B.2.1 Anholonomic bases

Although we have not said it explicitly, we have been using a particular basis for vectors, 1-forms and other tensors called "coordinate basis" because it is associated to our choice of coordinates:

$$\xi = \underset{\uparrow}{\xi^\mu} \underset{\uparrow}{\partial_\mu} \qquad \{\partial_\mu\}$$

components elements of the basis

$$\omega = \underset{\uparrow}{\omega_\mu} \underset{\uparrow}{dx^\mu} \qquad \{dx^\mu\}$$

components elements of the basis.

When we change coordinates our basis changes and this induces a change in the components.

It is convenient to describe our tensors using more general bases. Thus, we are going to consider sets of d vector fields labeled by $a, b, \ldots = 0, 1, \ldots, d-1$ $\{e_a\} = \{e_a{}^\mu(x)\,\partial_\mu\}$ which are linearly independent at any point x^μ and, therefore, provide a basis in which we can express all vectors:

$$\xi = \xi^a e_a = \xi^a e_a{}^\mu \partial_\mu = \xi^\mu \partial_\mu$$

$\mu, \nu \rightarrow$ "curved", "world", "coordinate"

$a, b \rightarrow$ "flat", "tangent space", "Lorentz" [in a more restricted sense]

The components ξ^a are related to the ξ^μ by the matrices $e_a{}^\mu$

$$\boxed{\xi^a e_a{}^\mu = \xi^\mu}$$

If we denote by $e^a{}_\mu$ the components of the inverse, transposed matrix

$$\boxed{\xi^a = e^a{}_\mu \xi^\mu}$$

The $e^a{}_\mu$ components can be interpreted as the components of a dual basis of 1-forms: $\{e^a\} = \{e^a{}_\mu(x)\, dx^\mu\}$

$$\boxed{e^a{}_\mu e_b{}^\mu = \delta^a{}_b}$$

Now we can use it to express general 1-forms

$$\omega = \omega_a e^a = \omega_a e^a{}_\mu dx^\mu = \omega_\mu dx^\mu$$

$$\boxed{\begin{aligned} \omega_a e^a{}_\mu &= \omega_\mu ; \\ \omega_a &= e_a{}^\mu \omega_\mu ; \end{aligned}}$$

and all tensors.

How do these components transform under a change of coordinates?

As scalars: $\xi'^a(x') = \xi^a(x(x'))$!

But we have to take into account that the components of the basis change:

$$e^a = e^a{}_\mu dx^\mu = e^a{}_\mu \frac{\partial x^\mu}{\partial x'^\nu} dx'^\nu ;$$

$$e_a = e_a{}^\mu \frac{\partial}{\partial x^\mu} = e_a{}^\mu \frac{\partial x'^\nu}{\partial x^\mu} \frac{\partial}{\partial x'^\nu} ;$$

Let us consider the Lie bracket of the vector fields of the basis

$$[e_a, e_b] = \text{another vector field} = -2\Omega_{ab}{}^c(x) e_c$$

$\Omega_{ab}{}^c(x)$: $\left\{\begin{array}{l}\text{Ricci rotation} \\ \text{anholonomy}\end{array}\right\}$ coefficient $= e_a{}^\mu e_b{}^\nu \partial_{[\mu} e^c{}_{\nu]}$

We can ask now the question: given a basis $\{e_a\}$ is there a coordinate system x'^μ such that, in that coordinate system $e'_a{}^\mu(x') = \delta_a{}^\mu$? In other words are the $\{e_a\}$ a coordinate basis?

The answer is that $\{e_a\}$ is a coordinate basis (for some coordinates) if and only if all the anholonomy coefficients vanish $C_{ab}{}^c = 0 \Rightarrow$ holonomic basis

$C_{ab}{}^c \neq 0 \Rightarrow$ anholonomic basis.

We are going to consider a generic case, not assuming $C_{ab}{}^c = 0$.

We are also interested in developing a formalism independent of the choice of basis $\{e_a\}$.

By definition, any other basis (in the same coordinate system)

$\{e'_a\}$ can be expressed in terms of $\{e_a\}$

$$\boxed{\begin{aligned} e'_a &= e_b \left(\Lambda^{-1}\right)^b{}_a(x) \;; \\ e'^a &= \Lambda^a{}_b(x)\, e^b \;; \end{aligned}}$$

Assuming $e'_a{}^\mu e'^b{}_\mu = \delta_a{}^b$ (orthonormality is preserved)

$\Lambda^a{}_b(x)$ is, at each point x^μ, a $GL(d,\mathbb{R})$ transformation.

Our formalism will be independent of the choice of basis if it is form-invariant under $\underbrace{\text{arbitrary, local}}_{\text{gauge}}$, $GL(d,\mathbb{R})$ transformations.

$\Rightarrow$ We can use the formalism of gauge theories with gauge group $GL(d,\mathbb{R})$.

A basis $e^a{}_\mu$ in $d=4$ is called a "tetrad" or Vierbein.

$\underset{4}{\text{Vier}}$

in $d=3$ Dreibein

$d=2$ Zweibein

$\vdots$

B.2.1.2 Review of the formalism of gauge theories

Consider a Lie group G. We are going to work with different linear representations r, Adj etc. and we are going to denote the matrix corresponding to an element $g \in G$ in those representations by

$$\Gamma_r(g)^i{}_j \, ; \quad i,j = 1, \dots, \dim r$$

Let $\{T_A\}$ be a basis of the Lie algebra of G, $\mathfrak{g}$. We can also assign matrices to them in the different representations

$$\Gamma_r(T_A)^i{}_j \, ;$$

$$[\Gamma_r(T_A), \Gamma_r(T_B)] = f_{AB}{}^C \, \Gamma_r(T_C) \quad \text{in any representation } r.$$

The adjoint action of the group on the algebra (representation space) defines a representation called "adjoint representation" which is $\dim \mathfrak{g}$-dimensional:

$$\Gamma_r(g)\, \Gamma_r(T_A)\, \Gamma_r(g^{-1}) \equiv \Gamma_r(T_B) \underbrace{\Gamma_{Adj}(g)^B{}_C}_{\dim \mathfrak{g} \times \dim \mathfrak{g} \text{ matrix}}$$

(You can check that it is a representation $\Gamma(gg') = \Gamma(g)\Gamma(g')$ etc.)

In any representation (around the identity) we can construct the elements of the group by exponentiation of the elements of the algebra

$$\Gamma_r(g(\sigma)) = \exp\{\sigma^A \, \Gamma_r(T_A)\} \sim 1 + \sigma^A \, \Gamma_r(T_A)$$

$$\Gamma_r(g)\, \Gamma_r(T_A)\, \Gamma_r(g^{-1}) \sim \Gamma_r(T_A) + \sigma^B \left[\Gamma_r(T_B), \Gamma_r(T_A)\right]$$

$$= \Gamma_r(T_A) + \sigma^B f_{BA}{}^C \Gamma_r(T_C) = \Gamma_r(T_B)\left(\delta^B{}_A + \sigma^C f_{CA}{}^B\right)$$

$$\Gamma_{Adj}(g)^B{}_A \sim \delta^B{}_A + \sigma^C \Gamma_{Adj}(T_C)^B{}_A ;$$

$$\Rightarrow \boxed{\Gamma_{Adj}(T_A)^C{}_B = f_{AB}{}^C ;}$$

Now, let us consider fields that transform in some representation r:

$$\psi'^i = \Gamma_r(g)^i{}_j \psi^j ; \quad \text{(contravariantly)}$$

$$\xi'_i = \xi_j \Gamma_r(g^{-1})^j{}_i ; \quad \text{(covariantly)}$$

Sometimes contravariant and covariant fields are related.
The transformation of a field in the adjoint representation can be written in a special way

$$\varphi'^A = \Gamma_{Adj}(g)^A{}_B \varphi^B ;$$

Defining $\varphi \equiv \varphi^A \Gamma_r(T_A)$;

$\varphi' = \Gamma_r(g)\, \varphi\, \Gamma_r(g^{-1})$; gives the same transformation for the components φ^A

The infinitesimal transformations are

$$\begin{cases} \delta_\sigma \psi^i = \sigma^A \Gamma_r(T_A)^i{}_j \psi^j \equiv \sigma_r \psi ; \\ \delta_\sigma \xi_i = -\xi_j \sigma^A \Gamma_r(T_A)^j{}_i \equiv -\xi \sigma_r ; \\ \delta_\sigma \varphi^A = \sigma^C f_{CB}{}^A \varphi^B \equiv \sigma^A{}_B \varphi^B = \sigma_{Adj} \varphi ; \end{cases}$$

$\sigma^A = \sigma^A(x)$ gauge parameters.

The covariant derivatives can be constructed generically as

$$D_\mu \phi = \partial_\mu \phi - g A^A_\mu \delta_A \phi ; \quad \text{if } \delta_\sigma \phi \equiv \sigma^A \delta_A \phi$$

$$D_\mu \psi = \partial_\mu \psi - g A^A_\mu \Gamma_r(T_A) \psi = \partial_\mu \psi - g A_{r\mu} \psi ;$$

$$D_\mu \xi = \partial_\mu \xi + g\, \xi A_\mu^A \Gamma_r(T_A) = \partial_\mu \xi + g\, \xi A_{r\mu} ;$$

$$D_\mu \varphi^A = \partial_\mu \varphi^A - g\, A_\mu^B\, \Gamma_{Adj}(T_B)^A{}_C\, \varphi^C = \partial_\mu \varphi^A - g\, f_{BC}{}^A A_\mu^B \varphi^C ;$$

The covariant derivative transforms covariantly under gauge transformations

$$(D_\mu \psi)' = \Gamma_r(g)\, D\psi \quad ; \quad (D_\mu \xi)' = (D\xi)\,\Gamma_r(g^{-1}) ;$$

$$(D_\mu \varphi)' = \Gamma_r(g)(D\varphi)\Gamma_r(g^{-1}) = \Gamma_{Adj}(g)\, D\varphi ;$$

if the connection A_μ transforms as follows:

$$A'_\mu = \Gamma_r(g) A_\mu \Gamma_r(g^{-1}) + \partial_\mu \Gamma_r(g)\, \Gamma_r(g^{-1})$$

Adjoint

$$\delta_\sigma A_\mu^A = \frac{1}{g}\left(\partial_\mu \sigma^A - g\, f_{BC}{}^A A^B \sigma^C\right) = \frac{1}{g} D_\mu \sigma^A ;$$

The curvature of the gauge field a.k.a. gauge field strength can be defined through the Ricci identities:

$$[D_\mu, D_\nu]\,\psi = -g\, F_{r\,\mu\nu}\,\psi = -g\, F^A{}_{\mu\nu}\, \Gamma_r(T_A)\,\psi ;$$

$$[D_\mu, D_\nu]\,\xi = +g\, \xi\, F_{r\,\mu\nu} = +g\, F^A{}_{\mu\nu}\, \xi\, \Gamma_r(T_A) ;$$

$$[D_\mu, D_\nu]\,\varphi = -g\,[F_{r\,\mu\nu}, \varphi] ;$$

$$F^A{}_{\mu\nu} = 2\partial_{[\mu} A^A{}_{\nu]} - g\, f_{BC}{}^A A^B_\mu A^C{}_\nu ;$$

$$F_{r\,\mu\nu} = 2\partial_{[\mu} A_{r\,\nu]} - g\,[A_{r\,\mu}, A_{r\,\nu}] ;$$

It satisfies the Bianchi identity $\quad D_{[\mu} F^A_{\nu\rho]} = 0 \quad ,$

B.2.1.3 GL(d,R) gauge theory

$GL(d,\mathbb{R})$ is the group of general linear transformations in d dimensions. The defining representation is the vector representation, d-dimensional: $d\times d$ non-singular matrices $\Lambda^a{}_b$ acting on vectors and 1-forms

$$\begin{cases} \xi'^a = \Lambda^a{}_b \xi^b; \\ \omega'_a = \omega_b (\Lambda^{-1})^b{}_a; \end{cases} \quad \text{in particular} \begin{cases} e'^a = \Lambda^a{}_b e^b; \\ e'_a = e_b (\Lambda^{-1})^b{}_a; \end{cases}$$

$GL(d,\mathbb{R})$ is d^2-dimensional: all the entries of the $d\times d$ matrix are independent parameters. We can use a pair of vector indices ab as adjoint index $A: 1 \ldots d^2$

$$\Lambda^a{}_b \sim \delta^a{}_b + \sigma^a{}_b \,; \qquad \sigma^a{}_b = \sigma^{cd} \underbrace{\delta^a{}_c \delta_{db}}_{(T_{cd})^a{}_b}$$

$(T_{cd})^a{}_b$ are the generators of $GL(d,\mathbb{R})$ in the vector representation

$$\Gamma_v (T_{cd})^a{}_b = \delta^a{}_c \delta_{db}$$

$$T_{ab}{}^e{}_f T_{cd}{}^f{}_g = \delta_a{}^e \delta_{bf} \delta^f{}_c \delta_{dg} = \delta_a{}^e \delta_{bc} \delta_{dg};$$

$$[T_{ab}, T_{cd}]^f{}_g = \delta_a{}^e \delta_{bc} \delta_{dg} - \delta_c{}^e \delta_{da} \delta_{bg}$$

$$= \underbrace{(\delta_a{}^h \delta_{bc} \delta_d{}^i - \delta_c{}^h \delta_{da} \delta_b{}^i)}_{f_{ab\ cd}{}^{hi}} \underbrace{\delta^e{}_h \delta_{ig}}_{(T_{hi})^e{}_g}$$

The $GL(d,\mathbb{R})$ connection (gauge field) A_μ^{ab} is denoted by $A_\mu{}^{ab} = -\omega_\mu{}^{ba}$. Observe that

conventional

$$-\omega_\mu{}^{ba} T_{ab}{}^c{}_d = -\omega_\mu{}_d{}^c$$

: the $GL(d,\mathbb{R})$ generators cannot be "seen"

Also $\omega_\mu{}^{ba}$ is neither symmetric nor antisymmetric on ab

Then, according to the general formalism $(g=1)$

$$D_\mu \xi^a = \partial_\mu \xi^a + \omega_\mu{}_b{}^a \xi^b ;$$
$$D_\mu w_a = \partial_\mu w_a - w_b \, \omega_\mu{}_a{}^b ;$$

The gauge field strength is denoted by $R_{\mu\nu}{}^{ab}(\omega)$:

$$R_{\mu\nu}{}^{ab}(A) = -R_{\mu\nu}{}^{ba}(\omega)$$

$$\boxed{R_{\mu\nu}{}^{ab}(\omega) = 2\,\partial_{[\mu}\,\omega_{\nu]}{}^{ab} - 2\,\omega_{[\mu}{}^{ac}\,\omega_{\nu]c}{}^{b}}$$

Observe that

1) $\omega_\mu{}^{ab}$ is a 1-form and $D_\mu \xi^a$ or $D_\mu w_a$ are also 1-forms from the point of view of g.c.t.s

2) $R_{\mu\nu}{}^{ab} = R_{[\mu\nu]}{}^{ab}$ is a 2-form from the point of view of g.c.t.s.

3) We have now two connections that play different roles
   $\Gamma_{\mu\nu}{}^{\rho}$: affine connection, not a tensor
   $\omega_\mu{}^{ab}$: $GL(d,\mathbb{R})$ connection, a tensor (1-form)

We can define the "total covariant derivative" ∇_μ with both connections. Then, for instance

$$\nabla_\mu e^a{}_\nu = \partial_\mu e^a{}_\nu - \Gamma_{\mu\nu}{}^{\rho} e^a{}_\rho + \omega_\mu{}_b{}^a e^b{}_\nu$$

transforms as a contravariant $GL(d,\mathbb{R})$ vector and as a $(0,2)$ tensor.

B.2.1.4 Tetrad postulate

In order to relate these two connections one can impose the "First Vielbein postulate" or "Tetrad postulate", which requires the Vielbein to be covariantly constant with respect to the total covariant derivative:

$$\nabla_\mu e_a{}^\nu = \partial_\mu e_a{}^\nu + \Gamma_{\mu\rho}{}^\nu e_a{}^\rho - e_b{}^\nu \omega_{\mu a}{}^b = 0$$

$$\Rightarrow \boxed{\omega_{\mu a}{}^b = \Gamma_{\mu a}{}^b - \underbrace{e_a{}^\nu \partial_\mu e^c{}_\nu}_{\text{Weitzenbock connection}}}$$

An immediate consequence of this relation is

$$\boxed{R_{\mu\nu a}{}^b(\omega) = R_{\mu\nu\rho}{}^\sigma(\Gamma)\, e_a{}^\rho e^b{}_\sigma}$$

⇒ We can compute the Riemann tensor from $R_{\mu\nu a}{}^b$

A second consequence is

$$0 = 2\nabla_{[\mu} e^a{}_{\nu]} = 2 D_{[\mu} e^a{}_{\nu]} - 2\Gamma_{[\mu\nu]}{}^a = 2D_{[\mu} e^a{}_{\nu]} + T_{\mu\nu}{}^a$$

$$\Rightarrow \boxed{2D_{[\mu} e^a{}_{\nu]} = -T_{\mu\nu}{}^a}$$ <u>1^{st} Cartan structure equation</u>

This equation can be solved for $\omega_\mu{}^a{}_b$. A convenient expression is

$$\omega_{ab}{}^c = \omega_{ab}{}^c(e) + K_{ab}{}^c + L_{ab}{}^c$$

Levi-Civita connection (related to $\begin{Bmatrix}\rho\\ \mu\nu\end{Bmatrix}$) — Contorsion tensor — "Co-non-metricity" tensor

$$\partial_\mu e^a{}_\nu - \begin{Bmatrix}\rho\\ \mu\nu\end{Bmatrix} e^a{}_\rho - \omega_\mu{}^a{}_b(e)\, e^b{}_\nu = 0$$

$$K_{\mu\nu}{}^\rho = \frac{1}{2} g^{\rho\sigma}\{T_{\mu\sigma\nu} + T_{\nu\sigma\mu} - T_{\mu\nu\sigma}\}$$

$$L_{\mu\nu}{}^\rho = \frac{1}{2} g^{\rho\sigma}\{Q_{\mu\nu\sigma} + Q_{\nu\mu\sigma} - Q_{\sigma\mu\nu}\}$$

where $Q_{\mu\nu\rho} \equiv -\nabla_\mu g_{\nu\rho}$ is the non-metricity tensor.

The Levi-Civita connection is given by

$$\omega_{ab}{}^{c}(e) = \left\{ {c \atop a\,b} \right\} - \Omega_{ab}{}^{c} + \Omega_{b}{}^{c}{}_{a} - \Omega^{c}{}_{ab}$$

$$\left\{ {c \atop a\,b} \right\} \equiv \frac{1}{2} g^{cd} \left\{ \partial_a g_{db} + \partial_b g_{da} - \partial_d g_{ab} \right\}$$

$$g_{ab} = g_{\mu\nu}\, e_a{}^{\mu} e_b{}^{\nu};$$

Observe that $\omega_{a(bc)} = \frac{1}{2}\left(Q_{abc} + \partial_a g_{bc}\right)$

$$\omega_{abc} \equiv \omega_{ab}{}^{d} g_{cd}$$

B.2.1.5 Metric compatibility. Spin connection

The metric-compatibility condition is, in this context, called "Second Vielbein postulate". If the first postulate is satisfied we can write it in two equivalent ways:

$$\boxed{\nabla_\mu g_{\nu\rho} = 0 \quad \Leftrightarrow \quad D_\mu g_{ab} = 0} \qquad \Rightarrow L = 0$$

Observe that this is not enough to imply $\omega_{a(bc)} = 0$ if g_{ab} is not constant.

There are two important cases

i) Coordinate basis $e^a{}_\mu = \delta^a{}_\mu \Rightarrow \boxed{\omega_{\mu a}{}^b = \Gamma_{\mu a}{}^b}$

They are identical. $(g_{ab} = g_{\mu\nu}\ ,\ \Omega = 0)$

ii) Orthonormal basis : $\boxed{e_a{}^\mu e_b{}^\nu g_{\mu\nu} = g_{ab} = \eta_{ab}}$

where η_{ab} in our case is the Minkowski metric in Cartesian coordinates (or any other coordinates in which it is constant).

This is a very important case. Several things happen at the same time:

1) $\omega_{a(bc)} = \omega_{a(b}{}^d \eta_{c)d} = 0$: the indices bc are Lorentz indices in $SO(1,d-1)$

2) The only $GL(d,\mathbb{R})$ transformations which are allowed are those that preserve this condition $\Rightarrow$ local Lorentz transformations. $SO(1,d-1)$ local transformations.

$$\Lambda_\mu{}^{ab} = -\omega_\mu{}^{ba} = +\omega_\mu{}^{ab}$$

$$\omega_\mu{}^a{}_b = \frac{1}{2}\omega_\mu{}^{cd}\underbrace{2\eta_{[c}{}^a\eta_{d]b}}_{\Gamma_v(M_{cd})^a{}_b}$$

$\Gamma_v(M_{cd})^a{}_b$: generators of the Lorentz group in the vector representation

$$R_{\mu\nu}{}^{ab}(\omega) = 2\partial_{[\mu}\omega_{\nu]}{}^{ab} - 2\omega_{[\mu|}{}^a{}_c\,\omega_{|\nu]}{}^{cb}$$

B.2.1.6 Covariant derivatives of spinor fields

The main difference between $GL(d,\mathbb{R})$ and $SO(1,d-1)$ (or any other orthogonal group) is that the latter admits spinorial transformations. In d dimensions these are $2^{[d/2]}$-dimensional and the generators of the Lorentz group in this representation are given by

$$\boxed{\begin{array}{l} \Gamma_S(M_{ab})^\alpha{}_\beta = \frac{1}{2}(\gamma_{ab})^\alpha{}_\beta \,; \qquad \gamma_{ab} = \gamma_{[a}\gamma_{b]} \\ \{\gamma_a, \gamma_b\} = 2\eta_{ab}\,; \end{array}}$$

A "contravariant" spinor transforms as

$$\delta_\sigma \psi^\alpha = \frac{1}{2}\sigma^{ab}\,\Gamma_S(M_{ab})^\alpha{}_\beta\,\psi^\beta \equiv \frac{1}{2}\sigma^{ab}\,\delta_{ab}\psi^\alpha;$$

$$\Rightarrow \quad D_\mu \psi^\alpha = \partial_\mu \psi^\alpha - \frac{1}{2}\omega_\mu{}^{ab}\,\Gamma_S(M_{ab})^\alpha{}_\beta\,\psi^\beta$$

$$D_\mu \psi = \partial_\mu \psi - \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\psi \equiv \partial_\mu\psi - \not{\omega}_\mu \psi\,;$$

$$[D_\mu, D_\nu]\psi = -\frac{1}{2}R_{\mu\nu}{}^{ab}\,\Gamma_S(M_{ab})\psi = -\frac{1}{4}R_{\mu\nu}{}^{ab}\gamma_{ab}\psi = -\frac{1}{4}\not{R}_{\mu\nu}\psi\,;$$

The Dirac conjugation matrix $D_{\alpha\beta}$ is defined by the property

$$\boxed{D^{-1}\,\Gamma_S(M_{ab})^\dagger\, D = -\Gamma_S(M_{ab})}$$

Then, the covariant (row) spinor defined by

$$\bar{\psi}_\alpha = \psi^{\beta *} D_{\beta\alpha}$$

transforms covariantly under the Lorentz group

$$\delta_\sigma \bar{\psi}_\alpha = (\delta_\sigma \psi^\beta)^* D_{\beta\alpha} = \frac{1}{2}\sigma^{ab}\,\Gamma_S(M_{ab})^{*\beta}{}_\gamma\,\psi^{\gamma *} D_{\beta\alpha}$$

$$= \frac{1}{2}\sigma^{ab}\psi^{*\gamma}\left(-D_{\gamma\delta}\,\Gamma_S(M_{ab})^\delta{}_\alpha\right) = -\bar{\psi}_\delta\, \frac{1}{2}\sigma^{ab}\,\Gamma_S(M_{ab})^\delta{}_\alpha$$

$$\Rightarrow D_\mu \bar{\Psi} = \partial_\mu \bar{\Psi} + \bar{\Psi} \frac{1}{4} \not{\omega}$$

The Vielbein formalism is essential to couple these fields to gravity.
The Dirac action is $\sim \bar{\Psi} \not{D} \psi$ in flat spacetime.

Also

$$\not{D} = \gamma^\mu D_\mu = \gamma^a e_a{}^\mu D_\mu$$

$$\Rightarrow \int d^4x \, e \, i \bar{\Psi} \not{D} \psi$$

$$\det e^a{}_\mu = \sqrt{|g|}$$

$\rightarrow$ Cartan – Sciama – Kibble theories

B.2.1.7 Example

We are going to use this formalism to compute the curvature of a metric with respect to the (torsionless) Levi-Civita connection.
This is given by

$$\omega_{abc} = -\Omega_{abc} + \Omega_{bca} - \Omega_{cab};$$

$$\Omega_{abc} \equiv e_a{}^\mu e_b{}^\nu \partial_{[\mu} e_{\nu]c};$$

Advantage: antisymmetric matrices always have less components!
Given a metric, we have to <u>choose</u> a Vielbein basis.
Consider the metric of the 2-sphere

$$ds^2 = d\theta^2 + \sin^2\theta\, d\varphi^2;$$

Simplest choice

$$e^1 = d\theta; \quad e^2 = \sin\theta\, d\varphi$$

$$e^1{}_\theta = 1; \quad e^2{}_\theta = 0; \quad \Big| \quad e_1{}^\theta = 1$$

$$e^1{}_\varphi = 0; \quad e^2{}_\varphi = \sin\theta; \quad \Big| \quad e_2{}^\varphi = \frac{1}{\sin\theta}$$

$$\partial . e^1 = 0; \quad \partial_\theta e^2_\varphi = \cos\theta; \quad \partial_{[\theta} e^2_{\varphi]} = \frac{1}{2}\cos\theta$$

$$e_1{}^\mu e_2{}^\nu \partial_{[\mu} e^2_{\varphi]} = \frac{1}{\sin\theta}\,\frac{1}{2}\cos\theta = \Omega_{122}$$

$$\underset{\omega_\varphi{}^{12}/\sin\theta}{\overset{\|}{\omega_{212}}} = -\Omega_{212} + \Omega_{122} - \cancel{\Omega_{221}} = -2\,\Omega_{212} = -\frac{\cos\theta}{\sin\theta};$$

$$R_{\theta\varphi}{}^{12} = \frac{1}{2}\partial_\theta \omega_\varphi{}^{12} = -\frac{1}{2}\partial_\theta\left(\cos\theta\right) = \frac{1}{2}\sin\theta,$$

$$R = R_{\theta\varphi}{}^{12} e_1{}^\theta e_2{}^\varphi = \frac{1}{2};$$

Consider now the static, spherically-symmetric metric

$$ds^2 = \lambda\, dt^2 - \lambda^{-1} dr^2 - R^2 d\Omega^2$$

$$e^0 = \lambda^{1/2}\, dt \qquad e_0 = \lambda^{-1/2}\partial_t$$
$$e^1 = \lambda^{-1/2}\, dr \qquad e_1 = \lambda^{1/2}\partial_r$$
$$e^2 = R\, d\theta \qquad e_2 = \frac{1}{R}\partial_\theta$$
$$e^3 = R\sin\theta\, d\varphi \qquad e_3 = \frac{1}{R\sin\theta}\partial_\varphi$$

$$\partial_r e^0{}_t = \frac{1}{2}\lambda^{-1/2}\lambda';$$

$$\partial_r e^2{}_\theta = R';$$
$$\partial_r e^3{}_\varphi = R'\sin\theta; \quad \partial_\theta e^3{}_\varphi = R\cos\theta;$$

$$\Omega_{010} = -\frac{1}{4}\lambda^{-1/2}\lambda';$$
$$\Omega_{122} = -\frac{1}{2}\frac{\lambda^{1/2}}{R}R';$$
$$\Omega_{133} = -\frac{1}{2}\lambda^{1/2}\frac{R'}{R};$$
$$\Omega_{233} = -\frac{1}{2}\frac{\cos\theta}{R\sin\theta};$$

$$\omega_{001} = -\cancel{\Omega_{001}} + \Omega_{010} - \Omega_{100} = 2\Omega_{010} = -\frac{1}{2}\lambda^{-1/2}\lambda';$$
$$\omega_{212} = -\Omega_{212} + \Omega_{122} - \cancel{\Omega_{221}} = 2\Omega_{122} = -\lambda^{1/2}\frac{R'}{R};$$
$$\omega_{313} = -\Omega_{313} + \Omega_{133} - \cancel{\Omega_{331}} = 2\Omega_{133} = -\lambda^{1/2}\frac{R'}{R};$$
$$\omega_{323} = 2\Omega_{233} = -\frac{\cos\theta}{R\sin\theta};$$

$$\omega_t{}^{01} = \frac{1}{2}\lambda';$$
$$\omega_\theta{}^{12} = -\lambda^{1/2}R';$$
$$\omega_\varphi{}^{13} = -\lambda^{1/2}R'\sin\theta;$$
$$\omega_\varphi{}^{23} = -\cos\theta;$$

$$R_{rt}{}^{01} = \partial_r\omega_t{}^{01} = \frac{1}{2}\lambda'';$$
$$R_{r\theta}{}^{12} = \partial_r\omega_\theta{}^{12} = -\left(\lambda^{1/2}R'\right)'$$
$$R_{r\varphi}{}^{13} = \partial_r\omega_\varphi{}^{13} = -\left(\lambda^{1/2}R'\right)'\sin\theta$$
$$R_{\theta\varphi}{}^{13} = \partial_\theta\omega_\varphi{}^{13} - \omega_\theta{}^{12}\omega_\varphi{}^{23} = -\lambda^{1/2}R'\cos\theta + \lambda^{1/2}R'\cos\theta = 0$$

$$R_{\theta\varphi}{}^{23} = d_\theta \omega_\varphi{}^{23} - \omega_\theta{}^{2} \wedge \omega_\varphi{}^{13} = \sin\theta - \lambda (R')^2 \sin\theta$$
$$= -\sin\theta \left[\lambda (R')^2 - 1 \right] ;$$

$$R_{t\theta}{}^{02} = - \omega_t{}^{0} \wedge \omega_\theta{}^{12} = -\frac{1}{2} \lambda^{1/2} \lambda' R' ;$$
$$R_{t\varphi}{}^{03} = - \omega_t{}^{0} \wedge \omega_\varphi{}^{13} = -\frac{1}{2} \lambda^{1/2} \lambda' R' \sin\theta ;$$

$$\boxed{\begin{aligned}
R_{01}{}^{01} &= -\frac{1}{2} \lambda'' ; \\
R_{02}{}^{02} &= -\frac{1}{2} \lambda' R'/R ; \\
R_{03}{}^{03} &= -\frac{1}{2} \lambda' R'/R ; \\
R_{12}{}^{12} &= -\frac{\lambda^{1/2}}{R} \left(\lambda^{1/2} R' \right)' ; \\
R_{13}{}^{13} &= -\frac{\lambda^{1/2}}{R} \left(\lambda^{1/2} R' \right)' ; \\
R_{23}{}^{23} &= -\frac{1}{R^2} \left[\lambda (R')^2 - 1 \right] ;
\end{aligned}}$$

B.2.1.8 Differential-form formulation of Cartan's structure equations

We can define a Lorentz-covariant exterior derivative D

$$De^a \equiv de^a - \omega^a{}_b \wedge e^b ;$$

$$e^a \equiv e^a{}_\mu dx^\mu ;$$
$$\omega^a{}_b \equiv \omega_\mu{}^a{}_b dx^\mu ;$$

$$De^a = \partial_{[\mu} e_{\nu]}{}^a dx^\mu \wedge dx^\nu - \omega_\mu{}^a{}_b e^b{}_\nu dx^\mu \wedge dx^\nu$$
$$= D_{[\mu} e^a{}_{\nu]} dx^\mu \wedge dx^\nu = -\frac{1}{2} T_{\mu\nu}{}^a dx^\mu \wedge dx^\nu$$
$$= -T^a ;$$

$$T^a \equiv \frac{1}{2} T_{\mu\nu}{}^a dx^\mu \wedge dx^\nu ; \qquad \boxed{De^a + T^a = 0} \quad 1^{st}$$

$$DDe^a = \frac{1}{2}[D, D]e^a = \frac{1}{2}\left(-R^a{}_b \wedge e^b\right) = -\frac{1}{2} R^a{}_b \wedge e^b$$

$$\boxed{DT^a - \frac{1}{2} R^a{}_b \wedge e^b = 0} \quad 2^{nd} \qquad R^a{}_b = \frac{1}{2} R_{\mu\nu}{}^a{}_b dx^\mu \wedge dx^\nu$$

$$R^a{}_b \wedge e^b = \frac{1}{2} R_{\mu\nu}{}^a{}_b e^b{}_\rho dx^\mu \wedge dx^\nu \wedge dx^\rho$$
$$= -\frac{1}{2} R_{\mu\nu\rho}{}^a dx^\mu \wedge dx^\nu \wedge dx^\rho$$

$$\boxed{T^a = 0 \quad \Rightarrow \quad R_{[\mu\nu\rho]}{}^a = 0} \quad \text{Bianchi identity}$$

$$-\frac{1}{2} R^a{}_b \wedge T^b - \cancel{\frac{1}{2} D R^a{}_b \wedge e^b}^{\text{Bianchi identity}} - \frac{1}{2} R^a{}_b \wedge \underbrace{De^b}_{=-T^b} = 0$$

$\Rightarrow$ automatically satisfied

B.2.1.9 The Einstein-Hilbert action in the Vielbein formalism