Lecture 5: Dualities

- S–Duality

- T–Duality

S-Duality

$$Q^i \stackrel{S}{\to} \left(e^{i\frac{\pi}{4}\tau_2}\right)^{ij} Q^j = \tfrac{1}{2}\sqrt{2}\begin{pmatrix} Q^1 + Q^2 \\ -Q^1 + Q^2 \end{pmatrix}$$

$$\begin{aligned} \left(Z^{(1)}_{\mu}\,, Z^{(1)}_{\mu_1\cdots\mu_5}\right) &\stackrel{S}{\to} \left(Z^{(3)}_{\mu}\,, Z^{(3)}_{\mu_1\cdots\mu_5}\right) \\ \left(Z^{(3)}_{\mu}\,, Z^{(3)}_{\mu_1\cdots\mu_5}\right) &\stackrel{S}{\to} -\left(Z^{(1)}_{\mu}\,, Z^{(1)}_{\mu_1\cdots\mu_5}\right) \end{aligned}$$

$$S^8 = 1$$

$$R \stackrel{S}{\to} R\,, \qquad g_s \stackrel{S}{\to} \frac{1}{g_s}\,, \qquad \ell_s \stackrel{S}{\to} (g_s)^{1/2}\ell_s$$

Example

$$\text{NS}-\text{1B} \stackrel{\text{S}}{\leftrightarrow} \text{D1}$$

$$\begin{aligned} ds^2 &= H^{-1}dx^2_{(2)} + dx^2_{(8)} & e^{2\phi} &= H^{-1} \\ ds^2 &= H^{-1/2}dx^2_{(2)} + H^{1/2}dx^2_{(8)} & e^{2\phi} &= H \end{aligned}$$

Exercise V

$$S \longrightarrow SL(2,\mathbb{R})$$

$$\lambda = \ell + i e^{-\phi}\,, \qquad \ell \equiv C^{(0)}$$

$$\lambda \stackrel{SL(2,\mathbb{R})}{\longrightarrow} \frac{a\lambda + b}{c\lambda + d}\,, \qquad ad - bc = 1$$

T–Duality: Bosonic

$$p^{25} = \frac{n}{R} : \qquad \text{Momentum}$$

$$X^{25}(\tau, \sigma + \ell) = X^{25}(\tau, \sigma) + 2\pi R m : \qquad \text{Winding}$$

$$\begin{aligned} X^{25} &= x^{25} + \frac{2\pi\alpha'}{\ell R}\tau n + \frac{2\pi R}{\ell}\sigma m \\ &+ \text{oscillators} \end{aligned}$$

$$m^2 = \frac{n^2}{R^2} + \frac{R^2 m^2}{\alpha'^2} + \frac{2}{\alpha'}(N_L + N_R - 2)$$

$$N_L - N_R + nm = 0$$

$$R \xrightarrow{T} \frac{\alpha'}{R}, \qquad n \overset{T}{\leftrightarrow} m$$

T–Duality: SUSY

$$
\begin{aligned}
X_L^9 &= \tfrac{1}{2}x^9 + \frac{\pi\alpha'}{\ell}\left(\frac{n}{R} + \frac{mR}{\alpha'}\right)(\tau+\sigma) \\
&\quad + \tilde{\alpha} \text{ oscillators} \\
X_R^9 &= \tfrac{1}{2}x^9 + \frac{\pi\alpha'}{\ell}\left(\frac{n}{R} - \frac{mR}{\alpha'}\right)(\tau-\sigma) \\
&\quad + \alpha \text{ oscillators}
\end{aligned}
$$

One Sided Parity Transformation

$$X_R^9 \xrightarrow{T} -X_R^9$$

Spacetime SUSY

$$S^a \xrightarrow{T} \gamma_9 S^a$$

$$\Downarrow$$

$$
\begin{aligned}
Q^+ &= Q^1 \\
Q^- &= \gamma_9 Q^2
\end{aligned}
$$

T–Duality: σ Model

$$S = -\frac{1}{4\pi\alpha'}\int d^2z(g_{\mu\nu} + B_{\mu\nu})\partial X^\mu\bar\partial X^\nu$$

$$X^\mu = \{X\,, X^\alpha\}$$

$$\begin{aligned} S_1 &= -\frac{1}{4\pi\alpha'}\int d^2z\Big[g_{xx}A\bar A + (g_{x\alpha} + B_{x\alpha})A\bar\partial X^\alpha \\ &+ (g_{\alpha x} + B_{\alpha x})\partial X^\alpha\bar A + (g_{\alpha\beta} + B_{\alpha\beta})\partial X^\alpha\bar\partial X^\beta \\ &+ \theta(\partial\bar A - \bar\partial A)\Big] \end{aligned}$$

$$\Downarrow$$

$$S = -\frac{1}{4\pi\alpha'}\int d^2z(g_{\mu\nu} + B_{\mu\nu})\partial X^\mu\bar\partial X^\nu$$

$$X^\mu = \{\theta\,, X^\alpha\}$$

$$\begin{aligned} g_{xx} &= 1/g_{xx}\,, \qquad g_{x\alpha} = B_{x\alpha}/g_{xx} \\ g_{\alpha\beta} &= g_{\alpha\beta} - (g_{x\alpha}g_{x\beta} - B_{x\alpha}B_{x\beta})/g_{xx} \\ B_{x\alpha} &= g_{x\alpha}/g_{xx}\,, \qquad B_{\alpha\beta} = B_{\alpha\beta} + 2g_{x[\alpha}B_{\beta]x}/g_{xx} \end{aligned}$$

T–Duality: N=1 SUGRA

$$\mathcal{L}_{S,D} = \sqrt{-\hat{g}}e^{-2\hat{\phi}}\left(-\hat{R} + 4\left(\partial\hat{\phi}\right)^2 - \frac{3}{4}\hat{H}^2\right)$$

$$\begin{aligned} \hat{g}_{\hat{\mu}\hat{\nu}} &\rightarrow \{g_{\mu\nu}, A_\mu, \chi\} \\ \hat{B}_{\hat{\mu}\hat{\nu}} &\rightarrow \{B_{\mu\nu}, B_\mu\} \\ \hat{\phi} &\rightarrow \phi \end{aligned}$$

$$\mathcal{L}_{S,D-1} = \sqrt{-g}e^{-2\phi}\left(-R + 4\left(\partial\phi\right)^2 - \frac{3}{4}H^2 - \frac{1}{16}\left(\partial\chi\right)^2 \right.$$
$$\left. -\frac{1}{4}e^{\chi/2}F^2(A) - \frac{1}{4}e^{-\chi/2}F^2(B)\right)$$

$$A_\mu \overset{T}{\leftrightarrow} B_\mu\,, \quad \chi \overset{T}{\rightarrow} -\chi \quad \Rightarrow \quad e^{2\hat{\phi}} \overset{T}{\rightarrow} e^{2\hat{\phi}}/\hat{g}_{\hat{x}\hat{x}}$$

$$R \overset{T}{\rightarrow} \frac{\ell_s^2}{R}\,, \qquad g_s \overset{T}{\rightarrow} \frac{g_s \ell_s}{R}\,, \qquad \ell_s \overset{T}{\rightarrow} \ell_s$$

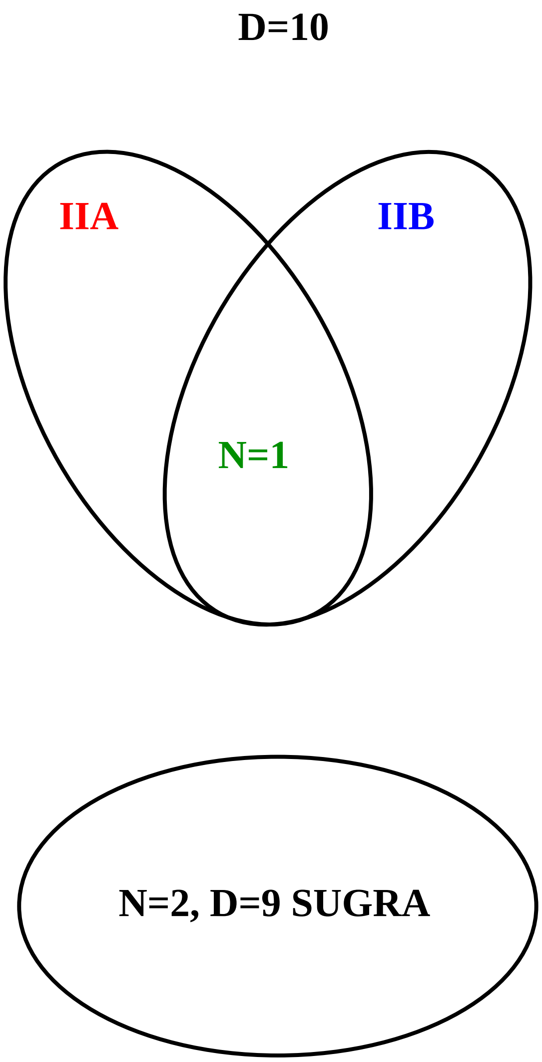
T-Duality: N=2 SUGRA

IIB			IIA		
$\mathbf{35_v}$	$\rightarrow$	$\mathbf{27}+\mathbf{7}+\mathbf{1}$	$\mathbf{35_v}$	$\rightarrow$	$\mathbf{27}+\mathbf{7}'+\mathbf{1}$
$\mathbf{1}$	$\rightarrow$	$\mathbf{1}'$	$\mathbf{1}$	$\rightarrow$	$\mathbf{1}'$
$\mathbf{28}$	$\rightarrow$	$\mathbf{21}+\mathbf{7}'$	$\mathbf{28}$	$\rightarrow$	$\mathbf{21}+\mathbf{7}$
$\mathbf{1}$	$\rightarrow$	$\mathbf{1}''$	$\mathbf{8_v}$	$\rightarrow$	$\mathbf{7}''+\mathbf{1}''$
$\mathbf{28}$	$\rightarrow$	$\mathbf{21}'+\mathbf{7}''$	$\mathbf{56_v}$	$\rightarrow$	$\mathbf{35}+\mathbf{21}'$
$\mathbf{35_c}$	$\rightarrow$	$\mathbf{35}$			

Unique N=2, D=9 SUGRA

$$\mathbf{27}+\mathbf{7}+\mathbf{7}'+\mathbf{7}''+\mathbf{1}+\mathbf{1}'+\mathbf{1}''$$
$$+\,\mathbf{35}+\mathbf{21}+\mathbf{21}'$$

T–Duality: N=1 $\Leftrightarrow$ N=2



Field Theory

IIB			IIA		
$g_{\mu\nu}$	$\rightarrow$	$g_{\mu\nu}, A^{(1)}, \kappa$	$g_{\mu\nu}$	$\rightarrow$	$g_{\mu\nu}, B^{(1)}, \kappa$
ϕ	$\rightarrow$	ϕ	ϕ	$\rightarrow$	ϕ
$B^{(2)}$	$\rightarrow$	$B^{(2)}, B^{(1)}$	$B^{(2)}$	$\rightarrow$	$B^{(2)}, A^{(1)}$
$B^{(10)}$	$\rightarrow$	$B^{(9)}$	$B^{(10)}$	$\rightarrow$	$B^{(9)}$
$C^{(0)}$	$\rightarrow$	$C^{(0)}, \underline{m}$	$C^{(1)}$	$\rightarrow$	$C^{(1)}, C^{(0)}$
$C^{(2)}$	$\rightarrow$	$C^{(2)}, C^{(1)}$	$C^{(3)}$	$\rightarrow$	$C^{(3)}, C^{(2)}$
$C^{(4),+}$	$\rightarrow$	$C^{(3)}$	$C^{(9)}$	$\rightarrow$	$C^{(9)}, \underline{C}^{(8)}$
$C^{(10)}$	$\rightarrow$	$C^{(9)}$			

N=2, D=9 SUGRA

$$\Big\{ g\,,\ A^{(1)}\,,\ B^{(1)}\,,\ C^{(1)}\,,\ C^{(0)}\,,\ \kappa\,,\ \phi$$
$$B^{(2)}\,,\ C^{(2)}\,,\ C^{(3)}\,,\ \underline{C}^{(8)}\,,\ B^{(9)}\,,\ C^{(9)} \Big\}$$

T–Duality: Central Charges

$$
\begin{array}{lcl|lcl}
 & \text{IIB} & & & \text{IIA} & \\
P_\mu & \to & P_\mu, P & P_\mu & \to & P_\mu, Z^{(3)} \\
Z^{(3)}_\mu & \to & Z^{(3)}_\mu, Z^{(3)} & Z_\mu & \to & Z^{(3)}_\mu, P \\
Z^{+}_{(5)} & \to & Z_{(4)} & Z_{(5)} & \to & Z^{(3)}_{(4)}, Z_{(4)} \\
Z^{(3),+}_{(5)} & \to & Z^{(3)}_{(4)} & & & \\
 & & & & & \\
Z^{(1)}_\mu & \to & Z^{(1)}_\mu, Z^{(1)} & Z & \to & Z^{(1)} \\
Z_{(3)} & \to & Z_{(3)}, Z_{\mu\nu} & Z_{\mu\nu} & \to & Z_{\mu\nu}, Z^{(1)}_\mu \\
Z^{(1),+}_{(5)} & \to & Z^{(1)}_{(4)} & Z_{(4)} & \to & Z^{(1)}_{(4)}, Z_{(3)}
\end{array}
$$

D=9 Central Charge Content

$$
\Big\{ Z^{(1)}\,,\; Z^{(3)}\,,\; P\,,\; Z^{(1)}_\mu\,,\; Z^{(3)}_\mu\,,\; P_\mu\,,\; Z_{\mu\nu}
$$

$$
Z_{\mu\nu\rho}\,,\; Z^{(1)}_{\mu_1\cdots\mu_4}\,,\; Z^{(3)}_{\mu_1\cdots\mu_4}\,,\; Z_{\mu_1\cdots\mu_4} \Big\}
$$

$$
\tfrac{1}{2}\times 32\times 33 \quad = \quad 3\times 1 + 3\times 9 + 36 + 84 + 3\times 126
$$

T–Duality Rules

$$
\begin{aligned}
P_\mu &= P_\mu \qquad\qquad \hat{\mu} = (\mu, x) \\
P_x &= Z^{(3)}_x \\
Z_\mu &= Z^{(3)}_\mu \\
Z_x &= P_x \\
Z_{\mu_1\cdots\mu_5} &= 2Z^{+}_{\mu_1\cdots\mu_5} \\
\tilde{Z}_{\mu_1\cdots\mu_5} &= 2Z^{(3),+}_{\mu_1\cdots\mu_5}
\end{aligned}
$$

$$
\begin{aligned}
Z &= -Z^{(1)}_x \\
Z_{\mu\nu} &= -Z_{\mu\nu x} \\
Z_{\mu x} &= -Z^{(1)}_\mu \\
Z_{\mu_1\cdots\mu_4} &= Z^{(1),+}_{\mu_1\cdots\mu_4 x} \\
Z_{\mu\nu\rho x} &= -Z_{\mu\nu\rho}
\end{aligned}
$$

T–Duality: Examples

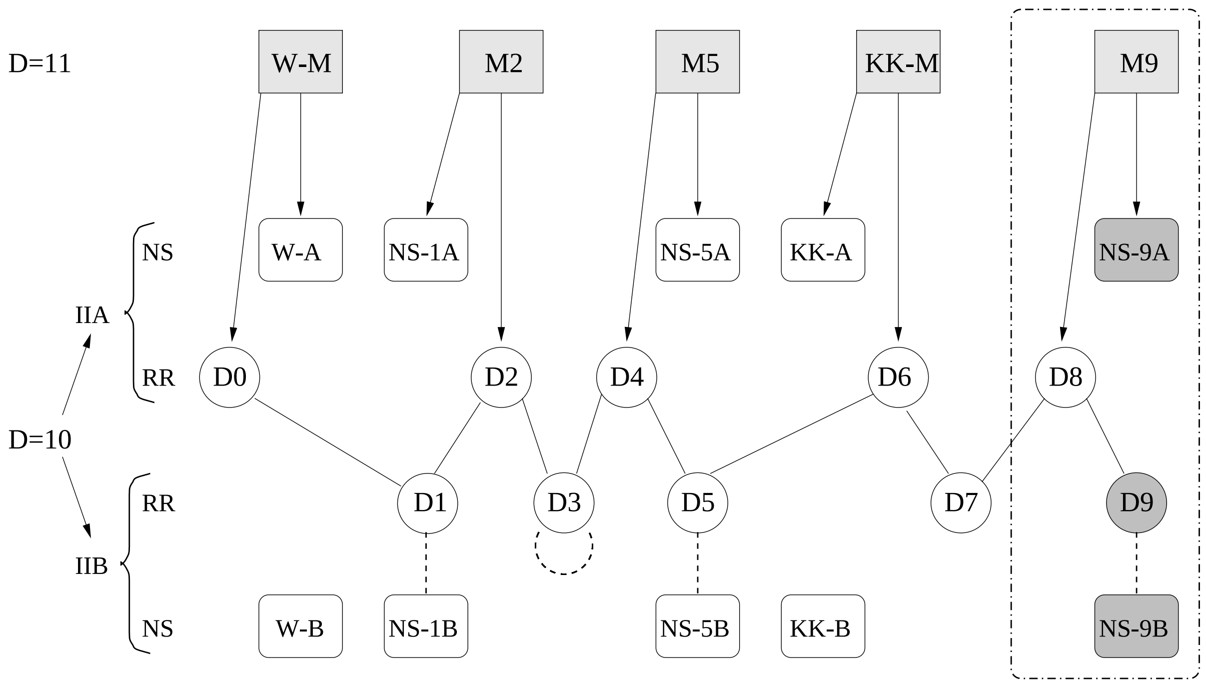
$$
\begin{array}{lcllcl}
P_\mu & = & P_\mu & : & \text{W–B} & \overset{T}{\leftrightarrow} & \text{W–A} \\
Z_\mu^{(3)} & = & Z_\mu & : & \text{NS–1B} & \overset{T}{\leftrightarrow} & \text{NS–1A} \\
P_x & = & Z_x & : & \text{W–B} & \overset{T}{\leftrightarrow} & \text{NS–1A} \\
Z_x^{(3)} & = & P_x & : & \text{NS–1B} & \overset{T}{\leftrightarrow} & \text{W–A}
\end{array}
$$

$$
\begin{array}{lcl}
\text{NS–1B/NS–1A} & : & ds^2 = H^{-1} dx_{(2)}^2 + dx_{(8)}^2 \\
\text{WB/WA} & : & ds^2 = -(2-H)dt^2 + H dz^2 \\
 & & -2(1-H)dtdz + dx_{(8)}^2
\end{array}
$$

$$
\begin{array}{lcllcl}
Z_x^{(1)} & = & Z & : & D1 & \overset{T}{\leftrightarrow} & D0 \\
Z_\mu^{(1)} & = & Z_{\mu x} & : & D1 & \overset{T}{\leftrightarrow} & D2 \\
 & \vdots & & & \vdots & &
\end{array}
$$

Dp/Dp: $ds^2 = H^{-1/2} dx_{(p+1)}^2 + H^{1/2} dx_{(9-p)}^2$

Summary



M–Branes and Branes in Ten Dimensions