B.1.1 More on the Schwarzschild black hole

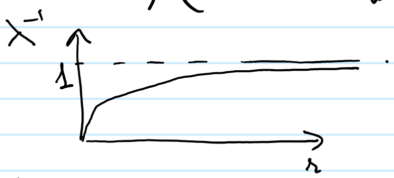
In this section we are going to use the Schwarzschild BH to learn several concepts that can be applied to more general black-hole space times. We will consider another example (the Reissner-Nordström black hole next)

B.1.1.1 The negative-mass Schwarzschild solution

If $M<0$ the only singular point is $r=0$, which is a timelike surface. There is no need to introduce a different set of coordinates. Nevertheless, it is useful to change to these:

$$ds^2 = \lambda (dt + dr_*)(dt - dr_*) - r^2 d\Omega^2 ;$$

$$\frac{dr_*}{dr} = \lambda^{-1} ;$$

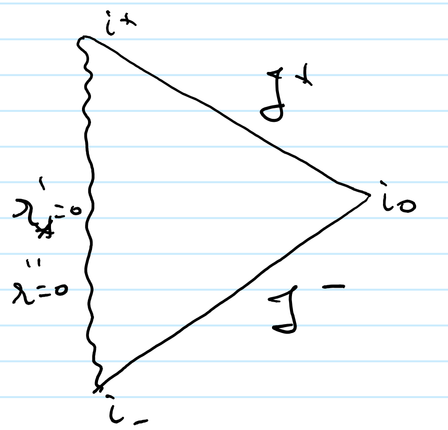


$r_* = r - R_S \log(r+R_S)$; → negative for small r

→ $r_*' = r_* + R_S \log R_S = r - R_S \log\left(1+\frac{r}{R_S}\right)$
$\sim \cancel{r} - R_S\left(\cancel{\frac{r}{R_S}} - \frac{1}{2}\left(\frac{r}{R_S}\right)^2\right) \sim \frac{1}{2}\frac{r^2}{R_S^2} > 0$

⇒ Choose the singularity is at $r_*' = 0$

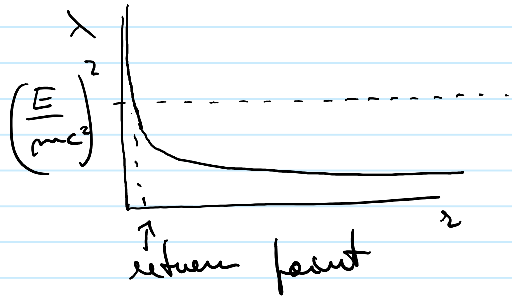
$$ds^2 = \lambda\left(dt^2 - dr_*'^2\right) - r^2 d\Omega^2 ;$$



Not so different from Minkowski but with a singularity ("naked")

You can scape from this kind of singularity! Study the geodesics. First, radial

$$\dot{r}^2 + \lambda = \left(\frac{E}{mc^2}\right)^2$$



This singularity is repulsive!

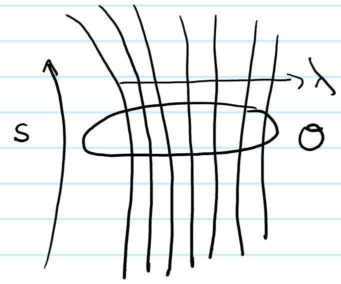
B.1.1.2 Apparent horizon and trapped surfaces

We have seen that the Schwarzschild spacetime has an event horizon, whose definition is local, so it has "teleological properties". There are other possible definitions of horizon, with different properties. The most important one is the "apparent horizon". In order to define it we need to introduce some concepts.

Def: Given an open region $\mathcal{O}$ of an spacetime, <u>a congruence</u> of curves in $\mathcal{O}$ is a family of curves such that through each point in $\mathcal{O}$ there passes one and only one curve of this family.

$u^\alpha(s,\lambda)$ — λ parametrises the family of curves

s is the evolution parameter of the curve (proper time if it is a timelike curve, for instance)



Def The <u>expansion</u> θ of a congruence of curves u^α is $\boxed{\theta \equiv \nabla_\alpha u^\alpha}$

It can be shown that, if $u^\alpha(s,\lambda)$ is a congruence of geodesics, θ is the fractional rate of change of the congruence's cross-sectional $\begin{cases} \text{volume} & \delta V \quad \text{(for timelike geodesics)} \\ \text{area} & \delta A \quad \text{(for null geodesics)} \end{cases}$

$$\theta = \frac{1}{\delta V}\frac{d}{ds}\delta V \; ; \qquad \theta = \frac{1}{\delta A}\frac{d}{ds}\delta A \; ;$$ ("A Relativist's toolkit" Eric Poisson C.U.P. 2004)

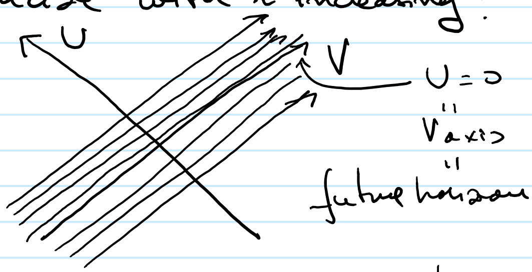
(Observe that the transverse metric to a timelike vector is 3-dimensional but the transverse metric to a null vector is only 2-dimensional in Lorentzian signature.

$$h_{\alpha\beta} = g_{\alpha\beta} - u_\alpha u_\beta \; ; \qquad h_{\alpha\beta} = g_{\alpha\beta} - u_\alpha N_\beta - u_\beta N_\alpha \; ; \quad u^\alpha N_\alpha = 1$$

$$ds^2 = du\,dv - dy^2 - dz^2 \xrightarrow[\substack{u = ct \\ v = ct}]{} -dy^2 - dz^2 \quad \text{2 dimensional.}$$

Let us consider now outgoing light rays in the neighborhood of the $r = R_S$ surface of the KS spacetime. We call outgoing here the curves $U = T - X$ = constant $\Rightarrow X = T$ - constant (X increases as T increases). Observe that this does not always coincide with r increasing!

$U = 0$: horizon
$U < 0$: exterior
$U > 0$: interior . r decreasing



We can show that the expansion of a congruence of light rays around $U = 0$ changes sign at $r = R_S$: the tangent vector to all of them is $u_\alpha = -\partial_\alpha U \left(= -\delta_\alpha^U\right)$

$$\theta = \nabla_\alpha u^\alpha = \frac{1}{\sqrt{|g|}}\,\partial_\alpha\left(\sqrt{|g|}\, g^{\alpha\beta} u_\beta\right)$$

$$\left.\begin{aligned}\sqrt{|g|} &= |g_{UV}|\, r^2 \sin^2\theta \\ g^{\alpha\beta} u_\beta &= \delta^{\alpha V}\, |g_{UV}|^{-1};\end{aligned}\right\}\quad \theta = \frac{2\,\partial_V r}{r\,|g_{UV}|} = -\frac{U}{2mr};$$

The surface $U = 0$ at which θ changes sign is an <u>apparent horizon</u>. Inside the apparent horizon $\theta < 0 \Rightarrow$ the cross-sectional area decreases. But, in order to give a more precise definition, we need to define the concept of "<u>trapped surface</u>"

<u>Def</u>: Let Σ be a spacelike hypersurface (timelike normal vector, spacelike tangent vectors). A trapped surface on Σ is a closed, 2-dimensional such that the congruences of outgoing and ingoing null geodesics orthogonal to S both have negative expansions everywhere on S.

For instance, each 2-sphere U, V = cte in $\underline{\text{II}}$ in the Kruskal-Szekeres spacetime is a trapped surface. $\theta = -\dfrac{U}{2mr} > 0$ in $\underline{\text{II}}$

V = constant $\Rightarrow$ ingoing $\qquad \theta = -\dfrac{V}{2mr} > 0$ in II as well.

Def Let $\mathcal{T}$ be the portion of Σ that contains trapped surfaces ($\mathcal{T}$ is known as the trapped region of Σ). Then, the boundary of $\mathcal{T}$, $\partial\mathcal{T}$, is the apparent horizon of Σ ($\Rightarrow$ one apparent horizon for each Σ)

In Schwarzschild the 2-spheres at $U=0$ ($r=R_S$) are apparent horizons.

The apparent horizon is a marginally trapped surface: there is a null congruence of geodesics orthogonal to $\partial\mathcal{T}$ such that $\theta=0$.

Def: The trapping horizon is the union of the apparent horizons. It is usually called apparent horizon.

In Schwarzschild the event horizon and the apparent horizon coincide. The same is true for stationary spacetimes, but not in general. We are going to see one example: the Vaidya spacetime.

The Vaidya spacetime

This spacetime has a metric whose general form is that of the Schwarzschild spacetime in ingoing Eddington-Finkelstein coordinates

$$ds^2 = \lambda\, dv^2 - 2\, dv\, dr - r^2 d\Omega^2 ;$$

but now with $\quad \lambda = 1 - \dfrac{2\,m(v)}{r}$; $\quad (G = c = 1)$

$m(v)$ is a mass function that depends on the ingoing coordinate v ("advanced time")

If we compute $G_{\mu\nu}$, we find that now

$$G_{\mu\nu} = 2\,\frac{m'}{r^2}\, l_\mu l_\nu ; \qquad l_\mu \equiv -\partial_\mu v$$

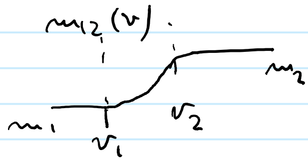
$\Rightarrow$ This can only be a solution of the Einstein equations if we find matter whose energy-momentum tensor is, precisely

$$T_{\mu\nu} = \frac{1}{8\pi}\, G_{\mu\nu} = \frac{m'}{4\pi r^2}\, l_\mu l_\nu ;$$

This kind of matter is <u>null dust</u>: a pressureless fluid with energy density $\dfrac{m'}{4\pi r^2}$ and velocity l^μ.

With this metric we can address the following situation: we irradiate a Schwarzschild BH with null dust in the time $v \in [v_1 ; v_2]$ $\Rightarrow$

$$m(v) = \begin{cases} m_1 & v \leq v_1 \\ m_{12}(v) & v_1 < v < v_2 \\ m_2 & v \geq v_2 \end{cases}$$



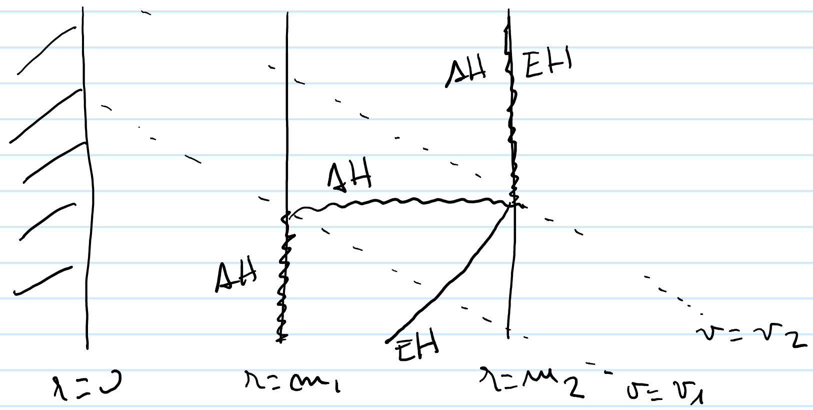
It can be shown that the apparent (trapped) horizon of the Vaidya spacetime is always at $\boxed{r = 2\,m(v)}$ (Poisson)

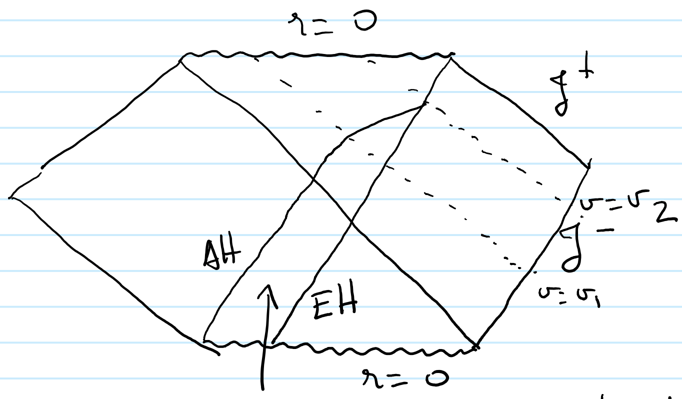
The vector normal to the hypersurface is $k_\alpha \equiv \partial_\alpha (r - 2\,m(v))$

and $\boxed{g^{\alpha\beta} k_\alpha k_\beta = 4\,m'}$ $\Rightarrow$ null for constant m (m_1, m_2)
timelike for $m'(v) > 0$.

Where is the event horizon of this spacetime?

For $v > v_2$ the event horizon should coincide with $r = 2m_2$. However it has to be null $\Rightarrow$ it cannot coincide with the apparent horizon





You can be doomed even though you are outside of the AH

B.1.1.3 Killing horizon and bifurcation sphere

The vector $k^\alpha = \frac{\partial x^\alpha}{\partial t}$ is a Killing vector of the Schwarzschild spacetime. It is timelike for $r > R_S$ or spacelike for $r < R_S$ and null in between, on the event horizon.

$r = R_S$ is a Killing horizon, where the norm of a Killing vector vanishes.

In static BH spacetimes event horizons, apparent horizons and Killing horizons all coincide.

The 2-sphere $U = V = 0$ at which the past and future horizons intersect is called bifurcation 2-sphere. It only exists in the maximally extended BH spacetime. The timelike Killing vector always vanishes there, which in Schwarzschild is a trivial statement.

B.1.2 The Reissner-Nordstrom black hole solution

In order to learn more general properties of BHs it is convenient to study more BH solutions. To keep things simple, we will only consider static, spherically symmetric solutions, but, then, we have to have some kind of matter because of Birkhoff's theorem.

We are going to consider the Einstein-Maxwell theory: gravity coupled to a massless electromagnetic field (but no charged matter).

The action is
$$S = \frac{1}{16\pi G}\int d^4x\sqrt{|g|}\left\{R - F^2\right\} \quad \text{Convenience}$$

(To simplify the calculations, it is customary to rescale A_μ so that G appears only as a global factor).

The equations of motion are

$$\begin{cases} G_{\mu\nu} = 2\left(F_\mu{}^\rho F_{\nu\rho} - \frac{1}{4} g_{\mu\nu} F^2\right) \\ \partial_\mu\left(\sqrt{|g|}\,F^{\mu\nu}\right) = 0; \qquad \left(+\ \partial_{[\mu}F_{\nu\rho]} = 0\right) \end{cases}$$

Again, we consider a static, spherically symmetric spacetime

$$ds^2 = \lambda(r)\,dt^2 - \lambda^{-1}(r)\,dr^2 - R^2(r)\,d\Omega^2,$$

and we also have to make a compatible "Ansatz" for A_μ.

$$A_t = \phi(r);$$

Let us consider the Maxwell equations first

$F_{rt} = \phi'$; $\quad F^{rt} = -\phi'$; (only non-vanishing component)

$\sqrt{|g|} = r^2\sin^2\theta$;

$$\partial_r\left(\sqrt{|g|}\,F^{rt}\right) = 0; \; -\sin^2\theta\,\partial_r\left(r^2\phi'\right) = 0; \quad \boxed{\phi' = \frac{a}{r^2}}; \; \boxed{\phi = -\frac{a}{r} + b}$$

$\partial_t\left(\sqrt{|g|}\,F^{tr}\right) = 0$; trivially satisfied.

Now we have to solve the Einstein equations. We compute first the energy-momentum tensor:

$$F^2 = 2F_{rt}F^{rt} = -2(F_{rt})^2 = -2\left(\frac{a}{r^2}\right)^2 = -2\frac{a^2}{r^4};$$

$$F_t{}^\mu F_{t\mu} = F_t{}^r F_{tr} = g^{rr}(F_{tr})^2 = -\lambda\,\frac{a^2}{r^4};$$

$$F_r{}^\mu F_{r\mu} = F_r{}^t F_{rt} = g^{tt}(F_{rt})^2 = \lambda^{-1}\,\frac{a^2}{r^4};$$

$$F_t{}^\mu F_{t\mu} - \frac{1}{4}g_{tt}F^2 = -\lambda\frac{a^2}{r^4} - \frac{1}{4}\lambda\left(-2\frac{a^2}{r^4}\right) = -\frac{1}{2}\lambda\frac{a^4}{r^4};$$

$$F_r{}^\mu F_{r\mu} - \frac{1}{4}g_{rr}F^2 = \lambda^{-1}\frac{a^2}{r^4} + \frac{1}{4}\lambda^{-1}\left(-2\frac{a^2}{r^4}\right) = \frac{1}{2}\lambda^{-1}\frac{a^4}{r^4};$$

$$F_\theta{}^\mu F_{\theta\mu} - \frac{1}{4}g_{\theta\theta}F^2 = +\frac{1}{4}r^2\left(-2\frac{a^2}{r^4}\right) = -\frac{1}{2}\frac{a^2}{r^2};$$

$$F_\varphi{}^\mu F_{\varphi\mu} - \frac{1}{4}g_{\varphi\varphi}F^2 = \sin^2\theta\,(\quad)_{\theta\theta};$$

We can simplify the Einstein equations using the tracelessness of the electromagnetic energy-momentum tensor:

$$T^\mu{}_\mu = 0 \Rightarrow G^\mu{}_\mu = 0 \Rightarrow R = 0$$

$$\Rightarrow R_{\mu\nu} = 2\left(F_\mu{}^\rho F_{\nu\rho} - \frac{1}{4}g_{\mu\nu}F^2\right);$$

$$tt \rightarrow \quad \frac{-\lambda}{2R^2}(R^2\lambda')' = -\lambda\frac{a^4}{r^4};$$

$$rr \rightarrow \quad -\lambda^{-2}\Big[\underbrace{\frac{-\lambda}{2R^2}(R^2\lambda')'}_{R_{tt}}\Big] + 2\frac{R''}{R} = \lambda\frac{a^4}{r^4};$$

$$\theta\theta \rightarrow \quad \frac{1}{2}\left[\lambda(R^2)'\right]' - 1 = -\frac{a^2}{r^2};$$

$$tt + rr \rightarrow \quad R'' = 0 \Rightarrow \boxed{R = r} \leftarrow \text{asymptotic flatness}$$

$$tt \rightarrow \quad \frac{-\lambda}{2r^2}(r^2\lambda')' = -\frac{\lambda}{r^4}a^2; \qquad (r^2\lambda')' = \frac{2a^2}{r^2}$$

$$r^2\lambda' = -\frac{2a^2}{r} + b; \qquad \lambda' = -\frac{2a^2}{r^3} + \frac{b}{r^2}; \quad \boxed{\lambda = c - \frac{b}{r} + \frac{a^2}{r^2}}$$

$$\infty \to \quad c = 1 \quad \Rightarrow \quad \boxed{\lambda = 1 - \frac{b}{r} + \frac{a^2}{r^2}}$$

$$\boxed{\begin{aligned} ds^2 &= \left(1 - \frac{b}{r} + \frac{a^2}{r^2}\right) dt^2 - \left(\quad \right)^{-1} dr^2 - r^2 d\Omega^2 \\ F_{rt} &= \frac{a}{r^2}; \end{aligned}}$$

Static, spherically symmetric and asymptotically flat.

Now, we have to interpret the integration constants a & b

Comparing, for large r, this metric with the Newtonian limit (or with Schwarzschild) $\boxed{b = 2GM}$ $(c=1)$

To identify a we study the definition of electric charge.

If the electromagnetic field is coupled to charged matter, the Maxwell equations take the form

$$*d*F = j \quad ; \quad j \to \text{charge, current density.}$$

or

$$\nabla_\mu F^{\mu\nu} = j^\nu$$

If we integrate the $\nu=0$ component of j^ν (charge density) over some 3-d volume (a spatial hypersurface with timelike unit vector n^μ, $n^2=+1$) we get the charge of that volume

$$Q_V = \int_V d^3x \sqrt{|g|}\, j^0 = \int_V d^3\Sigma\, n_\mu j^\mu = \int_V d^3\Sigma_\mu j^\mu$$

$$= \int_V d^3\Sigma_\mu \nabla_\nu F^{\nu\mu} = \int_{\partial V} d^2\Sigma_{\mu\nu} F^{\mu\nu} = \int_{\partial V} *F$$

Clearer: $\quad Q_V = \int_V *j = \int_V d*F = \int_{\partial V} *F$

The total charge of the spacetime is obtained by taking V as the whole space, whose boundary is a 2-sphere at infinity

$$Q = \int_{S^2_\infty} \star F = \int_{S^2_\infty} \frac{1}{4} \frac{\varepsilon_{\mu\nu\rho\sigma}}{\sqrt{|g|}} F^{\rho\sigma} dx^\mu \wedge dx^\nu =$$

$$= \int_{S^2_\infty} \frac{\varepsilon_{rt\theta\varphi}}{\sqrt{|g|}} F^{rt} \, d\theta \, d\varphi = \int_{S^2_\infty} -\frac{g}{\sqrt{|g|}} \overbrace{\varepsilon^{tr\theta\varphi}}^{1} \left(-F_{rt}\right) d\theta \, d\varphi$$

$$= -\int_{S^2_\infty} F_{rt} \, r^2 \sin\theta \, d\theta \, d\varphi = -a \int_{S^2_\infty} \sin\theta \, d\theta \, d\varphi =$$

$$= a \int_0^\pi d\cos\theta \int_0^{2\pi} d\varphi = 4\pi a ;$$

Usually, the charge is defined in rationalised units

$$\boxed{q = \frac{1}{4\pi} Q = a}$$

$$\boxed{\begin{array}{l} ds^2 = \left(1 - \frac{2GM}{r} + \frac{q^2}{r^2}\right) dt^2 - \left(\;\right)^{-1} dr^2 - r^2 d\Omega^2 ; \\ F_{rt} = \frac{q}{r^2} ; \end{array}}$$

Thus space has total mass M and total charge q. The mass (energy) is delocalised over the whole spacetime while the charge seems to come from $r=0$ (Abelian 1-form: the electromagnetic field itself is uncharged and there is no charged matter).

Setting $G=1$ for simplicity, we can rewrite the metric function λ in the form

$$\lambda = \frac{(r-r_+)(r-r_-)}{r^2} ; \quad r_\pm = M \pm \sqrt{M^2 - q^2} ;$$

$r_\pm$ are the roots of $r^2 \lambda = 0$ and $r_+ > r_-$ when they are real.

There are 3 possible cases:

i) $r_0^2 \equiv M^2 - q^2 > 0$ $\quad (M > |q|)$ "subextremal RN BH"

ii) $r_0^2 = 0$; $\quad (M = |q|)$ "extremal RN BH"

iii) $r_0^2 < 0$; $\quad (M < |q|)$ - "overextremal RN BH"

Subextremal $\quad g_{tt}$ vanishes at two radii $r = r_+$, $r = r_-$

Extremal $\quad g_{tt}$ vanishes at one radius $r = r_+ = r_- = M$

Over-extremal $\quad g_{tt}$ does not vanish anywhere

In all cases there is a curvature singularity at $r=0$ and in all cases the singularity is timelike (in Schwarzschild it is spacelike): signals emitted from it can be received by nearby observers. If there is an event horizon around it, the observers at ∞ cannot see it. This is the case where $r_0^2 \geq 0$ (i) and ii)) but not when $r_0^2 < 0$ (iii)). In the later case one says that the singularity is naked. It is expected (but not proved) that, starting with matter with good energy properties, a naked singularity will never form by gravitational collapse (Cosmic Censorship Conjecture)

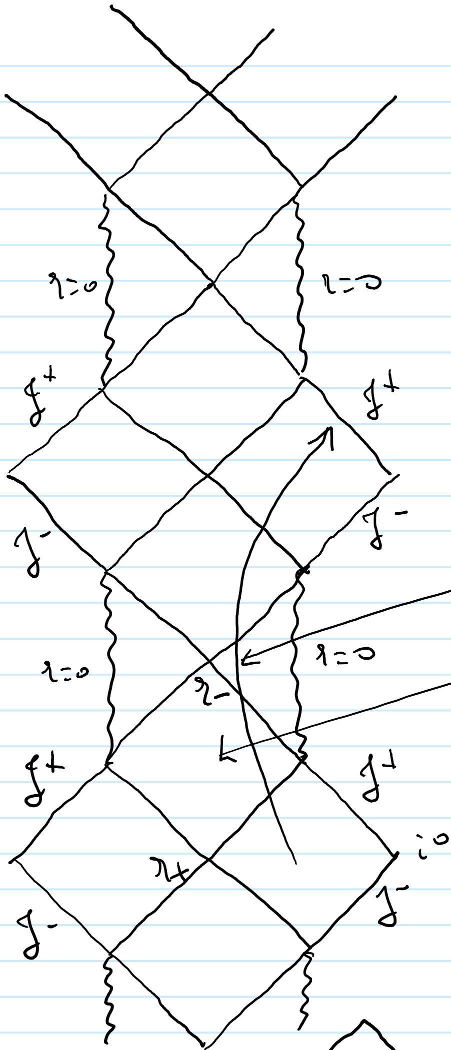
The points at which g_{tt} vanish look like horizons. They are indeed Killing horizons for the timelike Killing vector

$k^\mu = \delta^\mu{}_0$; $\quad k^2 = \lambda$

Observe that λ changes sign at r_+ and r_- so t is timelike again inside $r < r_-$.

To decide if the surfaces $r = r_\pm$ are event horizons we must find the maximal analytical extension. In this case the problem is more complicated than in Schwarzschild: a single coordinate patch is not enough.

The extension is studied in Townsend's book. We will just take the final result which is best represented by the Carter-Penrose

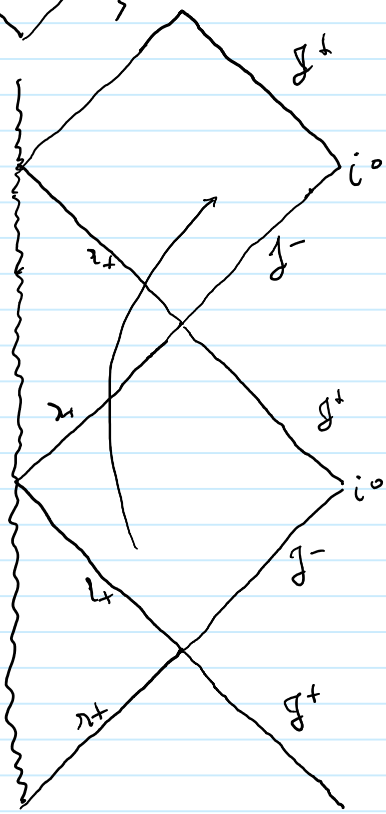


Timelike observers can avoid the singularity because it is timelike

In this region outgoing and ingoing light rays converge to the singularity $\Rightarrow$ apparent horizons

only r_+ is an event horizon

In the extremal case



Again, there is an infinite number of asymptotically-flat regions ("universes") and the observer can exit one and enter another one

Consider the metric $ds^2 = \left(1 - \frac{2m(r)}{r}\right) dt^2 - (\ )^{-1} dr^2 - r^2 d\Omega^2$ which is similar to the Schwarzschild metric but with the constant mass M replaced by a "mass function" $m(r)$.

$$R_{tt} = -\frac{\lambda}{r^2}\left(r^2\lambda'\right)' = -\frac{\lambda}{r^2}\left[r^2\left(-2\frac{m'}{r} + 2\frac{m}{r^2}\right)\right]' = -\frac{\lambda}{r^2}\left(-2m'r + 2m\right)'$$

$$= \frac{-\lambda}{r^2}\left(-2m''r \cancel{-2m'} \cancel{+2m'}\right) = 2m''\frac{\lambda}{r}$$

$R_{rr} = -\lambda^{-2} R_{tt}$;

$R_{\theta\theta} = \frac{1}{2}\left[\lambda\, 2r\right]' - 1 = \left[r - 2m\right]' - 1 = 1 - 2m' - 1 = -2m'$;

$R_{\varphi\varphi} = \sin^2\theta\, R_{\theta\theta}$;

$$R = g^{\mu\nu}R_{\mu\nu} = \lambda^{-1}R_{tt} - \lambda R_{rr} - r^{-2}R_{\theta\theta} - \left(r^2\sin^2\theta\right)^{-1}R_{\varphi\varphi}$$

$$= 4\frac{m''}{r} - 4\frac{m'}{r^2} = 4\left(\frac{m'}{r}\right)'$$

$$G_{tt} = \cancel{2m''\frac{\lambda}{r}} - \frac{1}{2}\lambda\, 4\left(\cancel{\frac{m''}{r}} - \frac{m'}{r^2}\right) = 2\lambda\frac{m'}{r^2};$$

$$G_{rr} = -\lambda^{-2}R_{tt} - \frac{1}{2}\left(-\lambda^{-1}\right)4\left(\frac{m''}{r} - \frac{m'}{r^2}\right) = \cancel{-2m''\frac{\lambda^{-1}}{r}} + 2\lambda^{-1}\left(\cancel{\frac{m''}{r}} - \frac{m'}{r^2}\right)$$

$$G_{\theta\theta} = \cancel{-2m'} - \frac{1}{2}\left(-r^2\right)4\left(\frac{m''}{r} \cancel{-\frac{m'}{r^2}}\right) = 2m''r;$$

$G_{\varphi\varphi} = \sin^2\theta\, G_{\theta\theta}$;

$\Rightarrow$ ($G=1$) $\quad T_{tt} = \frac{1}{8\pi}\frac{2\lambda m'}{r^2}$; $\quad T_{rr} = \frac{1}{8\pi}\left(-2\frac{\lambda^{-1}m'}{r^2}\right)$; $\quad T_{\theta\theta} = \frac{1}{8\pi}2m''r$

$T_{\varphi\varphi} = \sin^2\theta\, T_{\theta\theta}$;

The energy enclosed in an r = constant, t = constant 3-sphere is

$$E \sim \int_{B^3} d^3\Sigma_\mu T^\mu{}_t \sim \int \underbrace{\varepsilon_{\mu\nu\rho\sigma}}_{3!\sqrt{|g|}} dx^\nu dx^\rho dx^\sigma\, T^\mu{}_t$$

$r\ \theta\ \varphi$

$$\sim \int dr\, d\theta\, d\varphi\, r^2\sin\theta\, T^t{}_t = 4\pi\int_0 dr\, \cancel{r^2}\, \frac{1}{4\pi}\frac{m'}{\cancel{r^2}} = \boxed{m(r)}$$

For the R-N solution $m(r) = M - \frac{Q^2}{2r}$ and the sign depends on r: for r small enough m is negative. This implies that gravity becomes repulsive there. This always happens inside the inner horizon r_-.

(This is not a rigorous argument for anything. The energy density $T^t{}_t$ is not conserved because $\nabla_\mu T^\mu{}_t = 0$ is not a continuity equation. Only the total energy of the spacetime is well defined. It coincides with $\lim_{r\to\infty} m(r) = M$, but this does not need to be the case)

The repulsive nature of gravity manifests itself in the existence of turning points in the worldlines of free-falling observers inside r_-. Similar to the behaviour near the singularity of the Schwarzschild singularity with negative mass!

This can be checked in Eddington-Finkelstein-like coordinates

$$ds^2 = \lambda\, dv^2 - 2\,dv\,dr - r^2\, d\Omega^2$$

$$v \equiv t + r_*$$

$$\frac{dr_*}{dr} = \lambda^{-1}\ ; \quad r_* = \frac{(2M^2 - q^2)\, \mathrm{arctg}\left(\frac{r-M}{i r_0}\right)}{i r_0} + M \log r^2 \lambda + C$$

$$\left.\begin{array}{l} \sin i\theta = \dfrac{e^{-\theta} - e^{\theta}}{2i} = -\dfrac{1}{i}\, \mathrm{sh}\,\theta = i\, \mathrm{sh}\,\theta\ , \\[2ex] \cos i\theta = \dfrac{e^{\theta} + e^{-\theta}}{2} = \mathrm{ch}\,\theta\ , \end{array}\right\} \mathrm{tg}\, i\theta = i\, \mathrm{tgh}\,\theta$$

$$\mathrm{arctg}\left(\frac{\mathrm{tgh}\,\theta}{i}\right) = \mathrm{arctg}(-\mathrm{tg}\, i\theta) = -i\theta$$

$$\Rightarrow \mathrm{arctg}\, \frac{r-M}{i r_0} = \frac{1}{i}\, \mathrm{arctgh}\, \frac{r-M}{r_0}$$

$$\boxed{r_* = -\frac{(M^2 + r_0^2)\, \mathrm{arctgh}\left(\frac{r-M}{r_0}\right)}{r_0} + M \log r^2 \lambda + C}$$

$$\Gamma_{vv}{}^{v} = \frac{1}{2}\lambda'\ ; \quad \Gamma_{vv}{}^{r} = \frac{1}{2}\lambda\lambda'\ ; \quad \Gamma_{vr}{}^{r} = -\frac{1}{2}\lambda'\ ;$$

For timelike geodesics $\quad E = mc\, k^\mu g_{\mu\nu} \dot X^\nu = mc\, g_{v\mu} \dot X^\mu \qquad k^\mu = \delta^\mu_v$

$$E = mc\,(g_{vv}\dot v + g_{vr}\dot r)$$

$$\boxed{E = mc\,(\lambda \dot v - \dot r)} \qquad c=1$$

$\lambda \dot v - \dot r = E/m$; ← Energy conservation

$\dot v(\lambda \dot v - 2\dot r) = 1$; ← timelike (mass-shell) condition

$$\left.\begin{aligned} &\ddot v + \tfrac{1}{2}\lambda' (\dot v)^2 = 0 , \\ &\ddot r + \tfrac{1}{2}\lambda' \dot v (\lambda \dot v - 2\dot r) = 0 ; \end{aligned}\right\} \text{geodesic equation}$$

$\dot v\left[\lambda \dot v - 2\left(\lambda \dot v - E/m\right)\right] = 1$; $\quad +\lambda \dot v^2 - 2\frac{E}{m}\dot v + 1 = 0$

$\dot v = \lambda^{-1}(E/m + \dot r) \qquad \boxed{\dot v = \frac{E/m - \sqrt{(E/m)^2 - \lambda}}{\lambda}}$

$\lambda^{-1}(E/m + \dot r)\left[E/m + \cancel{\dot r} - \cancel{2}\dot r\right] = 1$; → signs for ingoing $\dot r < 0$

$\dot r^2 - (E/m)^2 = -\lambda$; $\quad \boxed{\dot r = -\sqrt{(E/m)^2 - \lambda}}$

or $\quad \dot r^2 + \lambda = (E/m)^2$

return point (as in M< Schwarzschild)

B.1.2.1 The Papapetrou-Majumdar solutions

We can rewrite the extremal RN solution in "isotropic coordinates"

$$ds^2 = \left(1-\frac{|q|}{r}\right)^2 dt^2 - \left(1-\frac{|q|}{r}\right)^{-2} dr^2 - r^2 d\Omega^2$$

$$\left(1-\frac{|q|}{r}\right)^{-2}\left(\frac{dr}{d\rho}\right)^2 = \frac{r^2}{\rho^2}, \qquad \frac{dr}{r-|q|} = \frac{d\rho}{\rho};$$

$$\log(r-|q|) = \log\rho + \cancel{c}; \quad \rho = r-|q|; \quad r = \rho + |q|$$

$$\boxed{\begin{aligned} &ds^2 = H^{-2}dt^2 - H^2\big(\underbrace{d\rho^2+\rho^2 d\Omega^2}_{d\vec{x}^2}\big), \\ &H = 1+\frac{|q|}{\rho}, \qquad A_t = \operatorname{sign} q\,(H^{-1}-1). \end{aligned}}$$

The horizon is now at $\rho=0$. Observe that, in the $\rho\to 0$ limit, the metric takes this form:

$$ds^2 \sim \underbrace{\left(\frac{\rho}{|q|}\right)^2 dt^2 - \left(\frac{|q|}{\rho}\right)^2 d\rho^2}_{AdS_2 \text{ with radius } |q|} - \underbrace{|q|^2 d\Omega^2}_{2\text{-sphere of radius } |q|}$$

This is the characteristic near-horizon limit of all static extremal asymptotically-flat BHs in $d=4$ dimensions. (Bertotti-Robinson)

It is an exact solution that we could have obtained by deleting the 1 in H. To check this, we use the above form of the solution as an Ansatz without imposing spherical symmetry

$$\begin{cases} R_{tt} = H^{-5}\partial^2 H - H^{-6}(\partial H)^2 \qquad \partial^2 \equiv \partial_i\partial_i \\ R_{ij} = -\delta_{ij}\left[H^{-2}(\partial H)^2 - H^{-1}\partial^2 H\right] + 2H^{-2}\partial_i H\partial_j H \end{cases}$$

$$F_{it} = -\operatorname{sign} q\, H^{-2}\partial_i H; \qquad F_{t}{}^{i} F_{ti} = -H^{-6}(\partial H)^2;$$

$$F_i{}^t F_{jt} = H^{-2}\partial_i H \partial_j H \,; \qquad F^2 = -2H^{-4}(\partial H)^2$$

$$F_t{}^i F_{ti} - \tfrac{1}{4} g_{tt} F^2 = -H^{-6}(\partial H)^2 - \tfrac{1}{4}H^{-2}\left(-2H^{-4}(\partial H)^2\right)$$
$$= -\tfrac{1}{2}H^{-6}(\partial H)^2 \,;$$

$$F_i{}^t F_{jt} - \tfrac{1}{4} g_{ij} F^2 = H^{-2}\partial_i H \partial_j H - \tfrac{1}{4}\left(-\delta_{ij}H^2\right)\left(-2H^{-4}(\partial H)^2\right)$$
$$= H^{-2}\left(\partial_i H \partial_j H - \tfrac{1}{2}\delta_{ij}(\partial H)^2\right);$$

$$tt \to -H^{-6}\cancel{(\partial H)^2} + H^{-5}\partial^2 H = -H^{-6}\cancel{(\partial H)^2} \Rightarrow \boxed{\partial^2 H = 0}$$

$$ij \to -\delta_{ij}\left[\cancel{H^{-2}(\partial H)^2} - H^{-1}\partial^2 H\right] + \cancel{2H^{-2}\partial_i H \partial_j H} =$$
$$= 2H^{-2}\left[\cancel{\partial_i H \partial_j H} - \cancel{\tfrac{1}{2}\delta_{ij}(\partial H)^2}\right]; \qquad \boxed{\partial^2 H = 0}$$

And the Maxwell equation is also solved for $\boxed{\partial^2 H = 0}$

For each choice of harmonic function on $\mathbb{E}^3$, $H(\vec{x})$ we have a different solution of the Einstein-Maxwell system.

If we want spherical symmetry

$$\partial_i \partial_i H = \frac{1}{\rho}\partial_\rho\left(\rho^2 \partial_\rho H\right) = 0 \,; \Rightarrow H' = \frac{a}{\rho^2}$$

$$\Rightarrow H = b - \frac{a}{\rho} \,; \to \begin{cases} \text{Extremal RN} & b \neq 0 \\ \text{Robinson-Bertotti} & b = 0 \end{cases}$$

If we expand in spherical harmonics, we only get a regular solution in the above case (monopole term). This agrees with the "no-hair theorem".

There is only one kind of non-spherically-symmetric solution which is regular:

$$\boxed{H = 1 + \sum_{i=1}^{N} \frac{|q_i|}{|\vec{x} - \vec{x}_i|}} \,;$$

N Extremal RN black holes with charges with the same sign in static equilibrium

<u>Papapetrou-Majumdar solutions.</u>

Each "point" $\vec{x} = \vec{x}_i$ is, actually, a sphere of radius $|q_i|$ and it is the event horizon of an extremal BH of charge $|q_i|$

$$\lim_{\vec{x} \to \vec{x}_i} ds^2 = \left(\frac{|q_i|}{\rho_i}\right)^{-2} dt^2 - \left(\quad\right)^2 d\rho_i^2 - |q_i|^2 d\Omega_i^2$$

where $\rho_i \equiv |\vec{x} - \vec{x}_i|$ and the angular coordinates are centered at $\rho_i = 0$.

The equilibrium between these extremal RN BHs can be "explained" as an equilibrium between gravitational attraction $\propto -M_i M_j$ and electrostatic repulsion $+|q_i q_j|$ with $M_i = |q_i|$. Strictly speaking only the global mass of the spacetime is well defined. Expanding for $|\vec{x}| \to \infty$

$|\vec{x} - \vec{x}_i| \sim |\vec{x}|$

$$g_{tt} \sim 1 - 2\frac{\sum_i |q_i|}{\rho} + \cdots \quad \Rightarrow \quad \boxed{M = \sum_i |q_i|}$$

The horizon of an extremal RN solution has a strange property: it is at infinite spatial distance from any observer.

B.1.3 Properties of the black-hole horizon

B.1.3.1 Surface gravity

Consider a static spherically-symmetric black hole metric

$$ds^2 = \lambda\, dt^2 - \lambda^{-1} dr^2 - R^2 d\Omega^2 ; \qquad \lambda = \lambda(r) ; \; R = R(r)$$

The event horizon coincides with the Killing horizon where $g_{tt} = \lambda = 0$ (the largest value of r for which $\lambda = 0$, r_+)

Near the event horizon we can always expand the metric as

$$\lambda = \lambda(r_+) + \lambda'(r_+)(r - r_+) + O(r-r_+)^2$$
$$R = R(r_+) + R'(r_+)(r - r_+) + O(r-r_+)^2$$

By assumption $\lambda(r_+) = 0$. We also assume that $\lambda'(r_+) \neq 0$. Then, to lowest order in $r - r_+$

$$ds^2 \sim \lambda'(r_+)(r-r_+)\, dt^2 - \underbrace{\frac{1}{\lambda'(r_+)(r-r_+)}\, dr^2}_{d\rho^2} - R^2(r_+)\, d\Omega^2$$

$$\Rightarrow \int \frac{dr}{\sqrt{\lambda'(r_+)(r-r_+)}} = \int d\rho$$

$$\frac{1}{\sqrt{\lambda'(r_+)}}\, 2\sqrt{(r-r_+)} = \rho + \cancel{C} ; \qquad (r - r_+) = \frac{1}{4}\lambda'(r_+)\, \rho^2$$

$$ds^2 \sim \underbrace{\left(\frac{\lambda'(r_+)}{2}\right)^2 \rho^2\, dt^2 - d\rho^2}_{\text{Rindler spacetime}} - R^2(r_+)\, d\Omega^2$$

with $g = \dfrac{\lambda'(r_+)}{2} \equiv \kappa$ surface gravity of the event horizon

The near-horizon limit of all the non-extremal (subextremal) black holes has this form: $\text{Rindler}_2 \times S^2$ with $g = \kappa$

If $\lambda'(r_+) = 0 \Rightarrow \kappa = 0$: extremal black hole

If $\lambda'(r_+)=0$, then $\lambda \sim \frac{1}{2}\lambda''(r_+)(r-r_+)^2$ and

$$ds^2 \sim \underbrace{\frac{\lambda''(r_+)}{2}(r-r_+)^2 dt^2 - \frac{1}{\frac{\lambda''(r_+)(r-r_+)^2}{2}} d(r-r_+)^2}_{AdS_2} - \underbrace{R^2(r_+)\, d\Omega^2}_{S^2}$$

as in the extremal RN case.

What is the physical meaning of the surface gravity?

Consider a particle of unit mass held at rest at some fixed radius $r=c$ by some means. Its worldline will be

$$\begin{cases} t = t(s); \\ r = c;\ \theta = c;\ \varphi = c; \end{cases} \qquad g_{tt}(\dot{t})^2 = 1; \quad \dot{t} = \lambda^{-1/2}(c)$$

The 4-speed is $u^\alpha = \dot{X}^\alpha = \lambda^{-1/2}(c)\,\delta^\alpha{}_t$;

The 4-acceleration is

$$a^\alpha = u^\beta \nabla_\beta u^\alpha = \cancel{\dot{u}^\alpha} + \Gamma_{\beta\gamma}{}^{\alpha} u^\beta u^\gamma$$

$$= \lambda^{-1}(c)\,\Gamma_{tt}{}^{\alpha} = \delta^\alpha_r\, \cancel{\lambda^{-1}}\, \tfrac{1}{2}\cancel{\lambda}\lambda' \Big|_{r=c}$$

$$= \frac{1}{2}\delta^\alpha{}_r\,\lambda'(c)$$

On the event horizon $a^\alpha(r_+) = \delta^\alpha{}_r\, \kappa$;

However, observe that $a^2(c) = a^\alpha a^\beta g_{\alpha\beta} = \lambda^{-1}(c)\left(\frac{1}{2}\lambda'(c)\right)^2$

$$a^2(c) \xrightarrow[c\to r_+]{} \infty$$

which is understandable since we know there is no way to stop the mass from falling into the black hole.

Suppose now that the particle is held in place by an observer at ∞ using an infinite, massless string. If the observer pulls the string δs he will do $\delta W_\infty = a_\infty \delta s$ where a_∞ is the force per unit mass he needs to do to hold the particle in place. Locally at $r=c$, $\delta W = a\,\delta s$. If this work was radiated to ∞, it would be redshifted with a factor $\lambda^{1/2}$: $\delta E = \lambda^{1/2} a_\infty \delta s$

$\rightarrow \delta E_\infty = \lambda^{1/2} a\, \delta s$ but $\delta E_\infty = \delta W_\infty$

$\Rightarrow a_\infty(c) = \lambda^{1/2}(c)\, a(c) = \frac{1}{2}\lambda'(c)$ ⇒ force exerted by the observer at ∞

$c \to r_+$ $\quad \boxed{a_\infty(r_+) = \frac{1}{2}\lambda'(r_+) = \kappa}$

κ can also be defined in terms of the timelike Killing vector $k^\alpha = \delta^\alpha{}_t$. The event horizon coincides with the Killing horizon.

$$\Phi \equiv k^\alpha k_\alpha \; ; \quad \Phi\big|_{r_+} = 0 \rightarrow \text{Killing horizon.}$$

k^α is normal to the event horizon (k^α is null and the event horizon is a null hypersurface. We can always choose k^α normal to the EH)

$$\partial_\alpha \Phi\Big|_{r_+} \propto k_\alpha\Big|_{r_+}$$

The proportionality constant is -2κ, by definition.

The advantage of this definition is that it does not depend on the choice of coordinates. If we choose Eddington-Finkelstein coordinates, $k^\alpha = \delta^\alpha{}_v$

$$\partial_\alpha \lambda = -2\kappa\, g_{\alpha v} \; ; \qquad \lambda'\Big|_{r_+} \partial_\alpha r = -2\kappa\, g_{\alpha v}\Big|_{r_+} = +2\kappa\, \delta_{\alpha r}$$

$$\Rightarrow \boxed{\kappa = \frac{1}{2}\lambda'(r_+)}$$

Using the above general definition it can be shown that the surface gravity is always constant over the event horizon of any stationary black hole metric that solves the Einstein equation with an energy-momentum tensor $T_{\mu\nu}$ that satisfies the <u>dominant energy condition</u>

The dominant energy condition basically says that matter should flow along timelike or null worldlines: if v^α is an arbitrary, future-directed, timelike vector field then $\quad -T^{\alpha}{}_{\beta} v^{\beta}$ which is the matter's energy-momentum density measured by an observer with worldline v^{α}, is always timelike or null.

For a perfect fluid $\Rightarrow \rho \geq 0; \ \rho \geq |p_i|$

Let us compute κ for

Schwarzschild $\qquad \kappa = \frac{1}{2}\left(+\frac{2M}{r^2}\right)\Big|_{r_+ = R_S} = \frac{1}{4M};$

Reissner–Nordström $\qquad \kappa = \frac{1}{2}\left(+\frac{2M}{r^2} - 2\frac{q^2}{r^3}\right)\Big|_{r_+} = \frac{M r_+ - q^2}{r_+^3} =$

$$= \frac{M^2 - q^2 + M\sqrt{M^2-q^2}}{\left(M+\sqrt{M^2-q^2}\right)^2} = \frac{\sqrt{M^2-q^2}}{\left(M+\sqrt{M^2-q^2}\right)^2}$$

$$\xrightarrow[\substack{\text{extremal limit}\\ M\to|q|}]{} 0$$

B.1.3.2 The area of the event horizon

The event horizon is always a null hypersurface and, therefore, it is 2-dimensional. We can compute the area of the event horizon easily taking first the near-horizon limit

$$ds^2 \sim \begin{cases} \text{Rindler}_2 \\ AdS_2 \end{cases} - R^2(r_+)\, d\Omega^2$$

$$\boxed{A = 4\pi R^2(r_+)}$$

$G = 1$

Schwarzschild $A = 4\pi R_S^2 = 16\pi M^2$

Reissner-Nordström $A = 4\pi r_+^2 = 4\pi\left(M + \sqrt{M^2 - q^2}\right)^2 \xrightarrow[\text{extremal limit}]{} 4\pi q^2$

Papapetrou-Majumdar $A = 4\pi \sum_i |q_i|^2$

The most interesting property of the event horizon is that, under some reasonable hypothesis the area of the intersection of the event horizon with spacelike surfaces always increases with time (Hawking (see Wald's book)).

B.1.4 Black-hole thermodynamics

We have found two properties of the event horizon of a stationary(*) black hole that resemble the 0^{th} and 2^{nd} laws of thermodynamics: the temperature is constant over a thermodynamical system in equilibrium and the entropy of such a system always grows spontaneously with time.

There are reasons to think that a black hole must have entropy (Bekenstein). Otherwise the total entropy of the Universe could be decreased by sending matter into the black hole. (Bekenstein also argued that there is a bound on the amount of entropy that a spherical region of radius R can contain $S \le \frac{2\pi k_B R E}{\hbar c}$)

One can also show that there is an analogue of the 1^{st} law of thermodynamics which holds for stationary black holes with M playing the role of energy (Bardeen, Carter, Hawking)

$$\boxed{\delta M = \underbrace{\frac{1}{8\pi G}}_{\frac{L^3T^{-2}}{M}} \underbrace{\kappa}_{LT^{-2}} \underbrace{\delta A}_{L^2} + \underbrace{\phi}_{\text{dimensionless}} \underbrace{\delta q}_{M}}$$

$$\phi(r_+) = \frac{1}{r_+}$$

The identification of κ with a temperature is problematic because, classically, black holes cannot radiate.

Quantum-mechanically they can, though (Hawking)

The temperature of the Hawking radiation is, precisely

$$T = \frac{1}{k_B}\frac{\hbar\kappa}{2\pi c} \Rightarrow \delta M c^2 = k_B \frac{2\pi c^3}{\hbar}\frac{1}{8\pi G} T\delta A + \phi\,\delta q\, c^2$$

$$\Rightarrow \boxed{S = k_B \frac{A}{4\,\hbar G/c^3} = k_B \frac{A/\ell_P^2}{4} = S_{BH}}$$

(*) Hawking's theorem is valid for more general black-hole spacetimes.

Finally there is an analogue of the third law: the surface gravity (temperature) of a black hole cannot be reduced to zero within a finite advanced time.

$\Rightarrow$ The extremal limit of the RN family of BHs cannot be reached from non-extremal BHs.

<u>Observations:</u> 1.- The specific heat of the Schwarzschild BH is negative $\Rightarrow$ instability

2.-