Strings and Branes

Madrid lectures, March 12-16, 2001

Exercise II : The M–Wave

Show, by following the steps indicated below, that the substitution of the D=11 Brinkmann wave solution

$$\begin{aligned} ds^2 &= -(2-H)dt^2 + Hdz^2 - 2(1-H)dtdz + dx^2_{(9)}\,, \\ H &= f(t \pm z)\left\{1 + \left(\frac{r_0}{r}\right)^7\right\} \end{aligned} \qquad (1)$$

into the Killing spinor equation

$$\delta\psi_\mu = \partial_\mu \epsilon + \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\,\epsilon = 0, \qquad (2)$$

with the spin-connection $\omega_\mu{}^{ab}$ given by

$$\omega_\mu{}^{ab}(e) = -e^{\nu[a}(\partial_\mu e_\nu^{b]} - \partial_\nu e_\mu^{b]}) - e^{\rho[a}e^{\sigma b]}(\partial_\sigma e_{c\rho})e_\mu^c\,, \qquad (3)$$

leads to the following conditions on the Killing spinor ϵ:

$$(1 \pm \gamma_{01})\,\epsilon = 0 \quad \text{with} \quad \gamma_1 = \gamma_z \quad \text{and} \quad \epsilon = \text{constant} \qquad (4)$$

(1) Introduce lightcone coordinates

$$u = \tfrac{1}{\sqrt{2}}(t+z)\,, \qquad v = \tfrac{1}{\sqrt{2}}(t-z)\,, \qquad (5)$$

and show that the Brinkmann wave in lightcone coordinates is given by

$$\begin{aligned} ds^2 &= -2dudv - Wdu^2 + dx^2_{(9)}\,, \qquad W = 2(1-H)\,, \\ H &= f(u)\left\{1 + \left(\frac{r_0}{r}\right)^7\right\}\,. \end{aligned} \qquad (6)$$

(2) Introduce a constant (lightlike) vector k_μ with $k_u = 1\,(k_v = 1)$ and all other components zero. Show that the elfbein $e_\mu{}^a$ and the inverse elfbein $e_a{}^\mu$ for the Brinkmann wave are given by

$$e_\mu{}^a = \delta_\mu{}^a - \tfrac{1}{2}W k_\mu k^a\,, \qquad e_a{}^\mu = \delta_a{}^\mu + \tfrac{1}{2}W k_a k^\mu\,, \tag{7}$$

with

$$e_\mu{}^a e_\nu{}^b \eta_{ab} = g_{\mu\nu}\,, \qquad e_a{}^\mu e_\mu{}^b = \delta_a{}^b\,. \tag{8}$$

(3) Show that the spin-connection is given by

$$\omega_\mu{}^{ab} = +k_\mu k^{[a}\left(\partial^{b]}W\right)\,. \tag{9}$$

(4) Show that the curvature

$$R_{\mu\nu}{}^{ab}(\omega) = \partial_\mu \omega_\nu{}^{ab} - \partial_\nu \omega_\mu{}^{ab} - \omega_\mu{}^{ac}\omega_{\nu c}{}^b + \omega_\nu{}^{ac}\omega_{\mu c}{}^b \tag{10}$$

is given by

$$R_{\mu\nu}{}^{ab} = +2k_{[\mu}\left(\partial_{\nu]}\partial^{[a}W\right)k^{b]} \tag{11}$$

(5) Show that the Killing spinor equation (2) leads to the integrability condition

$$R_{\mu\nu}{}^{ab}(\omega)\gamma_{ab}\;\epsilon = 0 \tag{12}$$

(6) Substitute the expression (11) for the curvature into the integrability condition (12) and derive that $\gamma_v\epsilon = 0$.

(7) Substitute the expression (9) for the spin-connection into the Killing spinor equation (2) and derive from this the conditions (4).

(8) Show that

$$R_{\mu\nu} = 0\,. \tag{13}$$

(9) Show that

$$\left(R_{\mu\nu}{}^{ab}\right)^n = 0\,. \tag{14}$$