Supersymmetric Gauged Mechanics

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Abstract

In this work, we obtain the supersymmetrization of a bosonic action describing the motion of a point-like particle with $\mathcal{N} = 1, 2$ worldline supersymmetry in the superspace framework, discussing the addition of a scalar potential, which will lead to a certain condition that must be satisfied for a potential to be supersymmetrizable. We will study the global symmetries of this action, and then the supersymmetric gauging of these symmetries. Finally, we apply this formalism to the supersymmetrization of effective actions describing black hole solutions.

Contents

Introduction

Symmetry has been present in the history of sciences since the ancient Greece as a proportion and harmony related to beauty. From the Platonic solid to Kepler's laws, geometrical shapes have been used to describe nature. After the studies of Emmy Noether, these symmetries were related to the existence of conserved quantities and conservation laws. In the 20th century and beyond, starting from Einstein's General Relativity, symmetry principles started to be seen as the primary feature of nature that constrains the physical laws. This thought was consolidated with the birth of Quantum Mechanics, and since then, symmetries have played a fundamental role in Physics.

A symmetry principle can be understood as a principle of equivalence which summarizes the regularities of the laws of physics that govern some physical system which do not depend on its specific state, endowing nature with coherence and structure. Symmetry principles constitute a guidance to construct theories.

A symmetry of a physical system is a transformation of dynamical variables that leaves its physical observables unchanged. In a classical system, it implies the invariance of the action and, thus, the equations of motion remain the same. For that reason symmetries can be used to derive new solutions. Symmetries are also useful because they lead to conservation laws and constraints. As the first Noether's theorem states, for each global, continuous symmetry of a physical system there exists a conserved quantity, such as the conservation of energy due to time translational invariance, or the conservation of momentum due to the invariance under spatial translations; and as the second Noether's theorem says, for each local, continuous symmetry of a physical system there exist a constraint between the equations of motion of the system.

But symmetries are sometimes hidden in nature. We can find some approximate symmetries, that is, symmetries which are slightly violated, and which lead to approximate conservation laws, such as the isotopic symmetry of the nuclear force, explaining the small difference between the masses of the up and down quarks. It is also possible to deal with a system in which the laws of physics are invariant under some symmetry, but the vacuum state of the system is not. This is what we call a spontaneously broken (or hidden) symmetry, which is not manifest, and then, not directly observable. An example of spontaneously broken symmetries is the invariance under rotations of a ferromagnet or the breaking of the Standard Model gauge symmetry $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ to $\mathrm{U}(1)_{\mathrm{em}}$ by the Higgs mechanism.

This existence of unobservable symmetries has lead to the search for new symmetries in order to unify the forces in nature. The first of these attempts took place around 1960. At that time, in the context of strong interactions, many hadrons had been successfully organized in multiplets of SO(3) of flavour by Gell-Mann and Ne'eman [1]. Then, many people started wondering about the existence of larger multiplets containing particles of different spins. The incompatibility of those new symmetries and relativistic invariance was highlighted by the so-called No-Go theorems. The most remarkable of these impossibility theorems is due to Coleman and Mandula (1967) [2], which proves that it is not possible to unify the internal and space-time symmetries within a quantum field theory, implying that most general symmetry of the S-matrix is the direct product of Poincaré and internal symmetries, which does not mix particles with different spins.

But Coleman and Mandula were considering symmetry groups with only bosonic generators, which satisfy commutation relations among themselves. In 1971, Gol'fand and Likhtman [3] proposed an extension of the Poincaré algebra known as superPoincaré algebra, in which some of the generators satisfied anticommuting relations. This transformations whose generators anticommute are what we call supersymmetry transformations. Supersymmetry is an extension of the ordinary spacetime symmetries obtained by adding $\mathcal{N}$ anticommuting generators $\mathcal{Q}_i$ to the usual set of translations and Lorentz generators constituting the Poincaré group. This set of commuting and anticommuting generators close a graded superalgebra which contains the usual relativistic symmetries of spacetime plus new symmetries.

This idea of considering anticommuting generators would inspire Haag, Łopuzański and Sohnius (1975) to extended the Coleman and Mandula theorem, proving that the largest possible symmetry of the S-matrix could also include those extra anticommuting symmetries [4].

Supersymmetry was first noticed in the context of string theory as a two-dimensional world sheet symmetry, and used as purely theoretical tool. In 1971, Ramond, Neveu and Schwarz built a supersymmetric action within the framework of string theory, in which they included transformations mixing scalar and spinorial fields [5]. This same year, Gervais and Sakita did something similar, obtaining a symmetry between bosons and fermions for a lineal supersymmetric action in two dimensions [6]. After some time, it was also considered to be a possible symmetry of four-dimensional quantum field theories connecting bosons and fermions into multiplets. This symmetry, if realized in nature, would have many relevant implications in elementary particle physics. Volkov and Akulov proposed in 1973 a non-linear Lagrangian invariant under supersymmetry transformations to study the possibility of neutrinos to be Goldstone bosons [7]. Simultaneously, Wess and Zumino constructed a linearly realized supersymmetric field theory in four dimensions describing spin 0 and 1/2 particles, the so-called Wess-Zumino model [8].

Supersymmetry, when linearly realized, connects fermions and bosons, i. e., for each particle with spin J there exists another particle of the same mass with spin $J \pm 1/2$. Due to the fact that these superpartners, degenerate in mass with the usual spectrum of particles, had not been observed so far, supersymmetry must be spontaneously broken at our scale of energy.

Supersymmetric models in particle physics are interesting since they heal divergences due to cancellations between fermion loops and boson loops. The most studied of those models might be the Minimal Supersymmetric Standard Model (MSSM), an extension to the Standard Model that considers each particle to have a supersymmetric partner associated. It was proposed in 1981 as an elegant way to stabilize the weak scale, solving the hierarchy problem [9], and also unifying fermions (matter) and bosons (carriers of force).

Supersymmetry also provides a candidate to a theory of quantum gravity, which could combine gravity and the Standard Model. This supersymmetric theory of gravity is called supergravity, and consists in a local version of supersymmetry, which allows to unify the four fundamental forces in nature. The non-renormalizability and the presence of anomalies in supergravity models has lead to discard these theories as a fundamental description of nature. One exception is the case of $\mathcal{N} = 8$ supergravity, which has been proved to be finite up to 4 loops [16]. However, supergravity theories appear as the low energy limit of string theories, which are supposed to be finite, so it is still interesting to study them.

Supersymmetry can be used to study certain bosonic systems as the bosonic sectors of a supersymmetric theory,[1] which enjoy special properties and take a more constrained form. One example can be the action describing a point-like particle moving in a spacetime with a metric $g_{\mu\nu}$, which takes the form:

$$S^{(0)}[x^\mu] = \int \mathrm{d}\tau \left\{ \tfrac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu - V(x) \right\}, \tag{1}$$

where we have also considered a scalar potential $V(x)$. As we are going to see, this action is the bosonic sector of a $\mathcal{N} = 1, 2$ supersymmetric action only if the potential has a certain form.

It is well known that the $\mathcal{N} = 1$ supersymmetric extensions of relativistic particle mechanics – known as spinning particle models – describe Dirac fermions after quantization en $d = 4$ dimensions. The first supersymmetric models describing spinning particles considered fermions moving in a flat spacetime [17–19], and were generalized to particles moving in an arbitrary background afterwards [20, 21].

Supersymmetry can also be useful to study more general mechanical systems. For example, let us consider now a mechanical effective action leading to black hole solutions from a supergravity-like theory of the form:

$$\begin{aligned} S[g, A, \varphi] = \int \mathrm{d}^4 x \sqrt{|g|} \big\{ & R + \mathcal{G}_{ij} \partial_\mu \varphi^i \partial^\mu \varphi^j \\ & + 2\mathrm{Im}[\mathcal{N}_{\Lambda\Sigma}] F^{\Lambda\ \mu\nu} F^{\Sigma}{}_{\mu\nu} - 2\mathrm{Re}[\mathcal{N}_{\Lambda\Sigma}] F^{\Lambda\ \mu\nu} \star F^{\Sigma}{}_{\mu\nu} \big\}, \end{aligned} \tag{2}$$

where $\mathcal{G}_{ij}$ and $\mathcal{N}_{\Lambda\Sigma}$ are symmetric matrices which depend on the scalar fields φ^i. The equations of motion of the solutions of this action describing the dynamics of a general single, static, spherically-symmetric black hole in $d = 4$ dimensions with scalar and vector fields can be found from the following effective action [22]:

$$S_{\text{eff}}[U, \varphi^i] = \int \mathrm{d}\tau \ \left\{ \dot{U}^2 + \tfrac{1}{2} \mathcal{G}_{ij} \dot{\varphi}^i \dot{\varphi}^j - e^{2U} V_{\text{bh}} \right\}, \tag{3}$$

where $\mathcal{Q}^M$ are the electric and magnetic charges of the vector fields and the black hole potential $V_{\text{bh}} = V_{\text{bh}}(\varphi, \mathcal{Q})$ is given by:

$$- V_{\text{bh}}(\varphi, \mathcal{Q}) = \tfrac{1}{2} \mathcal{Q}^M \mathcal{M}_{MN} \mathcal{Q}^N, \tag{4}$$

where $\mathcal{M} = \mathcal{M}(\varphi)$ is a matrix that depends on $\mathcal{N}_{\Lambda\Sigma}$.

[1] By bosonic sector we mean the system obtained by consistently setting all the fermionic variables to zero in a supersymmetric theory.

In $\mathcal{N}=2$ supergravity, this black hole potential can be rewritten in terms of the central charges (or fake central charges) [23] of the system $\mathcal{Z}(\varphi,\mathcal{Q})$ is the following way:

$$-V_{\rm bh}(\varphi,\mathcal{Q}) = |\mathcal{Z}|^2 + 2\mathcal{G}^{ij}\partial_i|\mathcal{Z}|\partial_j|\mathcal{Z}|. \tag{5}$$

If we make a correspondence between the coordinates appearing there and the components of the position variable $x^\mu = (U,\varphi^i)$. We can see that it takes the form of the action describing the motion of a point-like particle moving in a spacetime (1), where the metric tensor is defined as $g_{\mu\nu} = \begin{pmatrix} 2 & 0 \\ 0 & \mathcal{G}_{ij} \end{pmatrix}$ and the potential $V(x) = \frac{1}{2}g^{\mu\nu}\partial_\mu W\partial_\nu W$, with $W = 2e^U|\mathcal{Z}|$.

With these redefinitions the action can be rewritten as a perfect square up to a total derivative, which is known as the BPS form:

$$S^{(0)}[x^\mu] = \int \mathrm{d}\tau \left\{ \tfrac{1}{2}g_{\mu\nu}(\dot{x}^\mu \pm g^{\mu\rho}\partial_\rho W)(\dot{x}^\nu \pm g^{\nu\sigma}\partial_\sigma W) \right\}, \tag{6}$$

which can be trivially extremized[2] by the first order equation of motion:

$$\dot{x}^\mu = \mp g^{\mu\rho}\partial_\rho W, \tag{7}$$

those equations are known in the literatures as flow equations, and are used to develop the attractor mechanism in black hole solutions. This expression can be used to establish an analogy with the Hamilton-Jacobi formalism, which will be more deeply discussed in chapter 4.

As we will see, the form of the potential for the effective action describing a black hole is the same that will appear when studying $\mathcal{N}=1,2$ supersymmetric extensions of the action (1). For that reason we will extend this action by promoting the variables describing the position of the particle to a superfield by adding anticommuting variables, which will be related to the spin degrees of freedom of the particle.

In d=4, $\mathcal{N}=2$ supergravity theories, the scalar fields are complex, so we can redefine our coordinates as a real variable $x^\mu = (U,\varphi^i,\psi^i)$, being φ^i and ψ^i the real and imaginary part of the scalar fields, so that there appear $2n{+}1$ real fields. One can introduce an extra variable and a new set of variables H^M with $M = 1,\ldots,2n+2$ can be defined. These new variables have the property of transforming linearly under duality. The action then, can be written in terms of these variables in the following way:

$$S^{(0)}[H^M] = \int \mathrm{d}\tau \left\{ \tfrac{1}{2}g_{MN}\dot{H}^M\dot{H}^N - V(H) \right\}, \tag{8}$$

where this metric tensor g_{MN} is known to be singular [11]. When we introduced an extra variable, we implicitly included a local symmetry. Due to the singularity of the metric, not all the variables are dynamical, and this action can be thought off as coming from an action including a non-singular metric $\widetilde{g}_{MN}$ where this local symmetry has been gauged:

$$S^{(0)}[H^M] = \int \mathrm{d}\tau \left\{ \tfrac{1}{2}\widetilde{g}_{MN}\mathfrak{D}H^M\mathfrak{D}H^N - V(H) \right\}, \tag{9}$$

(the ordinary derivatives of the fields have been substituted by covariant derivatives.) In order to study systems of this form we will study the global symmetries of the supersymmetrized action describing a point-like particle within the superspace framework, we will gauge them and use them to eliminate degrees of freedom by fixing the gauge.

[2]In this case the metric tensor $g_{\mu\nu}$ is positive definite since $\mathcal{G}_{ij}$ is positive too.

Chapter 1

Worldline Supersymmetry and Superspace

We are interested in building actions describing particles which are invariant under supersymmetry through the extension of bosonic actions in spacetime. Since supersymmetry acts on fields in by interchanging bosons and fermions, working with supersymmetry transformations on spacetime might be a bit cumbersome. This task becomes much easier if we use the superspace approach, in which supersymmetry is built into the construction manifestly. This description will allow us to write directly supersymmetry-invariant actions and to develop a much more compact and elegant notation in which supersymmetry properties will be more transparent.

1.1 Superspace

Superspace is the arena in which the geometrical realization of supersymmetry takes place. It is constructed by extending spacetime to include $\mathcal{N}$ additional directions parametrized by a set of constant Grassmann numbers θ^i. Since the Poincaré group contains the symmetries which generate the ordinary spacetime (translations and rotations), we will extend it to a superPoincaré group by adding $\mathcal{N}$ supersymmetry generators $\mathcal{Q}_i$ which generate the translations along the new Grassmann directions, which in opposition to Poincaré generators, satisfy an anticommuting algebra.

Superspace was first introduced by Volkov and Akulov in 1973 as a way to geometrize supersymmetry, but the first definition of a superfield as a function of the superspace coordinates was due to Salam and Strathdee (1978) [10]. In 1977, Wess and Zumino interpreted the differential geometry of superspace as that of $\mathcal{N} = 1$ supergravity, opening the way to many other contributions to the field.

There are many kinds of superspace theories considering different supergroups and different numbers of Grassmann coordinates.[1] We will consider superspace to be parametrized by one spacetime coordinate τ and $\mathcal{N}$ Grassmann coordinates θ^i, all of them real: $\tau^* = \tau$, $(\theta^i)^* = \theta^i$.

[1] For example, Arnowitt and Nath (1975, 1978) introduced a larger supergroup Osp(3,1/4$\mathcal{N}$) with 4 bosonic and 4$\mathcal{N}$ fermionic coordinates to study local supersymmetry [12]. Ogievetski and Sokatchov (1978) [13] and Siegel and Gates (1981) [14] developed the chiral superspace approach where they considered two chiral complex superspaces related by complex conjugation .

They satisfy the following commutation and anticommutation relations:

$$[\tau,\tau] = [\tau,\theta^i] = \left\{\theta^i,\theta^j\right\} = 0, \tag{1.1}$$

where $[a,b]$ and $\{a,b\}$ are respectively the commutator and anticommutator of a and b.

Differentiation and integration over Grassmann variables is defined as follows:

$$\partial_i\theta^j = \delta_i{}^j, \qquad \int \mathrm{d}\theta^i\ \theta^j = \delta^{ij} \text{ where } \partial_i = \tfrac{\partial}{\partial\theta^i}, \tag{1.2}$$

which implies that integration and differentiation over Grassmann variables is the same operation. Using those properties, let us define

$$\widehat{\theta} = \tfrac{1}{\mathcal{N}!}\varepsilon_{i_1\ldots i_\mathcal{N}}\theta^{i_1}\ldots\theta^{i_\mathcal{N}} = \theta^1\ldots\theta^\mathcal{N}, \tag{1.3}$$

where $\varepsilon_{i_1,\ldots i_\mathcal{N}}$ is the completely antisymmetric Levi-Civita symbol. With this, we can define integration such that:

$$\int \mathrm{d}^\mathcal{N}\widehat{\theta}\ \widehat{\theta} = 1 \text{ where } \mathrm{d}^\mathcal{N}\widehat{\theta} = (-1)^{\mathcal{N}-1}\tfrac{1}{\mathcal{N}!}\varepsilon_{i_1\ldots i_\mathcal{N}}\mathrm{d}\theta^{i_\mathcal{N}}\wedge\cdots\wedge\mathrm{d}\theta^{i_1} = \mathrm{d}\theta^\mathcal{N}\wedge\cdots\wedge\mathrm{d}\theta^1. \tag{1.4}$$

As we can see, integration over $\widehat{\theta}$ selects the terms with the highest dependence in θ^i. This procedure – the so-called Berezin integration [15] – will be widely used, since we will define supersymmetric actions by integrating superfields over superspace.

Another interesting property of Grassmann coordinates is the fact that they square to zero. This implies that any analytic function $f(\tau,\theta)$ defined over superspace can be expanded in a finite power series in θ^i:

$$f(\tau,\theta^i) = f_0(\tau) + \theta^i f_i(\tau) + \tfrac{1}{2}\theta^i\theta^j f_{ij}(\tau) + \ldots + \widehat{\theta}\widehat{f}(\tau), \tag{1.5}$$

where the coefficients obtained in this decomposition are Grassmann even, bosonic, functions of τ, completely antisymmetric in the lower indices. Its higher component, then, will be proportional to the completely antisymmetric Levi-Civita symbol:

$$f_{i_1\ldots i_\mathcal{N}}(\tau) = \widehat{f}(\tau)\varepsilon_{i_1\ldots i_\mathcal{N}}. \tag{1.6}$$

We will apply this procedure to decompose superfields in superspace in terms of spacetime functions, such that the function $f(\tau,\theta^i)$ is the generalization to superspace of the spacetime function $f_0(\tau)$. This superfields will be used later on to build invariant actions in superspace, that, when expanded in components and setting all the fermions[2] to zero, will reduce to bosonic actions in spacetime.

1.2 Superalgebra

The definition of a Lie algebra can be extended to include, apart from the usual commuting relations, anticommuting relations [4]. These algebras are called superalgebras or graded Lie algebras. The generators of the symmetries of superspace satisfy a supersymmetric extension

[2]In this work the word "fermion" makes reference to an anticommuting variable, not a fermionic particle. This abuse of language should not lead to confusion.

of the Poincaré algebra, which is a $\mathbb{Z}_2$-graded algebra, i. e., being $\mathcal{O}_a$ an operator of a Lie algebra, then

$$\mathcal{O}_a\mathcal{O}_b - (-1)^{\eta_a\eta_b}\mathcal{O}_b\mathcal{O}_a = ic^c{}_{ab}\mathcal{O}_c, \tag{1.7}$$

where $c^c{}_{ab}$ are the structure coefficients and η_a carries the parity of the $\mathcal{O}_a$ operator:

$$\eta_a = \begin{cases} 0 & \text{for } \mathcal{O}_a \text{ bosonic,} \\ 1 & \text{for } \mathcal{O}_a \text{ fermionic.} \end{cases} \tag{1.8}$$

Functions in superspace, defined by $f(\tau, \theta^i)$, transform under the one-dimensional Poincaré group (translations, generated by $\mathcal{P}$) in the usual way:

$$\delta\tau = c, \tag{1.9}$$

where c is a constant. In order to extend the Poincaré algebra to a superPoincaré algebra we introduce $\mathcal{N}$ additional generators $\mathcal{Q}_i$ satisfying the supertranslation algebra. The explicit expression of the commutation and anticommutation relations between $\mathcal{P}$ and $\mathcal{Q}_i$ and the transformations of the fields under $\mathcal{Q}_i$ depends on the number of Grassmann coordinates considered. We are going to study the cases of $\mathcal{N} = 1, 2$, since they are the ones we are interested in.

Chapter 2

$\mathcal{N} = 1$ SUSY mechanics

As shown by several authors, the motion of a spinning point-like particle can be described by the supersymmetric extension of the worldline action for a point-like particle in a d-dimensional Minkowski spacetime [17–21]. We are going to generalize that action to superspace. We will construct a scalar, bosonic superfield, which is the supersymmetric extension of the x^μ variable (which describes the position of the particle), by adding a Grassmann variable ψ^a as its superpartner, and we will also endow the construction with a supercovariant derivative, the analogous to the ordinary τ derivative in superspace.

With those ingredients we will be ready to build an invariant action in superspace, that reduces to the original bosonic action in spacetime when setting the Grassmann variables ψ^a to zero. We will also discuss the addition of a scalar potential to the action. A function of the superfield $\mathcal{W}(\Phi)$ would be automatically supersymmetry invariant, it would not give a real, bosonic potential when going back to spacetime. In order to introduce a well behaved potential we will have to define a fermionic, scalar superfield. As we will see, the form of the potential will not be the most general, but it must satisfy a certain condition.

After that, we will study the global symmetries of this action by promoting the well known isometries of the bosonic part of the action to superisometries in superspace. The generalization of those transformations to local symmetries will allow us to set some fields to zero when fixing the gauge.

2.1 Superalgebra

In $\mathcal{N} = 1$ supersymmetry, superspace is parametrized by the worldline coordinate τ and a real Grassmann coordinate θ such that $\theta^* = \theta$. In this case, the Poincaré algebra can be expanded by adding a supersymmetry generator $\mathcal{Q}$ such that they satisfy the following commuting and anticommuting relations:

$$\{\mathcal{Q}, \mathcal{Q}\} = 2\mathcal{P}, \quad [\mathcal{Q}, \mathcal{P}] = 0. \tag{2.1}$$

As we can see there, two supersymmetry transformations yield a τ-space translation. This algebra is realized on superfields in terms of differential operators in superspace. Taking the representation of the translational generator to be $\mathcal{P} = i\partial/\partial\tau \equiv i\partial$, it is straightforward to check that the following expression provides a representation of the supersymmetry generator on superfields:

$$\mathcal{Q} = \frac{\partial}{\partial\theta} + i\theta\partial \equiv \partial_\theta + i\theta\partial, \tag{2.2}$$

where the i factor has been chosen such that $\mathcal{Q}$ is a Hermitean operator, since we have taken the convention of change of order of the Grassmann variables under complex conjugation:

$$(\phi\psi)^* = \psi^*\phi^* = \psi\phi = -\phi\psi \quad \text{for } \phi,\ \psi \text{ Grassman variables.} \tag{2.3}$$

We define the transformation of a superfield $F(\tau,\theta) = F_0(\tau) + i\theta F_1(\tau)$ under supersymmetries as the action of $\mathcal{Q}$ over it:

$$\delta_\epsilon F(\tau,\theta) = -\epsilon\mathcal{Q}F(\tau,\theta), \tag{2.4}$$

where ϵ is an anticommuting, real parameter. We can see that due to the structure of the supersymmetry generator, the highest component of a superfield – which is the one selected by Berezin integration, and therefore, the one that will be appearing in our actions – transforms under supersymmetry as a total derivative in τ:

$$\delta_\epsilon \int \mathrm{d}\tau\mathrm{d}\theta\ F = \int \mathrm{d}\tau\mathrm{d}\theta\ \left\{-i\epsilon F_1 + i\theta\epsilon\dot{F}_0\right\} = i\epsilon\int \mathrm{d}\tau\ \tfrac{\mathrm{d}}{\mathrm{d}\tau}F_0. \tag{2.5}$$

After integrating in τ, this will give a surface term which will not contribute to the action.

This implies that any action built by integrating superfields in superspace will be automatically supersymmetry invariant, just by construction. This is the point of working in the superspace framework: there is no need to check for the invariance of our actions under supersymmetry as long as our superfields are well defined.

However, an action constructed just by superfields leads to non-dynamical equations of motion for the fundamental variables in spacetime. In order to make our theory non-trivial we need kinetic terms to our action. Then, we would like to introduce τ derivatives of the superfields, but these are not superfields. It can be easily checked that τ derivatives of superfields are not invariant under supersymmetry transformations since this operator does not include derivatives in θ and therefore it does not commute with the supersymmetry transformations. This issue can be solved by defining a supercovariant derivative such that it commutes with the supersymmetry transformations $[\mathcal{D}, \delta_\epsilon] = 0$, i. e., such that it anticommutes with $\mathcal{Q}$, $\{\mathcal{D}, \mathcal{Q}\} = 0$:

$$\mathcal{D} = \partial_\theta - i\theta\partial, \tag{2.6}$$

and obeys this relation:

$$\{\mathcal{D}, \mathcal{D}\} = -2\mathcal{P}. \tag{2.7}$$

This definition makes the supercovariant derivative of a superfield transform as a superfield, and the transformation under supersymmetry of its higher term to transform as a total derivative:

$$\delta_\epsilon \int \mathrm{d}\tau\mathrm{d}\theta\ \mathcal{D}F = \epsilon\int \mathrm{d}\tau\tfrac{\mathrm{d}}{\mathrm{d}\tau}(F_1) = i\epsilon\int \mathrm{d}\tau\tfrac{\mathrm{d}}{\mathrm{d}\tau}(\mathcal{D}F)_0, \tag{2.8}$$

and the same happens for further covariant derivatives of a superfield.

Another property to be widely used is that the supercovariant derivative of a superfield itself also becomes a total derivative in τ once it has been integrated over θ:

$$\int \mathrm{d}\tau\mathrm{d}\theta\ \mathcal{D}F = -i\int \mathrm{d}\tau\ \tfrac{d}{d\tau}F_0. \tag{2.9}$$

Once we have proved that any action constructed from covariant derivatives of superfields and superfields is automatically invariant under supersymmetry we do not have to worry about the invariance of our action. The next step to take will be to define our superfields by their expansion in θ. This will allow us to expand our action by components and study it from the τ-space point of view.

2.2 Superfields

A scalar superfield[1] $\Phi(\tau, \theta)$ is defined as a function of the superspace coordinates which transforms under the superPoincaré group in the following way:

$$\delta_\xi \Phi = i\xi \mathcal{P} \Phi = -\xi \partial \Phi, \quad \delta_\epsilon \Phi = -\epsilon \mathcal{Q} \Phi = -\epsilon(\partial_\theta + i\theta\partial)\Phi, \tag{2.10}$$

where δ_ξ is a translation in τ. This implies that a product of superfields and the covariant derivative of a superfield are also superfields. Let us define a bosonic, real, scalar superfield Φ such that $\Phi^* = +\Phi$ as a function of the superspace coordinates by its expansion in the Grassmann coordinates θ as done in (1.5):

$$\Phi(\tau, \theta) = x(\tau) + i\theta\psi(\tau), \tag{2.11}$$

where $x(\tau)$ is a bosonic function and $\psi(\tau)$ is a Grassmann function, both real.

As we can see, a superfield is a collection of ordinary spacetime functions organized in what we call a multiplet. In general, the components of the superfields represent ordinary variables in spacetime. In this case, the bosonic component of the superfield x can be used as a coordinate $x(\tau)$ of the position of a point-like particle. The role played by the Grassmann variable will be seen later on, when introducing copies of the superfield.

However, there are two kinds of non-physical variables that can appear as components of the superfields: *auxiliary variables*, with non-derivative terms in the action, which can be integrated out from the action by using the equations of motion; and *compensating variables*, that only carry gauge degrees of freedom. They will appear when making the global symmetries of our action local.

The transformations of the components of the superfield under supersymmetry can be derived from the global transformations of the superfield under $\mathcal{Q}$, defined by (2.4). When expanding in components and equating powers of θ, we obtain [24]:

$$\delta_\epsilon x = -i\epsilon\psi, \quad \delta_\epsilon \psi = \epsilon \dot{x}, \tag{2.12}$$

where the overdot denotes an ordinary τ derivative. It is straightforward to verify that those components close the following supersymmetry algebra:

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] = \delta_\xi \text{ where } \xi = -2i\epsilon_1\epsilon_2. \tag{2.13}$$

In order to describe the motion of a spinning point-like particle in a spacetime of d dimensions with a metric $g_{\mu\nu}(x)$, we introduce d copies of the scalar superfield:

$$\Phi^\mu = x^\mu + i\theta\psi^\mu, \tag{2.14}$$

[1] Since we are not studying a field theory but a mechanical model, the term "field" in this work does not make reference to an actual field, but to a variable. If would be more correct to call our superfield Φ "supercoordinate", since it is a generalization of the coordinate x to superspace. This abuse of language should not lead to confusion.

where $\mu = 0, 1, \ldots, d-1$ is a Lorentz index. We are interested in dealing with the Grassmann functions ψ in tangent space because this is the usual treatment for spinors (although this variables ψ are *not* spinorial representations of Lorentz group). For that purpose, let us introduce a Vielbein $e^{\mu}{}_a(x)$ such that it connects curved spacetime indices μ to tangent indices a by $\eta_{ab} = e^a{}_\mu e^b{}_\nu g_{\mu\nu}$, where η_{ab} is the flat Minkowski metric. In those coordinates, the superfield takes the form:

$$\Phi^\mu = x^\mu + i\theta e^\mu{}_a \psi^a, \tag{2.15}$$

and its components transform as follows under supersymmetry:

$$\delta_\epsilon x^\mu = -i\epsilon e^\mu{}_a \psi^a, \quad \delta_\epsilon \psi = \epsilon \dot{x}^\mu - i\epsilon\psi^b \omega_b{}^a{}_c \psi^c = \epsilon \dot{x}^\mu + \delta_\epsilon x^\mu \omega_\mu{}^a{}_b \psi^b, \tag{2.16}$$

where $\omega_\mu{}^a{}_b$ is the Levi-Civita connection given by:

$$\omega_{ab}{}^c = -\Omega_{ab}{}^c + \Omega_b{}^c{}_a - \Omega_{ba}{}^c, \quad \Omega^c{}_{ab} = e^\mu{}_a e^\nu{}_b \partial_{[\mu} e_{\nu]}{}^c. \tag{2.17}$$

It is straightforward to check that those transformations still close the supersymmetry algebra (2.13).

In the next section we will define a supersymmetric action by integration of this superfield and its supercovariant derivative in superspace.

2.3 Action for a Point-like Particle

As commented before, the motion of a spin-1/2 fermion in a curved spacetime with metric $g_{\mu\nu}$ is known to be described by the supersymmetric extension of the action for an ordinary point-like particle in a one dimensional spacetime, which has the form[2]

$$S_{\text{kin}}[x^\mu] = \int \mathrm{d}\tau \; \tfrac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu, \tag{2.18}$$

where x^μ represents the position of the particle and τ is the proper time. We will also add a bosonic potential $V(x)$ at some point. We want to generalize this expression to superspace, building a supersymmetry invariant action from the previously defined bosonic superfield Φ in superspace such that when integrating over the supersymmetric coordinates θ and setting all the Grassmann variables to zero, it reduces to (2.18).

One can construct an action of this form with two covariant derivatives of the bosonic superfield, but this would not include terms in τ derivatives of the Grassmann variable ψ^a, so it will will lead to a trivial supersymmetrization of the bosonic action. The simplest way to construct a non-trivial supersymmetry invariant action is to also consider a term with the second covariant derivative of the superfield. After a short calculation, we can see that the supercovariant derivative of the superfields, when expanding in components, is given by:

$$\mathcal{D}\Phi^\mu = i e^\mu_a \psi^a - i\theta \dot{x}^\mu, \quad \mathcal{D}^2 \Phi^\mu = -i\dot{\Phi}^\mu. \tag{2.19}$$

With those ingredients one can construct the following action:

$$S^{(0)}_{\text{kin}}[\Phi] = \int \mathrm{d}\tau \; \mathrm{d}\theta \left\{ -\tfrac{1}{2} \mathcal{G}_{\mu\nu}(\Phi) \mathcal{D}^2 \Phi^\mu \mathcal{D}\Phi^\nu \right\}, \tag{2.20}$$

[2]Since we are using the mostly minuses convention for the metric $\eta = (+ - - -)$, there should be a normalization factor $-m$ in the action. We will omit it from now on. If we impose $g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 0$, this action can be used to describe a massless particle. In general, the constraint $g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$ is usually called Hamiltonian constraint.

where $\mathcal{G}_{\mu\nu}(\Phi)$ is the extension of the spacetime metric $g_{\mu\nu}$, and plays the role of a metric tensor in superspace. It can be expanded in θ as follows:

$$\mathcal{G}_{\mu\nu}(\Phi) = g_{\mu\nu}(x) + i\theta\partial_a g_{\mu\nu}\psi^a, \tag{2.21}$$

where we have defined $\partial_a = e^\mu{}_a\partial_\mu$. After splitting this action in components and integrating over θ, it takes the form of (2.18) plus a kinetic term for the supersymmetric partner of x, which is what we were looking for:

$$S^{(0)}_{\text{kin}}[x^\mu, \psi^a] = \int \mathrm{d}\tau \left\{\tfrac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + \tfrac{i}{2}\eta_{ab}\psi^a D\psi^b\right\}, \tag{2.22}$$

where $D\psi^a = \dot{\psi}^a - \dot{x}^\mu\omega_\mu{}^a{}_b\psi^b$ is the covariant derivative of the Grassmann variable ψ^a. This action, which is supersymmetry invariant, describes the motion of a point-like particle, being $x^\mu(\tau)$ its position. In $d = 4$ Minkowski spacetime, after quantization, it can be shown that the Grassmann functions $\psi^a(\tau)$ are related to the spin degrees of freedom of the particle [17].

We study now the variation of the action under arbitrary variations of the fields δx^μ and $\delta\psi^a$ vanishing at the endpoints, as done in [24], which is given by, up to total derivatives:

$$\delta S^{(0)}_{\text{kin}}[x^\mu, \psi^a] = \int \mathrm{d}\tau \left\{\delta x^\rho \left[-g_{\rho\mu}\nabla^2_\tau x^\mu - \frac{i}{2}\dot{x}^\mu R_{\mu\rho ab}\psi^a\psi^b\right] + i\eta_{ab}\Delta\psi^a D\psi^b\right\}, \tag{2.23}$$

where we have defined the covariantized variation of the Grassmann variable ψ^a:

$$\Delta\psi^a = \delta\psi^a - \delta x^\mu\omega_\mu{}^a{}_b\psi^b. \tag{2.24}$$

The equations of motion providing the invariance of the action can be read from the expression above:

$$D\psi^a = 0, \tag{2.25}$$

$$\nabla^2_\tau x^\mu + \frac{i}{2}\dot{x}^\nu R_\nu{}^\mu{}_{ab}\psi^a\psi^b = 0,, \tag{2.26}$$

where $R_\nu{}^\mu{}_{ab}$ is the Riemann tensor for the spacetime metric $g_{\mu\nu}$. As expected, if we set the Grassmann variables ψ^a to zero we recover the equations of motion for a free point-like particle moving in a geodesic in spacetime.

In the case of d = 4 we can find a physical interpretation for those expressions defining the following antisymmetric tensor, which describes the relativistic spin of the particle, and which is called *spin polarization tensor*:

$$S^{\mu\nu} = -i\psi^\mu\psi^\nu. \tag{2.27}$$

Written in terms of the spin polarization tensor $S^{\mu\nu}$, the equations above take the following form:

$$DS^{\mu\nu} = 0, \tag{2.28}$$

$$\nabla^2_\tau x^\mu + \frac{i}{2}\dot{x}^\nu R_\nu{}^\mu{}_{\rho\sigma}S^{\rho\sigma} = 0. \tag{2.29}$$

The first equation of motion implies the covariant conservation of the spin polarization tensor along the worldline, while the second one can be thought of as the spin-dependent gravitational interaction felt by a spinning particle, which is similar to the electromagnetic Lorentz force [25]:

$$m\ddot{x}^\mu = qF^\mu{}_\nu\dot{x}^\nu, \tag{2.30}$$

where a combination of the spin polarization tensor and the Riemann tensor replaces the electromagnetic tensor $F^{\mu\nu}$. As shown in many articles studying the electrodynamics of charged point-like particles [26], the space-like components $\varepsilon_{ijk}S^{ij}$ are proportional to the particle's magnetic dipole moment, while its time-like components S^{i3} represent its electric dipole moment.

If we were studying free fermions, we would be interested in setting the electric dipole moment to vanish in the rest frame, which can be expressed as the covariant constraint:

$$g_{\mu\nu}S^{\mu\rho}\dot{x}^{\nu} = 0, \tag{2.31}$$

or equivalently, in terms of the Grassmann variables ψ^a, as fixing to zero the conserved supercharge $\mathcal{Q}$:

$$\mathcal{Q} = g_{\mu\nu}e^{\nu}{}_{a}\dot{x}^{\mu}\psi^{a} = 0. \tag{2.32}$$

Another conserved quantity for this system is the worldline Hamiltonian H:

$$H = g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = 1, \tag{2.33}$$

which is the generator of the τ translations. We have fixed $H = \varpi m$ to describe a particle with mass ϖm, where $\varpi = 0$ corresponds to the case of a massless particle. In terms of the appropriate Poisson-Dirac brackets [25], it can be shown that those conserved quantities close the supersymmetry algebra:

$$\{\mathcal{Q},\mathcal{Q}\}_{\mathrm{PD}} = -2iH, \tag{2.34}$$

which is equivalent to (2.1).

2.4 Scalar Potential

Once we have constructed a supersymmetry invariant action, we are interested in adding a scalar supersymmetric potential to it. Naively, one could add a scalar function $\mathcal{W}(\Phi)$ to the action, but in order to reduce to a real, bosonic potential $V(x)$ when setting the Grassmann variables ψ^a to zero, it would have to be fermionic and imaginary, which is not possible. Instead of doing this, let us introduce N fermionic, real superfields Σ^n, with $n = 1,\ldots,N$:

$$\Sigma^n(\tau,\theta) = \eta^n(\tau) - \theta f^n(\tau), \tag{2.35}$$

where $\eta^n(\tau)$ and $f^n(\tau)$ are a fermionic and a bosonic real functions. They transform under supersymmetry in the following way:

$$\delta_\epsilon\eta^n = \epsilon f^n, \quad \delta_\epsilon f^n = -i\epsilon\dot{\eta}^n. \tag{2.36}$$

As we can see, the highest component of this fermionic superfield transforms under supersymmetry as a total derivative, and it is straightforward to check that the same thing will happen to its covariant derivative $\mathcal{D}\Sigma^n$. This means that we can build supersymmetry invariant actions by integrating Σ^n and its derivatives in superspace, as we previously did with the bosonic superfield Φ.

Let us consider an action of the following form:

$$S^{(0)}_{\mathrm{kin}}[\Sigma] = \tfrac{1}{2}\int \mathrm{d}\tau\ \mathrm{d}\theta\ \Sigma^n\mathcal{D}\Sigma^n = \int \mathrm{d}\tau\ \left\{\tfrac{1}{2}f^nf^n + \tfrac{i}{2}\eta^n\dot{\eta}^n\right\}. \tag{2.37}$$

When splitting in components and integrating over θ, we have obtained a quadratic term for the bosonic variable f^n (which will be eliminated using the equations of motion later on since it plays the role of an auxiliary field) and a kinetic term for the Grassmann variable η^n. It would be interesting to have a term coupling the fermionic and bosonic sector in order to have interaction. It can be constructed by adding another term of the form:

$$S^{(0)}_{\text{pot}}[\Phi, \Sigma] = \int \mathrm{d}\tau \ \mathrm{d}\theta \ \Sigma^n \mathcal{U}_n(\Phi) = - \int \mathrm{d}\tau \ \ \{f^n U_n(x) - i\psi^a \partial_a U_n(x) \eta^n\}\,, \tag{2.38}$$

where λ is a real positive constant and $\mathcal{U}_n(\Phi)$ is a bosonic, real function in superspace which can be expanded as follows:

$$\mathcal{U}_n(\Phi) = U_n(x) + i\theta\psi^a \partial_a U_n(x). \tag{2.39}$$

With these two terms, the potential takes the form:

$$\begin{aligned} S^{(0)}_{\text{kin+pot}}[\Phi, \Sigma] &= \int \mathrm{d}\tau \ \mathrm{d}\theta \left\{\tfrac{1}{2}\Sigma^n \mathcal{D}\Sigma^n + \lambda \Sigma^n \mathcal{U}_n(\Phi)\right\} \\ &= \int \mathrm{d}\tau \ \ \left\{\tfrac{1}{2} f^n f^n + \tfrac{i}{2}\eta^n \dot{\eta}^n + \lambda \left[U_n f^n - i\psi^a \partial_a U_n \eta^n\right]\right\}. \end{aligned} \tag{2.40}$$

As we can see, a new term coupling the fermionic ψ^a and bosonic η^n variables has appeared. Since f^n is a non-dynamical variable, we can use the equations of motion to integrate it out from the action:

$$\begin{aligned} S^{(0)}_{\text{tot}}[\Phi, \Sigma] &= \int \mathrm{d}\tau \mathrm{d}\theta \ \ \left\{\tfrac{1}{2}\Sigma^n \mathcal{D}\Sigma^n + \lambda \Sigma^n \mathcal{U}_n(\Phi)\right\} \\ &= \int \mathrm{d}\tau \ \ \left\{-\tfrac{\lambda^2}{2} U_n U_n + \tfrac{i}{2}\eta^n \dot{\eta}^n + i\lambda \psi^a \partial_a U_n \eta^n\right\}, \end{aligned} \tag{2.41}$$

and the full action reads:

$$\begin{aligned} S^{(0)}_{\text{tot}}[\Phi, \Sigma] &= \int \mathrm{d}\tau \ \mathrm{d}\theta \left\{-\tfrac{1}{2}\mathcal{G}_{\mu\nu} \mathcal{D}^2 \Phi^\mu \mathcal{D}\Phi^\nu + \tfrac{1}{2}\Sigma^n \mathcal{D}\Sigma^n + \lambda \Sigma^n \mathcal{U}_n(\Phi)\right\} \\ &= \int \mathrm{d}\tau \ \ \left\{\tfrac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + \tfrac{i}{2}\eta_{ab}\psi^a D\psi^b - \tfrac{\lambda^2}{2} U_n U_n + \tfrac{i}{2}\eta^n \dot{\eta}^n + i\lambda \psi^a \partial_a U_n \eta^n\right\}. \end{aligned} \tag{2.42}$$

If now we set the fermions to zero to recover the original bosonic action (2.18), we obtain:

$$S^{(0)}_{\text{tot}}[x^\mu] = \int \mathrm{d}\tau \ \ \left\{\tfrac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu - \tfrac{\lambda^2}{2} U_n U_n\right\}. \tag{2.43}$$

We have found the expected bosonic action plus a scalar potential $V(x)$ which is not the most general, but has the following form:

$$V(x) = \tfrac{\lambda^2}{2} U_n U_n \tag{2.44}$$

Due to the fact that $U_n(x)$ is a real function, this structure implies the positivity of the potential, meaning that only actions of this form with a positive definite potentials $V(x) \geq 0$ can be supersymmetrized in this way. In the following section we will consider the kinetic term for the fermionic field to be introduced with a metric tensor, finding a more general condition for the potential to be supersymmetrized.

2.4.1 Generalization of the Scalar Potential

As we will see in this section, the presence of a metric tensor $\mathcal{H}_{mn}$ in the kinetic term for the fermionic field Σ^n will result in a different condition for the potential to be supersymmetrizable. This tensor $\mathcal{H}_{mn}(\Phi)$ can be expanded in θ as follows:

$$\mathcal{H}_{mn}(\Phi) = h_{mn}(x) + i\theta\psi^a\partial_a h_{mn}(x). \tag{2.45}$$

Considering a term of this form the potential takes the form when splitting in components and integrating over θ:

$$\begin{aligned} S^{(0)}_{\text{kin+pot}}[\Phi,\Sigma] &= \int \mathrm{d}\tau\ \mathrm{d}\theta \left\{\tfrac{1}{2}\mathcal{H}_{mn}\Sigma^m\mathcal{D}\Sigma^n + \lambda\Sigma^n\mathcal{U}_n(\Phi)\right\} \\ &= \int \mathrm{d}\tau\ \left\{\tfrac{1}{2}h_{mn}f^mf^n + \tfrac{i}{2}h_{mn}\eta^m\dot{\eta}^n + \tfrac{i}{2}\psi^a\partial_a h_{mn}\eta^m f^n + \lambda\left[U_nf^n - i\psi^a\partial_aU_n\eta^n\right]\right\}. \end{aligned} \tag{2.46}$$

After using the equations of motion to integrate out the fermionic variable f^n as we did before, the action becomes:

$$\begin{aligned} S^{(0)}_{\text{kin+pot}}[\Phi,\Sigma] &= \int \mathrm{d}\tau\ \mathrm{d}\theta \left\{-\tfrac{1}{2}\mathcal{G}_{\mu\nu}\mathcal{D}^2\Phi^\mu\mathcal{D}\Phi^\nu + \tfrac{1}{2}\mathcal{H}_{mn}\Sigma^m\mathcal{D}\Sigma^n + \lambda\Sigma^n\mathcal{U}_n(\Phi)\right\} \\ &= \int \mathrm{d}\tau\ \Big\{\tfrac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + \tfrac{i}{2}\eta_{ab}\psi^aD\psi^b + \tfrac{i}{2}h_{mn}\eta^m\dot{\eta}^n - \tfrac{\lambda^2}{2}h^{mn}U_mU_n \\ &\qquad -\tfrac{1}{8}h^{mn}\partial_ah_{rn}\partial_bh_{ms}\psi^a\psi^b\eta^r\eta^s - i\lambda\psi^a\partial_aU_n\eta^n\Big\}, \end{aligned} \tag{2.47}$$

where h^{mn} is the inverse of h_{mn} such that $h^{mr}h_{mn} = \delta^r{}_n$. If we choose the indices $n, m, \ldots$ to be Lorentz indices and $\mathcal{H}_{mn}$ to be the metric tensor $\mathcal{G}_{\mu\nu}$, the action we have obtained is:

$$\begin{aligned} S^{(0)}_{\text{tot}}[\Phi,\Sigma] &= \int \mathrm{d}\tau\ \mathrm{d}\theta \left\{-\tfrac{1}{2}\mathcal{G}_{\mu\nu}\mathcal{D}^2\Phi^\mu\mathcal{D}\Phi^\nu + \tfrac{1}{2}\mathcal{G}_{\mu\nu}\Sigma^\mu\mathcal{D}\Sigma^\nu + \lambda\Sigma^\mu\mathcal{U}_\mu(\Phi)\right\} \\ &= \int \mathrm{d}\tau\ \Big\{\tfrac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + \tfrac{i}{2}\eta_{ab}\psi^aD\psi^b + \tfrac{i}{2}g_{\mu\nu}\eta^\mu\dot{\eta}^\nu - \tfrac{\lambda^2}{2}g^{\mu\nu}U_\mu U_\nu \\ &\qquad -\tfrac{1}{8}g^{\mu\nu}\partial_ag_{\rho\nu}\partial_bg_{\mu\sigma}\psi^a\psi^b\eta^\rho\eta^\sigma - i\lambda\psi^a\partial_aU_\mu\eta^\mu\Big\}. \end{aligned} \tag{2.48}$$

We have obtained an expression for the action which is not written in terms of spacetime tensors. This is due to the fact that the supercovariant derivative of the fermionic field $\mathcal{D}\Sigma^\mu$ does not transform as a spacetime tensor. In order to solve this problem we redefine this object as the pull-back of the spacetime covariant derivative in the worldline:

$$\mathcal{D}\Sigma^\mu \to \mathcal{D}\Sigma^\mu + \mathcal{D}\Phi^\nu\Gamma^\mu_{\nu\rho}\Sigma^\rho, \tag{2.49}$$

where $\Gamma^\mu_{\nu\rho}(\Phi)$ are the Christoffel symbols for the metric $\mathcal{G}_{\mu\nu}$. With this variation, the action takes the following form:

$$\begin{aligned} S^{(0)}_{\text{tot}}[\Phi,\Sigma] &= \int \mathrm{d}\tau\ \mathrm{d}\theta \left\{-\tfrac{1}{2}\mathcal{G}_{\mu\nu}\mathcal{D}^2\Phi^\mu\mathcal{D}\Phi^\nu + \tfrac{1}{2}\mathcal{G}_{\mu\nu}\Sigma^\mu(\mathcal{D}\Sigma^\nu + \mathcal{D}\Phi^\rho\Gamma^\nu_{\rho\sigma}\Sigma^\rho) + \lambda\Sigma^\mu\mathcal{U}_\mu(\Phi)\right\} \\ &= \int \mathrm{d}\tau\ \Big\{\tfrac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + \tfrac{i}{2}\eta_{ab}\psi^aD\psi^b + \tfrac{i}{2}g_{\mu\nu}\eta^\mu D\eta^\nu - \tfrac{\lambda^2}{2}g^{\mu\nu}U_\mu U_\nu \\ &\qquad +\tfrac{1}{16}R_{ab\rho\sigma}\psi^a\psi^b\eta^\rho\eta^\sigma - i\lambda\psi^a\partial_aU_\mu\eta^\mu\Big\}, \end{aligned} \tag{2.50}$$

where $R_{ab\mu\nu}$ is the Riemann tensor and the ordinary τ derivative of the variable η^μ has been substituted by its covariant derivative:

$$D\eta^\mu = \dot{\eta}^\mu + \dot{x}^\nu \Gamma^\mu_{\nu\rho}\eta^\rho. \tag{2.51}$$

If we set the Grassmann variables ψ^a and η^μ to zero, we obtain the following bosonic action:

$$S^{(0)}_{\text{tot}}[x^\mu] = \int \mathrm{d}\tau \ \left\{ \tfrac{1}{2} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu - \tfrac{\lambda^2}{2} g^{\mu\nu} U_\mu U_\nu \right\}, \tag{2.52}$$

which is the action describing a point-like particle (2.18) plus a potential which now has the form:

$$V(x) = \tfrac{\lambda^2}{2} g^{\mu\nu} U_\mu U_\nu. \tag{2.53}$$

Comparing with the previous form of the potential (2.44), we can see that we have arrived to a more general result. In this case, the potential does not have to be positive definite since the metric tensor $g_{\mu\nu}$ is not necessarily positive.

Also, we have to take into account that in the potential we previously considered, the fermionic field did not transform under superisometries, and therefore the potential was automatically invariant under them. But since the metric tensor $\mathcal{G}_{\mu\nu}$ depends on the superfield Φ, the new term we have added to the action will transform under them. Then, in order to ensure the invariance of the action, the fermionic field Σ^μ and the function $\mathcal{U}_\mu(\Phi)$ must transform under isometries, and new conditions will arise from the invariance of this term. This will be shown in the following section

2.5 Global Symmetries

Let us now study the symmetries of the supersymmetric action for a point-like particle obtained in the previous section. We will consider the potential to be introduced with a metric tensor since, as we have seen in the previous section, it leads to a more general condition.

The bosonic part of the action that we are generalizing (2.18) may be invariant (up to total derivatives) under isometries of the metric tensor $g_{\mu\nu}$, transformations of the coordinates x^μ of the form $\delta_\alpha x^\mu = \alpha^A k^\mu_A$ (being α^A constants, with A an index which labels the generators of the spacetime symmetries) if k^μ_A is a Killing vector of the metric, i. e. if the Lie derivative of the metric along k^μ_A vanishes:

$$\pounds_{k_A} g_{\mu\nu} = \nabla_{(\mu|} k_{A|\nu)} = 0. \tag{2.54}$$

For the purpose of examining the symmetries of this form under which our supersymmetric action is invariant, our concern will be to extend those isometries to superspace. In order to study these superisometries, let us consider the following variation of the superfield:

$$\delta_\chi \Phi^\mu = \chi^A \mathcal{K}^\mu_A(\Phi), \tag{2.55}$$

where $\mathcal{K}^\mu_A$ is a bosonic, real function resulting from trivially extending the Killing field k^μ_A to superspace. χ^A is a real, bosonic superparameter independent of τ, which is a generalization of α^A to superspace, and can be expanded in θ as follows:

$$\chi^A = \alpha^A + i\theta\beta^A, \tag{2.56}$$

where α^A is bosonic and β^A is fermionic, both constant, real parameters.

In order to provide the invariance under superisometries of the term proportional to λ in the action, we set the transformation of the fermionic superfield Σ^μ and the function $\mathcal{U}_\mu(\Phi)$ to be:

$$\delta_\chi \Sigma^\mu = \chi^A \partial_\nu \mathcal{K}^\mu_A \Sigma^\nu, \quad \delta_\chi \mathcal{U}_\mu = -\chi^A \partial_\mu \mathcal{K}^\nu_A \mathcal{U}_\nu. \tag{2.57}$$

If we expand every term in (2.55) and (2.57) we obtain the following expression for the transformations of the fields:

$$\delta_\chi x^\mu = \alpha^A k^\mu_A, \tag{2.58}$$

$$\delta_\chi \psi^a = \alpha^A \partial_\nu k^\mu_A \psi^a + \beta^A k^a_A, \tag{2.59}$$

$$\delta_\chi \eta^\mu = \alpha^A \partial_\nu k^\mu_A \eta^\mu. \tag{2.60}$$

As we can see, the transformation of x^μ has the form of the isometry of the bosonic part of the action that we were looking for, and the transformation of the fermionic variable η^μ was also expected. However, the transformation of the Grassmann variable ψ^a is different. While the δ_α term can be expected in similarity with the transformation of η^μ and x^μ, an extra transformation under the fermionic parameter β^A has been found. This δ_β transformation will be understood after fixing the gauge in next section.

It is straightforward to check that the first and second supercovariant derivatives of the bosonic superfield Φ^μ and the supercovariant derivative of the fermionic field Σ^μ transform under χ^A as:

$$\delta_\chi \mathcal{D}\Phi^\mu = \chi^A \mathcal{D}\Phi^\rho \partial_\rho \mathcal{K}^\mu_A, \tag{2.61}$$

$$\delta_\chi \mathcal{D}^2\Phi^\mu = \chi^A \mathcal{D}^2\Phi^\rho \partial_\rho \mathcal{K}^\mu_A, \tag{2.62}$$

$$\delta_\chi \mathcal{D}\Sigma^\mu = \chi^A \mathcal{D}(\partial_\rho \mathcal{K}^\mu_A \Sigma^\rho). \tag{2.63}$$

With these transformation properties we can obtain an expression for the variation of the action (2.22) under χ^A which, up to total derivatives, reads:

$$\delta_\chi S^{(0)}_{\text{tot}}[\Phi, \Sigma] = -\tfrac{1}{2}\chi^A \int \mathrm{d}\tau \ \mathrm{d}\theta \ \left\{ \pounds_{\mathcal{K}_A} \mathcal{G}_{\mu\nu} \left(\mathcal{D}^2\Phi^\mu \mathcal{D}\Phi^\nu + \Sigma^\mu \mathcal{D}\Sigma^\nu \right) \right\}. \tag{2.64}$$

We conclude that the transformations (2.55) are symmetries of the action (2.42) if they are isometries of the metric tensor $\mathcal{G}_{\mu\nu}$, i. e. if $\mathcal{K}_A$ is a Killing field of $\mathcal{G}_{\mu\nu}$:

$$\pounds_{\mathcal{K}_A} g_{\mu\nu} = \nabla_{(\mu|}\mathcal{K}_{A|\nu)} = 0. \tag{2.65}$$

The expansion of this condition in components can provide information about the τ-space functions. We find that (2.55) being a superisometry of the supersymmetric action implies $k_A(x)$ being a Killing vector of the spacetime metric $g_{\mu\nu}(x)$ – which was our starting point:

$$\pounds_{k_A} g_{\mu\nu} = \nabla_{(\mu|} k_{A|\nu)} = 0, \tag{2.66}$$

which is the spacetime version of the superspace condition we have found.

In order to make the term of the potential which is proportional to λ automatically invariant under superisometries, we have chosen the transformation of the function $\mathcal{U}_\mu(\Phi)$ under χ^A to be of the form $\delta_\chi \mathcal{U}_\mu = -\chi^A \partial_\mu \mathcal{K}^\nu_A \mathcal{U}_\nu$. However, the definition of $\mathcal{U}_\mu$ as a scalar function of the bosonic superfield Φ implies that it must transform under superisometries in the following way:

$$\delta_\chi \mathcal{U}_\mu = \chi^A \mathcal{K}^\nu_A \partial_\mu \mathcal{U}_\nu. \tag{2.67}$$

If we want to keep the invariance of this term consistent, these two transformation rules must be equivalent. After imposing this condition, we can conclude that the bosonic component of $\mathcal{U}_\mu$ must be a total derivative of a scalar function:

$$U_\mu(x) = \partial_\mu U(x). \tag{2.68}$$

This implies that the previously obtained condition to make the potential supersymmetrizable, (2.72), becomes:

$$V(x) = \tfrac{\lambda^2}{2} g^{\mu\nu} \partial_\mu U \partial_\nu U. \tag{2.69}$$

2.5.1 Summary

So far, we have constructed an action which is invariant under supersymmetry transformations and isometries, and which takes the following form when expanding in components:

$$\begin{aligned} S^{(0)}_{\text{tot}}[\Phi, \Sigma] &= \int \mathrm{d}\tau\ \mathrm{d}\theta \left\{ -\tfrac{1}{2}\mathcal{G}_{\mu\nu}\mathcal{D}^2\Phi^\mu \mathcal{D}\Phi^\nu + \tfrac{1}{2}\mathcal{G}_{\mu\nu}\Sigma^\mu(\mathcal{D}\Sigma^\nu + \mathcal{D}\Phi^\rho \Gamma^\nu_{\rho\sigma}\Sigma^\rho) + \lambda \Sigma^\mu \mathcal{U}_\mu(\Phi) \right\} \\ &= \int \mathrm{d}\tau\ \Big\{ \tfrac{1}{2} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu + \tfrac{i}{2}\eta_{ab}\psi^a D\psi^b + \tfrac{i}{2} g_{\mu\nu}\eta^\mu D\eta^\nu - \tfrac{\lambda^2}{2} g^{\mu\nu} U_\mu U_\nu \\ &\qquad + \tfrac{1}{16} R_{ab\rho\sigma}\psi^a\psi^b\eta^\rho\eta^\sigma - i\lambda\psi^a \partial_a U_\mu \eta^\mu \Big\}\,. \end{aligned} \tag{2.70}$$

If we set the Grassmann variables ψ^a and η^μ to zero, we obtain the bosonic action (2.18) which was our starting point and a bosonic potential:

$$S^{(0)}_{\text{tot}}[x^\mu] = \int \mathrm{d}\tau\ \left\{ \tfrac{1}{2} g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - V(x) \right\}, \tag{2.71}$$

where the potential takes the following form:

$$V(x) = \tfrac{\lambda^2}{2} g^{\mu\nu} \partial_\mu U \partial_\nu U. \tag{2.72}$$

We would like to re-express the action as a perfect square (up to a total derivative):

$$\int \mathrm{d}\tau \mathrm{d}\theta \left\{ \tfrac{1}{2} g_{\mu\nu}(\dot{x}^\mu \pm \tfrac{\lambda}{\sqrt{2}} g^{\mu\rho}\partial_\rho U)(\dot{x}^\nu \pm \tfrac{\lambda}{\sqrt{2}} g^{\nu\sigma}\partial_\sigma U) \right\}, \tag{2.73}$$

since this is the BPS form of the action. If the spacetime metric $g_{\mu\nu}$ is not singular, we can trivially find a first order equation which implies the second order equations of motion since it extremizes the action:

$$\dot{x}^\mu = \mp \tfrac{\lambda}{\sqrt{2}} g^{\mu\rho}\partial_\rho U. \tag{2.74}$$

A flow equation of this king appears for the FGK effective action (3) [22]. However, the minus sign in the potential does not allow us to put the action in that form. One could naively redefine the metric tensor for the fermionic superfield as $h_{mn} = \kappa g_{\mu\nu}$, with $\kappa = \pm 1$, but this will result in a change of sign of the kinetic term for the fermionic variable η^μ. At a classical level it seems consistent since the ψ^a dependent part and the η^μ dependent part of the action are separate. Otherwise this would present problems at a quantum level, since it could lead to an energy of a non-definite sign. Furthermore, this action may not be derived from local worldline supersymmetry models.

There is another way to look at this change of sign of the potential. If we rename the Grassmann variables $\psi^a = \psi_1^a$ and perform a change of variables of the form $\eta^\mu = e^\mu{}_a \psi_2^a$, the action can be written in a more compact way:

$$S_{\text{tot}}^{(0)}[\Phi, \Sigma] = \int \mathrm{d}\tau \, \left\{ \tfrac{1}{2} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu + \tfrac{i}{2}\delta^{ij}\eta_{ab}\psi_i^a D\psi_j^b - \tfrac{\lambda^2}{2} g^{\mu\nu} U_\mu U_\nu \right. \\ \left. + \tfrac{1}{16} R_{\mu\nu\rho\sigma}\psi_1^\mu \psi_1^\nu \psi_2^\rho \psi_2^\sigma - i\lambda \partial_\mu \partial_\nu U \psi^{\mu\nu} \right\}, \tag{2.75}$$

where we have defined $\psi^{\mu\nu} = \varepsilon^{ij}\psi_i^\mu \psi_j^\nu$. In this case, a change in the sign of the potential equivalent to changing the δ by the third Pauli matrix $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$:

$$S_{\text{tot}}^{(0)}[\Phi, \Sigma] = \int \mathrm{d}\tau \, \left\{ \tfrac{1}{2} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu + \tfrac{i}{2}(\sigma_3)^{ij}\eta_{ab}\psi_i^a D\psi_j^b + \tfrac{\lambda^2}{2} g^{\mu\nu} U_\mu U_\nu \right. \\ \left. - \tfrac{1}{16} R_{ab\rho\sigma}\psi_1^a \psi_1^b \psi_2^\rho \psi_2^\sigma - i\lambda \partial_\mu \partial_\nu U \psi^{\mu\nu} \right\}. \tag{2.76}$$

As we will see in the next chapter, this the form of the action in $\mathcal{N} = 2$ pseudo-supersymmetry, meaning that it this case, the addition of a potential is equivalent to considering an extra supersymmetry generator.

2.6 Gauging of the Symmetries

We are interested in extending the superparameter χ^A to be a general function of the spacetime coordinate τ, since it will be used later on to eliminate variables from the action:

$$\chi^A(\tau, \theta) = \alpha^A(\tau) + i\theta\beta^A(\tau). \tag{2.77}$$

Now, its bosonic and fermionic components will also depend on τ, which implies that in the transformations of the gauge supercovariant derivatives (2.61, 2.62) there will appear new terms depending on the τ derivatives of χ^A. These terms make the action non-invariant under isometries. In order to address this issue, let us define the gauge supercovariant derivatives for the bosonic and fermionic superfields:

$$\mathfrak{D}\Phi^\mu \equiv \mathcal{D}\Phi^\mu + \mathcal{A}^A \mathcal{K}_A^\mu, \tag{2.78}$$

$$\mathfrak{D}^2\Phi^\mu \equiv \mathcal{D}^2\Phi^\mu + \mathcal{D}\mathcal{A}^A \mathcal{K}_A^\mu, \tag{2.79}$$

$$\mathfrak{D}\Sigma^\mu \equiv \mathcal{D}\Sigma^\mu + \mathcal{A}^A \partial_\nu \mathcal{K}_A^\mu \Sigma^\nu, \tag{2.80}$$

where $\mathcal{A}^A(\tau)$ is a gauge superfield. Due to the fact that the supercovariant derivative of a bosonic, real superfield is a fermionic, imaginary superfield (which can be checked trivially),

$\mathcal{A}^A$ will also be a fermionic, imaginary function of the superspace coordinates. In order to keep the transformations of the gauge supercovariant derivatives of the superfields that provide the invariance of the action free from $\dot{\chi}^A$ terms, $\mathcal{A}^A$ must transform in the following way:

$$\delta_\chi \mathcal{A}^A = \mathcal{D}\chi^A - f_{BC}{}^A \mathcal{A}^B \chi^C, \tag{2.81}$$

where $f_{BC}{}^A$ are the structure coefficients. We chose the gauge superfield to take the form $\mathcal{A}^A = i\zeta^A - i\theta A^A$, with ζ^A and A^A a fermionic and a bosonic real parameters respectively.

When expanding as a power series in θ, the expression for the transformation of $\mathcal{A}^A$, one obtains the transformation rules for its components:

$$\delta_\chi \zeta^A = \beta^A - f_{BC}{}^A \zeta^B \alpha^C, \tag{2.82}$$

$$\delta_\chi A^A = \dot{\alpha}^A - f_{BC}{}^A A^B \alpha^C - i f_{BC}{}^A \zeta^B \beta^C. \tag{2.83}$$

Let us consider the total transformations of the components of $\mathcal{A}^A$ (supersymmetry and transformations under χ^A):

$$\delta \zeta^A = \epsilon A^A + \beta^A - f_{BC}{}^A \zeta^B \alpha^C, \tag{2.84}$$

$$\delta A^A = -i\epsilon\dot{\zeta} + \dot{\alpha}^A - f_{BC}{}^A A^B \alpha^C - i f_{BC}{}^A \zeta^B \beta^C. \tag{2.85}$$

We can fix $\beta^A = f_{BC}{}^A \zeta^B \alpha^C - \epsilon A^A$ in order to set the transformations of ζ^A to zero at every point of the superspace, and then consistently choose its value to be $\zeta^A = 0$ to eliminate it. It makes the transformations of the components of the gauge superfield to have the form:

$$\delta_\chi A^A = \dot{\alpha}^A - f_{BC}{}^A A^B \alpha^C. \tag{2.86}$$

Notice that after fixing the gauge, $\mathcal{A}^A = -i\theta A^A$ is a nilpotent superfield such that $(\mathcal{A}^A)^2 = 0$. In the last years, nilpotent superfields have been widely used in the literature, mainly in the constructions of supersymmetric models describing cosmological chaotic inflation, which also includes uplifting [27].

With this gauge election $\beta^A = -\epsilon A^A$ we can see that the δ_β transformations become supersymmetry transformations. Then, the total variation of the Grassmann functions ψ^a:

$$\delta_{\epsilon,\chi}\psi^a = \epsilon\dot{x}^\mu + i\epsilon\Omega_{bc}{}^a\psi^c\psi^b - \epsilon A^A k_A^a + \alpha^A \partial_\nu k_A^\mu \psi^a. \tag{2.87}$$

We can absorb the new term in the supersymmetry transformation of ψ^a so that its variation under isometries only depends on α^A. The δ_ϵ transformations then become:

$$\delta_\epsilon \psi^a = \epsilon(\dot{x}^\mu - A^A k_A^a) + i\epsilon\Omega_{bc}{}^a\psi^c\psi^b = \epsilon\mathfrak{D}x^\mu + i\epsilon\Omega_{bc}{}^a\psi^c\psi^b, \tag{2.88}$$

where $\mathcal{D}x^\mu = \dot{x}^\mu - A^A k_A^a$ is the covariant derivative of the spacetime coordinate x^μ, the supersymmetry transformation of x^μ has been covariantized. As we can see, gauging the superisometries in superspace is equivalent, at component level, to the gauging of the isometries in spacetime.

It is easy to verify that this transformation is consistent with the universality of the supersymmetry algebra (2.13), which now has a new term:

$$[\delta_{\epsilon_1}, \delta_{\epsilon_1}] = \delta_\xi + \delta_\alpha \quad \begin{cases} \xi = -2i\epsilon_1\epsilon_2, \\ \alpha^A = 2i\epsilon_1\epsilon_2 A^A. \end{cases} \tag{2.89}$$

After substituting the supercovariant derivatives by the gauge supercovariant ones, the action (2.42) takes the following form:

$$\begin{aligned} S^{(1)}_{\text{tot}}[\Phi,\Sigma] &= \int \mathrm{d}\tau\ \mathrm{d}\theta \left\{-\tfrac{1}{2}\mathcal{G}_{\mu\nu}\mathfrak{D}^2\Phi^\mu\mathfrak{D}\Phi^\nu + \tfrac{1}{2}\mathcal{G}_{\mu\nu}\Sigma^\mu\mathfrak{D}\Sigma^\nu + \lambda\Sigma^\mu\mathcal{U}_\mu(\Phi)\right\} \\ &= \int \mathrm{d}\tau \left\{\tfrac{1}{2}g_{\mu\nu}\mathfrak{D}x^\mu\mathfrak{D}x^\nu + \tfrac{i}{2}\eta_{ab}\psi^a\mathfrak{D}\psi^b - \tfrac{\lambda^2}{2}g^{\mu\nu}\partial_\mu U\partial_\nu U + \tfrac{i}{2}g_{\mu\nu}\eta^\mu\mathfrak{D}\eta^\nu + i\lambda\psi^a\partial_a U_\mu\eta^\mu\right\}, \end{aligned} \tag{2.90}$$

where the ordinary proper time derivatives of the spacetime fields x^μ and ψ^a have been substituted by covariant derivatives defined as:

$$\mathfrak{D}x^\mu = \dot{x}^\mu - A^A k^\mu_A, \tag{2.91}$$

$$\mathfrak{D}\psi^a = D\psi^a - A^A\partial_b k^a_A\psi^b, \tag{2.92}$$

$$\mathfrak{D}\eta^\mu = D\eta^\mu - A^A\partial_\nu k^\mu_A\eta^\nu. \tag{2.93}$$

As we have seen, by fixing the gauge we have managed to make the superisometry transformations dependent of just one spacetime parameter. With this gauge election, gauging the the χ^A transformations in superspace is equivalent to gauging the α^A transformations of the components of the bosonic and fermionic superfields in spacetime.

As we have seen, the covariant derivative of the Grassmann variable ψ^a is given by:

$$\mathfrak{D}\psi^a = \dot{\psi}^a - (A^A\lambda_A{}^a{}_b + \dot{x}^\mu\omega_\mu{}^a{}_b)\psi^b, \quad \text{where } \lambda_A{}^a{}_b = \nabla_b k^a_A = -\nabla^a k_{A\ a}. \tag{2.94}$$

While the term $\dot{x}^\mu\omega_\mu{}^a{}_b$ is the pull-back of the spin connection over the worldline, he term $\nabla_b k^a_A$ can be seen as a generalization of the momentum map. This $\lambda_A{}^a{}_b$ terms satisfy the following properties:

$$\text{i)} \quad [\lambda_A, \lambda_B]^a{}_b = f_{AB}{}^C\lambda_C{}^a{}_b - k^c_A k^d_B R_{cd}{}^a{}_b, \tag{2.95}$$

$$\text{ii)} \quad \nabla_a\lambda_A{}^b{}_c = -k^d_A R_{ad}{}^b{}_c, \tag{2.96}$$

where $f_{AB}{}^C$ are structure coefficients and $R_{ad}{}^a{}_b$ is the Riemann tensor. This second property is similar to the defining property if momentum maps:

$$\mathfrak{D}_\mu\mathcal{P}^i_A = -\mathcal{R}_{\mu\nu}{}^i k^\nu_A, \tag{2.97}$$

where i is a Lie algebra index. If we multiply both sides of this expression by $\Gamma(\mathcal{M}_i)^a{}_b$, (the generators of the relevant holonomy group in some representation) these relations are almost identical with the identification:

$$\nabla_a\lambda_A{}^b{}_c \equiv \mathcal{P}^i_A\Gamma(\mathcal{M}_i)^b{}_a \tag{2.98}$$

$$R_{\mu\nu}{}^a{}_b \equiv \mathcal{R}_{\mu\nu}{}^i\Gamma(\mathcal{M}_i)^b{}_a. \tag{2.99}$$

The deeper understanding of this term could lead to a generalization of the momentum map related not only with certain holonomy groups of the spacetime. This could be a future way of investigation.

Chapter 3

$\mathcal{N} = 2$ SUSY mechanics

Now we are interested in building a supersymmetric action in a superspace extended with $\mathcal{N} = 2$ Grassmann coordinates, repeating the procedure followed in the $\mathcal{N} = 1$ case. As we will see, although the form of the action is different, the results will be pretty similar.

3.1 Superalgebra

$\mathcal{N} = 2$ superspace is parametrized by one spacetime coordinate τ and two Grassmann coordinates θ^i with $i = 1, 2$. In this case, the Poincaré algebra is extended to superPoincaré by adding to the translation operator $\mathcal{P}$, two supersymmetry generators $\mathcal{Q}_i$ associated to the two θ^i coordinates. The form of the superPoincaré algebra is similar to (2.1):

$$\{\mathcal{Q}_i, \mathcal{Q}_j\} = 2\eta_{ij}\mathcal{P}, \tag{3.1}$$

where η is a real symmetric, 2×2, invertible matrix, which can therefore be taken to be the identity or the first or third Pauli matrices:

$$\delta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.2}$$

The cases of η being the first and third Pauli matrices are equivalent up to an unitary transformation of the supersymmetry generators. However, it is not possible to go from the case of $\eta = \delta$ to $\eta = \sigma_1, \sigma_3$ by performing an unitary transformation since this would imply a change in the determinant of η. For simplicity, we will consider η to be either the identity or σ_3, thus it will be diagonal, and recovering the $\mathcal{N} = 1$ case will be easier.

So we conclude that, in opposition to the case of $\mathcal{N} = 1$, there exist two inequivalent $\mathcal{N} = 2$ supersymmetry algebras. If we split (3.1) in components, we find two $\mathcal{N} = 1$ algebras: one is exactly the one we considered in the previous chapter (2.1), and the other one is also the same $\mathcal{N} = 1$ algebra up to a sign, which can be understood as a redefinition of the supersymmetry generator $\mathcal{Q}$.

While the first case corresponds to a positive definite energy $(\mathcal{Q}_1)^2 + (\mathcal{Q}_2)^2 = \mathcal{P}$, the second one corresponds to an energy with non-definite sign $(\mathcal{Q}_1)^2 - (\mathcal{Q}_2)^2 = \mathcal{P}$. In this case, the algebra is called a pseudo-supersymmetry algebra. In [28] it is shown how each solution of the Hamilton-Jacobi equation defines an $\mathcal{N} = 2$ pseudo-supersymmetric extension of the system.

This correspondence will be studied more deeply later on.

Now that we are considering two generators $\mathcal{Q}_i$ there appears a new symmetry transforming the supercharges in each other, the so-called R-symmetry. Any function carrying an internal index i will transform in the following way under this symmetry:

$$\delta_\Lambda f_i = f_j (\Lambda^{-1})^j{}_i, \tag{3.3}$$

where Λ is a unitary 2×2 matrix. In the case of $\mathcal{N}=2$ supersymmetric dimension this transformation leaves invariant only two matrices: η_{ij} and ε_{ij} (the completely antisymmetric Levi-Civita symbol). Since we will impose R-symmetry to be a symmetry of our action invariant, we will use this two matrices to built our model. We can see how do they transform under Λ:

$$\eta^{ij} = \Lambda^i{}_k \Lambda^j{}_l \eta^{kl}, \quad \varepsilon^{ij} = \Lambda^i{}_k \Lambda^j{}_l \varepsilon^{kl}. \tag{3.4}$$

Depending on the form chosen for η, the R-symmetry of our system will be either O(2) (for $\eta = \delta$, with $\det\eta = 1$) or O(1,1) (for $\eta = \sigma_3$, with $\det\eta = -1$). Taking into account that the invariance of ε^{ij} is equivalent to $\det\Lambda = 1$, the two different groups of this symmetry are found to be:

$$\Lambda \in \begin{cases} \mathrm{SO}(2) \text{ for } \eta = \delta \\ \mathrm{SO}(1,1) \text{ for } \eta = \sigma_3 \end{cases} \tag{3.5}$$

Let us now study the representation of our generators in terms of differential operators in superspace. Since no difference has been introduced in τ space, we can take the previously considered representation for the translation generator, $\mathcal{P} = i\partial$. With this election, the representation for the supersymmetric generators $\mathcal{Q}_i$ can be chosen to be:

$$\mathcal{Q}_i = \frac{\partial}{\partial\theta^i} + i\eta_{ij}\theta^j\partial_\tau \equiv \partial_i + i\eta_{ij}\theta^j\partial, \tag{3.6}$$

which corresponds to a generalization of (2.2) to superspace. Since we are considering $\mathcal{N}=2$ supersymmetry generators, there will also be two anticommuting parameters for a consistent definition of the supersymmetry transformations. The transformation of a superfield under supersymmetry is given by $\delta_\epsilon\Phi = -\epsilon^i\mathcal{Q}_i\Phi$. As in the $\mathcal{N}=1$ case, the highest component of a superfield transforms under supersymmetry as a total derivative:

$$\delta_\epsilon \int \mathrm{d}\tau \mathrm{d}^2\widehat{\theta}\ \Phi = -\epsilon^i \int \mathrm{d}\tau\mathrm{d}^2\widehat{\theta} \left\{ i\Phi_i + i\theta^j(\eta_{ij}\Phi_0 + \varepsilon_{ij}\widehat{\Phi}) + i\eta_{ij}\varepsilon^{jk}\dot{\Phi}_k \right\} = -i\epsilon^i\eta_{ij}\varepsilon^{jk}\int \mathrm{d}\tau\ \tfrac{\mathrm{d}}{\mathrm{d}\tau}\Phi_k. \tag{3.7}$$

So we can also construct supersymmetric actions by Berezin integration of superfields in superspace. In fact, this result is valid for any $\mathcal{N}$, which can be proved by defining a general expression for the supersymmetry generators.

Since τ derivatives of superfields do not transform as a superfield, we will also have to define two supercovariant derivatives $\mathcal{D}_i$, such that each of them anticommutes with one of the supersymmetric generators $\mathcal{Q}_i$:

$$\mathcal{D}_i = \partial_i - i\eta_{ij}\theta^j\partial \quad \text{such that } \{\mathcal{D}_i, \mathcal{Q}_j\} = [\mathcal{D}_i, \mathcal{P}] = 0,\ \{\mathcal{D}_i, \mathcal{D}_j\} = -2\eta_{ij}\mathcal{P}. \tag{3.8}$$

It is straightforward to check that, as happened in the $\mathcal{N}=1$ case, the supercovariant derivative of a superfield and its transformation under $\mathcal{Q}_i$ transform as a total derivative in τ once it has been integrated in superspace:

$$\int \mathrm{d}\tau\mathrm{d}^2\widehat{\theta}\ \mathcal{D}_i\Phi = -i\eta_{ij}\varepsilon^{jk}\int \mathrm{d}\tau\ \tfrac{\mathrm{d}}{\mathrm{d}\tau}\Phi_k, \quad \delta_\epsilon\int \mathrm{d}\tau\mathrm{d}^2\widehat{\theta}\ \mathcal{D}_i\Phi = \eta_{jk}\varepsilon^{kj}\int \mathrm{d}\tau\ \tfrac{\mathrm{d}}{\mathrm{d}\tau}\left\{\varepsilon_{il}\widehat{\Phi} - \eta_{il}\Phi_0\right\}, \tag{3.9}$$

so we can also use supercovariant derivatives of the superfields when constructing actions, as we did in $\mathcal{N}=1$. Then, the next step to take will be to define superfields as functions of the superspace coordinates.

3.2 Superfields

A scalar superfield $\Phi(\tau,\theta)$ is defined as a function of the superspace coordinates which transforms under the superPoincaré group in the following way:

$$\delta_\xi \Phi = -\xi \partial \Phi, \quad \delta_\epsilon \Phi = -\epsilon(\partial_i + i\eta_{ij}\theta^j \partial)\Phi, \tag{3.10}$$

where δ_ξ is a translation in τ. Let us define a real, bosonic, superfield Φ as a function of the superspace coordinates by its expansion in the supersymmetric coordinates. For the $\mathcal{N}=2$ case, it will have three components:

$$\Phi(\tau,\theta) = x(\tau) + i\theta^i \psi_i(\tau) + i\widehat{\theta} F(\tau), \tag{3.11}$$

where $x(\tau)$ is a bosonic, real position parameter and $\psi_i(\tau)$ are two real, fermionic functions. $F(\tau)$ is a bosonic, real field which will play the role of an auxiliary field. If we apply the expression for the transformation of Φ under supersymmetry we can obtain the transformations of its components by expanding in θ^i:

$$\delta_\epsilon x = -i\epsilon^i \psi_i, \quad \delta_\epsilon \psi_i = \epsilon^j \eta_{ij} \dot{x} - \epsilon^j \varepsilon_{ij} F, \quad \delta_\epsilon F = -\epsilon^i \eta_{ij} \varepsilon^{jk} \dot{\psi}_k. \tag{3.12}$$

It is straightforward to check that those supersymmetric transformations together with the τ translations, close the supersymmetry algebra (2.13). As we can see, if we set $\theta_2 = 0$ truncating the expansion in superspace at the first Grassmann generator, and also set to zero the auxiliary field F, we obtain the supersymmetric transformations satisfied by the fermionic and bosonic variables for the $\mathcal{N}=1$ case (2.16).

If we now introduce d copies of the scalar superfield Φ^μ:

$$\Phi^\mu = x^\mu + i\theta^i e^\mu{}_a \psi_i^a + i\widehat{\theta} F^\mu. \tag{3.13}$$

where $\mu = 0,\dots,d-1$ is a Lorentz index. Again, we have chosen to work with the fermionic variables in the tangent space, whose flat indices are $a = 0,\dots,d-1$. The supersymmetry transformations of its components become:

$$\delta_\epsilon x^\mu = -i\epsilon^i e^\mu{}_a \psi_i^a, \tag{3.14}$$

$$\delta_\epsilon \psi_i^a = \epsilon^j \eta_{ij} e^a{}_\mu \dot{x}^\mu + i\epsilon^j \Omega_{cb}{}^a \psi^c_{(j} \psi^b_{i)} - \epsilon^j \varepsilon_{ij}(F^a + iC_{bc}{}^a \psi^{bc}), \tag{3.15}$$

$$\delta_\epsilon F^\mu = -\epsilon^i \eta_{ij} \varepsilon^{jk} \dot{\psi}_k^\mu, \tag{3.16}$$

where we have defined $F^a = e^a{}_\mu F^\mu$,$\psi_i^\mu = e^\mu{}_a \psi_i^a$ and $\psi^{ab} = \varepsilon^{ij}\psi_i^a \psi_j^b$. In the next section we will use this superfield and its covariant derivative to build a $\mathcal{N}=2$ supersymmetry invariant action.

3.3 Action for a Point-like Particle

Now, we are going to supersymmetrize the worldline action for a point-like particle moving in a d-dimensional spacetime with a metric $g_{\mu\nu}$ (2.18) by integration on superspace of the bosonic superfield Φ^μ defined in the previous sections and its supercovariant derivative, which is given by the following expression:

$$\mathcal{D}_i\Phi^\mu = ie^\mu_a\psi^a_i + i\theta^j(\varepsilon_{ij}F^\mu - \eta_{ij}\dot{x}^\mu) + \eta_{ij}\varepsilon^{jk}\widehat{\theta}\dot{\psi}^\mu_k. \tag{3.17}$$

Unlike in the case of $\mathcal{N}=1$, there appears a term in τ derivatives of the Grassmann variables ψ^a_i in the first supercovariant derivative of the superfield. This implies that we can obtain a non-trivial supersymmetrization of the bosonic action (2.18) with two supercovariant derivatives, there is no need to use the second supercovariant derivative to obtain a kinetic term for the Grassmann variables when integrating over spacetime as we did in the $\mathcal{N}=1$ case. We also notice that there are not terms on τ derivatives of F^μ, so we will be able to eliminate it from the action by using the equations of motions since this function does not represent a physical field in ordinary τ space.

Our supersymmetric action, then, will have the form of the immediate supersymmetrization of (2.18):

$$\begin{aligned} S^{(0)}_{\text{kin}}[\Phi] &= \int \mathrm{d}\tau\ \mathrm{d}^2\widehat{\theta}\left\{-\tfrac{1}{4}\det\eta\ \eta^{ij}\mathcal{G}_{\mu\nu}(\Phi)\mathcal{D}_i\Phi^\mu\mathcal{D}_j\Phi^\nu\right\} \\ &= \int \mathrm{d}\tau\left\{\tfrac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + \tfrac{i}{2}\eta^{ij}\eta_{ab}\psi^a_i D\psi^b_j + \tfrac{1}{2}\det\eta\ (F^\mu - i\psi^{\rho\sigma}\Gamma^\mu_{\rho\sigma})g_{\mu\nu}(F^\nu - i\psi^{\alpha\beta}\Gamma^\nu_{\alpha\beta})\right. \\ &\qquad \left. - \tfrac{1}{2}\det\eta\ \psi^{\mu\nu}\psi^{\rho\sigma}(\partial_\mu\partial_\nu g_{\rho\sigma} - \Gamma^\alpha_{\mu\nu}\Gamma_{\rho\sigma\ \alpha})\right\}, \end{aligned} \tag{3.18}$$

where $\Gamma^\alpha_{\mu\nu}(x)$ are the Christoffel symbols obtained from the spacetime metric $g_{\mu\nu}$. $\mathcal{G}_{\mu\nu}(\Phi)$ is the generalization to superspace of the metric tensor in spacetime $g_{\mu\nu}(x)$, and can be expanded in θ as follows:

$$\mathcal{G}_{\mu\nu}(\Phi) = \mathcal{G}_{\mu\nu}(x) + i\partial_a\mathcal{G}_{\mu\nu}\theta^i\psi^a_i + i\widehat{\theta}\left[\partial_\rho\mathcal{G}_{\mu\nu}F^\rho + i\partial_\rho\partial_\sigma\mathcal{G}_{\mu\nu}e^\rho{}_a e^\sigma{}_b\psi^{ab}\right]. \tag{3.19}$$

As commented before, the variable F^μ plays the role of an auxiliary field, so one can integrate it out from the action by using its equations of motion:

$$\frac{\delta S^{(0)}}{\delta F^\mu} = F^\mu - i\psi^{\rho\sigma}\Gamma^\mu_{\rho\sigma} = 0 \tag{3.20}$$

This expression can also be used to eliminate the variable F^μ in (3.15), which gives the following supersymmetry transformation of the Grassmann variable ψ^a_i:

$$\delta_\epsilon\psi^a_i = \epsilon^j\eta_{ij}e^a{}_\mu\dot{x}^\mu - i\epsilon^j\psi^b_j\omega_b{}^a{}_c\psi^c_i = \epsilon^j\eta_{ij}e^a{}_\mu\dot{x}^\mu + \delta_\epsilon x^\mu\omega_\mu{}^a{}_c\psi^c_i, \tag{3.21}$$

which is the same transformation that for the $\mathcal{N}=1$ case. Substituting also the equations of motion in the action gives:

$$S^{(0)}_{\text{kin}}[x^\mu,\psi^a] = \int \mathrm{d}\tau\left\{\tfrac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + \tfrac{i}{2}\eta^{ij}\eta_{ab}\psi^a_i D\psi^b_j + \tfrac{1}{16}\det\eta\ \psi^\mu_i\psi^\nu_i\psi^\rho_j\psi^\sigma_j R_{\mu\nu\rho\sigma}\right\}. \tag{3.22}$$

where $R_{\mu\nu\rho\sigma}$ is the Riemann tensor for the metric $g_{\mu\nu}(x)$. If we compare with the supersymmetric action for a spinning point-like particle in $\mathcal{N}=1$ (2.42), we can see that there appears a new term proportional to the Riemann tensor of the spacetime.

In the next section we will discuss the addition of a scalar potential to the action, obtaining a condition for this potential to be supersymmetrized.

3.4 Scalar Potential

Now we are interested in adding a scalar potential to our supersymmetric action. In the case of $\mathcal{N}=2$, we can just add a real, bosonic function of the superfield $\mathcal{W}(\Phi)$ since it will lead to a real, bosonic potential $V(x)$ when integrating in superspace. We define the expansion of our potential in the following way:

$$\mathcal{W}(\Phi) = W(x) + i\theta^i \psi_i^a \partial_a W(x) + i\widehat{\theta}\left(F^a \partial_a W(x) - \tfrac{i}{2}\partial_a \partial_b W(x)\psi^{ab}\right), \tag{3.23}$$

where $W(x)$ is a bosonic, real function. Since the highest component of the potential is imaginary, we will have to put a i factor in order to make the full action real. With this new term, the action reads:

$$S^{(0)}_{\text{tot}}[\Phi] = \int d\tau\ d^2\widehat{\theta}\left\{\tfrac{1}{4}\det\eta\ \eta^{ij}\mathcal{G}_{\mu\nu}(\Phi)\mathcal{D}_i\Phi^\mu \mathcal{D}_j \Phi^\nu + i\mathcal{W}(\Phi)\right\}. \tag{3.24}$$

After integrating over superspace, the action takes the following form:

$$S^{(0)}_{\text{tot}}[x^\mu, \psi^a] = \int d\tau\left\{\tfrac{1}{2}g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu + \tfrac{i}{2}\eta^{ij}\eta_{ab}\psi_i^a D\psi_j^b + \tfrac{1}{16}\psi_i^\mu \psi_i^\nu \psi_j^\rho \psi_j^\sigma R_{\mu\nu\rho\sigma}\right.$$

$$\left. -\tfrac{1}{2}\det\eta\ g^{\mu\nu}\partial_\mu W(x)\partial_\nu W(x) + i\psi^{\mu\nu}\nabla_\mu \partial_\nu W(x)\right\}. \tag{3.25}$$

If we make the correspondence $W(x) = U(x)$ and chose η to be σ_3 we recover the action constructed in the $\mathcal{N}=1$ case. We can conclude that the introduction of a scalar potential in $\mathcal{N}=1$ supersymmetry in a certain way is equivalent to considering $\mathcal{N}=2$ pseudo-supersymmetry.

If now we set the Grassmann variables to zero in order to recover the bosonic action we obtain:

$$S^{(0)}_{\text{tot}}[x^\mu] = \int d\tau\left\{\tfrac{1}{2}g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu - \tfrac{1}{2}\det\eta\ g^{\mu\nu}\partial_\mu W(x)\partial_\nu W(x)\right\}, \tag{3.26}$$

which is the expected bosonic action plus a scalar potential:

$$V(x) = \tfrac{1}{2}\det\eta\ g^{\mu\nu}\partial_\mu W(x)\partial_\nu W(x), \tag{3.27}$$

which has the form of the potential appearing in a FGK action (3) when we chose $\eta = \sigma_3$.

3.5 Global Symmetries

In order to study the global symmetries of the action defined in the last section, let us consider a variation of the superfield of the same form as in the previous case, a generalization of the isometries of a bosonic action $\delta_\alpha x^\mu = \alpha^A k_A^\mu$:

$$\delta_\chi \Phi^\mu = \chi^A \mathcal{K}_A^\mu(\Phi), \tag{3.28}$$

where, as in the $\mathcal{N}=1$ case, $\mathcal{K}_A^\mu$ is a bosonic, real extension of the Killing field k_A^μ of the metric $g_{\mu\nu}$ to superspace and χ^A is a real, bosonic generalization of the parameter α^A to superspace. Its expansion in θ will have now an extra term:

$$\chi^A = \alpha^A + i\theta^i \beta_i^A + i\widehat{\theta}\gamma^A, \tag{3.29}$$

where α^A and γ^A are bosonic and β_i^A are fermionic, all of them real parameters. If we expand the expression for the transformation of the superfield (3.28) in θ^i we can obtain the transformations of the components of Φ^μ:

$$\delta_\chi x^\mu = \alpha^A k_A^\mu, \tag{3.30}$$

$$\delta_\chi \psi_i^a = \alpha^A \partial_b k_A^a \psi_i^b - \alpha^A k_A^\mu \Omega^\rho{}_{ba} \psi_i^b + \beta_i^A k_A^a. \tag{3.31}$$

where we have defined $k_A^a = e_\mu^a k_A^\mu$. As in the previous case, there appears a β_i^A extra transformation. Its meaning will be understood once the gauge is fixed.

The supercovariant derivative of the superfield Φ^μ transforms under χ^A in the following way:

$$\delta_\chi \mathcal{D}\Phi^\mu = \chi^A \mathcal{D}\Phi^\rho \partial_\rho \mathcal{K}_A^\mu. \tag{3.32}$$

With that expression we can compute also the variation of the action under χ^A:

$$\delta_\chi S_{\text{tot}}^{(0)}[\Phi] = \chi^A \int \mathrm{d}\tau\ \mathrm{d}\theta\ \left\{ -\tfrac{1}{4} \pounds_{\mathcal{K}_A} \mathcal{G}_{\mu\nu}(\Phi) \mathcal{D}\Phi^\mu \mathcal{D}\Phi^\nu + i \pounds_{\mathcal{K}_A} \mathcal{W} \right\}. \tag{3.33}$$

We have arrived to the same result as in the $\mathcal{N}=1$ case: the transformations (3.28) are symmetries of the action (3.22) if they are isometries of the metric tensor $\mathcal{G}_{\mu\nu}(\Phi)$ and if they leave the potential $\mathcal{W}(\Phi)$ invariant under χ^A:

$$\pounds_{\mathcal{K}_A} \mathcal{W}(\Phi) = \mathcal{K}_A^\mu \partial_\mu \mathcal{W}(\Phi) = 0, \tag{3.34}$$

which translate into its analogous expression in ordinary τ space:

$$\pounds_{k_A} W(x) = k_A^\mu \partial_\mu W(x) = 0. \tag{3.35}$$

Our next concern will be to generalize this transformations to be local, considering the gauge parameters to depend on the worldline parameter τ. In order to keep the action invariant under this local isometries, we will have to define a gauge supercovariant derivative for the superfield. This generalization will allow us to eliminate some degrees of freedom by fixing the gauge.

3.6 Gauging

In order to make the χ^A transformation local we let the components of this parameter to be general functions of τ:

$$\chi^A(\tau,\theta^i) = \alpha^A(\tau) + i\theta^i\beta_i^A(\tau) + i\widehat{\theta}\gamma^A(\tau). \tag{3.36}$$

This τ dependence of the parameters will appear in the expression for the transformation of the supercovariant derivatives of the superfield as τ derivatives of the parameters. Since this action is not invariant under this local transformations, we will have to define a gauge supercovariant derivative such that the action obtained by exchanging the supercovariant derivatives by these gauge supercovariant derivatives is invariant up to total derivatives under local isometries:

$$\mathfrak{D}_i\Phi^\mu = \mathcal{D}_i\Phi^\mu + \mathcal{A}_i^A\mathcal{K}_A^\mu, \tag{3.37}$$

where $\mathcal{A}_i^A$ is a fermionic, imaginary gauge superfield whose expansion in θ^i is:

$$\mathcal{A}_i^A = i\zeta_i^A - i\theta^j A_{ij}^A - \widehat{\theta}\Upsilon_i^A, \tag{3.38}$$

where ζ_i^A and Υ_i^A are fermionic variables and A_{ij}^A is a bosonic variable, all of them real. Due to the fact that $\mathcal{A}_i^A$ has to be invariant under R-symmetry, its bosonic component A_{ij}^A has to be proportional to the two invariant matrices, η_{ij} and ε_{ij}, so it will have the following form:

$$A_{ij}^A = \eta_{ij}A^A + \varepsilon_{ij}B^A. \tag{3.39}$$

In order to make the transformations of the gauge supercovariant derivatives free from τ derivatives of χ^A, the gauge superfield must transform under χ as follows:

$$\delta_\chi\mathcal{A}_i^A = \mathcal{D}_i\chi^A - f_{BC}{}^A\mathcal{A}_i^B\chi^C. \tag{3.40}$$

If we split this expression in components, we find:

$$\delta_\chi\zeta_i^A = \beta_i^A - f_{BC}{}^A\zeta_i^B\alpha^C, \tag{3.41}$$

$$\delta_\chi A_{ij}^A = \eta_{ij}\dot{\alpha}^A - f_{BC}{}^A A_{ij}^B\alpha^C - \varepsilon_{ij}\gamma^A - if_{BC}{}^A\zeta_i^B\beta_j^C, \tag{3.42}$$

$$\delta_\chi\Upsilon_i^A = \eta_{ij}\varepsilon^{jk}\dot{\beta}_k^A - \varepsilon^{jk}A_{ij}^B\beta_j^C + f_{BC}{}^A\zeta_i^B\gamma^C - f_{BC}{}^A\Upsilon_i^B\alpha^C. \tag{3.43}$$

We can use γ^A to eliminate the antisymmetric part of A_{ij}^A. With this, the total (isometries + supersymmetry) transformations of the components of the gauge superfield become:

$$\delta\zeta_i^A = \epsilon^k\eta_{ik}A^A + \beta_i^A - f_{BC}{}^A\zeta_i^B\alpha^C, \tag{3.44}$$

$$\delta_\chi A_{ij}^A = -i\epsilon^k\varepsilon_{kj}\Upsilon_i^A - i\epsilon^k\eta_{kj}\dot{\zeta}_i^A + \eta_{ij}\dot{\alpha}^A - f_{BC}{}^A\eta_{ij}A^B\alpha^C - if_{BC}{}^A\zeta_i^B\beta_j^C, \tag{3.45}$$

$$\delta_\chi\Upsilon_i^A = \epsilon^k\eta_{jk}\varepsilon^{jl}\dot{A}^A\eta_{il} + \eta_{ij}\varepsilon^{jk}\dot{\beta}_k^A - \varepsilon^{jk}\eta_{ij}A^B\beta_j^C + f_{BC}{}^A\zeta_i^B\gamma^C - f_{BC}{}^A\Upsilon_i^B\alpha^C. \tag{3.46}$$

Fixing $\beta_i^A = -\epsilon^j\eta_{ij}A^A$ we can consistently set $\zeta_i^A = 0$ and $\Upsilon_i^A = 0$. This relation between the β transformations and supersymmetries results in a redefinition of the supersymmetric transformation of the fermionic fields:

$$
\begin{aligned}
\delta_\epsilon \psi_i^a &= \epsilon^j \eta_{ij} e_\mu^a \dot{x}^\mu + \beta_i^A k_A^a + \delta_\epsilon x^\mu \omega_\mu{}^a{}_c \psi_i^c \\
&= \epsilon^j \eta_{ij} e_\mu^a (\dot{x}^\mu - A^A k_A^\mu) + \delta_\epsilon x^\mu \omega_\mu{}^a{}_c \psi_i^c \\
&= \epsilon^j \eta_{ij} \mathfrak{D} x^\mu + \delta_\epsilon x^\mu \omega_\mu{}^a{}_c \psi_i^c ,
\end{aligned}
\tag{3.47}
$$

which can be thought of as a covariantization of the supersymmetric transformations. When substituting the supercovariant derivatives in the action, ordinary derivatives for fermionic and bosonic fields will also be replaced by gauge covariant ones:

$$
\begin{aligned}
S^{(1)}_{\text{tot}}[\Phi] &= \int \mathrm{d}\tau \ \mathrm{d}^2\widehat{\theta} \left\{ -\tfrac{1}{4} \det\eta \ \eta^{ij} \mathcal{G}_{\mu\nu}(\Phi) \mathfrak{D}_i \Phi^\mu \mathfrak{D}_j \Phi^\nu + i\mathcal{W}(\Phi) \right\} \\
&= \int \mathrm{d}\tau \left\{ \tfrac{1}{2} g_{\mu\nu} \mathfrak{D} x^\mu \mathfrak{D} x^\nu + \tfrac{i}{2} \eta^{ij} \eta_{ab} \psi_i^a \mathfrak{D} \psi_j^b + \tfrac{1}{16} \det\eta \ \psi_i^\mu \psi_i^\nu \psi_j^\rho \psi_j^\sigma R_{\mu\nu\rho\sigma} \right. \\
&\qquad \left. - \tfrac{1}{2} \det\eta \ g^{\mu\nu} \partial_\mu W(x) \partial_\nu W(x) + i \psi^{\mu\nu} \nabla_\mu \partial_\nu W(x) \right\} .
\end{aligned}
\tag{3.48}
$$

When setting all the Grassmann variables to zero, we obtain the following action:

$$
S^{(1)}_{\text{tot}}[x^\mu] = \int \mathrm{d}\tau \left\{ \tfrac{1}{2} g_{\mu\nu} \mathfrak{D} x^\mu \mathfrak{D} x^\nu - \tfrac{1}{2} \det\eta \ g^{\mu\nu} \partial_\mu W(x) \partial_\nu W(x) \right\} . \tag{3.49}
$$

which corresponds to a bosonic gauged action with a scalar potential of the form:

$$
V(x) = \tfrac{1}{2} \det\eta \ g^{\mu\nu} \partial_\mu W(x) \partial_\nu W(x). \tag{3.50}
$$

Chapter 4

Supersymmetry and Hamilton-Jacobi formalism

As we have seen, the bosonic part of our $\mathcal{N}=2$ pseudo-supersymmetric action can be written as a sum of squares such that the second order Euler-Lagrange equations are contained in a first order equations, which in the literature are called *flow equations*:

$$\dot{x}^\mu = \mp\tfrac{\lambda}{\sqrt{2}} g^{\mu\nu}\partial_\nu U. \tag{4.1}$$

When the action takes the BPS form (6), the first order equations imply the second order Euler-Lagrange equations. The universality of this construction can be traced to the existence of another well known universal construction for mechanical systems: the Hamilton-Jacobi formalism.

In order to clarify this idea let us define the canonically conjugate momenta for the x^μ variable:

$$\pi_\mu = g_{\mu\nu}\dot{x}^\nu, \tag{4.2}$$

Then, the equations (4.1) can be compared with those of the Hamilton-Jacobi formalism, which can be obtained through a function $S(\tau, x^\mu)$ called Hamilton's principal function in the following way:

$$\pi_\mu = \frac{\partial S}{\partial x^\mu}, \quad H = -\frac{\partial S}{\partial \tau}, \tag{4.3}$$

where

$$H = \tfrac{1}{2} g^{\mu\nu}\pi_\mu\pi_\nu + V(x), \tag{4.4}$$

is the Hamiltonian. If we define Hamilton's principal function for this system as:

$$S(\tau, x) = \mp\tfrac{\lambda}{\sqrt{2}} U(x) - E\tau, \tag{4.5}$$

where E is the energy of the system. With this, (4.3) and (4.4) become:

$$\pi_\mu = \mp\tfrac{\lambda}{\sqrt{2}}\partial_\mu U, \quad \tfrac{\lambda^2}{2} g^{\mu\nu}\partial_\mu U\partial_\nu U + V = E. \tag{4.6}$$

If we find a solution for the last of these equations, then the solutions of the mechanical system satisfy the first order equations that can be rearranged in the form:

$$\dot{x}^\mu = \mp\tfrac{\lambda}{\sqrt{2}} g^{\mu\nu}\partial_\mu U, \tag{4.7}$$

which was our starting point. All mechanical systems admit this treatment for a number of functions S and U associated to different solutions. The Hamilton-Jacobi theory provides methods for computing them. If we consider a potential of the form:

$$V = -\tfrac{\lambda^2}{2} g^{\mu\nu} \partial_\mu U \partial_\nu U, \tag{4.8}$$

we can find a trivial solution with zero energy. As we have seen, this is the form of the potential in the $\mathcal{N} = 2$ pseudo-supersymmetry case. This correspondence between the solutions of Hamilton-Jacobi equations for a mechanical system and the $\mathcal{N} = 2$ pseudo-supersymmetric extension of this model was noticed by Townsend in [28].

Conclusions

In this work, we have constructed a $\mathcal{N} = 1, 2$ (pseudo-)supersymmetry invariant action whose bosonic sector describes a point-like particle moving in superspace with a metric, including a scalar potential. We have found that the inclusion of a potential in a certain way in the $\mathcal{N} = 1$ is equivalent to the $\mathcal{N} = 2$ pseudo-supersymmetric case.

We have studied the gauging, consistent with supersymmetry, of these models, obtaining a term in the definition of the covariant derivative of the fermionic fields that can be seen as a momentum map in tangent space. Finally, we have made a connection with Hamilton-Jacobi formalism, recovering the results obtained by Townsend in [28] in a different way.

This work can be extended by considering supersymmetry to be local, constructing a supergravity model. Also, the term identified as a momentum map can be studied to generalize the concept of momentum map to something related not only with certain holonomy groups of the spacetime.

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