Lecture 2: The M2–Brane

$$\{Q_\alpha, Q_\beta\} \;=\; \left(\gamma^\mu \mathcal{C}^{-1}\right)_{\alpha\beta} P_\mu$$

$$\det \not{P} = 0 \;\Rightarrow\; P^2 = 0$$

$$P^2 = 0 \;\Rightarrow\; (1 \pm \gamma_{01})\,\epsilon = 0$$

Half Unbroken Supersymmetry

$P^2 = 0 \;\Leftrightarrow\;$ Massless Particle

The Massless Superparticle

$$S_{\text{superparticle}}[X,\theta;e] \;=\; \frac{1}{2}\int d\tau e\; P^a P^b \eta_{ab}$$

$$\begin{aligned} P^\mu &= \dot{X}^\mu + i\,\dot{\bar{\theta}}\gamma^\mu\theta \\ \delta X^\mu &= a^\mu + i\bar{\epsilon}\gamma^\mu\theta \\ \delta\theta &= \epsilon \end{aligned}$$

Local Kappa-Symmetry: $\delta\theta = i\,\not{P}\kappa$

$$\{S_\alpha\,, S_\beta\} = \delta_{\alpha\beta} \Rightarrow$$

$$\{g_{\mu\nu}\,, \psi_\mu\,, C_{\mu\nu\rho}\} \quad : \quad 128 + 128 \text{ d.o.f.}$$

The M–Wave

$$S_{\mathrm{D=11\ SUGRA}} = \frac{1}{16\pi G_N}\int d^{11}x\sqrt{-g}\{-R+\cdots\}$$

$$\begin{aligned} ds^2 &= -(2-H)\,dt^2 + H\,dz^2 - 2\,(1-H)\,dt\,dz \\ &\quad + \left(dx_2^2 + \cdots dx_{10}^2\right) \end{aligned}$$

$$H = f(t \pm z)\left\{1 + \left(\frac{r_0}{r}\right)^7\right\}$$

$$\delta\psi_\mu = \partial_\mu\epsilon - \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\epsilon + \cdots = 0 \quad \Rightarrow$$

$$\left(1 \pm \gamma_{01}\right)\epsilon = 0$$

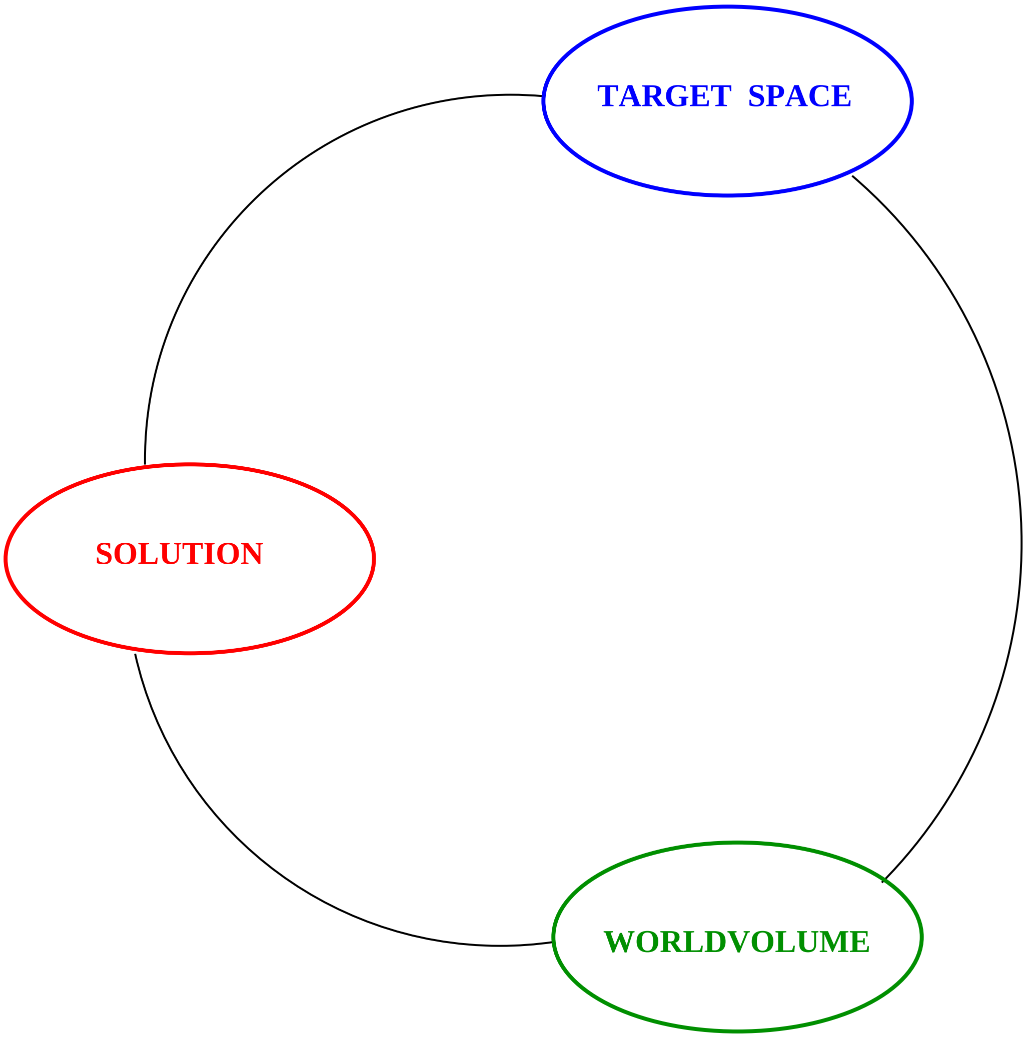
M–Wave $\Rightarrow$ Massless Superparticle

Exercise II

Substitute the Brinkmann wave solution into the Killing spinor equation $\delta\psi_\mu = 0$ and show that the Killing spinor ϵ satisfies

$$(1 \pm \gamma_{01})\,\epsilon \;=\; 0$$

Summary



What about the D=11 Supermembrane?

The M2–Brane

$$S_{11} = \frac{1}{16\pi G_N}\int d^{11}x\sqrt{-g}\{-R - \frac{1}{2\times 4!}F^2_{(4)} + \cdots\}$$

$$F^{(4)} = dC^{(3)}\,, \qquad 16\pi G_N = 2\kappa^2$$

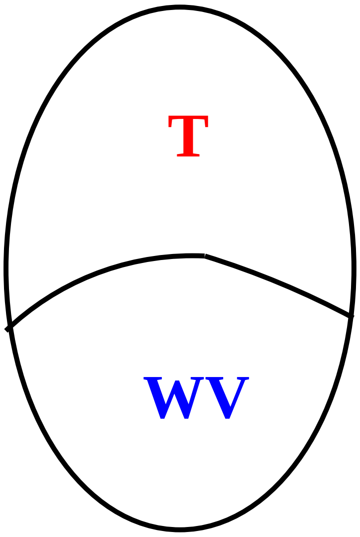
$$\begin{aligned} ds^2 &= H^{-2/3}dx^2_{(3)} + H^{1/3}dx^2_{(8)}\,, \\ F^{(4)}_{012m} &= \pm\partial_m H^{-1}\,, \qquad m = 3, 4, \cdots, 10 \end{aligned}$$

$$H = 1 + \left(\frac{r_0}{r}\right)^6$$

$$(1 \pm \gamma_{012})\,\epsilon_\infty = 0\,, \quad \epsilon = H^{-1/6}\epsilon_\infty$$

Notation

$$\mathrm{M2} \quad : \quad \times \,|\, \times \ \times \ - \ - \ - \ - \ - \ - \ - \ -$$



Likewise:

$$\mathrm{W-M} \quad : \quad \times \,|\, z \ - \ - \ - \ - \ - \ - \ - \ - \ -$$

Mass

τ_2 is Mass per unit 2–volume

$$\tau_2 = \frac{1}{2\kappa^2} \int_{\partial M^8} d^7\Sigma^m \left(\partial^n h_{mn} - \partial_m h^b{}_b\right)$$

$$m = 3, 4, \cdots, 10$$
$$a = 1, 2, \cdots, 10$$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,, \qquad \mu = 0, 1, \cdots, 10$$

$$h_{mn} = \tfrac{1}{3}\left(\frac{r_0}{r}\right)^6 \delta_{mn} \qquad h^b{}_b = \tfrac{4}{3}\left(\frac{r_0}{r}\right)^6$$

$$d^7\Sigma^m = r^6 y^m d\Omega^{(7)}$$

$$\Downarrow$$

$$\tau_2 = \frac{3}{\kappa^2} r_0^6 \Omega_7$$

Charge

μ_2 is Charge per unit 2–volume

$$\mu_2 = \frac{1}{2\kappa^2}\int_{\partial M^8} d^7\Sigma^{m_1\cdots m_7}\, {}^*F^{(4)}_{m_1\cdots m_7}$$

$$L^{-3} \qquad L^{-9} \qquad L^{7} \qquad L^{-1}$$

$${}^*F^{(4)}_{m_1\cdots m_7} = \pm 6 r_0^6 \epsilon_{m_1\cdots m_7 m_8}\frac{y^{m_8}}{r^8}$$

$$\Downarrow$$

$$\mu_2 = \pm\frac{3}{\kappa^2} r_0^6 \Omega_7$$

$$\Downarrow$$

$$T_2 = |\mu_2|$$

Worldvolume Action

$$S_{\mathrm{M2}} = \bar{\tau} \int d^3\xi \sqrt{|\det g|} + \bar{\mu} \int d^3\xi\, C^{(3)}$$

(1) ADM mass $\rightarrow \tau \sim r_0$

(2) $\Delta H = \bar{\tau}\delta \rightarrow r_0 \sim \bar{\tau} \quad \Rightarrow \bar{\tau} = \tau$

$$\kappa - \text{symmetry} \quad \Leftrightarrow \quad \bar{\tau} = \bar{\mu}$$

$$\begin{aligned} Z^M &= (X^\mu, \theta^\alpha) \\ \delta\theta &= (1+\Gamma)\,\kappa \end{aligned}$$

$$\begin{aligned} \Gamma &= \frac{1}{3!\sqrt{|g|}}\epsilon^{ijk}\gamma_{ijk}\,, \qquad \Gamma^2 = \mathbb{1}_{32}\,, \\ \gamma_i &= E_i{}^a\gamma_a\,, \qquad g_{ij} = E_i{}^a E_j{}^b \eta_{ab} \end{aligned}$$

WV d.o.f.: $8 + 8$

2–Form Central Charge

Symmetries $\rightarrow$ Noether charges $P_\mu, M_{\mu\nu}, Q_\alpha$

$$\delta_\alpha \mathcal{L}_{\mathrm{WZ}} = \partial_i \Delta_\alpha^i \,, \qquad i = 0, 1, \cdots, p \qquad \Rightarrow$$

$$j_\alpha^0 = \delta_\alpha Z^M \frac{\partial \mathcal{L}}{\partial(\partial_0 Z^M)} - \Delta_\alpha^0$$

$$\Downarrow$$

$$\{Q_\alpha, Q_\beta\} = \left(\gamma^\mu \mathcal{C}^{-1}\right)_{\alpha\beta} P_\mu + \tfrac{1}{2!}\left(\gamma^{\mu\nu}\mathcal{C}^{-1}\right)_{\alpha\beta} Z_{\mu\nu}$$

with

$$Z^{\mu\nu} = \mu \int d^2\sigma \epsilon^{0ij} \partial_i X^\mu \partial_j X^\nu$$

Alternatively: $\mathcal{L}_{\mathrm{WZ}} \rightarrow \partial b^{(2)} - \mathcal{L}_{\mathrm{WZ}}$

Supersymmetry

Use Majorana representation $\rightarrow \mathcal{C} = \gamma_0$

Also: $P^\mu = (M, 0, \cdots, 0)$ and $Z^{12} = Q$

$$\Downarrow$$

$$\{Q_\alpha \, , Q_\beta\} \quad = \quad \left(M \mathbb{1}_{32} + Q \gamma_{012}\right)_{\alpha\beta}$$

$$(\gamma_{012})^2 = \mathbb{1}_{32} \, , \mathrm{Tr}\, \gamma_{012} = 0 \Rightarrow \gamma_{012} = \begin{pmatrix} \mathbb{1}_{16} & \\ & -\mathbb{1}_{16} \end{pmatrix}$$

$$\Downarrow$$

$M \geq |Q|$: BPS bound

$M = |Q|$: BPS state $\Leftrightarrow$

$$\left(\mathbb{1}_{32} \pm \gamma_{012}\right) \epsilon_\infty = 0$$

D=4 Central Charges

$$\begin{aligned}
\{Q^i_\alpha\,, Q_{\beta j}\} &= \left(\gamma^\mu \mathcal{C}^{-1}\right)_{\alpha\beta} P_\mu \delta^i_j\,, \\
\{Q_{\alpha i}\,, Q_{\beta j}\} &= \mathcal{C}^{-1}_{\alpha\beta}\epsilon_{ij} Z\,, \\
[Z\,, \text{any generator}] &= 0\,, \qquad (N=2, D=4)
\end{aligned}$$

$$U(2) \quad \Rightarrow \quad USp(1)$$

$$A^{\star\alpha i} = Q^{\star\alpha i} + e^{-i\theta}\varepsilon^{ij} Q_{j\beta}\mathcal{C}^{\beta\alpha}$$

$$\left\{A_{\alpha i}, A^{\star\beta j}\right\} = \delta^j_i \delta^\beta_\alpha (2M + Ze^{-i\theta} + Z^\star e^{i\theta})$$

$$\Downarrow$$

$$\begin{aligned}
M \geq |Z| &\quad : \quad \text{BPS bound} \\
M = |Z| &\quad : \quad \text{BPS state}
\end{aligned}$$

Higher Dimensions

“Central Charges” $\Leftrightarrow$ BPS States

$$Z_{M_1 \cdots M_p} \rightarrow p\text{−brane}$$

$$Z_{M_1 \cdots M_p} \quad \Leftrightarrow \quad \tilde{Z}_{M_1 \cdots M_{(D-p)}}$$

$$\Downarrow$$

$$Z_{M_1 \cdots M_p} \rightarrow (D-p)\text{−brane}$$

Exceptions: P_M , $p = 0$, $p = D/2$ (self–dual)

M-theory central charges

$$\begin{aligned}\{Q_\alpha, Q_\beta\} &= \left(\gamma^{\mu}\mathcal{C}^{-1}\right)_{\alpha\beta} P_\mu \\ &\quad + \tfrac{1}{2!}\left(\gamma^{\mu\nu}\mathcal{C}^{-1}\right)_{\alpha\beta} Z_{\mu\nu} \\ &\quad + \tfrac{1}{5!}\left(\gamma^{\mu\nu\rho\sigma\tau}\mathcal{C}^{-1}\right)_{\alpha\beta} Z_{\mu\nu\rho\sigma\tau}\end{aligned}$$

$$\tfrac{1}{2} \times 32 \times 33 = 11 + 55 + 462$$

M–Branes

$$
\begin{aligned}
P_\mu &\rightarrow \text{W–M}\,,\\
Z_{\mu\nu} &\rightarrow \text{M2 and M9(k)}\,,\\
Z_{\mu_1\cdots\mu_5} &\rightarrow \text{M5 and KK–M}
\end{aligned}
$$

We find a gravitational wave (W–M), a membrane (M2), a five-brane (M5), a Kaluza-Klein monopole (KK–M) and a nine-brane (M9(k))