Strings and Branes

Madrid lectures, March 12-16, 2001

Exercise V : $SL(2,\mathbb{R})$

Consider the following action in D dimensions:

$$\mathcal{L} = -\tfrac{1}{2}\sqrt{-g}\left[(\partial\phi)^2 + e^{a\phi}(\partial\ell)^2\right] . \tag{1}$$

We are going to determine for which value of a this action exhibits an $SL(2,\mathbb{R})$ symmetry. Consider the general $SL(2,\mathbb{R})$ matrix:

$$\mathcal{M} = e^{\beta\phi}\begin{pmatrix} \ell^2 + e^{-2\beta\phi} & -\ell \\ -\ell & 1 \end{pmatrix} \tag{2}$$

with

$$\mathcal{M}' = (\Lambda^{-1})^T\mathcal{M}\Lambda^{-1}\,, \qquad \Lambda = \begin{pmatrix} d & c \\ b & a \end{pmatrix}, \qquad ad - bc = 1 \tag{3}$$

(1) Show that

$$\mathcal{L} = -\tfrac{1}{4}\sqrt{-g}\,\mathrm{Tr}\,\partial\mathcal{M}\partial\mathcal{M}^{-1}\,, \tag{4}$$

provided that

$$\beta = 1 \qquad \text{and} \qquad a = 2\,. \tag{5}$$

(2) Show that

$$\mathcal{L} = \tfrac{1}{2}\sqrt{-g}\frac{\partial\lambda\partial\bar{\lambda}}{(\mathrm{Im}\,\lambda)^2} \qquad \text{with} \qquad \lambda = \ell + ie^{-\phi}\,. \tag{6}$$

(3) Show that

$$\lambda \quad \xrightarrow{SL(2,\mathbb{R})} \quad \frac{a\lambda + b}{c\lambda + d} \,. \tag{7}$$

(4) Show that, for $\ell = 0$,

$$e^{\phi} \quad \xrightarrow{SL(2,\mathbb{R})} \quad e^{\phi}(c^2 e^{-2\phi} + d^2) \,. \tag{8}$$