1.1 Brief review of Special Relativity

lunes, 17 de agosto de 2015 22:03

Bibliography:

- Special Relativity: A First encounter
  Domenico Giulini, OUP (2005).
- An Illustrated Guide to Relativity
  Tatsu Takeuchi, CUP (2010)
- Special Relativity: From Einstein to Strings
  Patricia M. Schwarz & John H. Schwarz, CUP (2004)
- Einstein Gravity in a Nutshell
  A. Zee, PUP (2013) (Parts II-III-IV)

Main objectives

- Review the tensorial formulation of SR
- Review the Principle of Least Action, Noether theorems etc. for their later use in GR

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(310) 15:00 — 17:30

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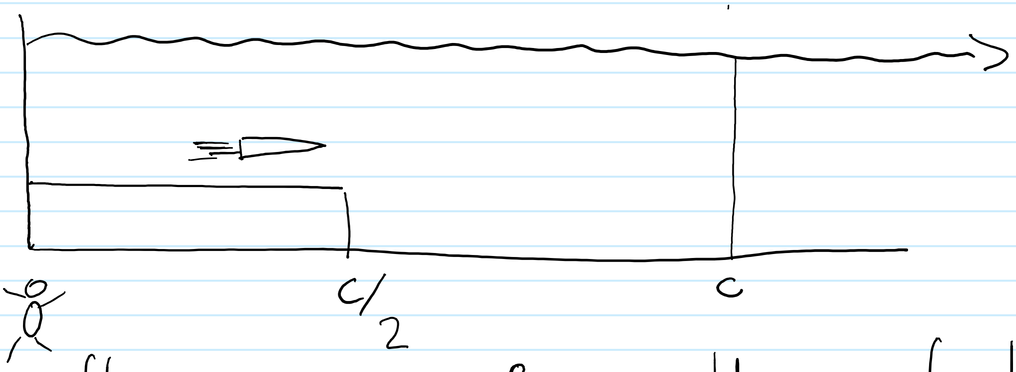
1.1.1 The Principle of Special Relativity

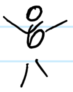
lunes, 17 de agosto de 2015 22:23

It is a generalization of Galileo's Relativity Principle to include electromagnetism formulated by Einstein in 1905:

"All the laws of Physics take the same form in any inertial reference" (1)

This principle includes Maxwell's laws of Electromagnetism. These laws imply that light is an electromagnetic wave which propagates in vacuum at speed $c \sim 3 \times 10^8$ m s^{-1}. Then, choosing Maxwell's laws to be good laws of Physics (but not Newton's!) c must be the same for any inertial observer independently of the motion of the source



After 1 second  sees the wavefront at c

 sees the wavefront at c

⇒ Their times must be different

(1) This is a reformulation of Einstein's 2 original postulates.

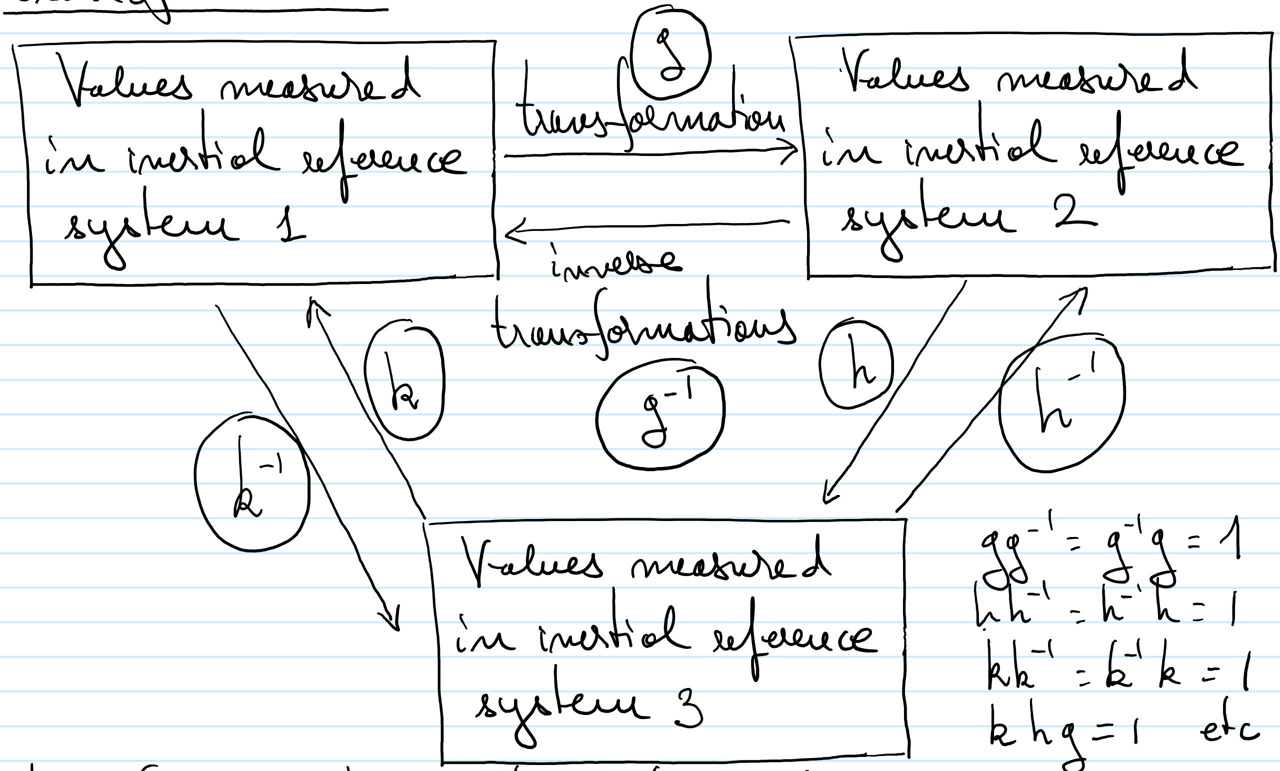
1.1.2 Lorentz transformations. The Lorentz and Poincaré groups

lunes, 17 de agosto de 2015 22:25

We want to find the relation between the measurements made by two inertial observers.

NB: The form of the laws of Physics must be the same but the values of the physical magnitudes involved must'nt.

The relation between measurements is expressed as a "transformation":



$gg^{-1} = g^{-1}g = 1$

$hh^{-1} = h^{-1}h = 1$

$kk^{-1} = k^{-1}k = 1$

$khg = 1$ etc

etc. for all the inertial reference frames. (∞)

$\Rightarrow$ The transformations that relate the measurements made in different inertial reference frames are a group.

This is also the group of transformations that leaves the laws of Physics invariant: it is a group of symmetry transformations.

All symmetry (or invariance) transformations $\rightarrow$ groups

To find the symmetry group of Special Relativity, the "Poincaré group" we have to study the invariance of the laws of Physics. For instance, the invariance of the speed of light:

Inertial reference frame ① : $(t, \underbrace{x^1, x^2, x^3}_{\equiv\, x^i;\ i=1,2,3}) \to$ event coordinates

A pulse of light is at (t_0, x_0^i) and at (t_1, x_1^i) as measured in this frame. Then $\Delta t = t_1 - t_0$; $\Delta x^i = x_1^i - x_0^i$;

$$\frac{\sqrt{(\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2}}{\Delta t} = c; \qquad c^2 \Delta t^2 - \Delta x^i \Delta x^i = 0$$

$\uparrow$ Einstein convention: repeated indices are summed over

$$\Delta x^i \Delta x^i = \sum_{i=1}^{3} \Delta x^i \Delta x^i$$

Defining $x^0 \equiv ct$

$$\Rightarrow \boxed{\Delta x^0 \Delta x^0 - \Delta x^i \Delta x^i = 0}$$

Inertial frame ② : (t', x'^i)

The same two events are, as measured in this frame (t'_0, x'^i_0) and (t'_1, x'^i_1); $\Delta t' = t'_1 - t'_0$; $\Delta x'^i = x'^i_1 - x'^i_0$;

$$\Rightarrow \boxed{\Delta x'^0 \Delta x'^0 - \Delta x'^i \Delta x'^i = 0}$$

(with primes!)

This law of Physics has the same form, it is form-invariant or just invariant:

$$\boxed{\Delta x^0 \Delta x^0 - \Delta x^i \Delta x^i = \Delta x'^0 \Delta x'^0 - \Delta x'^i \Delta x'^i;}$$

(It can be shown (Einstein clock, Zee page 166) that the invariance of the speed of light implies the above relation also for $\Delta x^0 \Delta x^0 - \Delta x^i \Delta^i \neq 0$. Otherwise...)

The transformations between the measurements express the coordinates of any event in frame 2 $(\underbrace{x'^0, x'^i}_{x'^\mu})$ as functions of those in frame 1 $(\underbrace{x^0, x^i}_{x^\mu})$

⇒ The Poincaré group is the group of functions $x'^\mu(x)$ which leave invariant $\underbrace{\Delta x^0 \Delta x^0 - \Delta x^i \Delta x^i}_{\text{"interval" between 2 events.}}$ for any two events x^μ_0, x^μ_1

- The simplest Poincaré transformations are constant <u>spacetime translations</u> (a subgroup)

$$\boxed{x'^\mu = x^\mu + a^\mu ; \quad a^\mu \text{ arbitrary}} \Rightarrow \Delta x'^\mu = \Delta x^\mu ;$$

⇓ Independence of the space and time origin of coordinates.

- Another subgroup of Poincaré transformations are the linear ones:

$$x \equiv \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} ; \qquad \boxed{x' = \Lambda x ; \quad \text{or} \quad x'^\mu = \Lambda^\mu{}_\nu x^\nu}$$

row column

$$\Lambda = \begin{pmatrix} \Lambda^0{}_0 & \Lambda^0{}_1 & \cdots \\ \Lambda^1{}_0 & & \end{pmatrix} = \begin{pmatrix} \Lambda^0{}_0 & \Lambda^0{}_i \\ \Lambda^i{}_0 & \Lambda^i{}_j \end{pmatrix} = (\Lambda^\mu{}_\nu)$$

What is Λ like?

$$\eta \equiv \begin{pmatrix} \eta_{00} & \eta_{01} & \cdots \\ \eta_{10} & \ddots & \\ \vdots & & \end{pmatrix} = \begin{pmatrix} \eta_{00} & \eta_{0j} \\ \eta_{i0} & \eta_{ij} \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$\eta_{00} = 1 \quad ; \quad \eta_{ij} = -\delta_{ij}$$

With this definition, the invariance of the interval is

$$\boxed{\Delta x'^T \eta \Delta x' = \Delta x^T \eta \Delta x ; \quad \text{or} \quad \Delta' x^\mu \eta_{\mu\nu} \Delta x'^\nu = \Delta x^\mu \eta_{\mu\nu} \Delta x^\nu ;}$$

sábado, 29 de agosto de 2015 15:50

$x' = \Lambda x$; $\Delta x' = \Lambda \Delta x$; $\Delta x'^T = \Delta x^T \Lambda^T$ $\Rightarrow$ $\Delta x^T \Lambda^T \eta \Lambda \Delta x = \Delta x^T \eta \, \Delta x$

$$\Downarrow$$

$$\boxed{\Lambda^T \eta \Lambda = \eta}$$

$x'^\mu = \Lambda^\mu{}_\rho x^\rho$; $\Delta x'^\rho = \Lambda^\mu{}_\rho \Delta x^\rho$; $\Lambda^\mu{}_\rho \Delta x^\rho \eta_{\mu\nu} \Lambda^\nu{}_\sigma \Delta x^\sigma =$

$= \Delta x^\rho \eta_{\rho\sigma} \Delta x^\sigma$

$$\Downarrow$$

$$\boxed{\Lambda^\mu{}_\rho \eta_{\mu\nu} \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}}$$

The group of transformations of this form is known as Lorentz group ($O(1,3)$) (Check Λ_1, Λ_2 Lorentz $\Rightarrow \Lambda_1 \Lambda_2$)

All the Poincaré transformations are of the form

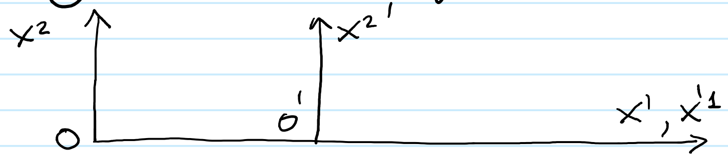
$$\boxed{x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu}$$

$\uparrow$ Lorentz $\uparrow$ traslation

Observe: $x''^\mu = \Sigma^\mu{}_\nu x'^\nu + b^\mu = \Sigma^\mu{}_\nu \left(\Lambda^\nu{}_\rho x^\rho + a^\nu \right) + b^\mu$

$$= \underbrace{\Sigma^\mu{}_\nu \Lambda^\nu{}_\rho x^\rho}_{\text{Lorentz}} + \underbrace{\Sigma^\mu{}_\nu a^\nu + b^\mu}_{\text{traslation}}$$

The Lorentz transformation between two inertial frames depends on their relative speed. Using spatial rotations ($SO(3) \subset O(1,3)$) and translations system 2 moves along the x^1 axis of system 1 with speed v and $x'^2 = x^2$; $x'^3 = x^3$



$x^{1'} = 0 \Rightarrow x^1 = vt = \frac{v}{c} x^0$

$\Rightarrow x^{1'} = \omega \left(x^1 - \frac{v}{c} x^0 \right)$

$(x'^0)^2 - (x'^1)^2 = (x^0)^2 - (x^1)^2$

$$\begin{pmatrix} x'^0 \\ x'^1 \end{pmatrix} = \begin{pmatrix} a & b \\ -\omega \frac{v}{c} & \omega \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \end{pmatrix} ;$$

$$b = a \xi \frac{v}{c} ; \quad \begin{cases} x'^0 = a \left(x^0 + \xi \frac{v}{c} x^1 \right) ; \\ x'^1 = \omega \left(x^1 - \frac{v}{c} x^0 \right) ; \end{cases}$$

jueves, 10 de septiembre de 2015 10:47

$$\left.\begin{aligned} x'^0 &= a\left(x^0 + \xi \tfrac{v}{c} x^1\right); \\ x'1 &= \omega\left(x^1 - \tfrac{v}{c} x^0\right); \end{aligned}\right\} \quad a^2\left(x^0 + \xi \tfrac{v}{c} x^1\right)^2 - \omega^2\left(x^1 - \tfrac{v}{c} x^0\right)^2 = (x^0)^2 - (x^1)^2$$

$$\begin{cases} a^2 - \omega^2 (v/c)^2 = 1; \\ a^2 \xi^2 (v/c)^2 - \omega^2 = -1; \\ 2\, a^2 \xi\, v/c + 2\omega^2 v/c = 0; \rightarrow a^2 \xi + \omega^2 = 0; \quad \boxed{\omega^2 = -\xi a^2} \end{cases}$$

$$\left.\begin{aligned} &a^2 + \xi a^2 (v/c)^2 = 1; && a^2 = 1/\left[1 + \xi\left(\tfrac{v}{c}\right)^2\right] \\ &a^2 \xi^2 (v/c)^2 + \xi a^2 = -1; && a^2 \xi = -1/\left[1 + \xi\left(\tfrac{v}{c}\right)^2\right] \end{aligned}\right\}$$

$$\boxed{\xi = -1; \quad a = \omega = \frac{1}{\sqrt{1-(v/c)^2}}}$$

$$\boxed{\begin{aligned} x'^0 &= \frac{x^0 - v/c\, x^1}{\sqrt{1-(v/c)^2}} \\ x'^1 &= \frac{x^1 - v/c\, x^0}{\sqrt{1-(v/c)^2}}; \end{aligned}}$$

This simple Lorentz transformation is called a "Lorentz boost". (3)
Spatial rotations are "the rest" of the Lorentz transformations (3)

3+3 = 6 independent transf.

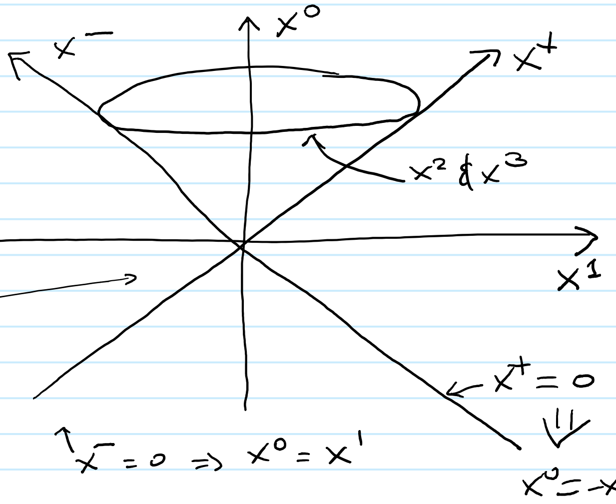
Light-cone coordinates

These coordinates are defined by $x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^1)$ (sometimes $x^+ \rightarrow u$, $x^- \rightarrow v$) so that

$$\boxed{(x^0)^2 - (x^1)^2 = 2x^+x^-}$$

Light-cone of a pulse of light emitted at $x^0 = x^1 = 0$ and $x^\mu x^\nu \eta_{\mu\nu} = 0$

$$\boxed{x^+x^- = 0}$$



In these coordinates the Lorentz t. are far easier to derive:

$$2x^+x^- = 2x'^+x'^- = 2(ax^+ + bx^-)(cx^+ + dx^-)$$

$ac = 0;$

$bd = 0;$ $\qquad ad - bc \neq 0$ (invertible) + connected to 1

$ad + bc = 1;$ $\qquad \Rightarrow b = c = 0 \; ; \; d = 1/a$

$$\boxed{\begin{array}{l} x'^+ = e^\phi x^+; \\ x'^- = e^{-\phi} x^-; \end{array}}$$

$$x'^0 = \frac{1}{\sqrt{2}}(x'^+ + x'^-) = \frac{1}{\sqrt{2}}(e^\phi x^+ + e^{-\phi} x^-) =$$

$$= \cosh\phi \, x^0 + \sinh x^1.$$

$$x'^1 = \frac{1}{\sqrt{2}}(x'^+ - x'^-) = \frac{1}{\sqrt{2}}(e^\phi x^+ - e^{-\phi}x^-) =$$

$$= \sinh\phi \, x^0 + \cosh\phi \, x^1;$$

$$\begin{cases} x'^0 = ch\phi \, x^0 + sh \, x^1; \\ x'^1 = sh\phi \, x^0 + ch \, x^1; \end{cases}$$

$$\begin{cases} x'^0 = \frac{1}{\sqrt{1-(v/c)^2}} x^0 - \frac{v/c}{\sqrt{1-(v/c)^2}} x^1; \\ x'^1 = \frac{-v/c}{\sqrt{1-(v/c)^2}} x^0 + \frac{1}{\sqrt{1-(v/c)^2}} x^1; \end{cases}$$

$$\cosh\phi = \frac{1}{\sqrt{1-(v/c)^2}} \; ; \quad \sinh\phi = \frac{-v/c}{\sqrt{1-(v/c)^2}} \; ;$$

$$\cosh^2\phi - \sinh^2\phi = \frac{1}{1-(v/c)^2} - \frac{(v/c)^2}{1-(v/c)^2} = 1;$$

$$SO(1,1) \simeq \mathbb{R} \simeq \mathbb{R}^+$$

viernes, 11 de septiembre de 2015 0:08

The Lorentz boosts when the coordinate axis of both systems are parallel but the speed $(v^i) \neq (v,0,0)$ take the rotation-invariant form

$$\boxed{v^2 \equiv v^i v^i}$$

$$\boxed{\begin{cases} x'^0 = \dfrac{x^0 - \dfrac{v^i x^i}{c}}{\sqrt{1 - v^2/c^2}} \, ; \\[2ex] x'^i = \left(\delta^{ij} - \dfrac{v^i v^j}{v^2}\right) x^j + \dfrac{\dfrac{v^i v^j x^j}{v^2} - \dfrac{v^i}{c} x^0}{\sqrt{1 - v^2/c^2}} \, ; \end{cases}}$$

$$\boxed{\begin{array}{ll} \Lambda^0{}_0 = \dfrac{1}{\sqrt{1-v^2/c^2}} \, ; & \Lambda^0{}_i = \dfrac{-v^i/c}{\sqrt{1-v^2/c^2}} \, ; \\[2ex] \Lambda^i{}_0 = \dfrac{-v^i/c}{\sqrt{1-v^2/c^2}} \, ; & \Lambda^i{}_j = \delta^i_j + \dfrac{v^i v_j}{v^2}\left(\dfrac{1}{\sqrt{\ }} - 1\right) ; \end{array}}$$

A different way to express them ;

$$\boxed{\begin{cases} x'^0 = \dfrac{x^0 - \dfrac{v}{c} x_{\parallel}}{\sqrt{1 - v^2/c^2}} \, ; \\[2ex] x'_{\parallel} = \dfrac{x_{\parallel} - \dfrac{v}{c} x^0}{\sqrt{1 - v^2/c^2}} \, ; \\[2ex] x'_{\perp} = x_{\perp} \, ; \end{cases}}$$

Immediate consequences of the Lorentz transformations

domingo, 30 de agosto de 2015 15:30

1.- Simultaneity

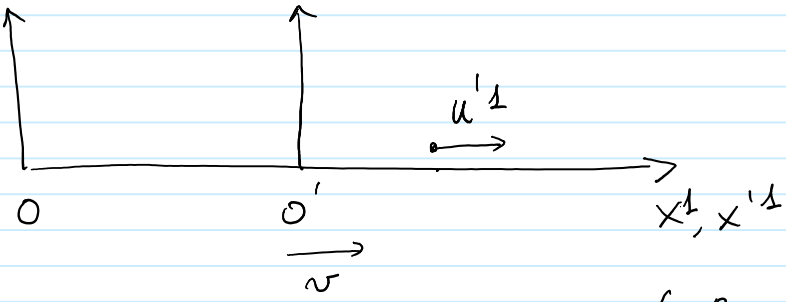
2.- Time dilation

3.- Space contraction (symmetric)

4.- Relativistic Doppler effect

1.1.3 The law of addition of velocities

domingo, 30 de agosto de 2015 13:01



Inverse Lorentz transformation

$$u^1 \equiv \frac{dx^1}{dt} \; ; \; \frac{u^1}{c} = \frac{1}{c}\frac{dx^1}{dt} = \frac{dx^1}{dx^0}$$

$$\begin{cases} x^0 = \frac{1}{\sqrt{1-(v/c)^2}}\, x'^0 + \frac{v/c}{\sqrt{1-(v/c)^2}}\, x'^1 ; \\ x^1 = \frac{+v/c}{\sqrt{1-(v/c)^2}}\, x'^0 + \frac{1}{\sqrt{1-(v/c)^2}}\, x'^1 ; \end{cases}$$

(idem for primed)

$$\frac{dx^1 = \operatorname{sh}\phi\, dx'^0 + \operatorname{ch}\phi\, dx'^1 ;}{dx^0 = \operatorname{ch}\phi\, dx'^0 + \operatorname{sh}\phi\, dx'^1 ;}$$

$$\frac{u^1}{c} = \frac{\operatorname{sh}\phi + \operatorname{ch}\phi\, u'^1/c}{\operatorname{ch}\phi + \operatorname{sh}\phi\, u'^1/c} = \frac{v/c + u'^1/c}{1 + v/c\, u'^1/c}$$

$$\boxed{u^1 = \frac{v + u'^1}{1 + \frac{v}{c} u'^1}}$$

Galilean addition rule

Special-Relativistic correction $\xrightarrow[c \to \infty]{} 0$

or $\frac{v}{c} \to 0$

We can use units such that $c = 1$, dimensionless. Convenient and the cs are easyly found, if necessary.

$$\boxed{u^1 = \frac{v + u'^1}{1 + v\, u'^1}}$$

i) Symmetric under $u'^1 \leftrightarrow v$

ii) Correct Galilean limit

iii) If $u'^1 = 1$ (light) $u^1 = 1 \; \forall v$

iv) If $v \to 1$ $u^1 \to 1 \; \forall u'^1$

The speed of light can't be reached in this way!

domingo, 30 de agosto de 2015 13:37

For the components which are orthogonal to v, u^2 and u^3

$$\boxed{u^{2,3} = \frac{dx^{2,3}}{dt} = \frac{dx'^{2,3}}{ch\phi\, dt' + sh\phi\, dx'^1} = \frac{u'^{2,3}}{ch\phi + sh\phi\, u'^1} = \sqrt{1-v^2}\,\frac{u'^{2,3}}{1 + v u'^1};}$$

Combining the results:

$$\begin{cases} u^1 = \dfrac{v^1 + u'^1}{1 + v^1 u'^1}, \\ u^{2,3} = \dfrac{\sqrt{1-v^2}\, u'^{2,3}}{1 + v^1 u'^1}, \end{cases}$$

perpendicular to v

parallel to v

$$u^i = u^j\left(\delta^{ij} - \frac{v^i v^j}{v^2}\right) + \frac{v^i v^j u^j}{v^2}$$

$$\boxed{u^i = \frac{v^i\,(1 + v^j u^j / v^2)}{1 + v^j u^j} + \frac{\sqrt{1-v^2}\; u^j(\delta^{ij} - v^i v^j / v^2)}{1 + v^j u^j}}$$

This is the general formula, compatible with spatial rotations.

Spacetime diagrams. Causality

domingo, 30 de agosto de 2015 15:35

1.1.4 Geometrization of Special Relativity: Minkowski space

lunes, 17 de agosto de 2015 22:29

The geometrization of Special Relativity provides a very convenient framework to derive many results, originally derived by Einstein in a different way. This framework is now considered more than just convenient, essential, because Special Relativity is a particular case of General Relativity, which is a geometrical theory.

Spacetime: the space of all events (points). Locally, they can be labeled by $(x^\mu) = (t, x^i)$

Distance between two events x^μ_1, x^μ_0 $\Delta x^\mu \equiv x^\mu_1 - x^\mu_0$ is a ("interval") function of the Δx^μ which gives a real number which can be positive, negative, or zero.

Metric: a real, symmetric bilinear form on the Δx^μs which can be used to construct a distance:

$$g(a\Delta x + b\Delta y, \Delta z) = a \cdot (\Delta x, \Delta z) + b\, g(\Delta y, \Delta z)$$

$$g(\Delta x, \Delta y) = g(\Delta y, \Delta x);$$

The distance between two events is $g(\Delta x, \Delta x)$

A metric is always associated to a symmetric matrix

$$g(\Delta x, \Delta y) = g_{\mu\nu}\Delta x^\mu \Delta y^\nu; \quad g_{\mu\nu} = g_{\nu\mu}$$

Minkowski metric is the $\eta_{\mu\nu}$ introduced before. Is constant in Cartesian coordinates. → $\mathbb{R}^4$

Minkowski spacetime is the spacetime equiped with the Minkowski metric, i.e. with the distance

$$\eta_{\mu\nu}\Delta x^\mu \Delta x^\nu$$

For infinitesimally close events one writes

$$\boxed{ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu}$$

(A more precise definition, later)

Only possible because $\eta_{\mu\nu}$ is constant. For $g_{\mu\nu} = g_{\mu\nu}(x)$ the infinitesimal one is the only one correct.

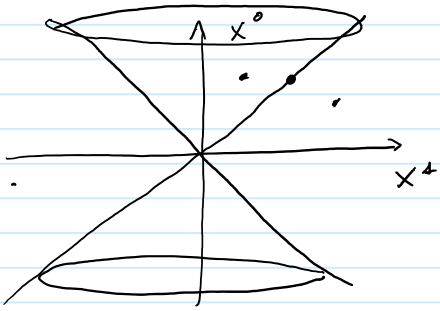
domingo, 30 de agosto de 2015 15:39

- The Minkowski metric is not positive definite, and pairs of events are classified by the sign of their distance:

$\eta_{\mu\nu}\Delta x^\mu \Delta x^\nu > 0$: timelike

$\eta_{\mu\nu}\Delta x^\mu \Delta x^\nu = 0$: lightlike or null

$\eta_{\mu\nu}\Delta x^\mu \Delta x^\nu < 0$: spacelike



⇒ inside, over or outside a lightcone centered at x_0^μ.

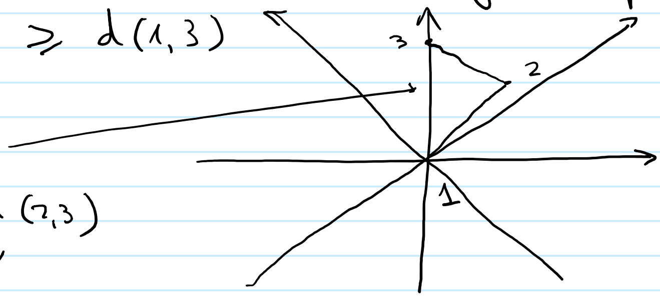
⇒ reachable by a signal slower than light, a light signal or unreachable (causally disconnected)

- The Minkowski distance violates the triangle inequality

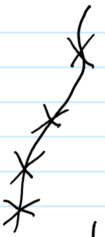
$d(1,2) + d(2,3) \geq d(1,3)$

Here d(1,3) is arbitrarily smaller than d(1,2) + d(2,3)

→ twins "paradox"



Worldline of a particle: its trajectory in spacetime. It is described in some inertial reference frame with coordinates x^μ it is a function $X^\mu(\lambda)$ $\lambda \in \mathbb{R}$.
The tangent always lies inside or over the lightcone at that point



Two main choices for the parameter λ

a) Coordinate time t $X^0(\lambda) = T(\lambda) = \lambda \to (T, X^i(T))$
(time measured by an observer at rest in the reference frame)

b) Proper time τ or s $\boxed{\frac{dX^\mu}{ds}\frac{dX^\nu}{ds}\eta_{\mu\nu} = 1}$
(Time measured by an observer comoving with the particle)

domingo, 30 de agosto de 2015 16:09

Why?

① s is the invariant interval between $X^\mu(0)$ and $X^\mu(s)$:

$$\int_0^s ds = \int_0^s \underbrace{\sqrt{\frac{dX^\mu}{ds}\frac{dX^\nu}{ds}\eta_{\mu\nu}}}_{\overset{\shortparallel}{1}} ds = \int_0^s \sqrt{dX^\mu dX^\nu \eta_{\mu\nu}}$$

② In a system x'^μ comoving with the particle $\frac{dX'^i}{ds} = 0$

$$ds^2 = dX'^\mu dX'^\nu \eta_{\mu\nu} = dT'^2$$

$\Rightarrow$ $s = T'$ in this system

→ Observe that the proper time can be computed using any inertial reference frame and any parameterization of the worldline

$$X'^\mu(\lambda) = \Lambda^\mu{}_\nu X^\nu(\lambda) + a^\mu;$$

and

$$X^\mu(\lambda') = X^\mu(\lambda'(\lambda));$$

→ Observe that when the velocities are << 1 in some system (t, x^i)

$$s = \int dt \sqrt{1 - \frac{dX^i}{dt}\frac{dX^i}{dt}} \sim \int dt \left(1 - \frac{1}{2}u^2\right)$$

$\Rightarrow$ up to a sign and a factor of m, the action for a free massive particle in Newtonian mechanics.

→ Observe that, out of all the possible worldlines joining two (t_0, x^i); (t_1, x^i), that with $u^i = 0$ is the worldline that maximizes s. (The twins "paradox" again).

We need more machinery ⋮

Vector (a.k.a. "contravariant vector) at a point P (event) of spacetime and in a given coordinate system x^μ is a set of 4 numbers V^μ_P such that, when we change the coordinates

$$\boxed{\begin{aligned} & x^\mu \to x'^\mu(x) \\ & V^\mu_P \to V'^\mu_P = \left.\frac{\partial x'^\mu}{\partial x^\nu}\right|_P V^\nu_P \end{aligned}}$$

Jacobian matrix

Contravariant transformation: $(\ ) \longrightarrow (\ )' = (\quad)(\ )$

lunes, 31 de agosto de 2015 15:40

Main example of vector: the vector tangent to a worldline $x^\mu(\lambda)$ at a point P $x^\mu(\lambda_P) = x^\mu(P)$, $\left.\frac{dx^\mu}{d\lambda}\right|_{\lambda_P}$, satisfies this definition:

$$\left.\frac{dx'^\mu}{d\lambda}\right|_{\lambda_P} = \left.\frac{\partial x'^\mu}{\partial x^\nu}\right|_P \left.\frac{dx^\nu}{d\lambda}\right|_{\lambda_P} \qquad \text{(chain rule)}$$

Main counterexample: the coordinates x^μ are not vectors. For instance, under Poincaré transformations

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + \boxed{a^\mu}$$

$$\frac{\partial x'^\mu}{\partial x^\nu} = \Lambda^\mu{}_\nu$$

Another example: $dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu$ (Δx^μ in Minkowski)

<u>Differential form</u> (a.k.a. covariant vector 1-form) at a point P (event) of spacetime in a given coordinate system is a set of 4 numbers ω^P_μ which, under coordinate transformations $x^\mu \to x'^\mu(x)$ transform as

$$\boxed{\omega^P_\mu \longrightarrow \omega'^P_\mu = \omega^P_\nu \left.\frac{\partial x^\nu}{\partial x'^\mu}\right|_P} \qquad \left(\frac{\partial x}{\partial x'}\right)^\nu{}_\mu = \left(\left(\frac{\partial x'}{\partial x}\right)^{-1}\right)^\nu{}_\mu$$

Covariant transformation: $(\ \ ) \longrightarrow (\ \ )' = (\ \ )\left(\ \ \right)^{-1}$ inverse jacobian matrix

Main example of covariant vector: the partial derivatives of any spacetime function at a point:

$$\left.\frac{\partial f}{\partial x^\mu}\right|_P \longrightarrow \frac{\partial f}{\partial x'^\mu} = \left.\frac{\partial x^\nu}{\partial x'^\mu}\right|_P \frac{\partial f}{\partial x^\nu} \qquad \text{(chain rule)}$$

With only a few exceptions (coordinates, for instance) we will assume that any object with 1 upper index is a vector and any object with 1 lower index is a 1-form.

⇒ The "contraction" of 1 upper and 1 lower index gives a function which is invariant under coordinate transformations:

jueves, 10 de septiembre de 2015 10:44

$$\delta^{\rho}{}_{\nu} \longrightarrow \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$\xi'^{\mu}_{p}\, \omega'^{p}_{\mu} = \xi^{\nu}_{p} \overbrace{\frac{\partial x'^{\mu}}{\partial x^{\nu}}\bigg|_{p} \frac{\partial x^{\rho}}{\partial x'^{\mu}}\bigg|_{p}}^{\delta^{\rho}{}_{\nu}} \omega^{p}_{\rho} = \xi^{\nu}_{p}\, \omega^{p}_{\nu};$$

Often vectors and 1-forms are expressed in this way

$$\boxed{\begin{aligned} \xi_p &= \xi^{\mu}_{p} \frac{\partial}{\partial x^{\mu}}; \\ \omega_p &= \omega^{p}_{\mu}\, dx^{\mu}; \end{aligned}}$$

ξ_p, ω^p intrinsic objects

$\xi^{\mu}_{p}, \omega^{p}_{\mu}$ coordinate-dependent representation

We can generalise these concepts to objects with more indices:

tensor fields: $T_p{}^{\mu_1 \ldots \mu_r}{}_{\nu_1 \ldots \nu_s}$ rank (r,s)

$$\boxed{T'_p{}^{\mu_1 \ldots \mu_s}{}_{\nu_1 \ldots \nu_s} = \frac{\partial x'^{\mu_1}}{\partial x^{\rho_1}}\bigg|_p \cdots \frac{\partial x'^{\mu_r}}{\partial x^{\rho_r}}\bigg|_p T_p{}^{\rho_1 \ldots \rho_r}{}_{\sigma_1 \ldots \sigma_s} \frac{\partial x^{\sigma_1}}{\partial x'^{\nu_1}}\bigg|_p \cdots \frac{\partial x^{\sigma_s}}{\partial x'^{\nu_s}}\bigg|_p}$$

$$T_p = T_p{}^{\mu_1 \ldots \mu_r}{}_{\nu_1 \ldots \nu_s}\, dx^{\nu_1} \ldots dx^{\nu_s} \frac{\partial}{\partial x^{\mu_1}} \ldots \frac{\partial}{\partial x^{\mu_r}}$$

Main example: the metric $g_{\mu\nu}$ is a $(2,0)$-rank tensor, symmetric

$$\boxed{g^p \ (\text{or } ds^2) = g^p_{\mu\nu}\, dx^{\mu} dx^{\nu}} \longleftarrow \text{invariant!}$$

A metric provides us with

i) A scalar product between vectors: $\xi \cdot \zeta = g_{\mu\nu} \xi^{\mu} \zeta^{\nu}$ which is invariant under (suppress p) coordinate transformations.

ii) A way to transform vectors into 1-forms

$$\xi^{\mu} \longrightarrow \xi_{\mu} \equiv g_{\mu\nu} \xi^{\nu} \qquad (\text{"lower the index"})$$

iii) If the metric is invertible (always assumed, otherwise "singular") defining $g^{\mu\nu} \equiv (g^{-1})^{\mu\nu}$ we can also transform 1-forms into vectors

$$\omega_{\mu} \longrightarrow \omega^{\mu} \equiv g^{\mu\nu} \omega_{\nu} \qquad (\text{"raise the index"})$$

The relation between vectors and 1-forms becomes 1-to-1 and the scalar product can be written as $\xi \cdot \zeta = \xi^{\mu} \zeta_{\mu} = \xi_{\mu} \zeta^{\mu}$.

The concepts of vector, differential form and rank (r,s)-tensor at a point can be extended to (vector, tensor) fields that assign (smoothly) one of these objects to each point of spacetime $\xi(x), \omega(x)$

Under
$$\begin{cases} x^\mu \to x'^\mu(x); \\ \xi^\mu(x) \to \xi'^\mu(x') = \dfrac{\partial x'^\mu}{\partial x^\nu}\,\xi^\nu\big(x(x')\big); \\ \omega_\mu(x) \to \omega'_\mu(x') = \dfrac{\partial x^\nu}{\partial x'^\mu}\,\omega_\nu\big(x(x')\big); \\ g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \dfrac{\partial x^\rho}{\partial x'^\mu}\dfrac{\partial x^\sigma}{\partial x'^\nu}\,g_{\mu\nu}\big(x(x')\big); \end{cases}$$

In Special Relativity we only consider one metric: the Minkowski metric η which we can write in different coordinates:

Cartesian : $ds^2 = \eta = \eta_{\mu\nu}\,dx^\mu dx^\nu = dt^2 - dx^i dx^i$;

Light-cone : $ds^2 = 2\,du\,dv - dy^2 - dz^2$;

Spherical : $ds^2 = dt^2 - dr^2 - r^2\underbrace{(d\theta^2 + \sin^2\theta\, d\varphi^2)}_{d\Omega^2_{(2)}}$

⋮

The Poincaré transformations preserve the form of the metric in any given set of coordinates. They have different expressions in different coordinate systems.

We will work mostly in Cartesian coordinates, but sometimes it is more convenient to use a different coordinate system.

1.1.5 Mechanics in Minkowski space: momentum, energy…

lunes, 17 de agosto de 2015 22:30

Here we want to study the laws of special-relativistic mechanics (i.e. mechanics whose laws are form-invariant under Poincaré transformations = compatible with the Principle of Special Relativity).

General warning: The laws of Physics cannot be "derived" from principles or experiences. They have to be invented and tested against experience. (Popper). Symmetry principles are often of great help to invent theories or laws because they restrict the space of all possible laws. Also, new laws and theories must reduce to the known and tested ones in the appropriate limits.

Let us start with the relativistic version of the

Law of inertia

Newtonian: $\frac{d^2X^i}{dt^2} = 0$; invariant under Galilean transformations

Relativistic: $\boxed{\frac{d^2X^\mu(s)}{ds^2} = 0}$; invariant under Poincaré transformations.

Observe that a) Lorentz invariance requires X^0 to enter in the same footing as the X^is

b) We can't use the velocity computed in the X^μ system $\frac{dX^\mu}{dX^0}$ → not a vector!

The only invariant parameter is the proper distance!

c) For small speeds $\frac{dX^\mu}{ds} << 1$ $s \sim x^0$

$$\frac{d^2X^0}{ds^2} = 0 \; ; \quad \frac{d^2X^i}{ds^2} \sim \frac{d^2X^i}{dt^2} \; ;$$ (correct Newtonian limit)

Newton's second law: $\boxed{F^\mu = m\, d^2X^\mu/ds^2}$

lunes, 31 de agosto de 2015 19:08

Observe that a) Relativistic (form) invariance forces us to introduce a component F^0 for the force.

b) This law is also a definition of force.

1.2 Relativistic particles and the Principle of Least Action

lunes, 17 de agosto de 2015 22:21

In order to use the 2nd law of dynamics

relativistic-invariant

we need examples of relativistic forces F^μ. The best way to find this forces is by constructing actions describing the interaction of particles with relativistic fields such as the electromagnetic field and then using the Principle of Least Action to find relativistic-invariant equations of motion

This procedure simplifies the invention of laws and theories obeying symmetry principles because it is always easier to deal with scalars.

A pre-requisite is that all the building blocks have well-defined transformation properties under the symmetry transformations.

In our case we only know some building blocks:

1) The worldline $X^\mu(\lambda)$
2) The velocities $\frac{dX^\mu}{d\lambda}$ → transforming as vectors

We also need the relativistic fields a particle can interact with.

We start by reviewing the non-relativistic case.

1.2.1 Non-relativistic particles and the Principle of Least Action (review)

lunes, 17 de agosto de 2015 22:30

1.2.1.1 Lagrangian and Euler-Lagrange equations

lunes, 17 de agosto de 2015 22:31

$L = T - V$ → energy.

For 1 massive particle $\begin{cases} T = \frac{1}{2} m \frac{dx^i}{dt} \frac{dx^i}{dt} = \frac{1}{2} m \dot{x}^i \dot{x}^i \\ V = V(x, \dot{x}, t); \end{cases}$ ← interactions with fields etc.

The Euler-Lagrange equations are

$$\boxed{\frac{\partial L}{\partial x^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}^i} = 0}$$

Galilean-invariant

For static force fields that do not interact with velocity

$V = V(x)$ only → $m \ddot{x}^i = -\frac{\partial V}{\partial x^i} \equiv F^i(x)$

2nd law of dynamics.

Momenta canonically conjugated to a coordinate x^i : By definition

$$\boxed{p^i \equiv \frac{\partial L}{\partial \dot{x}^i}}$$

⇒ The Euler-Lagrange equations can be rewritten as

$$\boxed{\dot{p}^i = \frac{\partial L}{\partial x^i} \equiv F^i}$$

Total derivatives : The equations of motion of two Lagrangians $L_1(x, \dot{x}, t)$; $L_2(x, \dot{x}, t)$; differing by a total derivative with respect to time

$$\frac{dB}{dt} = \frac{\partial B}{\partial x^i} \dot{x}^i + \frac{\partial B}{\partial t}$$

↖ total ↗ partial

$$L_2 = L_1 + \frac{d}{dt} B(x, t) ;$$

they give the same equations of motion

lunes, 28 de septiembre de 2015 13:48

$$L_2 = L_1 + \frac{\partial B}{\partial x^i}\dot{x}^i + \frac{\partial B}{\partial t}$$

$$\frac{\partial L_2}{\partial x^i} = \frac{\partial L_1}{\partial x^i} + \frac{\partial^2 B}{\partial x^i \partial x^j}\dot{x}^j + \frac{\partial^2 B}{\partial x^i \partial t} = \frac{\partial L_1}{\partial x^i} + \frac{d}{dt}\frac{\partial B}{\partial x^i};$$

$$\frac{\partial L_2}{\partial \dot{x}^i} = \frac{\partial L_1}{\partial \dot{x}^i} + \frac{\partial B}{\partial x^i}; \qquad (B = B(x,t) \text{ only})$$

$$\frac{\partial L_2}{\partial x^i} - \frac{d}{dt}\frac{\partial L_2}{\partial \dot{x}^i} = \frac{\partial L_1}{\partial x^i} + \cancel{\frac{d}{dt}\frac{\partial B}{\partial x^i}} - \frac{d}{dt}\left[\frac{\partial L_1}{\partial \dot{x}^i} + \cancel{\frac{\partial B}{\partial x^i}}\right]$$

⇒ The Euler-Lagrange equations are the same.

If B also depends on $\dot{x}$, then L_2 also depends on $\ddot{x}^i$ and we can show the equivalence of the Euler-Lagrange equations taking into account that the Euler-Lagrange equations for Lagrangians with second derivatives are

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{x}^i}\right) = 0$$

In general, the Euler-Lagrange equations are

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{x}^i} - \cdots + (-1)^n \frac{d^n}{dt^n}\left(\frac{\partial L}{\partial \frac{d^n x^i}{dt^n}}\right) \cdots = 0$$

and two Lagrangians differing in an arbitrary total derivative

$$L_2 = L_1 + \frac{d}{dt}B(x, \dot{x}, \ddot{x}, \dots, t);$$

give the same Euler-Lagrange equations of motion. To prove this, we must use

$$\left[\frac{d}{dt}, \frac{\partial}{\partial x^i}\right] B = 0; \qquad \left[\frac{d}{dt}, \frac{\partial}{\partial \dot{x}^i}\right] B = -\frac{\partial B}{\partial x^i};$$

$$\left[\frac{d}{dt}, \frac{\partial}{\partial \ddot{x}^i}\right] B = -\frac{\partial B}{\partial \dot{x}^i}; \qquad \text{etc.}$$

$$\sum_{n=0} (-1)^n \frac{d^n}{dt^n}\left(\frac{\partial L_2}{\partial \frac{d^n x^i}{dt^n}}\right) = \sum (-1)^n \frac{d^n}{dt^n}\left(\frac{\partial L_1}{\partial \frac{d^n x^i}{dt^n}} + \frac{\partial}{\partial \frac{d^n x^i}{dt^n}}\frac{d}{dt}B\right) =$$

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$$= \sum (-1)^n \frac{d^n}{dt^n} \left[\frac{\partial L_1}{\partial \frac{d^n x^i}{dt^n}} + \frac{d}{dt} \frac{\partial B}{\partial \frac{d^n x^i}{dt^n}} + \frac{\partial B}{\partial \frac{d^{n-1} x^i}{dt^{n-1}}} \right] =$$

$$= \sum \left\{ (-1) \frac{d^n}{dt^n} \frac{\partial L_1}{\partial \frac{d^n x^i}{dt^n}} + \cancel{(-1)^n \frac{d^{n+1}}{dt^{n+1}} \frac{\partial B}{\partial \frac{d^n x^i}{dt^n}}} + \cancel{(-1)^n \frac{d^n}{dt^n} \frac{\partial B}{\partial \frac{d^{n-1} x^i}{dt^{n-1}}}} \right\}$$

1.2.1.2 Conserved quantities

lunes, 17 de agosto de 2015 22:32

When a Lagrangian does not depend explicitly on a coordinate, say z, ("cyclic coordinate") there is a conserved quantity, a.k.a. constant of motion: a function of the coordinates and derivatives whose value remains constant over any solution:

$$\underset{\dot{p}_z(x,\dot{x},t)}{\underset{\parallel}{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right)}} = \frac{\partial L}{\partial z} = 0 \; ; \quad \Rightarrow \boxed{\dot{p}_z = 0} \; ; \Rightarrow \boxed{p_z(x,\dot{x},t) = \underset{\substack{\uparrow \\ \text{constant}}}{p_z}}$$

If Lagrangian is time-independent

$$H(x,\dot{x}) \equiv \dot{x}^i \frac{\partial L}{\partial \dot{x}^i} - L \; ;$$

$$\dot{H} = \cancel{\ddot{x}^i \frac{\partial L}{\partial \dot{x}^i}} + \dot{x}^i \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i}\dot{x}^i - \cancel{\frac{\partial L}{\partial \dot{x}^i}\ddot{x}^i}$$

$$= \dot{x}^i \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} \right) = 0 \; ;$$

$$\boxed{\dot{H} = 0} \; ; \quad \boxed{H(x,\dot{x}) = \underset{\substack{\uparrow \\ \text{constant}}}{E}} \longleftarrow \text{energy}$$

Sometimes, the existence of a cyclic coordinate is not manifest unless we use a particular coordinate system

$$L = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{mM}{(x^2+y^2+z^2)} =$$

$$= \frac{1}{2} m \left[\dot{r}^2 + r^2\left(\dot{\theta}^2 + \sin^2\theta\, \dot{\varphi}^2\right)\right] + \frac{mM}{r^2} \; ; \quad \longleftarrow \varphi\text{-independent}$$

$\nearrow$ φ-independent $\Rightarrow$ conservation of angular momentum.

How can we find them?

1.2.1.3 Action and Principle of Least Action

lunes, 17 de agosto de 2015 22:32

The action of a mechanical system along some trajectory $x^i(t)$ between two points $x^i(t_0)$, $x^i(t_1)$ is

$$S[x(t)] = \int_{t_0}^{t_1} dt\, L(x,\dot{x},t) ;$$

It is a functional that gives a real number for each function $x^i(t)$.

The number has dimensions energy × time $= ML^2T^{-1}$

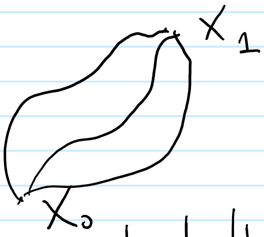
$=$ action $=$ angular momentum

$=$ the units of $\hbar$

Let's consider all the trajectories $x^i(t)$ such that

$x^i(t_0) = x_0^i$

$x^i(t_1) = x_1^i$



The Principle of Least Action says that the physical trajectory will be that with minimal action. If $x^i(t)$ is the physical trajectory, $S[x+\delta x] \sim S[x]$

$$\underset{\parallel}{S[x+\delta x]} = S[x]$$

By definition, the term linear in δx

$$S[x] + \frac{\delta S}{\delta x}\delta x = S[x] ; \Rightarrow \boxed{\frac{\delta S}{\delta x} = 0}$$

$$\delta S = \int dt \left\{ \frac{\partial L}{\partial x^i}\delta x^i + \frac{\partial L}{\partial \dot{x}^i}\delta \dot{x}^i \right\} = \quad \delta \frac{d}{dt} = \frac{d}{dt}\delta$$

$$= \int dt \left\{ \frac{\partial L}{\partial x^i}\delta x^i + \frac{\partial L}{\partial \dot{x}^i}\frac{d}{dt}\delta x^i \right\} =$$

$$= \int dt \left\{ \left[\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} \right]\delta x^i + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) \right\} =$$

The total derivative term vanishes:

$$\int_{t_0}^{t_1} dt \, \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\delta x^i\right) = \frac{\partial L}{\partial \dot{x}^i}\delta x^i \Big]_{t_0}^{t_1} = 0$$

because $\underline{\delta x^i(t_0) = \delta x^i(t_1) = 0}$. $\left(\text{Only trajectories with } x^i(t_{\substack{0\\1}}) = x^i_{\substack{0\\1}}\right)$

$$\Rightarrow \delta S = \int_{t_0}^{t_1} dt \underbrace{\left[\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i}\right]}_{\shortparallel} \delta x^i$$

$$\frac{\delta S}{\delta x^i} = 0 \quad \Rightarrow \text{Ecuaciones de Euler-Lagrange}$$

<u>Observe</u>:

1) It is crucial that δx^i vanishes on the boundary
2) It is crucial that there are no terms $\delta \dot{x}^i$ in the total derivative because we can't impose conditions on $\dot{x}^i$ on the boundary
3) If L depends on higher derivatives of x^i → generalised E-L equations

$$\sum_{n=0} (-1)^n \frac{d^n}{dt^n}\left(\frac{\partial L}{\partial \frac{d^n x^i}{dt^n}}\right) = 0;$$

4) Using the Principle of Least Action it is much easier to show that adding a total derivative does not change the equations of motion (integrate the total derivative and use $\delta x^\mu = 0$ at the boundary).

1.2.1.4 First Noether theorem. Conserved quantities

lunes, 17 de agosto de 2015 22:32

Let us consider some variations of the coordinates δX^i, $\delta X^i\big|_{t_{0,1}} \neq 0$
such that $\delta S = 0$
Then, we can compute δS as before

$$\delta S = \int dt \left\{ \frac{\delta S}{\delta X^i}\delta X^i + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot X^i}\delta X^i\right)\right\} = 0$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{\delta L}{\partial \dot X^i}\delta X^i\right) = -\frac{\delta S}{\delta X^i}\delta X^i = 0 \quad \underline{\text{on-shell}}$$

If $\delta X^i = \alpha^A \delta_A X^i$ $\quad \alpha^A$ constant parameters (independent)
we get a conserved quantity for each $\delta_A X^i$

$$\boxed{p_A \equiv \frac{\partial L}{\partial \dot X^i}\delta_A X^i}$$

Examples:

1) Free particle: $S = \int dt \left\{ \frac{1}{2} m \dot X^i \dot X^i \right\}$

Under translations $\delta_\alpha X^i = \alpha^i \quad (\delta_j X^i = \delta^i{}_j) \quad \delta_\alpha \dot X^i = 0$

$$\Rightarrow \delta_\alpha S = 0$$

Noether $\Rightarrow \dfrac{\partial L}{\partial \dot X^i}\delta_j X^i = \dfrac{\partial L}{\partial \dot X^j} = m \dot X^j = p_j$

Under rotations $\delta_\beta X^i = \varepsilon^{ijk}\beta^j X^k$; $\delta_j X^i = \varepsilon^{ijk} X^k$

$$\delta_\beta \dot X^i = \varepsilon^{ijk}\beta^j \dot X^k$$

$$\delta_\beta S = \int dt\, m\, \delta \dot X^i \dot X^i = \int dt\, m\, \varepsilon^{ijk}\beta^j \dot X^k \dot X^i = 0$$

Noether $\Rightarrow \dfrac{\partial L}{\partial \dot X^i}\delta_j X^i = m \dot X^i \varepsilon^{ijk} X^k$
$= m(\vec X \times \vec V)^j = l^j$

A small generalization: suppose δS does not vanish identically under δx^i, but it vanishes up to a total derivative:

$$\delta S = \int dt \frac{d}{dt}(Y_i \delta x^i)$$

Then $\delta S = \int dt \left\{ \frac{\delta S}{\delta x^i} \delta x^i + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i} \delta x^i \right) \right\} = \int dt \frac{d}{dt}(Y_i \delta x^i)$

$$\Rightarrow \frac{d}{dt}\left[\left(\frac{\partial L}{\partial \dot{x}^i} - Y_i \right) \delta x^i \right] = -\frac{\delta S}{\delta x^i} \delta x^i = 0 \quad \underline{\text{on-shell}}$$

$$\boxed{p_A = \left(\frac{\partial L}{\partial \dot{x}^i} - Y_i \right) \delta_A x^i}$$

1.2.1.5 The conservation of energy

lunes, 17 de agosto de 2015 22:33

The principle underlying the first Noether theorem can also be used to prove the conservation of energy: if the Lagrangian is independent of time $\frac{\partial L}{\partial t} = 0$, then, under constant shifts of time $\delta_\alpha t = \alpha$

$$\delta_\alpha S = \int \left\{ d\,\delta t\, L + dt \left[\frac{\partial L}{\partial x^i} \delta_\alpha x^i + \frac{\partial L}{\partial \dot{x}^i} \delta_\alpha \dot{x}^i \right] \right\} =$$

$$= \int d(\delta_\alpha t L) \cancel{- dL \delta_\alpha t} + dt \cancel{\left[\frac{\partial L}{\partial x^i} \dot{x}^i \delta_\alpha t + \frac{\partial L}{\partial \dot{x}^i} x^i \delta_\alpha t \right]} =$$

$$= \int d(L \delta_\alpha t) \rightarrow \text{a total derivative}$$

$$\delta_\alpha S = \int dt \left\{ \frac{\delta S}{\delta x^i} \delta_\alpha x^i + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^i} \delta_\alpha x^i \right) \right\}$$

$$\frac{d}{dt} \left\{ \underbrace{\left(\frac{\partial L}{\partial \dot{x}^i} \dot{x}^i - L \right)}_{H} \delta_\alpha t \right\} = - \frac{\delta S}{\delta x^i} \dot{x}^i \delta_\alpha t = 0 \text{ on shell}$$

$$\boxed{\dot{H} = 0 \Rightarrow H = \frac{\partial L}{\partial \dot{x}^i} \dot{x}^i - L = E} \quad \text{(the energy, constant)}$$

1.2.2 Actions for massive relativistic particles

lunes, 17 de agosto de 2015 22:34

The action for a free, massive, relativistic particle whose worldline is $X^\mu(\lambda)$ is equal to the proper distance:

$$\boxed{S[X] = -mc\int d\lambda \sqrt{\eta_{\mu\nu}\frac{dX^\mu}{d\lambda}\frac{dX^\nu}{d\lambda}}}$$

$X^\mu, \lambda \to [L]$

$\frac{dX^\mu}{d\lambda}$ dimensionless

and the equations of motion are determined by the Principle of Least Action $\frac{\delta S}{\delta X^\mu} = 0$

Properties: (they follow from those of the proper distance)

1) It is invariant under changes of parameter (reparametrization) $\lambda(\lambda') \xrightarrow{\text{Noether } 2^{nd}}$ constraints

2) It is form-invariant under Poincaré transformations $X'^\mu = \Lambda^\mu{}_\nu X^\nu + a^\mu \xrightarrow{\text{Noether } 1^{st}}$ conserved quantities

3) It has the right Newtonian limit (up to a total derivative)

Equations of motion

$$\delta S = -m\int d\lambda \frac{1}{\cancel{2}\sqrt{}}\cancel{2}\eta_{\mu\nu}\overset{\frac{d}{d\lambda}\delta X^\mu}{\overset{\parallel}{\delta\dot X^\mu}}\dot X^\nu = +m\int d\lambda \frac{d}{d\lambda}\left(\frac{\eta_{\mu\nu}\dot X^\nu}{\sqrt{}}\right)\delta X^\mu$$

$$\Rightarrow \boxed{\frac{\delta S}{\delta X^\mu} = m\frac{d}{d\lambda}\left(\frac{\eta_{\mu\nu}\dot X^\nu}{\sqrt{}}\right) = 0}$$ For an arbitrary parameter

If $\lambda = s$, the proper distance, $\sqrt{} = 1$ and $\boxed{\ddot X^\nu = 0}$

Conserved quantities

First we need δX^μ for Poincaré transformations, that is: we need an expression for infinitesimal Poincaré transformations

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \sigma^\mu{}_\nu \, ; \quad \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \eta_{\mu\nu} = \eta_{\rho\sigma}$$

$$\sigma_{\rho\sigma} + \sigma_{\sigma\rho} = 0$$

$\Rightarrow \sigma_{\mu\nu} = \eta_{\mu\rho}\sigma^\rho{}_\nu$ antisymmetric 4x4 $\Rightarrow$ 6 independent components.

$$\Rightarrow \begin{cases} \delta_\sigma X^\mu = \sigma^\mu{}_\nu X^\nu ; & \text{infinitesimal Lorentz transformations} \\ \delta_\alpha X^\mu = \alpha^\mu ; & \text{infinitesimal traslations} \end{cases}$$

$$\frac{d}{d\lambda}\left(\frac{\eta_{\mu\nu}\dot{X}^\nu}{\sqrt{\ }}\delta X^\mu\right) = 0 ; \quad \text{(exact invariance)}$$

$$-m\,\sigma_{\mu\nu}\frac{\dot{X}^\mu X^\nu}{\sqrt{\ }} = 0 ; \; m\,\dot{X}^{[\mu}X^{\nu]} = M^{\mu\nu}; \; \text{angular momentum}$$

$$-m\frac{\dot{X}^\mu}{\sqrt{\ }}\alpha_\mu = 0 ; \; -m\frac{\dot{X}^\mu}{\sqrt{\ }} = P^\mu ; \; \text{momentum (4-momentum)}$$

The constraint: 2nd Noether theorem

$\boxed{P^0 : \text{energy}}$

The 4 equations of motion $\frac{\delta S}{\delta X^\mu} = -\eta_{\mu\nu}\dot{P}^\nu = 0$

are not independent: they satisfy a relation (constraint)

$$\frac{\dot{X}^\mu}{\sqrt{\ }}\frac{\delta S}{\delta X^\mu} = -\eta_{\mu\nu}\frac{\dot{X}^\mu}{\sqrt{\ }}\frac{d}{d\lambda}\left(-m\frac{\dot{X}^\nu}{\sqrt{\ }}\right) = m\,\eta_{\mu\nu}\frac{\dot{X}^\mu}{\sqrt{\ }}\frac{d}{d\lambda}\left(\frac{\dot{X}^\nu}{\sqrt{\ }}\right)$$

$$= \frac{m}{2}\frac{d}{d\lambda}\left(\frac{\dot{X}^\mu\dot{X}^\nu\eta_{\mu\nu}}{(\sqrt{\ })^2}\right) = 0 ;$$

This is related to $\boxed{P^\mu P^\nu \eta_{\mu\nu} = m^2}$ "mass-shell condition"

which characterises massive particles

$$E = (P^0)^2 = \sqrt{m^2 + p^i p^i}$$

Restoring the c's $\quad E/c = \sqrt{m^2c^2 + p^i p^i}$

Why, if there are 4 variables X^μ, there are only 3 independent equations? Because there is a local symmetry, with infinitesimal parameters which are arbitrary functions of the integration variable/worldline coordinate

$$\lambda \to \lambda'(\lambda) \sim \lambda + \varepsilon(\lambda) ; \Rightarrow \boxed{\delta\lambda = \varepsilon(\lambda) ;}$$

martes, 1 de septiembre de 2015 19:30

$$\delta_\varepsilon S = -m\int\left\{d\delta_\varepsilon\lambda\,\sqrt{} + d\lambda\,\frac{\eta_{\mu\nu}\dot{X}^\nu\,\delta_\varepsilon\dot{X}^\mu}{\sqrt{}}\right\}$$

In order to compute $\delta_\varepsilon \dot{X}^\mu$ we have to be more precise in the definition of δ for general transformations (including λ)

$$\underbrace{\tilde{\delta}F(\lambda)}_{\substack{\text{total variation}\\ \shortparallel \\ F'(\lambda')-F(\lambda)}} = \underbrace{\delta F}_{\substack{\downarrow\\ \text{variation}\\ \text{at the same }\lambda\\ \shortparallel\\ F'(\lambda)-F(\lambda)}} + \underbrace{\delta\lambda\,\dot{F}}_{\text{transport term}}$$

$$X'^\mu(\lambda') = X^\mu(\lambda) \quad (\text{scalar}) \quad\Rightarrow\quad \tilde{\delta}X^\mu = 0$$

$$0 = \delta_\varepsilon X^\mu + \varepsilon\dot{X}^\mu\,; \quad\Rightarrow\quad \boxed{\delta_\varepsilon X^\mu = -\varepsilon\dot{X}^\mu}$$

$$\dot{X}'^\mu(\lambda') = \frac{dX'^\mu(\lambda')}{d\lambda'} = \frac{d\lambda}{d\lambda'}\,\frac{dX^\mu(\lambda)}{d\lambda} \qquad (1\text{-form})$$

$$\tilde{\delta}_\varepsilon\dot{X}^\mu = (1-\dot{\varepsilon})\dot{X}^\mu - \dot{X}^\mu = -\dot{\varepsilon}\dot{X}^\mu = \delta_\varepsilon\dot{X}^\mu + \varepsilon\ddot{X}^\mu$$

$$\boxed{\delta_\varepsilon\dot{X}^\mu = -\varepsilon\ddot{X}^\mu - \dot{\varepsilon}\dot{X}^\mu}$$

Show $\tilde{\delta}_\varepsilon S = 0$ identically. $\forall\,\varepsilon(\lambda)$!

We consider now transformations $\delta_\varepsilon X^\mu = -\varepsilon\dot{X}^\mu$ $(\delta\lambda = 0)$ inspired in the reparametrizations. $\hookrightarrow \delta_\varepsilon\dot{X}^\mu = \frac{d}{d\lambda}\delta_\varepsilon X^\mu$

We find

$$\delta_\varepsilon S = -m\int d\lambda\,\frac{\eta_{\mu\nu}\dot{X}^\nu\,\delta_\varepsilon\dot{X}^\mu}{\sqrt{}} = +m\int d\lambda\,\frac{\eta_{\mu\nu}\dot{X}^\nu\left(\varepsilon\ddot{X}^\mu + \dot{\varepsilon}\dot{X}^\mu\right)}{\sqrt{}}$$

$$= m\int d\lambda\,\frac{d}{d\lambda}\left\{\varepsilon\sqrt{}\right\} \Rightarrow \text{a total derivative}$$

martes, 1 de septiembre de 2015 20:11

Now, we compute $\delta_\varepsilon S$ through the equations of motion:

$$\delta_\varepsilon S = \int d\lambda \left\{ \frac{\delta S}{\delta X^\mu}\, \delta_\varepsilon X^\mu + t.d. \right\}$$

$$\Rightarrow \int d\lambda \left\{ \frac{\delta S}{\delta X^\mu}\, \delta_\varepsilon X^\mu + t.d.' \right\} = 0$$

$$\boxed{\int d\lambda \left\{ -\varepsilon\, \dot{X}^\mu \frac{\delta S}{\delta X^\mu} + t.d.' \right\} = 0} \qquad \forall \varepsilon(\lambda)$$

For $\varepsilon(\lambda)$ vanishing on the boundary $\int d\lambda \left\{ -\varepsilon\, \dot{X}^\mu \frac{\delta S}{\delta X^\mu} \right\} = 0$

$$\Rightarrow \boxed{\dot{X}^\mu \frac{\delta S}{\delta X^\mu} = 0}$$ <u>Noether (or gauge) identity.</u>

$$\boxed{\text{Gauge symmetry} \Leftrightarrow \text{less independent e.o.m.} \Leftrightarrow \text{less degrees of freedom}}$$

1.2.3 Actions for massless relativistic particles

lunes, 17 de agosto de 2015 22:34

The standard action for massive particles is non-linear. A more convenient, but, classically, equivalent action is

$$S = -\frac{1}{2}\int d\lambda \left[e^{-1} \eta_{\mu\nu} \dot{X}^\mu \dot{X}^\nu + e\, m^2 \right];$$

where $e(\lambda)$ is an auxiliary variable (not dynamical) transforming as $\boxed{e'(\lambda') = \frac{d\lambda'}{d\lambda} e(\lambda)}$ so that S is reparametrization-invariant and with dimensions of $[M]$ (or momentum in $c=1$ units!)

$$\frac{\delta S}{\delta e} = -e^{-2}\left(\sqrt{\ \ }\right)^2 + m^2 = 0 \ ; \qquad \left(e^{-1} = \frac{m}{\sqrt{\ \ }} \right.$$

same expression for p^ν)

$$\frac{\delta S}{\delta X^\mu} = \frac{d}{d\lambda}\left(e^{-1} \eta_{\mu\nu} \dot{X}^\nu \right) = \eta_{\mu\nu} \dot{p}^\nu = 0 \ ;$$

$$\Rightarrow \qquad p^\mu p_\mu = m^2$$

Idea: taking the $m^2 \to 0$ limit of the action, we will get $p^\mu p_\mu = 0$

$$\Rightarrow \boxed{S[X] = -\frac{1}{2}\int d\lambda\, e^{-1} \eta_{\mu\nu} \dot{X}^\mu \dot{X}^\nu \ ;}$$

1.2.4 The third law of dynamics. Interactions.

martes, 1 de septiembre de 2015 20:48

As we mentioned, the potential term $V(x, \dot{x}, t)$ contains all the information concerning the interactions of a particle. For 1 particle, its presence breaks, generically, the conservation of momentum. This is because the interaction implies an interchange of momentum with another system (field?) which is not included in the action as a dynamical system: it appears as a background which does not feel the back-reaction of the particle: "test particle".

If the action describes a complete system: particles interacting through dynamical fields, Poincaré-invariance ensures the conservation of the total momentum which leads to the action-reaction law.

Observe that reparametrization invariance of the action does not allow us to include a potential term $\int d\lambda\, V(x)$!

The only possibility is

$$\boxed{S[X] = -m \int d\lambda\, \underset{\substack{\uparrow \\ \text{scalar field (dilaton)}}}{\Phi(X)} \sqrt{\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}}$$

There is only one more possible interaction: with an electromagnetic field. But we have to study these fields first.

1.3 Relativistic fields

lunes, 17 de agosto de 2015 22:21

Fields: magnitudes defined in all points of spacetime.
These magnitudes must transform in a "well-defined way" under the Poincaré group → representations

→ Scalar fields (functions, Higgs) $\quad$ tensors + spinors
Vector fields (actually 1-forms → electromagnetism.)
Metric fields (gravitation) (why different?)
Spinorial fields (fermions!)

We want to study these fields, find suitable Poincaré-invariant actions for them, the associated conserved quantities and the simplest interactions between them and with point-like particles.

First, some generalities about field-theory actions:
The general form of a special-relativistic field theory is

$$S[\phi] = \int_{V^4} \underbrace{d^4x}_{\downarrow} \; \underbrace{\mathcal{L}(\phi, \partial\phi)}_{\text{Lagrangian density} \to \text{"Lagrangian"}} \qquad (\text{usually, no } \partial^2\phi)$$

$$dx^0 d^3x = dx^0 dx^1 dx^3 dx^4$$ volume element or measure

$$S = \int_{x^0_0}^{x^0_1} dx^0 \underbrace{\int_{V^3} d^3x \, \mathcal{L}(\phi, \partial\phi)}_{L \to \text{Lagrangian}}$$ (we won't use this name)

S should be Poincaré-invariant.
The measure d^4x is:

$$d^4x = dx^0 dx^1 dx^2 dx^3 = \underbrace{\det\left(\frac{\partial x}{\partial x'}\right)}_{\overset{\text{Jacobian determinant}}{} \; \overset{||}{1} \text{ for Poincaré transform.}} dx'^0 dx'^1 dx'^2 dx'^3$$

$\Rightarrow \mathcal{L}$ must be Poincaré-invariant.

The Principle of Least Action can be used to find the equations of motion of the field theory (this is the main use of the action in classical field theory): under general variations of the fields (indices etc not explicit) $\delta\phi$ vanishing on the boundary of V^4 $\delta\phi\big|_{\partial V^4}=0$

$$\delta S=\int_{V^4}d^4x\,\delta\mathcal{L}=\int_{V^4}d^4x\left[\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi+\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\overbrace{\delta\partial_\mu\phi}^{\partial_\mu\delta\phi}\right]=$$

$$=\int_{V^4}d^4x\left[\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi+\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\delta\phi\right)-\partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\delta\phi\right]$$

$$=\int_{V^4}d^4x\left\{\underbrace{\left[\frac{\partial\mathcal{L}}{\partial\phi}-\partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\right]}_{\frac{\delta S}{\delta\phi}}\delta\phi\right\}+\int_{\partial V^4}d^3\Sigma_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\underset{\overset{\|}{0}\ \delta\phi|_{\partial V^4}=0}{\delta\phi}\right)$$

Gauss' theorem $\int_{V^4}d^4x\,\partial_\mu W^\mu=\int_{\partial V^4}d^3\Sigma_\mu W^\mu=\int_{\partial V^4}d^3x\,\underset{\uparrow}{n_\mu}W^\mu$

outward-pointing unit vector normal to ∂V^4

$$\left(=\int_{\partial V^4}\varepsilon_{\mu\nu\rho\sigma}\frac{dx^\nu\wedge dx^\rho\wedge dx^\sigma}{3!}W^\mu\right)$$

$$\boxed{\frac{\delta S}{\delta\phi}=\frac{\partial\mathcal{L}}{\partial\phi}-\partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}=0\ ;}$$ Euler-Lagrange equations

If the fields ϕ transform in some representation of the Lorentz group, the equations of motion will be form-invariant.

If the Lagrangian depends on higher derivatives of ϕ: $\partial_\mu\partial_\nu\phi$, $\partial_\mu\partial_\nu\partial_\rho\phi$ etc. we find the generalized Euler-Lagrange equations

$$\boxed{\frac{\delta S}{\delta \phi} = \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} + \partial_\mu \partial_\nu \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi} - \cdots}$$

Adding a total derivative term to the Lagrangian does not change the Euler-Lagrange equations:

$$S' = \int d^4x \left[\mathcal{L} + \partial_\mu B^\mu\right] \quad ; \quad S = \int d^4x \, \mathcal{L}$$

$$\frac{\delta S'}{\delta \phi} = \frac{\delta S}{\delta \phi} \, ;$$

1.3.1 The scalar field

lunes, 17 de agosto de 2015 22:35

Assigns a scalar (a number, real for simplicity) to each point of spacetime. $x^\mu \longrightarrow \phi(x) \in \mathbb{R}$.

Warning: we only know of one relativistic scalar field: the Higgs field (complex). Many non-relativistic scalar fields (electrostatic potential, gravitational potential) are components of tensor fields, as we are going to see.

While it was clear from the beginning that the electrostatic potential is the A_0 component of the electromagnetic potential (the Lorentz transformations were found by studying the invariance of the Maxwell equations) it was not clear how to construct a relativistic field theory of gravity, whose existence was required by the Principle of Special Relativity.

The simplest possibility was to construct a theory in which the gravitational field was a scalar field. Nordström and Einstein himself tried this, but the theories constructed are not satisfactory for different reasons.

The next possibility would be to consider the gravitational field as the zeroth component of a 1-form field. This requires the addition of gravitomagnetic interactions (which exist). However, interactions mediated by 1-forms are repulsive between like charges.

The next possibility is that the Newtonian gravitostatic potential is a component of a rank-2 tensor. It has to be symmetric and in the end this leads to GR.

Historically, the Principles of Equivalence and General Covariance lead to $g_{\mu\nu}$.

1.3.1.1 Transformations

lunes, 17 de agosto de 2015 22:35

The value of $\phi(x)$ at a given point P is the same in any coordinate system:

$$\boxed{\phi'(x') = \phi(x)} \Rightarrow x'^{\mu} = x^{\mu} + \underset{\delta_{\xi} x^{\mu}}{\varepsilon^{\mu}(x)}\,; \quad \boxed{\tilde{\delta}_{\varepsilon}\phi = 0\,;}$$

Since $\tilde{\delta} = \delta + \delta x^{\mu}\partial_{\mu}$ $\rightarrow$ $\boxed{\delta_{\varepsilon}\phi = -\varepsilon^{\mu}\partial_{\mu}\phi\,;}$

Poincaré transformations: $\delta_{\varepsilon} x^{\mu} = \underbrace{\sigma^{\mu}{}_{\nu}x^{\nu} + \alpha^{\mu}}_{\varepsilon^{\mu}}\,;$

Traslations $\delta_{\alpha}\phi = -\alpha^{\mu}\partial_{\mu}\phi\,;$

Lorentz t. $\delta_{\sigma}\phi = -\sigma^{\mu}{}_{\nu}x^{\nu}\partial_{\mu}\phi\,;$

How does $\partial_{\mu}\phi$ transform?

$\partial_{\mu}\phi$ is a 1-form $\Rightarrow$ $\partial_{\mu}\phi'(x') = \frac{\partial x^{\nu}}{\partial x'^{\mu}}\partial_{\nu}\phi(x)$

$x'^{\mu} = x^{\mu} + \varepsilon^{\mu}(x)\,;$

$x^{\mu} = x'^{\mu} - \varepsilon^{\mu}(x')\,;$ $\rightarrow$ $\frac{\partial x^{\nu}}{\partial x'^{\mu}} = \delta^{\nu}{}_{\mu} - \partial_{\mu}\varepsilon^{\nu}(x') =$

$= \delta^{\nu}{}_{\mu} - \partial_{\mu}\varepsilon^{\nu}(x)$ to $O(\varepsilon)$

$\Rightarrow$ $\partial_{\mu}\phi'(x') = \partial_{\mu}\phi(x) - \partial_{\mu}\varepsilon^{\nu}\partial_{\nu}\phi$

$$\boxed{\tilde{\delta}_{\varepsilon}\partial_{\mu}\phi = -\partial_{\mu}\varepsilon^{\nu}\partial_{\nu}\phi} \;; \quad \boxed{\delta_{\varepsilon}\phi = -\varepsilon^{\nu}\partial_{\nu}\partial_{\mu}\phi - \partial_{\mu}\varepsilon^{\nu}\partial_{\nu}\phi}$$

Traslutions $\delta_{\alpha}\partial_{\mu}\phi = -\alpha^{\nu}\partial_{\nu}\partial_{\mu}\phi\,;$

Lorentz t. $\delta_{\sigma}\partial_{\mu}\phi = -\sigma^{\nu}{}_{\rho}x^{\rho}\partial_{\nu}\partial_{\mu}\phi - \sigma^{\nu}{}_{\mu}\partial_{\nu}\phi\,;$

1.3.1.2 Field strength. Scalar potential. Massive scalar field

lunes, 17 de agosto de 2015 22:36

Typically, scalar fields can be split into a constant value ("vacuum expected value") and a spacetime-dependent part $\phi(x) = \phi_0 + \varphi(x)$. The value of ϕ_0 is, a priori undetermined and we expect physical quantities such as the field strength, to be independent of ϕ_0.

$\partial_\mu \phi$ is independent of ϕ_0 and generalizes the idea that forces are the gradients of potentials.

The action of a free scalar field should contain $\partial_\mu \phi$ but no ϕ.

Self-interaction terms can be introduced via scalar potentials $V(\phi)$, whose minima may determine ϕ_0

A particular self-interaction term $V(\phi) = m^2 \phi^2$ gives a massive scalar field whose quantum excitations have mass m.

$$\boxed{-\Box\phi = V'(\phi)};$$

$\phi(x) = \phi_0$; $V'(\phi_0) = 0$ is always a solution ("vacuum" if minimum)

$V(\phi)$

Observe that we can always make field redefinitions

$$\phi = f(\varphi) \;; \quad \partial_\mu \phi = f'(\varphi)\, \partial_\mu \varphi$$

$$S[\varphi] = \int d^4x \;\; \frac{1}{2} \underbrace{\left[f'(\varphi)\right]^2}_{} (\partial\varphi)^2$$

This factor can always be eliminated when there are no more couplings.

1.3.1.3 Action and equations of motion

lunes, 17 de agosto de 2015 22:36

A Poincaré-invariant action for a free scalar field is (no more than 2 derivatives!)

$$\boxed{S_{free}[\phi] = \int d^4x \; \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi}$$

$$\frac{\partial\phi(x)}{\partial x^\mu} = \frac{\partial\phi(x(x'))}{\partial x'^\nu}\frac{\partial x'^\nu}{\partial x^\mu} = \partial_\nu\phi'\,\Lambda^\nu{}_\mu ;$$

$$S_{free} = \int d^4x' \; \frac{1}{2}\eta^{\mu\nu}\Lambda^\rho{}_\mu\Lambda^\sigma{}_\nu\,\partial_\rho\phi'\,\partial_\sigma\phi'$$

$$= \int d^4x' \; \frac{1}{2}\eta^{\rho\sigma}\partial_\rho\phi'\,\partial_\sigma\phi' \Rightarrow \text{Poincaré invariant}$$

$$\delta S = \int d^4x \, \frac{1}{\cancel{2}}\cancel{2}\,\eta^{\mu\nu}\,\delta\partial_\mu\phi\,\partial_\nu\phi = \int d^4x \; \eta^{\mu\nu}\partial_\mu\delta\phi\,\partial_\nu\phi$$

$$= \int d^4x \left\{ \partial_\mu\left(\eta^{\mu\nu}\delta\phi\,\partial_\nu\phi\right) - \underbrace{\eta^{\mu\nu}\partial_\mu\partial_\nu}_{\Box}\phi\,\delta\phi \right\}$$

$$\Rightarrow \boxed{\frac{\delta S}{\delta\phi} = -\Box\phi = 0} \leftarrow \text{Poincaré-invariant.}$$

The action is, by construction, invariant under $\delta_\alpha\phi = \alpha \in \mathbb{R}$

$$\delta_\alpha S = \int d^4x \left\{ \frac{\delta S}{\delta\phi}\underbrace{\delta_\alpha\phi}_{\alpha} + \partial_\mu\left(\eta^{\mu\nu}\partial_\nu\phi\,\underbrace{\delta_\alpha\phi}_{\alpha}\right) \right\}$$

$$\Rightarrow \partial_\mu\Big(\underbrace{\eta^{\mu\nu}\partial_\nu\phi}_{j^\mu}\Big) = -\frac{\delta S}{\delta\phi} = 0 \; , \text{ on-shell} \qquad \boxed{\partial_\mu j^\mu = 0}$$

The invariance under global shifts $\delta_\alpha\phi = \alpha$ leads to the existence of a conserved current j^μ (1st Noether theorem in classical field theory)

Why is $\partial_\mu j^\mu = 0$ a conservation law?
Let us consider a V^4 consisting in all the space between x^0_0 and x^0_1.

jueves, 3 de septiembre de 2015 17:58

$$0 = \int_{V^4} d^4x\, \partial_\mu j^\mu = \int_{x_0^0}^{x_1^0} dx^0 \int_{V^3} d^3x \big(\underset{\text{charge density}}{\partial_0 j^0} + \underset{\text{current density}}{\partial_i j^i}\big)$$

$$= \int_{x_0^0}^{x_1^0} dx^0\, \partial_0 \underbrace{\int_{V^3} d^3x\, j^0}_{\text{total charge in } V^3 \quad Q(x^0)} + \int_{x_0^0}^{x_1^0} dx^0 \int_{V^3} d^3x\, \partial_i j^i =$$

$$= \int_{x_0^0}^{x_1^0} dx^0 \frac{d}{dx^0} Q + \int_{x_0^0}^{x_1^0} dx^0 \int_{\partial V^3} d^2\Sigma_i \underset{\updownarrow\, -\text{flux through } \partial V^3}{j^i} =$$

$$= Q(x_1^0) - Q(x_0^0) - \int_{x_0^0}^{x_1^0} dx^0\, F(x^0)$$

$$\Rightarrow \Delta Q = \int_{x_0^0}^{x_1^0} dx^0\, F(x^0)$$

If ∂V^3 is infinity and the fields vanish there (there are no sources at ∞) $F(x^0) = 0$ and $\Delta Q = 0$: the total charge of the whole spacetime is conserved.

Since a total derivative term does not contribute to the equations of motion, a transformation leaving the action invariant <u>up to a total derivative</u> will also be a symmetry of the equations of motion.

The total derivative term will contribute to the Noether current, though:

$$\delta_\alpha S = \int d^4x \left\{ \frac{\delta S}{\delta \phi} \delta_\alpha \phi + \partial_\mu j^\mu_\alpha \right\} = \int d^4x\, \partial_\mu B^\mu_\alpha$$

$$\delta_\alpha \phi = \alpha^A \delta_A \phi$$

$$\partial_\mu \underbrace{\left[j^\mu_A - B^\mu_A \right]}_{j'^\mu_A \,=\, \partial\mathcal{L}/\partial\partial_\mu\phi\, \delta_A\phi \,-\, B^\mu_A} = -\frac{\delta S}{\delta \phi} \delta_A \phi = 0 \text{ on-shell}$$

1.3.1.4 The energy-momentum tensor of the scalar field

lunes, 17 de agosto de 2015 22:37

Let us now consider the spacetime symmetries: Poincaré and also $\begin{cases} x'^\mu = \Omega x^\mu ; & \Omega \text{ constant} \\ \phi' = \Omega^{-1/2}\phi ; & \end{cases}$

First, under Poincaré transformations $\begin{cases} x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu \\ \phi'(x') = \phi(x) ; \end{cases}$

$$\tilde\delta x^\mu \equiv x'^\mu - x^\mu = \omega^\mu{}_\nu x^\nu + a^\mu ;$$

$$\left.\begin{aligned} &\tilde\delta \phi \equiv \phi'(x') - \phi(x) = 0 ; \Rightarrow \tilde\delta \mathcal{L} = 0 \\ &\tilde\delta d^4x = d^4x\, \partial_\mu \tilde\delta x^\mu = d^4x\, \omega^\mu{}_\mu = 0 ; \end{aligned}\right\} \tilde\delta S = 0$$

$$\tilde\delta d^4x \;\, \| \;\, d^4x' - d^4x = d^4x \left[\det\left(\frac{\partial x'}{\partial x}\right) - 1\right] = d^4x\left[\det\left(1 + \partial_\mu \tilde\delta x^\nu\right) - 1\right]$$

$$= d^4x\, \partial_\mu \tilde\delta x^\mu$$

Ahora $\tilde\delta S = \int d^4x \left\{ \partial_\mu \tilde\delta x^\mu \mathcal{L} + \delta\mathcal{L} + \tilde\delta x^\mu \partial_\mu \mathcal{L} \right\} =$

$$= \int d^4x \left\{ \partial_\mu \tilde\delta x^\mu \mathcal{L} + \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\delta\partial_\mu\phi + \tilde\delta x^\mu \partial_\mu\mathcal{L} \right\} =$$

$$= \int d^4x \left\{ \partial_\mu \tilde\delta x^\mu \mathcal{L} + \tilde\delta x^\mu \partial_\mu \mathcal{L} + \left(\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\right)\delta\phi + \right.$$

$$\left. + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\delta\phi\right) \right\}$$

$$= \int d^4x \left\{ \frac{\delta S}{\delta\phi}\delta\phi + \partial_\mu\left[\mathcal{L}\tilde\delta x^\mu + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\delta\phi\right] \right\}$$

Translations: $\begin{cases} \tilde\delta_\alpha x^\mu = \alpha^\mu = \alpha^\nu \delta_\nu{}^\mu ; \\ \delta_\alpha \phi = -\alpha^\mu \partial_\mu\phi ; \end{cases}$

$$\partial_\mu T^\mu{}_\nu = + \frac{\delta S}{\delta\phi}\partial_\nu\phi = 0 \quad \text{on-shell}$$

$$\underbrace{T^\mu{}_\nu = \mathcal{L}\delta^\mu{}_\nu - \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\partial_\nu\phi = \frac{1}{2}(\partial\phi)^2\delta^\mu{}_\nu - \partial^\mu\phi\partial_\nu\phi}_{\text{energy-momentum tensor}} \quad T^\mu{}_\nu = j^\mu{}_{(\nu)}$$

$$T_{00} = -\frac{1}{2}\partial_0\phi\partial_0\phi = -\text{energy}$$

In this case $T_{\mu\nu} = T_{\nu\mu}$

$T^{\mu}{}_{0}$ → conserved current associated to time-translations

$\Downarrow$

$$\boxed{T^{0}{}_{0} = \text{energy density} \; ; \quad \int d^3x \, T^{0}{}_{0} = E}$$

Lorentz transformations $\quad \tilde{\delta}_{\sigma} x^{\mu} = \sigma^{\mu}{}_{\nu} x^{\nu} = \sigma^{\nu\rho} \, \eta_{[\nu}{}^{\mu} \eta_{\rho]\sigma} x^{\sigma}$

$$\delta_{\sigma} \phi = -\sigma^{\mu}{}_{\nu} x^{\nu} \partial_{\mu} \phi \; ;$$

$$\partial_{\mu} M^{\mu}{}_{\nu\rho} = \frac{\delta S}{\delta \phi} \eta_{[\nu}{}^{\mu} \eta_{\rho]\sigma} x^{\sigma} \partial_{\mu} \phi = 0 \quad \text{on shell}$$

$$\underbrace{M^{\mu}{}_{\nu\rho} = T^{\mu}{}_{[\nu} \eta_{\rho]\sigma} x^{\sigma}}_{\text{angular-momentum current density}} = j^{\mu}{}_{(\nu\rho)} \; ;$$

Dilatations $\quad \begin{cases} \tilde{\delta}_{\omega} x^{\mu} = \omega x^{\mu} \; ; \\ \tilde{\delta}_{\omega} \phi = -\omega \phi - \omega x^{\mu} \partial_{\mu} \phi \end{cases}$

$$\boxed{j^{\mu}_{D} = T^{\mu}{}_{\nu} x^{\nu} - \phi \partial^{\mu} \phi}$$ → Check that it is conserved on-shell.

1.3.1.5 Massive particle coupled to a scalar field

lunes, 17 de agosto de 2015 22:37

$$S[\phi, X^\mu] = \int d^4x \left[\frac{1}{2}(\partial\phi)^2 - V(\phi)\right] - m\int d\lambda\, \phi(X)\sqrt{\eta_{\mu\nu}\dot{X}^\mu\dot{X}^\nu}$$

$$= \int d^4x \left\{\frac{1}{2}(\partial\phi)^2 - V(\phi) - m\,\phi\, J\right\};$$

with $J = \delta^{(4)}(x - X(\lambda))\sqrt{\eta_{\mu\nu}\dot{X}^\mu\dot{X}^\nu}$

$$\boxed{\Box\phi = -\underset{\text{coupling constant}}{m}\;\underset{\text{source for the scalar field}}{J}} \qquad (V(\phi) = 0)$$

<u>Energy-momentum tensor of the coupled system.</u>

1.3.2 The vector field

lunes, 17 de agosto de 2015 22:35

1.4.1.1 Transformations

lunes, 17 de agosto de 2015 22:35

The vector field (1-form) $A_\mu(x)$ transforms, by definition, as

$$A'_\mu(x') = A_\nu(x(x'))\frac{\partial x^\nu}{\partial x'^\mu} \overset{\text{Poincaré}}{=} A_\nu[\Lambda^{-1}(x'-a)](\Lambda^{-1})^\nu{}_\mu$$

$$= A_\nu(\delta^\nu{}_\mu - \sigma^\nu{}_\mu);$$

$$\Rightarrow \boxed{\tilde{\delta} A_\mu = -A_\nu \sigma^\nu{}_\mu}$$

Relativistic fields are unitary representations of the Lorentz group but (except for scalars) not unitary representations of the Poincaré group (as particle states must be, according to Wigner's theorem of QM). This produces a mismatch between the particle states we want to describe (spin 1 in this case, ⇒ helicity $= \pm 1$ (2 d.o.f.) in the massless case and $s_z = +1, 0, -1$ (3 d.o.f.) in the massive case) and the field (4 components for A_μ in $d=4$). Moreover, as it can be shown, the particles described by the extra components of the relativistic fields have negative energies or kinetic terms with wrong signs ("ghosts"). Thus, they must be eliminated to construct consistent field theories. However, if we do this, Lorentz invariance will be explicitly broken.

The solution to this problem is to have local = gauge symmetry in the Poincaré-invariant theory

$$\boxed{\text{gauge symmetry} \Leftrightarrow \text{less independent e.o.m} \Leftrightarrow \text{less d.o.f}}$$

2nd Noether theorem

Noether (gauge) identities

In this case, we know Maxwell's theory describes photons and we will see that it has gauge symmetry.

1.4.1.2 Field strength. Gauge invariance

lunes, 17 de agosto de 2015 22:36

The field strength of a vector field A_μ is

$$\boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = 2\partial_{[\mu} A_{\nu]} .}$$

Under Poincaré transformations $F'_{\mu\nu} = F_{\rho\sigma} (\Lambda^{-1})^\rho{}_\mu (\Lambda^{-1})^\sigma{}_\nu$.

$$\Rightarrow \boxed{\delta F_{\mu\nu} = -F_{\mu\rho}\sigma^\rho{}_\nu - F_{\rho\nu}\sigma^\rho{}_\mu = -2F_{[\mu|\rho}\sigma^\rho{}_{|\nu]} ;}$$

Antisymmetric rank-2 covariant tensor : "<u>2-form</u>".

Due to this definition, the following identity is automatically satisfied (<u>Bianchi identity</u>) :

$$3\partial_{[\mu}F_{\nu\rho]} = \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} =$$

$$= \cancel{\partial_\mu \partial_\nu A_\rho} - \cancel{\partial_\mu \partial_\rho A_\nu} + \cancel{\partial_\nu \partial_\rho A_\mu} - \cancel{\partial_\nu \partial_\mu A_\rho}$$

$$+ \cancel{\partial_\rho \partial_\mu A_\nu} - \cancel{\partial_\rho \partial_\nu A_\mu} = 0$$

$$3\partial_{[\mu}F_{\nu\rho]} = \underset{0}{\underset{\shortparallel}{6\partial_{[\mu}\partial_\nu A_{\rho]}}} = 6\partial_{[\nu}\partial_\mu A_{\rho]} = -6\partial_{[\mu}\partial_\nu A_{\rho]} = 0$$

because

For similar reasons, if $A_\mu \to A'_\mu = A_\mu + \partial_\mu \overset{\text{arbitrary function of } x^\mu}{\chi(x)}$,

$$F_{\mu\nu}(A') = F_{\mu\nu}(A) + 2\partial_{[\mu}\partial_{\nu]}\chi = F_{\mu\nu}(A)$$

The field strength is invariant under local (gauge) transformations and, a theory defined in terms of just $F_{\mu\nu}(A)$ will also be gauge-invariant $\Rightarrow$ less d.o.f., as requested!

In a given inertial frame, the components of the 2-form F, $F_{\mu\nu}$ are of two different kinds : mixed time-space F_{0i} , $i=1,2,3$

pure space F_{ij}

$F_{0i} = -F_{i0} \longrightarrow$ 3 independent components like a vector in $\mathbb{E}^3$

$F_{ij} = -F_{ji} \longrightarrow$ 3 independent components like a vector in $\mathbb{E}^3$

jueves, 10 de septiembre de 2015 10:55

Define $E^i \equiv F^{i0}$;

$B^i \equiv -\frac{1}{2}\varepsilon^{ijk} F_{jk}$; → $F_{ij} = -\varepsilon_{ijk} B^k$;

$\boxed{\begin{matrix}\vec{E} \\ \vec{B}\end{matrix}}$ electric magnetic

Under Lorentz boosts

$$E'^i = F'^{i0} = \Lambda^i{}_\mu \Lambda^0{}_\nu F^{\mu\nu} = \Lambda^i{}_0 \Lambda^0{}_j F^{0j} + \Lambda^i{}_k \left[\Lambda^0{}_0 F^{k0} + \Lambda^0{}_j F^{kj}\right]$$

$$= -\Lambda^i{}_0 \Lambda^0{}_j E^j + \Lambda^i{}_k \left[\Lambda^0{}_0 E^k - \Lambda^0{}_j \varepsilon_{kjl} B^l\right];$$

To simplify this expression, we can project E^i and B^i directions parallel and perpendicular to v^i

$$E_{\parallel} \equiv \frac{v^i}{v} E^i; \quad E^i_{\perp} = \left(\delta^{ij} - \frac{v^i v^j}{v^2}\right) E^j; \quad \frac{1}{\sqrt{1 - v^2/c^2}} \equiv \gamma;$$

$$\Lambda^0{}_0 = \gamma; \quad \Lambda^0{}_i = -\frac{v^i}{c}\gamma; \quad \Lambda^i{}_0 = -\frac{v^i}{c}\gamma.$$

$$\Lambda^i{}_j = \delta^i_j + \frac{v^i v_j}{v^2}(\gamma - 1)$$

$$E'^i = \frac{v^i}{c}\gamma\left(-\frac{v^j}{c}\gamma\right)E^j + \left[\left(\delta^i_k - \frac{v^i v_k}{v^2}\right) + \frac{v^i v_k}{v^2}\gamma\right]\left[\gamma E^k + \gamma \frac{v^j}{c}\varepsilon_{kjl} B^l_{\perp}\right]$$

$$= -\gamma^2 \frac{v^2}{c^2}\frac{v^i}{v} E_{\parallel} + \gamma E^i_{\perp} + \gamma^2 \frac{v^i}{v} E_{\parallel} + \gamma\, \varepsilon_{ijk}\frac{v^j}{c} B^k_{\perp} =$$

$$= \cancel{\gamma^2}\cancel{\left(1 - \frac{v^2}{c^2}\right)}\frac{v^i}{v} E_{\parallel} + \gamma\left[E^i_{\perp} + \left(\frac{\vec{v}}{c}\times\vec{B}_{\perp}\right)^i\right];$$

$$\boxed{\begin{aligned} E'_{\parallel} &= E_{\parallel}; \\ E'^i_{\perp} &= \gamma\left[E^i_{\perp} + \left(\frac{\vec{v}}{c}\times\vec{B}_{\perp}\right)^i\right]; \end{aligned}}$$

Now, for the magnetic field, we have

$$B'^i = -\frac{1}{2}\varepsilon^{ijk} F'_{jk} = -\frac{1}{2}\varepsilon^{ijk}(\Lambda^{-1})^\mu{}_j (\Lambda^{-1})^\nu{}_k F_{\mu\nu} =$$

$$= -\frac{1}{2}\varepsilon^{ijk}\left[2(\Lambda^{-1})^l{}_j (\Lambda^{-1})^0{}_k F_{l0} + (\Lambda^{-1})^l{}_j (\Lambda^{-1})^m{}_k F_{lm}\right]$$

$$= -\varepsilon^{ijk}(\Lambda^{-1})^l{}_j \left(+\frac{v^k}{c}\gamma\right)\left(-E^l\right)$$

$$- \frac{1}{2}\left[\varepsilon^{ilm} + 2(\gamma - 1)\,\varepsilon^{[il|k|}\frac{v^{m]} v_k}{v^2}\right] F_{lm}$$

$$= \varepsilon^{ilk}\frac{v^k}{c}\gamma E^l_\perp +$$

$$+ \frac{1}{2}\left[\varepsilon^{ilm} + 2(\gamma-1)\,\varepsilon^{ilk}\,\frac{v^m\; v_k}{v^2}\right]\varepsilon_{lmp} B^p$$

$$= -\gamma\left(\frac{\vec{v}}{c}\times\vec{E}_\perp\right)^i + B^i + (\gamma-1)\left(-\delta^i{}_m\,\delta^k{}_p + \delta^i{}_p\delta^k{}_m\right)\frac{v^m v_k}{v^2}B^p$$

$$= -\gamma\left(\frac{\vec{v}}{c}\times\vec{E}_\perp\right)^i + B^i + (\gamma-1)\left(-\frac{v^i v_p}{v^2} + \delta^i{}_p\right)B^p$$

$$= -\gamma\left(\frac{\vec{v}}{c}\times\vec{E}_\perp\right)^i + B^i_\parallel + \cancel{B^i_\perp} + (\gamma-\cancel{1})\,B^i_\perp$$

$$\Rightarrow \boxed{\begin{array}{l} B'_\parallel = B_\parallel\,; \\ B'^i_\perp = \gamma\left[B^i_\perp - \left(\frac{\vec{v}}{c}\times\vec{E}_\perp\right)^i\right]; \end{array}}$$

In terms of the electric and magnetic fields, the Bianchi identities are: $\partial_{[i}F_{jk]}=0$; $\varepsilon^{ijk}\partial_i F_{jk}=0$; $\boxed{\vec{\nabla}\cdot\vec{B}=0}$

$$\partial_{[0}F_{jk]}=0 \;;\; \partial_0 F_{jk}+2\partial_{[j}F_{k]0}=0$$

$$\frac{1}{2}\varepsilon^{ijk}\{\varepsilon_{jkl}\partial_0 B^l-2\partial_{[j}E_{k]}=0\}$$

$$\partial_0 B^i+\varepsilon^{ijk}\partial_j E_k=0 \;;$$

$$\boxed{\partial_0\vec{B}+\vec{\nabla}\times\vec{E}=0} \;;$$

From this point of view, 2 of the 4 Maxwell equations in vacuum are the Bianchi identities of $F_{\mu\nu}$. They are solved by $F_{\mu\nu}=2\partial_{[\mu}A_{\nu]}$

<u>2 points of view</u>:

① $F_{\mu\nu}$ $(\vec{E},\vec{B})$ is the fundamental field ⇒ <u>impose</u> Bianchi id.

$$\Downarrow$$
$$\exists\; A_\mu \;/\; F_{\mu\nu}=2\partial_{[\mu}A_{\nu]}$$

② A_μ is the fundamental field ⇒ Bianchi id. trivially satisfied.

$$\Downarrow$$
We need e.o.m. for A_μ

They are <u>the rest</u> of Maxwell equations (known in this context as <u>the</u> Maxwell equations).

We will find them from a variational principle (Maxwell action)

1.4.1.3 Action and Maxwell equations in vacuum

lunes, 17 de agosto de 2015 22:36

The action for A_μ must be gauge invariant. Therefore we must build it with $F_{\mu\nu}$.

It must also be relativistic invariant, and so must be the Lagrangian. The only invariants are

$$F^{\mu\nu}F_{\mu\nu} \quad (= F^2)$$

$$F^{\mu\nu}F^{\rho\sigma}\varepsilon_{\mu\nu\rho\sigma} \quad \Big(\equiv 2F^{\mu\nu}\overbrace{*F_{\mu\nu}}^{\text{Hodge dual}}\Big)$$

Levi-Civita $\varepsilon^{0123}=+1$ fully antisym

However $F^{\mu\nu}F^{\rho\sigma}\varepsilon_{\mu\nu\rho\sigma} = \partial_\mu\left(2A^\nu F^{\rho\sigma}\varepsilon^{\mu}{}_{\nu\rho\sigma}\right)$ and a total derivative gives trivial equations of motion.

$$F^{\mu\nu}F_{\mu\nu} = -2F_{0i}F_{0i} + F_{ij}F_{ij}$$

In the gauge $A_0 = 0$ $\quad F_{0i} = \partial_0 A_i$

$$F^{\mu\nu}F_{\mu\nu} = -\underbrace{2\partial_0 A_i \partial_0 A_i}_{\text{kinetic term for } A_i} + F_{ij}F_{ij}$$

Canonical normalization: $\mathcal{L} = -\frac{1}{4}F^2$

$$\boxed{S_{\text{Maxwell}}[A_\mu] = \int d^4x\left\{-\frac{1}{4}F^2\right\}}$$

The equations of motion are found by extremizing this action:

$$\delta S_{\text{maxwell}} = \int d^4x\left\{-\frac{1}{4}\,2F^{\mu\nu}\delta F_{\mu\nu}\right\} = \int d^4x\left\{-\frac{1}{2}F^{\mu\nu}\,2\partial_\mu\delta A_\nu\right\}$$

$$= \int d^4x\left\{\partial_\mu\left[-F^{\mu\nu}\delta A_\nu\right] + \partial_\mu F^{\mu\nu}\delta A_\nu\right\}$$

$$\boxed{\frac{\delta S}{\delta A_\mu} = \partial_\mu F^{\mu\nu} = 0}$$ Maxwell equations

These equations are 2nd order partial differential equations for A_μ.

$$\partial_\mu F^{\mu\nu} = \partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu = \Box A^\nu - \partial^\nu \partial_\mu A^\mu = 0$$

In the gauge in which $\partial_\mu A^\mu = 0$ (covariant Lorentz gauge)

$$\boxed{\partial_\mu F^{\mu\nu} = \Box A^\mu = 0 ;}$$

⇒ Each component of A^μ satisfies the Klein-Gordon equation.

In terms of 3-vectors

$$\partial_\mu F^{\mu 0} = \partial_l F^{l0} = \partial_l E^l = \vec{\nabla}\cdot\vec{E} = 0$$

$$\partial_\mu F^{\mu i} = \partial_0 F^{0i} + \partial_j F^{ji} = -\partial_0 E^i - \varepsilon^{jik} \partial_j B^k$$
$$= -\partial_0 E^i + (\vec{\nabla}\times\vec{B})^i = 0$$

$$\boxed{\begin{array}{l} \vec{\nabla}\cdot\vec{E} = 0 ; \\ \vec{\nabla}\times\vec{B} + \frac{1}{c}\frac{\partial}{\partial t}\vec{E} = 0 ; \end{array}}$$

To find the Maxwell equations in presence of charges and currents we must couple A_μ to sources. See later

The existence of a gauge symmetry leads to a constraint between the equations of motion

$$\delta_\alpha S = \int d^4x \left\{ -\frac{1}{2} F^{\mu\nu} \delta_\alpha F_{\mu\nu} \right\} = \int d^4x \left\{ -F^{\mu\nu} \partial_\mu \delta_\alpha A_\nu \right\} =$$

$$= \int d^4x \left[\partial_\mu F^{\mu\nu} \delta_\alpha A_\nu - \partial_\mu F^{\mu\nu} \delta_\alpha A_\nu \right]$$

domingo, 13 de septiembre de 2015 13:01

$$\delta_\alpha S = \int d^4x \left\{ \partial_\mu F^{\mu\nu} \partial_\nu \alpha - \partial_\mu \left(F^{\mu\nu} \partial_\nu \alpha \right) \right\} =$$

$$= \int d^4x \left\{ \partial_\nu \left[\partial_\mu F^{\mu\nu} \alpha \right] - \partial_\nu \left(\partial_\mu F^{\mu\nu} \right) \alpha - \partial_\nu \left(F^{\nu\mu} \partial_\mu \alpha \right) \right\}$$

$$= \int d^4x \left\{ \partial_\nu \partial_\mu \cancel{\left(F^{\mu\nu} \alpha \right)} - \partial_\nu \left(\frac{\delta S}{\delta A_\nu} \right) \alpha \right\} = 0$$

$$\Rightarrow \partial_\nu \frac{\delta S}{\delta A_\nu} = 0 \quad \text{off-shell}$$

This is a trivial identity $\partial_\mu \partial_\nu F^{\nu\mu} = 0$. However, it implies that any consistent coupling $\partial_\nu F^{\nu\mu} = j^\mu$ must satisfy $\boxed{\partial_\mu j^\mu = 0}$ → conserved current.

1.4.1.4 The energy-momentum tensor of Maxwell's field. Gauge symmetry and positivity of the energy

lunes, 17 de agosto de 2015 22:37

The Maxwell action is invariant under Poincaré transformations. Let us consider infinitesimal translations $\delta x^\mu = \alpha^\mu = \alpha^\nu \delta_\nu{}^\mu$

According to the general result, the currents

$$j^\mu{}_{(\nu)} \equiv T^\mu{}_\nu = \mathcal{L}\,\delta^\mu{}_\nu - \frac{\partial \mathcal{L}}{\partial \partial_\mu A_\rho}\partial_\nu A_\rho =$$

$$= -\frac{1}{4}F^2\delta^\mu{}_\nu - \left(-F^{\mu\rho}\right)\partial_\nu A_\rho\,;$$

$$\partial_\mu j^\mu{}_{(\nu)} = -\frac{1}{4}\partial_\nu F^2 + \partial_\mu F^{\mu\rho}\partial_\nu A_\rho + F^{\mu\rho}\partial_\mu\partial_\nu A_\rho$$

$$= \cancel{-\frac{1}{4}\partial_\nu F^2 + \frac{1}{2}F^{\mu\rho}\partial_\nu F_{\mu\rho}} + \partial_\mu F^{\mu\rho}\partial_\nu A_\rho$$

$$= \partial_\nu A_\rho\,\partial_\mu F^{\mu\rho} = 0 \quad \text{on-shell}$$

$$\boxed{T^{\text{canonical}}_{\mu\nu} = F_\mu{}^\rho\,\partial_\nu A_\rho - \frac{1}{4}\eta_{\mu\nu}F^2}$$

Observe that this energy-momentum tensor is not symmetric. We can symmetrize it by adding, for instance

$$-F^{\mu\rho}\partial_\rho A_\nu = -\partial_\rho\left(F^{\mu\rho}A_\nu\right) + \partial_\rho F^{\mu\rho}A_\nu \quad \overset{\|}{0} \text{ on-shell}$$

$$\to \partial_\mu\left(-F^{\mu\rho}\partial_\rho A_\nu\right) = -\partial_\mu\partial_\rho\left(F^{\mu\rho}A_\nu\right) = 0$$

$$T^{\mu\,\text{canonical}}{}_\nu - F^{\mu\rho}\partial_\rho A_\nu = F^{\mu\rho}F_{\nu\rho} - \frac{1}{4}\eta^\mu{}_\nu F^2$$

$$= T^{\text{Belinfante}\,\mu}{}_\nu$$

Belinfante energy-momentum tensor

This is the energy-momentum tensor to which the gravitational field couples and it is gauge-invariant!

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$$F^2 = F^{\mu\nu} F_{\mu\nu} = 2 F^{0i} F_{0i} + F^{ij} F_{ij} =$$

$$= -2 E^i E^i + \varepsilon^{ijk} B^k \varepsilon_{ijl} B^l = -2 (E^i E^i - B^i B^i),$$

$$= -2 (\vec{E}^2 - \vec{B}^2);$$

$$T_{00} = F_0{}^i F_{0i} - \frac{1}{4} \eta_{00} \left(-2 (\vec{E}^2 - \vec{B}^2)\right) =$$

$$= -\vec{E}^2 + \frac{1}{2} (\vec{E}^2 - \vec{B}^2) = -(\vec{E}^2 + \vec{B}^2) = -$$ energy density

$$T_{i0} = F_i{}^j F_{0j} - \cancel{\frac{1}{4} \eta_{i0} (\quad)} = - F_{ij} F_{0j} =$$

$$= + \varepsilon_{ijk} B^k E^j = (\vec{E} \times \vec{B})^i =$$ Poynting vector (energy flux) or momentum density

$$T_{ij} = F_i{}^0 F_{j0} + F_i{}^k F_{jk} - \frac{1}{4} \eta_{ij} (-2) (\vec{E}^2 - \vec{B}^2)$$

$$= E^i E^j - \varepsilon_{ikl} \varepsilon_{jkm} B^l B^m - \frac{1}{2} \delta_{ij} (\vec{E}^2 - \vec{B}^2)$$

$$= E^i E^j - (\delta_{ij} \delta_{lm} - \delta_{im} \delta_{lj}) B^l B^m - \frac{1}{2} \delta_{ij} (\vec{E}^2 - \vec{B}^2)$$

$$= E^i E^j \cancel{- \delta_{ij} \vec{B}^2} + B^i B^j - \frac{1}{2} \delta_{ij} (\vec{E}^2 + \vec{B}^2)$$

$$= E^i E^j + B^i B^j - \frac{1}{2} \delta_{ij} (\vec{E}^2 + \vec{B})^2;$$ flux of momentum

1.4.1.5 Massive particle coupled to an electromagnetic field. Electric charge

lunes, 17 de agosto de 2015 22:37

To describe the interaction of a relativistic particle with some given electromagnetic field $A_\mu(x)$ we have to introduce into the particle action an interaction term. relativistic-invariant and built out of A_μ (or $F_{\mu\nu}$) and $\dot{X}^\mu$ (X^μ is not a tensor in general)

$F_{\mu\nu}\dot{X}^\mu\dot{X}^\nu = 0 \Rightarrow A_\mu \dot{X}^\mu$ is the only possibility (it is the "pullback" of the 1-form A over the worldline, always well defined). We also need a new coupling constant: the electric charge q

$$\Rightarrow \boxed{S[X^\mu] = -\int d\lambda \left\{ mc\sqrt{\eta_{\mu\nu}\dot{X}^\mu\dot{X}^\nu} + \frac{q}{c} A_\mu(x)\dot{X}^\mu \right\}}$$

$\lambda = x^0$ $\boxed{-qA_0}-qA_i v^i$

Is this interaction term gauge-invariant?

$$\delta_\alpha S = -\frac{q}{c}\int d\lambda\, \delta_\alpha A_\mu \dot{X}^\mu = -\frac{q}{c}\int d\lambda\, \partial_\mu \alpha\, \dot{X}^\mu$$

$$= -\frac{q}{c}\int d\lambda \frac{d}{d\lambda}\alpha = -\frac{q}{c}\,\alpha\Big]_{\lambda_0}^{\lambda_1}$$

It will vanish if we assume the initial and final points $X^\mu(\lambda_0)$ $X^\mu(\lambda_1)$ to be at infinity and the gauge parameter to vanish there

The equations of motion are now $(c=1)$

$$\delta S = -\int d\lambda \left\{ m \frac{\eta_{\mu\nu}\dot{X}^\mu \frac{d}{d\lambda}\delta X^\nu}{\sqrt{}} + q\,\partial_\nu A_\mu\, \delta X^\nu + q A_\mu \frac{d}{d\lambda}\delta X^\mu \right\}$$

by parts

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$$\delta S = -\int d\lambda \Big\{ \dot{P}_\mu \delta X^\mu + q \partial_\nu A_\mu \delta X^\nu \dot{X}^\mu - q \underbrace{\frac{d}{d\lambda} A_\mu}_{\partial_\nu A_\mu \dot{X}^\nu} \delta X^\mu + \frac{d}{d\lambda}\left(P_\mu \delta X^\mu + q A_\mu \delta X^\mu \right) \Big\}$$

where $P_\mu \equiv - \frac{\eta_{\mu\nu} \dot{X}^\nu}{\sqrt{}}$; $\qquad P_i = \frac{\dot{x}^i}{\sqrt{}} = \gamma \dot{x}^i$;

$$\delta S = -\int d\lambda \Big\{ \left(\dot{P}_\mu + q F_{\mu\nu} \dot{X}^\nu \right) \delta X^\mu + \frac{d}{d\lambda}\left[\left(P_\mu + q A_\mu \right) \delta X^\mu \right] \Big\}$$

$$\boxed{\frac{\delta S}{\delta X^\mu} = - \dot{P}_\mu - q F_{\mu\nu} \dot{X}^\nu = 0}$$

$\lambda = X^0 = t \rightarrow$

$$-\frac{d}{dt}\left(m \gamma \dot{x}^i \right) - q \left(F_{i0} + F_{ij} \dot{x}^j \right) = 0$$

$$-\frac{d}{dt}\left(m \gamma v^i \right) + q \left(+ E^i + \varepsilon_{ijk} B^k \dot{x}^j \right) = 0$$

$$\frac{d}{dt}\left(m \gamma \vec{v} \right) = \underbrace{q\left(\vec{E} + \vec{v} \times \vec{B} \right)}_{\text{Lorentz force}}$$

Here $A_\mu(x)$ has some given expression and the action's symmetries will depend on that. The energy-momentum tensor will not be conserved in general: in the interaction momentum and energy are transferred to and from the field, whose dynamics is ignored (test particle approximation).

We can consider the complete system particle + field + interaction between them.

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$$S[A_\mu, X^\mu] = \int d^4x \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right\} - \int d\lambda \left\{ m \sqrt{\eta_{\mu\nu} \dot{X}^\mu \dot{X}^\nu} - q A_\mu \dot{X}^\mu \right\}$$

$$= \int d^4x \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - A_\mu J^\mu \right\} - m \int d\lambda \sqrt{\eta_{\mu\nu} \dot{X}^\mu \dot{X}^\nu}$$

$$\boxed{j^\mu = -q \int d\lambda\, \delta^{(4)}(x - X)\, \dot{X}^\mu ;}$$

Electric current density

j^0 -> Charge density

j^i -> Current density

Observe that now $A_\mu(x)$ is a variable and it does not have any given expression.

The equations of motion of the particle do not change.

$$\frac{\delta S}{\delta A_\mu} = \boxed{\partial_\mu F^{\mu\nu} - j^\nu = 0}$$

Consistency requires $\partial_\mu j^\mu = 0$