7.1 The Schwazschild solution

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The concept of black hole is the product of many years of research trying to understand the properties of the simplest solution of Einstein's equations, obtained by Karl Schwarzschild soon after the publication of those equations. (I recommend you to read Kip S. Thorne's book "Black holes and time warps" for an excellent account of the story of the concept of black hole).

Before studying the properties of the solution I would like to review the construction of the solution from my own point of view.

We are interested in finding the gravitational field outside a body, i.e. in vacuum $T_{\mu\nu} = 0$. For simplicity, as a first approximation to more realistic situations (planets, stars) we can assume the matter distribution of the body to be static and spherically symmetric in a coordinate system centered in the body. In absence of any other perturbation, the gravitational field outside the body will also be static and spherically-symmetric.

What do "static" and "spherically symmetric" actually mean?

<u>Static</u>: invariant in time ⇒ nothing changes when we go from t to $t+\delta t$ ⇒ nothing changes under the coordinate transformation $\boxed{\delta x^\mu = \delta^\mu{}_0\, \alpha}$ (α constant).

The PGC tells us that the laws of Physics must keep their form under arbitrary changes of coordinates <u>but</u> the values of the physical magnitudes (the gravitational field, for instance) do change in general : $g'_{\mu\nu} = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}$

$\Rightarrow g'_{\mu\nu} \neq g_{\mu\nu}$, or, infinitesimally $\begin{cases} \delta_\varepsilon x^\mu = \varepsilon^\mu(x) \\ \delta_\varepsilon g_{\mu\nu} = -2\nabla_{(\mu}\varepsilon_{\nu)} \neq 0 \end{cases}$

In some cases, a metric $g_{\mu\nu}$ is invariant under a particular reparametrization $\varepsilon^\mu(x)$: $\boxed{\delta_\varepsilon g_{\mu\nu} = -2\nabla_{(\mu}\varepsilon_{\nu)} = 0}$;
When this happens, we say that $\varepsilon^\mu(x)$ is an isometry of $g_{\mu\nu}(x)$.
If we write $\varepsilon^\mu(x) = \underbrace{\varepsilon}_{\text{infinitesimal}} k^\mu(x)$ ↑finite the vector field satisfies the equation

$$\boxed{\begin{array}{ll} \nabla_{(\mu}k_{\nu)} = 0 & : \text{Killing equation} \\ k^\mu(x) : & \text{Killing vector field} \end{array}}$$

Isometry $\Longleftrightarrow$ $\exists$ Killing vector field

Example: the Minkowski metric is invariant under the Poincaré group of transformations $x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$

Infinitesimally $\delta x^\mu = \underbrace{\sigma^\mu{}_\nu x^\nu + a^\mu}_{\text{Killing vector fields}}$. $\sigma_{(\mu\nu)} = 0$

The Killing equation in Minkowski spacetime in Cartesian coordinates is $\partial_{(\mu}\varepsilon_{\nu)} = 0$

$$\partial_{(\mu}\left[\sigma_{\nu)\rho}x^\rho + \cancel{a_{\nu)}}\right] = \sigma_{(\nu\mu)} = 0.$$

Using this language, a static metric is a metric that admits a timelike Killing vector field $k^\mu(x)$ $k^2 > 0$
Given a Killing vector field we can always find new coordinates x'^μ called adapted coordinates such that $k'^\mu(x') = \delta^\mu{}_t, \ \delta^\mu{}_x, \ \delta^\mu{}_v$ (timelike, spacelike, null)

In adapted coordinates, the Killing equation takes the form

$$k^\rho \partial_\rho g_{\mu\nu} - \cancel{\partial_\mu k^\rho}\, g_{\rho\nu} - \cancel{\partial_\nu k^\rho}\, g_{\rho\mu} = 0$$

$$\Rightarrow \partial_t g_{\mu\nu} = 0; \text{ or } \partial_x g_{\mu\nu} = 0; \text{ or } \partial_\nu g_{\mu\nu} = 0;$$

Thus, in adapted coordinates, a static metric is simply time-independent.

Spherically symmetric: this means a metric invariant under the group of rotations in 3 dimensions: $SO(3)$.

According to the general discussion, there will be one Killing vector field $k_i^\mu(x)$ for each of the independent infinitesimal rotations. The infinitesimal transformations satisfy the Lie algebra of $SO(3)$:

$$[\delta_i, \delta_j]\, g_{\mu\nu} = -\varepsilon_{ijk}\, \delta_k\, g_{\mu\nu};$$

$$\delta_i g_{\mu\nu} \equiv \delta_{k_i} g_{\mu\nu} = -2\nabla_{(\mu|} k_{i|\nu)}$$

Here $[\cdot,\cdot]$ is the standard commutator of two operators.
There is a fundamental property of infinitesimal g.c.t.'s:
If δ_ε, δ_η are two infinitesimal g.c.t.s generated by the vector fields $\varepsilon^\mu(x)$, $\eta^\mu(x)$, then

$$[\delta_\varepsilon, \delta_\eta] = \delta_{[\varepsilon,\eta]}$$

$\Rightarrow$ another infinitesimal g.c.t. generated by the vector

$$[\varepsilon,\eta]^\mu \equiv \varepsilon^\nu \partial_\nu \eta^\mu - \eta^\nu \partial_\nu \varepsilon^\mu \quad : \text{ Lie bracket}$$

If ε and η are Killing vectors of $g_{\mu\nu}$ $\delta_\varepsilon g_{\mu\nu} = \delta_\eta g_{\mu\nu} = 0$
then $[\varepsilon, \eta]$ is a Killing vector too.
$\Rightarrow$ The Killing vector fields of a metric generate a Lie algebra with the Lie bracket ($\neq$ commutator)
Thus, a spherically-symmetric metric will admit 3 Killing vector fields k_i^μ $i = 1,2,3$ such that

$$\boxed{\begin{aligned} [k_i, k_j] &= -\varepsilon_{ijk} k_k ; \\ &\shortparallel \\ k_i^\mu \partial_\mu k_j^\nu &- k_j^\mu \partial_\mu k_i^\nu \end{aligned}}$$

In a static, spherically symmetric metric, we also demand

$$\boxed{[k, k_i] = 0 ;} \qquad k^\mu \partial_\mu = \partial_t$$

How do we find metrics (not necessarily solutions of the E. eq.) with these properties?

i) We have to take into account this theorem: If ε, η are two Killing vector fields, we cannot find coordinates which are simultaneously adapted to ε and η.
$\Rightarrow$ At most we can use coordinates adapted to k (t) and to, say, k_3 (φ) $\Rightarrow$ metrics independent of t and φ.

ii) A t-independent metric can be written in the form

$$ds^2 = g_{tt} (dt + \omega_m dx^m)^2 - \gamma_{mn} dx^m dx^n$$

Rotations modify ω_m. We may compensate with a transformation involving t. But only if $\omega_m dx^m = df$.

$$\Rightarrow \omega_m = 0 \quad \Rightarrow \quad ds^2 = g_{tt} dt^2 - g_{mn} dx^m dx^n ;$$

Our problem now is to find suitable spherically-symmetric 3-dimensional metrics $g_{mn}\, dx^m\, dx^n$

We can choose spherical coordinates r, θ, φ, defined by the action of $SO(3)$ on them: r invariant and θ, φ not complicated (exercise) $k_3{}^\mu \partial_\mu = \partial_\varphi$. We know that the $\mathbb{E}^3$ metric induced on the round S^2 is invariant under $SO(3)$: $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$

$$\Rightarrow \quad g_{mn}\, dx^m\, dx^n = g_{rr}(r)\, dr^2 - R^2(r)\, d\Omega^2$$

(No crossed terms due to spherical invariance)

$\Rightarrow$ Static, spherically-symmetric metrics have the form

$$ds^2 = g_{tt}(r)\, dt^2 + g_{rr}(r)\, dr^2 - R^2(r)\, d\Omega^2;$$

3 independent functions of r, but we can use a g.c.t. to eliminate 1, in different ways

a) $$g_{rr}\left(\frac{dr}{dr'}\right)^2 = -\frac{R^2(r')}{r'^2} = g_{r'r'}(r')$$

$$\boxed{ds^2 = g_{tt}(r)\, dt^2 + g_{r'r'}\underbrace{\left[dr'^2 + r'^2\, d\Omega^2\right]}_{d\vec{x}^2}}$$

b) $R^2(r') = r'^2 \quad (\to r^2)$

$$\boxed{ds^2 = g_{tt}(r)\, dt^2 + g_{rr}(r)\, dr^2 - r^2\, d\Omega^2}$$

In this case the meaning of r is that the surfaces of $r =$ ct. have an area $= 4\pi r^2$ $\Rightarrow$ singular coordinates at $r = 0$

c) $g_{rr}(r')\left(\frac{dr}{dr'}\right)^2 = -\frac{1}{g_{tt}(r')}$; $(r' \to r)$

$$\Rightarrow \boxed{ds^2 = g_{tt}(r)\,dt^2 - \frac{1}{g_{tt}(r)}\,dr^2 - R^2(r)\,d\Omega^2}$$

In order to find static, spherically symmetric solutions we have to compute the curvature and the Ricci tensor because the equations are just $R_{\mu\nu} = 0$.
The form c) is the simplest: defining $g_{tt} = \lambda$; $\lambda' = \frac{d\lambda}{dr}$

$$\boxed{\begin{aligned} &R_{tt} = -\frac{\lambda}{2R^2}\left(R^2\lambda'\right)' ; \qquad R_{rr} = -\lambda^{-2}R_{tt} + 2\frac{R''}{R} ; \\ &R_{\theta\theta} = \frac{1}{2}\left[\lambda\left(R^2\right)'\right]' - 1 ; \qquad R_{\varphi\varphi} = \sin^2\theta\, R_{\theta\theta} ; \end{aligned}}$$

We will use this result later, for other solutions.

$$R_{rr} = -\lambda^{-2}R_{tt} + 2\frac{R''}{R} \underset{R_{tt}=0}{\overset{\uparrow}{=}} 2\frac{R''}{R} = 0 \Rightarrow R'' = 0$$

$\Rightarrow \boxed{R = ar + b}$ But we can set $a = 1$, $b = 0$ (asymptotic flatness)

$$R_{tt} = -\frac{\lambda}{r^2}\left(r^2\lambda'\right)' = 0 \Rightarrow r^2\lambda' = -a ;\quad \lambda' = \frac{a}{r^2} ;$$

$$\boxed{\lambda = b + \frac{a}{r}}$$

$$R_{\theta\theta} = \frac{1}{2}\left[\left(b + \frac{a}{r}\right)2r\right]' - 1 = b - 1 = 0 \Rightarrow b = 1$$

$$\Rightarrow \boxed{ds^2 = \left(1 + \frac{a}{r}\right)dt^2 - \left(1 + \frac{a}{r}\right)^{-1}dr^2 - r^2\,d\Omega^2}$$

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This is the Schwarzschild solution in the form (coordinates) proposed by Droste much later.

Properties :

0) This is the only spherically symmetric solution of $R_{\mu\nu}=0$ (static or not) : Birkhoff's theorem.

00) It is stable under all kind of small perturbations.

1) When $r \longrightarrow \infty$ the metric approaches Minkowski in spherical coordinates $\quad ds^2 = dt^2 - dr^2 - r^2 d\Omega^2$

2) When $r \to \infty$ the gravitational field is weak and we can compare with the Newtonian potential associated to a mass M

$$\left.\begin{array}{l} g_{tt} \sim \left(1 + \dfrac{2\phi}{c^2}\right) c^2 \\ \phi = -\dfrac{GM}{r} \end{array}\right\} \quad \frac{a}{r} = -\frac{2\,GM}{r} \; ; \quad \boxed{a = -2GM}\,(*)$$

3) If $M > 0$ $\quad \left.\begin{array}{l} g_{tt}\,(r = 2\,GM) = 0 \\ g_{rr}\,(r = 2\,GM) = \infty \end{array}\right\}$ The metric is singular

$$\boxed{2\,GM \equiv R_S \quad : \quad \text{the Schwarzschild radius}}$$

(Classically if the radius of a planet (or star) of mass M is R_S, then the speed of scape is c...
Observe that, when $M \sim M_{Planck}$
$$\Rightarrow \quad \lambda_{compton} = \frac{\hbar}{Mc} \sim R_S \sim l_{Planck}\ )$$

Then, the Schwarzschild metric in these coordinates is singular

(*) A discussion of the definition of M is necessary here...

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at $r=0$ (*) and at $r=R_S$ ($r=R_S$ only for $M>0$).
This metric is meant to be valid from $r=\infty$ to r_E, the star's radius which in general should be $r_E > R_S > 0$. At $r=r_E$ it should be glued to another solution describing the star's interior, such as the interior Schwarzschild solution. Then, the singularity at $r=R_S > r_E$ is irrelevant. However, after many years of discussions it seems that certain stars may contract to radii smaller than the R_S corresponding to its mass. What happens, then at R_S? Which kind of singularities are there at $r=R_S$ and at $r=0$?

(*) At $r=0$ $\quad g_{tt}=-\infty \; ; \; g_{rr}=g_{\theta\theta}=g_{\varphi\varphi}=0$

7.2 The singularity at the Schwarzschild radius

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Our goal now is to study the nature of the singularities at $r=0$ and $r=R_S$. $r=0$ looks like the Coulomb singularity or the Newton singularity and we are used to it. Those are physical singularities at which gauge-invariant quantities diverge

Coulomb: $F_{rt} \sim \frac{q}{r^2}$; $\underset{\text{Minkowski}}{F^2} = -2(F_{rt})^2 \sim -2\frac{q^2}{r^4}$

In GR, we should study scalars constructed from the Riemann tensor: curvature invariants. If they diverge, we are dealing with a curvature singularity, which is always physical (any coordinate system)

But we know that some singularities are just due to a bad choice of coordinates: coordinate singularities.

For instance, the Minkowski metric in spherical coordinates is apparently singular at $r=0$, but we know it is a regular metric (spacetime). The change of coordinates $(x,y,z) \to (r,\theta,\varphi)$ is not invertible at $r=0$. All the points $r=0,\theta,\varphi$ correspond to a single point $x=y=z=0$.

To determine if a singularity which is not a curvature singularity is only a coordinate singularity we must find another coordinate system in which the metric is regular. Often, the new coordinates cover a patch larger than the original coordinates and the metric is extended analytically.

However, even if the metric has no curvature singularities at a given point, there can be geodesics corresponding to the worldlines of particles which end there and cannot be extended. The spacetime would be geodesically incomplete, which is another way of being physically singular or pathological.

Summarizing: we have to study the curvature invariants of the Schwarzschild metric first. If they diverge at some value of r, there will be a physical singularity there and that will be all. If there is no divergence, we must try to extend the metric analytically and, then, study the geodesics.

7.2.1 Curvature invariants

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In $d=4$, the Riemann tensor has 14 independent quantities at each point (20 components in a local inertial frame − 6 Lorentz transformations). These can be encoded in scalar quantities built out of the Riemann tensor, the metric and the covariant derivatives of the Riemann tensor. Checking that all of them are regular at a given point can be hard.

In our case (Schwarzschild) the simplest invariants vanish identically everywhere $R_{\mu\nu}=0 \Rightarrow R=0;\ R_{\mu\nu}R^{\mu\nu}=0;$

However, these don't:

Kretschmann $\quad R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \dfrac{48M^2\cos^2\theta}{r^6} + \dots$

Chern $\quad R_{\mu\nu\rho\sigma}R^{\mu\nu}{}_{\alpha\beta}\,\varepsilon^{\rho\sigma\alpha\beta} \dots$

Euler $\quad R_{\mu\nu\rho\sigma}R_{\alpha\beta\gamma\delta}\,\varepsilon^{\mu\nu\alpha\beta}\varepsilon^{\rho\sigma\gamma\delta} \dots$

$\Rightarrow$ The singularity at $r=0$ is physical.

The singularity at $r=R_s$ is not a curvature singularity.

The singularity at $r=0$ will be present in any coordinate system. We must study other coordinates to decide about the nature of the singularity at $r=R_s$.

7.2.2 Alternative coordinate systems

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Let us consider lightlike radial geodesics parametrized by ξ such that $e(\xi) = 1$. $(c=1)$

Radial $\Rightarrow$ $\begin{cases} t = t(\xi); \\ r = r(\xi); \end{cases}$ θ, φ constant

lightlike $\Rightarrow$ $ds^2 = \left[\left(1-\frac{R_S}{r}\right)(\dot t)^2 - \left(1-\frac{R_S}{r}\right)^{-1}(\dot r)^2\right] d\xi^2 = 0$

Geodesic $\Rightarrow$ $\ddot t + \cancel{\Gamma_{tt}{}^{t}(\dot t)^2} + 2\Gamma_{tr}{}^{t}\,\dot t\,\dot r + \cancel{\Gamma_{rr}{}^{t}}(\dot r)^2 = 0$

$\ddot r + \Gamma_{tt}{}^{r}(\dot t)^2 + 2\cancel{\Gamma_{tr}{}^{r}}\,\dot t\,\dot r + \Gamma_{rr}{}^{r}(\dot r)^2 = 0$

(The other two equations are automatically satisfied)

$\Gamma_{tr}{}^{t} = \frac{1}{2}\lambda^{-1}\lambda'$; $\Gamma_{tt}{}^{r} = \frac{1}{2}\lambda\lambda'$; $\Gamma_{rr}{}^{r} = -\frac{1}{2}\lambda^{-1}\lambda'$;

$\ddot t + \lambda^{-1}\lambda'\,\dot r\,\dot t = 0$;

$\ddot r + \frac{1}{2}\lambda\lambda'(\dot t)^2 - \frac{1}{2}\lambda^{-1}\lambda'(\dot r)^2 = 0$; $\ddot r = 0$; $r = a\xi + b$

$\lambda(\dot t)^2 = \lambda^{-1}(\dot r)^2$;

$\frac{d}{d\xi}\dot t + \frac{d}{d\xi}\log\lambda\,\dot t = 0$; $\frac{1}{\dot t}\frac{d}{d\xi}\dot t + \frac{d}{d\xi}\log\lambda = 0$

$\frac{d}{d\xi}\left(\log\dot t + \log\lambda\right) = 0$; $\log\dot t = -\log(C\lambda)$

$\frac{dt}{d\xi} = \frac{1}{C\lambda}$; Observe that C must be positive $r > R_S$

$t = \int \frac{d\xi}{C\left(1-\frac{R_S}{a\xi}\right)} + D = \frac{1}{C}\left[\frac{R_S}{a}\log\left|\xi - \frac{R_S}{a}\right| + \xi + D\right]$

Now, let us choose some convenient values for a, C, D

$$ct = \frac{R_s}{a} \log \frac{\xi}{R_s/a}\left(1 - \frac{R_s/a}{\xi}\right) - \cancel{\frac{R_s}{a} \log \frac{1}{R_s/a}} + \xi + \cancel{D}$$

$D \to$ v or u

integration const.

$$\underset{\substack{\| \\ r}}{a\xi} = aCt - R_s \log \frac{r}{R_s} \lambda$$

$$-aCt + \underbrace{r + R_s \log \frac{r}{R_s} \lambda}_{r_*} = \begin{pmatrix} v \\ u \end{pmatrix}$$

r_* Tortoise coordinate

Incoming light rays $\Rightarrow \dot{r} < 0$ (towards smaller r)

$\Rightarrow a < 0$

$-aC = +1$ $\qquad \boxed{t + r_* = v}$

Outgoing light rays $\Rightarrow \dot{r} > 0 \Rightarrow a > 0$

$-aC = -1$ $\qquad \boxed{t - r_* = u}$

This is similar to Minkowski $\quad dt^2 - dr_*^2$

Actually, the main property of the tortoise coordinate is

$$\frac{dr}{dr_*} = \left(\frac{dr_*}{dr}\right)^{-1} = \lambda$$

and, using it, the Schwarzschild metric takes the form

$$ds^2 = \lambda\left(dt^2 - dr_*^2\right) - r^2\, d\Omega^2$$

conformal factor preserve angles $\Rightarrow$ light cones.

Now, the idea is to use v or u as coordinates

$\Rightarrow (v, r, \theta, \varphi)$ or (u, r, θ, φ)

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$$dt = dv - dr_* = dv - \frac{dr_*}{dr}dr = dv - \lambda^{-1}dr$$

$$\Rightarrow ds^2 = \lambda\left(dv - \lambda^{-1}dr\right)^2 - \lambda^{-1}dr^2 - r^2 d\Omega^2$$

$$\boxed{ds^2 = \lambda\, dv^2 - 2\,dv\,dr - r^2 d\Omega^2}$$

Schwarzschild metric in ingoing Eddington-Finkelstein coordinates ($u = t - r_* \rightarrow$ outgoing)

Observe that none of the metric components diverges at $r = R_S$

The g_{vv} component vanishes there, but the determinant of the metric does not.

The inverse metric does not diverge at $r = R_S$ either

$$(g_{\mu\nu}) = \left(\begin{array}{cc|cc} \lambda & -1 & & \\ -1 & 0 & & \\ \hline & & -r^2 & \\ & & & -r^2\sin^2\theta \end{array}\right); \quad g = -r^4\sin^2\theta;$$

$$(g^{\mu\nu}) = \left(\begin{array}{cc|cc} 0 & -1 & & \\ -1 & -\lambda & & \\ \hline & & -\frac{1}{r^2} & \\ & & & -\frac{1}{r^2\sin^2\theta} \end{array}\right); \Rightarrow$$

The metric is now extended to the $r \leq R_S$ region.

$\Rightarrow$ The singularity of the Schwarzschild metric in Droste coordinates may just be a coordinate singularity. We have to study the geodesics that go through that point.

To avoid possible problems we should use the E-F coordinates, which are regular at $r = R_S$.

Thus, let us consider again null radial geodesics $\theta = ct.$, $\varphi = ct.$

$$\Gamma_{vv}^{\ v} = \frac{1}{2}\lambda' ; \quad \Gamma_{vv}^{\ r} = \frac{1}{2}\lambda\lambda' ; \quad \Gamma_{rv}^{\ r} = -\frac{1}{2}\lambda' ;$$

$$\ddot{v} + \frac{1}{2}\lambda'(\dot{v})^2 = 0 ;$$
$$\ddot{r} + \frac{1}{2}\lambda'\dot{v}(\lambda\dot{v} - 2\dot{r}) = 0 ;$$
$\Big\}$ geodesic $\Rightarrow \boxed{\ddot{\imath} = 0}$

$\dot{\imath} = a$

$\dot{v}(\lambda\dot{v} - 2\dot{r}) = 0 ;$ ($\leftarrow$ lightlike)

$\downarrow$

$\dot{v} = 0 \Rightarrow t + r_* = ct \Rightarrow$ ingoing.

or $\lambda\dot{v} - 2\dot{r} = 0 \Rightarrow t - r_* = ct \Rightarrow$ outgoing

Let us examine closely the outgoing ones: $\dot{r} = a > 0$; $\dot{v} > 0$ and $\lambda\dot{v} = 2\dot{r}$ not consistent for $\lambda < 0 \rightarrow r < R_S$!

$\Rightarrow$ <u>There are no outgoing null geodesics in the $r < R_S$ region</u>

The same is true for non radial null geodesics. The hypersurface $r = R_S$ is a sort of frontier.

7.2.3 Observers in free fall

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We can now study what happens for timelike geodesics

We can work with the Schwarzschild-Droste coordinates

We send a master student of mass m that will send periodically light signals. Its trajectory satisfies (timelike)

$$\left(g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = c^2\right)m^2 \rightarrow g_{\mu\nu}p^\mu p^\nu = m^2c^2$$

In order to find geodesics we are going to use the following result: if the metric $g_{\mu\nu}$ admits a Killing vector field $k^\mu(x)$, then the action for a massive particle

$$S[x^\mu] = -mc\int d\xi\sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}$$

has a global symmetry under $\delta x^\mu = \varepsilon k^\mu(x)$ and the associated conserved quantity (momentum) is

$$\boxed{-mc\, k^\mu g_{\mu\nu}\frac{\dot{x}^\nu}{\sqrt{g\dot{x}\dot{x}}} \equiv P}$$

If we use the proper time $\boxed{P = -mc\, k^\mu g_{\mu\nu}\dot{x}^\nu}$

(Exercise: prove this result)

The Schwarzschild metric is static and spherically-symmetric and, therefore, the motion of a massive test particle in that spacetime has 4 conserved quantities that can be used to integrate the equations of motion: the energy E and the 3 components of angular momentum l^i ($\rightarrow$ only one relevant in the end)

If we only consider radially-directed geodesics, the angular momentum will be zero, but we can still use the energy

$$E = mc\, k^\mu g_{\mu\nu} \dot{x}^\nu \overset{\text{Dirac}}{=} mc\, \delta^\mu{}_0\, g_{\mu\nu} \dot{x}^\nu = mc\, g_{00}\, c\dot{t}$$

$$= mc^2 \lambda \dot{t}\,; \qquad \dot{t} = \frac{E}{mc^2} \lambda^{-1}\,;$$

From the mass shell condition $\quad g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 1$

$$g_{00}\, c^2 (\dot{t})^2 + g_{rr} (\dot{r})^2 = 1\,;$$

$$\lambda c^2 \left(\frac{E}{mc^2} \lambda^{-1} \right)^2 - \lambda^{-1} (\dot{r})^2 = 1$$

$$\left(\frac{dr}{c\, ds} \right)^2 = \left(\frac{E}{mc^2} \right)^2 - \lambda\,;$$

$$ds = \frac{1}{c} \int_{r_1}^{r_2} \frac{dr}{\sqrt{\frac{E}{mc^2} - \lambda}} = \frac{1}{c} \int_{r_1}^{r_2} \frac{dr}{\sqrt{\frac{R_S}{r} - \frac{R_S}{R_0}}}$$

$$R_0 \equiv \frac{R_S}{1 - (E/mc^2)^2} \quad : \text{value of } r \text{ for which } \dot{r} = 0$$

We can use this formula to compute the proper time that it takes for the observer to go from $r_1 = R_0 > R_S$ to $r = 0$ crossing the surface $r = R_S$

$$\Delta s = \frac{\pi}{2c} R_0 \left(\frac{R_0}{R_S} \right)^{1/2} \quad : \text{finite!}$$

The relation between the time measured by the master student in free fall and the time measured by his supervisor is given by $\dot{t} = \dfrac{dt}{ds}$

$$\frac{dt}{ds} = E/mc^2\ \lambda^{-1} \xrightarrow[r \to R_S]{} \infty \quad + \text{ infinite redshift}$$

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His/her supervisor never observes the crossing of the $r = R_S$ surface.
This disagreement between the two observers cannot be reconciled. We can only argue that they will never be able to compare them, as we are going to see.
We can study more geodesics, but there is something better: the maximal analytical extension of the Schwarzschild solution provided by the Kruskal-Szekeres coordinates.

7.2.4 The Kruskal-Szekeres maximal analytical extension

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First, let us rewrite the Schwarzschild-Droste solution in a different form (same coordinates, though):

$$ds^2 = \lambda\, dt^2 - \lambda^{-1} dr^2 - r^2 d\Omega^2 =$$

$$= \lambda\left[dt^2 - \lambda^{-2} dr^2\right] - r^2 d\Omega^2 =$$

$$= \lambda \underbrace{\left(dt + \lambda^{-1} dr\right)}_{\equiv\, dv} \underbrace{\left(dt - \lambda^{-1} dr\right)}_{\equiv\, du} - r^2 d\Omega^2$$

(possible because $d^2 u = 0$ and $d^2 v = 0$)

$\lambda^{-1} dr = dr_*$, the tortoise coordinate

$v = t + r_*$ → the Eddington-Finkelstein ingoing coordinate

$u = t - r_*$; outgoing.

In $ds^2 = \lambda\, dv\, du - r^2 d\Omega^2$, r is to be regarded as a function of v and u.

Now we perform a second change of coordinates

$$U \equiv -e^{-u/2R_S}, \quad V \equiv e^{v/2R_S}; \qquad r \to r(U,V)$$

$$dU = -\frac{1}{2R_S} U\, du; \quad dV = \frac{1}{2R_S} V\, dv;$$

$$dU\, dV = -\frac{1}{4R_S^2} UV\, du\, dv = +\frac{1}{4R_S^2} e^{\frac{v-u}{2R_S}} du\, dv$$

$$= \frac{1}{4R_S^2} e^{r_*/R_S} du\, dv = \frac{1}{4R_S^2} e^{\frac{r}{R_S}} e^{\log\left|\frac{1}{R_S}\lambda\right|} du\, dv$$

$$= \frac{1}{4R_S^2} \left|\frac{r}{R_S}\lambda\right| e^{\frac{r}{R_S}} du\, dv$$

$$ds^2 = 4R_S^2 \frac{\lambda}{\left|\frac{r}{R_S}\lambda\right|} e^{-r/R_S} dU\, dV - r^2 d\Omega^2;$$

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$$ds^2 = \frac{4R_s^3}{r}\,\underbrace{\mathrm{sign}(r-R_s)}_{\uparrow}\, e^{-r/R_s}\, dU\,dV - r^2\, d\Omega^2 ;$$

Ahora, if we want to define standard $T\,X$ coordinates, we have to take into account that sign

$$\mathrm{sign}(r-R_s)\, dU\, dV = dT^2 - dX^2$$

$$\Rightarrow \boxed{ds^2 = \frac{4R_s^3}{r} e^{-r/R_s}\left[dT^2 - dX^2\right] - r^2\, d\Omega^2 ;} \quad r = r(T,X)$$

In these coordinates, the light cones are $\times$ as in Minkowski spacetime. We are going to see, through the explicit formulae of change of coordinates that they cover the $r>R_s$ and $r<R_s$ regions and <u>more</u>.

$$T^2 - X^2 = \mathrm{sign}(r-R_s)\, UV = -\mathrm{sign}(r-R_s)\, e^{\frac{v-u}{2R_s}} =$$

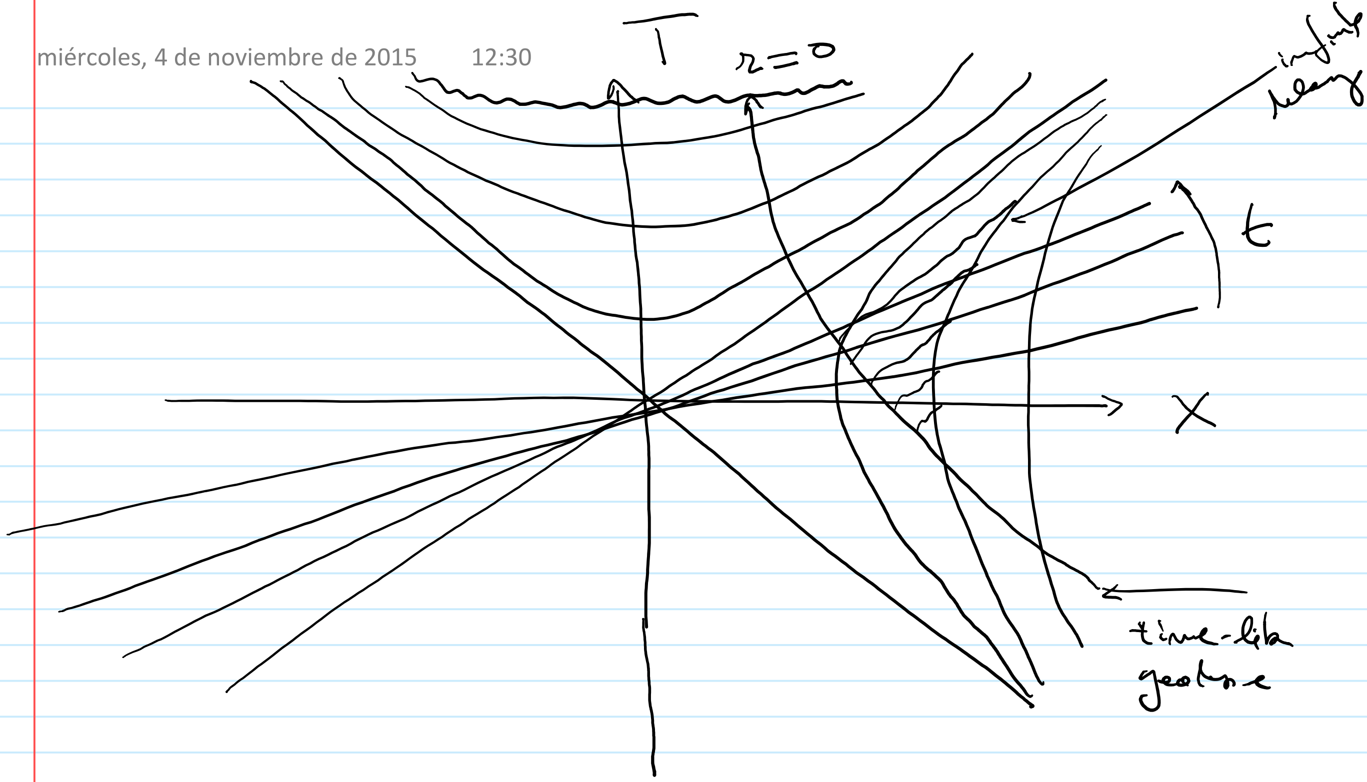
$$= -\mathrm{sign}(r-R_s)\, e^{r_*/R_s} = -\mathrm{sign}(r-R_s)\, \exp\left\{\frac{r}{R_s} + \log\frac{r}{R_s}|\lambda|\right\}$$

$$= -\mathrm{sign}(r-R_s)\, e^{r/R_s}\, \frac{\cancel{r}}{R_s}\, \left|\frac{r-R_s}{\cancel{r}}\right| = \left(1 - \frac{r}{R_s}\right) e^{r/R_s} ;$$

For $r > R_s$ $\quad T/X = \dfrac{V+U}{V-U} = \dfrac{e^{\frac{u+v}{2R_s}} - 1}{e^{\frac{u+v}{2R_s}} + 1} = \tanh\dfrac{t}{2R_s}$

$r < R_s$ $\quad T/X = \dfrac{V+U}{V-U} = \coth\dfrac{t}{2R_s}$

$$\Rightarrow \boxed{\begin{array}{ll} r>R_s : & T = \sqrt{\frac{r}{R_s}-1}\; e^{r/2R_s} \sinh\frac{t}{2R_s} ;\quad X = \sqrt{\frac{r}{R_s}-1}\; e^{r/2R_s} \cosh\frac{t}{2R_s} ; \\ r<R_s : & T = \sqrt{1-\frac{r}{R_s}}\; e^{r/2R_s} \cosh\frac{t}{2R_s} ;\quad X = \sqrt{1-\frac{r}{R_s}}\; e^{r/2R_s} \sinh\frac{t}{2R_s} \end{array}}$$



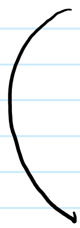
i) Lines of constant t: straight lines through the origin with some given slope:

$$T/X = \Theta(r-R_s)\tanh\frac{t}{2R_s} + \Theta(R_s-r)\coth\frac{t}{2R_s}$$

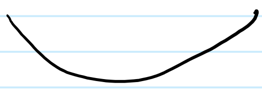
ii) Lines of constant r: hyperbolas

$$T^2-X^2 = \left(1-\frac{r}{R_s}\right)e^{r/R_s}$$

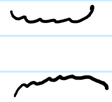
$r > R_s$:



$r < R_s$:



iii) $r=0$: 2 hyperbolas $T^2-X^2=1$



iv) $r=R_s$: 2 straight lines $T^2-X^2=0$: event horizons : BH

v) Extra asymptotically-flat region + "white hole"

→ Because this is an "eternal black hole", unphysical.

vi) The causal roles of r and t are reversed in the interior

vii) Physical BH: only to the right of the geodesic.

7.3 Carter-Penrose diagrams

miércoles, 16 de septiembre de 2015 16:26

These diagrams represent the complete spacetime using coordinates with a finite range so "infinity" and its properties can be visualised.

First, let us consider Minkowski spacetime in spherical coordinates

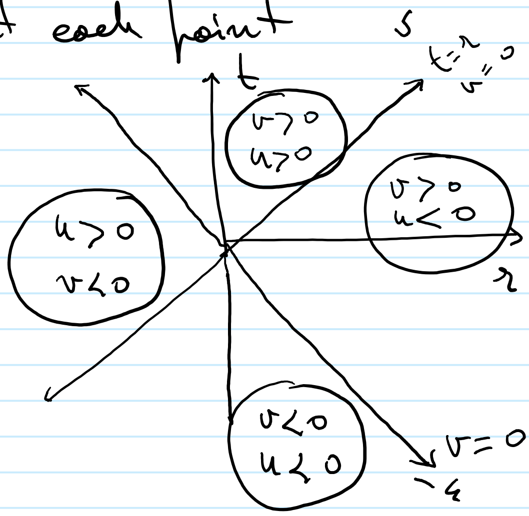
$$ds^2 = dt^2 - dr^2 - r^2 d\Omega^2;$$

$$\left.\begin{array}{l} v = t+r \\ u = t-r \end{array}\right\} = du\, dv - r^2 d\Omega^2;$$

We can represent it as the right half of the plane with a 2-sphere of radius r at each point

$t = r + u;$

$t = -r + v;$

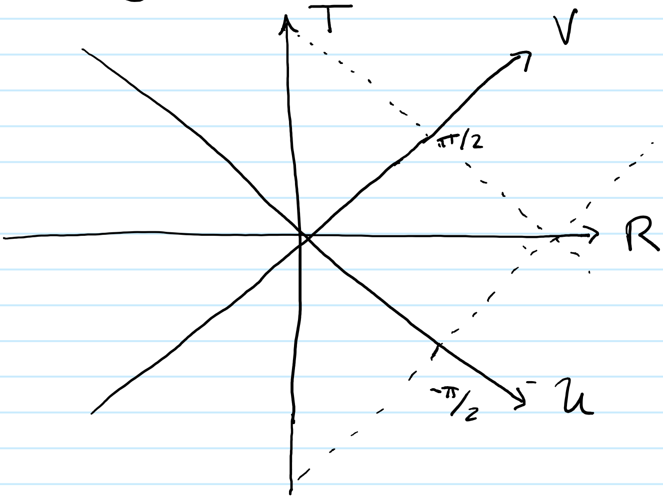


u and v take values between $-\infty$ and $+\infty$ and we can compactify them with the change of variables

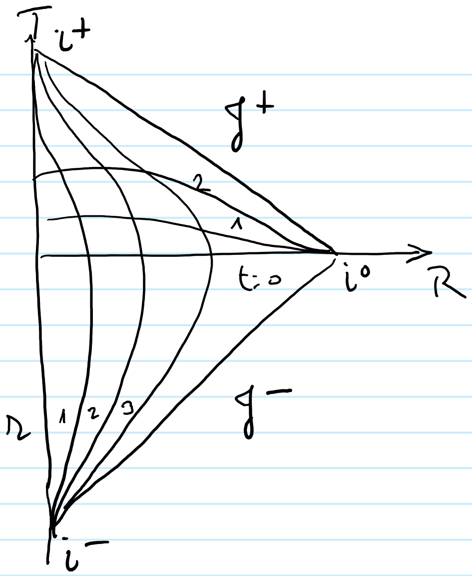
$$\left.\begin{array}{l} v \equiv \operatorname{tg} V \\ u \equiv \operatorname{tg} \mathcal{U} \end{array}\right\} \Rightarrow \mathcal{U}, V \in \left(-\pi/2, +\pi/2\right)$$

$T = V + \mathcal{U}$

$R = V - \mathcal{U}$



The whole Minkowski spacetime is in the interior of the triangle whose boundary is "infinity".
Light rays propagate at 45°



(There is a sphere at every point)

$\mathcal{J}^+$: future null infinity: all outgoing light rays go there $v=\infty, u<\infty$

$\mathcal{J}^-$: past null infinity: all incoming light rays come from there $u=-\infty, v<\infty$

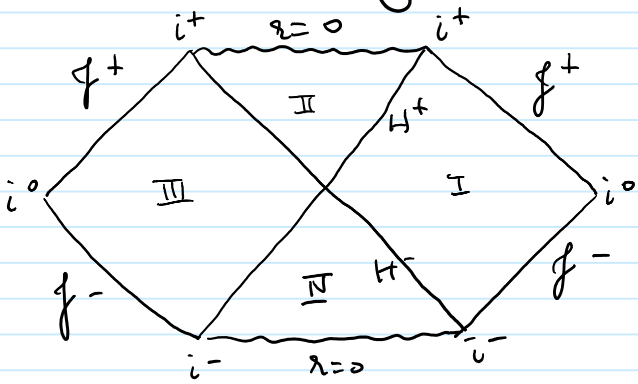
i^0 : spacelike infinity: all spacelike trajectories end there $r=\infty, t<\infty$

i^+ : future timelike infinity: all timelike trajectories end there, $t=\infty, r<\infty$

i^- : past timelike infinity: all timelike trajectories start there, $t=-\infty, r<\infty$

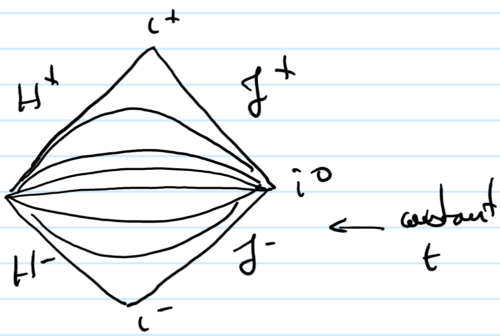
They are spheres at infinity!

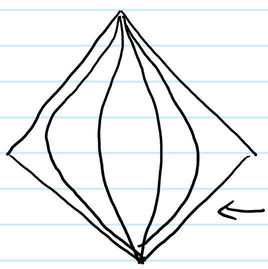
Doing the same for the Schwarzschild solution in KS coordinates we get the Penrose diagram



H^+ : future horizon
H^- : past horizon

In I

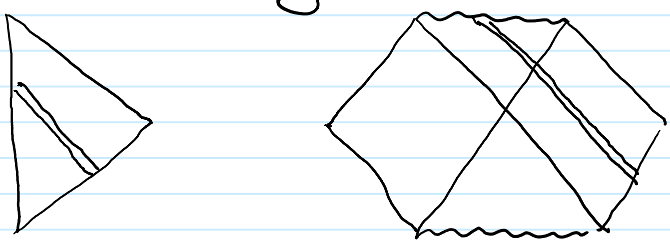




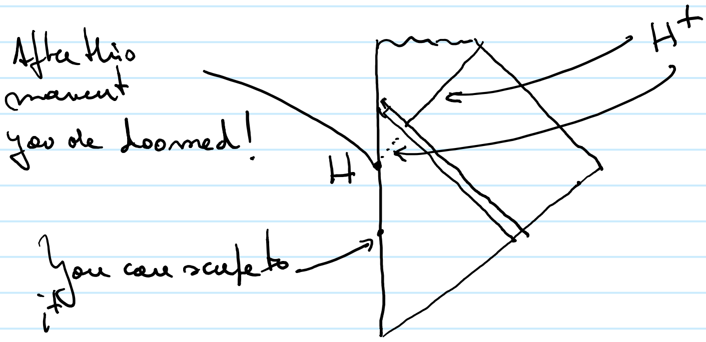
Using the Penrose diagrams of Minkowski and Schwarzschild spacetimes we can construct the Penrose diagram describing the simplest, very idealised, model of gravitational collapse and formation of a black hole: collapse of a thin, spherically symmetric shell of null dust (some kind of massless particles with no pressure).

Before the arrival of this shell the spacetime is described by Minkowski spacetime and, after its arrival, by the Schwarzschild spacetime (Birkhoff's theorem).

If the dust did not gravitate, its radial motion in Minkowski and Schwarzschild spacetime could be described by these Carter-Penrose diagrams:



However, if the shell has sufficient energy, it will give rise to a Schwarzschild BH and the upper part of the diagram on the left must be replaced by that of the diagram on the right:



The event horizon must have formed before the arrival of the shell: it is a global concept. We may be inside an event horizon (i.e. a black hole) and not be aware of it at all.

7.4 The Schwarzschild wormhole (a.k.a. Einstein-Rosen bridge)

Let us consider the $T=0$ slice of the Schwarzschild metric in Kruskal-Szekeres coordinates. The induced metric is

$$-ds^2 = \frac{4R_S^3}{r} e^{-r/R_S} dX^2 + r^2 d\Omega^2;$$

From $T^2 - X^2 = \left(1-\frac{r}{R_S}\right) e^{r/R_S} \Rightarrow X = \sqrt{\frac{r}{R_S}-1}\; e^{r/2R_S}$

$$dX = \left\{ \frac{1}{2\sqrt{\frac{r}{R_S}-1}} \frac{1}{R_S} e^{r/2R_S} + \sqrt{}\; \frac{1}{2R_S} e^{r/2R_S} \right\} dr$$

$$= \frac{1}{2R_S} \frac{1}{\sqrt{r/R_S-1}} e^{r/2R_S} \left(\cancel{1} + \frac{r}{R_S} \cancel{-1}\right) dr$$

$$= \frac{r}{2R_S^2} \frac{1}{\sqrt{r/R_S-1}} e^{r/2R_S} dr;$$

$$-ds^2 = \frac{4R_S^3}{r} e^{-r/R_S} \frac{r^2}{4R_S^4} \frac{1}{r/R_S-1} e^{r/R_S} dr^2 + r^2 d\Omega^2;$$

$$= \frac{r}{R_S} \frac{1}{\frac{r}{R_S}-1} dr^2 + r^2 d\Omega^2 = \frac{1}{1-\frac{R_S}{r}} dr^2 + r^2 d\Omega^2;$$

We can visualise this metric in this form: let us consider the surface $z^2 = 4R_S(r-R_S)$ in $\mathbb{E}^4$ with spherical coordinates

$$-ds^2 = dz^2 + dr^2 + r^2 d\Omega^2;$$

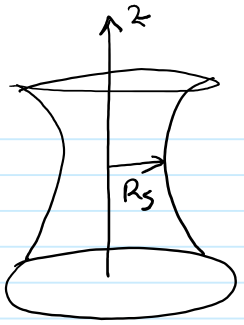
$$2z\,dz = 4R_S\,dr; \qquad dz = 2R_S \frac{dr}{z} = \frac{\cancel{2}R_S}{\cancel{2}R_S^{1/2}(r-R_S)^{1/2}} dr$$

$$= \frac{R_S^{1/2}}{(r-R_S)^{1/2}} dr = \frac{1}{\left(\frac{r}{R_S}-1\right)^{1/2}} dr$$

$$-ds^2 = \left(\frac{1}{\frac{r}{R_S}-1} + 1\right) dr^2 + r^2 d\Omega^2 = \frac{r/R_S}{\frac{r}{R_S}-1} dr^2 + r^2 d\Omega^2$$

$$= \frac{1}{1-\frac{R_S}{r}} dr + r^2 d\Omega^2;$$

The surface is



(actually, instead of a circle, we have a 2-sphere of radius r at each value of z).

($r = R_s$ coordinate singularity)

and it is completely regular. $r \to \infty$ when $z \to \pm\infty$ and this corresponds to the two asymptotic regions (I, III in KS spacetime) which are "connected" by this "wormhole".
However, as the Carter-Penrose diagram shows you cannot go from I to III. In II (the BH interior) you can receive signals both from I and III but you cannot send your signals to those regions.
This is a non-traversable wormhole. It looks traversable because we are not taking time into account.
We could study the evolution of $T = \text{constant} \equiv T_0$ slices of the KS spacetime, which are such that

$$X = \pm \left\{ T_0^2 - \left(1 - \frac{r}{R_S}\right) e^{r/R_S} \right\}^{1/2};$$

$$ds^2 = \frac{1}{1 - \frac{R_S}{r}\left(1 - T_0^2 e^{-r/R_S}\right)} dr^2 + r^2 d\Omega^2;$$

The width of the "throat" of the wormhole is the value of r for which $g_{rr} \to \infty$

$r = R_S$ for $V_0 = 0$ and decreasing until $r = 0$ for $V_0 = 1$: the physical singularity. The wormhole is pinched off there.