Universidad Autónoma de Madrid

Facultad de Ciencias

Departamento de Física Teórica

Supersymmetric solutions of ungauged $N = 2$ Supergravity in four dimensions

Mechthild Hübscher,

Madrid, March 2007.

Universidad Autónoma de Madrid

Facultad de Ciencias
Departamento de Física Teórica

Supersymmetric solutions of ungauged $N = 2$ Supergravity in four dimensions

Memoria del Trabajo de Iniciación a la Investigación realizado por
D$^{\underline{a}}$ Mechthild Hübscher,
presentada ante el Departamento de Física Teórica
de la Universidad Autónoma de Madrid
para la obtención del Diploma de Estudios Avanzados.

Trabajo de Iniciación a la Investigación tutelado por
Dr. D. Tomás Ortín Miguel,
Investigador Científico del Instituto de Física Teórica UAM-CSIC.

Madrid, Marzo 2007.

,,Es ist klar, daß man im allgemeinen eine Theorie als umso vollkommener beurteilen wird, eine je einfachere Struktur sie zugrunde legt. . . “

Albert Einstein

,,It is clear that, in general, one considers a theory more perfect, the simpler the underlying structure . . . “

Albert Einstein

Contents

Chapter 1

Introduction

To start with, we will give a short overview of the main motivations for introducing Supersymmetry, Supergravity and, last but not least, Superstring Theory. In this overview we summarize some "problems" of the Standard Model, then we shortly introduce supersymmetry and say a few words about Superstring Theory and its low energy limit, Supergravity. We sketch how including Supersymmetry, especially in the framework of Superstring Theory, might help to find a way out of the afore-mentioned problems and how Superstring Theory naturally includes gravity. The outline of this thesis is described in the last part of this introduction.

The Standard Model (SM) of elementary particle physics is a spectacularly successful theory of the known particles and their electroweak and strong interactions [1]. Experiments have verified its predictions with incredible precision, and all the particles predicted by this theory have been found apart from the Higgs boson, which is expected to be detected soon at the next generation of particle accelerators, e.g. at LHC at CERN. But it does not explain everything. For example, gravity is not included in the Standard Model of particle physics. Due to its weakness (at a typical energy-scale of particle physics, it is about 10^{-25} times weaker than the weak force, 10^{-38} times than the strong nuclear force[1]) gravity is irrelevant in describing the interactions of fundamental matter. While the electromagnetic, weak and strong force are transmitted by spin-1 particles, gravity is supposed to be transmitted by a particle which carries spin 2, and in contrast to the other forces, it acts on *every* particle. On the one hand, quantum field theory is used to explain the fundamental interactions at small distances, while on the other hand the large scale structure of the universe is governed by gravitational interactions described accurately by Einstein's General Relativity. Trying to add gravity to the Standard Model and in particular to combine General Relativity with Quantum Mechanics leads to inconsistencies [2]. This is fairly unsatisfactory from a theoretical and conceptual point of view since we assume that there should be a way to describe the four fundamental forces within the framework of a unique underlying theory.

- The SM is a Yang-Mills gauge theory, in which the gauge group $SU(3)_c \times SU(2)_L \times$

[1]The exact strengths depend on the particles and energies involved

$U(1)_Y$ is spontaneously broken to $SU(3)_c \times U(1)_{EM}$ by the non-vanishing vacuum expectation value (VEV) of a fundamental scalar field, the Higgs field. Phenomenologically, the mass of the Higgs boson associated with electroweak symmetry breaking must be in the electroweak range. However, the (mass)^2 of the Higgs boson receives radiative corrections from higher-order terms in perturbation theory and a fine tuning of 28 orders of magnitude is necessary in order to obtain a phenomenologically viable Higgs mass. This phenomenon is called the *hierarchy problem* and it is the main motivation for introducing supersymmetry at the weak scale.

The contribution of radiative corrections to the Higgs boson mass is *nonzero, divergent and positive*. While the corrections to the electron mass are themselves proportional to the electron mass and quite small, even if we use the Planck scale as cut-off ($\delta m_e(M_{Planck}) \approx 0.24 m_e$ [3]), the mass of Higgs particles is very sensitive to the scale. One has to make a fine adjustment in all orders of perturbation theory (PT) that gives the (mass)^2 of the Higgs boson a value 28 orders of magnitude or more below its natural value. This is possible but very unnatural.

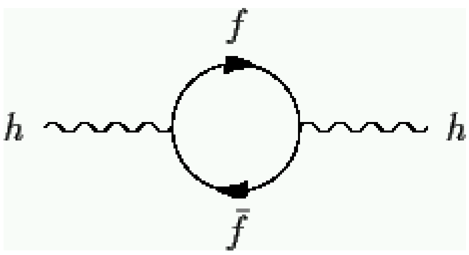

Figure 1.1: A Higgs boson dissociating into a virtual fermion-antifermion pair in the Standard Model [4].

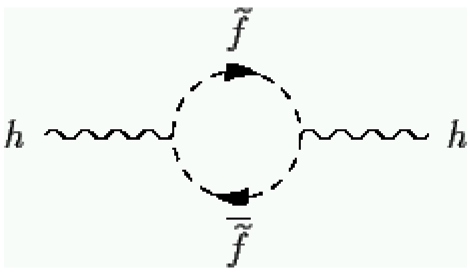

Figure 1.2: A Higgs boson dissociating into a virtual sfermion-antisfermion pair; this diagram cancels the one in Fig. 1.1 [4].

The best studied way of achieving this kind of cancellation of quadratic terms (also known as the cancellation of the quadratic divergencies) is supersymmetry (SUSY) [5]. Supersymmetry is a new kind of symmetry relating bosons and fermions. We will describe it later in further detail. In a supersymmetric theory every fermion is accompanied by a bosonic superpartner with the same mass [2]. For example, the quarks, which are fermions, are accompanied by squarks, which are bosons. Similarly, the gluons, which are bosons, are accompanied by gluinos, which are fermions [2].

[2]Since at today's particle accelerators none of the predicted superpartners has been found yet, it turn out that if supersymmetry is a symmetry of Nature, it must be broken (at least at low energy scale) by some appropriate mechanism.

Thus supersymmetric theories are characterized by equal numbers of bosonic and fermionic degrees of freedom. In SUSY the quadratic corrections to the Higgs boson mass are automatically canceled to all orders of PT. This is due to the contributions of superpartners of ordinary particles. The contributions from boson loops cancel those from the fermion ones because of an additional factor -1 arising from Fermi statistics, as shown in Figs.1.1 and 1.2.

- The Standard Model cannot describe accurately the unification of the gauge couplings in the framework of a Grand Unified Theory (GUT), which turns out to be much better in a supersymmetric theory.

  The philosophy of Grand Unification is based on a hypothesis: gauge symmetry increases with energy. Bearing in mind the unification of all forces of Nature on a common basis and neglecting gravity for the time being due to its weakness, the idea of GUTs is the following: all known interactions are different branches of a unique interaction associated with a simple gauge group.

Low energy					$\Longrightarrow$	**High energy**
$SU(3)_C$	$\otimes$	$SU(2)_L$	$\otimes$	$U(1)_Y$	$\longrightarrow$	G_{GUT}
g_3		g_2		g_1	$\longrightarrow$	g_{GUT}

Table 1.1: Unification of gauge couplings in a Grand Unified Theory.

  Although there is a big difference in the values of the couplings of strong, weak and electromagnetic interactions, a unification is possible at high energy [5]. The crucial point is the running of the coupling constants. After the precise measurement of the $SU(3) \times SU(2) \times U(1)$ coupling constants, it has become possible to check the unification numerically.

  It turns out that within the supersymmetric model a unification much better than in non-supersymmetric GUTs can be obtained, if the SUSY masses are of the order of 1 TeV [5]. The evolution of the gauge couplings in a supersymmetric generalization of the SM is shown in Fig. 1.4.

- Many attempts have been made to make General Relativity consistent with quantum field theory, especially within the framework of a theory which combines gravity with the strong and electroweak interactions. It is interesting that in some of the most successful attempts Supersymmetry is used, either as global symmetry or as local symmetry, that is containing Supergravity.

  "Super"-symmetry is described by "graded" Lie algebras (i.e. Lie algebras containing anticommutators as well as commutators). Thus, a partial unification of matter (fermions) with forces (bosons) naturally arises.

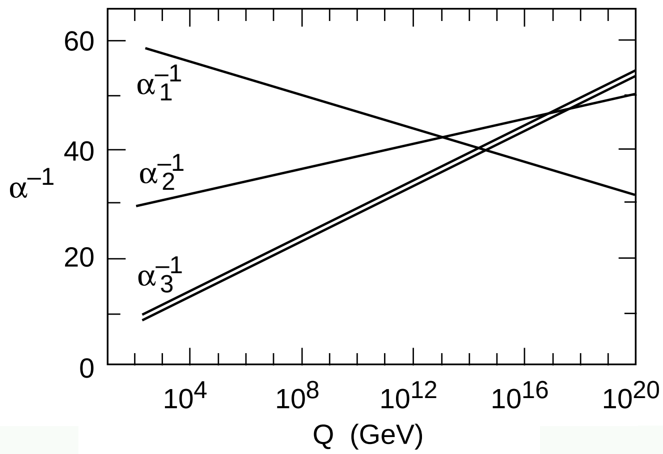

Figure 1.3: Evolution of the $SU(3) \times SU(2) \times U(1)$ gauge couplings to high energy scales, using the one-loop renormalization group equations of the Standard Model. The double line for α_3 indicates the experimental error in this quantity; the errors in α_1 and α_2 are too small to be visible [6].

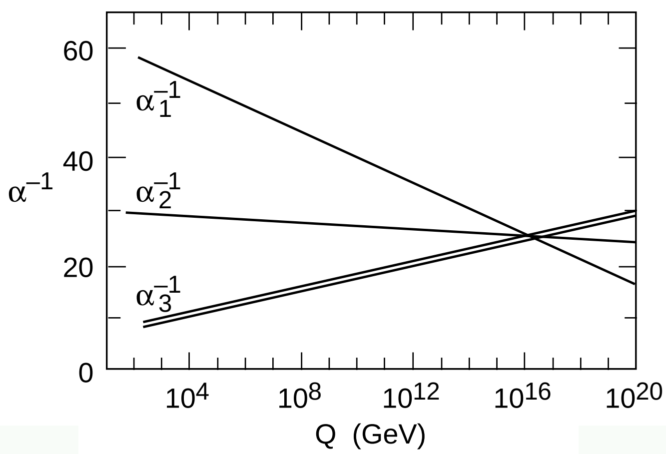

Figure 1.4: Evolution of the gauge couplings, using the one-loop renormalization group equations of the supersymmetric generalization of the Standard Model [6, 7].

The $N = 1$ SUSY algebra can be written as [8]

$$\{Q, \bar{Q}\} = 2\sigma^\mu P_\mu \tag{1.1}$$

and

$$[\xi Q, \overline{\eta}\overline{Q}] = 2\xi\sigma^\mu\overline{\eta}P_\mu. \tag{1.2}$$

Q are the generators of the SUSY transformation characterized by the anti-commuting Grassmann parameters ξ and η. In the case of global SUSY this describes a translation by the vector $\xi\sigma^\mu\overline{\eta}$. Choosing the parameters ξ and η to be local, i.e. functions of a space-time point, one finds that the right-hand side of Eq. (1.2) becomes $2\xi(x)\sigma^\mu\overline{\eta}(x)P_\mu$ which can be understood as a local coordinate transformation. We see that SUSY is not an internal symmetry, but a spacetime symmetry related through the SUSY algebra to spacetime translations. The theory which is invariant under a general coordinate transformation (GCTs) is General Relativity. Thus, making SUSY local, one obtains General Relativity, or a supersymmetric generalization thereof, Supergravity [5]. Note that the SM does include Special Relativity, but does not include General Relativity or gravity. Therefore we are lead to look for extensions of it and it seems natural to include supersymmetry.

- With only the ingredients of the Standard Model of particle physics we cannot understand why its particle content is the way it is: the existence of three families ...

The couplings of the Higgs field to fermions generate masses of quarks and leptons, however their values are free parameters of the SM. There seems to be no reason why the mass spectrum of quarks and leptons should stretch over six orders of magnitude between the masses of the electron and the top quark.

- Another evidence for the existence of physics beyond the Standard Model is the cold dark matter (CDM) of the universe, because the Standard Model does not provide a viable candidate for it. Under certain assumptions the lightest supersymmetric particle (LSP) is neutral and stable and hence provides an excellent candidate for CDM.

Thus, despite its spectacular success, the Standard Model of particle physics is not "The End of Science" [2] but could be just the low energy limit of some more fundamental underlying theory.

Apart from the arguments given above, there are also more theoretical motivations to study supersymmetry. The first to be mentioned is the Haag-Lopuszanski-Sohnius theorem, which states that supersymmetry is the most general extension of the Poincaré and Yang-Mills-type symmetries of the S-matrix.

Furthermore, supersymmetry often makes it possible to extrapolate results from weak coupling to strong coupling, thereby providing information about strongly coupled theories: Extended supersymmetry algebras with central charges have special representations, so-called short multiplets. The states in these representations, the BPS states, are annihilated by some of the generators of the supersymmetry algebra. They are characterized by the fact that they saturate the Bogomolny'i bound $M \leq |Z|$, an inequality between its mass and its charge. Even though both mass and charge may undergo renormalization, this definite mass-charge relationship for BPS states is expected to be protected from quantum corrections, since it is a consequence of the supersymmetry algebra assuming that the full theory is supersymmetric.[3] If it were violated, then new states would appear out of nowhere and quantum corrections are not expected to produce these new degrees of freedom. This property of BPS states means that supersymmetry plays a crucial role in the theory of supersymmetric black holes. It turns out that unbroken supersymmetry is an important ingredient in the stringy calculation of the black hole entropy by counting of microstates.

In the last decades of the past century a new theory, which for consistency requires supersymmetry, Superstring Theory, arose. It turned out to be well-suited to the construction of a quantum theory that unifies the description of gravity and the other fundamental forces of nature. Perhaps the most important feature of Superstring Theory is that gravity is naturally incorporated in the theory. The theory gets modified at very short distances/high energies but at ordinary distances and energies gravity is present in exactly the form proposed by Einstein. While ordinary quantum field theory does not seem to be compatible with gravity, String Theory *requires* gravity. Since the massive states of

[3] Thus the equality of mass and charge of BPS states is protected against quantum corrections, but mass and charge separately may receive corrections, which depend on the particular theory one is dealing with, especially on the number of supercharges.

String Theory have masses proportional to the Planck mass, massive states only start to play a role when considering processes at extremely high energies, far beyond the reach of any accelerator. Therefore one can restrict the analysis to the massless modes only and describe them by an effective theory. As long as one considers processes with energies far below the Planck mass, this is a good approximation. The low energy effective theories of superstring theories are supergravity (SUGRA) theories.

Supersymmetric solutions of supergravity theories have played a key role in many of the most important developments in string theory. For example, supersymmetric compactifications provide a promising setting for obtaining supersymmetric realistic models of particle physics: by compactifying down to four spacetime dimensions, one might hope to make contact with particle physics phenomenology. Another motivation to study supersymmetric solutions of supergravity theories is their importance for black hole thermodynamics: a microscopic interpretation of black hole entropy in string theory is best understood for supersymmetric black holes, and various kinds of supersymmetric solutions have transformed our understanding of quantum field theory via the AdS/CFT correspondence and its generalizations.

1.1 Outline of this thesis

This thesis deals with $N = 2$ supergravity theories in four dimensions, which are the effective string theories in Calabi-Yau compactifications. Our goal is to find all the supersymmetric solutions of the equations of motion and to characterize them by a minimal number of independent variables. The results of this thesis are based on [9].

The outline of this thesis is as follows: in Chapter 2 the higher dimensional origin of $N = 2$ $d = 4$ SUGRA from Calabi-Yau compactifications is explained. We will see how the matter content and prepotential of the four-dimensional theory, which describes the coupling of scalars to scalars in the vector multiplets and of those scalars to the vectors of the vector multiplets, is encoded in the geometry of the Calabi-Yau manifold. In Section 2.2 we explain what we mean by *supersymmetric configurations* and *supersymmetric solutions* and describe how demanding supersymmetry imposes constraints on the equations of motion. The procedure we use to find *supersymmetric solutions* of the four-dimensional theory, summarized in Section 2.3. In Chapter 3 we apply the afore-mentioned procedure to $N = 2$ $d = 4$ SUGRAs. In Section 3.1 we present the theories: field content, Lagrangian, equations of motion etc. It will turn out that the vector bilinear, which can be constructed out of Killing spinors, will always be timelike or null. These two cases are analyzed separately in Sections 3.3 and 3.4, respectively. In Section 3.5 we summarize the main results before we give an outlook on future research in Chapter 4, e.g. generalizations of the work presented in this thesis such as gaugings. . . .

Our conventions can be found in Appendix A. In Appendices B-D we summarize the main features of the geometry we are dealing with in this thesis, i.e. Kähler, special Kähler and quaternionic Kähler geometry. In the following Appendix F we resume the concept of holonomy, focussing on how it is related to supersymmetry breaking.

Chapter 2

Theoretical background

2.1 $N = 2$, $d = 4$ supergravity from string theory

In this chapter we are going to review the higher dimensional origin of $N = 2$, $d = 4$ Supergravity and how it arises from compactification of ten-dimensional Superstring Theory.

In String Theory the fundamental object is not a point particle, but a one-dimensional string sweeping out a two-dimensional *worldsheet* in the target spacetime. Elementary particles are identified with the oscillation modes of the string. Most of these have excitation energies far above the the energy scale one can presently probe in experiments. However, one of the massless modes turns out to be a spin-2 particle, which can be identified with the graviton, necessary as intermediating particle in any quantum theory of gravity. There seem to be only five consistent superstring theories: *Type I*, *Type IIA*, *Type IIB*, *Heterotic* $SO(32)$ and *Heterotic* $E_8 \times E_8$ which are related to each other by *dualities*. All these five theories live in ten spacetime dimensions and seem to be just special limits of a single underlying theory called *M-Theory*. This immediately leads to the idea of compactification, in order to make contact with our four-dimensional world. One of the problems arising in String Theory is the so-called *vacuum selection problem*: compactification of Superstring Theory down to four dimensions, may lead to very different physics described by the four-dimensional effective theory, because the spectrum (and gauge group) of the four-dimensional theory depends on the choice of six-dimensional internal manifold.

There is one fundamental (dimensionful) constant in String Theory that governs the scale of the massive string excitations. This constant can be expressed in terms of the *Regge slope parameter* α' which has mass dimension -2, the string tension (energy per unit length) $T = \frac{1}{2\pi\alpha'}$ or in terms of the string length scale $l_s^2 = 2\alpha'$.

Massive string excitations have masses of the order $M \sim \frac{1}{\sqrt{\alpha'}}$ which are typically of the order of the Planck mass. By definition, the low energy limit of string theory only involves processes at an energy scale E far below the Planck scale, i.e.

$$E^2\alpha' \ll 1. \tag{2.1}$$

This means that in the low energy approximation we only have to consider the massless modes. Remember that the *string coupling constant* g_s is given in terms of the vacuum expectation value (VEV) of the dilaton $g_s = e^{\langle\phi\rangle}$. In the small coupling and low energy limit we only need to consider string tree diagrams for the massless states which are well approximated by classical supergravity field theory. At energy scales much lower than the Planck scale, that is at length scales much larger than the string length $l_s = \sqrt{\alpha'}$, the string behaves like a pointlike particle. Effects due to the extension of the string are hidden in stringy α'-corrections. In flat Minkowski background the string coupling constant can be chosen to be small everywhere (and does not receive any α' corrections) and Supergravity constitutes a good approximation of Superstring Theory. However, on generic backgrounds, when the curvature becomes large, string loop corrections become very important and the supergravity limit is not anymore a good approximation. Furthermore, in some regions of a generic background the VEV of the dilaton and hence the string coupling constant may become large.

Let us now come back to the question of how $N = 2$, $d = 4$ Supergravity is related to ten-dimensional Supergravity and thereby to Superstring Theory. Type II Supergravity theories as low energy limits of type II superstring theory live in ten dimensions. To recover the four-dimensional spacetime of everyday experience, we have to compactify the ten-dimensional theory on a six-dimensional internal manifold. The four-dimensional theory obtained upon compactification heavily depends on the topology of the internal manifold (see below). If we choose a six-torus T^6 for example, we are left with $N = 8$ supersymmetry in four dimensions if we start from ten dimensional type II theory which has 32 supersymmetries, since, due to its trivial holonomy, a torus does not break any supersymmetry (for more details see Appendix F). Generically, compactification on a Calabi-Yau manifold CY_n, which by definition has $SU(n)$ holonomy breaks some fraction of supersymmetry. In case of compactification on a Calabi-Yau threefold CY_3 three quarters of the supersymmetries are broken. Thus, from the 32 supercharges we have in ten dimensions in case of type II supergravities, we are left with 8 in four dimensions. In this way CY_3 compactification of type II supergravity leads to $N = 2$, $d = 4$ supergravity coupled to n_V vector and n_H hypermultiplets, where the numbers of multiplets is given in terms of topological invariants of the Calabi-Yau manifold one is compactifying on.

Now let us see how the geometry of the internal manifold affects the number of unbroken supersymmetries in the lower dimensional theory. Schematically it can be explained in the following way: for an orientable six-dimensional manifold parallel transport of a spinor along a closed curve generically gives a rotation by a $SO(6) \sim SU(4)$ matrix, this is the generic holonomy group. The **16** Weyl representation of the ten dimensional Lorentz group $SO(1, 9)$ decomposes with respect to $SO(1, 3) \otimes SO(6)$ as

$$\mathbf{16} = (\mathbf{2_L}, \bar{\mathbf{4}}) + (\mathbf{2_R}, \mathbf{4}) \tag{2.2}$$

The largest subgroup of $SU(4)$ for which a spinor of definite chirality can be invariant is $SU(3)$. The reason is that the **4** has an $SU(3)$ decomposition

$$\mathbf{4} = \mathbf{3} \oplus \mathbf{1}, \tag{2.3}$$

i.e. it decomposes into a triplet and a singlet, which is invariant under $SU(3)$. Since the condition for $N = 1$ unbroken supersymmetry in four dimensions is the existence of a covariantly constant spinor on the internal six-dimensional manifold, and only the singlet pieces of $\mathbf{4}$ and $\bar{\mathbf{4}}$ in Eq. (2.3) lead to covariantly constant spinors, compactification on a manifold with $SU(3)$ holonomy breaks 3/4 of the original supersymmetries (see Appendix F). Imposing the Majorana condition in ten dimensions, it follows that type II supergravity on a CY_3 leads to $N = 2$ supergravity in four dimensions.

The massless Kaluza-Klein modes associated with various fields in ten dimensions, compactified on a Calabi-Yau space are given in Table 2.1. Let us see in some more detail how the massless scalars in four dimensions arise from the ten dimensional theory, taking IIB as example. The bosonic fields of IIB supergravity are: [1]

$$G_{MN},\, B_{MN},\, \phi,\, C,\, C_{MN},\, C_{MNPQ}\,. \tag{2.4}$$

Additionally the supergravity multiplet contains 2 gravitini and two dilatini with the same chirality. The metric G_{MN}, the dilaton ϕ and the two-form B_{MN} come from the NS-NS sector, whereas the axion C, the 2-form and 4-form C_{MN} and C_{MNPQ} come from the R-R sector.

The axion, the dilaton and the duals of $B_{\mu\nu}$ and $C_{\mu\nu}$ lead to 4 real scalars, combined in the so-called *universal hypermultiplet*, independently of the specific choice of Calabi-Yau manifold, since for any CY_3 $h^{0,0} = 1$. The Hodge numbers of a generic Calabi-Yau threefold are displayed in the so-called "Hodge diamond":

$$\begin{array}{ccccccc} & & & h^{3,3} & & & \\ & & h^{3,2} & & h^{2,3} & & \\ & h^{3,1} & & h^{2,2} & & h^{1,3} & \\ h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\ & h^{2,0} & & h^{1,1} & & h^{0,2} & \\ & & h^{1,0} & & h^{0,1} & & \\ & & & h^{0,0} & & & \end{array} \quad = \quad \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{1,2} & & h^{1,2} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

Now let us consider metric deformations of the Calabi-Yau manifold. After fixing the diffeomorphism invariance and taking into account the Ricci-flatness of Calabi-Yau manifolds, the deformations δg_{ij} and $\delta g_{i\bar{j}}$ decouple and thus can be considered separately. The purely holomorphic or antiholomorphic components g_{ij} and $g_{\bar{i}\bar{j}}$, respectively, are zero. However, one can consider variations to non-zero values, thereby changing the complex structure.

Thus metric deformations of the Calabi-Yau manifold give two types of moduli [10],[11]:

[1] In the following upper case Latin indices $M, L \ldots$ denote tendimensional indices, while Greek indices $\mu, \nu \ldots$ live in four dimensions and lower case Latin indices $i, j \ldots$ in the internal sixdimensional space.

- *Kähler moduli*: $h^{1,1}$ real scalars due to deformations of $g_{i\bar{j}}$:

$$\delta g_{i\bar{j}} = \sum_{\alpha=1}^{h^{1,1}} t^\alpha b^\alpha{}_{i\bar{j}}, \tag{2.5}$$

  where we expanded $\delta g_{i\bar{j}}$ in a basis of real $(1,1)$-forms, which we denoted by b^α, $\alpha = 1 \ldots h^{1,1}$, and t^α are the Kähler moduli, and

- *Complex structure moduli*: $h^{1,2}$ complex scalars due to the deformations of δg_{ij}:

$$\Omega_{ijk} \delta g^k{}_{\bar{l}} = \sum_{a=1}^{h^{2,1}} t^a b^a{}_{ij\bar{l}} \tag{2.6}$$

  where a complex $(2,1)$ form is associated to each variation of the complex structure. Here b^a, $a = 1 \ldots h^{2,1}$, denote a basis of harmonic $(2,1)$-forms and the complex parameters t^a are called the *complex structure moduli.* Ω denotes the unique holomorphic $(3,0)$-form of Calabi-Yau threefolds. It turns out that the metric on the complex structure moduli space is Kähler with Kähler potential given by [11]

$$\mathcal{K} = -\log(i \int \Omega \wedge \Omega^*). \tag{2.7}$$

The 2-forms lead to 2 $h^{1,1}$ scalars $B_{i\bar{j}}$ and $C_{i\bar{j}}$ and taking into account the self-duality of the 5-form field-strength of the 4-form, there are $h^{2,2} = h^{1,1}$ scalars $C_{ij\bar{k}\bar{l}}$ arising from C_{MNPQ}. These $4h^{1,1}$ scalars are part of $h^{1,1}$ additional hypermultiplets. Finally the $h^{1,2}$ complex scalars (complex structure moduli) are associated to $h^{1,2}$ vector multiplets.

Further, the spectrum of the low dimensional theory contains $h^{3,0} = 1$ vector $C_{\mu ijk}$ in the gravity multiplet and $h^{2,1} = h^{1,2}$ vectors $C_{\mu ij\bar{k}}$ associated to the vector multiplets.

In the case of the type IIA theory the massless bosonic fields in ten dimensions are

$$G_{MN},\, B_{MN},\, \phi,\, C_M,\, C_{MNP}\ . \tag{2.8}$$

Additionally the supergravity multiplet contains 2 gravitini and two dilatini with opposite chiralities. Note that just as for type IIB G_{MN}, B_{MN}, and ϕ arise from the NS-NS sector, whereas in the case at hand the R-R fields are forms of odd degree.

The NS-NS fields give the same number of massless scalars as in the IIB case, namely one real scalar from the dilaton, $2h^{1,2} + h^{1,1}$ real scalars from the metric and $h^{1,1} + 1$ real scalars from the NS-NS 2-form. Now the R-R 3-form leads to $h^{2,1} = h^{1,2}$ complex scalars $C_{ij\bar{k}}$ and $h^{3,0} = 1$ complex scalar C_{ijk}.

The 1-form leads to one vector field C_μ (which will be contained in the supergravity multiplet) and the 3-form to $h^{1,1}$ vectors $C_{\mu i\bar{j}}$, contained in the vector multiplets. Grouping all these fields again into multiplets, one obtains gravity coupled to $h^{1,1}$ vector multiplets and $h^{1,2}$ hypermultiplets in four dimensions. With these results it is easy to count the

A	B	field	spin-2	spin-1	spin-0
1	1	g_{MN}	1	0	$h^{1,1}$ real + $h^{1,2}$ complex
1	2	ϕ	0	0	1
1	0	A_M	0	1	0
1	2	A_{MN}	0	0	$(h^{1,1}+1)$ real
1	0	A_{MNP}	0	$h^{1,1}$	$(h^{1,2}+1)$ complex
0	1	$[A_{MNPQ}]_\pm$	0	$h^{1,2}+1$	$h^{1,1}$ real

Table 2.1: Massless Kaluza-Klein modes associated with various fields in ten dimensions, compactified on a Calabi-Yau space. The first two columns specify the number of these fields contained in IIA or IIB supergravity in ten space-time dimensions [12].

number of bosonic massless states that emerge in the compactification of IIA and IIB supergravity on a Calabi-Yau manifold [13]:

$$\text{nonchiral IIA SG}: \quad \begin{array}{ll} 1 \text{ spin-1} + 1 \text{ spin-2} & \text{gravity multiplet} \\ \left.\begin{array}{l} h^{1,1} \text{ spin-1} \\ h^{1,1} \text{ complex spin-0} \end{array}\right\} & h^{1,1} \text{ vector supermultiplets} \\ h^{1,2}+1 \text{ quaternionic spin-0} & h^{1,2}+1 \text{ hypermultiplets} \end{array} \tag{2.9}$$

$$\text{chiral IIB SG}: \quad \begin{array}{ll} 1 \text{ spin-1} + 1 \text{ spin-2} & \text{gravity multiplet} \\ \left.\begin{array}{l} h^{1,2} \text{ spin-1} \\ h^{1,2} \text{ complex spin-0} \end{array}\right\} & h^{1,2} \text{ vector supermultiplets} \\ h^{1,1}+1 \text{ quaternionic spin-0} & h^{1,1}+1 \text{ hypermultiplets} \end{array} \tag{2.10}$$

The field content of four-dimensional supergravity associated to the field content of ten-dimensional type IIA/B supergravity is summarized in Table 2.1.

The total target manifold parameterized by the various scalars factorizes as a product of vector and hypermultiplet manifolds:

$$\begin{aligned} \mathcal{M}_{scalar} &= \mathcal{SM} \otimes \mathcal{HM}, \\ \dim_{\mathbf{C}} \mathcal{SM} &= n_V, \\ \dim_{\mathbf{R}} \mathcal{HM} &= 4\, n_H, \end{aligned}$$

where $\mathcal{SM}$, $\mathcal{HM}$ are respectively *special Kähler* and *quaternionic Kähler* and n_V, n_H are respectively the number of *vector multiplets* and *hypermultiplets* contained in the theory. The direct product structure Eqn. (2.11) imposed by supersymmetry precisely reflects the fact that the quaternionic and special Kähler scalars belong to different supermultiplets [45]. $\mathcal{M}_V$ is a special Kähler manifold and $\mathcal{M}_H$ is a quaternionic Kähler manifold.

This is a very important result: since the string coupling constant is given by the vacuum expectation value of the dilaton $g_s \equiv e^{-\phi/2}$ and the the four-dimensional reduction of the dilaton always belongs to a hypermultiplet, the hypermultiplet sector receives both perturbative and non-perturbative g_s corrections [15]. Non-perturbative corrections arise from instantons and/or branes wrapping cycles in the Calabi-Yau. The vector multiplet geometry remains unaffected.

Up to now we were only considering the higher dimensional origin of the massless states in four dimension. However, also the coupling of the vector multiplet scalars to the vectors is encoded in the Calabi Yau geometry, namely in a holomorphic function called the *prepotential* (see also Appendix C.1). To start with we introduce a real symplectic basis $(\alpha_\Lambda, \beta^\Sigma)$ [16] of 3-forms of $H^3(CY) = H^{(3,0)} \oplus H^{(2,1)} \oplus H^{(1,2)} \oplus H^{(0,3)}$ chosen such that they satisfy

$$\int_{A^\Lambda} \alpha_\Sigma = \int \alpha_\Sigma \wedge \beta^\Lambda = \delta^\Lambda{}_\Sigma \ , \tag{2.11}$$

$$\int_{B_\Lambda} \beta^\Sigma = \int \beta^\Sigma \wedge \alpha_\Lambda = -\delta^\Sigma{}_\Lambda \ , \tag{2.12}$$

$$\int \alpha_\Lambda \wedge \alpha_\Sigma = \int \beta^\Lambda \wedge \beta^\Sigma = 0, \tag{2.13}$$

where (A^Λ, B_Σ) denotes the dual homology basis of 3-cycles [1] with intersection numbers

$$A^\Lambda \cap B_\Sigma = -B_\Sigma \cap A^\Lambda = \delta^\Lambda{}_\Sigma \ , \qquad \text{and} \qquad A^\Lambda \cap A^\Sigma = B_\Lambda \cap B_\Sigma = 0, \tag{2.14}$$

and $\Lambda, \Sigma = 0 \ldots h^{2,1}$. Now we can define coordinates on the moduli space by the periods of the holomorphic 3-form Ω

$$\mathcal{X}^\Lambda = \int_{A^\Lambda} \Omega = \int \Omega \wedge \beta^\Lambda. \tag{2.15}$$

In this way we define one more coordinate than we have moduli fields, but the additional degree of freedom is killed by fixing the $U(1)$ gauge freedom, as described in Appendix C.1. In order not to have more independent variables, the B periods

$$\mathcal{F}_\Lambda = \int_{B_\Lambda} \Omega = \int \Omega \wedge \alpha_\Lambda \tag{2.16}$$

must be functions of the $\mathcal{X}$ and now Ω (which from real point of view is just a 3-form) expanded in the basis of 3-forms reads

$$\Omega = \mathcal{X}^\Lambda \alpha_\Lambda - \mathcal{F}_\Lambda \beta^\Lambda \tag{2.17}$$

and using Eq. (2.7) the Kähler potential takes the form

$$\mathcal{K} = -\log\left(i(\mathcal{X}^{*\Lambda}\mathcal{F}_\Lambda - \mathcal{X}^\Lambda \mathcal{F}^*{}_\Lambda)\right) . \tag{2.18}$$

[1]Observe that the 3-form α_Λ is the Poincaré dual of the 3-cycle B_Λ and β^Σ of A^Σ, respectively.

Since under a change of the complex structure Eq. (2.6) dz becomes a linear combination of dz and $d\bar{z}$, the holomorphic $(3,0)$-form Ω becomes a linear combination of $(3,0)$ and $(2,1)$-forms [10]

$$\partial_\Lambda \Omega \in H^{(3,0)} \oplus H^{(2,1)}, \tag{2.19}$$

it follows

$$\Omega \wedge \partial_\Lambda \Omega = 0. \tag{2.20}$$

Integrating the last equation over the Calabi-Yau threefold and taking into account the basic properties of the basis of 3-forms Eqs. (2.11)-(2.13) this implies

$$\mathcal{F}_\Lambda = \mathcal{X}^\Sigma \partial_\Lambda \mathcal{F}_\Sigma, \tag{2.21}$$

where

$$\mathcal{F} = \tfrac{1}{2} \mathcal{X}^\Lambda \mathcal{F}_\Lambda. \tag{2.22}$$

This function is exactly the prepotential of $N = 2$ supergravity in four dimensions (Appendix C.1).

Notice that the results in case IIA/B are the same upon the exchange $h^{p,q} \longleftrightarrow h^{3-p,q}$. This phenomenon for Calabi-Yau threefolds is part of what is called *mirror symmetry*: type IIA theory compactified on a Calabi-Yau threefold M is equivalent to type IIB compactified on the mirror Calabi-Yau threefold W. The mirror map associates to a Calabi-Yau threefold M another one W such that

$$h^{p,q}(M) = h^{3-p,q}(W). \tag{2.23}$$

This means that mirror symmetry maps the complex structure moduli space of type IIB compactified on M to the Kähler structure moduli space of type IIA on W. But apart from the fact that the low energy spectrum of type IIA on M and IIB on the mirror manifold W are the same (up to now we were only considering the massless Kaluza-Klein modes), the mirror symmetry proposal implies much more. Actually mirror symmetry claims the two theories to be exactly equivalent to all orders of α', i.e. including stringy effects [17]. The α' corrections are controlled by the Kähler moduli, which for type $IIB(IIA)$ appear in the lower-dimensional theory through the scalars in a hypermultiplet (vector multiplet). This implies that the result obtained for type IIB on M, the vector multiplet moduli space, i.e. the complex structure moduli space, does not suffer from α' corrections, and the result obtained in the supergravity approximation is exact to all orders in α'. Mirror symmetry thus allows us to obtain information about the α'-corrections of the hypermultiplet sector in type IIA on the mirror manifold W, which are highly non-trivial.

Thence mirror symmetry is a very useful concept, e.g. to compute the holomorphic prepotential of the effective action, although it has not been proven yet [18].

2.2 Supersymmetric configurations and solutions

It is essential for the understanding of what follows to distinguish between supersymmetric *configurations* and supersymmetric *solutions* of a theory. In this section we will see what

Supersymmetry can tell us about solutions of the field equations and how it restricts the number of independent equations of motion. In general supersymmetric configurations of a supergravity theory are not invariant under all the supersymmetry transformations. Schematically, these SUSY transformations are generically of the form

$$\delta_\epsilon B \sim \bar{\epsilon} F \tag{2.24}$$

$$\delta_\epsilon F \sim \partial\epsilon + B\epsilon, \tag{2.25}$$

where B and F symbolically denote the bosonic and fermionic fields of the theory, respectively. A classical bosonic configuration (i.e. a configuration with vanishing fermionic fields $F = 0$) is invariant under the infinitesimal supersymmetry transformation generated by $\epsilon(x)$ if it satisfies

$$\delta_\epsilon F \sim \partial\epsilon + B\epsilon = 0. \tag{2.26}$$

These equations are called *Killing Spinor Equations (KSEs)* and the parameters $\epsilon(x)$ which generate the transformations accordingly *Killing spinors*. In supergravities (which may have one or more than one supercharge, $N \geq 1$) a configuration is called *supersymmetric* if there is at least one Killing spinor such that Eq. (2.26) is satisfied. Note that supersymmetry does not imply that the configurations are also solutions of the (classical) equations of motion and in what follows it is essential to distinguish between supersymmetric configurations and solutions. Actually, to reach our aim to find the supersymmetric solutions of a given supergravity theory, it is in general much simpler to start with finding supersymmetric configurations, since the equations of motion are second order differential equations, whereas the KSEs are only of first order. Further, the supersymmetric field configurations satisfy the so-called *Killing Spinor Identities (KSIs)*, which can be derived, for instance, from the integrability conditions of the KSEs. These equations relate the different (bosonic) equations of motion and their content is highly non-trivial, even if each term vanishes separately on-shell. Since in this way they reduce the number of independent equations to solve, they are of great avail in finding supersymmetric solutions. This is reflected by the fact that supersymmetric solutions are given in terms of a very small number of independent functions. This strategy, to exploit the KSIs in order to find supersymmetric solutions of a supergravity theory, was first applied in [19, 20] in the context of minimal five-dimensional and eleven-dimensional supergravity, respectively. However, the general Killing Spinor Identities, which the bosonic equations of motion have to satisfy in supersymmetric theories if the solutions admit Killing spinors, were found earlier in [21].

The Killing spinor identities can be derived from the supersymmetry variation of the action in the following way [22]: by hypothesis

$$\delta_\epsilon S = \int d^d x (\delta_B S \delta_\epsilon B + \delta_F S \delta_\epsilon F) = 0 \tag{2.27}$$

where B and F denote schematically the bosonic and fermionic fields, respectively, of the theory. $S_{,B} = \delta_B S = \frac{\delta S}{\delta B}$ is the equation of motion of the fermion field B and analogously for the fermions. Summation over the indices F, B is understood. Now we vary this equation w.r.t. the fermionic fields

$$\{S,_{BF_2}\,\delta_\epsilon B + S,_B\,(\delta_\epsilon B),_{F_2} + S,_{F_1F_2}\,\delta_\epsilon F_1 + S,_{F_1}\,(\delta_\epsilon F),_{F_2}\}|_{F=0} = 0. \tag{2.28}$$

Since we are only interested in bosonic backgrounds, we are now going to set the fermionic fields to zero. The bosonic equations of motion $S,_B$ and the supersymmetry variations of the fermions $\delta_\epsilon F$ are necessarily even in fermions and thus the first and the fourth term in Eq. (2.28) vanish:

$$S,_{BF_2}|_{F=0} = 0, \qquad (\delta_\epsilon F),_{F_2} = 0, \tag{2.29}$$

and we are left with

$$\{S,_B\,(\delta_\epsilon B),_{F_2} + S,_{F_1F_2}\,\delta_\epsilon F_1\}|_{F=0} = 0. \tag{2.30}$$

This equations is valid for arbitrary values of the bosonic fields and the supersymmetry parameter ϵ. We are interested in supersymmetric bosonic configurations, i.e. field configurations which admit (at least) one Killing spinor κ. A Killing spinor satisfies by definition the *Killing spinor equation*

$$\delta_\kappa F|_{F=0} = 0, \tag{2.31}$$

and thus it turns out that supersymmetric bosonic configurations always fullfill the *Killing spinor identities (KSIs)*

$$S,_B\,(\delta_\kappa B),_F|_{F=0} = 0. \tag{2.32}$$

Written in this form it is easy to see that the KSIs relate the bosonic equations of motion of the theory, as already mentioned in the previous paragraph.

Observe that the Bianchi identities (involving vector fieldstrengths, in the case treated in this thesis, or $p+1$-form field strengths in the general case) do not appear in these relations because the procedure used to derive them assumes the existence of the potentials and, therefore, the vanishing of the Bianchi identities. Since it is convenient to treat Maxwell equations and Bianchi identities on equal footing to preserve the electric-magnetic dualities of the theory, it is convenient to have the duality-covariant version of the above KSIs. These can be found by performing duality rotations of the above identities or from the integrability conditions of the KSEs. This will be done in detail in Section 3.2.2.

2.3 Statement of the problem and how to solve it

Since the main purpose of this thesis is to find systematically all the supersymmetric solutions of ungauged $N = 2$, $d = 4$ Supergravity, we should say some words about what we mean by "finding solutions" and how we are going to proceed in order to find the (complete) set of them. Finding supersymmetric configurations of the theory means expressing the bosonic fields of it in terms of a minimal set of independent variables in such a way that they admit Killing spinors, i.e. the Killing spinor equations are fullfilled for at least one Killing spinor whose existence is to be proved. The next step is to check which of these field configurations fullfill the equations of motions, viz. to find supersymmetric solutions.

The basic strategy to find supersymmetric solutions of a given supergravity theory is to assume the existence of at least one Killing spinor, and to derive consistency conditions (necessary conditions) in terms of bilinears constructed out of the Killing spinor(s). In more detail:[2]

I Translate the Killing spinor equations and KSIs into tensorial equations.

With the Killing spinor ϵ one can construct scalar, vector, and p- form bilinears $M \sim \bar{\epsilon}\epsilon\,, \quad V_\mu \sim \bar{\epsilon}\gamma_\mu\epsilon\,, \cdots$ that are related by Fierz identities. These bilinears satisfy certain equations because they are made out of Killing spinors, for instance, if the KSE is of the general form

$$\delta_\epsilon \psi_\mu = \tilde{\mathcal{D}}_\mu \epsilon = [\nabla_\mu + \Omega_\mu]\epsilon = 0\,, \ \Rightarrow \ \nabla_\mu M + 2\Omega_\mu M = 0\,, \tag{2.33}$$

The set of all such equations for the bilinears should be equivalent to the original spinorial equation or at least it should contain most of the information contained in it (but, certainly, not all of it).

II One of the vector bilinears (say V_μ) is always a Killing vector which can be timelike or null. These two cases are treated separately.

III One can get an expression of all the gauge field strengths of the theory using the Killing equation for those scalar bilinears: Ω_μ is usually of the form $F_{\mu\nu}V^\nu$ and, then Eq. (2.33) tells us that $F_{\mu\nu}V^\nu \sim \nabla_\mu \log M$. When V is timelike this determines completely F and, when it is null, it determines the general form of F. Of course, Eq. (2.33) is an oversimplified KSE and in real-life situations there are additional scalar factors, $SU(N)$ indices etc.

IV Up to now we found expressions for the bosonic fields of the theory which fullfill certain conditions, which we derived from the KSEs as necessary conditions for supersymmetry. Now we have to prove their sufficiency, that is we have to show the existence of the Killing spinor(s) we assumed to exist. This leads to additional conditions on the Killing spinors, which tell us the minimal amount of unbroken supersymmetry in the most general setup. Once the existence of the Killing spinor(s) is ensured, we have found all supersymmetric configurations of the theory.

V The KSEs have to fullfill some consistency conditions : the integrability conditions of the KSEs (the KSIs). These relate the Maxwell equations, Bianchi identities and the other bosonic equations of motion and guarantee that these sets of equations are combinations of a reduced number of simple equations involving a reduced number of scalar unknowns. solutions of the theory. The tricky part is, usually, identifying the right variables that satisfy simple equations and finding these equations as combinations of the Maxwell, Einstein etc. equations.

[2] We follow the procedure of [23], which we rewrite here for completeness

VI The equations of motion are imposed in order to find the supersymmetric solutions of the theory. In general, there are only the Maxwell and Bianchi identities and very few other independent equations left, since due to the KSIs supersymmetry already ensured some of the equations of motion to be fullfilled automatically. [3]

VII Find interesting examples

[3]In $N = 2$ $d = 4$ supergravity, which is the theory dealt with in this thesis, we will see that only the Maxwell equations, Bianchi identities and in the Null case one component of the Einstein equations have to be imposed.

Chapter 3

Supersymmetric solutions of ungauged $N = 2$, $d = 4$ supergravity

3.1 Field content, action and equations of motion

In this section we are going to describe briefly the theory we are going to work with. Our conventions for the metric, connection, curvature, gamma matrices and spinors are described in detail in Appendix A which also contains many identities and results that will be used repeatedly throughout the text. In four dimensions theories with up to 16 (real) supersymmetries allow matter multiplets. In this paper we are considering the coupling of supergravity to n_V vector multiplets and n_H hypermultiplets, thus we are dealing with the following fields:

Gravity multiplet

- Graviton ${e_a}^\mu$
- A pair of gravitinos $\Psi_{I\mu}$, $I = 1, 2$
- Vector field A_μ

n_V **Vector multiplets,** $i = 1 \dots n_V$

- Complex scalar Z^i
- A pair of gauginos λ^{Ii}, $I = 1, 2$
- Vector field ${A^i}_\mu$

n_H **Hypermultiplets**

- 4 real scalars q^u, $u = 1 \ldots 4n_H$
- 2 hyperinos ζ^α, $\alpha = 1 \ldots 2n_H$

In the coupled theory we denote the vector fields collectively by $A^\Lambda{}_\mu$, $\Lambda = 1 \ldots \bar{n}$ where $\bar{n} = n_V + 1$.

The action of the bosonic fields of the theory is

$$\begin{aligned} S &= \int d^4x \sqrt{|g|} \left[R + 2\mathcal{G}_{ij^*}\partial_\mu Z^i \partial^\mu Z^{*\,j^*} + 2\mathsf{h}_{uv}\partial_\mu q^u \partial^\mu q^v \right. \\ &\qquad \left. + 2\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma} F^{\Lambda\,\mu\nu} F^\Sigma{}_{\mu\nu} - 2\Re\mathrm{e}\mathcal{N}_{\Lambda\Sigma} F^{\Lambda\,\mu\nu\star} F^\Sigma{}_{\mu\nu} \right] , \end{aligned} \tag{3.1}$$

The coupling of scalars to scalars is described by a non-linear σ-model with Kähler metric $\mathcal{G}_{ij^*}(Z, Z^*)$ (see Appendix B), and the coupling to the vector fields by a complex scalar-field-valued matrix $\mathcal{N}_{\Lambda\Sigma}(Z, Z^*)$. These two couplings are related by a structure called special Kähler geometry, described in Appendix C. The symmetries of these two sectors will be related and this relation will be discussed shortly. The $4n_H$ hyperscalars parameterize a quaternionic Kähler manifold (defined and studied in Appendix D) with metric $\mathsf{h}_{uv}(q)$ [24]. Observe that the hypermultiplets do not couple to the vector multiplets.

For convenience, we denote the bosonic equations of motion by

$$\mathcal{E}_a{}^\mu \equiv -\frac{1}{2\sqrt{|g|}}\frac{\delta S}{\delta e^a{}_\mu}\,, \qquad \mathcal{E}_i \equiv -\frac{1}{2\sqrt{|g|}}\frac{\delta S}{\delta Z^i}\,, \tag{3.2}$$

$$\mathcal{E}_\Lambda{}^\mu \equiv \frac{1}{8\sqrt{|g|}}\frac{\delta S}{\delta A^\Lambda{}_\mu}\,, \qquad \mathcal{E}^u \equiv -\frac{1}{4\sqrt{|g|}}\mathsf{h}^{uv}\frac{\delta S}{\delta q^v}\,. \tag{3.3}$$

and the Bianchi identities for the vector field strengths by

$$\mathcal{B}^{\Lambda\,\mu} \equiv \nabla_\nu{}^\star F^{\Lambda\,\nu\mu}\,. \tag{3.4}$$

The explicit forms of the equations of motion can be found to be

$$\begin{aligned} \mathcal{E}_{\mu\nu} &= G_{\mu\nu} + 2\mathcal{G}_{ij^*}[\partial_\mu Z^i \partial_\nu Z^{*\,j^*} - \tfrac{1}{2}g_{\mu\nu}\partial_\rho Z^i \partial^\rho Z^{*\,j^*}] \\ &\quad + 8\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma} F^{\Lambda\,+}{}_\mu{}^\rho F^{\Sigma\,-}{}_{\nu\rho} + \ 2\mathsf{h}_{uv}\left[\partial_\mu q^u \partial_\nu q^v - \tfrac{1}{2}g_{\mu\nu}\partial_\rho q^u \partial_\rho q^v\right], \end{aligned} \tag{3.5}$$

$$\mathcal{E}_i = \nabla_\mu(\mathcal{G}_{ij^*}\partial^\mu Z^{*\,i^*}) - \partial_i\mathcal{G}_{jk^*}\partial_\rho Z^j \partial^\rho Z^{*\,k^*} + \partial_i[\tilde{F}_\Lambda{}^{\mu\nu\star} F^\Lambda{}_{\mu\nu}]\,, \tag{3.6}$$

$$\mathcal{E}_{\Lambda}{}^{\mu} = \nabla_{\nu}{}^{\star}\tilde{F}_{\Lambda}{}^{\nu\mu}\,, \tag{3.7}$$

$$\mathcal{E}^{u} = \mathfrak{D}_{\mu}\partial^{\mu}q^{u} = \nabla_{\mu}\partial^{\mu}q^{u} + \Gamma_{vw}{}^{u}\partial^{\mu}q^{v}\partial_{\mu}q^{w}\,, \tag{3.8}$$

where we have defined the dual vector field strength $\tilde{F}_{\Lambda}$ by

$$\tilde{F}_{\Lambda\,\mu\nu} \equiv -\frac{1}{4\sqrt{|g|}}\frac{\delta S}{\delta^{\star}F^{\Lambda}{}_{\mu\nu}} = \Re\mathrm{e}\mathcal{N}_{\Lambda\Sigma}F^{\Sigma}{}_{\mu\nu} + \Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}{}^{*}F^{\Sigma}{}_{\mu\nu}\,. \tag{3.9}$$

The symmetries of this set of equations of motion are the isometries of the Kähler manifold and those of the quaternionic manifold. A prerequisite to understand the following development is a study of the symplectic transformations. These are duality symmetries of four dimensions, which are a generalization of electromagnetic duality [25]. The Maxwell and Bianchi identities can be rotated into each other by $GL(2\bar{n}, \mathbb{R})$ transformations under which they are a $2\bar{n}$-dimensional vector:

$$\mathcal{E}^{\mu} \equiv \begin{pmatrix} \mathcal{B}^{\Lambda\,\mu} \\ \\ \mathcal{E}_{\Lambda}{}^{\mu} \end{pmatrix} \longrightarrow \begin{pmatrix} D & C \\ \\ B & A \end{pmatrix} \begin{pmatrix} \mathcal{B}^{\Lambda\,\mu} \\ \\ \mathcal{E}_{\Lambda}{}^{\mu} \end{pmatrix}\,, \tag{3.10}$$

where A, B, C and D are $\bar{n} \times \bar{n}$ matrices. These transformations act in the same form on the vector of $2\bar{n}$ 2-forms

$$F \equiv \begin{pmatrix} F^{\Lambda} \\ \\ \tilde{F}_{\Lambda} \end{pmatrix} \longrightarrow \begin{pmatrix} D & C \\ \\ B & A \end{pmatrix} \begin{pmatrix} F^{\Lambda} \\ \\ \tilde{F}_{\Lambda} \end{pmatrix}\,. \tag{3.11}$$

Now we are going to see, that consistency of this transformation rule with the definition of $\tilde{F}$ Eq. (3.9) requires the matrix

$$S = \begin{pmatrix} D & C \\ \\ B & A \end{pmatrix} \tag{3.12}$$

to belong to the symplectic subgroup of the general linear group:

$$S \in Sp(2\bar{n}, \mathbb{R}) \subset GL(2\bar{n}, \mathbb{R}). \tag{3.13}$$

While the duality rotation Eq. (3.11) is performed on the field strengths and their duals, also the scalar fields are transformed (since they belong to the same multiplets) by a diffeomorphism of the scalar manifold and, as a consequence, the matrix $\mathcal{N}_{\Lambda\Sigma}$ changes. By definition it is

$$\tilde{F}'_{\Lambda} = \Re\mathrm{e}\mathcal{N}'_{\Lambda\Sigma}F'^{\Sigma} + \Im\mathrm{m}\mathcal{N}'_{\Lambda\Sigma}{}^{\star}F'^{\Sigma}\,, \tag{3.14}$$

and for the transformations to be consistently defined, they must act on the period matrix $\mathcal{N}$ according to

$$\mathcal{N}' = (A\mathcal{N} + B)(C\mathcal{N} + D)^{-1} \equiv \mathcal{N}(Z', Z'^*)\,. \tag{3.15}$$

Furthermore, the transformations must preserve the symmetry of the period matrix, which requires

$$A^T C = C^T A\,, \qquad D^T B = B^T D\,, \qquad A^T D - C^T B = 1\,, \tag{3.16}$$

i.e. the transformations must belong to $Sp(2\bar{n}, \mathbb{R})$.

The above transformation rules for the vector field strength and period matrix imply

$$\Im\mathrm{m}\mathcal{N}' = (C\mathcal{N}^* + D)^{-1\,T}\Im\mathrm{m}\mathcal{N}(C\mathcal{N} + D)^{-1}\,, \qquad F'^{\Lambda\,+} = (C\mathcal{N}^* + D)_{\Lambda\Sigma}F^{\Sigma\,+}\,, \tag{3.17}$$

so the combination $\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,+}{}_{\mu}{}^{\rho}F^{\Lambda\,+}{}_{\nu\rho}$ that appears in the energy-momentum tensor is automatically invariant. These transformations have to be symmetries of the σ-model as well, which implies that only the isometries of the special Kähler manifold which are embedded in $Sp(2\bar{n}, \mathbb{R})$ and those of the quaternionic manifold parameterized by the hyperscalars are symmetries of all the equations of motion of the theory (dualities of the theory).

For vanishing fermions, the supersymmetry transformation rules of the fermions are

$$\delta_\epsilon \psi_{I\,\mu} \;=\; \mathfrak{D}_\mu \epsilon_I + \varepsilon_{IJ}\; T^{+}{}_{\mu\nu}\gamma^\nu\; \epsilon^J\,, \tag{3.18}$$

$$\delta_\epsilon \lambda^{iI} \;=\; i\; \not\partial Z^i \epsilon^I \;+\; \varepsilon^{IJ}\; \not G^{i\,+}\; \epsilon_J\,. \tag{3.19}$$

$$\delta_\epsilon \zeta_\alpha \;=\; -i\mathbb{C}_{\alpha\beta}\; \mathsf{U}^{\beta I}{}_u\; \varepsilon_{IJ}\;\; \not\partial q^u\; \epsilon^J\,, \tag{3.20}$$

Here $\mathfrak{D}$ is the Lorentz and Kähler-covariant derivative of Ref. [26] supplemented by (the pullback of) an $SU(2)$ connection $\mathsf{A}_I{}^J$ described in Appendix D, acting on objects with $SU(2)$ indices I, J and, in particular, on ϵ_I as:

$$\mathfrak{D}_\mu \epsilon_I = (\nabla_\mu \;+\; \tfrac{i}{2}\; \mathcal{Q}_\mu)\; \epsilon_I \;+\; \mathsf{A}_{\mu\,I}{}^J\; \epsilon_J\,. \tag{3.21}$$

This is the only place in which the hyperscalars appear in the supersymmetry transformation rules of the gravitinos and gauginos. $\mathsf{U}^{\beta I}{}_u$ is a Quadbein, *i.e.* a quaternionic Vielbein, and $\mathbb{C}_{\alpha\beta}$ the $Sp(m)$-invariant metric, both of which are described in Appendix D.

The supersymmetry transformations of the bosons are

$$\delta_\epsilon e^a{}_\mu \;=\; -\tfrac{i}{4}(\bar{\psi}_{I\,\mu}\gamma^a\epsilon^I + \bar{\psi}^I{}_\mu\gamma^a\epsilon_I)\,, \tag{3.22}$$

$$
\begin{aligned}
\delta_\epsilon A^\Lambda{}_\mu &= \tfrac{1}{4}(\mathcal{L}^{\Lambda\,*}\varepsilon^{IJ}\bar{\psi}_{I\,\mu}\,\epsilon_J \;+\; \mathcal{L}^\Lambda\varepsilon_{IJ}\bar{\psi}^I{}_\mu\,\epsilon^J) \\
&\quad + \tfrac{i}{8}(f^\Lambda{}_i\varepsilon_{IJ}\bar{\lambda}^{iI}\gamma_\mu\epsilon^J + f^{\Lambda *}{}_{i^*}\varepsilon^{IJ}\bar{\lambda}^{i^*}{}_I\gamma_\mu\epsilon_J)\,, \qquad (3.23)
\end{aligned}
$$

$$
\delta_\epsilon Z^i = \tfrac{1}{4}\bar{\lambda}^{iI}\epsilon_I\,, \tag{3.24}
$$

$$
\delta_\epsilon q^u = \mathsf{U}_{\alpha I}{}^u(\bar{\zeta}^\alpha\epsilon^I + \mathbb{C}^{\alpha\beta}\epsilon^{IJ}\bar{\zeta}_\beta\epsilon_J)\,. \tag{3.25}
$$

Observe that the fields of the hypermultiplet and the fields of the gravity and vector multiplets do not mix in any of these supersymmetry transformation rules. This means that the KSIs associated to the gravitinos and gauginos will have the same form as in Ref. [26] and in the KSIs associated to the hyperinos only the hyperscalars equations of motion will appear.

3.2 Supersymmetric configurations: general setup

Our first goal is to find all the bosonic field configurations $\{g_{\mu\nu}, F^\Lambda{}_{\mu\nu}, Z^i, q^u\}$ for which the Killing spinor equations (KSEs) admit at least one solution. It must be stressed that the configurations considered need not be classical solutions of the equations of motion. Furthermore, we will not assume that the Bianchi identities are satisfied by the field strengths of a configuration.

Our second goal will be to identify among all the supersymmetric field configurations those that satisfy all the equations of motion (including the Bianchi identities).

3.2.1 Killing spinor equations

The supersymmetry variations of the fermionic field of the theory are given by:

$$
\delta_\epsilon\psi_{I\,\mu} = \mathfrak{D}_\mu\epsilon_I + \epsilon_{IJ}T^+{}_{\mu\nu}\gamma^\nu\epsilon^J = 0\,, \tag{3.26}
$$

$$
\delta_\epsilon\lambda^{Ii} = i\,\not{\partial}Z^i\epsilon^I + \epsilon^{IJ}\,\not{G}^{i\,+}\epsilon_J = 0\,, \tag{3.27}
$$

$$
\delta_\epsilon\zeta_\alpha = -i\mathbb{C}_{\alpha\beta}\,\mathsf{U}^{\beta I}{}_u\,\varepsilon_{IJ}\;\not{\partial}q^u\,\epsilon^J = 0\,. \tag{3.28}
$$

3.2.2 Killing spinor identities

Using the supersymmetry transformation rules of the bosonic fields Eqs. (3.22)-(3.25) and using the procedure described in Section 2.2 we can derive the following relations (*Killing* *spinor identities*, KSIs) between the (off-shell) equations of motion of the bosonic fields that are satisfied by any field configuration $\{e^a{}_\mu, A^\Lambda{}_\mu, Z^i, q^u\}$ admitting Killing spinors:

$$\mathcal{E}_a{}^\mu \gamma^a \epsilon^I - 4i\epsilon^{IJ}\mathcal{L}^\Lambda \mathcal{E}_\Lambda{}^\mu \epsilon_J \;=\; 0\,, \tag{3.29}$$

$$\mathcal{E}^i \epsilon^I - 2i\epsilon^{IJ} f^{*\,i\Lambda}\; \not{\mathcal{E}}_\Lambda \epsilon_J \;=\; 0\,. \tag{3.30}$$

The vector field Bianchi identities Eq. (3.4) do not appear in these relations because the procedure used to derive them, assumes the existence of the vector potentials, and hence the vanishing of the Bianchi identities.

It is convenient to treat the Maxwell equations and Bianchi identities on an equal footing as to preserve the electric-magnetic dualities of the theory, for which it is convenient to have a duality-covariant version of the above KSIs. This can be found by performing duality rotations on the above identities or from the integrability conditions of the KSEs Eqs. (3.26),(3.27) and (3.28), which is the method we are going to use.

Using the Kähler special geometry machinery, we obtain

$$\begin{aligned}
\mathfrak{D}_{[\mu}\delta_\epsilon\psi_{I\,\nu]} \;&=\; -\tfrac{1}{8}\{[R_{\mu\nu}{}^{ab} - 8\mathcal{T}_\Lambda\mathcal{T}^*_\Sigma F^{\Lambda\,+}{}_{[\mu|}{}^a F^{\Sigma\,-}{}_{|\nu]}{}^b]\gamma_{ab} + 4\mathcal{G}_{ij^*}\partial_{[\mu}Z^i\partial_{\nu]}Z^{*\,j^*}\}\epsilon_I \\
&\quad +\epsilon_{IJ}\mathfrak{D}_{[\mu}T^+{}_{\nu]\rho}\gamma^\rho\epsilon^J + \left(\partial_{[\mu}\mathsf{A}_{\nu]I}{}^J + \mathsf{A}_{[\mu I}{}^K\mathsf{A}_{\nu]K}{}^J\right)\epsilon_J\,, \qquad (3.31)\\
&=\; 0.
\end{aligned}$$

where, using Eq. (D.29)

$$\begin{aligned}
\left(\partial_{[\mu}\mathsf{A}_{\nu]I}{}^J + \mathsf{A}_{[\mu I}{}^K\mathsf{A}_{\nu]K}{}^J\right)\epsilon_J \;&=\; \tfrac{1}{2}\Omega_{\mu\nu\;I}{}^J \qquad (3.32)\\
&=\; \lambda\mathsf{U}_{uI\alpha}\mathsf{U}_v{}^{J\alpha}\partial_{[\mu}q^u\partial_{\nu]}q^v \qquad (3.33)
\end{aligned}$$

Setting $\lambda = -1$ this can be simplified to

$$4\gamma^\nu\mathfrak{D}_{[\mu}\delta_\epsilon\psi_{I\,\nu]} = (\mathcal{E}_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}\,\mathcal{E}_\sigma{}^\sigma)\gamma^\nu\epsilon_I - 2i\epsilon_{IJ}\mathcal{L}^\Lambda(\not{\mathcal{E}}_\Lambda - \mathcal{N}_{\Lambda\Sigma}\,\not{\mathcal{B}}^\Sigma)\gamma_\mu\epsilon^J = 0\,. \tag{3.34}$$

Contracting this identity with γ^μ we get another one involving only the trace $\mathcal{E}_\sigma{}^\sigma$, which can be used in the above identity to eliminate completely that trace. The result is the duality-covariant version of (the complex conjugate of) Eq. (3.29) we were after:

$$\mathcal{E}_a{}^\mu\gamma^a\epsilon_I - 4i\epsilon_{IJ}\mathcal{L}^\Lambda(\mathcal{E}_\Lambda{}^\mu - \mathcal{N}_{\Lambda\Sigma}\mathcal{B}^{\Sigma\mu})\epsilon^J = 0\,. \tag{3.35}$$

The $SU(2)$ connection acts on objects with vector $SU(2)$ indices such as the chiral spinors we are dealing with, as follows:

$$\begin{aligned}
\mathfrak{D}\xi_I \;&\equiv\; d\xi_I + \mathsf{A}_I{}^J\xi_J\,,\\
\mathfrak{D}\chi^I \;&\equiv\; d\chi^I + \mathsf{A}^I{}_J\chi^J\,.
\end{aligned} \tag{3.36}$$

Consistency with the raising and lowering of vector $SU(2)$ indices via complex conjugation requires

$$\mathsf{A}^I{}_J = (\mathsf{A}_I{}^J)^* = -\mathsf{A}_J{}^I\,, \tag{3.37}$$

It is convenient to define the combination

$$\mathcal{H}^{\Lambda\mu} \equiv (\Im\mathrm{m}\mathcal{N})^{-1|\Lambda\Sigma}(\mathcal{E}_\Sigma{}^\mu - \mathcal{N}_{\Sigma\Omega}\mathcal{B}^{\Sigma\mu})\,. \tag{3.38}$$

Using it, the above KSIs Eqs. (3.34) and (3.35) take the form

$$(\mathcal{E}_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}\,\mathcal{E}_\sigma{}^\sigma)\gamma^\nu\epsilon_I - \mathcal{T}_\Lambda\,\not{\mathcal{H}}^\Lambda\gamma_\mu\epsilon_{IJ}\epsilon^J \;=\; 0\,, \tag{3.39}$$

$$\mathcal{E}_a{}^\mu\gamma^a\epsilon_I - 2\mathcal{T}_\Lambda\mathcal{H}^{\Lambda\mu}\epsilon_{IJ}\epsilon^J \;=\; 0\,. \tag{3.40}$$

Observe that the graviphoton-projected combination $\mathcal{T}_\Lambda\mathcal{H}^{\Lambda\,\mu}$ can be written in the form

$$\mathcal{T}_\Lambda\mathcal{H}^{\Lambda\,\mu} = 2i[\mathcal{L}^\Lambda\mathcal{E}_\Lambda{}^\mu - \mathcal{M}_\Lambda\mathcal{B}^{\Lambda\,\mu}] = 2i\,\langle\,\mathcal{E}^\mu\mid\mathcal{V}\,\rangle\,, \tag{3.41}$$

where $\mathcal{E}$ is the symplectic vector defined in Eq. (3.10).

We get in a similar way

$$-i\,\not{\mathcal{D}}\delta_\epsilon\lambda^{Ii} = \mathcal{E}^i\epsilon^I - 2i\mathcal{T}^i{}_\Lambda\,\not{\mathcal{H}}^\Lambda\epsilon^{IJ}\epsilon_J = 0\,. \tag{3.42}$$

The KSIs involving the equations of motion of the bosonic fields of the gravity and vector multiplets take, of course, the same form as in absence of hypermultiplets Now we are going to consider the supersymmetry variation of the hyperino. It is

$$\not{\mathfrak{D}}\delta_\epsilon\zeta_\alpha = -i\mathbb{C}_{\alpha\beta}\,\mathsf{U}^{\beta I}{}_u\,\varepsilon_{IJ}\,\gamma^\mu\gamma^\nu(\mathfrak{D}\partial_\nu q^u\epsilon^J + \partial_\nu q^u\mathfrak{D}_\mu\epsilon^J) \tag{3.43}$$

where we used the covariant constancy of the Quadbein, Eq. (D.26). Now we use

$$\mathfrak{D}_{[\mu}\partial_{\nu]}q^u = \partial_{[\mu}\partial_{\nu]}q^u - \partial_\rho q^u\Gamma_{[\mu\nu]}{}^\rho + \Gamma_{vw}{}^u\partial_{[\nu}q^v\partial_{\mu]}q^w = 0 \tag{3.44}$$

and

$$F^{-\Lambda}{}_{\mu\rho}\gamma^\rho\epsilon_K = -\tfrac{1}{4}\,\not{F}^{-\Lambda}\gamma_\mu\epsilon_K \tag{3.45}$$

and we obtain

$$\mathcal{E}^u\mathsf{U}_u{}^{I\alpha}\epsilon_I = 0. \tag{3.46}$$

.

3.2.3 KSEs for the bilinears

In four dimensions we can construct the following tensor bilinears out of the Killing spinors:

1. A complex matrix of scalars

$$M_{IJ} \equiv \bar{\epsilon}_I \epsilon_J \,, \qquad M^{IJ} \equiv \bar{\epsilon}^I \epsilon^J = (M_{IJ})^* \qquad M_{IJ} = -M_{JI}, \tag{3.47}$$

2. A complex hermitean matrix of vectors

$$V^I{}_{J\,a} \equiv i\bar{\epsilon}^I \gamma_a \epsilon_J \,, \qquad V_I{}^J{}_a \equiv i\bar{\epsilon}_I \gamma_a \epsilon^J = (V^I{}_{J\,a})^* \,, \tag{3.48}$$

3. A complex matrix of 2-forms

$$\Phi_{IJ\,ab} \equiv \bar{\epsilon}_I \gamma_{ab} \epsilon_J \,, \qquad \Phi^{IJ}{}_{ab} \equiv \bar{\epsilon}^I \gamma_{ab} \epsilon^J = (\Phi_{IJab})^* \,, \tag{3.49}$$

It follows that the real $SU(2)$-invariant combination of vectors $V_a \equiv V^I{}_{I\,a}$ is always non-spacelike:

$$V^2 = -V^I{}_J \cdot V^J{}_I = 2M^{IJ} M_{IJ} \geq 0 \,. \tag{3.50}$$

In the timelike case we further define $X = \frac{1}{2}\epsilon^{IJ} M_{IJ}$ which is an $SU(2)$ scalar.

As is usual, it is convenient to consider the case in which the vector bilinear $V^\mu \equiv i\bar{\epsilon}^I \gamma^\mu \epsilon_I$ is timelike and the case in which it is null, separately.

As mentioned before, the presence of hypermultiplets only introduces an $SU(2)$ connection in the covariant derivative $\mathfrak{D}_\mu \epsilon_I$ in $\delta_\epsilon \psi_{I\mu} = 0$ and has no effect on the KSE $\delta_\epsilon \lambda^{iI} = 0$. Following the same steps as in Ref. [26], by way of the gravitino supersymmetry transformation rule Eq. (3.18), we arrive at

$$\mathfrak{D}_\mu X \;=\; -iT^+{}_{\mu\nu} V^\nu \,, \tag{3.51}$$

$$\mathfrak{D}_\mu V_J{}^I{}_\nu \;=\; i\delta^I{}_J (XT^{*-}{}_{\mu\nu} - X^* T^+{}_{\mu\nu}) - i(\epsilon^{IK} T^{*\,-}{}_{\mu\rho} \Phi_{KJ}{}^\rho{}_\nu - \epsilon_{JK} T^+{}_{\mu\rho} \Phi^{IK}{}_\nu{}^\rho) \,. \tag{3.52}$$

The $SU(2)$ connection does not occur in the first equation, simply because $X = \frac{1}{2}\epsilon^{IJ} M_{IJ}$ is an $SU(2)$ scalar, but it does occur in the second, although not in its trace. This means that V^μ is, once again, a Killing vector and the 1-form $\hat{V} = V_\mu dx^\mu$ satisfies the equation

$$d\hat{V} = 4i(XT^{*\,-} - X^* T^+) \,. \tag{3.53}$$

The remaining 3 independent 1-forms[1]

$$\hat{V}^x \;\equiv\; \tfrac{1}{\sqrt{2}} (\sigma_x)_I{}^J \; V_J{}^I{}_\mu \; dx^\mu \,, \tag{3.54}$$

however, are only $SU(2)$-covariantly exact

[1] $\sigma_{x\,J}{}^I$, $(x = 1, 2, 3)$ are the Pauli matrices satisfying Eq. (D.13).

$$d\hat{V}^x \ + \ \varepsilon^{xyz}\ \mathsf{A}^y \wedge \hat{V}^z \ = \ 0\,. \tag{3.55}$$

From $\delta_\epsilon \lambda^{iI} = 0$ we get exactly the same equations as in absence of hypermultiplets. In particular

$$V^\mu \partial_\mu Z^i \ = \ 0\,, \tag{3.56}$$

$$2iX^* \partial_\mu Z^i + 4iG^{i\,+}{}_{\mu\nu} V^\nu \ = \ 0\,. \tag{3.57}$$

Combine Eqs. (3.51) and (3.57), we get

$$V^\nu F^{\Lambda\,+}{}_{\nu\mu} = \mathcal{L}^{*\,\Lambda}\mathfrak{D}_\mu X + X^* f^\Lambda{}_i \partial_\mu Z^i = \mathcal{L}^{*\,\Lambda}\mathfrak{D}_\mu X + X^*\mathfrak{D}_\mu \mathcal{L}^\Lambda\,. \tag{3.58}$$

3.3 The timelike case

In the timelike case at hand, Eq. (3.58) is enough to completely determine the field strength through the identity

$$C^{\Lambda\,+}{}_\mu \equiv V^\nu F^{\Lambda\,+}{}_{\nu\mu} \ \Rightarrow \ F^{\Lambda\,+} = V^{-2}[\hat{V} \wedge \hat{C}^{\Lambda\,+} + i\,{}^\star(\hat{V} \wedge \hat{C}^{\Lambda\,+})]\,. \tag{3.59}$$

Observe that this equation does not involve the hyperscalars in any explicit way, as was to be expected due to the absence of couplings between the vector fields and the hyperscalars.

When V^μ is timelike one can derive the following identities:

$$\mathcal{E}^{\mu\nu} \ = \ \mathcal{E}^{\rho\sigma} v_\rho v_\sigma v^\mu v^\nu\,, \tag{3.60}$$

$$\mathcal{T}_\Lambda \mathcal{H}^{\Lambda\,\mu} \ = \ -\tfrac{i}{2} e^{i\alpha} \mathcal{E}^{\rho\sigma} v_\rho v_\sigma v^\mu\,, \tag{3.61}$$

$$\mathcal{T}^i{}_\Lambda \mathcal{H}^{\Lambda\,\mu} \ = \ \tfrac{1}{2} e^{-i\alpha} \mathcal{E}^i v^\mu\,, \tag{3.62}$$

where we have defined the unit vector and the (local) phase

$$v^\mu \equiv V^\mu/2|X|\,, \qquad e^{i\alpha} \equiv X/|X|\,. \tag{3.63}$$

These identities contain a large amount of information about the supersymmetric configurations. In particular, they contain the necessary information about which equations of motion need to be checked explicitly in order to determine whether a given configuration solves the equations of motion: the first identity Eq. (3.60) tells us that the only components of the Einstein equations that do not vanish automatically for supersymmetric configurations are those in the direction of $v^\mu v^\nu$; the rest vanish automatically. That is, once supersymmetry is established, one does not need to check that those components of the Einstein equations are satisfied. Further, the second and third identities state that the only components of the combination of Maxwell equations and Bianchi identities $\mathcal{H}^{\Lambda\,\mu}$ that do not vanish automatically are the ones in the direction v^μ. For the graviphoton (second equation), they are related to the only non-trivial components of the Einstein equations and for the matter vector fields (third equation), they are related to the equations of motion of the scalars.

Let us now consider the new equation $\delta_\epsilon\zeta_\alpha = 0$. Acting on it from the left with $\bar{\epsilon}^K$ and $\bar{\epsilon}^K\gamma_\mu$ we get, respectively

$$\mathsf{U}^{\alpha I}{}_u\, \varepsilon_{IJ}\, V^J{}_K{}^\mu\, \partial_\mu q^u \;\;=\;\; 0\,, \tag{3.64}$$

$$X^*\mathsf{U}^{\alpha K}{}_u\, \partial_\mu q^u \,+\, \mathsf{U}^{\alpha I}{}_u\, \varepsilon_{IJ}\, \Phi^{KJ}{}_\mu{}^\rho\, \partial_\rho q^u \;\;=\;\; 0\,. \tag{3.65}$$

Using $\varepsilon_{IJ}\, V^J{}_K \;=\; \varepsilon_{KJ}\, V^J{}_I \;+\; \varepsilon_{IK}\, V$ in the first equation we get

$$\mathsf{U}^{\alpha I}{}_u\, V^J{}_I{}^\mu \partial_\mu q^u \;-\; \mathsf{U}^{\alpha J}{}_u\, V^\mu \partial_\mu q^u \;=\; 0\,. \tag{3.66}$$

It is not difficult to see that the second equation can be derived from this one using the Fierz identities that the bilinears satisfy in the timelike case (see Ref. [28]), whence the only equations to be solved are (3.66).
Acting with $\bar{\epsilon}^J$ from the left on the hyperino KSI Eq. (3.46) we get

$$X\mathcal{E}^u\, \mathsf{U}^{\alpha I}{}_u \;=\; 0\,, \tag{3.67}$$

which implies, in the timelike $X \neq 0$ case, that all the supersymmetric configurations satisfy the hyperscalars equations of motion automatically:

$$\mathcal{E}^u = 0\,. \tag{3.68}$$

Remember that Eq. (3.42) already told us that supersymmetric configurations automatically fullfill the equations of motions of the scalars from the vector multiplets, if the Bianchi identities and Maxwell equations are satisfied. Therefore, we see that iff the Maxwell equation and Bianchi identities are satisfied, then the equations of motion of the scalars (from both the vector and hypermultiplets) and the Einstein equations are satisfied identically. The conclusion then must be that, in the timelike case, one only needs to solve the Maxwell equation and the Bianchi identities in order to be sure that a supersymmetric configuration is an actual (supersymmetric) solution of the equations of motion.

3.3.1 The metric

If we define the time coordinate t by

$$V^\mu \partial_\mu \equiv \sqrt{2}\partial_t \,, \tag{3.69}$$

then $V^2 = 4|X|^2$ implies that $\hat{V}$ must take the form

$$\hat{V} = 2\sqrt{2}|X|^2(dt + \omega) \,, \tag{3.70}$$

where

$$\omega = \omega_{\underline{m}} dx^m = \tfrac{1}{2\sqrt{2}|X|^2} V_{\underline{m}} dx^m \tag{3.71}$$

is a time-independent 1-form to be determined later.

Since the $\hat{V}^x$s are not exact, we cannot simply define coordinates by putting $\hat{V}^x \equiv dx^x$ as we could in the absence of hypermultiplets. We can, however, still use them to construct the metric: using

$$g_{\mu\nu} = 2V^{-2}[V_\mu V_\nu - V_J{}^I{}_\mu V_I{}^J{}_\nu] \,, \tag{3.72}$$

and the decomposition

$$V_J{}^I{}_\mu \;=\; \tfrac{1}{2} V_\mu \; \delta_J{}^I \;+\; \tfrac{1}{\sqrt{2}} \; (\sigma_x)_J{}^I \; V^x{}_\mu \,, \tag{3.73}$$

we find that the metric can be written in the "conformastationary" form

$$ds^2 = \frac{1}{4|X|^2} \hat{V} \otimes \hat{V} - \frac{1}{2|X|^2} \delta_{xy} \hat{V}^x \otimes \hat{V}^y \,. \tag{3.74}$$

The $\hat{V}^x$ are mutually orthogonal and also orthogonal to $\hat{V}$, which means that they can be used as a Dreibein for a 3-dimensional Euclidean metric

$$\delta_{xy} \hat{V}^x \otimes \hat{V}^y \;\equiv\; \gamma_{\underline{mn}} dx^m dx^n \,, \tag{3.75}$$

and the 4-dimensional metric takes the form

$$ds^2 \;=\; 2|X|^2(dt + \omega)^2 - \frac{1}{2|X|^2} \gamma_{\underline{mn}} dx^m dx^n \,, \tag{3.76}$$

where $\gamma_{\underline{ij}}$ is a time-independent (positive-definite!) metric on constant t hypersurfaces. The presence of a non-trivial Dreibein and the corresponding 3D metric $\gamma_{\underline{mn}}$ is the main (and only) novelty brought about by the hyperscalars!

In what follows we will use the Vierbein basis

$$e^0 = \frac{1}{2|X|} \hat{V} \,, \qquad e^x = \frac{1}{\sqrt{2}|X|} \hat{V}^x \,, \tag{3.77}$$

that is

$$(e^a{}_\mu) = \begin{pmatrix} \sqrt{2}|X| & \sqrt{2}|X|\omega_{\underline{m}} \\ 0 & \frac{1}{\sqrt{2}|X|} V^x{}_{\underline{m}} \end{pmatrix} , \qquad (e^\mu{}_a) = \begin{pmatrix} \frac{1}{\sqrt{2}|X|} & -\sqrt{2}|X|\omega_x \\ 0 & \sqrt{2}|X| V_x{}^{\underline{m}} \end{pmatrix} . \tag{3.78}$$

where $V_x{}^{\underline{m}}$ is the inverse Dreibein $V_x{}^{\underline{m}} V^y{}_{\underline{m}} = \delta^y{}_x$ and $\omega_x = V_x{}^{\underline{m}}\omega_{\underline{m}}$. We shall also adopt the convention that all objects with flat or curved 3-dimensional indices refer to the above Dreibein and the corresponding metric.

Our choice of time coordinate Eq. (3.56) means that the scalars Z^i are time-independent, whence $\imath_V \mathcal{Q} \;= V^\mu \mathcal{Q}_\mu\; 0$. Contracting Eq. (3.51) with V^μ we get

$$V^\mu \mathfrak{D}_\mu\, X = V^\mu(\partial_\mu + i\mathcal{Q})X = 0\,, \;\;\Rightarrow\;\; V^\mu \partial_\mu X = 0\,, \tag{3.79}$$

so that also X is time-independent.

We know the $\hat{V}^x$s to have no time components. If we choose the gauge for the pullback of the $SU(2)$ connection $\mathsf{A}^x{}_t = 0$, then the $SU(2)$-covariant constancy of the $\hat{V}^x$ (Eq. (3.55)) states that the pullback of A^x, the $\hat{V}^x$s and, therefore, the 3-dimensional metric $\gamma_{\underline{mn}}$ are also time-independent. Eq. (3.55) can then be interpreted as Cartan's first structure equation for a torsionless connection ϖ in 3-dimensional space

$$d\hat{V}^x - \varpi^{xy} \wedge \hat{V}^y = 0\,, \tag{3.80}$$

which means that the 3-dimensional spin connection 1-form $\varpi_x{}^y$ is related to the pullback of the $SU(2)$ connection A^x by

$$\varpi_{\underline{m}}{}^{xy} = \varepsilon^{xyz}\mathsf{A}^z{}_u\; \partial_{\underline{m}} q^u\,, \tag{3.81}$$

implying the embedding of the internal group $SU(2)$ into the Lorentz group of the 3-dimensional space as discussed in the introduction.

The $\mathfrak{su}(2)$ curvature will also be time-independent and Eq. (D.29) implies that the pullback of the Quadbein is also time-independent and its time component vanishes:

$$\mathsf{U}^{\alpha I}{}_u\; V^\mu \partial_\mu q^u = 0\,. \tag{3.82}$$

This together with Eq. (3.66) implies

$$\mathsf{U}^{\alpha I}{}_u\; V^J{}_I{}^\mu \partial_\mu q^u = 0\,, \tag{3.83}$$

and multiplying Eq. (3.82) with the inverse Quadbein $\mathsf{U}_{\alpha I}{}^v$ immediately yields

$$\dot{q}^u \equiv \frac{\partial}{\partial t} q^u = 0. \tag{3.84}$$

Let us then consider the 1-form ω: following the same steps as in Ref. [26], we arrive at

$$(d\omega)_{xy} = -\frac{i}{2|X|^4}\varepsilon_{xyz}(X^*\mathfrak{D}^z X - X\mathfrak{D}^z X^*)\,. \tag{3.85}$$

This equation has the same form as in the case without hypermultiplets, but now the Dreibein is non-trivial and, in curved indices, it takes the form

$$(d\omega)_{\underline{mn}} = -\frac{i}{2|X|^4\sqrt{|\gamma|}}\varepsilon_{\underline{mnp}}(X^*\mathfrak{D}^{\underline{p}}X - X\mathfrak{D}^{\underline{p}}X^*)\,. \tag{3.86}$$

Introducing the real symplectic sections $\mathcal{I}$ and $\mathcal{R}$

$$\mathcal{R} \equiv \Re\mathrm{e}(\mathcal{V}/X)\,, \qquad \mathcal{I} \equiv \Im\mathrm{m}(\mathcal{V}/X)\,, \tag{3.87}$$

where $\mathcal{V}$ is the symplectic section defined in Eq. (C.1)

$$\mathcal{V} = \begin{pmatrix} \mathcal{L}^\Lambda \\ \mathcal{M}_\Sigma \end{pmatrix}\,, \qquad \langle \mathcal{V} \mid \mathcal{V}^* \rangle \equiv \mathcal{L}^{*\,\Lambda}\mathcal{M}_\Lambda - \mathcal{L}^\Lambda\mathcal{M}^*_\Lambda = -i\,, \tag{3.88}$$

we can rewrite the equation for ω to the alternative form

$$(d\omega)_{xy} = 2\epsilon_{xyz}\langle \mathcal{I} \mid \partial^z\mathcal{I}\,\rangle\,. \tag{3.89}$$

To obtain this result the following identities are useful:

$$\langle\mathcal{V}|\partial_i\mathcal{V}\rangle \quad = 0, \tag{3.90}$$

$$\langle\mathcal{V}|\partial_{i^*}\mathcal{V}\rangle \quad = 0, \tag{3.91}$$

$$\langle\mathcal{V}^*|\partial_i\mathcal{V}\rangle \quad = -\tfrac{i}{2}\partial_i\mathcal{K}, \tag{3.92}$$

$$\langle\mathcal{V}^*|\partial_{i^*}\mathcal{V}\rangle \quad = \tfrac{i}{2}\partial_{i^*}\mathcal{K}, \tag{3.93}$$

and from these

$$\langle\mathcal{V}|\partial^z\mathcal{V}\rangle \quad = 0, \tag{3.94}$$

$$\langle\mathcal{V}^*|\partial^z\mathcal{V}\rangle \quad = \mathcal{Q}^z. \tag{3.95}$$

The integrability condition of Eq. (3.89) is

$$\langle\,\mathcal{I} \mid \nabla_{\underline{m}}\partial^{\underline{m}}\mathcal{I}\,\rangle = 0\,, \tag{3.96}$$

and will be satisfied by harmonic functions on the 3-dimensional space, *i.e.* by those real symplectic sections satisfying $\nabla_{\underline{m}}\partial^{\underline{m}}\mathcal{I} = 0$.

3.3.2 Solving the Killing spinor equations

We are now going to see that it is always possible to solve the KSEs for field configurations with metric of the form (3.76) where the 1-form ω satisfies Eq. (3.85) and the 3-dimensional metric has spin connection related to the $SU(2)$ connection by Eq. (3.81), vector fields of the form (3.58) and (A.18), time-independent scalars Z^i and, most importantly, hyperscalars satisfying

$$\mathsf{U}^{\alpha J}{}_{x}\,(\sigma_x)_J{}^I \;=\; 0\,, \tag{3.97}$$

which results from Eqs. (3.73) and (3.82) and where

$$\mathsf{U}^{\alpha J}{}_{x} \;= e^{\mu}{}_{x}\partial_\mu q^u \mathsf{U}^{\alpha J}{}_{u} \;=\; \sqrt{2}|X|V_x{}^{\underline{m}}\partial_{\underline{m}}q^u\;. \tag{3.98}$$

Let us consider first the $\delta_\epsilon\zeta_\alpha = 0$ equation. Due to the time-independence of the hyperscalars it is

$$\partial_\mu q^u\gamma^\mu = \partial_x q^u\gamma^x. \tag{3.99}$$

Using the Vierbein Eq. (3.78) and multiplying by γ^0 it can be rewritten in the form

$$\mathsf{U}_{\alpha I\,x}\;\gamma^{0x}\;\epsilon^I \;=\; 0\,, \tag{3.100}$$

which can be solved using Eq. (3.97) if the spinors satisfy a constraint

$$\Pi^x{}_I{}^J\;\epsilon_J \;=\; 0 \quad ,\quad \Pi^x{}_I{}^J \;\equiv\; \tfrac{1}{2}[\;\delta_I{}^J \;-\; \gamma^{0(x)}\;(\sigma_{(x)})_I{}^J\;] \qquad \text{(no sum over } x), \tag{3.101}$$

for each non-vanishing $\mathsf{U}_{\alpha I\,x}$. These three operators are projectors, *i.e.* they satisfy $(\Pi^x)^2 = \Pi^x$, and commute with each other. From $(\sigma_{(x)})_I{}^K\;\Pi^{(x)}{}_K{}^J\;\epsilon_J \;=\; 0$ we find

$$(\sigma_{(x)})_I{}^J\epsilon_J \;=\; \gamma^{0(x)}\epsilon_I\,, \tag{3.102}$$

which solves $\delta_\epsilon\zeta_\alpha = 0$ together with Eq. (3.97) and tells us that the embedding of the $SU(2)$ connection in the Lorentz group requires the action of the generators of $\mathfrak{su}(2)$ to be identical to the action of the three Lorentz generators $\frac{1}{2}\gamma^{0x}$ on the spinors. When we impose these constraints on the spinors, each of the first two reduces by a factor of 1/2 the number of independent spinors, but the third condition is implied by the first two and does not reduce any further the number of independent spinors.

Observe that

$$\Pi^{x\,I}{}_J \;\equiv\; (\Pi^x{}_I{}^J)^* \;=\; -\varepsilon^{IK}\;\Pi^x{}_K{}^L\;\varepsilon_{LJ}\,. \tag{3.103}$$

Let us now consider the gaugino supersymmetry variation $\delta_\epsilon\lambda^{iI} = 0$. Using Eqs. (A.18) and $i\epsilon_{\mu\nu\rho\sigma}\gamma^{\mu\nu} = -2\gamma_{\rho\sigma}\gamma_5$, we find

$$\not{F}^{\Lambda\,+} = -\frac{1}{|X|^2}C^{\Lambda\,+}{}_{\rho}V_\sigma\gamma^{\rho\sigma}\tfrac{1}{2}(1-\gamma_5)\,. \tag{3.104}$$

On the other hand, using the properties Eqs. (C.19) and (C.20), we find that

$$\mathcal{T}^{i}{}_{\Lambda} C^{\Lambda +}{}_{\mu} = \tfrac{1}{2} X^{*} \partial_{\mu} Z^{i} \,, \tag{3.105}$$

and, combining this with the previous result we get

$$\mathcal{T}^{i}{}_{\Lambda} \not{F}^{\Lambda +} \epsilon^{IJ} \epsilon_{J} = -\frac{M^{IJ}}{2|X|^{2}} \partial_{\rho} Z^{i} V_{\sigma} \gamma^{\rho\sigma} \epsilon_{J} = i \not{\partial} Z^{i} (i\gamma_{0} e^{-i\alpha} \epsilon^{IJ} \epsilon_{J}) \,, \tag{3.106}$$

where α is the phase of the complex scalar bilinear X and we have used that in our Vierbein basis $\hat{V} = 2|X| e_{0}$, Eq. (3.56).

Eq. (3.27) takes, then the form

$$i \not{\partial} Z^{i} (\epsilon^{I} + i\gamma_{0} e^{-i\alpha} \epsilon^{IJ} \epsilon_{J}) = 0 \,, \tag{3.107}$$

and can always be solved by imposing the constraint

$$\epsilon_{I} + i\gamma_{0} e^{i\alpha} \epsilon_{IJ} \epsilon^{J} = 0 \,, \tag{3.108}$$

which can be seen to commute with the projections Π^{x} since, by virtue of Eq. (3.103),

$$\Pi^{x\,K}{}_{I} \; (\epsilon^{I} + i\gamma_{0} e^{-i\alpha} \varepsilon^{IJ} \epsilon_{J}) \;=\; (\Pi^{x\,K}{}_{I} \epsilon^{I}) + i\gamma_{0} e^{-i\alpha} \varepsilon^{KJ} (\Pi^{x}{}_{J}{}^{L} \epsilon_{L}) \,. \tag{3.109}$$

Let us finally consider the gravitino supersymmetry rule $\delta_{\epsilon} \psi_{I\mu} = 0$:

Using Eq. (3.85) if it takes the form

$$\mathfrak{D}_{0} \epsilon_{I} = \frac{1}{\sqrt{2}|X|} \{\partial_{t} - X^{*} \mathfrak{D}_{m} X \gamma_{0m}\} \epsilon_{I} \; + \mathsf{A}_{0I}{}^{J} \epsilon_{J}. \tag{3.110}$$

The last term vanishes in the $SU(2)$ gauge $\mathsf{A}^{x}{}_{t} = 0$, since from Eq. (D.11) we obtain

$$\mathsf{A}_{0I}{}^{J} = 0. \tag{3.111}$$

On the other hand, using Eqs. (C.18) and (C.19), we find

$$\mathcal{T}_{\Lambda} F^{\Lambda +}{}_{0m} = \tfrac{i}{\sqrt{2}} \mathfrak{D}_{m} X \,, \tag{3.112}$$

and combining this with the previous result we find that the 0^{th} component of Eq. (3.26) takes, up to a global factor, the form

$$\partial_{t} \epsilon_{I} - \frac{X^{*} \mathfrak{D}_{m} X}{|X|^{2}} \gamma_{0m} \left[\epsilon_{I} + i\gamma_{0} e^{i\alpha} \epsilon_{IJ} \epsilon^{J}\right] = 0 \,, \tag{3.113}$$

which is always solved by time-independent spinors satisfying the constraint (3.108).

Thus in the $SU(2)$ gauge $\mathsf{A}^{x}{}_{t} = 0$ the 0th component of the gravitino KSE is automatically solved by time-independent Killing spinors using the above constraint. In the same gauge, the spatial (flat) components of the $\delta_{\epsilon} \psi^{I\mu} = 0$ equation can be written, upon use of the above constraint and the relation Eq. (3.81) between the $SU(2)$ and spatial spin connection, in the form

$$X^{1/2}\partial_y(X^{-1/2}\epsilon_I) \;+\; \tfrac{i}{2}\mathsf{A}^x{}_y\;[(\sigma_x)_I{}^J\epsilon_J - \gamma^{0x}\epsilon_I] \;=\; 0 \quad , \quad \mathsf{A}^x{}_y \;=\; \mathsf{A}^x{}_u\partial_{\underline{m}}q^u\; V_y{}^{\underline{m}} , \tag{3.114}$$

which is solved by

$$\epsilon_I = X^{1/2}\epsilon_{I\,0}\,, \qquad \partial_\mu\epsilon_{I\,0} = 0\,, \qquad \epsilon_{I\,0} + i\gamma_0\varepsilon_{IJ}\epsilon^J{}_0 = 0\,, \qquad \Pi^x{}_I{}^J\;\epsilon_{J\,0} = 0\,, \tag{3.115}$$

where the constraints Eq. (3.101) are imposed for each non-vanishing component of the $SU(2)$ connection.

3.3.3 Equations of motion

According to the KSI (3.42), all the vector scalars equations of motion of the supersymmetric solutions will be satisfied if the Maxwell equations and Bianchi identities of the vector fields are satisfied. Furthermore Eq. (3.68) told us that all supersymmetric configurations in the timelike case fullfill the hyperscalars equation of motion automatically. This means, we only have to solve the Maxwell equations and Bianchi identities of the vectorfields. Before studying these equations it is important to notice that supersymmetry requires Eqs. (3.97) to be satisfied. We will assume here that this has been done and we will study possible solutions to these equations.

Using Eqs. (3.58) and (A.18) we can write the symplectic vector of 2-forms in the form

$$F = \frac{1}{2|X|^2}\{\hat{V}\wedge d[|X|^2\mathcal{R}] - {}^\star[\hat{V}\wedge \Im\mathrm{m}(\mathcal{V}^*\mathfrak{D}X + X^*\mathfrak{D}\mathcal{V})]\}\,, \tag{3.116}$$

which can be rewritten in the form

$$F = -\tfrac{1}{2}\{d[\mathcal{R}\hat{V}] + {}^\star[\hat{V}\wedge d\mathcal{I}]\}\,. \tag{3.117}$$

The Maxwell equations and Bianchi identities $dF = 0$ are, therefore, satisfied if

$$d^\star[\hat{V}\wedge d\mathcal{I}] = 0\,, \;\;\Rightarrow \nabla_{\underline{m}}\partial^{\underline{m}}\mathcal{I} = 0\,, \tag{3.118}$$

i.e. if the $2\bar{n}$ components of $\mathcal{I}$ are as many real harmonic functions in the 3-dimensional space with metric $\gamma_{\underline{mn}}$.

Summarizing, the timelike supersymmetric solutions are determined by a choice of Dreibein and hyperscalars such that Eq. (3.97) is satisfied and a choice of $2\bar{n}$ real harmonic functions in the 3-dimensional metric space determined by our choice of Dreibein $\mathcal{I}$. This choice determines the 1-form ω. The full $\mathcal{V}/X$ is determined in terms of $\mathcal{I}$ by solving the stabilization equations and with $\mathcal{V}/X$ one constructs the remaining elements of the solution as explained in Ref. [26].

3.4 The null case

In the null case[2] the two spinors ϵ_I are proportional $\epsilon_I = \phi_I \epsilon$. The complex scalar functions ϕ_I carry $U(1)$ charge -1 w.r.t. the purely imaginary connection

$$\zeta_\mu \equiv \phi^I \mathfrak{D}_\mu \phi_I \,, \tag{3.119}$$

opposite to that of the spinor ϵ, so the ϵ_I are neutral. On the other hand, the ϕ_Is are neutral with respect to the Kähler connection, and the Kähler weight of the spinor ϵ is the same as that of the spinor ϵ_I, i.e. 1/2. The $SU(2)$-action is the one implied by the I-index structure.

We are now going to substitute $\epsilon_I = \phi_I \epsilon$ into the KSEs and we are going to use the normalization condition of the scalars $\phi_I \phi^I = 1$ to split the KSEs into three algebraic and one differential equation for ϵ. One of the algebraic equations for ϵ will be a differential equation for ϕ_I.

The substitution yields immediately

$$\mathfrak{D}_\mu \phi_I \epsilon + \phi_I \mathfrak{D}_\mu \epsilon + \epsilon_{IJ} \phi^J T^+{}_{\mu\nu} \gamma^\nu \epsilon^* \;=\; 0 \,, \tag{3.120}$$

$$\phi^I \, \partial\!\!\!/ Z^i \epsilon^* + \epsilon^{IJ} \phi_J \, G\!\!\!/^{i\,+} \epsilon \;=\; 0 \,, \tag{3.121}$$

$$\mathbb{C}_{\alpha\beta} \mathcal{U}^{I\beta}{}_u \epsilon_{IJ} \, \partial\!\!\!/ q^u \phi^J \epsilon^* \;=\; 0 \tag{3.122}$$

and from the hyperino KSI Eq. (3.46) we get

$$\mathcal{E}^u \; \mathsf{U}^{\alpha I}{}_u \; \phi_I \; \epsilon \;=\; 0 \,. \tag{3.123}$$

Acting on Eq. (3.120) with ϕ^I leads to

$$\mathfrak{D}_\mu \epsilon = -\phi^I \mathfrak{D}_\mu \phi_I \epsilon \,, \tag{3.124}$$

which takes the form

$$\tilde{\mathfrak{D}}_\mu \epsilon \equiv (\mathfrak{D}_\mu + \zeta_\mu) \epsilon = 0 \,, \tag{3.125}$$

and becomes the only differential equation for ϵ. Observe that the covariant derivative $\tilde{\mathcal{D}}_\mu$ contains, apart from the connection ζ, the spin and Kähler connections and the $SU(2)$ connection. Using Eq. (3.125) into Eq. (3.120) to eliminate $\mathcal{D}_\mu \epsilon$ we obtain

$$\tilde{\mathfrak{D}} \phi_I \epsilon + \epsilon_{IJ} \phi^J T^+{}_{\mu\nu} \gamma^\nu \epsilon^* = 0 \,, \qquad \tilde{\mathfrak{D}} \phi_I \equiv (\mathfrak{D}_\mu - \zeta_\mu) \phi_I \,, \tag{3.126}$$

which is one of the algebraic constraints for ϵ and is a differential equation for ϕ_I.

[2]The technical details concerning the normalization of the spinors and the construction of the bilinears in this case are explained in the Appendix of Ref. [28].

Acting with ϕ^I on Eq. (3.121) we see that it splits into two algebraic constraints for ϵ:

$$\partial\!\!\!/ Z^i \epsilon^* = 0\,, \tag{3.127}$$

$$G\!\!\!/^{i\,+}\epsilon = 0\,. \tag{3.128}$$

Finally, we add to the system an auxiliary spinor η, with the same chirality as ϵ but with all $U(1)$ charges opposite to those of ϵ and normalized by the condition

$$\bar{\epsilon}\eta = \tfrac{1}{2}\,. \tag{3.129}$$

This normalization condition will be preserved if and only if η satisfies

$$\tilde{\mathfrak{D}}_\mu \eta + a_\mu \epsilon = 0\,, \tag{3.130}$$

for some a_μ with $U(1)$ charges -2 times those of ϵ, i.e.

$$\tilde{\mathfrak{D}}_\mu a_\nu = (\nabla_\mu - 2\zeta_\mu - i\mathcal{Q}_\mu)a_\nu\,, \tag{3.131}$$

to be determined by the requirement that the integrability conditions of this differential equation have to be compatible with those of the differential equation for ϵ.

Notice that the null tetrad of vector bilinears that one constructs from ϵ and η will in general have non-trivial charges and, in particular, non-trivial Kähler weight.

The definition of the bilinear vectors is:

$$l_\mu = i\sqrt{2}\bar{\epsilon^*}\gamma_\mu\epsilon\,, \quad n_\mu = i\sqrt{2}\bar{\eta^*}\gamma_\mu\eta\,, \quad m_\mu = i\sqrt{2}\bar{\epsilon^*}\gamma_\mu\eta = i\bar{\eta}\gamma_\mu\epsilon^*\,, \quad m^*_\mu = i\sqrt{2}\bar{\epsilon}\gamma_\mu\eta^* = i\bar{\eta^*}\gamma_\mu\epsilon\,. \tag{3.132}$$

we see that l and n have 0 $U(1)$ charges but m has -2 times the charges of ϵ and m^* has $+2$ times the charges of ϵ. The metric

$$ds^2 = 2\hat{l}\otimes\hat{n} - 2\hat{m}\otimes\hat{m}^*\,, \tag{3.133}$$

is, however, invariant.

The orientation of the complex null tetrad is important: we choose the relation between a standard Cartesian tetrad $\{e^0, e^1, e^2, e^3\}$ and the complex null tetrad $\{e^u, e^v, e^z, e^{z^*}\} = \{\hat{l}, \hat{n}, \hat{m}, \hat{m}^*\}$ to be

$$\begin{pmatrix} e^u \\ e^v \\ e^z \\ e^{z^*} \end{pmatrix} = \frac{1}{\sqrt{2}} \left(\begin{array}{cc|cc} 1 & 1 & & \\ 1 & -1 & & \\ \hline & & 1 & i \\ & & 1 & -i \end{array}\right) \begin{pmatrix} e^0 \\ e^1 \\ e^2 \\ e^3 \end{pmatrix}\,. \tag{3.134}$$

This translates into identical relations between gamma matrices:

$$
\begin{pmatrix} \gamma^u \\ \gamma^v \\ \gamma^z \\ \gamma^{z^*} \end{pmatrix} = \begin{pmatrix} \not{l} \\ \not{n} \\ \not{m} \\ \not{m}^* \end{pmatrix} = \frac{1}{\sqrt{2}} \left(\begin{array}{cc|cc} 1 & 1 & & \\ 1 & -1 & & \\ \hline & & 1 & i \\ & & 1 & -i \end{array}\right) \begin{pmatrix} \gamma^0 \\ \gamma^1 \\ \gamma^2 \\ \gamma^3 \end{pmatrix} . \tag{3.135}
$$

This choice implies for the chirality matrix

$$
\gamma_5 \equiv -i\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^{uv}\gamma^{zz^*} . \tag{3.136}
$$

3.4.1 Killing equations for the vector bilinears and first consequences

We are now ready to derive equations involving the bilinears, in particular the vector bilinears which we construct with ϵ and the auxiliary spinor η introduced above. First we deal with the equations that do not involve derivative of the spinors. Acting with $\bar{\epsilon}$ on Eq. (3.126) and with $\bar{\epsilon}\gamma^\mu$ on Eq. (3.128) we get, respectively

$$
T^+{}_{\mu\nu} l^\nu \;=\; 0 \,, \tag{3.137}
$$

$$
G^{i\,+}{}_{\mu\nu} l^\nu \;=\; 0 \,, \tag{3.138}
$$

which, together, imply

$$
F^{\Lambda\,+}{}_{\mu\nu} l^\nu = 0 \,, \tag{3.139}
$$

which, in turn, implies

$$
F^{\Lambda\,+} = \tfrac{1}{2}\phi^\Lambda \hat{l} \wedge \hat{m}^* . \tag{3.140}
$$

for some complex functions ϕ^Λ. This form of $F^{\Lambda\,+}$ solves completely Eq. (3.128), as can be seen using the Fierz identity

$$
l_\mu \gamma^{\mu\nu} \epsilon^* = 3 l^\nu \epsilon^* . \tag{3.141}
$$

Acting with $\bar{\eta}$ on Eq. (3.126) we get

$$
\tilde{\mathfrak{D}}_\mu \phi_I + i\sqrt{2}\epsilon_{IJ}\phi^J T^+{}_{\mu\nu} m^\nu = 0 \,, \tag{3.142}
$$

and substituting Eq. (3.140) into it, we get

$$
\tilde{\mathfrak{D}}_\mu \phi_I - \tfrac{i}{\sqrt{2}}\epsilon_{IJ}\phi^J \mathcal{T}_\Lambda \phi^\Lambda l_\mu = 0 \,. \tag{3.143}
$$

Finally, acting with $\bar{\epsilon}$ and $\bar{\eta}$ on Eq. (3.127) we get

$$l^{\mu}\partial_{\mu}Z^{i} = 0\,, \tag{3.144}$$

$$m^{\mu}\partial_{\mu}Z^{i} = 0\,, \tag{3.145}$$

which imply

$$dZ^{i} = A^{i}\hat{l} + B^{i}\hat{m}\,, \tag{3.146}$$

for some functions A^i and B^i that do not depend on v. Observe that, since dZ^i and $\hat{l}$ have no Kähler weight and $\hat{m}$ has Kähler weight $+2$, B^i must have Kähler weight -2. As shown in Refs. [28, 27], for a single scalar $dZ = A\hat{l} + B\hat{m}$ we can always assume that either B is zero (case A) or A is zero (case B). However, for more than one scalar, it is not possible to remove all the A^is and we are going to have, in general, non-vanishing A^is and B^is, although we can consider simple particular cases in which all the A^is or all the B^is vanish.

Observe that this expression for dZ^i solves completely Eq. (3.127) owing to the Fierz identities

$$\not{l}\epsilon^{*} = \not{m}\epsilon^{*} = 0\,. \tag{3.147}$$

These are all the algebraic equations for the bilinears. Now, from Eqs. (3.125) and (3.130) we find the differential equations

$$\nabla_{\mu}l_{\nu} = 0\,, \tag{3.148}$$

$$\tilde{\mathfrak{D}}_{\mu}n_{\nu} = \nabla_{\mu}n_{\nu} = -a^{*}_{\mu}m_{\nu} - a_{\mu}m^{*}_{\nu}\,, \tag{3.149}$$

$$\tilde{\mathfrak{D}}_{\mu}m_{\nu} = (\nabla_{\mu} - 2\zeta_{\mu} - i\mathcal{Q}_{\mu})m_{\nu} = -a_{\mu}l_{\nu}\,. \tag{3.150}$$

3.4.2 Equations of motion and integrability constraints

When V^{μ} is null (we denote it by l^{μ}), using the auxiliary spinor η to construct a standard complex null tetrad $\{l^{\mu}, n^{\mu}, m^{\mu}, m^{*\,\mu}\}$ (see Appendix E we can derive the following identities:

$$(\mathcal{E}_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}\,\mathcal{E}_{\sigma}{}^{\sigma})l^{\nu} = (\mathcal{E}_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}\,\mathcal{E}_{\sigma}{}^{\sigma})m^{\nu} = 0\,, \tag{3.151}$$

$$\mathcal{E}_{\mu\nu}l^{\nu} = \mathcal{E}_{\mu\nu}m^{\nu} = 0\,, \tag{3.152}$$

$$\mathcal{T}_{\Lambda}\mathcal{H}^{\Lambda\,\mu} = 0\,, \tag{3.153}$$

$$\mathcal{T}^{i}{}_{\Lambda}\mathcal{H}^{\Lambda\,\mu}l_{\mu} = \mathcal{T}^{i}{}_{\Lambda}\mathcal{H}^{\Lambda\,\mu}m_{\mu} \;=\; 0\,, \tag{3.154}$$

$$\mathcal{E}^{i} \;=\; 0\,, \tag{3.155}$$

$$\mathcal{E}^{u}\mathsf{U}_{u}{}^{\alpha I}\phi_{I} \;=\; 0\,, \tag{3.156}$$

Thus, in the null case, just as in the timelike case, the equations of motion of the scalars Z^i are always automatically satisfied in presence of supersymmetry. Only a few components of the Einstein and Maxwell equations and Bianchi identities may also be non-zero and these are the only ones that need to be checked, in addition to the hyperscalars equation of motion, if we want to have classical solutions. However, as we shall see in a few pages, the hyperscalar equation of motion is anyhow identically satisfied. Observe that the vanishing of the graviphoton-projected combination $\mathcal{T}_{\Lambda}\mathcal{H}^{\Lambda\,\mu}$ does not imply the vanishing of the Maxwell equations or Bianchi identities.

The Einstein equation takes the form

$$\begin{aligned}\mathcal{E}_{\mu\nu} \;=\;& G_{\mu\nu} + 2\mathcal{G}_{ij^*}[\partial_{\mu}Z^{i}\partial_{\nu}Z^{*\,j^*} - \tfrac{1}{2}g_{\mu\nu}\partial_{\rho}Z^{i}\partial^{\rho}Z^{*\,j^*}]\\ &+8\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,+}{}_{\mu}{}^{\rho}F^{\Sigma\,-}{}_{\nu\rho} + 2\mathsf{h}_{uv}[\partial_{\mu}q^{u}\partial_{\nu}q^{v} - \tfrac{1}{2}g_{\mu\nu}\partial_{\rho}q^{u}\partial_{\rho}q^{v}]\,.\end{aligned} \tag{3.157}$$

It can be rewritten in the form

$$\begin{aligned}\mathcal{E}_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}\mathcal{E}^{\rho}{}_{\rho} \;=\;& R_{\mu\nu} + \left[2\mathcal{G}_{ij^*}A^{i}A^{*\,j^*} - 8\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}\phi^{\Lambda}\phi^{*\,\Sigma}\right]l_{\mu}l_{\nu}\\ &+2\mathcal{G}_{ij^*}B^{i}B^{*\,j^*}m_{(\mu}m^{*}_{\nu)} + 2\mathcal{G}_{ij^*}A^{i}B^{*\,j^*}l_{(\mu}m^{*}_{\nu)} + 2\mathcal{G}_{ij^*}B^{i}A^{*\,j^*}l_{(\mu}m_{\nu)}\\ &+2\mathsf{h}_{uv}\partial_{\mu}q^{u}\partial_{\nu}q^{v}\,.\end{aligned} \tag{3.158}$$

Substituting into the KSI Eq. (3.151), we find the two conditions

$$[R_{\mu\nu} + 2\mathsf{h}_{uv}\partial_{\mu}q^{u}\partial_{\nu}q^{v}]l^{\nu} \;=\; 0\,, \tag{3.159}$$

$$[R_{\mu\nu} + 2\mathsf{h}_{uv}\partial_{\mu}q^{u}\partial_{\nu}q^{v}]m^{\nu} - \mathcal{G}_{ij^*}(A^{i}l_{\mu} + B^{i}m_{\mu})B^{*\,j^*} \;=\; 0\,. \tag{3.160}$$

Commuting the derivative and projecting with gamma matrices and spinors in the usual way, and using

$$(d\mathcal{Q})_{\mu\nu}m^{*\nu} = i\mathcal{G}_{ij^*}B^{i}B^{*\,j^*}m^{*}_{\mu}\,, \qquad (d\mathcal{Q})_{\mu\nu}l^{\nu} = (d\mathcal{Q})_{\mu\nu}n^{\nu} = 0\,, \tag{3.161}$$

which follows from the definition of the Kähler connection and form Eq. (3.146), it is easy to find, from Eq. (3.125)

$$\{R_{\mu\nu} + 2(d\zeta)_{\mu\nu}\}l^{\nu} \;=\; 0\,, \tag{3.162}$$

$$\{R_{\mu\nu} + 2(d\zeta)_{\mu\nu}\}m^{*\,\nu} - \mathcal{G}_{ij^*}B^i(A^{*\,j^*}l_{\mu} + B^{*\,j^*}m^*_{\mu}) \;=\; 0\,, \tag{3.163}$$

and from Eq. (3.130)

$$\{R_{\mu\nu} - 2(d\zeta)_{\mu\nu}\}m^{\nu} - \mathcal{G}_{ij^*}(A^i l_{\mu} + B^i m_{\mu})B^{*\,j^*} + 2(\tilde{\mathfrak{D}}a)_{\mu\nu}l^{\nu} \;=\; 0\,, \tag{3.164}$$

$$\{R_{\mu\nu} - 2(d\zeta)_{\mu\nu}\}n^{\nu} + 2(\tilde{\mathfrak{D}}a)_{\mu\nu}m^{*\,\nu} \;=\; 0\,. \tag{3.165}$$

Comparing these three sets of equations, we find that they are compatible if

$$\mathsf{h}_{uv}\partial_{\mu}q^u\partial_{\nu}q^v l^{\nu} \;=\; (d\zeta)_{\mu\nu}l^{\nu}\,, \tag{3.166}$$

$$\mathsf{h}_{uv}\partial_{\mu}q^u\partial_{\nu}q^v m^{*\nu} \;=\; (d\zeta)_{\mu\nu}m^{*\nu}\,, \tag{3.167}$$

and

$$(\tilde{\mathfrak{D}}a)_{\mu\nu}l^{\nu} = 0. \tag{3.168}$$

Please observe that, due to the positive definiteness of h, Eq. (3.166) implies $l^{\nu}\partial_{\nu}q^v = 0$, but that Eq. (3.167) need not imply $m^{*\nu}\partial_{\nu}q^v = 0$.

3.4.3 Metric

To go on and check the KSIs that involve the Ricci tensor it is helpful to have an explicit form of the metric. Its form is dictated by the existence of a covariantly constant null Killing vector, according to Eq. (3.148), which tells us that the spacetime is a Brinkmann *pp*-wave, [29, 30]. Since l^{μ} is a Killing vector and $d\hat{l} = 0$ we can introduce the coordinates u and v

$$\hat{l} = l_{\mu}dx^{\mu} \;\equiv\; du\,, \tag{3.169}$$

$$l^{\mu}\partial_{\mu} \;\equiv\; \frac{\partial}{\partial v}\,. \tag{3.170}$$

We can also define a complex coordinate z by

$$\hat{m} = e^{U}dz\,, \tag{3.171}$$

where U may depend on z, z^* and u. Then, Eq. (3.144) tells us that the scalars Z^i are just functions of z and u:

$$Z^i = Z^i(z,u)\,, \tag{3.172}$$

and, thus, the functions A^i and B^i defined in Eq. (3.146) are

$$A^i = \partial_{\underline{u}} Z^i\,, \qquad e^U B^i = \partial_{\underline{z}} Z^i\,, \;\Rightarrow \partial_{\underline{z}^*}(e^U B^i) = 0\,. \tag{3.173}$$

Finally, the most general form that $\hat{n}$ can take in this case is

$$\hat{n} = dv + Hdu + \hat{\omega}\,, \qquad \hat{\omega} = \omega_{\underline{z}} dz + \omega_{\underline{z}^*} dz^*\,, \tag{3.174}$$

where all the functions in the metric are independent of v and where either H or the 1-form $\hat{\omega}$ could, in principle, be removed by a coordinate transformation but we have to check that the tetrad integrability equations (3.148)-(3.150) are satisfied by our choices of e^U, H and $\hat{\omega}$.

Then, Eq. (3.133) and the above form of the null tetrad components, lead to the metric[3]

$$ds^2 = 2du(dv + Hdu + \hat{\omega}) - 2e^{2U} dz dz^*\,. \tag{3.175}$$

Having a metric, we can now check the integrability conditions Eqs. (3.159), (3.163). Since for a Brinkmann metric it is

$$R_{\mu\nu} l^\nu = 0, \tag{3.176}$$

the first of these implies

$$\mathsf{h}_{uv} \partial_\mu q^u \partial_\nu q^v l^\nu = (d\zeta)_{\mu\nu} l^\nu = 0. \tag{3.177}$$

This means

$$\mathsf{h}_{uv} (l \cdot \partial q^u)(l \cdot \partial q^v) = 0, \tag{3.178}$$

and since the metric h_{uv} is positive definite this directly implies

$$l \cdot \partial q^u = 0. \tag{3.179}$$

For the connection ζ we can write

$$\zeta = i\zeta_l \hat{l} + \zeta_m \hat{m} - \zeta_m^* \hat{m}^* \tag{3.180}$$

where ζ_l and ζ_n are real functions (whereas ζ_m is complex) and further

$$\hat{a} = a_l \hat{l} + a_m \hat{m} + a_{m^*} \hat{m}^* + a_n \hat{n}, \tag{3.181}$$

and

$$\mathcal{Q} = \mathcal{Q}_l \hat{l} + \mathcal{Q}_m \hat{m} + \mathcal{Q}_{m^*} \hat{m}^* + \mathcal{Q}_n \hat{n}, \tag{3.182}$$

[3] The components of the connection and the Ricci tensor of this metric can be found in Appendix E.

where $\mathcal{Q}^*_m = \mathcal{Q}_{m^*}$ since $\mathcal{Q}$ is real. Let us now consider the tetrad integrability equations (3.148)-(3.150). The first equation is solved because the metric does not depend on v. The third equation, with the choice of coordinate z Eq. (3.171) implies

$$
\begin{aligned}
e^{-U}\partial_{\underline{z}^*}U + 2\zeta^*_m - i\mathcal{Q}_{m^*} &= 0, &(3.183)\\
\dot{U} - 2i\zeta_l - i\mathcal{Q}_l &= a_m &(3.184)\\
a_{m^*} &= 0 &(3.185)\\
-2i\zeta_n - i\mathcal{Q}_n &= 0 &(3.186)\\
a_n &= 0. &(3.187)\\
\left[\dot{U} - 2i\zeta_l - i\mathcal{Q}_l\right]\hat{m} + a_l\hat{l} &= \hat{a}\,, &(3.188)
\end{aligned}
$$

where $a_l(z, z^*, u)$ is a functions to be determined and overdots indicate partial derivation w.r.t. u. Eq. (3.172) tells us that $\zeta_n = \mathcal{Q}_n = 0$ and from Eq. (3.183) we obtain

$$
\partial_{\underline{z}^*}(U + \tfrac{1}{2}\mathcal{K}) = -2\zeta_{\underline{z}^*}. \tag{3.189}
$$

This last equation tells us that that ζ^*_m and hence ζ_m can be eliminated by a gauge transformation and thus

$$
\hat{\zeta} = i\zeta_l\hat{l}. \tag{3.190}
$$

Finally, the second tetrad integrability equation (3.149) implies

$$
\begin{aligned}
a_l &= e^{-U}(\partial_{\underline{z}^*}H - \dot{\omega}_{\underline{z}^*})\,, &(3.191)\\
(d\hat{\omega})_{\underline{zz}^*} &= 2ie^{2U}(2\zeta_l + \mathcal{Q}_l)\,, &(3.192)
\end{aligned}
$$

and thus

$$
a_m = \dot{U} - \tfrac{1}{2}e^{-2U}(d\hat{\omega})_{\underline{zz}^*}\,, \tag{3.193}
$$

so, finally, $\hat{a}$ is given by

$$
\hat{a} = (\dot{U} - \tfrac{1}{2}e^{U}(d\hat{\omega})_{\underline{zz}^*})\hat{m} + e^{-U}(\partial_{\underline{z}^*}H - \dot{\omega}_{\underline{z}^*})\hat{l}\,. \tag{3.194}
$$

Now lets see wether we can get some additional information from Eq. (3.167):

$$
\begin{aligned}
(d\zeta)_{\mu\nu}m^{*\nu} &= 2e^{-U}(\partial_{\underline{z}}\zeta_l m_{[\mu}l_{\nu]} + \partial_{\underline{z}^*}\zeta_l m^*_{[\mu}l_{\nu]})m^{*\nu} && (3.195)\\
&= e^{-U}\partial_{\underline{z}^*}\zeta_l l_\mu && (3.196)\\
&= \mathsf{h}_{uv}\partial_\mu q^u m^* \cdot \partial q^v. && (3.197)
\end{aligned}
$$

This equation tells us that dq^u has only a component in direction u, i.e. the hyperscalars can only depend on the spacetime coordinate u: $q = q(u)$,

$$dq^u = n \cdot \partial q^u \hat{l} \tag{3.198}$$

or

$$m \cdot \partial q^u = m^* \cdot \partial q^u = l \cdot \partial q^u = 0. \tag{3.199}$$

This means that

$$(d\zeta)_{\mu\nu}m^{*\nu} = (d\zeta)_{\mu\nu}l^\nu = 0 \tag{3.200}$$

and so $d\zeta = 0$ and ζ can be completely eliminated. Eq. (3.188) now tells us that [4]

$$U = -\mathcal{K}/2\,. \tag{3.201}$$

The integrability condition Eq. (3.163) splits into

$$
\left.\begin{aligned}
R_{uz^*} + \mathcal{G}_{ij^*}A^i B^{*\,j^*} &= 0\,,\\
R_{zz^*} + \mathcal{G}_{ij^*}B^i B^{*\,j^*} &= 0\,.
\end{aligned}\right. \tag{3.202}
$$

Now let's check Eq. (3.165) where now $\tilde{\mathfrak{D}}a = \mathfrak{D}a = da - i\mathcal{Q} \wedge a$. It is

$$
\begin{aligned}
\mathcal{Q}_l &= -\tfrac{i}{2}e^{-2U}f_{\underline{zz}^*} && (3.203)\\
\mathcal{Q}_m &= ie^{-U}\partial_{\underline{z}}U = (\mathcal{Q}_{m^*})^* && (3.204)\\
a_l &= e^{-U}(\partial_{\underline{z}^*}H - \dot{\omega}_{\underline{z}^*}) && (3.205)\\
a_m &= \dot{U} - \tfrac{1}{2}e^{-2U}f_{\underline{zz}^*}. && (3.206)
\end{aligned}
$$

We obtain

$$
\begin{aligned}
-2(\mathfrak{D}a)_{\mu\nu}m^{*\nu} &= R_{uu}l_\mu + 2e^{-U}\partial_{\underline{z}^*}[\dot{U} - \tfrac{1}{2}e^{-2U}f_{\underline{zz}^*}]m^*_\mu\\
&= R_{uu}l_\mu + 2R_{z^*u}m^*_\mu. && (3.207)
\end{aligned}
$$

[4]Actually, the most general solution is $U = -\mathcal{K}/2 + h(u)$, but we can always eliminate $h(u)$ by a redefinition of z that does not change the structure of the metric.

On the other hand

$$
\begin{aligned}
-2(\mathfrak{D}a)_{\mu\nu}m^{*\nu} &= R_{\mu\nu}n^{\nu} = R_{\mu\nu}e_{u}{}^{\nu} = R_{\mu u} = R_{au}e_{\mu}{}^{a} \\
&= R_{zu}e_{\mu}{}^{z} + R_{z^{*}u}e_{\mu}{}^{z^{*}} + R_{uu}e_{\mu}{}^{u} \\
&= R_{zu}m_{\mu} + R_{z^{*}u}m^{*}_{\mu} + R_{uu}l_{\mu}, \qquad (3.208)
\end{aligned}
$$

so we obtain

$$R_{zu} = 0 \qquad (3.209)$$

from the integrability condition, which is incompatible with the actual value of $R_{uz^{*}}$ for the Brinkmann metric unless

$$\partial_{\underline{u}}Z^{i}\partial_{\underline{z}^{*}}Z^{*j^{*}}\mathcal{G}_{ij^{*}} = 0\,. \qquad (3.210)$$

Another consequence of the elimination of ζ is

$$\tilde{\mathfrak{D}}\phi_{I} = \mathfrak{D}\phi_{I} \qquad (3.211)$$

Taking into account $\mathsf{A}_{\mu I}{}^{J} = \partial_{\mu}q^{u}\mathsf{A}_{uI}{}^{J} \sim l_{\mu}$ Eq. (3.143) now tells us that ϕ_{I} and consequently φ are functions only of u and the graviphoton combination

$$\phi \equiv \mathcal{T}_{\Lambda}\phi^{\Lambda}\,, \qquad d\phi \sim \hat{l}\,. \qquad (3.212)$$

Observe that a similar statement cannot be made about the matter combinations

$$\psi^{i} \equiv \mathcal{T}^{i}{}_{\Lambda}\phi^{\Lambda}\,. \qquad (3.213)$$

We can derive

$$i\mathcal{L}^{*\,\Lambda}T^{+} + 2f^{\Lambda}{}_{i}G^{i\,+} = F^{\Lambda\,+}\,, \qquad (3.214)$$

The variables ϕ, ψ^{i} will be convenient for further calculations, and the relation between them and the ϕ^{Λ} can be obtained from Eq. (3.214):

$$\phi^{\Lambda} = i\mathcal{L}^{*\,\Lambda}\phi + 2f^{\Lambda}{}_{i}\psi^{i}\,. \qquad (3.215)$$

Using these variables, the symplectic vector of field strengths defined in Eq. (3.11) takes the form

$$F = \left(\mathcal{U}_{i}\psi^{i} + \tfrac{i}{2}\mathcal{V}^{*}\phi\right)\hat{l}\wedge\hat{m}^{*} + \text{c.c.}\,, \qquad (3.216)$$

and the symplectic vector containing the Bianchi identities and Maxwell equations, defined in Eq. (3.10) is, in differential-form language

$${}^{\star}\hat{\mathcal{E}} = dF = -\hat{l}\wedge\left[d\left(\mathcal{U}_{i}\psi^{i} + \tfrac{i}{2}\mathcal{V}^{*}\phi\right)\wedge\hat{m}^{*} + \left(\mathcal{U}_{i}\psi^{i} + \tfrac{i}{2}\mathcal{V}^{*}\phi\right)d\hat{m}^{*} + \text{c.c.}\right]\,. \qquad (3.217)$$

Since $d\phi \sim \hat{l}$, it drops out of the above equations. Next, we substitute

$$d\mathcal{V}^* = \mathcal{U}^*{}_{i^*} dZ^{*\,i^*} + \tfrac{1}{2}\mathcal{V}^* d\mathcal{K}\,. \tag{3.218}$$

Finally, using Eqs. (3.150) and (3.146) we find

$$\hat{l} \wedge d\hat{m}^* = \hat{l} \wedge (-\tfrac{1}{2} d\mathcal{K}) \wedge \hat{m}^*\,, \tag{3.219}$$

which, after substituting and assuming independence of v, leads to

$$^\star\hat{\mathcal{E}} = e^{\mathcal{K}/2} d(e^{-\mathcal{K}/2}\psi^i\mathcal{U}_i) \wedge \hat{l} \wedge \hat{m}^* + \text{c.c.} \tag{3.220}$$

The only component of these equations is, then,

$$m^{*\,\mu}\partial_\mu(e^{-\mathcal{K}/2}\psi^i\mathcal{U}_i) - \text{c.c.} = 0\,. \tag{3.221}$$

Finally, let us consider the scalar equation of motion, which takes the form

$$\mathcal{E}^{i^*} = l^{*\,\mu}\mathfrak{D}_\mu A^{*\,i^*} + m^{*\,\mu}\mathfrak{D}_\mu B^{*\,i^*} - B^{*\,i^*} l^\mu a^*_\mu\,, \tag{3.222}$$

but since A^i has Kähler weight 0 it is $\mathfrak{D}_\mu A^{*\,i^*} = \partial_\mu A^{*\,i^*}$ and supposing independence of v the first term vanishes and we have

$$\mathcal{E}^{i^*} = m^{*\,\mu}\mathfrak{D}_\mu B^{*\,i^*} - B^{*\,i^*} l^\mu a^*_\mu\,, \tag{3.223}$$

According to Eq. (3.155), this combination has to vanish in order to have supersymmetry, and we are going to see that this happens if the B^is are covariantly holomorphic in a complex coordinate, denoted by z, and $l^\mu a_\mu = 0$.

3.4.4 Solving the Killing spinor equations

We are now going to see that field configurations given by a metric of the form (Eqs. (3.175) and (3.201))

$$ds^2 = 2du(dv + H du + \hat{\omega}) - 2e^{-\mathcal{K}} dz dz^*\,, \tag{3.224}$$

where $\hat{\omega}$ satisfies (Eq. (3.192))

$$(d\omega)_{\underline{zz^*}} = 2ie^{-\mathcal{K}}\mathcal{Q}_{\underline{u}}\,, \tag{3.225}$$

scalars of the form (Eqs. (3.146), (3.198))

$$dZ^i = A^i\hat{l} + B^i\hat{m}\,, \tag{3.226}$$

$$dq^u = n\cdot\partial q^u\hat{l}, \tag{3.227}$$

and vector field strengths of the form (Eq. (3.140))

$$F^{\Lambda\,+} = \tfrac{1}{2}\phi^{\Lambda}\hat{l}\wedge\hat{m}^{*}\,, \tag{3.228}$$

are always supersymmetric, even though we derived these equations as necessary conditions for supersymmetry.

With the above form of the scalars and vector field strengths the KSE $\delta_\epsilon\lambda^{iI}=0$ takes the form

$$iA^{i}\ \not{l}\epsilon^{I} + iB^{i}\ \not{m}\epsilon^{I} - \tfrac{1}{2}\epsilon^{IJ}\mathcal{T}^{i}{}_{\Lambda}\phi^{\Lambda}\ \not{m}^{*}\not{l}\epsilon_{J} = 0\,, \tag{3.229}$$

and can be solved by imposing two conditions on the spinors:

$$\not{l}\epsilon^{I} = 0\,, \qquad \not{m}\epsilon^{I} = 0\,, \tag{3.230}$$

which formally coincide with the Fierz identities Eqs. (3.147), although now, since there is no *priori* relation between l, m and ϵ^{I}, they are not identities but *constraints on* ϵ^{I}. This fact should be enough to show that they are compatible, but we are going to go further and show that they are equivalent. Multiplying the first condition by $\not{n}$ and the second by $\not{m}^{*}$ we obtain the more conventional-looking conditions

$$\begin{aligned} \not{n}\ \not{l}\epsilon^{I} &= (1-\gamma^{uv})\epsilon^{I} = 0\,, \\ \not{m}^{*}\ \not{m}\epsilon^{I} &= -(1+\gamma^{zz^{*}})\epsilon^{I} = 0\,. \end{aligned} \tag{3.231}$$

If ϵ^{I} satisfies the second condition, using $\gamma_5 = -\gamma^{uv}\gamma^{zz^{*}}$

$$\gamma^{zz^{*}}\epsilon^{I} = -\epsilon^{I}\,,\ \Rightarrow\ \gamma^{uv}\gamma^{zz^{*}}\epsilon^{I} = \gamma^{uv}\epsilon^{I}\,,\ \Rightarrow -\gamma^{5}\epsilon^{I} = \gamma^{uv}\epsilon^{I}\,, \tag{3.232}$$

which, due to the chirality of ϵ^{I}, leads to the first condition.

The supersymmetry variation of the hyperinos Eq. (3.28)

$$\begin{aligned} \delta_\epsilon\zeta_\alpha &= -i\mathbb{C}_{\alpha\beta}\mathcal{U}^{I\beta}{}_{u}\epsilon_{IJ}\ \not{\partial}q^{u}\epsilon^{J}\,, \\ &= -i\mathbb{C}_{\alpha\beta}\mathcal{U}^{I\beta}{}_{u}\epsilon_{IJ}\ \not{l}\dot{q}^{u}\epsilon^{J} \end{aligned} \tag{3.233}$$

vanishes if we demand

$$\not{l}\epsilon^{I} = 0, \tag{3.234}$$

which coincides with the Fierz identity Eq. (3.147) and thus does not lead to any additional broken supersymmetries.

Let us now consider the KSE $\delta_\epsilon\psi_{I\,a}=0$. Taking into account Eqs. (3.230), our tetrad choice and Eq. (3.192), we find

$$\begin{aligned} \mathfrak{D}_\mu\epsilon_I &= (\nabla_\mu + \tfrac{i}{2}\mathcal{Q}_\mu)\epsilon_I + \mathsf{A}_{\mu I}{}^{J}\epsilon_J \\ &= (\partial_\mu - \tfrac{1}{4}\omega_{\mu ab}\gamma^{ab} + \tfrac{i}{2}\mathcal{Q}_\mu)\epsilon_I + \mathsf{A}_{\mu I}{}^{J}\epsilon_J, \end{aligned} \tag{3.235}$$

and if we now impose the constraint Eq. (3.230) we get

$$\begin{aligned} \mathfrak{D}_\mu \epsilon_I &= \partial_\mu \epsilon_I + \partial_\mu q^u \mathsf{A}_{uI}{}^J \epsilon_J \\ &= -\tfrac{1}{2}\epsilon_{IJ}\phi l_\mu \not{m}^* \epsilon^J . \end{aligned} \tag{3.236}$$

However, the fact that the hyperscalars can only depend on u, means that we can eliminate the connection A from the initial set-up and we are left with

$$\partial_\mu \epsilon_I + \tfrac{1}{2}\epsilon_{IJ}\phi l_\mu \not{m}^* \epsilon^J = 0. \tag{3.237}$$

This tells us that the Killing spinors ϵ_I must be independent of v, z, z^* and must satisfy

$$\dot{\epsilon}_I + \tfrac{1}{2}\epsilon_{IJ}\phi\gamma^{z^*}\epsilon^J = 0\,, \tag{3.238}$$

where $\phi = \mathcal{T}_\Lambda \phi^\Lambda = \phi(u)$. Observe that this equation can always be integrated, even though the explicit form of the ϵ_I may be hard to find, and thus does not break any additional supersymmetry.

If $\phi(u)$ is a real function, however, the general solution is readily found to be

$$\epsilon_I = e^{i\Phi}\epsilon_{I0} + \tfrac{1}{\sqrt{2}}\epsilon_{IJ}\gamma^{z^*} e^{-i\Phi}\epsilon^J{}_0,\,, \tag{3.239}$$

$$\gamma^{z^*}\epsilon_{I\,0} = \gamma^u \epsilon_{I\,0} = 0\,, \qquad (\epsilon_{I\,0})^* = \epsilon^I{}_0 \qquad \dot{\Phi} = -i\phi/\sqrt{2}\,. \tag{3.240}$$

Now we still have to consider the hyperino supersymmetry rule Eq. (3.28). In the case at hand it reduces to

$$0 \;=\; \mathsf{U}_v^{\alpha I}\varepsilon_{IJ}\,\partial_u q^v\,\gamma^u \epsilon^J\,, \tag{3.241}$$

so that either we take the hyperscalars to be constant or impose the condition $\gamma^u\epsilon^I = \not{l}\epsilon^I = 0$. This last condition is however always satisfied by any non-maximally supersymmetric solution of the null case. Thus, all the configurations identified are supersymmetric and preserve, at least 1/2 of the available supersymmetries.

One can see, moreover, that the only configurations that preserve more than 1/2 are in fact maximally supersymmetric: Minkowski space and the maximally supersymmetric wave of minimal $N = 2$ $D = 4$ supergravity (KG4) found by Kowalski-Glikman [31], embedded such that only the graviphoton is non-trivial. To be more precise, the embedding of KG4 is obtained by assuming all matter fields to be constant and $U = \omega_{\underline{z}} = \omega_{\underline{z}^*} = \psi^i = 0$. Then, as we are going to see in the next chapter, from Eq. (3.248) it follows that the wave-profile H has to take the form

$$H = 2|\phi|^2|z|^2 \tag{3.242}$$

in order for the configuration to be a solution of the equations of motion, where now the function $\phi(u)$ is related to the field strength by

$$F^{\Lambda +} = \tfrac{i}{2}\mathcal{L}^{*\Lambda}\phi\hat{l}\wedge\hat{m}^*. \tag{3.243}$$

3.4.5 Equations of motion

Let us start with the Maxwell equations and Bianchi identities, given in Eq. (3.220). There is only one non-trivial component which is not automatically satisfied for supersymmetric configurations, namely Eq. (3.221), and we can rewrite it as

$$e^{\mathcal{K}/2}\mathfrak{D}_{\underline{z}}(e^{-\mathcal{K}/2}\psi^i)\mathcal{U}_i + \psi^i\partial_{\underline{z}}Z^j\mathfrak{D}_j\mathcal{U}_i - \text{c.c.} = 0\,, \tag{3.244}$$

where one should keep in mind that the combination $e^{-\mathcal{K}/2}\psi^i$ is a weight -1 vector field. Taking the symplectic product with $\mathcal{U}_k$ and using Eqs. (C.10), (C.11) and (C.13), one finds

$$\mathfrak{D}_{\underline{z}^*}(e^{-\mathcal{K}/2}\psi^{*\,i^*}) - ie^{-\mathcal{K}/2}\psi^j\partial_{\underline{z}}Z^k\mathcal{C}_{jk}{}^{i^*} = 0\,. \tag{3.245}$$

A somewhat lighter equation can be derived by defining

$$\psi^i \;=\; e^{\mathcal{K}}\mathcal{G}^{ij^*}\;P_{j^*} \;\;\rightarrow\;\; \partial_{\underline{z}^*}P^*_i \;=\; i\mathcal{C}_{ij}{}^{k^*}\;\partial_z Z^j\;P_{k^*}\;, \tag{3.246}$$

where P_{i^*} is of Kähler weight $(0,2)$. This equation determines ψ^i, but it is extremely difficult to find a general solution, although we will give some examples.

The only non-automatically satisfied component of the Einstein equations is the uu one

$$\mathcal{E}_{uu} = R_{uu} + 2\mathcal{G}_{ij^*}A^iA^{*\,j^*} - 8\Im\text{m}\mathcal{N}_{\Lambda\Sigma}\phi^\Lambda\phi^\Sigma + 2\mathsf{h}_{uv}n\cdot\partial q^u n\cdot\partial q^v = 0\,. \tag{3.247}$$

Using Eq. (3.215), and the value of R_{uu} this equation takes the form

$$\begin{aligned}&-2e^{-2U}\partial_{\underline{z}}\partial_{\underline{z}^*}H + \tfrac{1}{2}e^{-4U}(\partial_{\underline{z}^*}\omega_{\underline{z}} - \partial_{\underline{z}}\omega_{\underline{z}^*})^2 + e^{-2U}(\partial_{\underline{z}^*}\dot{\omega}_{\underline{z}} + \partial_{\underline{z}}\dot{\omega}_{\underline{z}^*})\\ &+2(\ddot{U}+\dot{U}\dot{U}) + 2\mathcal{G}_{ij^*}(A^iA^{*\,j^*} + 8\psi^i\psi^{*\,j^*}) + 4|\phi|^2 + 2\mathsf{h}_{uv}n\cdot\partial q^u n\cdot\partial q^v \;=\; 0\,.\end{aligned} \tag{3.248}$$

This differential equations determines the form of the waveprofile H. If we take into account that the hyperscalars depend only on the coordinate u and that $g^{uu} = 0$ since we are dealing with a pp-wave metric, it turns out that the equation of motion of the hyperscalars Eq. (3.8) is automatically satisfied.

A supersymmetric solution in this class is, then, fully determined by the real function $H(z,z^*,u)$ and the complex functions $\omega_{\underline{z}}(z,z^*,u), \phi(u), \psi^i(z,z^*,u), Z^i(z,u)$ satisfying Eqs. (3.225), (3.244) and (3.248). There are two simple and interesting families of solutions

1. $Z^i = Z^i(z)$. $(A^i = 0)$. This implies that $\mathcal{Q}_{\underline{u}} = 0$ and we can safely take $\hat{\omega} = 0$. The Einstein equation takes the form

$$e^{\mathcal{K}}\partial_{\underline{z}}\partial_{\underline{z}^*}H = 8\mathcal{G}_{ij^*}\psi^i\psi^{*\,j^*} + 2|\phi|^2 + \mathsf{h}_{uv}n\cdot\partial q^u n\cdot\partial q^v\,. \tag{3.249}$$

If we now set the vector field strengths to zero, i.e. $\phi^\Lambda = 0 \rightarrow \phi = \psi^i = 0$ we obtain a solutions which is a *deformation of the cosmic string*:

$$\left\{ \begin{array}{rcl} ds^2 & = & 2\, du\, \Big(dv \;+\; \tilde{H}(\dot{q},\dot{q})\; |z|^2 du\Big) \;-\; 2e^{-(\mathcal{K}-f-f^*)}\; dz\; dz^*\,, \\ Z^i & = & Z^i(z)\,, \\ F^\Lambda & = & 0\,, \\ q^w & = & q^w(u)\,, \\ f & = & f(z)\,, \end{array} \right. \tag{3.250}$$

Note that for constant hyperscalars one recovers the *cosmic string* solution of [26] where $H = \tilde{H}(\dot{q},\dot{q})$ is a real harmonic function on $\mathbb{C}$: $\partial_{\underline{z}}\partial_{\underline{z}^*}H = 0$.

2. $Z^i = Z^i(u) = 0$. This implies that $\mathcal{K}$ and, therefore, U are functions of u only, whence the latter can be eliminated from the metric by a change of coordinates. Since the pullback of Kähler 1-form depends on u only, we can solve Eq. (3.192) for $\hat{\omega}$:

$$\hat{\omega} = ie^{-\mathcal{K}}\mathcal{Q}_{\underline{u}}(zdz^* - z^*dz)\,, \tag{3.251}$$

which can, however, be eliminated by further change of coordinates. The remaining Einstein equation takes the form

$$2\partial_{\underline{z}}\partial_{\underline{z}^*}H = 2\mathcal{G}_{ij^*}(A^iA^{*j^*} + 8\psi^i\psi^{*j^*}) + 4|\phi|^2 + 2\mathsf{h}_{uv}n\cdot\partial q^u n\cdot\partial q^v\,. \tag{3.252}$$

We have now *plane waves*, which in the simplest case are given by

$$\left\{ \begin{array}{rcl} ds^2 & = & 2du(dv + Hdu) - 2dzdz^*\,, \\ F^{\Lambda\,+} & = & \left[\frac{i}{2}\mathcal{L}^{*\,\Lambda}\phi(u)\right] du\wedge dz^*\,, \\ Z^i & = & Z^i(u)\,, \\ q^w & = & q^w(u)\,, \end{array} \right. \tag{3.253}$$

where Z^i, ϕ are arbitrary functions of u and the wave profile H is given by

$$H = \{\mathcal{G}_{ij^*}\dot{Z}^i\dot{Z}^{*\,j^*} + 2|\phi|^2 + \tilde{\mathsf{H}}(\dot{q},\dot{q})\}|z|^2 + f(z,u) + f^*(z^*,u)\,. \tag{3.254}$$

3.5 Summary of the main results

In this thesis we have studied the supersymmetric solutions of ungauged $N = 2$ supergravity in four dimensions coupled to an arbitrary number of vector and hypermultiplets. Let us summarize our results:

1. In the timelike case the supersymmetric configurations are completely determined by

   (a) A 3-dimensional space metric

   $$\gamma_{\underline{mn}} dx^m dx^n \,, \quad m, n = 1, 2, 3 \,, \tag{3.255}$$

   and a mapping $q^u(x)$ from it to the quaternionic hyperscalar manifold such that the 3-dimensional spin connection[5] $\varpi_x{}^y$ is related to the pullback of the quaternionic $SU(2)$ connection A^x by

   $$\varpi_{\underline{m}}{}^{xy} = \varepsilon^{xyz} \mathsf{A}^z{}_u \, \partial_{\underline{m}} q^u \,, \tag{3.256}$$

   and such that

   $$\mathsf{U}^{\alpha J}{}_x \, (\sigma_x)_J{}^I \; = \; 0 \,, \qquad \mathsf{U}^{\alpha J}{}_x \; \equiv \; V_x{}^{\underline{m}} \partial_{\underline{m}} q^u \; \mathsf{U}^{\alpha J}{}_u \,, \tag{3.257}$$

   where $\mathsf{U}^{\alpha I}{}_u$ is the *Quadbein* defined in Appendix D.

   (b) A choice of a symplectic vector $\mathcal{I} \equiv \Im\mathrm{m}(\mathcal{V}/X)$ whose components are real harmonic functions with respect to the above 3-dimensional metric:

   $$\nabla_{\underline{m}} \partial^{\underline{m}} \mathcal{I} = 0 \,. \tag{3.258}$$

   Given $\mathcal{I}$, $\mathcal{R} \equiv \Re\mathrm{e}(\mathcal{V}/X)$ can in principle be found by solving the generalized stabilization equations and then the metric is given by

   $$ds^2 = |M|^2 (dt + \omega)^2 - |M|^{-2} \gamma_{\underline{mn}} dx^m dx^n \,, \tag{3.259}$$

   where

   $$|M|^{-2} \;\; = \;\; \langle \mathcal{R} \mid \mathcal{I} \rangle \,, \tag{3.260}$$

   $$(d\omega)_{xy} \;\; = \;\; 2\epsilon_{xyz} \langle \mathcal{I} \mid \partial^z \mathcal{I} \rangle \,. \tag{3.261}$$

   The second equation implicitly contains the Dreibein of the 3-dimensional metric γ and its integrability condition is

[5] In this thesis we use $x, y, z = 1, 2, 3$ as (flat three-dimensional) tangent-space indices.

$$\langle \mathcal{I} \mid \nabla_{\underline{m}} \partial^{\underline{m}} \mathcal{I} \rangle = 0 \, . \tag{3.262}$$

As is discussed in *e.g.* Refs. [32, 33], this condition will lead to non-trivial constraints. The vector field strengths are given by

$$F = -\tfrac{1}{\sqrt{2}} \{ d[|M|^2 \mathcal{R}(dt + \omega)] - {}^\star[|M|^2 d\mathcal{I} \wedge (dt + \omega)] \} \, , \tag{3.263}$$

and the scalar fields Z^i can be computed by taking the quotients

$$Z^i = (\mathcal{V}/X)^i / (\mathcal{V}/X)^0 \, . \tag{3.264}$$

The hyperscalars $q^u(x)$ are just the mapping whose existence we assumed from the onset.

These solutions can therefore be seen as deformations of those devoid of hypers, originally found in Ref. [34].

As for the number of unbroken supersymmetries, the presence of non-trivial hyperscalars breaks 1/2 or 1/4 of the supersymmetries of the related solution without hypers, which may have all or 1/2 of the original supersymmetries. Therefore, we will have solutions with $1/2, 1/4$ and $1/8$ of the original supersymmetries. The Killing spinors take the form

$$\epsilon_I = X^{1/2} \epsilon_{I\,0} \, , \qquad \partial_\mu \epsilon_{I\,0} = 0 \, , \qquad \epsilon_{I\,0} + i\gamma_0 \varepsilon_{IJ} \epsilon^J{}_0 = 0 \, , \qquad \Pi^x{}_I{}^J \, \epsilon_{J\,0} = 0 \, , \tag{3.265}$$

where the first constraint is imposed only if there are non-trivial vector multiplets and each of the other three constraints is imposed for each non-vanishing component of the $SU(2)$ connection. Each constraint breaks 1/2 of the supersymmetries independently, but the third constraint $\Pi^x{}_I{}^J \, \epsilon_{J\,0} = 0$ is implied by the first two. Finally, the meaning of these last three constraints is that they enforce the embedding of the gauge connection into the gauge connection since they are in different representations.

2. In the null case the hyperscalars can only depend on the null coordinate u and

$$ds^2 = 2du(dv + Hdu + \hat{\omega}) - 2e^{-\mathcal{K}} dz dz^* \, . \tag{3.266}$$

where $\mathcal{K}$ is the Kähler potential and $\hat{\omega}$ is determined by the equation

$$(d\hat{\omega})_{\underline{zz^*}} = 2ie^{-\mathcal{K}} \mathcal{Q}_{\underline{u}} \, , \tag{3.267}$$

where $\mathcal{Q}_\mu$ is the pullback of the Kähler 1-form connection (See Eq. (B.22)).

The scalar fields can be defined through a symplectic section with arbitrary dependence on u and z and the vector fields are determined by complex arbitrary functions $\phi(u)$ and functions $\psi^i(z, z^*, u)$ which fullfill Eq. (3.244) through

$$F = e^{-\mathcal{K}/2} \left(\mathcal{U}_i \psi^i + \tfrac{i}{2}\mathcal{V}^* \phi\right) du \wedge dz^* + \text{c.c.}\,. \tag{3.268}$$

Furthermore, the wave profile H follows from integrating Eq. (3.248).

The solutions of this case are harder to determine completely. There are, however, two interesting families of solutions, namely deformed cosmic strings and plane waves, described in Eqs. (3.250) and (3.253), respectively.

Chapter 4

Conclusions and outlook

In this thesis we have discussed supersymmetric solutions of ungauged $N = 2$ supergravity in four dimensions. We assumed the existence of - at least one - Killing spinor and found differential and algebraic equations satisfied by the tensors that can be built as bilinears of the Killing spinor. We then derived consistency conditions for these equations to admit solutions and determined necessary conditions for the backgrounds to be supersymmetric. Subsequently we showed that the conditions are also sufficient, meaning that we had identified all the supersymmetric *configurations* of the theory. Finally we imposed the equations of motion in order to find the supersymmetric *solutions*. The solutions fell into two subgroups, depending on whether the vector bilinear constructed out of Killing spinors was timelike or null. We found that in the timelike case non-trivial hyperscalars lead to a non-trivial metric on the constant-time hypersurfaces and the $SU(2)$ connection A needs to be embedded in the three-dimensional Lorentz group (Eq. (3.81)). It turned out that solutions of this class preserve 1/8, 1/4 or 1/2 of the original supersymmetries. In the null case we found that solutions always preserve at least 1/2 of the supersymmetries. Solutions in this class which do not break any supersymmetries are flat Minkowski spacetime and the KG4 wave solution, which are two of the three maximally supersymmetric solutions of pure $N = 2$ $d = 4$ supergravity.[1] The solutions in this case are more complicated to determine explicitely, but we saw two simple examples: waves and deformations of the cosmic string of the hyper-free case.

It is worth mentioning that among the solutions we found, there is a subclass (in the timelike case) describing black-hole type solutions.

Single, static, asymptotically flat, spherically symmetric, black-hole-type solutions of $N = 2, d = 4$ supergravity coupled to vector multiplets are given by real harmonic functions of the form [33]

$$\mathcal{I} = \mathcal{I}_\infty + \frac{q}{r}\,, \tag{4.1}$$

[1]The third maximally supersymmetric solution of pure $N = 2$ $d = 4$ supergravity, namely the Robinson-Bertotti solution, which has $AdS_2 \times S^2$ geometry, together with Minkowski space falls into the timelike class

where the hyperscalars where set to zero. The metric can be conveniently written in spherical coordinates as

$$ds^2 = 2|X|^2 dt^2 - \frac{1}{2|X|^2}[dr^2 + r^2 d\Omega^2_{(2)}] \,. \tag{4.2}$$

This metric describes black holes if

$$-g_{rr} = \frac{1}{2|X|^2} \stackrel{r\to\infty}{\longrightarrow} 1 + \frac{2M}{r} \,, \tag{4.3}$$

is always finite for finite r, whence M, which is the mass, must be positive. Further, we have to require

$$\frac{1}{2|X|^2} \stackrel{r\to 0}{\longrightarrow} \frac{A}{4\pi r^2} > 0 \,, \tag{4.4}$$

which imposes the existence of an event horizon with area $A > 0$ at $r = 0$ instead of a naked singularity. It was shown in [33] that in this subclass of solutions requiring supersymmetry *everywhere* ensures the absence of naked singularities (in most cases). Recently it was shown in [35] that in $N = 1$ Supergravity in five dimensions unfrozen hyperscalars lead to naked singularities. How far this is true when turning on the hypermultiplets in $N = 2$ $d = 4$ Supergravity is object of further investigation. However, should this be generically the case, then we are obliged to find out how string theory gets rid of them, the usual suspects being quantum and non-perturbative corrections.

Another type of solution of particular interest is the cosmic string solution in the null case. This solution has some features in common with the 7-branes solutions of type IIB supergravity, recently discussed in [36]. Both types of solutions are supersymmetric and describe (p=d-3)-branes with 2-dimensional transverse space. The 7-branes naturally couple to 8-form potentials, the duals of the 2-forms appearing in type IIB. This is motivation enough to think about whether one could dualize the complex scalars in the four-dimensional theory discussed in this thesis into 2-form potentials, to which the string-like solutions can couple, and in a next step how to introduce source-terms in the action. Work along this direction is in progress.

Furthermore, one could ask how to generalize the results obtained to more complicated cases. It is known that, in certain limits, the low-energy dynamics of superstring/M-theory compactified in the presence of internal fluxes is encoded in a lower-dimensional gauged supergravity. The fluxes, which may include not just VEV of higher dimensional field strengths across cycles of the internal manifold, but also background quantities related to the geometry of the compactification manifold itself, totally define the local internal symmetry of the lower-dimensional supergravity and, as a consequence of supersymmetry, mass-term deformations and a scalar potential. Thus, one interesting possibility to generalize the analysis presented in this thesis is to consider gaugings. Work in this direction is in progress.

Appendix A

Conventions

The conventions we use in this thesis can essentially be found in [28] and [37]. For convenience we summarize here the most important ones.

A.1 Tensors

We use Greek letters $\mu, \nu, \rho, \ldots$ as (*curved*) tensor indices in a coordinate basis and Latin letters $a, b, c \ldots$ as (*flat*) tensor indices in a tetrad basis. Underlined indices are always curved indices. We symmetrize () and antisymmetrize [] with weight one (i.e. dividing by $n!$). We use mostly minus signature $(+ - --)$. η is the Minkowski metric and a general metric is denoted by g. Flat and curved indices are related by tetrads $e_a{}^\mu$ and their inverses $e^a{}_\mu$, satisfying

$$e_a{}^\mu e_b{}^\nu g_{\mu\nu} = \eta_{ab}\,, \qquad e^a{}_\mu e^b{}_\nu \eta_{ab} = g_{\mu\nu}\,. \tag{A.1}$$

∇ is the total (general- and Lorentz-) covariant derivative, whose action on tensors and spinors (ψ) is given by

$$\begin{aligned}
\nabla_\mu \xi^\nu &= \partial_\mu \xi^\nu + \Gamma_{\mu\rho}{}^\nu \xi^\rho\,, \\
\nabla_\mu \xi^a &= \partial_\mu \xi^a + \omega_{\mu b}{}^a \xi^b\,, \\
\nabla_\mu \psi &= \partial_\mu \psi - \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\psi\,,
\end{aligned} \tag{A.2}$$

where γ_{ab} is the antisymmetric product of two gamma matrices (see next section), $\omega_{\mu b}{}^a$ is the spin connection and $\Gamma_{\mu\rho}{}^\nu$ is the affine connection. The respective curvatures are defined through the Ricci identities

$$
\begin{aligned}
[\nabla_\mu, \nabla_\nu]\ \xi^\rho &= R_{\mu\nu\sigma}{}^\rho(\Gamma)\,\xi^\sigma + T_{\mu\nu}{}^\sigma \nabla_\sigma \xi^\rho\,, \\
[\nabla_\mu, \nabla_\nu]\ \xi^a &= R_{\mu\nu b}{}^a(\omega)\xi^b\,, \\
[\nabla_\mu, \nabla_\nu]\ \psi &= -\tfrac{1}{4} R_{\mu\nu}{}^{ab}(\omega)\gamma_{ab}\psi\,.
\end{aligned}
\tag{A.3}
$$

and given in terms of the connections by

$$
\begin{aligned}
R_{\mu\nu\rho}{}^\sigma(\Gamma) &= 2\partial_{[\mu}\Gamma_{\nu]\rho}{}^\sigma + 2\Gamma_{[\mu|\lambda}{}^\sigma \Gamma_{\nu]\rho}{}^\lambda\,, \\
R_{\mu\nu a}{}^b(\omega) &= 2\partial_{[\mu}\,\omega_{\nu]a}{}^b - 2\omega_{[\mu|a}{}^c\,\omega_{|\nu]c}{}^b\,.
\end{aligned}
\tag{A.4}
$$

These two connections are related by the tetrad postulate

$$\nabla_\mu e_a{}^\mu = 0\,, \tag{A.5}$$

by

$$\omega_{\mu a}{}^b = \Gamma_{\mu a}{}^b + e_a{}^\nu \partial_\mu e_\nu{}^b\,, \tag{A.6}$$

which implies that the curvatures are, in turn, related by

$$R_{\mu\nu\rho}{}^\sigma(\Gamma) = e_\rho{}^a e^\sigma{}_b R_{\mu\nu a}{}^b(\omega)\,. \tag{A.7}$$

Finally, metric compatibility and torsionlessness fully determine the connections to be of the form

$$
\begin{aligned}
\Gamma_{\mu\nu}{}^\rho &= \tfrac{1}{2} g^{\rho\sigma}\left\{\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right\}\,, \\
\omega_{abc} &= -\Omega_{abc} + \Omega_{bca} - \Omega_{cab}\,, \qquad \Omega_{ab}{}^c = e_a{}^\mu e_b{}^\nu \partial_{[\mu} e^c{}_{\nu]}\,.
\end{aligned}
\tag{A.8}
$$

The 4-dimensional fully antisymmetric tensor is defined in flat indices by tangent space by

$$\epsilon^{0123} = +1\,, \quad \Rightarrow \epsilon_{0123} = -1\,, \tag{A.9}$$

and in curved indices by

$$\epsilon^{\mu_1\cdots\mu_3} = \sqrt{|g|}\, e^{\mu_1}{}_{a_1}\cdots e^{\mu_3}{}_{a_3}\epsilon^{a_3\cdots a_3}\,, \tag{A.10}$$

so, with upper indices, is independent of the metric and has the same value as with flat indices.

Contractions of two ϵ symbols in $d = 4$ dimensions:

$$\epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu\nu\alpha\beta} = -4\delta_{[\rho}{}^\alpha \delta_{\sigma]}{}^\beta \tag{A.11}$$

$$\epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu\alpha\beta\gamma} = -3!\delta_{[\nu}{}^\alpha \delta_\rho{}^\beta \delta_{\sigma]}{}^\gamma \tag{A.12}$$

We define the (Hodge) dual of a completely antisymmetric tensor of rank k, $F_{(k)}$ by

$$ {}^{\star}F_{(k)}{}^{\mu_1\cdots\mu_{(d-k)}} = \frac{1}{k!\sqrt{|g|}}\epsilon^{\mu_1\cdots\mu_{(d-k)}\mu_{(d-k+1)}\cdots\mu_d}F_{(k)\mu_{(d-k+1)}\cdots\mu_d}\,. \tag{A.13} $$

Differential forms of rank k are normalized as follows:

$$ F_{(k)} \equiv \tfrac{1}{k!}F_{(k)}{}^{\mu_1\cdots\mu_k}dx^1\wedge\cdots dx^k\,. \tag{A.14} $$

For any 4-dimensional 2-form, we define

$$ F^{\pm} \equiv \tfrac{1}{2}(F \pm i\,{}^{\star}F)\,, \qquad \pm i{}^{\star}F^{\pm} = F^{\pm}\,. \tag{A.15} $$

For any two 2-forms F, G, we have

$$ F^{\pm}\cdot G^{\mp} = 0\,, \qquad F^{\pm}{}_{[\mu}{}^{\rho}\cdot G^{\mp}{}_{\nu]\rho} = 0\,. \tag{A.16} $$

Given any 2-form $F = \frac{1}{2}F_{\mu\nu}dx^{\mu}\wedge dx^{\nu}$ and a non-null 1-form $\hat{V} = V_{\mu}dx^{\mu}$, we can express F in the form

$$ F = V^{-2}[E\wedge\hat{V} - {}^{\star}(B\wedge\hat{V})]\,, \qquad E_{\mu}\equiv F_{\mu\nu}V^{\nu}\,, \qquad B_{\mu}\equiv {}^{\star}F_{\mu\nu}V^{\nu}\,. \tag{A.17} $$

For the complex combinations $F^{\pm}$ we have

$$ F^{\pm} = V^{-2}[C^{\pm}\wedge\hat{V} \pm i\,{}^{\star}(C^{\pm}\wedge\hat{V})]\,, \qquad C^{\pm}_{\mu}\equiv F^{\pm}_{\mu\nu}V^{\nu}\,. \tag{A.18} $$

If we have a (real) null vector l^{μ}, we can always add three more null vectors $n^{\mu}, m^{\mu}, m^{*\,\mu}$ to construct a complex null tetrad as described in Appendix E. The general expansion in the dual basis of 1-forms $\left(\hat{l},\hat{n},\hat{m},\hat{m}^*\right)$ of F^+ depends on three arbitrary complex functions a, b, c

$$ F^+ = a\left(\hat{l}\wedge\hat{n} + \hat{m}\wedge\hat{m}^*\right) + b\hat{l}\wedge\hat{m}^* + c\hat{n}\wedge\hat{m}\,, \qquad F^- = (F^+)^*\,. \tag{A.19} $$

Then, in this case, F is not completely determined by its contraction with the null vector l, but

$$ F^+ = L^{\pm}\wedge\hat{n} \pm {}^{\star}(L^{\pm}\wedge\hat{n}) + b\hat{l}\wedge\hat{m}\,, \qquad L^{\pm}_{\mu}\equiv F^{\pm}{}_{\mu\nu}l^{\nu} = al_{\mu} - cm_{\mu}\,. \tag{A.20} $$

A.2 Gamma matrices and spinors

We work with a purely imaginary representation

$$ \gamma^{a\,*} = -\gamma^{a}\,, \tag{A.21} $$

and our convention for their anticommutator is

$$\{\gamma^a, \gamma^b\} = +2\eta^{ab}\,. \tag{A.22}$$

Thus,

$$\gamma^0\gamma^a\gamma^0 = \gamma^{a\,\dagger} = \gamma^{a\,-1} = \gamma_a\,. \tag{A.23}$$

The chirality matrix is defined by

$$\gamma_5 \equiv -i\gamma^0\gamma^1\gamma^2\gamma^3 = \tfrac{i}{4!}\epsilon_{abcd}\gamma^a\gamma^b\gamma^c\gamma^d\,, \tag{A.24}$$

and satisfies

$$\gamma_5{}^\dagger = -\gamma_5{}^* = \gamma_5\,, \qquad (\gamma_5)^2 = 1\,. \tag{A.25}$$

With this chirality matrix, we have the identity

$$\gamma^{a_1\cdots a_n} = \frac{(-1)^{[n/2]}i}{(4-n)!}\epsilon^{a_1\cdots a_n b_1\cdots b_{4-n}}\gamma_{b_1\cdots b_{4-n}}\gamma_5\,. \tag{A.26}$$

Our convention for Dirac conjugation is

$$\bar\psi = i\psi^\dagger\gamma_0\,. \tag{A.27}$$

Using the identity Eq. (A.26) the general $d=4$ Fierz identity for *commuting* spinors takes the form

$$\begin{aligned}(\bar\lambda M\chi)(\bar\psi N\varphi) \;=\;& \tfrac14(\bar\lambda MN\varphi)(\bar\psi\chi) + \tfrac14(\bar\lambda M\gamma^a N\varphi)(\bar\psi\gamma_a\chi) - \tfrac18(\bar\lambda M\gamma^{ab}N\varphi)(\bar\psi\gamma_{ab}\chi)\\ &-\tfrac14(\bar\lambda M\gamma^a\gamma_5 N\varphi)(\bar\psi\gamma_a\gamma_5\chi) + \tfrac14(\bar\lambda M\gamma_5 N\varphi)(\bar\psi\gamma_5\chi)\,.\end{aligned} \tag{A.28}$$

We use 4-component chiral spinors whose chirality is related to the position of the $SU(2)$ index:

$$\gamma_5\epsilon_I = -\epsilon_I\,. \tag{A.29}$$

Both (chirality and position of the $SU(2)$ index) are reversed under complex conjugation:

$$\gamma_5\epsilon_I^* \equiv \gamma_5\epsilon^I = +\epsilon^I\,. \tag{A.30}$$

We take this fact into account when Dirac-conjugating chiral spinors:

$$\bar\epsilon^I \equiv i(\epsilon_I)^\dagger\gamma_0\,, \quad \bar\epsilon^I\gamma_5 = -\bar\epsilon^I\,, \quad \text{etc.} \tag{A.31}$$

Talking into account the chirality of the spinor ϵ_I and it is (for commuting spinors):

$$\bar\epsilon_I\epsilon_J = -\bar\epsilon_J\epsilon_I \tag{A.32}$$

$$\bar{\epsilon}^I \gamma^\mu \epsilon_J = \bar{\epsilon}_J \gamma^\mu \epsilon^I \tag{A.33}$$

$$\bar{\epsilon}_I \gamma^{\mu\nu} \epsilon_J = \bar{\epsilon}_J \gamma^{\mu\nu} \epsilon_I \tag{A.34}$$

and

$$\bar{\epsilon}_I \epsilon^J = \bar{\epsilon}^I \gamma^\mu \epsilon^J = \bar{\epsilon}_I \gamma^{\mu\nu} \epsilon^J = 0. \tag{A.35}$$

Further we obtain the following useful relations for any 2-forms F and G:

$$\begin{aligned} \not{F}^- \gamma^\mu \epsilon_I &= -4F^{-\mu\nu} \gamma_\nu \epsilon_I && \text{(A.36)} \\ \not{F}^- \gamma^\mu \epsilon^I &= 0 && \text{(A.37)} \end{aligned}$$

$$\begin{aligned} \not{F}^+ \gamma^\mu \epsilon_I &= 0 && \text{(A.38)} \\ \not{F}^+ \gamma^\mu \epsilon^I &= -4F^{+\mu\nu} \gamma_\nu \epsilon^I && \text{(A.39)} \end{aligned}$$

$$\begin{aligned} \gamma^\mu \not{F}^- \epsilon_I &= 0 && \text{(A.40)} \\ \gamma^\mu \not{F}^- \epsilon^I &= 4F^{-\mu\nu} \gamma_\nu \epsilon^I && \text{(A.41)} \end{aligned}$$

$$\begin{aligned} \gamma^\mu \not{F}^+ \epsilon^I &= 0 && \text{(A.42)} \\ \gamma^\mu \not{F}^+ \epsilon_I &= 4F^{+\mu\nu} \gamma_\nu \epsilon_I && \text{(A.43)} \end{aligned}$$

$$[\gamma_\rho, \not{F}^\pm] = 4F^\pm{}_{\rho\nu} \gamma^\nu \tag{A.44}$$

$$\begin{aligned} \not{F}^+ \epsilon^I &= 0 && \text{(A.45)} \\ \not{F}^- \epsilon_I &= 0 && \text{(A.46)} \end{aligned}$$

A.3 Antisymmetric tensor in d=2

$I, J = 1 \dots 2$: $SU(2)$-indices

$$\epsilon_{IJ} = \epsilon^{IJ} \tag{A.47}$$

$$\epsilon^{12} = 1 \tag{A.48}$$

$$\epsilon_{12} \epsilon^{12} = 1 \tag{A.49}$$

$$\epsilon^{IJ} \epsilon_{IK} = \delta^J{}_K \tag{A.50}$$

$$\epsilon^{IJ} \epsilon_{JK} = -\delta^I{}_K \tag{A.51}$$

$$\epsilon^{IJ} \epsilon_{IJ} = 2 \tag{A.52}$$

$$\epsilon_{IJ} \epsilon_{KL} = \delta_{IJ,KL} \tag{A.53}$$

$$\epsilon_{IJ} \epsilon^{KL} = 2\delta_{[I}{}^K \delta_{J]}{}^L \tag{A.54}$$

Appendix B

Kähler geometry

A *Kähler manifold* is a complex manifold on which there exist complex coordinates z^i and $z^{*i^*} = (z^i)^*$ and a function $\mathcal{K}$ called the *Kähler potential* such that the line element is

$$ds^2 = 2\mathcal{G}_{ii^*}\ dz^i dz^{*i^*}\,, \tag{B.1}$$

with

$$\mathcal{G}_{ii^*} = \partial_i \partial_{i^*} \mathcal{K}\,. \tag{B.2}$$

The *Kähler (connection) 1-form* $\mathcal{Q}$ is defined by

$$\begin{aligned} \mathcal{Q} &\equiv (2i)^{-1}(dz^i \partial_i \mathcal{K} - dz^{*i^*} \partial_{i^*} \mathcal{K})\,, & \text{(B.3)} \\ &= (2i)^{-1}(\partial - \bar{\partial})\mathcal{K}\,, & \text{(B.4)} \end{aligned}$$

and the *Kähler 2-form* $\mathcal{J}$ is its exterior derivative

$$\begin{aligned} \mathcal{J} &\equiv d\mathcal{Q} & \text{(B.5)} \\ &= i\mathcal{G}_{ii^*} dz^i \wedge dz^{*i^*} & \text{(B.6)} \\ &= i\partial\bar{\partial}\mathcal{K}\,. & \text{(B.7)} \end{aligned}$$

Note that this yields immediately that the Kähler 2-form is closed: [1]

$$d\mathcal{J} = 0. \tag{B.13}$$

[1] Actually there is an alternative way to define a Kähler manifold:
Definition: A Kähler manifold is an Hermitean manifold whose Kähler form is closed.
This then implies

$$\begin{aligned} d\mathcal{J} &= (\partial + \bar{\partial}) i\mathcal{G}_{ii^*} dz^i \wedge dz^{*i^*} & \text{(B.8)} \\ &= i\partial_j \mathcal{G}_{ii^*} dz^j \wedge dz^i \wedge dz^{*i^*} + i\partial_{j^*} \mathcal{G}_{ii^*} dz^{*j^*} \wedge dz^i \wedge dz^{*i^*} & \text{(B.9)} \\ &= \tfrac{i}{2}(\partial_j \mathcal{G}_{ii^*} - \partial_i \mathcal{G}_{ji^*}) dz^j \wedge dz^i \wedge dz^{*i^*} + \tfrac{i}{2}(\partial_{j^*} \mathcal{G}_{ii^*} - \partial_{i^*} \mathcal{G}_{ij^*}) dz^{*j^*} \wedge dz^i \wedge dz^{*i^*}, & \text{(B.10)} \end{aligned}$$

The Levi-Cività connection is given by

$$\Gamma_{jk}{}^{i} = \mathcal{G}^{ii^*}\partial_j \mathcal{G}_{i^*k}\,, \qquad \Gamma_{j^*k^*}{}^{i^*} = \mathcal{G}^{i^*i}\partial_{j^*}\mathcal{G}_{k^*i}\,. \tag{B.14}$$

The Riemann curvature tensor has as only non-vanishing components $R_{ij^*kl^*}$, but we will not need their explicit expression. The Ricci tensor is given by

$$R_{ii^*} = \partial_i \partial_{i^*}\left(\tfrac{1}{2}\log\det\mathcal{G}\right)\,, \tag{B.15}$$

and the Ricci 2-form by

$$\mathcal{R} = iR_{ii^*}dz^i \wedge dz^{*i^*}. \tag{B.16}$$

The Kähler potential is not unique: it is defined only up to *Kähler transformations* of the form

$$\mathcal{K}'(z,z^*) = \mathcal{K}(z,z^*) + f + f^*\,, \tag{B.17}$$

where f is any holomorphic function of the complex coordinates z^i. Under these transformations, the Kähler metric and Kähler 2-form are invariant, while the components of the Kähler connection 1-form transform according to

$$\mathcal{Q}'_i = \mathcal{Q}_i - \tfrac{i}{2}\partial_i f\,. \tag{B.18}$$

By definition, objects X with Kähler weight $(q,\bar{q})$ transform under the above Kähler transformations like:

$$X' = Xe^{-(qf+\bar{q}f^*)/2} \tag{B.19}$$

and the Kähler-covariant derivative $\mathfrak{D}$ acting on them is given by

$$\mathfrak{D}_i \equiv \nabla_i + iq\mathcal{Q}_i\,, \qquad \mathfrak{D}_{i^*} \equiv \nabla_{i^*} - i\bar{q}\mathcal{Q}_{i^*}\,, \tag{B.20}$$

where ∇ is the standard covariant derivative associated to the Levi-Cività connection.

This defines a complex line bundle $L^1 \to \mathcal{M}$ over the Kähler manifold $\mathcal{M}$ whose first, and only, Chern class equals the Kähler 2-form $\mathcal{J}$. A complex line bundle with this property is known as a *Kähler-Hodge (KH) manifold* and provides the formal starting point for the definition of a special Kähler manifold[2] that is explained in the next Appendix.

We will often use the spacetime pullback of the Kähler-covariant derivative on fields with Kähler weight $(q,-q)$ (weight q, for short) for which it takes the simple form

---

leading to the following relations

$$\partial_j \mathcal{G}_{ii^*} = \partial_i \mathcal{G}_{ji^*}, \qquad \partial_{j^*}\mathcal{G}_{ii^*} = \partial_{i^*}\mathcal{G}_{ij^*}, \tag{B.11}$$

whose solutions is (locally) given by

$$\mathcal{G}_{ii^*} = \partial_i \partial_{i^*}\mathcal{K}, \tag{B.12}$$

and the converse is also true locally [38] (see definition above).

[2]Some basic references for this material are [39, 40, 41] and the review [42]. The definition of special Kähler manifold was made in Ref. [43], formalizing the original results of Ref. [44].

$$\mathfrak{D}_\mu = \nabla_\mu + iq\mathcal{Q}_\mu \,, \tag{B.21}$$

where ∇_μ is the standard spacetime covariant derivative associated to the Levi-Cività connection and $\mathcal{Q}_\mu$ is the pullback of the Kähler 1-form

$$\mathcal{Q}_\mu = (2i)^{-1}(\partial_\mu z^i \partial_i \mathcal{K} - \partial_\mu z^{*\,i^*} \partial_{i^*} \mathcal{K})\,. \tag{B.22}$$

Note that for a Kähler manifold the torsion vanishes, and since it is proportional to the exterior derivative of the Ricci 2-form $\mathcal{R}$ defined in Eq. (B.16), $\mathcal{R}$ is closed and hence a representative of $H^{(1,1)}$ and the *first Chern class* of a Kähler manifold is given by

$$c_1(M) = \tfrac{1}{2\pi}[\mathcal{R}]. \tag{B.23}$$

B.1 Definition of a Kähler-Hodge manifold

A Kähler manifold is a *Hodge-Kähler manifold* if and only if there exists a line bundle $\mathcal{L} \longrightarrow \mathcal{M}$ such that its first Chern class equals the cohomology class of the Kähler 2-form $\mathcal{J}$:

$$c_1(\mathcal{L}) \;=\; [\,\mathcal{J}\,] \tag{B.24}$$

In local terms this means that there is a holomorphic section $\Omega(z)$ such that we can write [45]

$$\mathcal{J} \;=\; i\,\mathcal{G}_{ij^\star}\,dz^i \wedge d\bar{z}^{j^\star} \;=\; i\,\bar{\partial}\,\partial \log \parallel \Omega(z) \parallel^2 . \tag{B.25}$$

B.2 Kähler weights of certain frequently used objects

The Kähler weights $(q, \bar{q})$ of an object as defined in Eq. (B.19):

	ϵ_I	ϵ^I	$\bar{\epsilon}_I$	$\bar{\epsilon}^I$	λ^{Ii}	$\psi_{I\mu}$	ϵ	η
q	$1/2$	$-1/2$	$1/2$	$-1/2$	$-1/2$	$1/2$	$1/2$	$-1/2$
$\bar{q}$	$-1/2$	$1/2$	$-1/2$	$1/2$	$1/2$	$-1/2$	$-1/2$	$1/2$

Table B.1: Kähler weights of certain fermionic fields

	Z^i	F^Λ	G^{i+}	T^+	$\mathcal{V}$	$\mathcal{U}_i$	${\mathcal{T}^i}_\Lambda$	$\mathcal{T}_\Lambda$	$\mathcal{N}_{\Lambda\Sigma}$	$\mathfrak{D}_i\mathcal{U}_j$	$\mathfrak{D}_{i^*}\mathcal{U}_j$	$\mathcal{C}_{ijk}$	Ω
q	0	0	-1	1	1	1	-1	1	0	1	1	2	2
$\bar{q}$	0	0	1	-1	-1	-1	1	-1	0	-1	-1	-2	0

Table B.2: Kähler weights of certain bosonic fields

Appendix C

Special Kähler geometry

Let $\mathcal{L} \longrightarrow \mathcal{M}$ denote the complex line bundle whose first Chern class equals the Kähler form K of an n-dimensional Hodge–Kähler manifold $\mathcal{M}$. Let $\mathcal{SV} \longrightarrow \mathcal{M}$ denote a holomorphic flat vector bundle of rank $2\bar{n}$ with structure group $Sp(2\bar{n}, \mathbb{R})$. Consider tensor bundles of the type $\mathcal{H} = \mathcal{SV} \otimes \mathcal{L}$. A possible definition of a special Kähler manifold is given by constructing a covariantly holomorphic section $\mathcal{V}$ of the bundle $\mathcal{H}$ such that [26, 45]:

$$\mathcal{V} = \begin{pmatrix} \mathcal{L}^\Lambda \\ \mathcal{M}_\Sigma \end{pmatrix} \rightarrow \begin{cases} \langle \mathcal{V} \mid \mathcal{V}^* \rangle & \equiv & \mathcal{L}^{*\,\Lambda} \mathcal{M}_\Lambda - \mathcal{L}^\Lambda \mathcal{M}^*_\Lambda = -i\,, \\ \mathfrak{D}_{i^*} \mathcal{V} & = & (\partial_{i^*} - \frac{1}{2} \partial_{i^*} \mathcal{K}) \mathcal{V} = 0\,, \\ \langle \mathfrak{D}_i \mathcal{V} \mid \mathcal{V} \rangle & = & 0\,. \end{cases} \tag{C.1}$$

We can alternatively write the symplectic product in the form

$$\langle \mathcal{V} | \mathcal{V}^* \rangle \equiv \mathcal{V}^T \Sigma \mathcal{V}, \tag{C.2}$$

where the $2\bar{n} \times 2\bar{n}$ matrix S fullfills the symplectic condition

$$\mathcal{S} \Sigma \mathcal{S}^T = \Sigma, \qquad \text{where} \qquad \Sigma = \begin{pmatrix} 0 & -\mathbb{I}_{\bar{n}} \\ \mathbb{I}_{\bar{n}} & 0 \end{pmatrix}. \tag{C.3}$$

Some relations following from the basic definitions are

$$\mathfrak{D}_{i^*} \mathcal{V} = (\partial_{i^*} - \tfrac{1}{2} \partial_{i^*} \mathcal{K}) \mathcal{V} = 0 \tag{C.4}$$

$$\mathfrak{D}_i \mathcal{V} = (\partial_i + \tfrac{1}{2} \partial_i \mathcal{K}) \mathcal{V} = \mathcal{U}_i \tag{C.5}$$

$$\mathfrak{D}_{i^*} \mathcal{U}_i = (\partial_{i^*} - \tfrac{1}{2} \partial_{i^*} \mathcal{K}) \mathcal{U}_i = \mathcal{G}_{ii^*} \mathcal{V} \tag{C.6}$$

$$\mathfrak{D}_i \mathcal{U}_j = (\partial_i + \tfrac{1}{2} \partial_i \mathcal{K}) \mathcal{U}_j = i \mathcal{C}_{ijk} \mathcal{G}^{kl^*} \mathcal{U}^*{}_{l^*} \tag{C.7}$$

$$\tag{C.8}$$

If we then define

$$\mathcal{U}_i \equiv \mathfrak{D}_i \mathcal{V} = \begin{pmatrix} f^{\Lambda}{}_i \\ h_{\Sigma\, i} \end{pmatrix}\,, \quad \mathcal{U}^*{}_{i^*} = (\mathcal{U}_i)^*\,, \tag{C.9}$$

then it follows from the basic definitions that

$$\begin{aligned} \mathfrak{D}_{i^*}\, \mathcal{U}_i &= \mathcal{G}_{ii^*}\, \mathcal{V} & \qquad \langle \mathcal{U}_i \mid \mathcal{U}^*{}_{i^*} \rangle &= i\mathcal{G}_{ii^*}\,, \\ \langle \mathcal{U}_i \mid \mathcal{V}^* \rangle &= 0\,, & \langle \mathcal{U}_i \mid \mathcal{V} \rangle &= 0\,. \end{aligned} \tag{C.10}$$

Taking the covariant derivative of the last identity $\langle \mathcal{U}_i \mid \mathcal{V} \rangle = 0$ we find immediately that $\langle \mathfrak{D}_i \mathcal{U}_j \mid \mathcal{V} \rangle = -\langle\, \mathcal{U}_j \mid \mathcal{U}_i \rangle$. It can be shown that the r.h.s. of this equation is antisymmetric while the l.h.s. is symmetric, so that

$$\langle \mathfrak{D}_i \mathcal{U}_j \mid \mathcal{V} \rangle = \langle \mathcal{U}_j \mid \mathcal{U}_i \rangle = 0\,. \tag{C.11}$$

The importance of this last equation is that if we group together $\mathcal{E}_{\Lambda} = (\mathcal{V}, \mathcal{U}_i)$, we can see that $\langle \mathcal{E}_{\Sigma} \mid \mathcal{E}^*{}_{\Lambda} \rangle$ is a non-degenerate matrix. This then allows us to construct an identity operator for the symplectic indices, such that for a given section of $\mathcal{A} \ni \Gamma\,(E, \mathcal{M})$ we have

$$\mathcal{A} = i\langle \mathcal{A} \mid \mathcal{V}^* \rangle \mathcal{V} - i\langle \mathcal{A} \mid \mathcal{V} \rangle\, \mathcal{V}^* + i\langle \mathcal{A} \mid \mathcal{U}_i \rangle \mathcal{G}^{ii^*}\, \mathcal{U}^*{}_{i^*} - i\langle \mathcal{A} \mid \mathcal{U}^*{}_{i^*} \rangle \mathcal{G}^{ii^*} \mathcal{U}_i\,. \tag{C.12}$$

As we have seen $\mathfrak{D}_i \mathcal{U}_j$ is symmetric in i and j, but what more can be said about it: as one can easily see, the inner product with $\mathcal{V}^*$ and $\mathcal{U}^*{}_{i^*}$ vanishes due to the basic properties. Let us then define the Kähler-weight 2 object

$$\mathcal{C}_{ijk} \equiv \langle \mathfrak{D}_i\, \mathcal{U}_j \mid \mathcal{U}_k \rangle \quad \rightarrow \quad \mathfrak{D}_i\, \mathcal{U}_j = i\mathcal{C}_{ijk} \mathcal{G}^{kl^*} \mathcal{U}^*{}_{l^*}\,, \tag{C.13}$$

where the last equation is a consequence of Eq. (C.12). Since the $\mathcal{U}$'s are orthogonal, however, one can see that $\mathcal{C}$ is completely symmetric in its 3 indices. Furthermore one can show that

$$\mathfrak{D}_{i^*}\, \mathcal{C}_{jkl} = 0\,, \qquad \mathfrak{D}_{[i}\, \mathcal{C}_{j]kl} = 0\,. \tag{C.14}$$

Observe that these equations imply the existence of a function $\mathcal{S}$, such that

$$\mathcal{C}_{ijk} = \mathfrak{D}_i \mathfrak{D}_j \mathfrak{D}_k\, \mathcal{S}\,. \tag{C.15}$$

The function $\mathcal{S}$ is given by [46]

$$\mathcal{S} \sim \mathcal{L}^{\Lambda} \Im\mathrm{m} \mathcal{N}_{\Lambda\Sigma} \mathcal{L}^{\Sigma}\,, \tag{C.16}$$

where $\mathcal{N}$ is the period or monodromy matrix. This matrix is defined by the relations

$$\mathcal{M}_{\Lambda} = \mathcal{N}_{\Lambda\Sigma} \mathcal{L}^{\Sigma}\,, \qquad h_{\Lambda\, i} = \mathcal{N}^*{}_{\Lambda\Sigma} f^{\Sigma}{}_i\,. \tag{C.17}$$

The relation $\langle \mathcal{U}_i \mid \overline{\mathcal{V}} \rangle = 0$ then implies that $\mathcal{N}$ is symmetric, which then also trivializes $\langle \mathcal{U}_i \mid \mathcal{U}_j \rangle = 0$.

From the other basic properties in (C.10) we find

$$\begin{aligned} \mathcal{L}^\Lambda \Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}\mathcal{L}^{*\,\Sigma} &= -\tfrac{1}{2}\,, &\text{(C.18)}\\ \mathcal{L}^\Lambda \Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma} f^\Sigma{}_i &= \mathcal{L}^\Lambda \Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma} f^{*\,\Sigma}{}_{i^*} = 0\,, &\text{(C.19)}\\ f^\Lambda{}_i\, \Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma} f^{*\,\Sigma}{}_{i^*} &= -\tfrac{1}{2}\mathcal{G}_{ii^*}\,. &\text{(C.20)} \end{aligned}$$

Further identities that can be derived are

$$\begin{aligned} (\partial_i \mathcal{N}_{\Lambda\Sigma})\mathcal{L}^\Sigma &= -2i\Im\mathrm{m}(\mathcal{N})_{\Lambda\Sigma}\, f^\Sigma{}_i = 2i\mathcal{T}^*{}_{i\Lambda}\,, &\text{(C.21)}\\ &= \\ \partial_i \mathcal{N}^*{}_{\Lambda\Sigma}\, f^\Sigma{}_j &= -2\mathcal{C}_{ijk}\mathcal{G}^{kk^*}\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma} f^{*\,\Sigma}{}_{k^*}\,, &\text{(C.22)}\\ n_V \mathcal{C}_{ijk} &= f^\Lambda{}_i f^\Sigma{}_j \partial_k \mathcal{N}^*_{\Lambda\Sigma} \qquad (n_V = \text{number of vectormultiplets})\,, &\text{(C.23)}\\ \mathcal{L}^\Sigma \partial_{i^*}\mathcal{N}_{\Lambda\Sigma} &= 0\,, &\text{(C.24)}\\ \partial_{i^*}\mathcal{N}^*{}_{\Lambda\Sigma}\, f^\Sigma{}_i &= 2i\mathcal{G}_{ii^*}\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}\mathcal{L}^\Sigma\,. &\text{(C.25)} \end{aligned}$$

An important identity one can derive, and that will be used various times in the main text, is given by

$$U^{\Lambda\Sigma} \equiv f^\Lambda{}_i \mathcal{G}^{ii^*} f^{*\,\Sigma}{}_{i^*} = -\tfrac{1}{2}\Im\mathrm{m}(\mathcal{N})^{-1|\Lambda\Sigma} - \mathcal{L}^{*\,\Lambda}\mathcal{L}^\Sigma\,, \tag{C.26}$$

whence $(U^{\Lambda\Sigma})^* = U^{\Sigma\Lambda}$.

We can define the graviphoton and matter vector projectors

$$\begin{aligned} \mathcal{T}_\Lambda &\equiv 2i\mathcal{L}_\Lambda = 2i\mathcal{L}^\Sigma \Im\mathrm{m}\,\mathcal{N}_{\Sigma\Lambda}\,, &\text{(C.27)}\\ \mathcal{T}^i{}_\Lambda &\equiv -f^*{}_\Lambda{}^i = -\mathcal{G}^{ij^*} f^{*\,\Sigma}{}_{j^*}\Im\mathrm{m}\,\mathcal{N}_{\Sigma\Lambda}\,. &\text{(C.28)} \end{aligned}$$

This immediately implies

$$\mathcal{T}_\Lambda \mathcal{L}^{*\Lambda} = -i \tag{C.29}$$

and

$$\mathcal{T}^i{}_\Lambda \mathcal{L}^\Lambda = \mathcal{T}^i{}_\Lambda \mathcal{L}^{*\Lambda} = 0. \tag{C.30}$$

Using these definitions and the above properties one can show the following identities for the derivatives of the period matrix:

$$\begin{aligned} \partial_i \mathcal{N}_{\Lambda\Sigma} &= 4\mathcal{T}_{i(\Lambda}\mathcal{T}_{\Sigma)}\,, \\ \partial_{i^*} \mathcal{N}_{\Lambda\Sigma} &= 4\mathcal{C}^*{}_{i^*j^*k^*}\mathcal{T}^{i^*}{}_{(\Lambda}\mathcal{T}^{j^*}{}_{\Sigma)}\,. \end{aligned} \tag{C.31}$$

$$\mathcal{T}^i{}_{(\Lambda}\mathcal{T}_{\Sigma)} = \tfrac{1}{4}\mathcal{G}^{ij^*}\partial_{j^*}\mathcal{N}^*_{\Lambda\Sigma} \tag{C.32}$$

$$4\mathcal{T}^i{}_{(\Lambda}\mathcal{T}_{\Sigma)}\mathcal{L}^{*\Sigma} = -2i\mathcal{T}^i{}_{\Lambda} \tag{C.33}$$

C.1 Prepotential: Existence and more formulae

Let us start by introducing the explicitly holomorphic section $\Omega = e^{-\mathcal{K}/2}\mathcal{V}$, which allows us to rewrite the system Eqs. (C.1) as

$$\Omega = \begin{pmatrix} \mathcal{X}^\Lambda \\ \mathcal{F}_\Sigma \end{pmatrix} \quad \rightarrow \quad \begin{cases} \langle \Omega \mid \Omega^* \rangle & \equiv \; \mathcal{X}^{*\Lambda}\mathcal{F}_\Lambda - \mathcal{X}^\Lambda \mathcal{F}^*_\Lambda = -i\; e^{-\mathcal{K}}\,, \\ \partial_{i^*}\Omega & = \; 0\,, \\ \langle \partial_i \Omega \mid \Omega \rangle & = \; 0\,. \end{cases} \tag{C.34}$$

This allows us to give an alternative definition of a special Kähler manifold, following [45]:

A Hodge-Kähler manifold is a special Kähler manifold if there exists a bundle $\mathcal{H}$ as defined in Section C such that for some holomorphic section Ω the Kähler 2-form $\mathcal{J}$ is given by

$$\mathcal{J} = i\partial\bar{\partial}\log(i\langle\Omega|\Omega^*\rangle). \tag{C.35}$$

From this definition follows immediately the first equation of (C.34).

Observe that the first of Eqs. (C.34) together with the definition of the period matrix $\mathcal{N}$ imply the following expression for the Kähler potential:

$$e^{-\mathcal{K}} = -2\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}\mathcal{X}^\Lambda\mathcal{X}^{*\,\Sigma}\,. \tag{C.36}$$

If we now assume that $\mathcal{F}_\Lambda$ depends on Z^i through the $\mathcal{X}$'s, then from the last equation we can derive that

$$\partial_i \mathcal{X}^\Lambda \left[2\mathcal{F}_\Lambda - \partial_\Lambda\left(\mathcal{X}^\Sigma\mathcal{F}_\Sigma\right)\right] = 0\,. \tag{C.37}$$

If $\partial_i\mathcal{X}^\Lambda$ is invertible as an $n \times \bar{n}$ matrix, then we must conclude that

$$\mathcal{F}_\Lambda = \partial_\Lambda \mathcal{F}(\mathcal{X})\,, \tag{C.38}$$

where $\mathcal{F}$ is a homogeneous function of degree 2 $\mathcal{F}(\lambda\mathcal{X}) = \lambda^2\mathcal{F}(\Lambda)$, called the *prepotential.*

Making use of the prepotential and the definitions (C.17), we can calculate

$$\mathcal{N}_{\Lambda\Sigma} = \mathcal{F}^*_{\Lambda\Sigma} + 2i \frac{\Im\mathrm{m}\mathcal{F}_{\Lambda\Lambda'}\mathcal{X}^{\Lambda'}\Im\mathrm{m}\mathcal{F}_{\Sigma\Sigma'}\mathcal{X}^{\Sigma'}}{\mathcal{X}^{\Omega}\Im\mathrm{m}\mathcal{F}_{\Omega\Omega'}\mathcal{X}^{\Omega'}} \,. \tag{C.39}$$

Having the explicit form of $\mathcal{N}$, we can also derive an explicit representation for $\mathcal{C}$ by applying Eq. (C.24). One finds

$$\mathcal{C}_{ijk} = e^{\mathcal{K}}\partial_i\mathcal{X}^{\Lambda}\partial_j\mathcal{X}^{\Sigma}\partial_k\mathcal{X}^{\Omega}\mathcal{F}_{\Lambda\Sigma\Omega} \;, \tag{C.40}$$

so that the prepotential really determines all structures in special geometry. Two different functions $\mathcal{F}(\Lambda)$ may correspond to equivalent equations of motion and to the same geometry. This relation is made by certain symplectic transformations.

The physical scalar fields of this system parameterize an n_V-dimensional complex hypersurface, defined by the above constraint

$$\langle \Omega \mid \Omega^* \rangle = -i \; e^{-\mathcal{K}} \,. \tag{C.41}$$

Note that under Kähler transformations $\mathcal{K} \to \mathcal{K} + f + f^*$ the holomorphic sections Ω transforms as $\Omega \to \Omega e^{-f}$ and hence $\mathcal{X}^{\Lambda} \to \mathcal{X}^{\Lambda} e^{-f}$. A convenient choice of coordinates Z^i are the *special coordinates*, defined by fixing the $U(1)$ gauge such that

$$Z^i = \mathcal{L}^i/\mathcal{L}^0, \qquad i = 1 \ldots n_V \tag{C.42}$$

or equivalently

$$\mathcal{X}^0 = 1, \qquad \mathcal{X}^i = Z^i. \tag{C.43}$$

A last remark has to be made about the existence of a prepotential: clearly, given a holomorphic section Ω a prepotential need not exist. It was shown in Ref. [41], however, that one can always apply an $Sp(\bar{n}, \mathbb{R})$ transformation such that a prepotential exists. Clearly the $N = 2$ SUGRA action is not invariant under the full $Sp(\bar{n}, \mathbb{R})$, but the equations of motion and the supersymmetry equations are. This means that for the purpose of this thesis we can always, even if this is not done, impose the existence of a prepotential.

Appendix D

Quaternionic Kähler geometry

So far we have discussed geometry relevant for the vector multiplets of an $N = 2$ supergravity theory in four dimensions. Next we turn our attention to the geometry relevant for the hypermultiplet sector of such a theory. The $4n_H$ (real) scalars in the hypermultiplet sector can be considered the coordinates on a quaternionic Kähler manifold, which is defined in the following.

A *quaternionic Kähler manifold* is a real $4m$-dimensional Riemannian manifold HM endowed with a metric $ds^2 = \mathsf{h}_{uv}(q)dq^u \otimes dq^v \quad ; \quad u, v = 1, \ldots, 4n_H$, a triplet of complex structures $\mathsf{J}^x : T(\mathsf{HM}) \to T(\mathsf{HM})\,,\ \ (x = 1, 2, 3)$ that satisfy the quaternionic algebra [47]

$$(\mathsf{J}^x)^u{}_v(\mathsf{J}^y)^v{}_w = -\delta^{xy}\delta^u{}_w + \varepsilon^{xyz}(\mathsf{J}^z)^u{}_w\,, \tag{D.1}$$

and with respect to which the metric, denoted by h, is Hermitean:

$$\mathsf{h}(\ \mathsf{J}^x X,\ \mathsf{J}^x Y\ ) = \mathsf{h}(X, Y)\,, \qquad \forall X, Y \in T(\mathsf{HM})\,. \tag{D.2}$$

One can always introduce a triplet of $\mathfrak{su}(2)$ Lie–algebra valued 2-forms $\mathsf{K}^x(X, Y) \equiv \mathsf{h}(\ \mathsf{J}^x X, Y)$ [49], i.e.

$$\mathsf{K}^x \;=\; \mathsf{K}^x_{uv} dq^u \wedge dq^v \quad ; \quad \mathsf{K}^x_{uv} \;=\; h_{uw}(\mathsf{J}^x)^w{}_v \tag{D.3}$$

globally known as the *hyperKähler 2-forms* (in the same way as the Kähler form is a $\mathfrak{u}(1)$ Lie–algebra valued 2-form).

The structure of quaternionic Kähler manifold requires an $SU(2)$ bundle to be constructed over HM with connection 1-form A^x with respect to which the hyperKähler 2-form is covariantly closed, i.e.

$$\mathfrak{D}\mathsf{K}^x \equiv d\mathsf{K}^x + \varepsilon^{xyz}\mathsf{A}^y \wedge \mathsf{K}^z = 0 \tag{D.4}$$

Then, depending on whether this bundle is flat or its curvature

$$\mathsf{F}^x \;\equiv\; d\mathsf{A}^x \;+\; \tfrac{1}{2}\varepsilon^{xyz}\ \mathsf{A}^y \wedge \mathsf{A}^z\,, \tag{D.5}$$

is proportional to the hyperKähler 2-form

$$\mathsf{F}^x = \lambda \mathsf{K}^x\,, \qquad \lambda \in \mathbb{R}_{/\{0\}}\,, \tag{D.6}$$

the manifold is a *hyperKähler manifold* (rigid supersymmetry case) or a *quaternionic Kähler manifold* (supergravity case at hand), respectively. Note that only for $\lambda = -1$ coupling to gravity is possible.

The $4n_H$ hyperscalars can be regarded, at least locally, as the four components of a quaternion

$$q = q_0 + q_x \mathsf{J}^x, \quad q_0, q_x \in \mathbb{R}. \tag{D.7}$$

The $SU(2)$ connection acts on objects with vectorial $SU(2)$ indices, such as the chiral spinors in this article, as follows:

$$\begin{aligned} \mathfrak{D}\xi_I &\equiv d\xi_I + \mathsf{A}_I{}^J \xi_J\,, \\ \mathfrak{D}\chi^I &\equiv d\chi^I + \mathsf{A}^I{}_J \chi^J\,. \end{aligned} \tag{D.8}$$

Consistency with the raising and lowering of vector $SU(2)$ indices via complex conjugation requires

$$\mathsf{A}^I{}_J = (\mathsf{A}_I{}^J)^*\,. \tag{D.9}$$

If we, following Ref. [49], put

$$\mathsf{A}_I{}^J \equiv \tfrac{i}{2}\, \mathsf{A}^x\, (\sigma_x)_I{}^J\,, \tag{D.10}$$

we get

$$\mathsf{A}^I{}_J = \tfrac{i}{2}\, \mathsf{A}^x\, (\varepsilon \sigma_x \varepsilon^{-1})^I{}_J = -\tfrac{i}{2}\, \mathsf{A}^x \varepsilon^{IK}\, (\sigma_x)_K{}^L\, \varepsilon_{LJ}\,. \tag{D.11}$$

Consistency between the above definitions of $SU(2)$-covariant derivatives, $\mathsf{A}_I{}^J$ and $SU(2)$ curvature[1] F^x requires that the 3 matrices $(\sigma_x)_I{}^J$ satisfy

$$[\,\sigma_x\,,\,\sigma_y\,]_I{}^J = -2i\varepsilon_{xyz}\,(\sigma_z)_I{}^J\,, \tag{D.12}$$

whence we can take them to be the (Hermitean, traceless) Pauli matrices satisfying

$$(\sigma_x \sigma_y)_I{}^J = \delta_{xy}\,\delta_I{}^J - i\varepsilon_{xyz}\,(\sigma_z)_I{}^J\,. \tag{D.13}$$

Note that this is nothing else than the statement, that the quaternionic algebra Eq. (D.1) is realized by the Pauli matrices times the imaginary unit.

It is convenient to use a Vielbein on HM having as "flat" indices a pair αI consisting of one $SU(2)$-index I and one $Sp(m)$-index $\alpha = 1, \cdots, 2m$

$$\mathsf{U}^{\alpha I} = \mathsf{U}^{\alpha I}{}_u\, dq^u\,, \tag{D.14}$$

[1] Of course, $\mathsf{F}_I{}^J \equiv \frac{i}{2}\,\mathsf{F}^x\,(\sigma_x)_I{}^J$.

where $u = 1, \ldots, 4m$ and from now on we shall refer to this object as the Quadbein. This Quadbein is related to the metric h_{uv} by

$$\mathsf{h}_{uv} = \mathsf{U}^{\alpha I}{}_u \, \mathsf{U}^{\beta J}{}_v \, \varepsilon_{IJ}\mathbb{C}_{\alpha\beta} \,, \tag{D.15}$$

and, further, it is required that

$$\begin{aligned} 2 \, \mathsf{U}^{\alpha I}{}_{(u} \, \mathsf{U}^{\beta J}{}_{v)} \, \mathbb{C}_{\alpha\beta} &= \mathsf{h}_{uv}\varepsilon^{IJ} \,, \\ 2n_H \, \mathsf{U}^{\alpha I}{}_{(u} \, \mathsf{U}^{\beta J}{}_{v)} \, \varepsilon_{IJ} &= \mathsf{h}_{uv} \, \mathbb{C}^{\alpha\beta} \,, \\ \mathsf{U}_{\alpha I \, u} \equiv (\mathsf{U}^{\alpha I}{}_u)^* &= \varepsilon_{IJ}\mathbb{C}_{\alpha\beta} \, \mathsf{U}^{\beta J}{}_u \,. \end{aligned} \tag{D.16}$$

Using these vielbeins, we can construct the triplet complex structure as [47]

$$(\mathsf{J}^x)^u{}_v = -i\mathsf{U}_v{}^{I\alpha}(\sigma_x)_I{}^J \mathsf{U}^u{}_{J\alpha}. \tag{D.17}$$

The inverse Quadbein $\mathsf{U}^u{}_{\alpha I}$ satisfies

$$\mathsf{U}_{\alpha I}{}^u \, \mathsf{U}^{\alpha I}{}_v = \delta^u{}_v \,, \tag{D.18}$$

and, therefore,

$$\mathsf{U}_{\alpha I}{}^u = \mathsf{h}^{uv} \, \varepsilon_{IJ}\mathbb{C}_{\alpha\beta} \, \mathsf{U}^{\beta J}{}_v \,. \tag{D.19}$$

Further relations which can be derived from the former ones are:

$$\mathbb{C}_{\alpha\beta} = (\mathbb{C}^{\alpha\beta})^* = -\mathbb{C}_{\beta\alpha} \tag{D.20}$$

$$\mathsf{U}_{\alpha I}{}^u \mathsf{U}^{\alpha J}{}_v = \mathsf{U}^{\alpha L u}\mathsf{U}_{\alpha K w}\epsilon_{LI}\epsilon^{KJ} \tag{D.21}$$

$$\mathsf{U}^{\alpha I u}\mathsf{U}_{\alpha K w} = \delta^u{}_w \delta^I{}_K - \mathsf{U}_{\alpha K}{}^u \mathsf{U}^{\alpha I}{}_w \tag{D.22}$$

$$\mathsf{U}^{\alpha I}{}_u = \epsilon^{IJ}\mathbb{C}^{\alpha\beta}\mathsf{U}_{\beta J u} \tag{D.23}$$

$$\mathsf{U}^{\alpha I}{}_u \mathsf{U}^{\beta J}{}_v \mathsf{h}^{uv} = \epsilon^{IJ}\mathbb{C}^{\alpha\beta} \tag{D.24}$$

$$\mathsf{U}^{\alpha I}{}_u \mathsf{U}_{\beta J}{}^u = \delta^I{}_J \, \delta^\alpha{}_\beta \tag{D.25}$$

The Quadbein satisfies a *Vielbein postulate*, *i.e.* they are covariantly constant with respect to the standard Levi-Cività connection $\Gamma_{uv}{}^w$, the $SU(2)$ connection $\mathsf{A}_{u\,I}{}^J$ and the $Sp(m)$ connection $\Delta_u{}^{\alpha\beta}$:

$$\mathsf{D}_u \, \mathsf{U}^{\alpha I}{}_v = \partial_u \mathsf{U}^{\alpha I}{}_v - \Gamma_{uv}{}^w \, \mathsf{U}^{\alpha I}{}_w + \mathsf{A}_u{}^I{}_J \, \mathsf{U}^{\alpha J}{}_v + \Delta_u{}^{\alpha\beta} \, \mathsf{U}^{\gamma I}{}_v \mathbb{C}_{\beta\gamma} = 0 \,. \tag{D.26}$$

This postulate relates the three connections and the respective curvatures, leading to the statement that the holonomy of a quaternionic Kähler manifold is contained in $Sp(1){\cdot}Sp(m)$ (see Appendix F), *i.e.*

$$R_{ts}{}^{uv}\ \mathsf{U}^{\alpha I}{}_{u}\ \mathsf{U}^{\beta J}{}_{v}\ +\ \varepsilon^{IK}\ \mathsf{F}_{ts\,K}{}^{J}\ \mathbb{C}^{\alpha\beta}\ -\ 2\ \overline{R}_{ts}{}^{\alpha\beta}\varepsilon^{IJ}\ =\ 0\,, \tag{D.27}$$

where

$$\overline{R}_{ts}{}^{\alpha\beta}\ =\ 2\partial_{[t}\Delta_{s]}{}^{\alpha\beta}+2\Delta_{[t}{}^{\alpha\gamma}\ \Delta_{s]}{}^{\delta\beta}\mathbb{C}_{\gamma\delta}\,, \tag{D.28}$$

is the curvature of the $Sp(m)$ connection. Note that $SU(2)$ is isomorphic to $Sp(1)$ and $Sp(2m,\mathbb{R})$ is isomorphic to $Sp(m)$ [50].

A useful relation is

$$\mathsf{F}_{\mu\nu\,I}{}^{J}=2\lambda\mathsf{U}_{uI\alpha}\mathsf{U}_{v}{}^{J\alpha}\partial_{[\mu}q^{u}\partial_{\nu]}q^{v}\,. \tag{D.29}$$

Appendix E

Null tetrad and Brinkmann *pp*-wave metrics

pp-waves are metrics, that, by definition, admit a covariantly constant null Killing vector field, i.e. a vector satisfying

$$\nabla_{(\mu}l_{\nu)} = 0, \qquad l^2 = l_\mu l^\mu = 0, \tag{E.1}$$

To describe *pp*-waves, , we define light-cone coordinates u and v in terms of the usual Cartesian coordinates

$$u = \frac{1}{\sqrt{2}}(t - z) \tag{E.2}$$

$$v = \frac{1}{\sqrt{2}}(t + z), \tag{E.3}$$

which are related to the Killing vector by

$$l_\mu = \partial_\mu u \quad \Rightarrow \quad l_\mu dx^\mu = \partial_\mu u \, dx^\mu \quad \Rightarrow \quad l = du \tag{E.4}$$

$$l^\mu \partial_\mu v = 1 \quad \Rightarrow \quad l^\mu \partial_\mu = \frac{\partial}{\partial v} \tag{E.5}$$

The metric of any spacetime admitting a covariantly constant null Killing vector l_μ can always put into the Brinkmann metric form:

$$ds^2 = 2du(dv + Hdu + \omega) - 2e^{2U} dzdz^*\,, \qquad \omega = \omega_{\underline{z}} dz + \omega_{\underline{z}^*} dz^*\,, \tag{E.6}$$

where all the functions in the metric are independent of v.

The only non-vanishing components of l are $l_u = l^v = 1$.

Using also light-cone coordinates in tangent space, a natural Vielbein basis is

$$\begin{array}{rclclrclcl} e^u &=& du &=& \hat{l}\,, & e_u &=& \partial_{\underline{u}} - H\partial_{\underline{v}} &=& n^\mu\partial_\mu\,, \\ e^v &=& dv + H du + \omega &=& \hat{n}\,, & e_v &=& \partial_{\underline{v}} &=& l^\mu\partial_\mu\,, \\ e^z &=& e^U dz &=& \hat{m}\,, & e_z &=& e^{-U}(\partial_{\underline{z}} - \omega_{\underline{z}}\partial_{\underline{v}}) &=& -m^{*\mu}\partial_\mu\,, \\ e^{z^*} &=& e^U dz^* &=& \hat{m}^*\,, & e_{z^*} &=& e^{-U}(\partial_{\underline{z}^*} - \omega_{\underline{z}^*}\partial_{\underline{v}}) &=& -m^\mu\partial_\mu\,. \end{array} \tag{E.7}$$

The local metric in this basis takes the form

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{E.8}$$

with the ordering (l, n, m, m^*). For the local volume element we obtain $\epsilon^{lnmm^*} = i$.

If we chose (du, dv, dz, dz^*) as Vierbein basis, the metric takes the form

$$\begin{pmatrix} 2H & 1 & \omega_{\underline{z}} & \omega_{\underline{z}^*} \\ 1 & 0 & 0 & 0 \\ \omega_{\underline{z}} & 0 & 0 & -e^{2U} \\ \omega_{\underline{z}^*} & 0 & -e^{2U} & 0 \end{pmatrix}, \tag{E.9}$$

and its inverse

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -2e^{-2U}(e^{2U}H + \omega_{\underline{z}}\omega_{\underline{z}^*}) & e^{-2U}\omega_{\underline{z}^*} & e^{-2U}\omega_{\underline{z}} \\ 0 & e^{-2U}\omega_{\underline{z}^*} & 0 & -e^{-2U} \\ 0 & e^{-2U}\omega_{\underline{z}} & -e^{-2U} & 0 \end{pmatrix}. \tag{E.10}$$

The components of the spin connection are

$$\begin{array}{rclrcl} \omega_{uzu} &=& e^{-U}(\partial_{\underline{z}} H - \dot{\omega}_{\underline{z}})\,, & \omega_{uzz^*} &=& \frac{1}{2}e^{-2U} f_{\underline{z}\underline{z}^*}\,, \\ \omega_{zz^*u} &=& -\frac{1}{2}e^{-2U} f_{\underline{z}\underline{z}^*} + \dot{U}\,, & \omega_{zzz^*} &=& -e^{-U}\partial_{\underline{z}} U\,, \end{array} \tag{E.11}$$

where $f_{\underline{z}\underline{z}^*} = 2\partial_{[\underline{z}}\omega_{\underline{z}^*]}$ and a dot stands for partial derivation with respect to u.

The components of the Ricci tensor are

$$
\begin{aligned}
R_{zz^*} &= 2e^{-2U}\partial_{\underline{z}}\partial_{\underline{z}^*}U\,, \\
R_{zu} &= \tfrac{1}{2}e^{-U}\partial_{\underline{z}}(e^{-2U}f_{\underline{z}\underline{z}^*}) + e^{-U}\partial_{\underline{z}}\dot{U}\,, \\
R_{uu} &= -2e^{-2U}\partial_{\underline{z}}\partial_{\underline{z}^*}H + \tfrac{1}{2}e^{-4U}(f_{\underline{z}\underline{z}^*})^2 + e^{-2U}(\partial_{\underline{z}}\dot{\omega}_{\underline{z}^*} + \partial_{\underline{z}^*}\dot{\omega}_{\underline{z}}) + 2(\ddot{U} + \dot{U}\dot{U})\,,
\end{aligned}
\tag{E.12}
$$

and the Ricci scalar is just

$$
R = -4e^{-2U}\partial_{\underline{z}}\partial_{\underline{z}^*}U\,. \tag{E.13}
$$

Appendix F

A few words about holonomy

Riemannian manifolds with special holonomy play an important role in string theory compactifications. This is because special holonomy manifolds admit covariantly constant spinors and thus preserve some fraction of the original supersymmetry. Consider an oriented manifold X of real dimension n and a vector $\vec{v}$ at some point on this manifold. One can explore the geometry of X by parallel transporting $\vec{v}$ along a closed contractible path in X, see Figure F.1. Under such an operation the vector $\vec{v}$ may not come back to itself. In fact, generically it will transform into a different vector that depends on the geometry of X, on the path, and on the connection which was used to transport $\vec{v}$. For a Riemannian manifold X with metric $g(X)$, the natural connection is the Levi-Cività connection. With this connection the length of the vector covariantly transported along a closed path should be the same as the length of the original vector. But the direction may be different, and this is precisely what leads to the concept of holonomy.

The relative direction of the vector after parallel transport relative to that of the original vector $\vec{v}$ is described by *holonomy*. It is not hard to see that the set of all holonomies themselves form a group, called the *holonomy group*, where the group structure is induced by the composition of paths and its inverse corresponds to a path traversed in the opposite

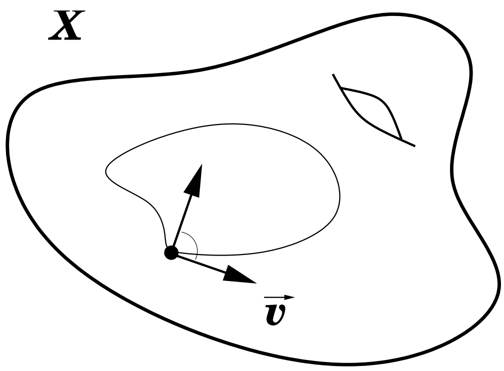

Figure F.1: Parallel transport of a vector $\vec{v}$ along a closed path on the manifold X [51].

orientation. From the way we introduced the holonomy group, $\mathcal{H}(X)$, it seems to depend upon the choice of the base point. However, for generic choices of base points the holonomy group is in fact the same, and therefore $\mathcal{H}(X)$ becomes a true geometric characteristic of the space X with metric $g(X)$. By definition, we have

$$\mathcal{H}(X) \subseteq SO(n) \tag{F.1}$$

where the equality holds for sufficiently generic metric on X.

Now let us consider irreducible (compact, simply-connected) Riemannian manifolds. Among these are the symmetric spaces of the form G/H. These spaces are completely classified, and their geometry is well-known; the holonomy group is H itself. Excluding this case, we get a set of manifolds, which were classified by Berger:

Let M be an irreducible (simply-connected) Riemannian manifold, which is not isomorphic to a symmetric space. Then the holonomy group $\mathcal{H}$ of M belongs to the following list [51]:

Holonomy	$dim_{\mathbb{R}}$	Geometry
$SO(n)$	n	Riemannian manifolds
$U(n)$	$2n$	Kähler manifolds
$SU(n)$	$2n$	Calabi-Yau manifolds
$Sp(n)$	$4n$	hyperkähler manifolds
$Sp(n) \times Sp(1)$	$4n$	quaternionic-Kähler manifolds
G_2	7	G_2-manifolds
$Spin(7)$	8	$Spin(7)$-manifolds

Since in this thesis we focus on Calabi-Yau threefolds, let's see why in this case the holonomy group turns out to be $SU(3)$. To do so we start with a Kähler manifold. For an affine connection ∇ corresponding to a Kähler metric, vectors with holomorphic indices remain with holomorphic indices under parallel transport, i.e. elements of $T^{(1,0)}$ and $T^{(0,1)}$ do not mix. Moreover, ∇ preserves the length of a vector and hence the holonomy group of a Kähler manifold of complex dimension n turns out to be $U(n)$ (or a subgroup thereof) [52]. If we now consider the change of a vector $V^i \in T^{(1,0)}$ under parallel transport[1] around a loop with area a, it turns out that [10]

$$\delta V^i = -\delta a^{kl^*} R_{kl^*}{}^i{}_j V^j, \tag{F.2}$$

that is

$$V^{i'} = (\delta^i{}_j - \delta a^{kl^*} R_{kl^*}{}^i{}_j) V^j, \tag{F.3}$$

where the matrices $\delta^i{}_j - \delta a^{kl^*} R_{kl^*}{}^i{}_j$ are by definition the elements of the holonomy group

[1]Note that elements of $T^{(1,0)}$ transform in the fundamental representation $\mathbf{n}$ and those of $T^{(0,1)}$ in the antifundamental $\bar{\mathbf{n}}$

(in the fundamental representation **n**) which are infinitesimally close to the identity. For a Kähler manifold they are elements of $U(n)$ Due to the exponential map which provides a local parameterization of the group in a neighborhood of the identity $g(\sigma) = 1 + \sigma^a T_a + \ldots$ and where T_a denote the elements of the corresponding Lie algebra, in the case at hand $-\delta a^{kl^*} R_{kl^*}$ is an element of the Lie algebra $\mathfrak{u}(n)$. In a neighborhood of the identity we can now decompose $U(n) = SU(n) \times U(1)$ where the $U(1)$ factor is generated by the trace of the matrix $-\delta a^{kl^*} R_{kl^*}{}^i{}_i$. Thus a Calabi-Yau manifold, which by definition has vanishing first Chern class, i.e. it is Ricci flat, has a holonomy group contained in $SU(n)$. The converse is also true: if the holonomy group of a Kähler manifold is contained in $SU(n)$, then its Kähler metric is Ricci-flat [52].

As we already saw in Section 2.1, the holonomy of the manifold we compactify on has important implications on the amount of unbroken supersymmetry in the lower dimensional theory. Now let us see which implications the geometry of the internal manifold has on the existence of a Killing spinor. Since, as usual, we assume the ten-dimensional spacetime to decompose into a product of a four-dimensional and a compact six-dimensional internal manifold, a ten-dimensional spinor can be decomposed into a product structure

$$\epsilon(x, y) = \zeta(x) \otimes \eta(y), \tag{F.4}$$

where $\zeta(x)$ lives in four dimensions and $\eta(y)$ on the internal manifold.

If one assumes for simplicity that the dilaton is constant and sets all bosonic fields apart from the graviton to zero, it turns out, that the supersymmetry variations of the gravitino and dilatino vanish, i.e. the Killing spinor equations are fullfilled, only if the background admits a covariantly constant spinors

$$\nabla_M \epsilon = 0. \tag{F.5}$$

This means that demanding some amount of unbroken supersymmetry implies

$$[\nabla_M, \nabla_N]\epsilon = -\tfrac{1}{4} R_{MNKL} \Gamma^{KL} \epsilon = 0. \tag{F.6}$$

Due to the decomposition Eq. (F.4) this implies that demanding some unbroken supersymmetry in four dimensions leads to some restrictions coming from the internal components of Eq. (F.7):

$$[\nabla_m, \nabla_n]\eta = -\tfrac{1}{4} R_{mnkl} \Gamma^{kl} \eta = 0. \tag{F.7}$$

This directly implies in analogy to Eq. (F.3) that $(\Gamma_S(g) - \mathbb{I})\eta = 0$, where $\Gamma_S(g)$ denotes an element of the holonomy group $\mathcal{H}$, now in the spinorial representation, which means that a Killing spinor must be invariant under the holonomy group $\mathcal{H} \subset SO(D-d)$ generated by $R_{mnkl}\Gamma^{kl}$, i.e. it must be a singlet under the decomposition of the spinor representation of $SO(D-d)$ into $\mathcal{H}$

$$\Gamma_S(g)\eta = \eta. \tag{F.8}$$

Now it is obvious that the amount of unbroken supersymmetry in the lower dimension, one is left with after compactification, depends on the decomposition of $SO(D-d)$ into

$\mathcal{H}$. In Section 2.1 we already saw that for a Calabi-Yau threefold CY^3 this leads to 1/4 unbroken supersymmetry in four dimensions. Compactification on a torus, which has trivial holonomy group $\mathcal{H} = \mathbb{I}$, does not break any supersymmetry since every spinor on T^6 transforms as a singlet due to

$$\mathbf{4}_{SU(4)} = (\mathbf{1} + \mathbf{1} + \mathbf{1} + \mathbf{1})_{\mathbb{I}}. \tag{F.9}$$

If we chose an internal six-dimensional manifold with holonomy group $\mathcal{H} = SU(2)$ such as $CY_2 \times T^2$ this would break one half of the supersymmetries due to the decomposition

$$\mathbf{4}_{SU(4)} = (\mathbf{2} + \mathbf{1} + \mathbf{1})_{SU(2)}. \tag{F.10}$$

Now let us summarize these results. Compactification on a Calabi-Yau threefold, which has 6 real dimensions, breaks 3/4 of the original supersymmetry. Thus, Calabi-Yau compactification of the heterotic string results in $N = 1$ SUSY in four dimensions, while for type II strings we end up with $N = 2$. The G_2 and $Spin(7)$ manifolds play an important role when compactifying M-theory and F-theory, respectively, to four dimensions. It turns out that compactification on G_2 breaks 7/8 of the supersymmetry and $Spin(7)$ 15/16. Calabi-Yau fourfolds, which are also eightdimensional, break 7/8 of the supersymmetry. The corresponding decomposition of spinors of $SO(8)$ under the respective holonomy groups can be found in [53] and some useful tables can be found in [54].

The results for some of the most important manifolds are summarized in the following table:

Manifold X	T^n		CY_2		CY_3		X_{G_2}		CY_4		$X_{Spin(7)}$
$\dim_{\mathbb{R}}(X)$	n		4		6		7		8		8
$Hol(X)$	$\mathbf{1}$	$\subset$	$SU(2)$	$\subset$	$SU(3)$	$\subset$	G_2		$SU(4)$	$\subset$	$Spin(7)$
SUSY	1	$>$	1/2	$>$	1/4	$>$	1/8	$=$	1/8	$>$	1/16

Table F.1: Relation between holonomy and supersymmetry for certain manifolds.

Bibliography

[1] D. J. H. Chung, L. L. Everett, G. L. Kane, S. F. King, J. Lykken, Lian-Tao Wang, [arXiv:hep-ph/0312378].

[2] N. Seiberg, "The superworld," arXiv:hep-th/9802144.

[3] N. R. Shah [http://theory.uchicago.edu/ sethi/Teaching/P487/MSSMnausheen.pdf]

[4] S. Heinemeyer, "The Higgs boson sector of the complex MSSM in the Feynman-diagrammatic approach," Eur. Phys. J. C **22** (2001) 521 [arXiv:hep-ph/0108059].

[5] D. I. Kazakov [arXiv:hep-ph/0012288v2];

[6] M. E. Peskin [arXiv:hep-ph/9705479v1] SLAC-PUB-7479;

[7] U. Amaldi, W. de Boer and H. Furstenau, "Comparison of grand unified theories with electroweak and strong coupling constants measured at LEP," Phys. Lett. B **260** (1991) 447.

[8] D. Bailin and A. Love, "Supersymmetric gauge field theory and string theory," *Bristol, UK: IOP (1994) 322 p. (Graduate student series in physics)*

[9] M. Huebscher, P. Meessen and T. Ortín, "Supersymmetric solutions of N = 2 d = 4 SUGRA: The whole ungauged shebang," Nucl. Phys. B **759** (2006) 228 [arXiv:hep-th/0606281].

[10] Anamaría Font, Stefan Theisen: "Introduction to String Compactification", lecture notes

[11] Katrin Becker, Melanie Becker, John Schwarz: "String Theory and M-Theory. A Modern Introduction", Cambridge University Press 2007

[12] B. de Wit and A. Van Proeyen, "Hidden symmetries, special geometry and quaternionic manifolds," Int. J. Mod. Phys. D **3** (1994) 31 [arXiv:hep-th/9310067].

[13] S. Ferrara and S. Sabharwal, "Quaternionic manifolds for Type II superstring vacua of Calabi-Yau spaces," Nucl. Phys. B **332** (1990) 317.

[14] R. D'Auria and P. Fré, "BPS black holes in supergravity: Duality groups, p-branes, central charges and the entropy," arXiv:hep-th/9812160.

[15] D. Robles-Llana, M. Rocek, F. Saueressig, U. Theis and S. Vandoren, "Some exact results in four-dimensional non-perturbative string theory," arXiv:hep-th/0612027.

[16] T. W. Grimm and J. Louis, "The effective action of N = 1 Calabi-Yau orientifolds," Nucl. Phys. B **699** (2004) 387 [arXiv:hep-th/0403067].

[17] Ángel M. Uranga: Graduate Course in String Theory, `http://gesalerico.ft.uam.es/paginaspersonales/angeluranga/firstpage.html`

[18] J. Louis, "Compactifications on generalized geometries," Fortsch. Phys. **54** (2006) 146.

[19] J. P. Gauntlett, J. B. Gutowski, C. M. Hull, S. Pakis and H. S. Reall, "All supersymmetric solutions of minimal supergravity in five dimensions," Class. Quant. Grav. **20** (2003) 4587 [arXiv:hep-th/0209114].

[20] J. P. Gauntlett and S. Pakis, "The geometry of D = 11 Killing spinors. ((T)," JHEP **0304** (2003) 039 [arXiv:hep-th/0212008].

[21] R. Kallosh and T. Ortín, "Killing spinor identities," arXiv:hep-th/9306085.

[22] J. Bellorín and T. Ortín, "A note on simple applications of the Killing spinor identities," Phys. Lett. B **616** (2005) 118 [arXiv:hep-th/0501246].

[23] T. Ortín, "Supersymmetry and the supergravity landscape," AIP Conf. Proc. **841** (2006) 162 [arXiv:gr-qc/0601003].

[24] J. Bagger and E. Witten, "Matter Couplings In N=2 Supergravity ," Nucl. Phys. B **222** (1983) 1.

[25] P. Fré, "Lectures on Special Kähler Geometry and Electric–Magnetic Duality Rotations," Nucl. Phys. Proc. Suppl. **45BC** (1996) 59 [arXiv:hep-th/9512043].

[26] P. Meessen and T. Ortín, "The supersymmetric configurations of N = 2, d = 4 supergravity coupled to vector supermultiplets," Nucl. Phys. B **749** (2006) 291 [arXiv:hep-th/0603099].

[27] K.P. Tod, "More on supercovariantly constant spinors," Class. Quant. Grav. **12** (1995) 1801.

[28] J. Bellorín and T. Ortín, "All the supersymmetric configurations of N = 4, d = 4 supergravity," Nucl. Phys. B **726** (2005) 171 [arXiv:hep-th/0506056].

[29] H. W. Brinkmann, *Proc. Natl. Acad. Sci. U.S.* **9** (1923) 1.

[30] H. W. Brinkmann, "Einstein spapces which are mapped conformally on each other," Math. Ann. **94** (1925) 119.

[31] J. Kowalski-Glikman, "Positive Energy Theorem And Vacuum States For The Einstein-Maxwell System," Phys. Lett. B **150** (1985) 125.

[32] F. Denef, "Supergravity flows and D-brane stability," JHEP **0008** (2000) 050 [arXiv:`hep-th/0005049`];

[33] J. Bellorin, P. Meessen and T. Ortin, "Supersymmetry, attractors and cosmic censorship," Nucl. Phys. B **762** (2007) 229 [arXiv:hep-th/0606201].

[34] K. Behrndt, D. Lüst and W.A. Sabra, "Stationary solutions of N = 2 supergravity," Nucl. Phys. B **510** (1998) 264 [arXiv:`hep-th/9705169`].

[35] P. Meessen, "Unfrozen hyperscalars and supersymmetry," to be published, Contribution to the proceedings of the RTN ForcesUniverse Network Workshop, Napoli, October 9th - 13th, 2006

[36] E. A. Bergshoeff, J. Hartong, T. Ortin and D. Roest, "Seven-branes and supersymmetry," arXiv:hep-th/0612072.

[37] T. Ortín, "Gravity And Strings," *Cambridge Unversity, Cambridge University Press, 2004*

[38] Edoardo di Napoli, 2003, Lecture notes, "The Role of Kähler and Special Kähler Geometry in Supersymmetric Field Theory," `http://www.ma.utexas.edu/ hausel/m392cr/dinapoli.pdf`

[39] A. Ceresole, R. D'Auria and S. Ferrara, "The Symplectic Structure of N=2 Supergravity and its Central Extension," Phys. Rev. D **66** (2002) 010001 arXiv:hep-th/9509160.

[40] A. Ceresole, R. D'Auria, S. Ferrara and A. van Proeyen, "Duality transformations in supersymmetric Yang-Mills theories coupled to supergravity," Nucl. Phys. B **444** (1995) 92 [`hep-th/9502072`].

[41] B. Craps, F. Roose, W. Troost and A. Van Proeyen, "What is special Kaehler geometry?," Nucl. Phys. B **503** (1997) 565 [arXiv:`hep-th/9703082`].

[42] A. van Proeyen, "N=2 supergravity in d=4,5,6 and its matter couplings," lectures given at the Institute Henri Poincaré, Paris, November 2000. `http://itf.fys.kuleuven.ac.be/ toine/LectParis.pdfhttp://itf.fys.kuleuven.ac.be/~`

[43] A. Strominger, "Special Geometry," Commun. Math. Phys. **133** (1990) 163.

[44] B. de Wit, P.G. Lauwers and A. van Proeyen, "Lagrangians Of N=2 Supergravity - Matter Systems," Nucl. Phys. B **255** (1985) 569.

[45] R. D'Auria and P. Fré, "BPS black holes in supergravity: Duality groups, p-branes, central charges and the entropy," Phys. Rev. D **66** (2002) 010001 arXiv:hep-th/9812160.

[46] L. Castellani, R. D'Auria and S. Ferrara, "Special Geometry Without Special Coordinates," Class. Quant. Grav. **7** (1990) 1767.

[47] S. Aoyama, "More on the triplet Killing potentials of quaternionic Kaehler manifolds," Phys. Lett. B **625** (2005) 127 [arXiv:hep-th/0506248].

[48] L. Andrianopoli, M. Bertolini, A. Ceresole, R. D'Auria, S. Ferrara, P. Fré and T. Magri, "*N = 2 supergravity and N = 2 super Yang-Mills theory on general scalar manifolds: Symplectic covariance, gaugings and the momentum map*", J. Geom. Phys. **23** (1997) 111 [arXiv:`hep-th/9605032`].

[49] L. Andrianopoli, M. Bertolini, A. Ceresole, R. D'Auria, S. Ferrara, P. Fré and T. Magri, "N = 2 supergravity and N = 2 super Yang-Mills theory on general scalar manifolds: Symplectic covariance, gaugings and the momentum map," J. Geom. Phys. **23** (1997) 111 [arXiv:hep-th/9605032].

[50] M. Davidse, "Membrane and fivebrane instantons and quaternonic geometry," arXiv:hep-th/0603073.

[51] B. S. Acharya and S. Gukov, "M theory and Singularities of Exceptional Holonomy Manifolds," Phys. Rept. **392** (2004) 121 [arXiv:hep-th/0409191].

[52] V. Bouchard, "Lectures on complex geometry, Calabi-Yau manifolds and toric geometry," arXiv:hep-th/0702063.

[53] J. P. Gauntlett, "Branes, calibrations and supergravity," arXiv:hep-th/0305074.

[54] R. Slansky, "Group Theory For Unified Model Building," Phys. Rept. **79** (1981) 1.

[55] P. Fré and P. Soriani, "The N=2 wonderland: From Calabi-Yau manifolds to topological field theories," *Singapore, Singapore: World Scientific (1995) 468 p*