Non-Abelian Black Holes in String Theory

Tomás Ortín IFT-UAM/CSIC

In collaboration with

Pablo Bueno (KUL)
Pablo A. Cano (IFT)
Patrick (U. Oviedo)
Pedro F. Ramírez (IFT)

Introduction/Motivation

- Many interactions in Nature are non-Abelian.
- String Theory describes non-Abelian interactions:
  - → Fenomenology
  - → Anomaly cancellation

  and Gravity.
- At some level the interplay between Gravity and non-Abelian fields must be relevant.

  Black holes...?
- Very little is known: the focus, so far, has been on gravitating solutions carrying Abelian charges.

- Working with non-Abelian fields is much more complicated.
- Previous work in Einstein-Yang-Mills (EYM) and EYM-Higgs: interesting (non-Abelian hair) but discouraging (only numerical solutions) (Bartnik-McKinnon's particle, Bizon's black hole etc.)
- Supersymmetry, the main tool in the construction of gravitating solutions with Abelian fields has not been used in EYM or EYMH

Consider the supersymmetrization

~~$N=1$~~

$N=2$

SEYM Theories

(gauged SUGRAS)

SEYM Theories:

- Supersymmetric solution-generating techniques ("Classifications")
- Embedding in Superstring Theories.

N=2 SEYM Theories ($d = 4,5$)

Simplest supersymmetrization of EYM theories (Higgses are determined by SUSY)

$\mathcal{N}=1$: No "timelike" (black holes, monopoles) solutions.

$\mathcal{N}=2$: Simplest with timelike supersymmetric solutions.

Options

a) NA gauging of isometries

iso hypers $\Rightarrow$ iso vectors

b) NA gauging of R-symmetry (SU(2))

Fayet-Iliopoulos $\Rightarrow$ iso vectors

· Simplest possibility:

gauge SU(2) ⊂ iso vectors

w/o { FI terms
hypers

$\mathcal{N}=2,\ d=4,5,\ SU(2)$ SEYM Theories

Scalar potentials?

$d=4$; $\quad V(Z,\bar{Z}) = -\frac{1}{4} g^2\, \mathrm{Im}\, \mathcal{N}^{\Lambda\Sigma} P_\Lambda P_\Sigma \geq 0$

$d=5$; $\quad V(\phi) = 0 \geq 0$

⇒ Asymptotically-flat susy solutions.

Equations for timelike susy solutions

Hübscher, Meessen, O., Vaulà 2008
Bellorín, O., 2007, Bellorín 2008
Bueno, Meessen, O., Ramírez 2015
Meessen, O., Ramírez 2015

1 Timelike susy solutions of $\mathcal{N}=2, d=4$, SEYM

2 Timelike &
3 Null

susy solutions of $\mathcal{N}=2, d=5$, SEYM with one additional isometry

SAME EQUATIONS!

+ 3 sets of rules

Bogomol'nyi eqs: $$\frac{1}{2}\varepsilon_{\underline{rsw}}\breve{F}^{\Lambda}{}_{\underline{sw}} - \breve{\mathfrak{D}}_{\underline{r}}\Phi^{\Lambda} = 0\,,$$

Dyon eqs: $$\breve{\mathfrak{D}}_{\underline{r}}\breve{\mathfrak{D}}_{\underline{r}}\Phi_{\Lambda} - g^{2} f_{\Lambda\Sigma}{}^{\Omega} f_{\Delta\Omega}{}^{\Gamma}\Phi^{\Sigma}\Phi^{\Delta}\Phi_{\Gamma} = 0\,,$$

Bubble eqs: $$\Phi_{\Lambda}\breve{\mathfrak{D}}_{\underline{r}}\breve{\mathfrak{D}}_{\underline{r}}\Phi^{\Lambda} - \Phi^{\Lambda}\breve{\mathfrak{D}}_{\underline{r}}\breve{\mathfrak{D}}_{\underline{r}}\Phi_{\Lambda} = 0\,,$$

Variables: Φ^{Λ}, Φ_{Λ}, $\breve{A}^{\Lambda}{}_{\underline{r}} \longrightarrow \breve{\mathfrak{D}}_{\underline{r}}$

$\breve{F}^{\Lambda}{}_{\underline{r}}$

$\Lambda = 0, 1, \ldots, n_{V_4}$

$n_{V_4} = n_{V_5} + 1$

$\underline{r}, \underline{s} = 1, 2, 3$

$\mathbb{E}^{3}$ indices

① Bogomol'nyi Equations

$$\tfrac{1}{2}\varepsilon_{\underline{rsw}}\breve{F}^{\Lambda}{}_{\underline{sw}} - \breve{\mathfrak{D}}_{\underline{r}}\Phi^{\Lambda} \;=\; 0\,,$$

Magnetic gauge field in Minkowski 1+3

Time-independent adjoint Higgs field

YMH action in 1+3: $\int d^4x \left\{ -\frac{1}{4} \breve{F}^{\Lambda} \breve{F}^{\Lambda} + \frac{1}{2} \breve{\mathfrak{D}}\Phi^{\Lambda} \breve{\mathfrak{D}}\Phi^{\Lambda} \right\}$

$\parallel$

Time-independent magnetic configuration

$$-\frac{1}{2}\int d^4x \left[*_3 \breve{F}^{\Lambda} \pm \breve{\mathfrak{D}}\Phi^{\Lambda} \right]^2$$

1st order B. eqs $\Rightarrow$ 2nd order YMH e.o.m.

The solutions are BPS magnetic monopoles.

Relation to $d=5$:

a) Kronheimer 1985: selfdual instantons in GH spaces
$\parallel$
BPS monopoles in $\mathbb{E}^3$

GH spaces: $ds^2 = H^{-1}(dz+\alpha)^2 + H\, d\vec{x}^{\,3}$

$$dH = *_3\, d\alpha$$
$$\downarrow \qquad\qquad \downarrow$$
$$\Phi^0 \qquad\qquad A^0$$

(Abelian B. eqs)

b) Gauntlett et al. 2002
Bellorín, O. 2007

$\mathcal{N}=2$, $d=5$, SEYM timelike susy solutions with one additional isometry have GH base spaces and selfdual YM fields

Solutions to the SU(2) Bogomol'nyi Eqs.

a) Spherically symmetric (Protogenov 1977)

General form
$$\begin{cases} \breve{A}^A = -h(r)\, \varepsilon^A{}_{rs}\, x^r\, dx^s ; \\ \breve{\Phi}^A = -f(r)\, \delta^A{}_r\, x^r ; \end{cases}$$

BPS 't Hooft-Polyakov magnetic monopole

$$f = -\frac{1}{g r^2}\left[1 - \mu r \coth(\mu r + s)\right];$$

$$h = \frac{1}{g r^2}\left[\frac{\mu r}{\sinh(\mu r + s)} - 1\right];$$

Protogenov's hair parameter

Coloured monopoles → BPST instantons $\left(H = \frac{1}{r}\right)$

$$f = -\frac{1}{g r^2 (1 + \lambda^2 r)} ; \quad h = -f ;$$

f) Multicenter solutions

Ramirez's multimonopole solution 2015

$$\Phi^A = -\delta^{Ar} \frac{1}{gP} \partial_{\underline{r}} P \; ; \qquad \breve{A}^A{}_{\underline{r}} = -\varepsilon^A{}_{rs} \frac{1}{gP} \partial_{\underline{s}} P \; ;$$

$$\partial_{\underline{r}} \partial_{\underline{r}} P = 0 \quad ; \qquad P = \lambda^2 + \frac{1}{r}$$ → Coloured monopole

No more simple solutions known

② Dyon Equations

$$\breve{\mathfrak{D}}_{\underline{r}}\breve{\mathfrak{D}}_{\underline{r}}\Phi_\Lambda - g^2 f_{\Lambda\Sigma}{}^{\Omega} f_{\Delta\Omega}{}^{\Gamma}\Phi^\Sigma\Phi^\Delta\Phi_\Gamma = 0,$$

Determined by the B. Eqs.

a) Trivial solution: $\Phi_\Lambda = 0$;

b) Dyon solution: $\Phi_\Lambda = K\,\Phi^\Lambda$; (compact groups)

c) Ramírez's dyon:

$$\left(\begin{array}{l} \Phi^A = -\delta^{Ar}\dfrac{1}{gP}\partial_{\underline{r}}P; \\ \breve{A}^A{}_{\underline{r}} = -\varepsilon^A{}_{rs}\dfrac{1}{gP}\partial_{\underline{s}}P; \end{array}\right)$$

$$\boxed{\begin{array}{l} \Phi_A = -\dfrac{1}{gP}\delta_A{}^{\underline{r}}\partial_{\underline{r}}Q; \\ \dfrac{1}{P}\partial_{\underline{r}}\partial_{\underline{r}}Q \stackrel{?}{=} 0; \end{array}}$$

3 Bubble Equations

$$\underbrace{\Phi_\Lambda \breve{\mathfrak{D}}_{\underline{r}} \breve{\mathfrak{D}}_{\underline{r}} \Phi^\Lambda}_{=0} - \underbrace{\Phi^\Lambda \breve{\mathfrak{D}}_{\underline{r}} \breve{\mathfrak{D}}_{\underline{r}} \Phi_\Lambda}_{=0} \;=\; 0\,,$$

$$\tfrac{1}{2}\varepsilon_{\underline{rsw}} \breve{F}^\Lambda{}_{\underline{sw}} - \breve{\mathfrak{D}}_{\underline{r}} \Phi^\Lambda \;=\; 0\,, \quad \xrightarrow{\text{Bianchi YM}} \quad \breve{\mathfrak{D}}_{\underline{r}} \breve{\mathfrak{D}}_{\underline{r}} \Phi^\Lambda = 0\,;$$

$$\breve{\mathfrak{D}}_{\underline{r}} \breve{\mathfrak{D}}_{\underline{r}} \Phi_\Lambda - g^2 f_{\Lambda\Sigma}{}^{\Omega} f_{\Delta\Omega}{}^{\Gamma} \Phi^\Sigma \Phi^\Delta \Phi_\Gamma \;=\; 0\,, \quad \xrightarrow{f_{(\Lambda\Sigma)}{}^{\Gamma} = 0} \quad \Phi^\Lambda \breve{\mathfrak{D}}_{\underline{r}} \breve{\mathfrak{D}}_{\underline{r}} \Phi_\Lambda = 0\,;$$

except at the singularities:

Fixed relative positions in Abelian multicenter (Denef, Bates)

For Ramirez's multicenter dyon

$$P = P_0 + \sum_\alpha \frac{P_\alpha}{|\vec{x}-\vec{x}_\alpha|}\,; \quad Q = Q_0 + \sum_\alpha \frac{Q_\alpha}{|\vec{x}-\vec{x}_\alpha|}$$

NO RESTRICTIONS

The bubble eqs are the integrability conditions of:

$$\partial_{[\underline{r}}\omega_{\underline{s}]} = 2\varepsilon_{rsw}\left(\Phi_\Lambda \breve{\mathfrak{D}}_{\underline{w}}\Phi^\Lambda - \Phi^\Lambda \breve{\mathfrak{D}}_{\underline{w}}\Phi_\Lambda\right)$$

For Ramirez's multicenter dyon $\omega_{\underline{r}}^{NA} = -4\varepsilon_{rsw}\frac{\partial_{\underline{s}}P}{P}\frac{\partial_{\underline{w}}Q}{P}$

and it does not contribute at the horizons $r \to 0$

Asymptotically $\omega_{\underline{r}}^{NA} \sim 1/r^5$

The solutions $\Phi^{\wedge}$, $\tilde{\Phi}_{\wedge}$, $\tilde{A}^{\wedge}_{\underline{n}}$ of these equations are the building blocks of the SEYM solutions

Let's build some!

N=2,d=4 SEYM solutions

Rules to construct solutions of the $\mathbb{CP}^3$ model:

Metric:

$$ds^2 = e^{2U}(dt+\omega)^2 - e^{-2U}dx^r dx^r\,,$$

where

$$\begin{cases} e^{-2U} = \mathsf{W}(\mathcal{I})\,. \qquad \mathsf{W}(\mathcal{I}) = \tfrac{1}{2}\eta_{\Lambda\Sigma}\mathcal{I}^\Lambda\mathcal{I}^\Sigma + 2\eta^{\Lambda\Sigma}\mathcal{I}_\Lambda\mathcal{I}_\Sigma\,. \\ \partial_{[\underline{r}}\omega_{\underline{s}]} = 2\varepsilon_{rsw}\left(\Phi_\Lambda\breve{\mathfrak{D}}_{\underline{w}}\Phi^\Lambda - \Phi^\Lambda\breve{\mathfrak{D}}_{\underline{w}}\Phi_\Lambda\right) \end{cases}$$

0 $\quad A$

$$\left(+\,\underbrace{-\,-\,-}_{SO(3)}\right)$$

and

$$\mathcal{I}^\Lambda = -\sqrt{2}\Phi^\Lambda\,, \qquad \mathcal{I}_\Lambda = -\sqrt{2}\Phi_\Lambda\,,$$

(To simplify the presentation we focus on the metrics and ignore scalars and vector fields)

1) Spherically-symmetric solutions: monopole
Abelian BH + monopole
coloured BH

2) Multicenter coloured BHs

1.- Global monopole

Abelian sector $\Lambda = 0 \rightarrow \Phi^0 = \text{constant}, \Phi_0 = 0$

Non-Abelian sector $\Lambda = A \rightarrow \begin{cases} \Phi^A \text{ Higgs field of BPS 't Hooft-Polyakov} \\ \Phi_A = 0 \end{cases}$

$$\Rightarrow \quad \omega = 0; \quad \boxed{e^{-2U} = 1 + (\mu/g)^2 - \frac{1}{g^2 r^2}\left[1 - \frac{\mu r}{\sinh \mu r} \cosh \mu r\right]^2}$$

$$e^{-2U} \in \Big[\underbrace{1 + \tfrac{1}{2}(\mu/g)^2}_{r=0}, \underset{\substack{\uparrow \\ r \sim \infty}}{1}\Big), \quad M = \frac{\mu}{g^2 G_N^{(4)}};$$

Globally regular, horizonless solution.

2.- Global monopole + RN BH

$\Phi_0 = 0$

Abelian sector $\Lambda = 0 \rightarrow \Phi^0 = \text{constant} + \frac{\hat{p}{}^0/2}{r}$;

Non-Abelian sector $\Lambda = A \rightarrow \begin{cases} \Phi^A \text{ Higgs field of BPS 't Hooft-Polyakov} \\ \Phi_A = 0 \end{cases}$

$$\Rightarrow \omega = 0 \,; \quad \boxed{e^{-2U} = \left[\sqrt{1+(\mu/g)^2} + \frac{p^0/2}{r}\right]^2 - \frac{1}{g^2 r^2}\left[1 - \frac{\mu r}{\sinh \mu r}\cosh \mu r\right]^2}$$

$$e^{-2U} \in (\underset{r\to 0}{\infty}, \underset{r\sim\infty}{1}), \quad G_N^{(4)} M = \sqrt{\frac{1+(\mu/g)^2}{4}}\, p^0 + \frac{\mu}{g^2} \,;$$

$$e^{-2U} \underset{r\sim 0}{\sim} \frac{(p^0)^2/4}{r^2}, \quad S = \pi (p^0)^2/4 \,;$$

No contribution to the entropy.

3.- Coloured Black Hole

Non-Abelian sector: $\begin{cases} \Phi^A = \dfrac{1}{g r^2(1+\lambda^2 r)} \delta^A{}_r x^r ; \\ \Phi_A = 0 ; \end{cases}$

Since $\Phi^A \Phi^A \underset{r\sim 0}{\sim} \dfrac{1}{g^2 r^2}$, we need a charge in the Abelian sector

Abelian sector: $\Phi^0 = 1 + \dfrac{p^0/2}{r}$; $\Phi_0 = 0$;

$$\Rightarrow \omega = 0 ; \quad \boxed{e^{-2U} = \left(1 + \frac{p^0/2}{r}\right)^2 - \frac{1}{g^2 r^2 (1+\lambda^2 r)^2} ;}$$

$$e^{-2U} \underset{r\to\infty}{\sim} 1 + \frac{p^0}{r} ; \quad M = \frac{p^0}{2\, G_N^{(4)}} ;$$

The non-Abelian field only at horizon?

$$e^{-2U} \sim \left[(p^0/2)^2 - \frac{1}{g^2} \right] \frac{1}{r^2} ; \quad S = \frac{\pi}{G_N^{(4)}} \left[(p^0/2)^2 - \frac{1}{g^2} \right] ;$$

4.- Dumbbell solution

Non-Abelian sector : $\begin{cases} \Phi^A = \dfrac{1}{g r^2(1+\lambda^2 r)}\, \delta^A{}_r\, x^r ; \\ \Phi_A = 0 ; \end{cases}$

Will not be asymptotically flat

Abelian sector : $\Phi^0 \doteq \cancel{1} + \dfrac{p^0/2}{r}$; $\Phi_0 = 0$;

$$\Rightarrow \omega = 0 ; \quad \boxed{e^{-2U} = \frac{(p^0/2)^2}{r^2} - \frac{1}{g^2 r^2 (1+\lambda^2 r)^2} ;}$$

$$e^{-2U} \underset{r\sim\infty}{\sim} \frac{(p^0/2)^2}{r^2} \Rightarrow AdS_2 \times S^2 \text{ at } r=\infty$$

Different radii

$$e^{-2U} \underset{r\sim 0}{\sim} \left[(p^0/2)^2 - \frac{1}{g^2}\right]\frac{1}{r^2} ; \Rightarrow AdS_2 \times S^2 \text{ at } r=0$$

5.- Multicenter Coloured Black Hole

$$H = h + \sum_{\alpha=1}^{N} \frac{p_\alpha}{r_\alpha}\,, \qquad P = \lambda + \sum_{\alpha=1}^{N} \frac{s_\alpha}{r_\alpha}\,, \qquad Q = -\sum_{\alpha=1}^{N} \frac{\eta_\alpha s_\alpha/2}{r_\alpha}\,,$$

$$\Phi^0 = -H\,, \qquad \vec{\Phi} = -\frac{1}{gP}\vec{\nabla}P\,, \qquad \vec{\mathcal{J}} = \frac{2}{gP}\vec{\nabla}Q\,,$$

$\vec{\Phi} = (\Phi^A)$

$\vec{\mathcal{J}} = 2(\Phi_A)$

Non-Abelian sector:

Abelian sector:

Ramírez's dyon

Papapetrou-Majumdar (multi-Reissner-Nordström)

$$e^{-2U} = H^2 - \vec{\Phi}^2 - \vec{\mathcal{J}}^2\,, \qquad \vec{Z} = e^{-i\gamma}\frac{\vec{\Phi} + i\vec{\mathcal{J}}}{H}\,,$$

$$\vec{\omega} = 2g^2\,\vec{\Phi} \times \vec{\mathcal{J}}\,, \qquad V = 2g^2 e^{4U}\,|\vec{\Phi} \times \vec{\mathcal{J}}|^2$$

The metric function e^{-2U} can be written like this:

$$e^{-2U} = h + \sum_{\alpha=1}^{N} \frac{2M_\alpha}{r_\alpha} + \sum_{\alpha=1}^{N} \left[E_\alpha + (1+\eta_\alpha^2)\, R_\alpha\right] \frac{1}{r_\alpha^2}$$

$$+ \sum_{\alpha>\beta}^{N} \left[E_{\alpha\beta} - E_\alpha - E_\beta + 2(1+\eta_\alpha\eta_\beta)\, R_{\alpha\beta}\right] \frac{1}{r_\alpha r_\beta}$$

Manifestly positive functions

where

Mass of α^{th} BH

$$M_\alpha \equiv h p_\alpha\,, \qquad > 0$$

Entropy of α^{th} BH

$$E_\alpha \equiv p_\alpha^2 - (1+\eta_\alpha^2)/g^2\,, \qquad > 0$$

$E(\alpha+\beta)$

$$E_{\alpha\beta} \equiv (p_\alpha + p_\beta)^2 - 4/g^2 - (\eta_\alpha + \eta_\beta)^2/g^2 \qquad > E_\alpha + E_\beta$$

$$\Rightarrow e^{-2U} > 0$$

$\vec{\omega}$ is regular at each r_α and there are no CTCs.

N=2,d=5 SEYM solutions

ST[2,n] model

$$\left.\begin{array}{l} A^0_\mu,\ A^x_\mu \\ \phi^x \end{array}\right\} x = 1, 2 \ldots n$$

$$C_{0xy} = \frac{1}{6}\eta_{xy}$$

$$\eta = (\underbrace{\overset{1}{+}\ \overset{2}{-}}_{k,\,\phi}\ \underbrace{- \cdots}_{\ell^A\ (A \leftarrow su(2))})$$

$$\begin{aligned} S \;=\; & \int d^5x\sqrt{g}\,\Big\{R + \partial_\mu\phi\partial^\mu\phi + \tfrac{4}{3}\partial_\mu \log k\partial^\mu \log k + 2e^{-\phi}k^{-2}\mathfrak{D}_\mu\ell^A\mathfrak{D}^\mu\ell^A \\ & -\tfrac{1}{12}e^{2\phi}k^{-4/3}F^0\cdot F^0 + \tfrac{1}{12}\left(\eta_{xy}e^{-\phi}k^{2/3} - 9h_xh_y\right)F^x\cdot F^y \\ & +\frac{1}{24\sqrt{3}}\frac{\varepsilon^{\mu\nu\rho\sigma\alpha}}{\sqrt{g}}A^0{}_\mu\eta_{xy}F^x{}_{\nu\rho}F^y{}_{\sigma\alpha}\Big\}, \end{aligned}$$

These models can be obtained from Heterotic Supergravity compactified on T^5 and truncated to $\mathcal{N}=2$

Rules to construct timelike solutions of the ST[2,n] model:

Metric:

$$ds^2 = \hat{f}^{\,2}(dt+\hat{\omega})^2 - \hat{f}^{\,-1}\left[H^{-1}(dz+\chi)^2 + H dx^r dx^r\right]$$

(only)

where

$$\left\{\begin{aligned}
\hat{f}^{\,-1} &= H^{-1}\Big\{\tfrac{1}{4}\left(6HL_0 + 8\eta_{xy}K^x K^y\right)\left[9H^2\eta^{xy}L_x L_y + 48HK^0 L_x K^x\right.\\
&\qquad \left.+64(K^0)^2\eta_{xy}K^x K^y\right]\Big\}^{1/3}.\\
\hat{\omega} &= \omega_5(dz+\chi) + \omega\,,\\
\omega_5 &= M + 16\sqrt{2}H^{-2}C_{IJK}K^I K^J K^K + 3\sqrt{2}H^{-1}L_I K^I\,,\\
\partial_{[\underline{r}}\omega_{\underline{s}]} &= 2\varepsilon_{rsw}\left(\Phi_\Lambda \breve{\mathfrak{D}}_{\underline{w}}\Phi^\Lambda - \Phi^\Lambda \breve{\mathfrak{D}}_{\underline{w}}\Phi_\Lambda\right)
\end{aligned}\right.$$

and $K^I = \delta^I{}_\Lambda \Phi^{\Lambda+1}\,, \quad L_I = -\frac{2\sqrt{2}}{3}\delta_I{}^\Lambda \Phi_{\Lambda+1}\,, \quad H = -2\sqrt{2}\Phi^0\,, \quad M = +\sqrt{2}\Phi_0\,,$

Simplest non-Abelian Black Hole

Abelian sector:
(3-charge BH)

$$\begin{cases} L_0 = -\frac{2\sqrt{2}}{3}\,\Phi_1 = B_0 + q_0/\rho^2 ; \\ L_\pm = L_1 \pm L_2 = -\frac{2\sqrt{2}}{3}\left(\Phi_2 \pm \Phi_3\right) = B_\pm + q_\pm/\rho^2 ; \end{cases}$$

Non-Abelian sector:
("Coloured monopole")

$$\Phi^A = \frac{1}{g\, r^2 (1 + \lambda^2 r)}\, \delta^A{}_r\, x^r$$

GH metric:
$(\mathbb{R}^4 - \{0\})$

$$H = 1/r ; \qquad \underset{\mathbb{R}^3}{r} = \underset{\mathbb{R}^4}{\rho^2/4} ;$$

Kronheimer

BPST instanton in $\mathbb{R}^4$

$$\Phi^2 = \Phi^A \Phi^A = \frac{2\kappa^4}{3 g^2 \rho^4 (\rho^2 + \kappa^2)^2} ;$$

$$\hat{\omega} = 0 \; ; \quad \hat{f}^{-3} = \underbrace{\left(L_0 - \frac{\rho^2}{3}\Phi^2\right)}_{\tilde{L}_0} L_+ L_- \; ;$$

$$\tilde{L}_0 B_0 + \frac{q_0}{\rho^2} - \frac{2}{9g^2}\frac{\kappa^2}{\rho^2(\rho^2+\kappa^2)^2} \; ;$$

$O\left(\frac{1}{\rho^2}\right)$ on the horizon $\rho \to 0$

$O\left(\frac{1}{\rho^2}\right)$ at $\rho \to \infty$

$O\left(\frac{1}{\rho^6}\right)$ at $\rho \to \infty$

$\longrightarrow$ Same puzzle as in $d=4$.

BUT

$$\tilde{L}_0 = B_0 + \left(q_0 - \frac{2}{9g^2}\right)\frac{1}{\rho^2} + \frac{2}{9g^2}\frac{\rho^2 + 2\kappa^2}{(\rho^2+\kappa^2)^2} ,$$

!!

It is remarkable that we can rewrite $\tilde{L}_0$ like this:

$$\tilde{L}_0 = B_0 + \left(q_0 - \frac{2}{9g^2}\right)\frac{1}{\rho^2} + \frac{2}{9g^2}\frac{\rho^2 + 2\kappa^2}{(\rho^2+\kappa^2)^2},$$

Suggests: $q_0 - \frac{2}{9g^2}$ some standard brane charge $\frac{1}{\rho^2}$

What is $\frac{2}{9g^2}$?

Let's switch off everything else: $\begin{cases} q_0 - \frac{2}{9g^2} = 0 \\ q_\pm = 0 \end{cases}$

What do we get?

The full solution has this form:

$$\boxed{\begin{aligned}
ds^2 &= \hat{f}^2 dt^2 - \hat{f}^{-1}(d\rho^2 + \rho^2 d\Omega^2_{(3)})\,, \\
\hat{f}^{-3} &= 1 + \frac{2e^{-\phi_\infty}k_\infty^{2/3}}{3g^2}\frac{\rho^2 + 2\kappa^2}{(\rho^2+\kappa^2)^2}\,, \\
A^0 &= -\frac{1}{\sqrt{3}}\hat{f}^3 dt\,, \qquad A^A = \frac{\kappa^2}{g(\rho^2+\kappa^2)}v^A_L\,, \\
e^{2\phi} &= e^{2\phi_\infty}\hat{f}^{-3}\,, \qquad k = k_\infty \hat{f}^{3/4}\,,
\end{aligned}}$$

Spherically symmetric, globally regular, horizon-less, asymptotically flat

"GLOBAL INSTANTON"

What is a "global instanton"?

let's uplift the solution to $d=10$ Heterotic Supergravity (other uplifts more difficult or impossible)

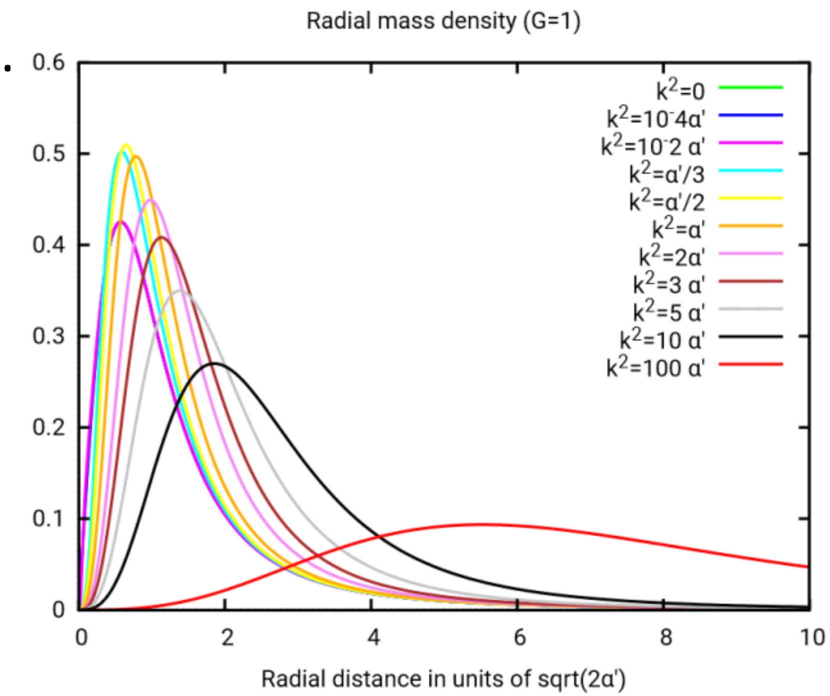
$$g_5 = k_\infty^{1/3}\, e^{-\phi_\infty/2} / \sqrt{12\alpha'} \;;$$

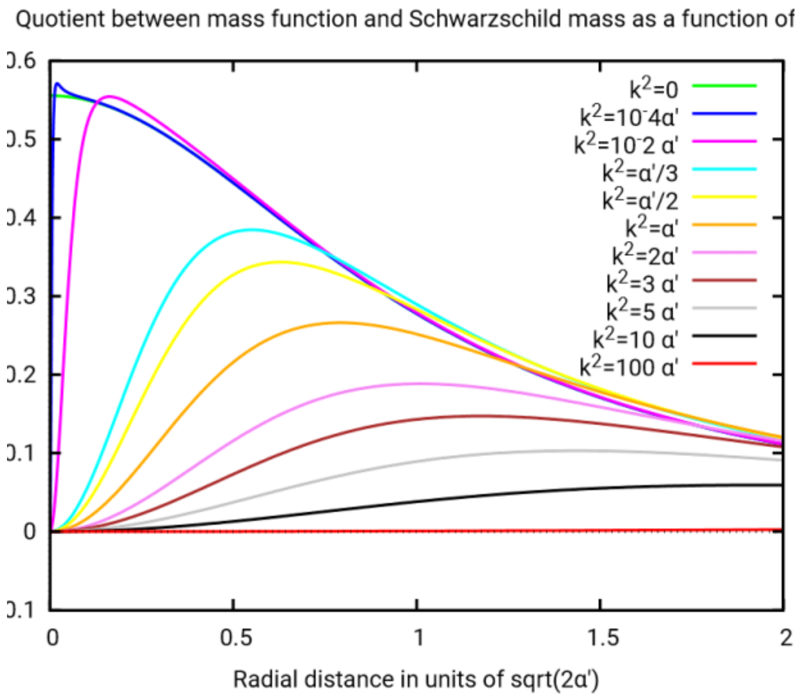
$$g_s = e^{\phi_\infty} \;;$$

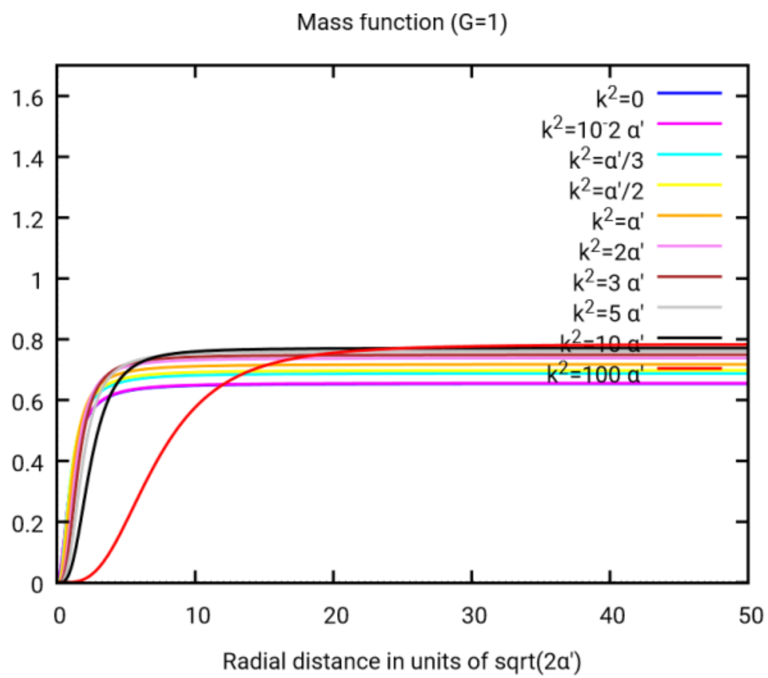
$$l_s = \sqrt{\alpha'} \;;$$

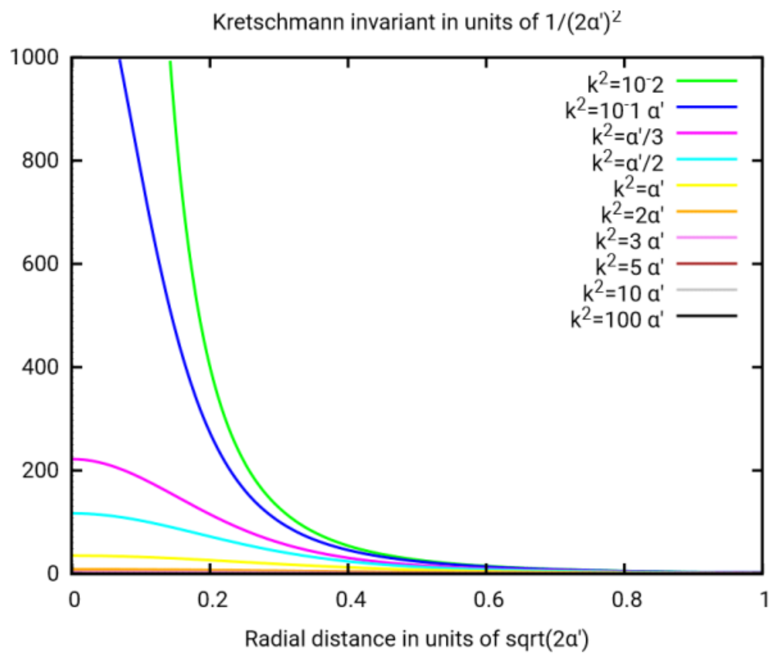
$$\Rightarrow \quad e^{2\phi} = e^{2\phi_\infty} \hat{f}^{-3} = e^{2\phi_\infty} \left\{ 1 + 8\alpha' \frac{\rho^2 + 2\kappa^2}{(\rho^2 + \kappa^2)^2} \right\}$$

Characteristic of the GAUGE FIVEBRANE


Radial mass density (G=1)
k2=0
k2=10-4α'
k2=10-2 α'
k2=α'/3
k2=α'/2
k2=α'
k2=2α'
k2=3 α'
k2=5 α'
k2=10 α'
k2=100 α'
Radial distance in units of sqrt(2α')


Quotient between mass function and Schwarzschild mass as a function of
k2=0
k2=10-4α'
k2=10-2 α'
k2=α'/3
k2=α'/2
k2=α'
k2=2α'
k2=3 α'
k2=5 α'
k2=10 α'
k2=100 α'
Radial distance in units of sqrt(2α')


Mass function (G=1)
k2=0
k2=10-2 α'
k2=α'/3
k2=α'/2
k2=α'
k2=2α'
k2=3 α'
k2=5 α'
k2=10 α'
k2=100 α'
Radial distance in units of sqrt(2α')


Kretschmann invariant in units of 1/(2α')2
k2=10-2
k2=10-1 α'
k2=α'/3
k2=α'/2
k2=α'
k2=2α'
k2=3 α'
k2=5 α'
k2=10 α'
k2=100 α'
Radial distance in units of sqrt(2α')

These non-Abelian black holes consist of 3 standard branes plus a gauge 5-brane.

$\underbrace{\text{3 standard branes}}$ → Standard 3-charge BH → D1 D5 W up to dualities (Strominger-Vafa)

$\underbrace{\text{gauge 5-brane}}$ → Contributes to the mass but not to the entropy

⇒ We can explain the entropy.

Correct identification of charges is essential.

After some redefinitions, this is the full solution

$$ds^2 = f^2 dt^2 - f^{-1}(d\rho^2 + \rho^2 d\Omega^2_{(3)}),$$

$$A^0 = -\sqrt{3}e^{-\phi_\infty}k_\infty^{2/3}\frac{dt}{\tilde{\mathcal{Z}}_0}, \qquad A^1 + A^2 = -\sqrt{3}e^{\phi_\infty}k_\infty^{2/3}\frac{dt}{\mathcal{Z}_+},$$

$$A^A = -\frac{1}{g}\frac{\rho^2}{(\kappa^2+\rho^2)}v_R^A, \qquad A^1 - A^2 = -2\sqrt{3}k_\infty^{-4/3}\frac{dt}{\mathcal{Z}_-},$$

$$e^{2\phi} = e^{2\phi_\infty}\frac{\tilde{\mathcal{Z}}_0}{\mathcal{Z}_+}, \qquad k = k_\infty(f\mathcal{Z}_-)^{3/4},$$

$$f^{-3} = \tilde{\mathcal{Z}}_0\,\mathcal{Z}_+\,\mathcal{Z}_-\,,$$

$$\left(q_0 - \frac{q}{2g^2}\right)$$

$$\tilde{\mathcal{Z}}_0 = 1 + \frac{\tilde{\mathcal{Q}}_0}{\rho^2} + \frac{2e^{-\phi_\infty}k_\infty^{2/3}}{3g^2}\frac{\rho^2+2\kappa^2}{(\rho^2+\kappa^2)^2}\,, \qquad \mathcal{Z}_\pm = 1 + \frac{\mathcal{Q}_\pm}{\rho^2}.$$

.

$$
\begin{aligned}
M &= \frac{\pi}{4G_N^{(5)}}\left[\tilde{\mathcal{Q}}_0 + \frac{2e^{-\phi_\infty}k_\infty^{2/3}}{3g^2} + \mathcal{Q}_+ + \mathcal{Q}_-\right] \\
S &= \frac{\pi^2}{2G_N^{(5)}}\sqrt{\tilde{\mathcal{Q}}_0\mathcal{Q}_+\mathcal{Q}_-}\,.
\end{aligned}
$$

5-brane contribution

absent

$$\tilde{Q}_0 \sim \int_{S^3_\infty} \left(\star F^0 - \omega_{CS}\right); \qquad \left(d\star F^0 - F^A \wedge F^A = 0\right)$$

Embedding In String Theory

Strategy: 1.- Uplift the solution to $d=10$

(A theory with 10-dimensional Yang-Mills fields is needed → Heterotic Superstring)

2.- Dualize it to a type-II configuration with D-branes

→ Reduction of Heterotic Supergravity to $d=5$

$\downarrow$ T^4 + truncation

$\mathcal{N}=(1,0)$, $d=6$ + 1 tensor + vectors

$\downarrow$ S^1

$\mathcal{N}=1$, $d=5$ SEYM

Heterotic Supergravity

$$\hat{S} = \frac{g_s^2}{16\pi G_N^{(10)}} \int d^{10}x \sqrt{|\hat{g}|}\, e^{-2\hat{\phi}} \left[\hat{R} - 4(\partial\hat{\phi})^2 + \tfrac{1}{2\cdot 3!}\hat{H}^2 - \alpha' \hat{F}^A \hat{F}^A\right]$$

$e^{2\phi_\infty}$ $\quad$ $\to 8\pi^6 g_s^2 \ell_s^8$ $\quad$ ℓ_s^2

$$\hat{F}^A = d\hat{A}^A + \tfrac{1}{2}\epsilon^{ABC}\hat{A}^B \wedge \hat{A}^C ,$$

$\to$ SU(2) only

$$\hat{H} = d\hat{B} + 2\alpha' \omega_{\text{CS}} ,$$

$$\omega_{\text{CS}} \equiv \hat{F}^A \wedge \hat{A}^A - \tfrac{1}{3!}\epsilon^{ABC}\hat{A}^A \wedge \hat{A}^B \wedge \hat{A}^C , \qquad d\omega_{\text{CS}} = \hat{F}^A \wedge \hat{F}^A . \Rightarrow d\hat{H} = 2\alpha' \hat{F}^A \wedge \hat{F}^A$$

We are ignoring other α' corrections like the Lorentz Chern-Simons term and the R^2 term in the action

$\to$ Good approximation for small curvatures

Consistent with supersymmetry and gauge invariance.

- Reduction on a trivial T^4 truncating all KK fields

$$\hat{S} = \frac{(2\pi\ell_s)^4 g_s^2}{16\pi G_N^{(10)}} \int d^6x \sqrt{|g|}\, e^{-2\hat{\phi}} \left[\hat{R} - 4(\partial\hat{\phi})^2 + \frac{1}{2\cdot 3!}\hat{H}^2 - \alpha' \hat{F}^A \hat{F}^A\right]$$

w/ same definitions for the field strengths

$$\hat{g}_{\hat{\mu}\hat{\nu}} = g_s^{-1} e^{\hat{\phi}} \hat{g}_{E\,\hat{\mu}\hat{\nu}} \longrightarrow \hat{S} = \frac{(2\pi\ell_s)^4}{16\pi G_N^{(10)}} \int d^6x \sqrt{|g|} \left[\hat{R}_E + (\partial\hat{\phi})^2 + \frac{1}{2\cdot 3!} g_s^2 e^{-2\hat{\phi}} \hat{H}^2 - \alpha' g_s e^{-\hat{\phi}} \hat{F}^A \hat{F}^A\right]$$

"modified Einstein frame"

$\mathcal{N} = 2A$, $d = 6$ upon the redefinitions

$$\hat{\phi} = -\tilde{\varphi}/\sqrt{2}\,, \qquad g_s \hat{H}/2 = \tilde{H}\,, \qquad \sqrt{g_s \alpha'}\hat{F}^A = \tilde{F}^A$$

$\Rightarrow \quad g_6 = (g_s \alpha')^{-1/2}$ 6-dimensional Yang-Mills coupling constant.

A new 6-dimensional duality

(Cano, Ó., Santoli, 1607.02595)

$\mathcal{N}=2A,\ d=6$ $\quad$ $(\mathcal{N}=(2,0),\ d=6$ w/ $n_T=1,\ n_V)$

$$\hat{g}_{\hat{\mu}\hat{\nu}}, \quad \hat{B}^{-}_{\hat{\mu}\hat{\nu}} \leftarrow (\hat{B}_{\hat{\mu}\hat{\nu}}) \rightarrow \hat{B}^{+}_{\hat{\mu}\hat{\nu}}, \quad \phi \qquad \hat{V}^{I}_{\hat{\mu}}$$

$\mathcal{N}=1,\ d=5,$ $\quad g_{\mu\nu},\ A_\mu,\ k \qquad C_\mu \qquad B_\mu \qquad \phi \qquad V^I_\mu \quad l^I$

$$\check{g}_{\check{\mu}\check{\nu}} \qquad \check{B}^{-}_{\check{\mu}\check{\nu}} \qquad \check{B}^{+}_{\check{\mu}\check{\nu}} \qquad \phi \qquad \check{B}^{I+}_{\check{\mu}\check{\nu}},\ l^I$$

$\mathcal{N}=2B,\ d=6$ $\quad$ $(\mathcal{N}=(2,0),\ d=6$ w/ $n_T=1+n_V,\ n_V=0)$

The relation is T-duality-type but more involved.

Reduction on S^1 with radius $R_z = k_\infty \ell_s$ leads to the SU(2)-gauged, ST[2,6] model of $\mathcal{N}=1, d=5$ SUGRA w/

$$G_N^{(5)} = \frac{G_N^{(10)}}{(2\pi)^5 \ell_s^4 R_z} = \frac{\pi g_s^2 \ell_s^4}{4R_z}, \quad \text{and} \quad g = \frac{g_6 k_\infty^{1/3}}{\sqrt{12}} = \frac{k_\infty^{1/3}}{\sqrt{12 g_s \ell_s^2}}$$

5-dimensional Yang Mills coupling constant

Inverting this procedure we can uplift any solution of the SU(2)-gauged, ST[2,6] model of $\mathcal{N}=1, d=5$ SUGRA to a solution of d=10 Heterotic Supergravity.

Any solution of the ST[2,6] model of $\mathcal{N}=1$, $d=5$ SUGRA
$ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, A^0, A^1, A^2, A^A, k, ϕ, ℓ^A can be uplifted
to a solution of $d=10$ Heterotic Supergravity:

$$
\left\{
\begin{aligned}
d\hat{s}^2 &= e^{\phi-\phi_\infty}\left[(k/k_\infty)^{-2/3}ds^2 - k^2\mathcal{A}^2\right] - dy^i dy^i\,,\\
\hat{\phi} &= \phi\,,\\
\hat{A}^A &= \frac{k_\infty^{1/3}}{\sqrt{12 g_s\alpha'}}A^A - \frac{k^2\ell^A}{\sqrt{\alpha' g_s}}\mathcal{A}\,,\\
\hat{H} &= -\frac{k_\infty^{2/3}}{g_s\sqrt{3}}e^{2\phi}k^{-4/3}\star_{(5)}F^0 + \frac{k_\infty^{1/3}}{g_s\sqrt{3}}\mathcal{A}\wedge\mathcal{F}\,,
\end{aligned}
\right.
$$

Auxiliary definitions
$$
\left\{
\begin{aligned}
&\mathcal{A} \equiv dz + \frac{k_\infty^{1/3}}{\sqrt{12}}A^+\,, \qquad A^+ \equiv A^1 + A^2\,,\\
&\mathcal{F} \equiv F^- + \ell^2 F^+ - 2\ell^A F^A\,.
\end{aligned}
\right.
$$

In d = 10 Heterotic Supergravity

$$d\hat{s}^2 = \frac{2}{\mathcal{Z}_+} du \left(dv - \tfrac{1}{2} \mathcal{Z}_- du \right) - \tilde{\mathcal{Z}}_0 (d\rho^2 + \rho^2 d\Omega^2_{(3)}) - dz^i dz^i \,,$$

F1

S5 + G5

$$\hat{B} = -\frac{1}{\mathcal{Z}_+} dv \wedge du + \tfrac{1}{4} \mathcal{Q}_0 \cos\theta d\psi \wedge d\phi \,,$$

$$\hat{A}^A = -\frac{\rho^2}{(\kappa^2 + \rho^2)} v_R^A \,,$$

→ BPST instanton

$$e^{-2\hat{\phi}} = e^{-2\hat{\phi}_\infty} \frac{\mathcal{Z}_+}{\tilde{\mathcal{Z}}_0} \,,$$

$Q_0 = \tilde{Q}_0 + 8\alpha'$

→ This is the field produced by F1s NS5, w/ momentum flowing along the F1s plus a "gauge 5-brane"

To confirm the interpretation we must compute the charges of the solution:

(i) S5 charge: coupling N_{S5} branes to the bulk action:

$$\hat{S} = \frac{g_s^2}{16\pi G_N^{(10)}} \int \Big\{ e^{-2\hat{\phi}} \left[\star\hat{R} - 4d\hat{\phi}\wedge\star d\hat{\phi} + 2\alpha'\hat{F}^A\wedge\star\hat{F}^A\right]$$

$$+\tfrac{1}{2}e^{2\hat{\phi}}\hat{\tilde{H}}\wedge\star\hat{\tilde{H}} + 2\alpha'\hat{\tilde{B}}\wedge\hat{F}^A\wedge\hat{F}^A\Big\}\,.$$

$$+ N_{S5} T_5 g_s^2 \left\{ \int d^6\xi\, e^{-2\phi}\sqrt{|g|} + \int \hat{\tilde{B}} \right\}$$

→ The e.o.m. of the NS 6-form $\hat{\tilde{B}}$ is

$$\frac{g_s^2}{16\pi G_N^{(10)}}\left\{d(\star e^{2\hat{\phi}}\hat{\tilde{H}}) - 2\alpha'\hat{F}^A\wedge\hat{F}^A\right\} = g_s^2 N_{S5} T_{S5} \star_{(4)} \delta^{(4)}(\rho)\,,$$

$$\Longrightarrow \quad \tilde{\mathcal{Q}}_0 = \mathcal{Q}_0 - 8\alpha' = \ell_s^2 N_{S5}\,, \qquad \Longrightarrow \boxed{\mathcal{Q}_0 = \ell_s^2(N_{S5} + 8N_{G5})\,.}$$

(ii) F1 charge: by the same procedure we arrive at

$$T_{F1} N_{F1} = \frac{g_s^2}{16\pi G_N^{(10)}} \int_{V^8} d(\star e^{-2\hat{\phi}} \hat{H}) \,, \qquad \text{where} \qquad T_{F1} = \frac{1}{2\pi\alpha'} \,,$$

$\partial V^8 = T^4 \times S^3_\infty \Rightarrow$ $\boxed{\mathcal{Q}_- = \ell_s^2 g_s^2 N_{F1} \,.}$

(iii) Momentum: T-duality along $x^5 = z$ interchanges $\mathcal{Z}_+$ & $\mathcal{Z}_-$

$$\Rightarrow \boxed{\mathcal{Q}_+ = \ell_s^2 g_s'^2 N'_{F1} = \ell_s^2 \left(g_s \ell_s / R_z\right)^2 N_W = \frac{g_s^2 \ell^4}{R_z^2} N_W \,,}$$

In terms of the numbers of branes, the mass and entropy are

$$\boxed{\begin{aligned} M &= \frac{R_z}{g_s^2 \ell_s^2}(N_{S5} + 8N_{G5}) + \frac{R_z}{\ell_s^2} N_{F1} + \frac{1}{R_z} N_W \,, \\ S &= 2\pi \sqrt{N_{F1} N_W N_{S5}} \,. \end{aligned}}$$

Heterotic $\longrightarrow$ Type-I (Type IIB + 16 D9 + O9)

$$\hat{g}_{\hat{\mu}\hat{\nu}} = e^{-(\hat{\varphi}-\hat{\varphi}_\infty)}\hat{\jmath}_{\hat{\mu}\hat{\nu}}\,, \quad \hat{\phi} = -\hat{\varphi}\,, \quad \hat{C}^{(2)}_{\hat{\mu}\hat{\nu}} = e^{-\hat{\varphi}_\infty}\hat{B}_{\hat{\mu}\hat{\nu}} \quad \hat{A}^A_{\hat{\mu}} = g_I^{1/2}\hat{\mathcal{A}}^A_{\hat{\mu}}\,,$$

BI term of D9s/O9

$$g_I^{-4}\hat{S}_I = \frac{g_I^2}{16\pi G^{(10)}_{N,I}}\int\left\{e^{-2\hat{\varphi}}\left[\star\hat{R} - 4d\hat{\varphi}\wedge\star d\hat{\varphi}\right] + \tfrac{1}{2}\hat{G}^{(3)}\wedge\star\hat{G}^{(3)} + 2\alpha' e^{-\hat{\varphi}}\hat{\mathcal{F}}^A\wedge\star\hat{\mathcal{F}}^A\right\},$$

Observe that the WZ of the D9s contains the term

$$\hat{C}^{(6)}\wedge\hat{F}^A\wedge\hat{F}^A$$

so we have similar interpretation for the charges.

$N_{F1}\longrightarrow N_{D1}\;;\; N_{S5}\longrightarrow N_{D5}\;;\; N_W\longrightarrow N_W$

$G5 \longrightarrow GD5$

$$\boxed{\begin{aligned} M &= \frac{R_z}{g_I\ell_s^2}(N_{D5}+8N_{GD5}) + \frac{R_z}{g_I\ell_s^2}N_{D1} + \frac{1}{R_z}N_W\,,\\ S &= 2\pi\sqrt{N_{D1}N_{D5}N_W}\,.\end{aligned}}$$

Strominger-Vafa

+ GD5 → no $\cancel{S}$

Conclusions

- Non-Abelian solutions are interesting but still poorly understood.
- The $d=5$ case is very rich and easier to understand than the $d=4$ (which are the right charges?)
- Non-extremal
  Asymptotically AdS
  ⋮
  } ?
- Type II embeddings?
- Dumbbells & RG flows?

Thanks!

Some comments on the 4-d case

- Main difference: 1 more object (KK monopole in simplest version)
  → Different instantons on hyperKähler manifolds
  → Problems to dualize to D-brane configurations: solitonic and maybe exotic branes are needed