoring the non-perturbative corrections. These solutions turned out to be purely *quantum*[8], in the sense that not only the classical limit $c \to 0$ was ill-defined, but also the truncated theory became inconsistent and therefore no classical limit could be assigned to such solutions. An interesting question to ask now is whether the non-perturbative contributions could actually be able to cure or at least improve this behaviour. On the other hand, it is also interesting *per se* to explore the existence of black hole solutions when the subleading contribution to the prepotential is not given by c, but has a non-perturbative origin. In order to tackle these two questions, let us restrict ourselves to C.Y. three-folds with vanishing Euler characteristic ($c = 0$), the so-called self-mirror C.Y. three-folds. Under this assumption, and considering only the subleading contribution in (7.37), which is now given by the fourth addend in (7.39), such equation becomes[9]

$$-\frac{1}{3!}\kappa^0_{ijk}\frac{H^i H^j H^k}{(\mathcal{R}^0)^3} + \frac{1}{4\pi^3}\sum_{\{d_i\}} n_{\{d_i\}} e^{-2\pi d_i \frac{H^i}{\mathcal{R}^0}}\left[\frac{\pi d_i H^i}{\mathcal{R}^0}\right] = 0. \tag{7.40}$$

The sum over $\{d_i\}$ in (7.40) will be dominated in each case by a certain term corresponding to a particular vector $\left\{\hat{d}_i\right\}$ (and, as a consequence, to a particular $n_{\hat{d}_i} \equiv \hat{n}$), which, consistently with the assumption $\Im\text{m}z^i >> 1$, is the only one that we are going to consider. Therefore, (7.40) becomes

$$-\frac{1}{3!}\kappa^0_{ijk}\frac{H^i H^j H^k}{(\mathcal{R}^0)^3} + \frac{\hat{n}}{4\pi^3} e^{-2\pi \hat{d}_i \frac{H^i}{\mathcal{R}^0}}\left[\frac{\pi \hat{d}_i H^i}{\mathcal{R}^0}\right] = 0\,, \quad \Im\text{m}z^i >> 1\,. \tag{7.41}$$

[8]It is worth pointing out that, in this context, the term *quantum* does not refer to space-time but to world-sheet properties [219]. In this respect, although such denomination is widely spread in the literature, the adjective *stringy* might result more acqurate.

[9]$e^{2\pi i d_i z^i} << \pi|d_i \Im\text{m}z^i| e^{2\pi i d_i z^i}$ for $\Im\text{m}z^i >> 1$.

This equation is solved by[10]

$$\mathcal{R}^0 = \frac{\pi \hat{d}_l H^l}{W_a\left(s_a\sqrt{\frac{3\hat{n}(\hat{d}_n H^n)^3}{2\kappa^0_{ijk}H^iH^jH^k}}\right)}\,, \tag{7.42}$$

where $W_a(x)$, $(a = 0, -1)$ stands for (any of the two real branches of) the *Lambert W function*[11] (also known as *product logarithm*), and $s_a = \pm 1$. Using now Eqs. (7.42) and (7.36) we can obtain $\mathcal{R}^i$. The result is

$$\mathcal{R}_i = \frac{1}{2}\kappa^0_{ijk}\frac{H^jH^k}{\pi \hat{d}_l H^l}W_a\left(s_a\sqrt{\frac{3\hat{n}(\hat{d}_m H^m)^3}{2\kappa^0_{pqr}H^pH^qH^r}}\right)\,. \tag{7.43}$$

The physical fields can now be written as a function of the H^i as

$$e^{-2U} = \mathsf{W}(H) = \frac{\kappa^0_{ijk}H^iH^jH^k}{2\pi\hat{d}_m H^m}W_a\left(s_a\sqrt{\frac{3\hat{n}(\hat{d}_l H^l)^3}{2\kappa^0_{pqr}H^pH^qH^r}}\right)\,, \tag{7.44}$$

$$z^i = i\frac{H^i}{\pi\hat{d}_m H^m}W_a\left(s_a\sqrt{\frac{3\hat{n}(\hat{d}_l H^l)^3}{2\kappa^0_{pqr}H^pH^qH^r}}\right)\,. \tag{7.45}$$

In order to have a regular solution, we need to have a positive definite metric warp factor e^{-2U}. Since, as explained in section 7.3.2, $\text{sign}\,[W_a(x)] = \text{sign}\,[x]\,,\ a = 0, -1,\ x \in D^a_{\mathbb{R}}$, we have to require that

$$s_0 \equiv sign\left[\kappa^0_{ijk}\frac{H^iH^jH^k}{\hat{d}_m H^m}\right]\,, \tag{7.46}$$

for the real branch W_0 and

$$s_{-1} \equiv -1\,. \tag{7.47}$$

for the real branch W_{-1}. On the other hand, since $W_0(x) = 0$ for $x = 0$ and $W_{-1}(x)$ is a real function only when $x \in \left[-\frac{1}{e}, 0\right)$, we have to impose that the argument $\text{Arg}\,[W_a]$ of $W_a(x)$ lies entirely either in $\left[-\frac{1}{e}, 0\right)$ or in $(0, \infty)$ for all $\tau \in (-\infty, 0)$, since e^{2U} cannot be zero in a regular black hole solution for any $\tau \in (-\infty, 0)$. This condition must be imposed in a case by case basis, since it depends on the specific form of the symplectic vector $H^M = H^M(\tau)$ as a function of τ. Notice that if $\text{Arg}\,[W_a] \in \left[-\frac{1}{e}, 0\right)\ \ \forall\ \ \tau \in (-\infty, 0)$ we can in principle[12] choose either W_0 or W_{-1} to build the solution, whereas if $\text{Arg}\,[W_a] \in (0, +\infty)\ \ \forall\ \ \tau \in (-\infty, 0)$, only W_0 is available.

Needless to say, in order to construct actual solutions, we have to solve the H-FGK equations of motion (7.8) (plus hamiltonian constraint (7.9)) using the Hessian potential given by (7.44). Fortunately, such equations admit a *model-independent* solution which is obtained choosing the H^i to be harmonic functions in the flat transverse space, with one of the poles given in terms of the corresponding charge

$$H^i = a^i - \frac{p^i}{\sqrt{2}}\tau, \qquad r_0 = 0\,. \tag{7.48}$$

This corresponds to a supersymmetric black hole.

7.3.1 The general supersymmetric solution

As we have said, plugging (7.48) into (7.45) and (7.44) provides us with a supersymmetric solution without solving any further equation. The entropy of such solution reads

$$S = \frac{1}{2}\kappa^0_{ijk}\frac{p^ip^jp^k}{\hat{d}_m p^m}W_a\left(s_a\beta\right)\,, \tag{7.49}$$

[10]Henceforth we will be using W for the Hessian potential, and W for the Lambert function. We hope this is not a source of confusion.

[11]See section 7.3.2 for more details.

[12]As we will see in section 7.3.1, the possibility $s_0 = s_{-1} = -1$ will not be consistent with the large volume approximation we are considering.

$$\beta = \sqrt{\frac{3\hat{n}(\hat{d}_l p^l)^3}{2\kappa^0_{pqr}p^p p^q p^r}}\,,$$

and the mass is given by

$$M = \dot{U}(0) = \frac{1}{2\sqrt{2}}\left[\frac{3\kappa^0_{ijk}p^i a^j a^k}{\kappa^0_{pqr}a^p a^q a^r}\left[1 - \frac{1}{1+W_a(s_a\alpha)}\right] - \frac{d_l p^l}{d_n a^n}\left[1 - \frac{3}{2\,(1+W_a(s_a\alpha))}\right]\right], \tag{7.50}$$

$$\alpha = \sqrt{\frac{3\hat{n}(d_l a^l)^3}{2\kappa^0_{pqr}a^p a^q a^r}}\,. \tag{7.51}$$

In the approximation under consideration, we are neglecting terms $\sim e^{-2\pi d_i \Im m z^i}$ with respect to those going as $\sim |d_i \Im m z^i| e^{-2\pi d_i \Im m z^i}$. Taking into account (7.45), this assumption is translated into the condition

$$W_a(x) >> 1\,, \tag{7.52}$$

where x stands for the argument of the Lambert function (see (7.42)). It is clear that this condition is satisfied for $a = 0$ if $\mathrm{Arg}[W_0] \in [\alpha, \beta]$ for positive and suficiently large values of α and β. However, it is not satisfied at all for $\mathrm{Arg}[W_a] \in [-\frac{1}{e}, 0)$, which is the range for which both branches of the Lambert function are available.

If we assume $\mathrm{Arg}\,[W_a] \in [\alpha, \beta]$ for suficiently large $\alpha, \beta \in \mathbb{R}^+$, $a = 0$ and W_0 is the only real branch of the Lambert function. In that case, $s = s_0 = 1$, and we have

$$e^{-2U} = \frac{\kappa^0_{ijk}H^i H^j H^k}{2\pi \hat{d}_m H^m} W_0\left(\sqrt{\frac{3\hat{n}(\hat{d}_l H^l)^3}{2\kappa^0_{pqr}H^p H^q H^r}}\right)\,, \tag{7.53}$$

$$z^i = i\frac{H^i}{\pi \hat{d}_m H^m} W_0\left(\sqrt{\frac{3\hat{n}(\hat{d}_l H^l)^3}{2\kappa^0_{pqr}H^p H^q H^r}}\right)\,. \tag{7.54}$$

In the conformastatic coordinates we are working with, the metric warp factor e^{-2U} is expected to diverge at the event horizon $(\tau \to -\infty)$ as τ^2. In addition, we have to require $e^{-2U} > 0\ \forall \tau \in (-\infty, 0]$, and impose asymptotic flatness $e^{-2U(\tau=0)} = 1$. The last two conditions read

$$\frac{\kappa^0_{ijk}H^i H^j H^k}{2\pi \hat{d}_n H^n} > 0 \quad \forall \tau \in (-\infty, 0]\,, \tag{7.55}$$

$$\frac{\kappa^0_{ijk}a^i a^j a^k}{2\pi \hat{d}_m a^m} W_0\,(\alpha) = 1\,, \tag{7.56}$$

whereas the first one turns out to hold, since

$$e^{-2U} \overset{\tau \to -\infty}{\longrightarrow} \frac{\kappa^0_{ijk}p^i p^j p^k}{8\pi \hat{d}_m p^m} W_0\,(\beta)\,\tau^2\,. \tag{7.57}$$

(7.55) and (7.56) can in general be safely imposed in any particular model we consider. Finally, the condition for a well-defined and positive mass $M > 0$ can be read off from (7.50).

7.3.2 Multivalued functions and black hole uniqueness theorems

As we explained in the previous section, our approximation is not consistent with a solution such that $\mathrm{Arg}[W_a] \in [-\frac{1}{e}, 0)$. This forbids the domain in which $W(x)$ is a multivalued function (both W_0 and W_{-1} are real there). However, it seems legitimate to ask what the consequences of having two different branches would have been, if this constraint had not been present. In principle, we could have tried to assign the asymptotic $(\tau \to 0)$ and near horizon $(\tau \to -\infty)$ limits to any particular pair of values of the arguments of W_0 and W_{-1} through a suitable election of the parameters available in the solution. In particular, if we had chosen $\mathrm{Arg}\,[W_0]\,|_{\tau=0} = \mathrm{Arg}\,[W_{-1}]\,|_{\tau=0} = -1/e$

and $\text{Arg}\,[W_0]\,|_{\tau\to-\infty} = \text{Arg}\,[W_{-1}]\,|_{\tau\to-\infty} = \beta$, $\beta \in (-1/e, 0)$, both solutions would have had exactly the same asymptotic behavior (and therefore the scalars of both solutions would have coincided at spatial infinity), and we would have been dealing with two completely different regular solutions with the same mass[13], charges and asymptotic values of the scalar fields, in flagrant contradiction[14] with the corresponding black hole uniqueness theorem (conjecture). At this point, and provided that our approximation is not consistent with such presumable two-branched solution (therefore, we could say that ST forbids such possibility), the feasibility of this reasoning in a different context can only be catalogued as *speculative* at the very least. However, as a matter of fact, a violation of the black hole uniqueness theorem (and, in turn, of the No-Hair conjecture) in four-dimensional ungauged Supergravity would have far-reaching consequences independently of whether the solution is embedded in ST or not. In this regard, the very possibility that the stabilization equations may admit (for certain more or less complicated prepotentials) solutions depending on multivalued functions seems to open up a window for possible violations of the black hole uniqueness theorems in the context of $\mathcal{N}=2$ $d=4$ ungauged Supergravity. The question (whose answer is widely assumed to be "no") is now: is it possible to find a four-dimensional (Super)gravity theory with a physically-admisible matter content admitting more than one stable black hole solution with the same mass, electric, magnetic and scalar charges? This question will be addressed in [225].

The polylogarithm

The *polylogarithmic function* or *polylogarithm* $Li_w(z)$ (see e.g. [226] for an exhaustive study) is a special function defined through the power series

$$Li_w(z) = \sum_{j=1}^{\infty} \frac{z^j}{j^w}\,, \quad z, w \in \mathbb{C}\,. \tag{7.58}$$

This definition is valid for arbitrary complex numbers w and z for $|z| < 1$, but can be extended to $z's$ with $|z| \geq 1$ by analytic continuation. From its definition, it is easy to find the recurrence relation

$$Li_{w-1}(z) = z\frac{\partial Li_w(z)}{\partial z}\,. \tag{7.59}$$

The case $w = 1$ corresponds to

$$Li_1(z) = -\log(1-z)\,, \tag{7.60}$$

and from this it is easy to see that for $w = -n \in \mathbb{Z}^- \cup \{0\}$, the polylogarithm is an elementary function given by

$$Li_0(z) = \frac{z}{1-z}\,, \quad Li_{-n}(z) = \left(z\frac{\partial}{\partial z}\right)^n \frac{z}{1-z}\,. \tag{7.61}$$

The special cases $w = 2, 3$ are called *dilogarithm* and *trilogarithm* respectively, and their integral representations can be obtained from $Li_1(z)$ making use of

$$Li_w(z) = \int_0^z \frac{Li_{w-1}(s)}{s}ds\,. \tag{7.62}$$

The Lambert W function

The *Lambert W function* $W(z)$ (also known as *product logarithm*) is named after Johann Heinrich Lambert (1728-1777), who was the first to introduce it in 1758 [227]. During its more than two hundred years of history, it has found numerous applications in different areas of physics (mainly during the 20th century) such as electrostatics, thermodynamics (e.g. [228]), statistical physics (e.g. [229]), QCD (e.g. [230], [231], [232], [233], [234]), cosmology (e.g. [235]), quantum mechanics (e.g. [236]) and general relativity (e.g. [237]).

$W(z)$ is defined implicitly through the equation

[13] Although $W'_{0,-1}(x)$ are divergent at $x = -1/e$ (as explained in section 7.3.2), and the definition of M would involve derivatives of the Lambert function at that point, it would not be difficult to cure this behaviour and get a positive (and finite) mass by imposing $\dot{x}(\tau) \overset{\tau\to 0}{\longrightarrow} 0$ faster than $|W'_{0,-1}(x)| \overset{x\to-1/e}{\longrightarrow} \infty$.

[14] Up to possible stability issues, which should be carefully studied.

$$z = W(z)e^{W(z)}, \quad \forall z \in \mathbb{C}. \tag{7.63}$$

Since $f(z) = ze^z$ is not an injective mapping, $W(z)$ is not uniquely defined, and $W(z)$ generically stands for the whole set of branches solving (7.63). For $W : \mathbb{R} \to \mathbb{R}$, $W(x)$ has two branches $W_0(x)$ and $W_{-1}(x)$ defined in the intervals $x \in [-1/e, +\infty)$ and $x \in [-1/e, 0)$ respectively (See Figure 1). Both functions coincide in the branching point $x = -1/e$, where $W_0(-1/e) = W_{-1}(-1/e) = -1$. As a consequence, the defining equation $x = W(x)e^{W(x)}$ admits two different solutions in the interval $x \in [-1/e, 0)$.

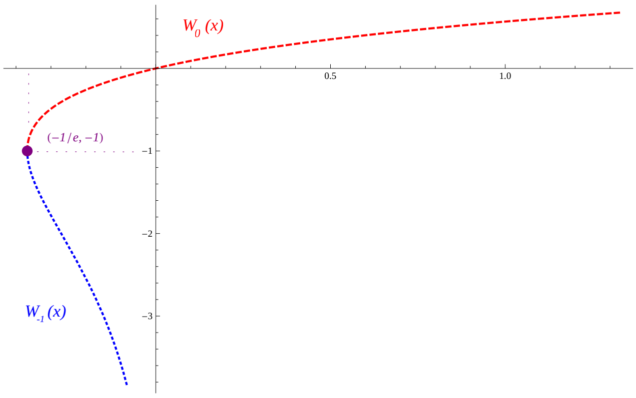

Figure 7.1: The two real branches of $W(x)$.

The derivative of $W(z)$ reads

$$\frac{dW(z)}{dz} = \frac{W(z)}{z(1+W(z))}, \quad \forall z \notin \{0, -1/e\}; \quad \left.\frac{dW(z)}{dz}\right|_{z=0} = 1, \tag{7.64}$$

and is not defined for $z = -1/e$ (the function is not differentiable there). At that point we have

$$\lim_{x \to -1/e} \frac{dW_0(x)}{dx} = \infty, \quad \lim_{x \to -1/e} \frac{dW_{-1}(x)}{dx} = -\infty. \tag{7.65}$$

Appendix A

Resumen

Esta tesis ha sido dedicada a la obtención y estudio de soluciones de tipo agujero negro en teorías supersimétricas de la gravedad sin gaugeos y en cuatro dimensiones. Las teorías supersimétricas de la gravedad, llamadas Supergravedades [27–31], son teorías que describen la gravedad a partir de una métrica definida en una variedad diferenciable, al estilo de la Relatividad General, y que incorporan una simetría concreta, la Supersimetría, que relaciona los bosones y los fermiones de la teoría. La Supersimetría [29, 32–34] es una simetría hipotética de la naturaleza que se postuló en los anõs setenta y que aún no ha sido observada en la naturaleza. No obstante, ha sido y sigue siendo intensamente estudiada por su atractivo teórico. En concreto, Supersimetría presenta una unificación no trivial de las simetrías internas de una teoría de campos con las espaciotemporales, dadas por el grupo de Poincaré. Asimismo permite resolver algunos de los problemas presentes actualmente en física de partículas, como por ejemplo el problema de la jerarquía, y además mejora en general el comportamiento ultravioleta de las teorías cuánticas de campos.

En Supergravedad, la Supersimetría aparece irremediablemente gaugeada, es decir, las transformaciones que relacionan los bosones y los fermiones de la teoría son locales y por tanto dependen del punto del espaciotiempo. Supergravedad surgió a finales de los años setenta y desde entonces ha sido estudiada con intensidad por parte de la comunidad de físicos teóricos. Su relevancia viene dada fundamentalmente, por dos motivos:

1. Presenta una extensión de la Relatividad General de Einstein incorporando una nueva simetría, la Supersimetría.
2. La Supergravedad es el límite de baja energía de la Teoría de Cuerdas, y contiene además importante información no perturbativa sobre la misma.

Teoría de Cuerdas [1, 2] es una teoría que se comenzó a desarrollar a finales de los años sesenta y hoy en día se ha consolidado como un *approach* consistente al *problema* de la unificación de las interacciones fundamentales y al *problema* de la Gravedad Cuántica.

Como se cree que Teoría de Cuerdas contiene una teoría de Gravedad Cuántica consistente, es preciso estudiar las predicciones de dicha teoría en situaciones gravitacionalmente no triviales y donde los efectos cuánticos sean importantes, como en un agujero negro. Dado que la Supergravedad contiene soluciones de tipo agujero negro, y es a su vez el límite de baja energía de la Teoría de Cuerdas, el estudio de agujeros negros en Supergravedad es de extrema importancia para poder estudiar los aspectos cuánticos de los mismos en el contexto de la Teoría de Cuerdas.

Por tanto, el estudio de agujeros negros en Teoría de Cuerdas y Supergravedad es de gran importancia, y a ello se ha dedicado un gran esfuerzo en la literatura [104, 107–110, 119, 120, 124, 132, 165]. En esta tesis pretendemos dar un pequeño paso en esa dirección, desarrollando un formalismo para obtener soluciones de tipo agujero negro en Supergravedad, y obteniendo explítamente una clase de soluciones de tipo agujero negro que se puede embeber en Teoría de Cuerdas y es, por tanto, relevante para estudiar los aspectos cuántico-gravitacionales de la Teoría de Cuerdas.

Appendix B

Conclusiones

En esta tesis hemos estudiado soluciones de tipo agujero negro de Supergravedades en cuatro dimensiones y sin gaugeos. En concreto, hemos caracterizado primero la solución de tipo agujero negro más general de una clase de teorías de gravedad, acopladas a escalares y campos vectoriales, que incluye cualquier Supergravedad sin gaugeos en cuatro dimensiones. Una vez obtenida la forma general de dicha solución, la hemos utilizado para estudiar su simetría *escondida* en el límite cercano al horizonte. Es decir, hemos estudiado las simetrías de la ecuación de Klein-Gordon en el espaciotiempo dado por la métrica general y hemos concluido la existencia de una simetría $SL(2,\mathbb{R})$ en el límite cercano al horizonte. Asimismo, hemos demostrado, mediante la correspondiente construcción explícita, que dicha simetría se puede extender de manera canónica a un álgebra de Virasoro, lo que proporciona la buscada conexión con la teoría conforme dual que describe los grados de libertad microscópicos de la entropía. Éste ha sido el contenido del capítulo 4.

En el capítulo 5, construimos la métrica de todos los agujeros negros supersimétricos de Supergravedades extendidas, de uno y varios centros, usando el formalismo presentado en [131] así como las propiedades de los grupos *de tipo* E_7. Estos agujeros negros son muy relevantes en Teoría de Cuerdas, ya que al ser supersimétricos no reciben correcciones y sus propiedades físicas pueden creerse más allá del límite de baja energía de Supergravedad donde fueron obtenidos.

En el capítulo 6 desarrollamos un nuevo formalismo, llamado el formalismo H-F.G.K., que facilita la construcción de soluciones de tipo agujero negro no supersimétricas en Supergravedad sin gaugeos $\mathcal{N}=2$. El formalismo está basado en la utilización de un nuevo tipo de variables, que se transforman en una representación lineal y simpléctica del grupo de U-dualidad de la teoría y que son funciones armónicas en $\mathbb{R}^3$ en el caso de agujeros negros supersimétricos.

En el capítulo 7 aplicamos el formalismo H-F.G.K. a la Teoría de Cuerdas tipo IIA compactificada en una variedad Calabi-Yau, obteniendo una clase de soluciones llamada *agujeros negros cuánticos*, que sólo existen en presencia de correcciones cuánticas en el prepotencial y para los que no existe, ni puede ser asignado, un límite clásico. Asimismo, obtenemos la primera clase de soluciones en presencia de correcciones cuánticas no perturbativas. Dichas soluciones vienen dadas en función de funciones multivaluadas, lo cual podría dar lugar a una violación de la conjetura del no-pelo. No obstante, tal violación no es posible en el contexto de Teoría de Cuerdas, aunque la posibilidad permanece abierta en el contexto de Supergravedad.

En el capítulo 2 damos una breve introducción a la geometría Special Kähler y a los espacios homogéneos, relevantes para la formulación de Supergravedad en cuatro dimensiones, que se resume en el capítulo 3.

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