Strings and Branes

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Exercise III : The PST-Action

We are going to show how the selfduality condition for the D=6 two-form follows from the equations of motion corresponding to the PST-action.

The starting point is the PST-Action

$$S_{\rm PST} = \int d^6x \left[-\tfrac{1}{2} H^-_{\mu\nu} H^{*\mu\nu} \right] , \tag{1}$$

where we have used the following definitions

$$\begin{aligned}
H_{\mu\nu\rho} &\equiv 3\partial_{[\mu} B_{\nu\rho]} \,, \\
v_\mu &\equiv \frac{\partial_\mu a}{\sqrt{(\partial_\rho a)(\partial^\rho a)}} \quad \rightarrow \quad v^2 = 1 \,, \\
H^*_{\mu\nu\rho} &\equiv \tfrac{1}{6} \epsilon_{\mu\nu\rho\sigma\tau\phi} H^{\sigma\tau\phi} \,, \\
H^-_{\mu\nu} &\equiv \tfrac{1}{2} \left(H_{\mu\nu\rho} - H^*_{\mu\nu\rho} \right) v^\rho = H^-_{\mu\nu\rho} v^\rho \,, \\
H^*_{\mu\nu} &\equiv H^*_{\mu\nu\rho} v^\rho \,.
\end{aligned} \tag{2}$$

The action (1) is invariant under the local symmetries I, II and III (without proof):

$$\begin{array}{lll}
\mathrm{I)} & \delta_{\rm I} B_{\mu\nu} = 2\partial_{[\mu} \Lambda_{\nu]} \,, & \delta_{\rm I} a = 0 \,, \\
\mathrm{II)} & \delta_{\rm II} B_{\mu\nu} = 2 H^-_{\mu\nu} \dfrac{1}{\sqrt{u^2}} \phi \,, & \delta_{\rm II} a = \phi \,, \\
\mathrm{III)} & \delta_{\rm III} B_{\mu\nu} = \psi_{[\mu} v_{\nu]} \,, & \delta_{\rm III} a = 0 \,,
\end{array} \tag{3}$$

with $u_\mu = \partial_\mu a$.

The field equations corresponding to $B_{\mu\nu}$ and a are given by (without proof)

$$\begin{aligned}\partial_{[\mu}\left(v_\nu H^-_{\rho\sigma]}\right) &= 0\,,\\ \epsilon^{\mu\nu\rho\sigma\tau\phi}\partial_\phi\left(H^-_{\mu\nu}H^-_{\rho\sigma}v_\tau\right) &= 0\,,\end{aligned} \tag{4}$$

respectively.

Choose a basis in which

$$u_\mu = \partial_\mu a = \delta_{\mu 0}\,, \qquad\qquad u_\mu \text{ is constant}\,, \tag{5}$$

and write $\mu = (0, i)$ with $i = 1 \cdots 5$.

(1) Show that the equation of motion for $B_{\mu\nu}$, see first equation of (4), can be rewritten as

$$\partial_{[i}H^-_{jk]} = 0 \qquad \Rightarrow \qquad H^-_{ij} = \partial_{[i}\Phi_{j]}\,. \tag{6}$$

(2) Show that the term $\partial_{[i}\Phi_{j]}$ takes the form of a type III gauge transformation with $\Psi_i = \Phi_i$. Hence we can take $H^-_{ij} = 0$.

(3) Show that

$$H^-_{ij} = 0 \qquad \Rightarrow \qquad H^-_{\mu\nu\rho} = 0\,. \tag{7}$$

(4) Show that the choice of basis (5) exists.

(5) Show that for $u_\mu = \delta_{\mu 0}$ the action (1) is given by

$$S_{\rm PST} = \int d^6x \left[-\tfrac{1}{24}\epsilon^{0ijklm}H_{0ij}H_{klm} - \tfrac{1}{12}H^{ijk}H_{ijk}\,.\right]\,. \tag{8}$$