The Strong Equivalence Principle in Higher Order Gravities

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Do HOGs of the form

$$S[g_{\mu\nu}] = \frac{1}{\kappa^2}\int d^4x\sqrt{|g|}\left\{R + F(g, R, \nabla R)\right\}$$

satisfy the SEP?

Common lore: NO!

i) Extra d.o.f.

ii) Spacetime-dependent coupling constants from cosmological curvature.

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i) Extra d.o.f. ← (Not always: Cubic Einstein Gravity (Bueno & Cano 2016))

ii) Spacetime-dependent coupling constants from cosmological curvature. ← (maybe not so easy to test)

A more basic/fundamental question:

Does gravity couple to gravitational energy in the same way it couples to other kinds of energy? (Through $T^{\mu\nu}$)

We are going to give a general answer in the "post-Minkowskian" approx. $g_{\mu\nu} = \eta_{\mu\nu} + \chi h_{\mu\nu}$

$$g_{\mu\nu} = \eta_{\mu\nu} + \chi h_{\mu\nu};$$

$$S[g] = S^{(0)}[h] + \chi S^{(1)}[h] + \chi^2 S^{(2)}[h] + \dots;$$

$$-2\frac{\delta S}{\delta h_{\mu\nu}} = D^{(0)\mu\nu} - \chi t^{(0)\mu\nu} + \dots$$

wave operator

energy-momentum tensor of the gravitational field derived from $S^{(0)}$?

??

Our goal is to answer this question and determine the general properties of $t^{(0)\mu\nu}$ for HOGs.

The energy-momentum tensor

$$S[\phi] = \int d^d x\, \mathcal{L}(\partial^\mu \phi)$$

1st Noether th.: invariance under constant δx^μ

$$\partial_\mu t_{can}{}^\mu{}_\nu = \frac{\delta S}{\delta \phi} \partial_\nu \phi \; ; \quad (= 0 \text{ on-shell})$$

$$t_{can}{}^\mu{}_\nu = \eta^\mu{}_\nu \mathcal{L} - \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \partial_\nu \phi - \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\alpha \phi} \partial_\nu \partial_\alpha \phi$$

$$+ \partial_\alpha \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\alpha \phi} \right) \partial_\nu \phi + \ldots$$

We can add to $t_{can}{}^{\mu}{}_{\nu}$ terms of the form

$$\partial_\rho \psi^{\rho\mu}{}_{\nu} \; ; \quad \text{with} \quad \psi^{\rho\mu}{}_{\nu} = \psi^{[\rho\mu]}{}_{\nu} \; ;$$

For instance $t_{can}{}^{\mu}{}_{\nu} + \partial_\rho \psi^{\rho\mu}{}_{\nu \, \text{Belinfante}} = t^{\mu}{}_{\nu \, \text{Belinfante}}$

$$t^{\mu}{}_{\nu \, \text{Belinfante}} = t_{\nu}{}^{\mu}{}_{\text{Belinfante}}$$

We can also add terms proportional to the e.o.m.

$$A^{\mu}{}_{\nu} \frac{\delta S}{\delta \phi}$$

We want to show that

$$t^{(0)\mu\nu} \equiv 2\,\frac{\delta S^{(1)}}{\delta h_{\mu\nu}} = t_{can}^{(0)\,\mu\nu} + \partial_\rho \underbrace{\psi^{\rho\mu}{}_{\nu}}_{\text{Some}} + \underbrace{A^{\mu\nu}{}_{\rho\sigma}}_{\text{Some}} \frac{\delta S^{(0)}}{\delta h_{\rho\sigma}} ;$$

with

$$t_{can}^{(0)\,\mu\nu} \equiv \eta^{\mu\nu}\mathcal{L}^{(0)} - \frac{\partial \mathcal{L}^{(0)}}{\partial_\mu h_{\alpha\beta}} \partial^\nu h^{\alpha\beta} \ldots\ldots$$

$$\Rightarrow \quad \partial_\mu t_{can}^{(0)\,\mu\nu} = \frac{\delta S^{(0)}}{\delta h_{\rho\sigma}} \partial^\nu h_{\rho\sigma} ;$$

In GR it can be checked that

$$t^{(0)\mu\nu}_{GR} \equiv 2\,\frac{\delta S^{(1)}}{\delta h_{\mu\nu}} = t^{(0)\mu\nu}_{can} + \partial_\rho \psi^{\rho\mu\nu}_{GR} \,;$$

but $\psi^{\rho\mu\nu}_{GR} \neq \psi^{\rho\mu\nu}_{Belinfante}$;

$t^{(0)\mu\nu}_{GR}$ is unambiguosly defined by $S^{(1)}_{E-H}$ and changes are physically relevant! ☿

In GR $t^{(0)\mu\nu}$ is completely determined by the gauge invariance associated to g.c.t.s because $S^{(1)}$ is.

What happens in HOGs?

Under infinitesimal g.c.t.s $\delta_\xi x^\mu = \xi^\mu(x)$

$$\delta_\xi S_{HOG}[g] = 0\,;$$

In the post-Minkowskian approx.

$$\delta_\xi = \delta_\xi^{(0)} + \alpha\,\delta_\xi^{(1)} + \alpha^2\,\delta_\xi^{(2)} + \dots$$

For $h_{\mu\nu}$

$$\delta_\xi h_{\mu\nu} = \left(\delta_\xi^{(0)} + \alpha\,\delta_\xi^{(1)}\right) h_{\mu\nu}\,; \quad \left(\text{no } \mathcal{O}(\alpha^2)\right)$$

$$\delta_\xi^{(0)} h_{\mu\nu} = 2\partial_{(\mu}\xi_{\nu)}\,; \qquad \delta_\xi^{(1)} h_{\mu\nu} = \mathcal{L}_\xi h_{\mu\nu} =$$

$$= \xi^\rho \partial_\rho h_{\mu\nu} + 2\partial_{(\mu}\xi^\rho h_{\nu)\rho}\,;$$

$$\delta_\xi S_{HOG}[g] = 0 \;;\quad \Rightarrow \quad \begin{cases} \delta_\xi^{(0)} S^{(0)}[h] = 0; \\ \delta_\xi^{(0)} S^{(n)}[h] + \delta_\xi^{(1)} S^{(n-1)}[h] = 0; \quad n \geq 1 \end{cases}$$

(Order by order in α)

2nd Noether th:

$$\begin{cases} \partial_\mu \dfrac{\delta S^{(0)}}{\delta h_{\mu\nu}} = 0; \\ \partial_\mu \dfrac{\delta S^{(1)}}{\delta h_{\mu\nu}} = -\partial_\mu \left(\dfrac{\delta S^{(0)}}{\delta h_{\mu\rho}} h_\rho{}^\nu \right) + \dfrac{1}{2} \dfrac{\delta S^{(0)}}{\delta h_{\rho\sigma}} \partial^\nu h_{\rho\sigma}; \end{cases}$$

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1st Noether th: $\partial_\mu t^{(0)}_{can}{}^{\mu\nu} = \dfrac{\delta S^{(0)}}{\delta h_{\rho\sigma}} \partial^\nu h_{\rho\sigma};$

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$\times 2$

$$\partial_\mu \left\{ 2 \frac{\delta S^{(1)}}{\delta h_{\mu\nu}} - t_{con}^{(0)\,\mu\nu} + 2 \frac{\delta S^{(0)}}{\delta h_{\mu\rho}} h_\rho{}^\nu \right\} = 0$$

$$\delta_\xi S_{HOG}[g] = 0; \Rightarrow \begin{cases} \delta_\xi^{(0)} S^{(0)}[h] = 0; \\ \delta_\xi^{(0)} S^{(n)}[h] + \delta_\xi^{(1)} S^{(n-1)}[h] = 0; \quad n \geq 1 \end{cases}$$

(Order by order in κ)

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×2

$$\partial_\mu \left\{ 2 \frac{\delta S^{(1)}}{\delta h_{\mu\nu}} - t^{(0)\,\mu\nu}_{can} + 2 \frac{\delta S^{(0)}}{\delta h_{\mu\rho}} h_\rho{}^\nu \right\} = 0$$

$$2 \frac{\delta S^{(1)}}{\delta h_{\mu\nu}} = t^{(0)\,\mu\nu}_{can} - 2 \frac{\delta S^{(0)}}{\delta h_{\mu\rho}} h_\rho{}^\nu + \partial_\rho \psi^{\rho\mu\nu};$$

$$t^{(0)\mu\nu}$$

$$\|$$

$$2\frac{\delta S^{(1)}}{\delta h_{\mu\nu}} = t_{\text{can}}^{(0)\ \mu\nu} - 2\frac{\delta S^{(0)}}{\delta h_{\mu\rho}} h_\rho{}^\nu{}^\mu + \partial_\rho \psi^{\rho\mu\nu};$$

Q.E.D.

This result can be extended to higher orders in $\mathcal{K}$:

n^{th} Noether identity:

$$\partial_\mu \frac{\delta S^{(n)}}{\delta h_{\mu\nu}} = -\partial_\mu \left(\frac{\delta S^{(n-1)}}{\delta h_{\mu\rho}} h_\rho{}^\nu \right) + \frac{1}{2} \frac{\delta S^{(n-1)}}{\delta h_{\rho\sigma}} \partial^\nu h_{\rho\sigma} \, ;$$

This result can be extended to higher orders in $\varkappa$:

n^{th} Noether identity:

$$\partial_\mu \frac{\delta S^{(n)}}{\delta h_{\mu\nu}} = -\partial_\mu \left(\frac{\delta S^{(n-1)}}{\delta h_{\mu\rho}} h_\rho^{\ \nu} \right) + \frac{1}{2} \frac{\delta S^{(n-1)}}{\delta h_{\rho\sigma}} \partial^\nu h_{\rho\sigma} ;$$

$$\partial_\mu t_{con}^{(n-1)\,\mu\nu}$$

This result can be extended to higher orders in α:

n^{th} Noether identity:

$$\partial_\mu \frac{\delta S^{(n)}}{\delta h_{\mu\nu}} = -\partial_\mu\left(\frac{\delta S^{(n-1)}}{\delta h_{\mu\rho}} h_\rho{}^\nu\right) + \frac{1}{2}\frac{\delta S^{(n-1)}}{\delta h_{\rho\sigma}}\partial^\nu h_{\rho\sigma};$$

$\times 2 \Rightarrow$

$$\partial_\mu\left(2\frac{\delta S^{(n)}}{\delta h_{\mu\nu}} - t_{con}^{(n-1)\mu\nu} + 2\frac{\delta S^{(n-1)}}{\delta h_{\mu\rho}} h_\rho{}^\nu\right) = 0$$

This result can be extended to higher orders in χ:

n^{th} Noether identity:

$$\partial_\mu \frac{\delta S^{(n)}}{\delta h_{\mu\nu}} = -\partial_\mu\left(\frac{\delta S^{(n-1)}}{\delta h_{\mu\rho}} h_\rho{}^\nu\right) + \frac{1}{2}\frac{\delta S^{(n-1)}}{\delta h_{\rho\sigma}}\partial^\nu h_{\rho\sigma};$$

$\times 2$

$$\partial_\mu\left(2\frac{\delta S^{(n)}}{\delta h_{\mu\nu}} - t_{con}^{(n-1)\mu\nu} + 2\frac{\delta S^{(n-1)}}{\delta h_{\mu\rho}} h_\rho{}^\nu\right) = 0$$

$$2\frac{\delta S^{(n)}}{\delta h_{\mu\nu}} = t_{con}^{(n-1)\mu\nu} - 2\frac{\delta S^{(n-1)}}{\delta h_{\mu\rho}} h_\rho{}^\nu + \partial_\rho \phi^{(n-1)\rho\mu\nu}$$

$$\chi\left\{2\frac{\delta S^{(1)}}{\delta h_{\mu\nu}} = t_{con}^{(0)\mu\nu} - 2\frac{\delta S^{(0)}}{\delta h_{\mu\rho}} h_\rho{}^\nu + \partial_\rho \phi^{(0)} g^{\mu\nu}{}_{;\rho}\right\}$$

$$+\;\chi^2\left\{2\frac{\delta S^{(2)}}{\delta h_{\mu\nu}} = t_{con}^{(1)\mu\nu} - 2\frac{\delta S^{(1)}}{\delta h_{\mu\rho}} h_\rho{}^\nu + \partial_\rho \phi^{(1)} g^{\mu\nu}{}_{;\rho}\right\}$$

$$+\;\chi^3\left\{2\frac{\delta S^{(3)}}{\delta h_{\mu\nu}} = t_{con}^{(2)\mu\nu} - 2\frac{\delta S^{(2)}}{\delta h_{\mu\rho}} h_\rho{}^\nu + \partial_\rho \phi^{(2)} g^{\mu\nu}{}_{;\rho}\right\} \ldots$$

$$\alpha\left\{2\frac{\delta S^{(1)}}{\delta h_{\mu\nu}} = t_{con}^{(0)\mu\nu} - 2\frac{\delta S^{(0)}}{\delta h_{\mu\rho}} h_\rho{}^\nu + \partial_\rho \psi^{(0)\rho\mu\nu};\right\}$$

$$+\ \alpha^2\left\{2\frac{\delta S^{(2)}}{\delta h_{\mu\nu}} = t_{con}^{(1)\mu\nu} - 2\frac{\delta S^{(1)}}{\delta h_{\mu\rho}} h_\rho{}^\nu + \partial_\rho \psi^{(1)\rho\mu\nu};\right\}$$

$$+\ \alpha^3\left\{2\frac{\delta S^{(3)}}{\delta h_{\mu\nu}} = t_{con}^{(2)\mu\nu} - 2\frac{\delta S^{(2)}}{\delta h_{\mu\rho}} h_\rho{}^\nu + \partial_\rho \psi^{(2)\rho\mu\nu};\right\} \ldots$$

$$2\left(\alpha\frac{\delta S^{(1)}}{\delta h_{\mu\nu}} + \alpha^2\frac{\delta S^{(2)}}{\delta h_{\mu\nu}} + \cdots + \alpha^m\frac{\delta S^{(m)}}{\delta h_{\mu\nu}}\right) =$$

$$= \alpha\, t_{con}^{(0)\mu\nu} + \alpha^2 t_{con}^{(1)\mu\nu} + \cdots + \alpha^m t_{con}^{(m)\mu\nu}$$

$$-2\alpha\left(\frac{\delta S^{(0)}}{\delta h_{\mu\nu}} + \alpha\frac{\delta S^{(1)}}{\delta h_{\mu\nu}} + \cdots + \alpha^{m-1}\frac{\delta S^{(m-1)}}{\delta h_{\mu\nu}}\right) h_\rho{}^\nu$$

$$+\partial_\rho\left(\alpha\,\psi^{(0)\rho\mu\nu} + \alpha^2\psi^{(1)\rho\mu\nu} + \cdots + \alpha^m\psi^{(m-1)\rho\mu\nu}\right)$$

To this order, the e.o.m. are

$$\frac{\delta S^{(0)}}{\delta h_{\mu\nu}} + \gamma \frac{\delta S^{(1)}}{\delta h_{\mu\nu}} + \cdots + \gamma^{n-1} \frac{\delta S^{(n-1)}}{\delta h_{\mu\nu}} = -\gamma^{n} \frac{\delta S^{(n)}}{\delta h_{\mu\nu}}$$

and this term vanishes on-shell to $\mathcal{O}(\gamma^{n})$

$$t^{\mu\nu}(h) = t^{\mu\nu}_{can} + \partial_\rho \psi^{\rho\mu\nu} + \mathcal{O}(\gamma^{n+1})$$

$$D^{(0)\mu\nu} = \gamma\, t^{\mu\nu} \; ; \quad t^{\mu\nu} = t^{(0)\mu\nu} + \gamma\, t^{(1)\mu\nu} + \cdots$$

$$D^{(0)\mu\nu} = -2\frac{\delta S^{(0)}}{\delta h_{\mu\nu}}, \quad t^{(0)\mu\nu} = 2\frac{\delta S^{(1)}}{\delta h_{\mu\nu}}, \quad t^{(1)\mu\nu} = 2\frac{\delta S^{(2)}}{\delta h_{\mu\nu}} \cdots$$

$$t^{\mu\nu}_{can} = t^{(0)\,\mu\nu}_{can} + \gamma\, t^{(1)\,\mu\nu}_{can} + \cdots \text{ from } S^{(0)} + \gamma S^{(1)} + \cdots$$

Finally, using similar variational methods we can determine the gauge transformation properties of $t^{\mu\nu}(h)$ to any order in α:

$$\delta^{(0)}_{\xi} \frac{\delta S^{(n)}}{\delta h_{\alpha\beta}} = - \delta^{(1)}_{\xi} \frac{\delta S^{(n-1)}}{\delta h_{\alpha\beta}} + \partial_{\rho}\left(\xi^{\rho} \frac{\delta S^{(n-1)}}{\delta h_{\alpha\beta}}\right) - 2 \partial^{(\alpha} \xi_{\rho} \frac{\delta S^{(n-1)}}{\delta h_{\beta)\rho}} ;$$

At lowest order (Checked for GR)

$$\begin{cases} \delta^{(0)}_{\xi} D^{(0)\mu\nu} = 0 ; \\ \delta^{(0)}_{\xi} t^{(0)\mu\nu} = \delta^{(1)}_{\xi} D^{(0)\mu\nu} + \partial_{\rho}\left(\xi^{(0)} D^{(0)\mu\nu}\right) - 2 \partial^{(\mu} \xi_{\rho} D^{(0)\nu)\rho} \end{cases}$$