Strings and Branes

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Exercise IV : κ–Symmetry

Our starting point is the following κ–symmetry rule (we only consider the IIA case)

$$\delta_\kappa Z^M E_M{}^\alpha = [\bar{\kappa}(1+\Gamma)]^\alpha \tag{1}$$

where Γ is defined by

$$\begin{aligned}\Gamma \;=\; & \frac{\sqrt{|g|}}{\sqrt{|g+\mathcal{F}|}}\sum_{n=0}^{\infty}\frac{1}{2^n n!}\gamma^{j_1k_1\cdots j_nk_n}\mathcal{F}_{j_1k_1}\ldots\mathcal{F}_{j_nk_n}\times \\ & \times(\Gamma_{11})^{n+\frac{p-2}{2}}\Gamma_{(0)}\,, \qquad p \text{ even}\,, \quad i=0,1,\cdots p.\end{aligned} \tag{2}$$

The matrix $\Gamma_{(0)}$ is given by

$$\Gamma_{(0)} = \frac{1}{(p+1)!\sqrt{|g|}}\epsilon^{i_1\cdots i_{p+1}}\gamma_{i_1\cdots i_{p+1}}\,, \tag{3}$$

with the 32×32 matrices γ_i defined as

$$\gamma_i = E_i^a\Gamma_a\,, \tag{4}$$

where Γ_a $(a=0,1,\cdots 9)$ are the space-time gamma matrices.

(1) Show that

$$(\Gamma_{(0)})^2 = (-)^{(p-1)(p-2)/2} \tag{5}$$

and that $\Gamma^2 = \mathbb{1}_{32}$ for $\mathcal{F}=0$.

We rewrite Γ as

$$\Gamma = \frac{\sqrt{|g|}}{\sqrt{|g+\mathcal{F}|}}\,\mathrm{se}^{\frac{1}{2}\mathcal{F}_{ij}\gamma^{ij}\Gamma_{11}}\Gamma'_{(0)}\,, \tag{6}$$

where

$$\mathrm{s}e^{\frac{1}{2}\mathcal{F}_{ij}\gamma^{ij}} := \sum_{n=0}^{\infty} \frac{1}{2^n n!}\gamma^{i_1 j_1 \cdots i_n j_n}\mathcal{F}_{i_1 j_1}\cdots\mathcal{F}_{i_n j_n} \tag{7}$$

and

$$\Gamma'_{(0)} := (\Gamma_{11})^{\frac{p-2}{2}}\,\Gamma_{(0)}\,. \tag{8}$$

(2) Show that

$$(\Gamma'_{(0)})^2 = \mathbb{1}_{32}\,. \tag{9}$$

We next rewrite Γ as

$$\Gamma = \frac{1}{\sqrt{|1+X|}}\,\mathrm{s}e^{\frac{1}{2}X_{\underline{i}\underline{j}}\gamma^{\underline{i}\underline{j}}\Gamma_{11}}\,\Gamma'_{(0)}\,, \tag{10}$$

where $\underline{i},\underline{j}$ are worldvolume frame indices and

$$X^i{}_j = g^{ik}\mathcal{F}_{kj}\,. \tag{11}$$

In a special Lorentz frame, X can be written as

$$X = \tanh\phi_0\, e^0\wedge e^1 + \sum_{r=1}^{\frac{1}{2}(p-2)} \tan\phi_r\, e^{2r}\wedge e^{2r+1}\,, \tag{12}$$

where $\{\phi_0, \phi_r; r = 1\cdots\frac{1}{2}(p-2)\}$ are "angles" and $\{e^i\} = \{e^0, e^s; s = 1\cdots, p\}$ is a Lorentz basis.

(3) Show that in this special frame, we have

$$\sqrt{|1+X|} = \frac{1}{\cosh\phi_0\,\Pi_{r=1}^{\frac{1}{2}(p-2)}\cos\,\phi_r}\,. \tag{13}$$

(4) Substitute (12) and (13) in (6) and show that Γ can be rewritten as

$$
\begin{aligned}
\Gamma &= \left(\cosh\phi_0\,\Pi_{s=1}^{\frac{1}{2}(p-2)}\ \cos\phi_s\right)\ \mathrm{se}^{\tanh\phi_0\gamma^{\underline{01}}\Gamma_{11}}\ \left(\Pi_{r=1}^{\frac{1}{2}(p-2)}\mathrm{se}^{\tan\phi_r\gamma^{\underline{2r2r+1}}\Gamma_{11}}\right)\ \Gamma'_{(0)} \\
&= \left(\cosh\phi_0 + \sinh\phi_0\ \gamma^{\underline{01}}\Gamma_{11}\right)\times \\
&\quad \left[\Pi_{r=1}^{\frac{1}{2}(p-2)}\left(\cos\phi_r + \sin\phi_r\ \gamma^{\underline{2r2r+1}}\Gamma_{11}\right)\right]\ \Gamma'_{(0)} \qquad (14) \\
&= e^{\phi_0\gamma^{\underline{01}}\Gamma_{11}}\ \left(\Pi_{r=1}^{\frac{1}{2}(p-2)}e^{\phi_r\gamma^{\underline{2r2r+1}}\Gamma_{11}}\right)\ \Gamma'_{(0)} \\
&= \exp\left\{\left(\phi_0\gamma^{\underline{01}} + \sum_{r=1}^{\frac{1}{2}(p-2)}\phi_r\gamma^{\underline{2r2r+1}}\right)\Gamma_{11}\right\}\Gamma'_{(0)} \\
&= e^{\frac{1}{2}Y_{\underline{ik}}\gamma^{\underline{ik}}\Gamma_{11}}\Gamma'_{(0)}\,, \qquad (15)
\end{aligned}
$$

where

$$
Y = \phi_0\, e^0\wedge e^1 + \sum_{r=1}^{\frac{1}{2}(p-2)}\phi_r e^{2r}\wedge e^{2r+1}\,. \tag{16}
$$

or

$$
\mathcal{F} = \text{``}\tan\text{''}\,Y\,. \tag{17}
$$

(5) Use the above result to show that

$$
\Gamma = e^{-a/2}\Gamma'_{(0)}e^{a/2}\,, \tag{18}
$$

with

$$
a = -\tfrac{1}{2}Y_{ij}\gamma^{ij}\Gamma_{11}\,. \tag{19}
$$

(6) Finally, show that

$$
\Gamma^2 = \mathbb{1}_{32}\,. \tag{20}
$$