4.0 Differential forms and integration

miércoles, 30 de septiembre de 2015 16:07

In what follows we need more precise definitions of integration. We start by stating that the only objects that can be integrated over a d-dimensional space (-time) are p-forms

A p-form, or differential form of rank p is a tensor of rank (0, p) completely antisymmetric in all its indices. A special notation is used with them:

exterior product of forms

1-forms $\omega^{(1)} = \omega^{(1)}_\mu dx^\mu$;

2-forms $\omega^{(2)} = \frac{1}{2}\omega^{(2)}_{\mu\nu}\, dx^\mu \wedge dx^\nu$

3-forms $\omega^{(3)} = \frac{1}{3!}\omega^{(3)}{}_{\mu\nu\rho}\, dx^\mu \wedge dx^\nu \wedge dx^\rho$

p-forms $\omega^{(p)} = \frac{1}{p!}\omega^{(p)}{}_{\mu_1\cdots\mu_p}\, dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}$

In a spacetime of dimension d; $p \le d$

For instance $d=2$

$$\omega^{(1)} = \omega^{(1)}_1 dx^1 + \omega^{(2)}{}_2 dx^2;$$

$$\omega^{(2)} = \frac{1}{2}\omega^{(2)}{}_{12}\, dx^1\wedge dx^2 + \frac{1}{2}\underset{-\omega_{12}}{\underbrace{\omega^{(2)}{}_{21}}}\,\underset{-dx^1\wedge dx^2}{\underbrace{dx^2\wedge dx^1}} = \omega^{(2)}_{12}\, dx^1\wedge dx^2$$

The components of $\omega^{(3)}$ must be repeated: ω_{111}, ω_{122}, ω_{121} ... and they must vanish due to antisymmetry.

The linear combination of differential forms of the same rank with functions as coefficients gives a differential form of the same rank:

$$f\eta^{(p)} + g\xi^{(p)} \rightarrow p\text{-form}$$

Functions can be understood as "0-forms".

4.0.1 Exterior product and exterior derivative

miércoles, 30 de septiembre de 2015 16:41

The exterior product of a p- and a q-form is a $(p+q)$-form defined by

$$\omega^{(p)} \wedge \omega^{(q)} = \frac{1}{p!\,q!}\, \omega^{(p)}_{\mu_1 \dots \mu_p}\, \omega^{(q)}_{\nu_1 \dots \nu_q}\, dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} \wedge dx^{\nu_1} \wedge \dots \wedge dx^{\nu_q}$$

$$= \frac{1}{(p+q)!} \left[\frac{(p+q)!}{p!\,q!}\, \omega^{(p)}_{[\mu_1 \dots \mu_p}\, \omega^{(q)}_{\nu_1 \dots \nu_q]} \right] dx^{\mu_1} \wedge \dots \wedge dx^{\nu_q}$$

For example, 2 1-forms $\eta \wedge \xi = \frac{1}{2}\left(2\eta_{[\mu}\xi_{\nu]}\right) dx^\mu \wedge dx^\nu$

The exterior product of a function ("0-form") by a p-form is equal to the standard product.

Observe that $\omega^{(p)} \wedge \omega^{(q)} = (-1)^{pq}\, \omega^{(q)} \wedge \omega^{(p)}$

The exterior derivative of a p-form is a $(p+1)$-form given by

$$d\,\omega^{(p)} = \frac{1}{p!}\, \partial_\mu\, \omega^{(p)}_{\nu_1 \dots \nu_p}\, dx^\mu \wedge dx^{\nu_1} \wedge \dots \wedge dx^{\nu_p}$$

$$= \frac{1}{(p+1)!} \left[(p+1)\, \partial_{[\mu_1} \omega^{(p)}_{\mu_2 \dots \mu_{p+1}]} \right] dx^{\mu_1} \wedge \dots \wedge dx^{\mu_{p+1}}$$

Observe that $\partial_{[\mu_1} \omega_{\mu_2 \dots \mu_{p+1}]} = \nabla_{[\mu_1} \omega_{\mu_2 \dots \mu_{p+1}]}$ because $\Gamma^\rho_{[\mu\nu]} = 0$. This is why $d\omega^{(p)}$ transforms as a $(p+1)$-form. The exterior derivative of forms is a "covariant derivative".

Observe that the exterior derivative of a 0-form (function) f is the 1-form $df = \partial_\mu f\, dx^\mu$.

Properties:
1) $d(\omega + \eta) = d\omega + d\eta$
2) $d\left(\omega^{(p)} \wedge \omega^{(q)}\right) = \left(d\omega^{(p)}\right) \wedge \omega^{(q)} + (-1)^p\, \omega^{(p)} \wedge d\omega^{(q)}$
3) $\boxed{d^2 = 0}$ - fundamental

4.0.2 Volume forms and integration

miércoles, 30 de septiembre de 2015 17:15

In d-dimensions d-forms are called "volume forms" or "top forms" and they are the objects that can be integrated there:

$$\int_{M_d} \omega^{(d)} = \int_{Md} \omega_{12\dots d} \overbrace{\underbrace{dx^1 \wedge \dots \wedge dx^d}_{d^d x}}^{\text{given order} = \underline{\text{orientation}}} = \int_{Md} d^d x \;\omega_{12\dots d}$$

Observe that $\omega_{12\dots d}$, which is just one component of $\omega^{(d)}$ is not a function because it transforms under g.c.t.'s

$$\omega'_{12\dots d} = \frac{\partial x^{\mu_1}}{\partial x'^1} \cdots \frac{\partial x^{\mu_d}}{\partial x'^d} \omega_{\mu_1 \dots \mu_d}$$

$$= \det\left(\frac{\partial x}{\partial x'}\right) \omega_{1\dots d} \, ;$$

$$d^d x' = dx'^1 \wedge \dots \wedge dx'^d = \frac{\partial x'^1}{\partial x^{\mu_1}} \cdots \frac{\partial x'^d}{\partial x^{\mu_d}} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_d}$$

$$= \det\left(\frac{\partial x'}{\partial x}\right) d^d x$$

$\Rightarrow$ the value of the integral does not depend on the coordinates chosen.

If we have a metric $g_{\mu\nu}$ we can measure the volume of the spacetime M_d by integrating <u>the</u> volume form.

To define this volume form we use the Levi-Civita symbol

$$\varepsilon^{\mu_1 \dots \mu_d} = \varepsilon^{[\mu_1 \dots \mu_d]} \, ; \quad \varepsilon^{01\dots d-1} = +1$$

<u>in any coordinate system</u>

(There are other definitions which differ by powers of $\sqrt{|g|}$)

Then, $\varepsilon_{\mu_1 \dots \mu_d} \equiv g_{\mu_1 \nu_1} \cdots g_{\mu_d \nu_d} \varepsilon^{\nu_1 \dots \nu_d}$

$$\varepsilon_{1 \dots d} = g\, \varepsilon^{1 \dots d} = g;$$

Observe that for Lorentzian metrics sign $\det g = (-1)^{d-1}$

Observe that $\varepsilon^{\mu_1 \dots \mu_d}$ and $\varepsilon_{\mu_1 \dots \mu_d}$ are not tensors, but tensor densities:

$$\varepsilon'^{\mu_1 \dots \mu_d} = \underbrace{\left(\det \frac{\partial x'}{\partial x}\right)^{-1}} \frac{\partial x'^{\mu_1}}{\partial x^{\nu_1}} \cdots \frac{\partial x'^{\mu_d}}{\partial x^{\nu_d}} \varepsilon^{\nu_1 \dots \nu_d}$$

Since $g' = \left(\frac{\partial x}{\partial x'}\right)^2 g$; $\sqrt{|g'|} = \left|\frac{\partial x}{\partial x'}\right| \sqrt{|g|}$

$\frac{\varepsilon^{\mu_1 \dots \mu_d}}{\sqrt{|g|}}$ <u>almost</u> transforms as a tensor: if

$$\det \frac{\partial x'}{\partial x} = -1 \qquad \frac{\varepsilon'^{\mu_1 \dots \mu_d}}{\sqrt{|g'|}} = -\frac{\varepsilon^{\mu_1 \dots \mu_d}}{\sqrt{|g|}}$$

The g.c.t.'s with this property are said to <u>change the orientation</u> of the spacetime which is defined by $\varepsilon^{1 \dots d} = +1$ ($1 \leftrightarrow 2$, $1 \to -2 \dots$)

$\frac{\varepsilon^{\mu_1 \dots \mu_d}}{\sqrt{|g|}}$ is called a <u>pseudotensor</u> (there are pseudoscalars, pseudovectors etc.)

sign$(g) \frac{\varepsilon_{\mu_1 \dots \mu_d}}{\sqrt{|g|}}$ is the volume form (except under orientation-changing g.c.t.'s)

$$\boxed{\text{sign}(g) \frac{1}{d!} \frac{\varepsilon_{\mu_1 \dots \mu_d}}{\sqrt{|g|}} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_d}}$$

$$= \mathrm{sign}(g) \frac{\varepsilon_{1\ldots d}}{\sqrt{|g|}} dx^1 \wedge \cdots \wedge dx^d = \mathrm{sign}(g) \frac{g}{\sqrt{|g|}} d^d x$$

$$= \frac{|g|}{\sqrt{|g|}} d^d x = \boxed{d^d x \sqrt{|g|}}$$

$$\int_{M_d} d^d x \sqrt{|g|} \equiv \text{volume of } M_d = \int_{M_d} d^d x' \sqrt{|g'|}$$

How do we "integrate a function" over M_d?
We integrate the top form that we obtain multiplying the function $f(x)$ by the volume form:

$$\int_{M_d} d^d x' \sqrt{|g'|} f'(x') = \int_{M_d} d^d x \sqrt{|g|} f(x)$$

4.0.2 The Hodge dual

miércoles, 30 de septiembre de 2015 17:25

The Hodge dual of a p-form ω in d dimensions is a $(d-p)$-form denoted by $*\omega$ whose components are obtained from those of ω according to the rule

$$(*\omega)_{\mu_1 \cdots \mu_{d-p}} \equiv \frac{1}{p!} g_{\mu_1\nu_1} \cdots g_{\mu_{d-p}\nu_{d-p}} \frac{\varepsilon^{\nu_1 \cdots \nu_{d-p}\sigma_1 \cdots \sigma_p}}{\sqrt{|g|}} \omega_{\sigma_1 \cdots \sigma_p}$$

a metric is necessary to define $*\omega$

$$(*^2\omega)_{\mu_1 \cdots \mu_p} = \frac{1}{(d-p)!} \frac{\varepsilon_{\mu_1 \cdots \mu_p \nu_1 \cdots \nu_{d-p}}}{\sqrt{|g|}} \frac{1}{p!} \frac{\varepsilon^{\nu_1 \cdots \nu_{d-p}\rho_1 \cdots \rho_p}}{\sqrt{|g|}} \omega_{\rho_1 \cdots \rho_p}$$

$$= \frac{1}{p!(d-p)!} \frac{1}{|g|} (-1)^{p(d-p)} \underbrace{\varepsilon_{\mu_1 \cdots \mu_p \nu_1 \cdots \nu_{d-p}} \varepsilon^{\rho_1 \cdots \rho_p \nu_1 \cdots \nu_{d-p}}}_{p!(d-p)!\, g\, \delta^{[\rho_1}_{\mu_1} \cdots \delta^{\rho_p]}_{\mu_p}} \omega_{\rho_1 \cdots \rho_p}$$

$$= (-1)^{p(d-p)} \operatorname{sign}(g)\, \omega_{\mu_1 \cdots \mu_p}$$

From a function, we get a top form:

$$(*f)_{\mu_1 \cdots \mu_d} = \frac{\varepsilon_{\mu_1 \cdots \mu_d} f}{\sqrt{|g|}} \Rightarrow *f = \operatorname{sign}(g)\, d^dx \sqrt{|g|}\, f$$

$$\boxed{\int d^dx \sqrt{|g|}\, f = \operatorname{sign}(g) \int *f}$$

Observe that $\left(dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_d} = \omega\, \varepsilon^{\mu_1 \cdots \mu_d}\right) \frac{\varepsilon_{\mu_1 \cdots \mu_d}}{d! \sqrt{|g|}}$

$$\operatorname{sign}(g)\, d^dx \sqrt{|g|} = \omega \frac{g}{\sqrt{|g|}} \Rightarrow \omega = d^dx$$

$$\Rightarrow \boxed{dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_d} = d^dx\, \varepsilon^{\mu_1 \cdots \mu_d}}$$

4.0.4 Stokes' theorem

miércoles, 30 de septiembre de 2015 17:26

If $\omega^{(d-1)}$ is a $(d-1)$ form in a d-dimensional space M_d whose boundary is ∂M_d (always $(d-1)$-dimensional) then

$$\boxed{\int_{M_d} d\,\omega^{(d-1)} = \int_{\partial M_d} \omega^{(d-1)}}$$

Let's see how this works:

i) If $\omega^{(d-1)} = d\,\omega^{(d-2)}$, we can apply the theorem again!

$$0 = \int_{M_d} \underbrace{d^2}_{\overset{\shortparallel}{0}} \omega^{(d-2)} = \int_{\partial M_d} d\,\omega^{(d-1)} = \int_{\underset{\shortparallel\ \emptyset}{\partial^2 M_d}} \omega^{(d-1)} = 0$$

$\Rightarrow$ The theorem is consistent.

ii) For $d=1$, $M_1 = [x_0, x_1]$; $\partial M_d = \{x_0, x_1\}$

$$\int_{x_0}^{x_1} df = f(x_1) - f(x_0) \Rightarrow \text{Barrow's rule}$$

iii) For general d

$$\int_{M_d} d\,\omega^{(d-1)} = \int \frac{1}{(d-1)!} \partial_{[\mu_1} \omega^{(d-1)}_{\mu_2 \dots \mu_d]} \, dx^{\mu_1} \wedge \dots \wedge dx^{\mu_d} =$$

$$= \int d^d x \, \frac{1}{(d-1)!} \varepsilon^{\mu_1 \dots \mu_d} \partial_{\mu_1} \omega^{(d-1)}_{\mu_2 \dots \mu_d} = \int d^d x \sqrt{|g|} \frac{1}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|} * \omega^\mu\right)$$

$$= \int d^d x \sqrt{|g|} \, \nabla_\mu \left(* \omega^\mu \right);$$

$$\int_{\partial M_d} \omega^{(d-1)} = \int \frac{1}{(d-1)!} \omega_{\mu_1 \dots \mu_{d-1}} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_{d-1}} =$$

$$= \operatorname{sign}(g) \, (-1)^{d-1} \int \frac{1}{(d-1)} \frac{\varepsilon_{\mu_1 \dots \mu_{d-1} \sigma}}{\sqrt{|g|}} (*\omega)^\sigma \, dx^{\mu_1} \wedge \dots dx^{\mu_{d-1}}$$

$$= (-1)^{d-1} \, \mathrm{sign}(g) \int \underbrace{dx^{\mu_1} \wedge \dots \wedge dx^{\mu_{d-1}} \frac{\varepsilon_{\mu_1 \dots \mu_{d-1} \sigma}}{(d-1)! \sqrt{|g|}}}_{\mathrm{sign}(g)\, d^{d-1}\Sigma_\sigma} * \omega^\sigma$$

$$\Rightarrow \boxed{\int d^d x \sqrt{|g|} \, \nabla_\mu v^\mu = (-1)^{d-1} \int d^{d-1} \Sigma_\mu v^\mu}$$

Gauss-Ostrogradski theorem

iii) More general case: $T^{\mu_1 \dots \mu_m} = T^{[\mu_1 \dots \mu_m]}$

Let us define

$$d^{d-m} \Sigma_{\mu_1 \dots \mu_m} = \mathrm{sign}(g) \, dx^{\nu_1} \wedge \dots \wedge dx^{\nu_{d-m}} \frac{\varepsilon_{\nu_1 \dots \nu_{d-m} \mu_1 \dots \mu_m}}{(d-m)! \sqrt{|g|}}$$

$$\frac{1}{m!} d^{d-m} \Sigma_{\mu_1 \dots \mu_m} T^{\mu_1 \dots \mu_m} = \mathrm{sign}(g) * T$$

$$\frac{1}{(m-1)!} d^{d-m+1} \Sigma_{\mu_1 \dots \mu_{m-1}} \nabla_\rho T^{\rho \mu_1 \dots \mu_{m-1}} =$$

$$= \mathrm{sign}(g) \, dx^{\nu_1} \wedge \dots \wedge dx^{\nu_{d-m+1}} \frac{\varepsilon_{\nu_1 \dots \nu_{d-m+1} \mu_1 \dots \mu_{m-1}}}{(d-m+1)! \, (m-1)!} \frac{1}{\sqrt{|g|}} \partial_\rho \left(\sqrt{|g|} T^{\rho \mu_1 \dots \mu_{m-1}} \right)$$

$$= \cancel{\mathrm{sign}(g)} \, dx^{\nu_1} \wedge \dots \wedge dx^{\nu_{d-m+1}} \frac{\varepsilon_{\nu_1 \dots \nu_{d-m+1} \mu_1 \dots \mu_{m-1}}}{(d-m+1)! (m-1)! \sqrt{|g|}} \frac{\cancel{\mathrm{sign}}(g) (-1)^{(d-m)m}}{(d-m)!} \frac{\varepsilon^{\rho \mu_1 \dots \mu_{m-1} \sigma_1 \dots \sigma_{d-m}}}{\sqrt{|g|}} \quad (-1)^{(d-m)(m-1)}$$

$$\partial_\rho (*T)_{\sigma_1 \dots \sigma_{d-m}}$$

$$= (-1)^{d-m} \frac{\cancel{(m-1)!} \cancel{(d-m+1)!}}{\cancel{(d-m+1)!} \cancel{(m-1)!} (d-m)!} \frac{g}{(\sqrt{|g|})^2} \delta^{\rho \sigma_1 \dots \sigma_{d-m}}_{\nu_1 \dots \nu_{d-m+1}} dx^{\nu_1} \wedge \dots \wedge dx^{\nu_{d-m+1}}$$

$$\partial_\rho (*T)_{\sigma_1 \dots \sigma_{d-m}}$$

$$= (-1)^{d-m} \, \mathrm{sign}(g) \frac{1}{(d-m)!} \partial_{[\nu_1} * T_{\nu_2 \dots \nu_{d-m}]} \, dx^{\nu_1} \wedge \dots \wedge dx^{\nu_{d-m}}$$

$$= (-1)^{d-m} \, \mathrm{sign}(g) \quad d * T$$

$$\int_{M_{d-m+1}} d * T = \int_{\partial M_{d-m+1}} * T$$

$$\| $$

$$(-1)^{d-m}\,\mathrm{sign}(g) \int_{M_{d-m+1}} \frac{1}{(d-m+1)!} d^{d-m+1}\Sigma_{\mu_1 \cdots \mu_{m-1}} \nabla_\rho T^{\rho \mu_1 \cdots \mu_{m-1}}$$

$$\| $$

$$\mathrm{sign}(g) \int_{\partial M_{d-m+1}} \frac{1}{(d-m)!} d^{d-m}\Sigma_{\mu_1 \cdots \mu_m} T^{\mu_1 \cdots \mu_m}$$

$$\boxed{\int_{M_{d-m+1}} \frac{1}{(d-m+1)!} d^{d-m+1}\Sigma_{\mu_1 \cdots \mu_{m-1}} \nabla_\rho T^{\rho \mu_1 \cdots \mu_{m-1}} = (-1)^{d-m} \int_{\partial M_{d-m+1}} \frac{1}{(d-m)!} d^{d-m}\Sigma_{\mu_1 \cdots \mu_m} T^{\mu_1 \cdots \mu_m},}$$

The surface elements $d^{d-m}\Sigma_{\mu_1 \cdots \mu_m}$ are usually rewritten using m unit vectors $n_i{}^\mu$ normal to M_{d-m}

$n_i{}^\mu n_{j\mu} = \eta_{ij}$ $(+1$ or $-1)$

$$d^{d-m}\Sigma_{\mu_1 \cdots \mu_m} = \frac{d^{d-m}\Sigma_{\nu_1 \cdots \nu_m} n_1^{\nu_1} \cdots n_m^{\nu_m}}{\det \eta_{ij}} n_{1\mu_1} \cdots n_{m\mu_m}$$

$\|_4$

$d^{d-m}\Sigma \to$ to be expressed in intrinsic coordinates

4.1 Invariant actions in metric manifolds

miércoles, 16 de septiembre de 2015 16:02

The Principle of General Covariance requires all the laws of Physics to be covariant (= form-invariant = "invariant") under general transformations of coordinates ($\Rightarrow$ they have the same form in all reference frames = for all observers).

The simplest way to construct laws of Physics (equations) satisfying invariance principles is to construct invariant actions.(*)

Thus, we want to study how to construct actions for field theories in generic spacetimes invariant under general coordinate transformations (g.c.t.'s).

We have to take into account that the gravitational field (the metric $g_{\mu\nu}$) will always be present. This makes the construction of invariant actions much easier.

The action will have the general form

$$S = c \int d^4x \, \mathcal{L}$$

Absolute value of the determinant of the Jacobian matrix $\frac{\partial x'^\mu}{\partial x^\nu}$

Under g.c.t.s $\quad d^4x' = \left|\frac{\partial x'}{\partial x}\right| d^4x$

(For Poincaré transformations $x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$; $\frac{\partial x'^\mu}{\partial x^\nu} = \Lambda^\mu{}_\nu$ and $|\det \Lambda| = +1$)

Then, we need $\quad \mathcal{L}' = \left|\frac{\partial x'}{\partial x}\right|^{-1} \mathcal{L}$. ("scalar density")

Observe that

Absolute value of the determinant of $g_{\mu\nu}$

$$g'_{\mu\nu} = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}; \Rightarrow |g'| = \left|\frac{\partial x}{\partial x'}\right|^2 |g| = \left|\frac{\partial x'}{\partial x}\right|^{-2} |g|$$

(*) <u>Warning</u>: not all equations of motion can be obtained from an action. Most can, though.

and $\sqrt{|g'|} = \left|\frac{\partial x'}{\partial x}\right|^{-1} \sqrt{|g|}$

(If the metric is pseudoRiemannian, with signature $+---$ or $-+++$ $g = \det g < 0$ and sometimes people write $\sqrt{|g|} = \sqrt{-g}$)

Then,

$$S = c \int \underbrace{d^4x \sqrt{|g|}}_{\text{invariant}} \frac{\mathcal{L}}{\sqrt{|g|}} \rightarrow \text{invariant}$$

$d^4x\sqrt{|g|}$: invariant volume element $d^4\Sigma$

$\int_M d^4x \sqrt{|g|}$ = volume of M measured with the metric $g_{\mu\nu}$

We only have to find $\frac{\mathcal{L}}{\sqrt{|g|}}$ transforming as scalar (invariant).

Consider, for instance, a scalar field ϕ.

ϕ scalar $\Rightarrow$ $\partial_\mu \phi$ 1-form ; $\underbrace{\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi}_{\mathcal{L}/\sqrt{|g|}}$ invariant

$$\Rightarrow S = C \int d^4x \sqrt{|g|} \left\{ \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right\}$$

Observe that this action can be obtained from the special-relativistic one $\int d^4x \, \frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$ by replacing $d^4x \to d^4x\sqrt{|g|}$ and $\eta^{\mu\nu} \to g^{\mu\nu}$.

This is known as "<u>minimal coupling</u>" and it is the simplest way to make a special-relativistic theory to conform to the PGC.

4.1.1 Scalar field in a background metric

Let us find the equations of motion of the scalar field ϕ assuming for the moment that the metric $g_{\mu\nu}$ is not a dynamical field and has some fixed value in the coordinates x^μ (like $\eta_{\mu\nu}$).

We fix the value of ϕ on the boundary ∂M $\Rightarrow$ $\delta\phi\big|_{\partial M} = 0$

$$\delta S = C\int_M d^4x \sqrt{|g|}\, g^{\mu\nu}\,\partial_\mu\delta\phi\,\partial_\nu\phi =$$

total derivative $+\ \delta\phi|_{\partial M}=0$

$$= C\int_M d^4x\left\{\cancel{\partial_\mu\left[\sqrt{|g|}\,g^{\mu\nu}\partial_\nu\phi\,\delta\phi\right]} - \partial_\mu\left(\sqrt{|g|}\,g^{\mu\nu}\partial_\nu\phi\right)\delta\phi\right\}$$

$$\equiv \int d^4x\,\frac{\delta S}{\delta\phi}\,\delta\phi; \quad \text{(by definition)}$$

$$\frac{\delta S}{\delta\phi} = -\partial_\mu\left[\sqrt{|g|}\,g^{\mu\nu}\partial_\nu\phi\right] = -\sqrt{|g|}\,\nabla_\mu\partial^\mu\phi \equiv -\sqrt{|g|}\,\nabla^2\phi$$

$$\equiv -\sqrt{|g|}\,\Box\phi;$$

We have used $\nabla_\mu V^\mu = \partial_\mu V^\mu + \Gamma_{\mu\nu}{}^{\mu}V^\nu = \partial_\mu V^\mu + \frac{1}{\sqrt{|g|}}\partial_\nu\sqrt{|g|}\,V^\nu$

$$= \frac{1}{\sqrt{|g|}}\partial_\nu\left(\sqrt{|g|}\,V^\nu\right);$$

Observe that a total derivative term $\int d^4x\,\partial_\mu B^\mu$, which, as we have discussed, does not change the equations of motion, can be rewritten in this form

$$\int_M d^4x\,\partial_\mu B^\mu = \int_M d^4x\sqrt{|g|}\,\frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}\,\frac{B^\mu}{\sqrt{|g|}}\right) = \int_M d^4x\sqrt{|g|}\,\nabla_\mu\left(\frac{B^\mu}{\sqrt{|g|}}\right)$$

General covariance requires $B^\mu/\sqrt{|g|}$ to transform as a contravariant vector.

We can add a scalar potential term $V(\phi)$ respecting the PGC:

$$S[\phi] = \int d^4x \sqrt{|g|} \left\{ \frac{1}{2}(\partial\phi)^2 - V(\phi) \right\}$$

Now

$$\delta S = \int d^4x \sqrt{|g|} \left\{ -\delta\phi \,\Box\phi - \frac{dV}{d\phi}\delta\phi \right\} + t.d.$$

$$\frac{\delta S}{\delta \phi} = -\left[\Box\phi + V'(\phi)\right] = 0 \,;$$

4.1.2 Higher derivatives and boundary terms

lunes, 28 de septiembre de 2015 16:47

The special-relativistic action for a free scalar can also be written in this form (the difference is a total derivative)

$$S = \int d^4x \left\{ -\frac{1}{2}\phi\, \eta^{\mu\nu}\partial_\mu\partial_\nu\phi \right\}$$

$$\delta S = \int d^4x \left\{ -\frac{1}{2}\delta\phi\, \eta^{\mu\nu}\partial_\mu\partial_\nu\phi - \frac{1}{2}\phi\,\eta^{\mu\nu}\partial_\mu\partial_\nu\delta\phi \right\} =$$

$$= \int d^4x \left\{ -\frac{1}{2}\delta\phi\,\eta^{\mu\nu}\partial_\mu\partial_\nu\phi - \frac{1}{2}\partial_\mu\left(\phi\,\eta^{\mu\nu}\partial_\nu\delta\phi\right) + \frac{1}{2}\partial_\mu\phi\,\eta^{\mu\nu}\partial_\nu\delta\phi \right\}$$

$$= \int d^4x \left\{ -\frac{1}{2}\delta\phi\,\eta^{\mu\nu}\partial_\mu\partial_\nu\phi + \frac{1}{2}\partial_\nu\left(\partial_\mu\phi\,\eta^{\mu\nu}\delta\phi - \phi\,\eta^{\nu\mu}\partial_\mu\delta\phi\right) - \frac{1}{2}\partial_\nu\partial_\mu\phi\,\eta^{\mu\nu}\delta\phi \right\}$$

$$= \int d^4x \left\{ -\delta\phi\,\eta^{\mu\nu}\partial_\mu\partial_\nu\phi + \partial_\nu\left[\partial_\mu\phi\,\eta^{\mu\nu}\delta\phi - \frac{1}{2}\partial_\mu\left(\phi\,\eta^{\mu\nu}\delta\phi\right)\right] \right\}$$

using $\delta\phi\big|_{\partial M} = 0$

$$= \int_M d^4x \left\{ -\delta\phi\,\eta^{\mu\nu}\partial_\mu\partial_\nu\phi \right\} - \int_{\partial M} d^3\Sigma_\mu \left[\cancel{\partial^\mu\phi\,\delta\phi} - \frac{1}{2}\partial^\mu(\phi\,\delta\phi) \right]$$

If the boundary term vanished, we would obtain the same equations of motion as we did with $\mathcal{L} = \frac{1}{2}(\partial\phi)^2$. However, the boundary term contains $\partial_\mu\delta\phi$, which does not vanish on the boundary (we cannot impose <u>both</u> $\delta\phi\big|_{\partial M} = \partial_\mu\delta\phi\big|_{\partial M} = 0$)

In order to formulate a well-posed variational problem we must add a boundary term (i.e. a total derivative term in $\mathcal{L}$ so the field equations do not change) whose variation cancels that one:

The boundary term is $\Delta \mathcal{L} = \frac{1}{4}\Box\phi^2$

$$\delta(S+\Delta S) = \int d^4x\left\{-\delta\phi\Box\phi + \cancel{\frac{1}{2}\Box(\phi\delta\phi)}\right\} + \cancel{\frac{1}{2}\int d^3\Sigma_\mu \partial^\mu(\phi\delta\phi)}$$

and the action whose variation gives equations of motion via the PLA with $\delta\phi|_{\partial M}=0$

$$S'[\phi] = \int_M d^4x\left\{-\frac{1}{2}\phi\Box\phi + \frac{1}{4}\Box\phi^2\right\}$$

This example is a bit academic because

$$S'[\phi] = \int d^4x\left\{\cancel{-\frac{1}{2}\phi\Box\phi} + \cancel{\frac{1}{2}\phi\Box\phi} + \frac{1}{2}(\partial\phi)^2\right\}$$
$$= S[\phi];$$

However, it illustrates a possible problem in the application of the PLA.

4.1.3 More in minimal coupling to the gravitational field

martes, 29 de septiembre de 2015 12:07

How do we couple minimally ϕ to the metric now?
We must replace all derivatives by covariant derivatives simultaneously with $\eta_{\mu\nu} \to g_{\mu\nu}$

$$\to S = \int d^4x \sqrt{|g|} \left\{ -\frac{1}{2} \phi g^{\mu\nu} \nabla_\mu \nabla_\nu \phi + \frac{1}{4} g^{\mu\nu} \nabla_\mu \nabla_\nu \phi^2 \right\}$$

$$\underbrace{g^{\mu\nu} \nabla_\mu \nabla_\nu \phi}_{\Box\phi} = g^{\mu\nu} \nabla_\mu \partial_\nu \phi = \nabla_\mu \left(g^{\mu\nu} \partial_\nu \phi \right) = \frac{1}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|} g^{\mu\nu} \partial_\nu \phi \right);$$

$|||$ ← general definition of $\Box$

metric compatibility of the connection

Another example: the electromagnetic field. The special-relativistic action is

$$S_{\text{Maxwell}}[A] = \int d^4x \left\{ -\frac{1}{4} \eta^{\mu\rho} \eta^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} \right\}$$

with $F_{\mu\nu} = 2 \partial_{[\mu} A_{\nu]}$

Minimal coupling $\to S_M[A] = \int d^4x \sqrt{|g|} \left\{ -\frac{1}{2} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} \right\}$

with $F_{\mu\nu} = 2 \nabla_{[\mu} A_{\nu]} = 2 \partial_{[\mu} A_{\nu]} - \cancel{2 \Gamma_{[\mu\nu]}{}^{\rho} A_\rho}$

The Bianchi identities are

$$\nabla_{[\mu} F_{\nu\rho]} = \partial_{[\mu} F_{\nu\rho]} - \cancel{2 \Gamma_{[\mu\nu|}{}^{\sigma} F_{\sigma|\rho]}}$$

and the Maxwell equations are

$$\delta S_{\text{Max}} = \int d^4x \sqrt{|g|} \left\{ -\frac{1}{4} 2 g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} \delta F_{\rho\sigma} \right\} =$$

$$= \int d^4x \sqrt{|g|} \left\{ -\frac{1}{\cancel{2}} F^{\rho\sigma} \cancel{2} \partial_\rho \delta A_\sigma \right\} =$$

$$= \int d^4x \left\{ \cancel{-\partial_\rho\left(\sqrt{|g|}\, F^{\rho\sigma} \delta A_\sigma\right)} + \partial_\rho\left(\sqrt{|g|} F^{\rho\sigma}\right) \delta A_\sigma \right\}$$

$$\Rightarrow \frac{\delta S_{maxwell}}{\delta A_\sigma} = \partial_\rho\left(\sqrt{|g|}\, F^{\rho\sigma}\right) = \sqrt{|g|}\ \nabla_\rho F^{\rho\sigma} = 0.$$

Observe that, if we define the 1- and 2-forms

$$A \equiv A_\mu\, dx^\mu ;$$

$$F \equiv \frac{1}{2} F_{\mu\nu}\, dx^\mu \wedge dx^\nu = \frac{1}{\cancel{2}} \cdot \cancel{2}\, \partial_\mu A_\nu\, dx^\mu \wedge dx^\nu = dA ;$$

the Bianchi identity is a 3-form equation

$$\frac{1}{2} \partial_{[\mu} F_{\nu\rho]}\, dx^\mu \wedge dx^\nu \wedge dx^\rho = dF = ddA = 0 ;$$

due to the fundamental property of the exterior derivative.
According to the general formulae, the Maxwell equations are equivalent to

$$d \star F = 0 ;$$

Finally, in this language, the gauge transformations take the form

$$\delta A = d\chi \ ; \quad \delta F = d\,\delta A = dd\chi = 0 ;$$

We will see more examples later.
<u>Warning</u>: minimal coupling cannot be used with spinors. Something more sofisticated is needed.

4.1.4 The gravitational energy-momentum tensor

lunes, 28 de septiembre de 2015 17:03

Let us now consider the variation of the action of the scalar field with respect to the metric: $g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = \partial^\mu\phi\partial_\mu\phi$

$$\delta S = \int d^4x \left\{ \delta\sqrt{|g|}\, \frac{1}{2}(\partial\phi)^2 + \sqrt{|g|}\,\frac{1}{2}\delta g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \right\}$$

$$\delta\sqrt{|g|} = \frac{1}{2\sqrt{|g|}}\delta|g| = \frac{1}{2\sqrt{|g|}}|g|g^{\mu\nu}\delta g_{\mu\nu} = \frac{1}{2}\sqrt{|g|}\,g^{\mu\nu}\delta g_{\mu\nu};$$

$$\left(0 = \delta\delta^\mu{}_\nu = \delta(g^{\mu\rho}g_{\rho\nu}) = \delta g^{\mu\rho}g_{\rho\nu} + g^{\mu\rho}\delta g_{\rho\nu}\right) g_{\sigma\mu}$$

$$\delta g_{\sigma\nu} = -g_{\sigma\mu}\delta g^{\mu\rho}g_{\rho\nu}; \qquad \delta g^{\mu\nu} = -g^{\mu\rho}\delta g_{\rho\sigma}g^{\sigma\nu};$$

→ We can vary the action w.r.t. $g_{\mu\nu}$ or $g^{\mu\nu}$ and it is equivalent

$$\delta\sqrt{|g|} = -\frac{1}{2}\sqrt{|g|}\,g_{\mu\nu}\delta g^{\mu\nu} = +\frac{1}{2}\sqrt{|g|}\,g^{\mu\nu}\delta g_{\mu\nu};$$

Let us use $\delta g^{\mu\nu}$ only

$$\delta S = \int d^4x \frac{1}{2}\sqrt{|g|}\underbrace{\left\{\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(\partial\phi)^2\right\}}\delta g^{\mu\nu}$$

$$\frac{2}{\sqrt{|g|}}\frac{\delta S}{\delta g^{\mu\nu}} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(\partial\phi)^2 \xrightarrow[g_{\mu\nu}\to\eta_{\mu\nu}]{} -T_{\mu\nu}^{\text{canonical}}$$

The energy-momentum tensor can be <u>defined</u> in this way:

<u>Rosenfeld's prescription</u>: the special-relativistic energy-momentum tensor can be found by coupling the fields to an arbitrary metric $g_{\mu\nu}$ $S[\phi] \to S[g,\phi]$ and then

$$\boxed{T_{\mu\nu} = \frac{-2}{\sqrt{|g|}}\frac{\delta S}{\delta g^{\mu\nu}}\bigg|_{g_{\mu\nu}=\eta_{\mu\nu}}}$$

standard normalization

Automatically symmetric in $\mu\nu$.

Is this $T_{\mu\nu}$ conserved $\partial_\mu T^{\mu\nu} = 0$? Yes, we know it...

If we do not set $g_{\mu\nu} = \eta_{\mu\nu}$ we find

$$-\nabla_\mu T^{\mu\nu} = \nabla_\mu \left(\partial^\mu \phi \partial^\nu \phi - \frac{1}{2} g^{\mu\nu} \partial_\rho \phi \partial^\rho \phi \right) -$$

$$= \nabla^2 \phi \partial^\nu \phi + \partial^\mu \phi \nabla_\mu \partial^\nu \phi - \frac{1}{2} \nabla^\nu \left(\partial_\rho \phi \partial^\rho \phi \right) =$$

$$= \nabla^2 \phi \partial^\nu \phi + \cancel{\partial^\mu \phi \nabla^\nu \partial_\mu \phi} - \cancel{\frac{1}{2} \cdot 2 \nabla^\nu \partial_\rho \phi \partial^\rho \phi} =$$

$$= \nabla^2 \phi \partial^\nu \phi = 0 \quad \text{on-shell}$$

$$\Rightarrow \underbrace{\partial_\mu T^{\mu\nu} + \Gamma_{\mu\rho}{}^{\mu} T^{\rho\nu}}_{} + \Gamma_{\mu\rho}{}^{\nu} T^{\mu\rho} = 0 \quad \text{on-shell}$$

$$\frac{1}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|} T^{\mu\nu} \right) = - \Gamma_{\mu\rho}{}^{\nu} T^{\mu\rho}$$

$$\boxed{\partial_\mu \left(\sqrt{|g|} T^{\mu\nu} \right) = - \Gamma_{\mu\rho}{}^{\nu} \sqrt{|g|} T^{\mu\rho}} \Bigg|_{g=\eta} \Rightarrow \partial_\mu T^{\mu\nu} = 0$$

Neither $T^{\mu\nu}$ nor $\sqrt{|g|} T^{\mu\nu}$ are conserved. This is to be expected because the field ϕ interacts with the background metric $g_{\mu\nu}$, interchanging energy and momentum: only the total energy-momentum tensor is expected to be conserved.

$\Rightarrow$ $\nabla_\mu T^{\mu\nu} = 0$ is not a conservation law and it is not associated to any global symmetry via the first Noether theorem.

Then, where does this identity come from?
It is associated to general covariance (a local symmetry) through the second Noether theorem: let us consider infinitesimal g.c.t.s

$$\delta_\varepsilon x^\mu = \varepsilon^\mu(x);$$

and let us consider a field theory coupled to the metric $g_{\mu\nu}$

$$S[\phi, g] = \int d^4x \, \mathcal{L}(g, \phi, \partial\phi \ldots)$$

according to the PGC. $\Rightarrow$ $\delta_\varepsilon S = 0$

$$0 = \underset{\varepsilon}{\delta} S = \int \left\{ \delta_\varepsilon d^4x\, \mathcal{L} + d^4x \left[\frac{\partial \mathcal{L}}{\partial\phi}\tilde\delta_\varepsilon \phi + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\tilde\delta_\varepsilon\partial_\mu\phi + \frac{\partial\mathcal{L}}{\partial g_{\mu\nu}}\tilde\delta_\varepsilon g_{\mu\nu}\right]\right\}$$

$$= \int d^4x \left\{ \partial_\mu \varepsilon^\mu \mathcal{L} + \frac{\partial \mathcal{L}}{\partial\phi}\Big[\delta_\varepsilon\phi + \underbrace{\varepsilon^\mu\partial_\mu\phi}_{1}\Big] + \frac{\partial \mathcal{L}}{\partial\partial_\mu\phi}\Big[\delta_\varepsilon\partial_\mu\phi + \underbrace{\varepsilon^\nu\partial_\nu\partial_\mu\phi}_{2}\Big] \right.$$

$$\left. + \frac{\delta S}{\delta g_{\mu\nu}}\Big[\delta_\varepsilon g_{\mu\nu} + \underbrace{\varepsilon^\rho\partial_\rho g_{\mu\nu}}_{3}\Big]\right\}$$

$[\delta_\varepsilon, \partial_\mu] = 0$

$[\tilde\delta_\varepsilon, \partial_\mu] \neq 0$

$$= \int d^4x \left\{\partial_\mu\varepsilon^\mu \mathcal{L} + \underbrace{\varepsilon^\mu\partial_\mu\mathcal{L}}_{1+2+3} + \left[\frac{\partial \mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial\partial_\mu\phi}\right)\right]\delta_\varepsilon\phi \right.$$

$$\left. + \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial\partial_\mu\phi}\delta_\varepsilon\phi\right) + \frac{\delta S}{\delta g_{\mu\nu}}\delta_\varepsilon g_{\mu\nu}\right\}$$

$$= \int d^4x \left\{ \frac{\delta S}{\delta\phi}\delta_\varepsilon\phi + \frac12\sqrt{|g|}\,T^{\mu\nu}\delta_\varepsilon g_{\mu\nu} + \partial_\mu\left[\varepsilon^\mu\mathcal{L} + \frac{\partial \mathcal{L}}{\partial\partial_\mu\phi}\delta_\varepsilon\phi\right]\right\}$$

Remember $\tilde\delta_\varepsilon\phi = 0 = \delta_\varepsilon\phi + \varepsilon^\mu\partial_\mu\phi \Rightarrow \delta_\varepsilon\phi = -\varepsilon^\mu\partial_\mu\phi$

$$\tilde\delta_\varepsilon g_{\mu\nu} = g'_{\mu\nu}(x') - g_{\mu\nu}(x) = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma} - g_{\mu\nu}$$

$$= (\delta^\rho{}_\mu - \partial_\mu\varepsilon^\rho)(\delta^\sigma{}_\nu - \partial_\nu\varepsilon^\sigma)\,g_{\rho\sigma} - g_{\mu\nu}$$

$$= \cancel{g_{\mu\nu}} - \partial_\mu\varepsilon^\rho g_{\rho\nu} - \partial_\nu\varepsilon^\rho g_{\mu\rho} - \cancel{g_{\mu\nu}}$$

$$= -2\,\partial_{(\mu}\varepsilon^\rho g_{\nu)\rho} = \delta_\varepsilon g_{\mu\nu} + \varepsilon^\rho\partial_\rho g_{\mu\nu}$$

$$\Rightarrow \delta_\varepsilon g_{\mu\nu} = -\varepsilon^\rho\partial_\rho g_{\mu\nu} - 2\,\partial_{(\mu}\varepsilon^\rho g_{\nu)\rho} = \cdots =$$

$$= -2\,\nabla_{(\mu}\varepsilon_{\nu)}$$

$$0 = \int d^4x \left\{ -\frac{\delta S}{\delta\phi}\varepsilon^\mu\partial_\mu\phi - \sqrt{|g|}\,T^{\mu\nu}\nabla_\mu\varepsilon_\nu + \partial_\mu\left[\varepsilon^\mu\mathcal{L} - \varepsilon^\rho\frac{\partial \mathcal{L}}{\partial\partial_\mu\phi}\partial_\rho\phi\right]\right\}$$

$$= \int d^4x \left\{-\varepsilon^\mu\left[\frac{\delta S}{\delta\phi}\partial_\mu\phi - \sqrt{|g|}\,\nabla_\nu T^\nu{}_\mu\right] \right.$$

$$\left. + \partial_\mu\left[-\sqrt{|g|}\,T^{\mu\nu}\varepsilon_\nu + \varepsilon^\mu\mathcal{L} - \varepsilon^\rho\frac{\partial \mathcal{L}}{\partial\partial_\mu\phi}\partial_\rho\phi\right]\right\}$$

True for any $\varepsilon^\mu(x)$. For $\varepsilon^\mu\big|_{\partial M} = 0$

$$\Rightarrow \boxed{\nabla_\mu T^{\mu\nu} = + \frac{1}{\sqrt{|g|}} \frac{\delta S}{\delta \phi} \partial^\nu \phi} = 0 \quad \text{on-shell}$$

Let us consider the Maxwell action in a spacetime with metric $g_{\mu\nu}$ (we found it before):

$$S_{maxwell} = \int d^4x \sqrt{|g|} \left\{ -\frac{1}{4} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} \right\}$$

and let us find the energy-momentum tensor:

$$\delta S_{maxwell} = \int d^4x \left\{ \delta\sqrt{|g|} \left[-\frac{1}{4} F^2 \right] + \sqrt{|g|} \left[-\frac{1}{4} \cdot 2\, \delta g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} \right] \right\}$$

$$= \int d^4x \sqrt{|g|} \left\{ -\frac{1}{2} \delta g^{\mu\nu} g_{\mu\nu} \left[-\frac{1}{4} F^2 \right] - \frac{1}{2} \delta g^{\mu\nu} g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} \right\}$$

$$= \int d^4x\, \delta g^{\mu\nu} \sqrt{|g|} \left(-\frac{1}{2} \right) \left\{ F_\mu{}^\rho F_{\nu\rho} - \frac{1}{4} g_{\mu\nu} F^2 \right\}$$

$$\boxed{T^{maxwell}_{\mu\nu} = F_\mu{}^\rho F_{\nu\rho} - \frac{1}{4} g_{\mu\nu} F^2 ;}$$

Now we can check the general result:

$$\nabla_\mu T^{\mu\nu}_{maxwell} = \nabla_\mu F^{\mu\rho} F_{\nu\rho} + F^{\mu\rho} \nabla_\mu F_{\nu\rho} - \frac{1}{4} g_{\mu\nu} \nabla^\mu F^2 =$$

$$= \nabla_\mu F^{\mu\rho} F_{\nu\rho} + F^{\mu\rho} \nabla_\mu F_{\nu\rho} - \frac{1}{4} 2 F^{\mu\rho} \nabla_\nu F_{\mu\rho} =$$

$$= \nabla_\mu F^{\mu\rho} F_{\nu\rho} + \frac{1}{2} F^{\mu\rho} \left[\nabla_\mu F_{\nu\rho} + \nabla_\rho F_{\mu\nu} + \nabla_\nu F_{\rho\mu} \right] =$$

$$= \nabla_\mu F^{\mu\rho} F_{\nu\rho} + \frac{3}{2} F^{\mu\rho} \underbrace{\nabla_{[\mu} F_{\nu\rho]}}_{=0 \text{ (Bianchi identity)}} \qquad \boxed{= -\frac{1}{\sqrt{|g|}} \frac{\delta S}{\delta A_\rho} F_{\nu\rho}}$$

As a final example, we can compute the energy-momentum tensor of a scalar field with the higher-derivative action

$$S[\phi] = \int d^4x \sqrt{|g|}\left[-\frac{1}{2}\phi\nabla^2\phi\right] = \int d^4x\left\{-\frac{1}{2}\phi\partial_\mu\left(\sqrt{|g|}g^{\mu\nu}\partial_\nu\phi\right)\right\}$$

$$\delta S[\phi] = \int d^4x\,\phi\left\{-\frac{1}{2}\partial_\mu\left[\left(-\frac{1}{2}\sqrt{|g|}\,\delta g^{\alpha\beta}g_{\alpha\beta}g^{\mu\nu} + \sqrt{|g|}\,\delta g^{\mu\nu}\right)\partial_\nu\phi\right]\right\}$$

$$= \int d^4x\,\phi\left\{-\frac{1}{2}\partial_\mu\left[\sqrt{|g|}\,\delta g^{\alpha\beta}\left(-\frac{1}{2}g_{\alpha\beta}g^{\mu\nu} + g_\alpha{}^\mu g_\beta{}^\nu\right)\partial_\nu\phi\right]\right\}$$

$$= \int d^4x\sqrt{|g|}\,\delta g^{\alpha\beta}\frac{1}{2}\left(\partial_\alpha\phi\partial_\beta\phi - \frac{1}{2}g_{\alpha\beta}(\partial\phi)^2\right) +$$

$$+ \underbrace{\int_{\partial M} d^3\Sigma_\mu\left[\frac{1}{2}\delta g^{\alpha\beta}\ \cdots\right]}_{=0 \text{ because } \delta g^{\alpha\beta}|_{\partial M}=0}$$

(We must always impose the boundary conditions used to derive the Einstein equations

We have not yet considered the dynamics of the gravitational field itself. The equations of motion of the gravitational field can also be derived from an action principle. The construction of an action for the gravitational field (i.e. for the field $g_{\mu\nu}$) will be our next task. The coupling to matter is described by the covariantised actions that we have studied here.

4.1.5 The Principle of General Covariance and the second Noether theorem

martes, 29 de septiembre de 2015 12:41

The PGC and the second Noether theorem give us some interesting information even if we do not know the gravitational action, though: let $S_{grav}[g]$ be the action of the ("free") gravitational field. If it is invariant under g.c.t.'s

$$0 = \delta_\xi S_{grav}[g] = \int d^4x \, \frac{\delta S_{grav}}{\delta g_{\mu\nu}} \delta_\xi g_{\mu\nu} + \text{total derivatives}$$

$$= \int d^4x \, 2 \nabla_\mu \frac{\delta S_{grav}}{\delta g_{\mu\nu}} \xi^\nu + \text{total derivatives}$$

$$\Rightarrow \boxed{\nabla_\mu \frac{\delta S_{grav}}{\delta g_{\mu\nu}} = 0 \qquad \text{off-shell}}$$

gauge identity or Noether identity relating the field equations.

Let us now consider the scalar field coupled to dynamical gravity

$$S[g,\phi] = S_{grav}[g] + S[g,\phi]$$

minimal coupling

$$\frac{\delta S}{\delta g_{\mu\nu}} = \frac{\delta S_{grav}}{\delta g_{\mu\nu}} + \frac{\delta S[\phi,g]}{\delta g_{\mu\nu}} = \frac{\delta S_{grav}}{\delta g_{\mu\nu}} + \frac{1}{2}\sqrt{|g|}\, T^{\mu\nu} \underset{\text{on-shell}}{=} 0$$

$$\nabla_\mu \frac{\delta S}{\delta g_{\mu\nu}} = \underbrace{\nabla_\mu \frac{\delta S_{grav}}{\delta g_{\mu\nu}}}_{=0 \text{ off-shell}} + \frac{1}{2}\sqrt{|g|}\,\nabla_\mu T^{\mu\nu} \underset{\text{on-shell}}{=} 0 \Rightarrow \boxed{\nabla_\mu T^{\mu\nu} = 0 \quad \text{on-shell}}$$

Since we know that $\nabla_\mu T^{\mu\nu} = \frac{1}{\sqrt{|g|}} \frac{\delta S}{\delta \phi} \partial^\nu \phi$ we conclude that, if the PGC is satisfied, the equations of the gravitational field (whatever they are) "almost" imply the equations of the matter fields.

4.2 The Einstein-Hilbert action

miércoles, 19 de agosto de 2015 13:20

According to the general arguments, in order to construct an action for the gravitational field $g_{\mu\nu}$

$$S \sim \int d^4x \sqrt{|g|}\ \mathcal{L}/\sqrt{|g|}$$

We must find suitable $\mathcal{L}/\sqrt{|g|}$:

i) $\mathcal{L}/\sqrt{|g|}$ must transform as a scalar under g.c.t.'s so that S is invariant and the field equations $\frac{\delta S}{\delta g_{\mu\nu}}$ transform covariantly.

ii) $\mathcal{L}/\sqrt{|g|}$ can depend on $g_{\mu\nu}$, $\partial_\mu g_{\nu\rho}$, $\partial_\mu \partial_\nu g_{\rho\sigma}$ ... but we "prefer" field equations which are second-order in derivatives of the field $\Rightarrow$ no $\partial^3 g_{\mu\nu}$ or higher. ($g\partial^2 g$ & $(\partial g)^2$ are allowed)

$\Rightarrow$ Which tensors can be constructed with $g_{\mu\nu}$ and its derivatives? All the tensors that can be constructed from $R_{\mu\nu\rho}{}^{\sigma}$ and $g_{\mu\nu}$ by multiplication, contraction and covariant differentiation:

$R_{\mu\nu\rho}{}^{\sigma}$; $R_{\mu\rho} = R_{\mu\nu\rho}{}^{\nu}$; $R = g^{\mu\rho} R_{\mu\rho}$; $g_{\alpha\beta} R$
$R R_{\mu\nu}$; $\nabla_\mu R_{\nu\rho\sigma\delta}$; etc.

We are interested in the scalars:

R : $\partial g \partial g$ & $g \partial^2 g$

$\left.\begin{array}{l} R^2 ; \quad R_{\mu\nu} R^{\mu\nu} ; \quad R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \\ \nabla_\mu R \nabla^\mu R ; \ \text{etc etc.} \end{array}\right\}$ higher-order in derivatives

The only candidate satisfying our "preference" is $\boxed{\frac{\mathcal{L}}{\sqrt{|g|}} = R}$
We will consider more general possibilities later.

The resulting action $S[g] \sim \int d^4x \sqrt{|g|}\, R$, first proposed by Hilbert, is known as the <u>Einstein-Hilbert action</u>.

Before we derive the field equations, let us study the normalization.

$$d^4x \to [L^4]$$

$$g_{\mu\nu} \to \text{dimensionless}, \text{ so } ds^2 = g_{\mu\nu}\, \underset{[L]}{dx^\mu}\, \underset{[L]}{dx^\nu} \to [L^2]$$

$$\Gamma_{\mu\nu}^{\;\;\rho} = \left\{ {\rho \atop \mu\nu} \right\} = \frac{1}{2} g^{\rho\sigma} \left\{ \partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu} \right\} \to [L^{-1}]$$

$$\left.\begin{array}{l} \partial\Gamma \to [L^{-2}] \\ \Gamma\Gamma \to [L^{-2}] \end{array}\right\} \Rightarrow R \quad [L^{-2}] \Rightarrow \int d^4x \sqrt{g}\, R \to [L^2]$$

In natural units $\hbar = c = 1$ the action must be dimensionless $\Rightarrow$ we have to multiply by a constant with $[L^{-2}]$

The natural choice is the <u>Planck length squared</u>

Up to a normalization factor

$$\boxed{\frac{S_{EH}[g]}{\hbar} = \frac{1}{16\pi \ell_P^2} \int d^4x \sqrt{|g|}\, R} \qquad \boxed{\ell_P^2 = \frac{\hbar G_N}{c^3}}$$

$$\boxed{S_{EH}[g] = \frac{c^3}{16\pi G_N} \int d^4x \sqrt{|g|}\, R\,;} \qquad \boxed{\ell_P^{-1} = \frac{M_P c}{\hbar}}$$

This action, being the simplest candidate, is a very complicated, non-linear action $\Rightarrow$ it describes complicated self-couplings of the gravitational field to itself, as it must because it couples to all kinds of energy and it carries energy (Principle of Equivalence, strong form)

4.2.1 Einstein's equations in vacuum

lunes, 28 de septiembre de 2015 22:56

If we try to use the E-L equations we will get a huge number of terms in g, ∂g, $\partial^2 g$ etc and recovering tensors from them is difficult and time-consuming.

It is much better to vary directly the action:

$$\delta S_{EH} \sim \int d^4x \left[\delta\sqrt{|g|}\, R + \sqrt{|g|}\, \delta R \right] =$$

$$= \int d^4x \left\{ -\frac{1}{2}\sqrt{|g|}\, g_{\mu\nu}\, \delta g^{\mu\nu} R + \sqrt{|g|} \left(\delta g^{\mu\nu} R_{\mu\nu} + g^{\mu\nu}\, \delta R_{\mu\nu} \right) \right\}$$

$$= \int d^4x \sqrt{|g|} \left\{ \delta g^{\mu\nu} \underbrace{\left[R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right]}_{G_{\mu\nu}} + g^{\mu\nu}\, \delta R_{\mu\nu} \right\}$$

$R_{\mu\nu} = R_{\mu\rho\nu}{}^{\rho}$ depends on $g_{\mu\nu}$ only through the connection $\Gamma_{\mu\nu}{}^{\rho}$

$$R_{\mu\nu} = \partial_\mu \Gamma_{\rho\nu}{}^{\rho} - \partial_\rho \Gamma_{\mu\nu}{}^{\rho} + \Gamma_{\mu\lambda}{}^{\rho}\, \Gamma_{\rho\nu}{}^{\lambda} - \Gamma_{\rho\lambda}{}^{\rho}\, \Gamma_{\mu\nu}{}^{\lambda}$$

$$\delta R_{\mu\nu} = \partial_\mu \delta\Gamma_{\rho\nu}{}^{\rho} + \Gamma_{\mu\lambda}{}^{\rho}\, \delta\Gamma_{\rho\nu}{}^{\lambda} - \Gamma_{\mu\nu}{}^{\lambda}\, \delta\Gamma_{\rho\lambda}{}^{\rho} - \Gamma_{\mu\rho}{}^{\lambda}\, \delta\Gamma_{\lambda\nu}{}^{\rho}$$
$$- \left(\partial_\rho \delta\Gamma_{\mu\nu}{}^{\rho} + \Gamma_{\rho\lambda}{}^{\rho}\, \delta\Gamma_{\mu\nu}{}^{\lambda} - \Gamma_{\rho\nu}{}^{\lambda}\, \delta\Gamma_{\mu\lambda}{}^{\rho} \right) + \Gamma_{\rho\mu}{}^{\lambda}\, \delta\Gamma_{\lambda\nu}{}^{\rho}$$
$$= \nabla_\mu \delta\Gamma_{\rho\nu}{}^{\rho} - \nabla_\rho \delta\Gamma_{\mu\nu}{}^{\rho}$$

(Observe that $\delta\Gamma$ transforms as a tensor even though Γ does not)

$$\delta S[g] \sim \int d^4x \sqrt{|g|} \left\{ \delta g^{\mu\nu} G_{\mu\nu} + g^{\mu\nu} \left(\nabla_\mu \delta\Gamma_{\rho\nu}{}^{\rho} - \nabla_\rho \delta\Gamma_{\mu\nu}{}^{\rho} \right) \right\}$$

$$= \int d^4x \sqrt{|g|} \left\{ \delta g^{\mu\nu} G_{\mu\nu} + \underbrace{\nabla_\mu \left(g^{\mu\nu}\, \delta\Gamma_{\rho\nu}{}^{\rho} - g^{\rho\nu}\, \delta\Gamma_{\rho\nu}{}^{\mu} \right)}_{\text{total derivative}} \right\}$$

If we can argue that the total derivative gives a boundary term that vanishes when $\delta g_{\mu\nu} = 0$ on the boundary, we will have proven that the E-H action is extremized by the field configurations $g_{\mu\nu}$ satisfying the

$$\boxed{\underline{\text{Einstein equations (in vacuum)}} \quad G_{\mu\nu} = 0}$$

Then, we will have to show that these equations really describe the gravitational field (Newtonian limit + predictions).

4.2.2The York-Gibbons-Hawking boundary term

martes, 29 de septiembre de 2015 11:57

The total derivative gives

$$\int d^4x \sqrt{|g|}\, \nabla_\mu \left(g^{\mu\nu} \delta \Gamma_{\rho\nu}^{\ \rho} - g^{\rho\nu} \delta \Gamma_{\rho\nu}^{\ \mu} \right) =$$

$$= -\int d^3 \Sigma_\mu \left(g^{\mu\nu} \delta \Gamma_{\rho\nu}^{\ \rho} - g^{\rho\nu} \delta \Gamma_{\rho\nu}^{\ \mu} \right)$$

Now

$$\delta \Gamma_{\mu\nu}^{\ \rho} = \frac{1}{2} \delta g^{\rho\sigma} \{ \quad \} + \frac{1}{2} g^{\rho\sigma} \{ \partial_\mu \delta g_{\sigma\nu} + \partial_\nu \delta g_{\sigma\mu} - \partial_\sigma \delta g_{\mu\nu} \}$$

$$= - g^{\rho\alpha} \delta g_{\alpha\beta} \Gamma_{\mu\nu}^{\ \beta} + \dots$$

$$= \frac{1}{2} g^{\rho\sigma} \{ \partial_\mu \delta g_{\sigma\nu} + \partial_\nu \delta g_{\sigma\mu} - \partial_\sigma \delta g_{\mu\nu} - 2 \delta g_{\sigma\lambda} \Gamma_{\mu\nu}^{\ \lambda} \}$$

$$= \frac{1}{2} g^{\rho\sigma} \{ \nabla_\mu \delta g_{\sigma\nu} + \nabla_\nu \delta g_{\sigma\mu} - \nabla_\sigma \delta g_{\mu\nu} \}$$

(Check: $\frac{1}{2} g^{\rho\sigma} \{ \partial_\mu \delta g_{\sigma\nu} - \cancel{\Gamma_{\mu\sigma}^{\ \lambda} \delta g_{\lambda\nu}} - \Gamma_{\mu\nu}^{\ \lambda} \delta g_{\sigma\lambda}$

$+ \partial_\nu \delta g_{\sigma\mu} - \cancel{\Gamma_{\nu\sigma}^{\ \lambda} \delta g_{\lambda\mu}} - \Gamma_{\nu\mu}^{\ \lambda} \delta g_{\sigma\lambda}$

$- (\partial_\sigma \delta g_{\mu\nu} - \cancel{\Gamma_{\sigma\mu}^{\ \lambda} \delta g_{\lambda\nu}} - \cancel{\Gamma_{\sigma\nu}^{\ \lambda} \delta g_{\mu\lambda}}) \}$

$$\Rightarrow g^{\mu\nu} \delta \Gamma_{\rho\nu}^{\ \rho} - g^{\rho\nu} \delta \Gamma_{\rho\nu}^{\ \mu} = g^{\mu\rho} g^{\sigma\nu} \left(\nabla_\rho \delta g_{\sigma\nu} - \nabla_\sigma \delta g_{\rho\nu} \right)$$

$\Rightarrow$ The boundary term depends on derivatives of the variations of the gravitational field and it will not vanish unless we require those derivatives to vanish on the boundary, which is not allowed. Therefore, we must introduce a boundary term whose own variation cancels the above term. Since it is a boundary term, the Einstein field equations will not be modified.

The boundary term plays a very important role in different contexts and, therefore, it is not simply a purely mathematical construction.

First, let us massage a little bit more the boundary term.

$$-\int d^3\Sigma_\mu v^\mu = -\int d^3\Sigma \, n_\mu v^\mu$$

$(d^3\Sigma = n^2 \, d^3\Sigma_\mu n^\mu \; ; \; n^2 = \pm 1$ normal to the boundary$)$

$$n_\mu v^\mu = n_\mu g^{\mu\rho} g^{\sigma\nu} \left(\nabla_\rho \delta g_{\sigma\nu} - \nabla_\sigma \delta g_{\rho\nu}\right) =$$
$$= n^\rho g^{\sigma\nu} \left(\nabla_\rho \delta g_{\sigma\nu} - \nabla_\sigma \delta g_{\rho\nu}\right)$$
$$= n^\rho h^{\sigma\nu} \left(\nabla_\rho \delta g_{\sigma\nu} - \nabla_\sigma \delta g_{\rho\nu}\right) \qquad (\rho \leftrightarrow \sigma \text{ antisymmetric})$$

where $h_{\mu\nu} \equiv g_{\mu\nu} - n^2 n_\mu n_\nu$ is the <u>induced metric</u> on the hypersurface normal to n^μ

$$n^\mu h_{\mu\nu} = n_\nu - \cancel{n^2}\,\cancel{n^2}\, n_\nu = 0$$

$\Rightarrow h^{\mu\nu}$ projects onto the boundary $\Rightarrow h^{\sigma\nu} \nabla_\sigma \delta g_{\rho\nu} = 0$

(if $\delta g_{\rho\nu}$ vanishes (is constant) over the boundary, its covariant derivative along the boundary is)

$$\Rightarrow \delta S_{EH} = \frac{1}{16\pi G} \int_{M_4} d^4x \sqrt{|g|}\, \delta g^{\mu\nu} G_{\mu\nu} - \int_{\partial M_4} d^3\Sigma \, n^\mu h^{\sigma\nu} \nabla_\mu \delta g_{\sigma\nu}$$

The boundary term whose variation cancels the last term is constructed from the <u>extrinsic curvature</u> of the boundary which measures how the boundary is curved inside M_4

$$K^{\mu\nu} = h^{\mu\alpha} h^{\nu\beta} \nabla_{(\alpha} n_{\beta)} \; ; \; K = h_{\mu\nu} K^{\mu\nu} = h^{\alpha\beta} \nabla_\alpha n_\beta$$

$$\delta K\Big|_{\partial M} = \cancel{\delta h^{\mu\alpha}} \nabla_\mu n_\alpha - h^{\mu\alpha} \delta \Gamma_{\mu\alpha}{}^{\rho} n_\rho =$$
$$= -h^{\mu\alpha} \left(\cancel{\nabla_\mu \delta g_{\rho\alpha}} - \frac{1}{2} \nabla_\rho \delta g_{\mu\alpha}\right) n^\rho = \frac{1}{2} n^\rho h^{\mu\alpha} \nabla_\rho g_{\mu\alpha}$$

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$$\boxed{S_{EH} = \frac{1}{16\pi G}\int_{M_4} d^4x \sqrt{|g|}\, R + \frac{1}{8\pi G}\int_{\partial M_4} d^3\Sigma\, K}$$

York-Gibbons-Hawking boundary term

$$\boxed{\frac{\delta S_{EH}}{\delta g^{\mu\nu}} = \frac{1}{16\pi G}\sqrt{|g|}\, G_{\mu\nu};}$$

with $\delta g_{\mu\nu}\Big|_{\partial M_4} = 0$

4.2.3 Noether's second theorem and the contracted Bianchi identity

miércoles, 16 de septiembre de 2015 16:05

The Einstein-Hilbert action is invariant under g.c.t.'s by construction. Therefore, it must satisfy off-shell

$$\nabla_\mu \frac{\delta S_{EH}}{\delta g_{\mu\nu}} = 0 \ ;$$

Indeed $\nabla_\mu G^{\mu\nu} = 0$ (contracted Bianchi identity) for any metric, whether or not it satisfies the Einstein equations in vacuum $G_{\mu\nu} = 0$.

4.2.4 The cosmological constant term

miércoles, 16 de septiembre de 2015 16:08

When we discussed the possible terms that we could include in an action for the gravitational field we focused in terms containing derivatives of the metric because they are necessary to get dynamical (differential) field equations.

However we may add to them terms without derivatives.

There is only one scalar term with no derivatives of the metric: a constant. This constant is known as the cosmological constant, Λ, and it is usually normalised as follows:

$$\boxed{S_{EH}[g] = \frac{c^3}{16\pi G_N}\int d^4x\sqrt{|g|}\left\{R - 2\Lambda\right\} + \text{boundary term.}}$$

$\Lambda \to [L^{-2}] \Rightarrow$ sometimes people write $\Lambda \sim 1/\ell^2 \leftarrow$ some length scale

Now, the "cosmological" Einstein equations are $(c=1)$

$$\frac{1}{\sqrt{|g|}}\frac{\delta S}{\delta g^{\mu\nu}} = \frac{1}{16\pi G}\left[G_{\mu\nu} - g_{\mu\nu}\Lambda\right] - \frac{1}{2}T_{\mu\nu}^{\text{matter}} = 0$$

$$\boxed{G_{\mu\nu} = 8\pi G\, T_{\mu\nu}^{\text{matter}} + g_{\mu\nu}\Lambda}$$

"additional contribution to the e-m tensor"

$\Lambda > 0$ de Sitter-like cosmological constant

$\Lambda < 0$ anti-de Sitter-like cosmological constant

$\nabla_\mu\left(G^{\mu\nu} - g^{\mu\nu}\Lambda\right) = 0$ off-shell, as it must!

Whether or not $\Lambda \neq 0$ and its actual value is an experimental question.

4.2.5 Coupling to matter fields

miércoles, 16 de septiembre de 2015 16:07

We have studied how to couple matter fields to a non-dynamical gravitational field $g_{\mu\nu}$ through a recipe called "minimal coupling".

We have also constructed an action from which one can derive candidate field equations for the gravitational field in vacuum ($\neq$ "free").

Now we can combine these ingredients and construct actions for the complete gravity + matter systems

some generic field

$$S[g,\phi] = S_{EH}[g] + S_{matter}[\phi, g]$$

$$\frac{1}{\sqrt{|g|}}\frac{\delta S}{\delta g^{\mu\nu}} = \frac{1}{16\pi G} G^{\mu\nu} - \frac{1}{2} T^{\mu\nu}_{matter}(\phi, g) = 0 ;$$

$$\Rightarrow \boxed{G_{\mu\nu} = 8\pi G \, T_{\mu\nu}^{matter}}$$ Einstein equations.

$\oplus$

$$\boxed{\frac{\delta S}{\delta \phi} = 0 ;}$$ matter field equations.

We have shown that

$$\boxed{0 \underset{\text{Bianchi}}{=} \nabla_\mu G^{\mu\nu} \underset{\text{Einstein}}{=} 8\pi G \, \nabla_\mu T^{\mu\nu} \underset{\text{Noether}}{=} 8\pi G \frac{1}{\sqrt{|g|}} \frac{\delta S}{\delta \phi} \partial^\nu \phi = 0 \quad \text{on-shell.}}$$

Remember that, with our conventions $T^{00} < 0$

4.2.6 Coupling to fluids

miércoles, 16 de septiembre de 2015 16:13

Often we want to couple to gravity macroscopic systems that cannot be described by fundamental fields. In these cases we have a statistical or thermodynamical description only and we do not have an action for the statistical or thermodynamical variables. We have to find equations of motion and an energy-momentum tensor for the Einstein equations.

However, as we have discussed, we can only couple to the Einstein equations energy-momentum tensors satisfying $\nabla_\mu T^{\mu\nu} = 0$, and we have also shown that this equation is satisfied only on-shell because it is proportional to the matter field equations.

This suggests the following way to proceed: construct a suitable candidate for the matter, containing the relevant variables and with the correct Newtonian limit and then derive the equations of motion for those variables by requiring $\nabla_\mu T^{\mu\nu} = 0$.

In many cases the matter that we want to describe can be approximated by a <u>fluid</u>: continuous medium that can <u>flow</u>. This flow is described by a velocity field $\vec{v}(x,t)$ which tells us the velocity of the infinitesimal volume of fluid at position $\vec{x}$ and time t. (The special-relativistic generalization will be a vector field $u^\mu(x)$ normalized $u^\mu u_\mu = +1$ $(c=1)$)

We are only going to deal with <u>perfect fluids</u>, which can be described by $u^\mu(x)$, their <u>rest mass density</u> $\rho(x)$ and their <u>isotropic pressure</u> $p(x)$. Both $\rho(x)$ and $p(x)$ behave as scalar relativistic fields. For comoving observers the fluid is isotropic. Other stresses (heating, viscosity) are ignored.

4.2.6.1 Newtonian perfect fluids

domingo, 4 de octubre de 2015 16:28

A Newtonian perfect fluid is governed by 4 equations:

i) The <u>continuity equation</u> for the mass density $\rho(x)$

$\rho =$ constant $\Downarrow$ $\frac{\partial v^i}{\partial x^i} = 0$ incompressible fluid

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial (v^i \rho)}{\partial x^i} = 0} \quad ; \qquad v^i \rho \rightarrow \text{matter density current.}$$

which expresses the conservation of mass.

ii) The <u>Euler equation</u>

hydrostatics: $f^i = \rho g^i = \frac{\partial p}{\partial x^i}$

$$\boxed{\frac{\partial v^i}{\partial t} + v^j \frac{\partial v^i}{\partial x^j} = -\frac{1}{\rho} \frac{\partial p}{\partial x^i} + \frac{f^i}{\rho}}$$

f^i: density of external forces

which is the expression of Newton's 2nd law for fluids:

Let's consider an infinitesimal volume element of fluid and follow its trajectory. Newton's law:

$$\frac{d}{dt}(\text{momentum per unit mass}) = \text{force per unit mass} = \frac{\text{force per unit volume}}{\text{mass density}} = \frac{-\text{gradient of pressure} + \text{density of ext. forces}}{\text{density}}$$

MTW § 5.5

$$\frac{dv^i}{dt} = -\frac{1}{\rho}\left(\frac{\partial p}{\partial x^i} + f^i\right)$$

$$\begin{pmatrix}\text{rate of change}\\ \text{with time}\\ \text{following fluid}\end{pmatrix} = \begin{pmatrix}\text{rate of change}\\ \text{with time}\\ \text{at fixed location}\end{pmatrix} + \begin{pmatrix}\text{velocity}\\ \text{of}\\ \text{fluid}\end{pmatrix} \cdot \begin{pmatrix}\text{rate of change}\\ \text{with}\\ \text{position}\end{pmatrix}$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v^i \frac{\partial}{\partial x^i} \rightarrow \text{Euler equation}$$

Weinberg 2-10

iii) The equation that expresses the absence of heat transfer: the specific entropy (entropy per unit mass) s is constant along the flow:

(or 2nd law of thermodynamics)

$$\boxed{\frac{ds}{dt} = \frac{\partial s}{\partial t} + v^i \frac{\partial s}{\partial x^i} = 0}$$

iv) An equation of state for the fluid relating, for instance, p and ρ $\quad f(p,\rho)=0 \rightarrow p=p(\rho)$

An important case is $\boxed{p=0}$ called "dust" or "incoherent matter".

Some of these equations can be combined using the Newtonian stress-energy tensor T_{ij}, which is associated to the conservation of energy and momentum.

The density of momentum of the fluid is ρv^i and its rate of change is given by ($f^i=0$) Euler + continuity

$$\frac{\partial}{\partial t}\underbrace{(\rho v^i)}_{\equiv T^{0i}} = \frac{\partial\rho}{\partial t}v^i+\rho\frac{\partial v^i}{\partial t} = -\frac{\partial(\rho v^j)}{\partial x^j}v^i+\rho\left[-v^j\frac{\partial v^i}{\partial x^j}-\frac{1}{\rho}\frac{\partial p}{\partial x^i}\right]$$

$$= -\frac{\partial}{\partial x^j}(\rho v^i v^j)-\frac{\partial p}{\partial x^i} =$$

$$= -\frac{\partial}{\partial x^j}\Big[\underbrace{\rho v^i v^j + p\delta^{ij}}_{\equiv T^{ij}\ (\Pi^{ij})}\Big] \quad ; \quad T^{00}\equiv\rho$$

$$\left(T^{\mu\nu}\right) \equiv \left(\begin{array}{c|c} \rho & \rho v^j \\ \hline \rho v^i & \rho v^i v^j+\delta^{ij}p \end{array}\right);$$

$\partial_\mu T^{\mu 0}=0 \rightarrow$ continuity equation

$\partial_\mu T^{i0}=0 \rightarrow$ continuity $\oplus$ Euler $\Big\} \rightarrow$ Euler equation

The conservation of the Newtonian stress-energy tensor leads to the fundamental equations of the fluid. We want to do the same in the relativistic setting but, before we do that, let's mention the "imperfections".

domingo, 4 de octubre de 2015 17:09

Non-perfect fluids are governed by Euler equations with additional terms (the continuity equation holds for any fluid). The possible corrections can be studied as corrections to the Newtonian stress tensor Π^{ij}

$$\partial_\mu T^{i\mu} = 0 \longrightarrow \frac{\partial}{\partial t}(\rho v^i) = -\partial_j \Pi^{ij}$$

$\Pi^{ij} = p\delta^{ij} + \rho v^i v^j$ (perfect fluid)

The possible additions are analysed for instance in Landau & Lifschitz Ch II. They depend on several new constants: η, ξ and the modified Euler equation is known as the <u>Navier-Stokes equation</u>

$$\boxed{\frac{\partial v^i}{\partial t} + v^j \frac{\partial v^i}{\partial x^j} = -\frac{1}{\rho}\frac{\partial p}{\partial x^i} + \eta \frac{\partial^2 v^i}{\partial x^j \partial x^j} + \left(\xi + \frac{\eta}{3}\right)\frac{\partial^2 v^j}{\partial x^i x^j}}$$

$\eta, \xi \rightarrow$ <u>viscosity coefficients</u>

4.2.6.2 Relativistic perfect fluids

domingo, 4 de octubre de 2015

Using the Newtonian stress-energy tensor Π^{ij} we have constructed a $T^{\mu\nu}$ which can be seen as the Newtonian limit of a relativistic stress-energy tensor of a perfect fluid.

$$(-T^{\mu\nu}) = \left(\begin{array}{c|c} \rho & \rho v^j \\ \hline \rho v^i & \rho v^i v^j + \delta^{ij} p \end{array}\right) ;$$

ρ is a scalar $\neq$ 00-component of a tensor

In a comoving coordinate system $(u^\mu) = (u^0, \vec{0}) = (1, \vec{0})$ and we only see isotropic pressure

$$(-T^{\mu\nu}) = \left(\begin{array}{c|ccc} \rho & & & \\ \hline & p & & \\ & & p & \\ & & & p \end{array}\right) = \left(-p\eta^{\mu\nu} + (p+\rho) u^\mu u^\mu\right)$$

$$\Rightarrow \boxed{-T^{\mu\nu} = (p+\rho)\, u^\mu u^\nu - p\,\eta^{\mu\nu}}$$

In the Newtonian limit $(u^\mu) \sim (1, v^i)$ $\quad (v^2 \ll 1)$

and we recover our previous result.

Now, the conservation of $T^{\mu\nu}$ must give the relativistic version of the continuity and Euler's equations. Observe first that

$$-T^{00} = u^0 u^0 \left\{\rho + p \left[\underbrace{1 - \frac{1}{(u^0)^2}}_{v^2}\right]\right\} = w \qquad u^0 = \frac{1}{1-v^2}$$

$$-T^{0i} = u^0 u^0 \left\{ (p+\rho) \overbrace{\frac{u^i}{u^0}}^{v^i} \right\} = v^i \Big[\overbrace{u^0 u^0 (\rho + p) - p}^{w} + p\Big] = v^i (w+p)$$

$$\boxed{\partial_t w + \partial_i \left[(w+p)\, v^i\right] = 0}$$

Main lesson: pressure contributes to the energy density w

$$-T^{ij} = u^0 u^0 (\rho + p) v^i v^j + p\,\delta^{ij} = (w+p) v^i v^j + p\,\delta^{ij}$$

$$\partial_t\left[(\omega+p)v^i\right]+\partial_j\left[(\omega+p)v^iv^j+p\delta^{ij}\right]=0$$

$$v^i\partial_t(\omega+p)+(\omega+p)\partial_t v^i+(\omega+p)v^j\partial_j v^i+v^i\partial_j\left[(\omega+p)v^j\right]+\partial_i p=0$$

$$v^i\Big\{\underbrace{\partial_t(\omega+p)+\partial_j\left[(\omega+p)v^j\right]}_{=\,\partial_t p}\Big\}+(\omega+p)\left[\partial_t v^i+v^j\partial_j v^i\right]=-\partial_i p$$

$$\boxed{\partial_t v^i+v^j\partial_j v^i=-\frac{1}{\omega+p}\left(\partial_i p-v^i\partial_t p\right)}$$ Special-relativistic Euler equation

In the Newtonian limit $v^2 \ll 1$ $\quad p v^i \ll 1$ we recover the Euler equation

In GR, applying the PGC ($\eta_{\mu\nu}\to g_{\mu\nu}$), we find that a perfect fluid has the energy-momentum tensor

$$\boxed{T_{\mu\nu}=(p+\rho)u_\mu u_\nu-p\,g_{\mu\nu}}$$

and the general relativistic fluid equations are derived from $\nabla_\mu T^{\mu\nu}=0$;

$$\underbrace{\partial_\mu\left(\sqrt{|g|}\,T^{\mu\nu}\right)}_{\text{not conserved}}=-\sqrt{|g|}\,\underbrace{\Gamma_{\rho\sigma}{}^{\nu}\,T^{\rho\sigma}}_{\text{interaction with gravity}}$$

A particular case which arises in cosmology:

$ds^2=dt^2-a^2(t)\,d\vec{x}^{\,2}$: homogeneous and isotropic universe

$\Rightarrow p=p(t)$; $\rho=\rho(t)$; $u^\mu=(1,\vec{0})$

$\Gamma_{ij}{}^{t}=a\dot{a}\,\delta_{ij}$; $\quad \Gamma_{jt}{}^{i}=\frac{\dot{a}}{a}\,\delta^i_j$; $\quad \sqrt{|g|}=a^3$;

$T^{tt}=\rho+\cancel{p}-\cancel{p\,g^{tt}}=\rho$; $\quad T^{ij}=-p\,g^{ij}=p\,a^{-2}\delta^{ij}$;

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$$T^{\mu\nu}\Gamma^{t}_{\mu\nu} = T^{ij}\Gamma^{t}_{ij} = \frac{p}{a^2}\delta^{ij}\, a\dot{a}\,\delta_{ij} = 3p\frac{\dot{a}}{a};$$

$$T^{\mu\nu}\Gamma^{i}_{\mu\nu} = 0;$$

$$\begin{cases} \partial_\mu\left(\sqrt{|g|}\,T^{\mu t}\right) = -\sqrt{|g|}\,T^{\mu\nu}\Gamma^{t}_{\mu\nu}; \\ \partial_\mu\left(\sqrt{|g|}\,T^{\mu i}\right) = 0; \end{cases}$$

$$\begin{cases} \partial_t\left(a^3 T^{tt}\right) = 3p\frac{\dot{a}}{a}\cdot a^3; \\ \cancel{\partial_j\left(a^3 T^{ji}\right) = 0}; \end{cases} \quad \left\{ -\partial_t\left(a^3\rho\right) = 3p\,a^2\dot{a}; \right.$$

$$\boxed{\frac{d(a^3\rho)}{dt} = -p\frac{d a^3}{dt}}$$

$a^3 \sim \Delta\,\text{Volume}$

$\rho a^3 \sim \Delta E$

$\Rightarrow d\Delta E \sim -p\, d\Delta V$

(first law of thermodynamics. Adiabatic process)

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Which equations of state $p = p(\rho)$ can we expect?
It turns out that a large class of equations of state are of the form

$$\boxed{p = w\rho} \qquad w: \text{some constant}$$

$p = 0 \rightarrow$ incoherent or cold matter ("dust")
$p = \rho/3 \rightarrow$ hot, relativistic matter (such as radiation)
$p = -\rho \rightarrow$ "vacuum energy"

4.2.6.1 Fundamental fields as fluids

domingo, 4 de octubre de 2015 13:46

Many material systems can be modelled as fluids. For instance, fundamental fields. We are going to review some examples.

(i) The cosmological constant term

It is not a fundamental field (although it can be replaced by a 3-form field) but it can be viewed as a form of matter. Its contribution to the Einstein equations is

$$G_{\mu\nu} = \Lambda g_{\mu\nu} \equiv 8\pi G\, T^{(\Lambda)}_{\mu\nu} \; ; \; T_{\mu\nu} = \frac{\Lambda}{8\pi G} g_{\mu\nu} \; ;$$

Comparing with the stress-energy tensor for a perfect fluid we arrive to the identifications : $-p = \dfrac{\Lambda}{8\pi G} = +\rho$

$p = -\rho \rightarrow$ "equation of state"

De-Sitter-like cosmological constant $\dfrac{\Lambda}{8\pi G} > 0 \; ; \; \rho > 0 \; ; \; p < 0$

(ii) A real scalar field

$$S[\phi] = \int d^4x \sqrt{|g|} \left\{ \frac{1}{2} (\partial\phi)^2 - V(\phi) \right\} ; \quad T_{\mu\nu} = \frac{-2}{\sqrt{|g|}} \frac{\delta S}{\delta g^{\mu\nu}}$$

$$-T_{\mu\nu} = 2\partial_\mu \phi \partial_\nu \phi - g_{\mu\nu}\left[(\partial\phi)^2 - 2V(\phi) \right] \}$$

$$u_\mu \sim \partial_\mu \phi \; ; \qquad p = 2\left[\frac{1}{2}(\partial\phi)^2 - V(\phi) \right] ;$$

$$u^\mu u_\mu = 1 \Rightarrow u^\mu = \frac{\partial^\mu \phi}{\sqrt{(\partial\phi)^2}}$$

$$-T_{\mu\nu} = 2(\partial\phi)^2 u_\mu u_\nu - p g_{\mu\nu} \; ; \quad 2(\partial\phi)^2 = p + \rho \; ;$$

$$\rho = 2(\partial\phi)^2 - p = 2\left[\frac{1}{2}(\partial\phi)^2 + V(\phi) \right] ;$$

$\nabla_\mu T^{\mu\nu} = 0 \longleftrightarrow$ fluid equations $\longleftrightarrow$ scalar field equations

$V(\phi) = 0 \Rightarrow \boxed{p = \rho}$ equation of state.

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(iii) Electromagnetic field

$$-T_{\mu\nu} = \underbrace{-F_{\mu}{}^{\rho} F_{\nu\rho}} + \frac{1}{4} g_{\mu\nu} F^2 \, ; \qquad p = +\frac{1}{4} F^2$$

We can't always find u^μ such that $F_\mu{}^\rho F_{\nu\rho} \propto u_\mu u_\nu$.

When it is possible we find a $T_{\mu\nu}$ such that $T_\mu{}^\mu = 0$

$$-T_\mu{}^\mu = (p+\rho)\overbrace{u_\mu u^\mu}^{1} - p\, g_\mu{}^\mu = (p+\rho) - 4p = \rho - 3p = 0$$

$$\boxed{p = \frac{1}{3}\rho}$$

4.2.6 Coupling to point-like particles. Motion of test particles in gravitational fields

miércoles, 16 de septiembre de 2015 16:08

The action of a point-particle coupled to the gravitational field can be found by applying the PGC (minimal coupling) to the special-relativistic actions that we know

Massive particle : $S[X,g] = -Mc\int d\xi \sqrt{g_{\mu\nu}\dot{X}^\mu \dot{X}^\nu}$

Massless particle : $S[X,e,g] = -\frac{1}{2}\int d\xi\, e^{-1} g_{\mu\nu}\dot{X}^\mu \dot{X}^\nu$

The equations of motion are

Massive particle

$$\delta S = -Mc\int d\xi \frac{1}{2\sqrt{\ }}\delta(g_{\mu\nu}\dot{X}^\mu\dot{X}^\nu) =$$

$$= -Mc\int d\xi \frac{1}{2\sqrt{\ }}\left(\partial_\rho g_{\mu\nu}\,\delta X^\rho\, \dot{X}^\mu\dot{X}^\nu + 2 g_{\mu\nu}\dot{X}^\mu \frac{d}{d\xi}\delta X^\nu\right)$$

$$= -Mc\int d\xi \left[\frac{1}{2\sqrt{\ }}\partial_\rho g_{\mu\nu}\,\delta X^\rho \dot{X}^\mu\dot{X}^\nu - \frac{d}{d\xi}\left(\frac{g_{\rho\nu}\dot{X}^\nu}{\sqrt{\ }}\right)\delta X^\rho + t.d.\right]$$

$$= -Mc\int d\xi\, \delta X^\rho \left[-\frac{d}{d\xi}\left(\frac{g_{\rho\sigma}\dot{X}^\sigma}{\sqrt{\ }}\right) + \frac{1}{2}\partial_\rho g_{\mu\nu}\frac{\dot{X}^\mu}{\sqrt{\ }}\dot{X}^\nu\right]$$

$$\frac{1}{Mc\sqrt{\ }}\frac{\delta S}{\delta X^\rho} = \frac{1}{\sqrt{\ }}\frac{d}{d\xi}\left(\frac{g_{\rho\sigma}\dot{X}^\sigma}{\sqrt{\ }}\right) - \frac{1}{2}\partial_\rho g_{\mu\nu}\frac{\dot{X}^\mu}{\sqrt{\ }}\frac{\dot{X}^\nu}{\sqrt{\ }} =$$

$$= \frac{1}{\sqrt{\ }}g_{\rho\sigma}\frac{d}{d\xi}\left(\frac{\dot{X}^\sigma}{\sqrt{\ }}\right) + \partial_\mu g_{\rho\sigma}\frac{\dot{X}^\mu}{\sqrt{\ }}\frac{\dot{X}^\sigma}{\sqrt{\ }} - \frac{1}{2}\partial_\rho g_{\mu\nu}\dot{X}^\mu\dot{X}^\nu$$

$$= g_{\rho\sigma}\underbrace{\left\{\frac{1}{\sqrt{\ }}\frac{d}{d\xi}\left(\frac{\dot{X}^\sigma}{\sqrt{\ }}\right) + \Gamma_{\mu\nu}{}^{\sigma}\frac{\dot{X}^\mu}{\sqrt{\ }}\frac{\dot{X}^\nu}{\sqrt{\ }}\right\}}_{\text{geodesic equation in arbitrary parametrization}} = 0$$

$\xi = s$ (proper time) $\Rightarrow \sqrt{\ } = 1$ $\qquad \boxed{\ddot{X}^\sigma + \Gamma_{\mu\nu}{}^{\sigma}\dot{X}^\mu\dot{X}^\nu = 0}$

By construction $\dot{X}^\mu\dot{X}^\nu g_{\mu\nu} = +1 \Rightarrow$ timelike geodesics.

For massless particles

$$\delta S \sim \int d\xi \Big\{ -e^{-2}\delta e\, g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu + e^{-1}\Big[\partial_\rho g_{\mu\nu}\delta x^\rho \dot{x}^\mu\dot{x}^\nu + 2 g_{\mu\nu}\dot{x}^\mu \frac{d}{d\xi}\delta x^\nu\Big\}$$

$$= \int d\xi \Big\{ -e^{-2} g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu \delta e + 2e\Big[-\frac{1}{e}\frac{d}{d\xi}\Big(g_{\mu\nu}\frac{\dot{x}^\mu}{e}\Big) + \frac{1}{2}\partial_\nu g_{\mu\rho}\frac{\dot{x}^\mu}{e}\frac{\dot{x}^\rho}{e}\Big]\delta x^\nu\Big\}$$

$$-e^2\frac{\delta S}{\delta e} = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0 \rightarrow \text{null}$$

$$-\frac{1}{2e}\frac{\delta S}{\delta x^\nu} = g_{\nu\mu}\Big\{ \underbrace{\frac{1}{e}\frac{d}{d\xi}\Big(\frac{\dot{x}^\mu}{e}\Big) + \Gamma^\mu_{\rho\sigma}\frac{\dot{x}^\rho}{e}\frac{\dot{x}^\sigma}{e}}_{\text{geodesic in an arbitrary parametrization}}\Big\}$$

$g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0 \Rightarrow$ we can't use the proper time.

Now we can also compute the energy-momentum tensor. Observe that we can rewrite these actions as spacetime integrals using $\delta^{(4)}(x - X(\xi))$:

$$S = \int d^4x\, d\xi \sqrt{|g|}\Big\{ -Mc\sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}\; \frac{\delta^{(4)}(x - X(\xi))}{\sqrt{|g|}}\Big\}$$

$$\frac{2c\,\delta S}{\sqrt{|g|}\,\delta g_{\mu\nu}} = -Mc^2\int d\xi \frac{1}{\sqrt{\ }}\dot{x}^\mu\dot{x}^\nu \frac{\delta^{(4)}(x - X(\xi))}{\sqrt{|g|}} = T^{\mu\nu}$$

In a comoving coordinate system using the proper time

$$T^{\mu\nu} = -Mc^2\int d\xi\, \frac{\delta^{(4)}(x - X(\xi))}{\sqrt{|g|}}\,\delta^{\mu 0}\delta^{\nu 0}$$

Consistency requires that we can derive the equations of

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motion from $\nabla_\mu T^{\mu\nu} = 0$

$$\sqrt{|g|}\,\nabla_\mu T^{\mu\nu} = \partial_\mu\left(\sqrt{|g|}\,T^{\mu\nu}\right) + \sqrt{|g|}\,T^{\rho\sigma}\,\Gamma_{\rho\sigma}{}^{\nu}$$

$$\sqrt{|g|}\,T^{\mu\nu} = -Mc^2\int d\xi\,\frac{\dot X^\mu \dot X^\nu}{\sqrt{\ }}\,\delta^{(4)}(x-X)$$

$$\partial_\mu\left(\sqrt{|g|}\,T^{\mu\nu}\right) = -Mc^2\int d\xi\,\frac{\dot X^\mu \dot X^\nu}{\sqrt{\ }}\,\partial_\mu\delta^{(4)}(x-X) =$$

$$= -Mc^2\int d\xi\,\frac{\dot X^\nu}{\sqrt{\ }}\,\frac{d}{d\xi}\delta^{(4)}(x-X)$$

$$= -Mc^2\int d\xi\left\{\frac{d}{d\xi}\left[\frac{\dot X^\nu}{\sqrt{\ }}\,\delta^{(4)}(x-X)\right] - \frac{d}{d\xi}\left(\frac{\dot X^\nu}{\sqrt{\ }}\right)\delta^{(4)}(x-X)\right\}$$

$$\sqrt{|g|}\,T^{\rho\sigma}\,\Gamma_{\rho\sigma}{}^{\nu} = -Mc^2\int d\xi\;\Gamma_{\rho\sigma}{}^{\nu}\,\frac{\dot X^\rho \dot X^\sigma}{\sqrt{\ }}\,\delta^{(4)}(x-X)$$

$$\nabla_\mu T^{\mu\nu} = -Mc^2\int d\xi\,\sqrt{\ }\,\underbrace{\left\{\frac{1}{\sqrt{\ }}\frac{d}{d\xi}\left(\frac{\dot X^\nu}{\sqrt{\ }}\right) + \Gamma_{\rho\sigma}{}^{\nu}\,\frac{\dot X^\rho}{\sqrt{\ }}\frac{\dot X^\sigma}{\sqrt{\ }}\right\}}_{=\,0 \text{ on shell}}\delta^{(4)}(x-X)$$

$$\underbrace{-Mc^2\,\frac{\dot X^\nu}{\sqrt{\ }}\,\delta^{(4)}(x-X)\Big]_{\xi=-\infty}^{\xi=+\infty}}_{\text{a source and a sink at } \infty}$$

4.3 The weak field limit of the Einstein equations

miércoles, 16 de septiembre de 2015 16:05

After the construction of a set of field equations to describe the gravitational field and its interaction with matter we have to study if it really does. This implies

1) Recovering the Newtonian results in the appropriate limits. (There is no special-relativistic limit).

2) Finding predictions of the theory to be tested experimentally.

The Newtonian theory of gravity is non-relativistic and also linear (Newtonian gravity does not gravitate), and we should recover it from GR in a linearized (weak field) limit + a non-relativistic limit.

Here we want to study (a bit) this limit, which is extremely important because most gravitational phenomena in the Solar System are weak-field phenomena.

In GR weak field means $g_{\mu\nu}$ close to the Minkowski metric $\eta_{\mu\nu}$

$$g_{\mu\nu} = \eta_{\mu\nu} + \delta g_{\mu\nu} ; \qquad \delta g_{\mu\nu} \equiv \alpha\, h_{\mu\nu} ; \qquad \alpha^2 = \frac{16\pi G}{c^3} ;$$

We are interested in the equations satisfied by $h_{\mu\nu}$ which will be assumed to be small enough so we can ignore h^2 terms. Thus, we cannot raise the indices of $h_{\mu\nu}$ with anything proportional to h $\Rightarrow$ we deal with it as if it is just a Lorentz tensor

$$h^{\mu\nu} \equiv \eta^{\mu\rho}\eta^{\nu\sigma} h_{\rho\sigma}$$

Equivalently $O(\alpha^2)$

$$\Rightarrow g^{\mu\nu} = \eta^{\mu\nu} - \alpha\, h^{\mu\nu} + O(h^2)$$

$$g^{\mu\nu} g_{\nu\rho} = \delta^{\mu}{}_{\rho} - \cancel{\alpha\, h^{\mu}{}_{\rho}} + \cancel{\alpha\, h_{\rho}{}^{\mu}} + O(h^2)$$

$$g = \alpha\, h_{\mu}{}^{\mu} ;$$

The equations satisfied by $h_{\mu\nu}$ follow from those satisfied by $g_{\mu\nu}$: Einstein equations. We have to compute $G_{\mu\nu}$

$$\Gamma_{\mu\nu}{}^{\rho} = \cancel{\Gamma_{\mu\nu}{}^{\rho}(\eta)} + \delta\Gamma_{\mu\nu}{}^{\rho} = \frac{1}{2}\eta^{\rho\sigma}\{\partial_\mu \delta g_{\sigma\nu} + \partial_\nu \delta g_{\sigma\mu} - \partial_\sigma \delta g_{\mu\nu}\} =$$

$$= \frac{1}{2}\chi\,\eta^{\rho\sigma}(\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu});$$

$\Rightarrow \Gamma\Gamma$ is $\mathcal{O}(h^2)$

$$R_{\mu\nu\rho}{}^{\sigma} = 2\partial_{[\mu}\Gamma_{\nu]\rho}{}^{\sigma} + \mathcal{O}(h^2) =$$

$$= \chi\,\eta^{\sigma\lambda}\partial_{[\mu|}\{\cancel{\partial_{|\nu]}h_{\lambda\rho}} + \partial_\rho h_{\lambda|\nu]} - \partial_\lambda h_{\rho|\nu]}\} =$$

$$= \chi\left[\partial_\rho\partial_{[\mu}h^{\sigma}{}_{\nu]} - \partial^{\sigma}\partial_{[\mu}h_{\nu]\rho}\right];$$

$$R_{\mu\rho} = R_{\mu\nu\rho}{}^{\nu} = \frac{1}{2}\chi\left[\partial_\rho\partial_\mu h^{\nu}{}_{\nu} - \partial_\rho\partial_\nu h^{\nu}{}_{\mu} - \partial^{\nu}\partial_\mu h_{\nu\rho} + \partial^2 h_{\mu\rho}\right]$$

Define $h \equiv h^{\rho}{}_{\rho} = \eta^{\rho\sigma}h_{\rho\sigma}$

$$R = \frac{1}{2}\chi\{\partial^2 h - 2\partial_\mu\partial_\nu h^{\mu\nu} + \partial^2 h] = \chi[\partial^2 h - \partial_\mu\partial_\nu h^{\mu\nu}]$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R + \mathcal{O}(h^2)$$

$$= \frac{1}{2}\chi\Big\{\underbrace{\partial_\mu\partial_\nu h - 2\partial_{(\mu|}\partial^{\rho}h_{\rho|\nu)} + \partial^2 h_{\mu\nu} - \eta_{\mu\nu}(\partial^2 h - \partial_\rho\partial_\sigma h^{\rho\sigma})}_{D^{(1)}_{\mu\nu}(h)}\Big\}$$

<u>Fierz-Pauli field equations</u>

Let us consider now the coupling to matter

$$G_{\mu\nu} = \frac{\chi^2}{2c}T_{\mu\nu} \quad ; \quad \frac{1}{2}\chi D_{\mu\nu}(h) = \frac{\chi^2}{2c}\left(T_{\mu\nu}\Big|_{g=\eta} + \mathcal{O}(\chi)\right)$$

Rosenfeld

$$\Rightarrow \boxed{D_{\mu\nu}(h) = \frac{\chi}{c}T_{\mu\nu}^{\text{Rosenfeld}}}$$

Linear theory $\Rightarrow$ no $T_{\mu\nu}(h)$!

martes, 6 de octubre de 2015 18:57

The Fierz-Pauli equations are invariant under the following gauge transformations:

$$\delta h_{\mu\nu} = -2\partial_{(\mu}\varepsilon_{\nu)} \quad ; \quad \varepsilon_\mu(x) \text{ arbitrary infinitesimal}$$

This is the weak-field limit of

$$\delta g_{\mu\nu} = -2\nabla_{(\mu}\varepsilon_{\nu)} \quad \Rightarrow \quad \text{g.c.t.s.}$$

The contracted Bianchi identity $\nabla_\mu G^{\mu\nu} = 0$ has the limit

$$\partial_\mu \mathcal{D}^{\mu\nu}(h) = 0 \, ; \qquad \Rightarrow \quad \partial_\mu T^{\mu\nu} = 0 \quad \text{(which we know holds)}$$

We can use the gauge freedom to impose conditions on $h_{\mu\nu}$:

① Transverse, traceless gauge $\partial_\mu h^{\mu\nu} = h = 0$

② DeDonder gauge $\partial_\mu \bar{h}^{\mu\nu} \equiv \partial_\mu\left(h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h\right) = 0$

$\Downarrow$

$$\mathcal{D}_{\mu\nu}(h) = \partial^2 \bar{h}_{\mu\nu} \quad \text{(usual wave operator for a massless field)}$$

$\Rightarrow$ There will be gravitational waves

$$dx^0\,\delta(x^0)$$
$$dt''\,\delta(t)$$

In general $\boxed{\partial^2 \bar{h}_{\mu\nu} = \frac{\chi}{c} T_{\mu\nu}}$

If we have a massive particle at rest $s = ct$, in $\vec{x} = \vec{0}$

$$T^{\mu\nu} = -Mc^2 \int dt \;\; \delta(t)\,\delta^{(3)}(\vec{x}) \quad \delta^{\mu 0}\delta^{\nu 0}$$
$$= -Mc^2 \;\; \delta^{(3)}(\vec{x}) \cdot \delta^{\mu 0}\delta^{\nu 0}$$

$$\Rightarrow \; +\partial_i\partial_i \bar{h}_{\mu\nu} = +\chi M c \;\; \delta^{(3)}(\vec{x})\, \delta_{\mu 0}\delta_{\nu 0} ;$$

$$\bar{h}_{\mu\nu} = -\frac{\chi M c}{4\pi} \frac{1}{r} \delta_{\mu 0}\delta_{\nu 0} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h ;$$

$$\bar{h} = \eta^{\mu\nu}\bar{h}_{\mu\nu} = -h \quad \Rightarrow \quad h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h} ;$$

$$\bar{h} = -\frac{\alpha M c}{4\pi}\frac{1}{r} ; \quad h_{\mu\nu} = -\frac{\alpha M c}{4\pi}\frac{1}{r}\delta_{\mu 0}\delta_{\nu 0} - \frac{1}{2}\eta_{\mu\nu}\left(-\frac{\alpha M c}{4\pi}\frac{1}{r}\right)$$

$$h_{00} = -\frac{\alpha M c}{8\pi}\frac{1}{r} ; \quad \frac{c^2\alpha}{2} h_{00} = -\frac{GM}{r} = \phi ;$$

$$h_{ij} = -\frac{\alpha M c}{8\pi}\frac{1}{r}\delta_{ij} ; \quad \frac{c^2\alpha}{2} h_{ij} = -\frac{GM}{r}\delta_{ij} = \phi\delta_{ij} ;$$

where ϕ is the Newtonian potential created by a mass M

$$\Rightarrow \boxed{ds^2 = \left(1 + \frac{2\phi}{c^2}\right)c^2 dt^2 - \left(1 - \frac{2\phi}{c^2}\right) d\vec{x}^2 \qquad \phi = -\frac{GM}{r} ;}$$

4.3.1 Point particles coupled to weak gravitational fields

martes, 6 de octubre de 2015 16:57

The action that describes the motion of a test point particle moving in a gravitational field is, according to the PGE

$$S[X^\mu] = -mc\int d\xi \sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}$$

For weak gravitational fields $g_{\mu\nu} = \eta_{\mu\nu} + \alpha h_{\mu\nu}$

$$S[X^\mu] = -mc\int d\xi \left\{ \underbrace{\sqrt{\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}}_{\text{kinetic term}} + \underbrace{\frac{1}{2}\frac{\alpha h_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}{\sqrt{\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}}}_{\text{interaction term}} \right\}$$

In the static gauge $\xi = X^0 = ct$

$$S[X^\mu] = -mc^2\int dt \left\{ \sqrt{1-v^2/c^2} + \frac{\alpha/2}{\sqrt{1-v^2/c^2}}\left(h_{00} + 2h_{0i}\frac{v^i}{c} + h_{ij}\frac{v^i v^j}{c^2}\right)\right\}$$

$\frac{hv^2}{c^2} \ll 1$; $\left(\frac{v}{c}\right)^4 \ll 1 \Rightarrow \sim -mc^2\int dt\left\{1 - \frac{1}{2c^2}v^2 + \alpha/2\left(1+\frac{1}{2c}v^2\right)\left(h_{00} + 2h_{0i}\frac{v^i}{c}\right)\right\}$

$$\sim \int dt \left\{ \overset{\text{t.d.}}{\cancel{-mc^2}} + \frac{1}{2}mv^2 - \frac{\alpha}{2}\underbrace{\left(mc^2 + \frac{1}{2}mv^2\right)}_{\text{all energy couples to gravity}}\left(h_{00} + 2h_{0i}\frac{v^i}{c}\right)\right\}$$

$$\sim \int dt\left\{\frac{1}{2}mv^2 - m\phi - \alpha m c\, h_{0i}v^i\right\} \qquad A_i \equiv \alpha c^2 h_{0i}$$

$$= \int dt\left\{\frac{1}{2}mv^2 - m\phi - A_i\frac{v^i}{c}\right\}$$

gravitostatic "electric" potential

"gravitomagnetic" potential

If ϕ is created by a mass M $\phi = -\frac{MG}{r}$ and the interaction term is $V = m\phi = -\frac{mMG}{r}$

$$\boxed{F_i = -\partial_i V = -\frac{mMG}{r^3}x_i}$$ Newton's law.

We see that there will be relativistic corrections $\mathcal{O}(v/c)^2$ but also new "gravitomagnetic" effects.

4.4 Covariant generalizations of GR and higher curvature and derivative terms

miércoles, 16 de septiembre de 2015 16:14

The Einstein-Hilbert action is constructed with the simplest invariant: R. If we allow for terms of higher orders in derivatives, there are many other possibilities. Also, we can couple matter (specially scalars) in more general ways, both to gravity and to other matter fields (vectors and scalars), introducing new interactions. (String theory, quantum corrections)

Let us start with scalars. Since they are scalars, we can multiply by functions of them $\mathcal{L}/\sqrt{|g|}$

$$-\frac{1}{4}F^2 \longrightarrow -\frac{1}{4}f(\phi)F^2$$

$$R \longrightarrow g(\phi)R$$

$$\frac{1}{2}(\partial\varphi)^2 \longrightarrow \frac{1}{2}h(\phi)(\partial\varphi)^2$$

We can always redefine $g(\phi)\to\phi'$, but $f(\phi')$, $h(\phi')$ remain there.

A very well studied theory that makes use of this possibility is the Jordan-Brans-Dicke theory, whose action is given by

Brans-Dicke parameter.

$$S[\phi, g, \text{matter}] = \frac{1}{16\pi}\int d^4x\sqrt{|g|}\left\{\phi R - \omega\frac{(\partial\phi)^2}{\phi} + 16\pi\frac{\mathcal{L}_{\text{matter}}}{\sqrt{|g|}}\right\}$$

ϕ plays the role of spacetime-dependent inverse Newton constant.

To see how GR is modified, let us derive the equations of motion.

$$\frac{16\pi}{\sqrt{|g|}}\frac{\delta S}{\delta\phi} = R + \omega\frac{(\partial\phi)^2}{\phi^2} + 2\omega\,\nabla_\mu\left(\partial^\mu\phi/\phi\right) = R - \omega\left(\frac{\partial\phi}{\phi}\right)^2 + 2\omega\frac{\nabla^2\phi}{\phi} = 0$$

martes, 6 de octubre de 2015 21:03

The variation w.r.t. $g_{\mu\nu}$ is more complicated

$$\delta S = \frac{1}{16\pi}\int d^4x \Big\{ -\frac{1}{2}\sqrt{|g|}\,\delta g^{\mu\nu} g_{\mu\nu}\,\phi R + \sqrt{|g|}\,\phi\left[\delta g^{\mu\nu} R_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}\right]$$

$$-\frac{\omega}{\phi}\sqrt{|g|}\,\delta g^{\mu\nu}\left[\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(\partial\phi)^2\right] - 8\pi\sqrt{|g|}\,\delta g^{\mu\nu}\, T^{\text{matt}}_{\mu\nu}\Big\}$$

$$= \frac{1}{16\pi}\int d^4x\sqrt{|g|}\Big\{\delta g^{\mu\nu}\phi\left[G_{\mu\nu} - \frac{\omega}{\phi^2}\left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(\partial\phi)^2\right) - \frac{8\pi}{\phi}T_{\mu\nu}\right]$$

$$+\phi\nabla_\mu v^\mu\Big\}$$ no longer a total derivative

$$v^\mu = 2g^{\mu\nu,\rho\sigma}\nabla_\rho\delta g_{\sigma\nu}\ ;$$

Let us focus on the last term only:

$$\int d^4x\sqrt{|g|}\,\phi\nabla_\mu v^\mu = \int d^4x\sqrt{|g|}\left[-v^\mu\nabla_\mu\phi + t.d.\right] =$$

$$= \int d^4x\sqrt{|g|}\left[-2g^{\mu\nu,\rho\sigma}\nabla_\rho\delta g_{\sigma\nu}\nabla_\mu\phi + t.d.\right]$$

$$= \int d^4x\sqrt{|g|}\left[2\,\delta g_{\sigma\nu}\, g^{\mu\nu,\rho\sigma}\nabla_\rho\nabla_\mu\phi + t.d.\right]$$

$$= \int d^4x\sqrt{|g|}\left[\delta g_{\sigma\nu}\left(g^{\sigma\nu}\nabla^2\phi - \nabla^\sigma\nabla^\nu\phi\right) + t.d.\right]$$

$$= \int d^4x\sqrt{|g|}\left[\delta g^{\mu\nu}\left(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\nabla^2\phi\right)\right.$$

$$\frac{16\pi}{\phi\sqrt{|g|}}\frac{\delta S}{\delta g^{\mu\nu}} = \boxed{G_{\mu\nu} - \omega\left[\frac{\partial_\mu\phi}{\phi}\frac{\partial_\nu\phi}{\phi} - \frac{1}{2}g_{\mu\nu}\left(\frac{\partial\phi}{\phi}\right)^2\right] - \frac{8\pi}{\phi}T^{\text{matter}}_{\mu\nu} + \frac{1}{\phi}\left(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\nabla^2\phi\right) = 0}$$

$$g^{\mu\nu} \quad -R + \omega\left(\frac{\partial\phi}{\phi}\right)^2 - \frac{8\pi}{\phi}T + 3\frac{\nabla^2\phi}{\phi} = 0\ ;$$

$$R - \omega\left(\partial\phi/\phi\right)^2 + 2\omega\nabla^2\phi/\phi = 0\ ;$$

Combining both equations we get

$$\boxed{\nabla^2 \phi = \frac{8\pi}{3+2\omega} T}$$

$$\boxed{G_{\mu\nu} - \omega\left[\frac{\partial_\mu \phi}{\phi}\frac{\partial_\nu \phi}{\phi} - \frac{1}{2} g_{\mu\nu}\left(\frac{\partial \phi}{\phi}\right)^2\right] - \frac{8\pi}{\phi} T_{\mu\nu}^{\text{matter}} + \frac{1}{\phi}\left(\nabla_\mu \nabla_\nu \phi - g_{\mu\nu}\nabla^2 \phi\right) = 0}$$

Comment: ϕ should be strictly positive $\phi > 0$. A better variable is $e^{-2\varphi} = \phi$, because φ could be unconstrained and ϕ always > 0. String theory is JBD with $\omega = 1$

Einstein frame

Performing local rescalings $g_{\mu\nu} \to \Omega^2(x) g_{\mu\nu}$ (which are not symmetries of the EH action nor of Physics, as far as we know) we can change the prefactor of R in the EH action. Different factors are said to characterize different "conformal frames". The Einstein conformal frame is the one in which there is no prefactor. The frame in which we have defined the JBD theory is usually called "Jordan frame".

To change frames we use

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}. \qquad \tilde{g}^{\mu\nu} = \Omega^{-2} g^{\mu\nu}$$

$$\Rightarrow \sqrt{|\tilde{g}|} = \Omega^4 \sqrt{|g|}$$

$$\tilde{R} = \Omega^{-2}\left[R + 6(\partial \ln \Omega)^2 + 6\nabla^2 \ln \Omega\right]$$

$$\sqrt{|\tilde{g}|}\,\tilde{R} = \Omega^2 \sqrt{|g|}\left[R + 6(\partial \ln \Omega)^2 + 6\nabla^2 \ln \Omega\right]$$

If we want to cancel ϕ

$$\phi\sqrt{|\tilde{g}|}\,\tilde{R} = \phi\,\Omega^2\sqrt{|g|}\left[R + 6(\partial \ln\Omega)^2 + 6\nabla^2\ln\Omega\right]$$

$$\Omega = \phi^{-1/2} \longrightarrow = \sqrt{|g|}\left[R + \frac{3}{2}\left(\frac{\partial\phi}{\phi}\right)^2 - 3\nabla^2\ln\phi\right].$$

$$\phi\sqrt{|\tilde{g}|}\,\tilde{g}^{\mu\nu}\frac{\partial_\mu\phi\,\partial_\nu\phi}{\phi^2} = \sqrt{|g|}\,g^{\mu\nu}\frac{\partial_\mu\phi\,\partial_\nu\phi}{\phi}$$

$$\Rightarrow S[\tilde{g},\phi] = \frac{1}{16\pi}\int d^4x\sqrt{|\tilde{g}|}\,\phi\left[\tilde{R} - \omega\left(\frac{\partial\phi}{\phi}\right)^2\right] =$$

total derivative

$$= \frac{1}{16\pi}\int d^4x\sqrt{|g|}\left[R + \left(\frac{3}{2}-\omega\right)\left(\frac{\partial\phi}{\phi}\right)^2 \cancel{- 3\nabla^2\ln\phi}\right]$$

The Weyl scaling of the matter Lagrangian depends on the matter fields under consideration.

Physically, these frames are clearly not equivalent. However, one can map all the results obtained in one of them to the other.

Let us now consider invariants built with the curvature tensor: $f(R)$, $R_{\mu\nu}R^{\mu\nu}$, $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ etc.
Let us take, for example $f(R)$ (inflation etc)

$$S[g] = \frac{1}{16\pi G}\int d^4x\sqrt{|g|}\, f(R)$$

$$\delta S = \frac{1}{16\pi G}\int d^4x \left\{-\frac{1}{2}\sqrt{|g|}\,\delta g^{\mu\nu} g_{\mu\nu} f(R) + \sqrt{|g|}\, f'(R)\,\delta R\right\}$$

$$= \frac{1}{16\pi G}\int d^4x\sqrt{|g|}\left\{\delta g^{\mu\nu}\left[R_{\mu\nu} f'(R) - \frac{1}{2} g_{\mu\nu} f(R)\right] + \underbrace{f'(R)}_{\phi}\nabla_\mu v^\mu\right\}$$

$$= \frac{1}{16\pi G}\int d^4x\sqrt{|g|}\,\delta g^{\mu\nu}\left[R_{\mu\nu}\phi - \frac{1}{2}g_{\mu\nu}f(R) + \nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\nabla^2\phi\right]$$

$$\frac{16\pi G}{\phi\sqrt{|g|}}\frac{\delta S}{\delta g^{\mu\nu}} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\frac{f(R)}{\phi} + \frac{1}{\phi}\left(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\nabla^2\phi\right) = \frac{8\pi G T_{\mu\nu}}{\phi}$$

$$\boxed{G_{\mu\nu} = \frac{8\pi G}{\phi}T_{\mu\nu} + \frac{1}{2}g_{\mu\nu}\left(\frac{f(R)}{\phi} - R\right) - \frac{1}{\phi}\left(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\nabla^2\phi\right)}.$$

This equation is similar to that of the Jordan-Brans-Dicke theory. In fact $f(R)$ theories are equivalent to JBD theories with a scalar potential and $\omega = 0$.

$$S = \frac{1}{16\pi}\int d^4x\sqrt{|g|}\left[\phi R - \omega\frac{(\partial\phi)^2}{\phi} - V(\phi)\right]$$

with $\phi = f'(R)$ and $V(\phi) = R f'(R) - f(R)$;
$\Downarrow$
$R = R(\phi)$