Supersymmetric configurations and solutions

miércoles, 30 de noviembre de 2016 18:23

The symmetry of configurations or solutions has to be addressed in the framework of a given theory and its symmetry group.

Any given configuration of a theory breaks ("spontaneously") most of the symmetries of the theory and "preserves" (or leaves unbroken) some subgroup. If the configuration is a solution and is used as a background (or vacuum state) on which the perturbations evolve, these are described by theories whose symmetries are those left unbroken by the background solution. This is the basis of the Higgs mechanism and of the "spontaneous compactification" mechanism invoked in Kaluza-Klein supergravity/superstring theories.

The main difference between these two mechanisms is the absence or presence of gravity. In absence of gravity (i.e. in the fixed Minkowski background of Special Relativistic QFTs) there is a common energy scale for all the vacuum states and the vacuum selection mechanism is the minimization of energy. In presence of gravity only the energy of solutions with common asymptotics that allow for a well-defined notion of energy can be compared. Actually, the asymptotics are associated to a common vacuum solution and, solutions with the same asymptotics can be understood as perturbations over the common vacuum. Thus the relative energies of different vacua cannot be compared and there is no clear vacuum selection mechanism.

This is the root of the "Landscape problem" of superstring theory and, from the previous discussion, it follows that it is a problem of all theories of gravity.

Although there is no vacuum selection mechanism, there is, however, a criterion to distinguish vacua from other solutions: the amount of unbroken symmetry of the solution: vacua have a high degree of symmetry and are, typically (products of) homogeneous spaces with no distinguished points while the solutions that represent excitations over that vacuum have globally less symmetry although it can be recovered asymptotically or in other limits.

Solutions that preserve many symmetries are also relevant for other reasons (they can be used to describe idealized situations with great success (Schwarzschild, Kerr...)) and, moreover, they are simpler to find. This makes them very interesting objects of study.

In the context of theories which are invariant under (local or global) supersymmetry transformations we can consider solutions that preserve some of them. They are called supersymmetric solutions (or configurations, if we do not impose the e.o.m.) or BPS solutions because, often, they satisfy 1st-order differential equations similar to the Bogomol'nyi equations satisfied by the 't Hooft-Polyakov monopole in the Bogomol'nyi-Prasad-Sommerfield (BPS) limit.

The subject of these lectures is the construction and study of the supersymmetric solutions of a particularly interesting class of theories which are invariant under local supersymmetry transformations: $\mathcal{N}=1$, $d=5$ supergravities. Before studying the solutions (and how they are found) we need to understand well the theories and, more generally, what are the main features of supersymmetric solutions.

Killing spinor equations

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Field theories invariant under linear supersymmetry transformations must have bosonic ϕ^B and fermionic ϕ^F fields (commuting and anti-commuting). They are, respectively, tensor and spinor (or tensor-spinor) fields and their quantum excitations are particle states of integer or half-integer spin.

The infinitesimal parameters of the supersymmetry transformations are spinors ε. If they are only allowed to be constants or constants times fixed functions, the supersymmetry is said to be global. When they can be arbitrary spacetime functions, the supersymmetry is local.

Local linear supersymmetry always implies invariance under g.c.t.s and the presence of a dynamical metric: gravity $\Rightarrow$ supergravity

One of the reasons for this is that supersymmetry transformations alone do not form a closed algebraic structure (group) but close on standard bosonic spacetime symmetry transformations. Supersymmetry is a spacetime symmetry and not just an internal symmetry. Global supersymmetry is the Special Relativity of superspace (an extension of Minkowski spacetime with odd coordinates). Local supersymmetry is the General Relativity of curved superspace. Finding supersymmetry could be much more than finding another internal symmetry because, gravity couples to everything and everything exists in spacetime.

Generically, infinitesimal susy transf. have the form

$$\begin{cases} \delta_\varepsilon \phi^b = \varepsilon \phi^f (\times \phi^b) ; \\ \delta_\varepsilon \phi^f = \partial \varepsilon + \left(\phi^b + \phi^f \phi^f\right) \varepsilon ; \end{cases} \quad \partial\varepsilon = 0 \text{ in the global case}$$

A configuration ϕ would be supersymmetric if, for some ε (Killing spinor)

$$\boxed{\begin{aligned} \delta_\varepsilon \phi^f &= 0 ; \\ \delta_\varepsilon \phi^b &= 0 ; \end{aligned}}$$

Classically only purely bosonic configurations are considered $\phi^f = 0$

The condition reduces then to $\boxed{\delta_\varepsilon \phi^f = \partial \varepsilon + \phi^b \varepsilon = 0}$
Killing spinor equations (KSE)
ε: Killing spinor of the field configuration $\phi^b, \phi^f = 0$.
The KSE that contains the derivative of ε can be written in the form $\mathcal{D}\varepsilon = 0$ where $\mathcal{D}$ is a linear differential operator called supercovariant derivative. It is covariant w.r.t. all the transformations acting on ε because the spinor in the l.h.s. $\delta_\varepsilon \phi^f$ must transform in the same way. Thus, it is, at least Lorentz-covariant (contains the spin connection $\omega_\mu{}^{ab}$) but can contain additional connections: gauge, if the spinors are charged ("gauged supergravities") and the pullbacks of the connections of the scalar manifolds (if any), related to R-symmetry.

The Killing spinors span a finite-dimensional vector space of unbroken supersymmetries. Maximally supersymmetric if the dimension is that of the spinor. (½ BPS, ¼ BPS ...) These are the candidates to vacua of the supergravity theory.

The Killing spinors of a bosonic field configuration are the odd generators of the super(Lie) group of symmetry of the configuration (the theory is invariant under the infinite-dimensional supergroup of general superreparametrizations) How can we find it?

i) Find all the Killing spinors $\kappa_I{}^\alpha \xrightarrow{\text{associate}} Q_I$ odd

ii) Find all the Killing vectors $k_A{}^\mu \longrightarrow P_A$ even

iii) Find all the Killing gauge parameters $\chi_M \longrightarrow B_M$ even
(internal transf. leaving the ϕ^b invariant)

We want to find the structure constants of the (anti-)commutators of all the generators.

The k_A span a closed Lie algebra by themselves

$$[k_A, k_B] = -f_{AB}{}^C k_C \Rightarrow [P_A, P_B] = f_{AB}{}^C P_C \,;$$

$$\text{"}\pounds_{k_A} k_B\text{"} \Rightarrow [B_M, B_N] = f_{MN}{}^P B_P \,;$$

$$[B_M, P_A] = f_{MA}{}^N B_N \ ?$$

From $\pounds_{k_A} B_M = -f_{MA}{}^N B_N$; $(T_A)^N{}_M$ a rep. of the isometry algebra on the B's

The difficult part is to compute

$$\{Q_I, Q_J\} = f_{IJ}{}^A P_A + f_{IJ}{}^M B_M \,;$$

$$\left.\begin{aligned} [Q_I, P_A] &= f_{IA}{}^J Q_J \,; \\ [Q_I, B_M] &= f_{IM}{}^J Q_J \,; \end{aligned}\right\}$$ We can interpret them as the action of the bosonic generators on the fermionic ones.

→ <u>Easy</u>: the spinors of the theory (in particular the susy t. parameters) have well-defined transformation rules under internal symmetries.

→ <u>More difficult</u>: the standard Lie derivative is not compatible with ("covariant under") Lorentz (spin) or internal transformations. → <u>Kosmann Lie derivative</u>

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Kosmann defined a generalisation of the Lie derivative for spinors.
A generalisation for any object T transforming in the local representation r of the Lorentz group is, for any vector V^μ

$$\mathbb{L}_V T = V^\mu \underbrace{\nabla_\mu}_{\text{Full (affine + Lorentz) covariant derivative}} T + \underbrace{\tfrac{1}{2} \nabla_{[a} V_{b]} \Gamma_r (M^{ab})}_{\text{compensating Lorentz transformation}} T$$

$$= \underbrace{V^\mu \partial_\mu T}_{\mathcal{L}_V T} - \tfrac{1}{4}\left(V^\mu \omega_\mu{}^{ab} - 2 \nabla^{[a} V^{b]}\right) \Gamma_r (M_{ab}) T$$

$$\mathbb{L}_V T_{\mu_1 \cdots \mu_m}{}^{\nu_1 \cdots \nu_m} = \mathcal{L}_V T_{\mu_1 \cdots \mu_m}{}^{\nu_1 \cdots \nu_m} - \tfrac{1}{4}\left(\quad \right) \Gamma_r (M_{ab}) T_{\mu \cdots}{}^{\nu \cdots}$$

The desired properties only hold when V^μ is a KV or CKV

1) Leibnitz rule
2) $[\mathbb{L}_{k_1}, \mathbb{L}_{k_2}] = \mathbb{L}_{[k_1,k_2]} T$ (Lie bracket) } Lie algebra structure
3) Linearity in V

Further

4) $\mathbb{L}_k \gamma^a = 0$
5) Preserves the Clifford action of vectors V^μ on spinors ψ $V\cdot\psi =$
$V\cdot\psi = V^\mu \gamma_\mu \psi = V^\mu e^a{}_\mu \gamma_a \psi = V^a \gamma_a \psi = \not{V} \psi$
$[\mathbb{L}_{k}, \not{V}] \psi = [k, V]\cdot\psi$

6) (Only for KV) $[\mathbb{L}_{k}, \nabla_V] T = \nabla_{[k,V]} T$

7) If we include additional connections $[\mathbb{L}_k, \mathcal{D}_V]\varepsilon = \mathcal{D}_{[k,V]}\varepsilon$

$\Rightarrow$ It follows that it transforms KS into KS.

$\Rightarrow \; -\mathbb{L}_{k_A} \varepsilon_I = f_{AI}{}^J \varepsilon_J \quad \rightarrow \quad [Q_I, P_A] = -f_{AI}{}^J Q_J$;

Finally $\varepsilon_I^\alpha \to \bar{\varepsilon}_\alpha$ (conjugation (Dirac, Majorana))

$$\bar{\varepsilon}_{I\alpha}\, \varepsilon_J{}^\beta = C_{IJ}\, \delta^\beta{}_\alpha + C_{IJ}{}^a \gamma_a{}^\beta{}_\alpha + \frac{1}{2} C_{IJ}{}^{ab} \gamma_{ab}{}^\beta{}_\alpha + \dots$$

Which terms do actually occur on the r.h.s. depends on the dimension etc. However, we always have $C_{IJ}{}^a \neq 0$ and, furthermore

$$C_{IJ}{}^a = f_{IJ}{}^A\, k_A{}^a$$

$$\bar{\varepsilon}\gamma^a\varepsilon \equiv \bar{\varepsilon}_{I\alpha}\gamma^{a\,\alpha}{}_\beta\, \varepsilon_J{}^\beta \sim f_{IJ}{}^A\, k_A{}^a$$

The rest of the terms correspond to parameters of gauge transformations of the fields of the theory $\quad \delta_\chi A_\mu = \partial_\mu \chi^{(0)},\ \delta_\chi B_{\mu\nu} = 2\partial_{[\mu}\chi^{(1)}_{\nu]}$ etc

$\Rightarrow \quad f_{IJ}{}^M\, B_M$

The tensor bilinears that one constructs from the KSs contain a great deal of information about the bosonic fields of the configuration through their spacetime and internal symmetries.

Demanding unbroken supersymmetry is much more restrictive than demanding some spacetime symmetry.

On the other hand, the bilinears contain the same information as the spinors <u>with respect to the bosonic fields</u>.

Killing spinor identities

miércoles, 30 de noviembre de 2016 18:39

These identities show how powerful the requirement of unbroken supersymmetry is compared with standard symmetry.

Consider a theory with linearly realised local supersymmetry. The action S is invariant under $\delta_\varepsilon \phi^b$, $\delta_\varepsilon \phi^f$:

$$\delta_\varepsilon S = 0 = \int \left\{ \frac{\delta S}{\delta \phi^b} \delta_\varepsilon \phi^b + \frac{\delta S}{\delta \phi^f} \delta_\varepsilon \phi^f \right\} \text{ (up tot. d.)}$$

Let us take the functional derivative of this identity w.r.t. the fermionic fields:

$$0 = \frac{\delta}{\delta \phi^{f^1}} \left\{ \frac{\delta S}{\delta \phi^b} \delta_\varepsilon \phi^b + \frac{\delta S}{\delta \phi^{f^2}} \delta_\varepsilon \phi^{f^2} \right\} =$$

$$= \frac{\delta^2 S}{\delta \phi^{f^1} \delta \phi^b} \delta_\varepsilon \phi^b + \frac{\delta S}{\delta \phi^b} \frac{\delta}{\delta \phi^{f^1}} \left(\delta_\varepsilon \phi^b \right) +$$

$$+ \frac{\delta^2 S}{\delta \phi^{f^1} \delta \phi^{f^2}} \delta_\varepsilon \phi^{f^2} + \frac{\delta S}{\delta \phi^{f^2}} \frac{\delta}{\delta \phi^{f^2}} \left(\delta_\varepsilon \phi^{f^1} \right) = 0$$

Now we set to zero all the fermionic fields $\phi^f = 0 \Rightarrow \begin{cases} \delta_\varepsilon \phi^b = 0 \\ \dfrac{\delta S}{\delta \phi^{f^2}} = 0 \end{cases}$

$$\left. \left(\frac{\delta S}{\delta \phi^b} \frac{\delta}{\delta \phi^{f^1}} \left(\delta_\varepsilon \phi^b \right) + \frac{\delta^2 S}{\delta \phi^{f^1} \delta \phi^{f^2}} \delta_\varepsilon \phi^{f^2} \right) \right|_{\phi^f = 0} = 0$$

Then, require the field configuration to be supersymmetric $\delta_\varepsilon \phi^f = 0$

$$\boxed{ \left. \frac{\delta S}{\delta \phi^b} \right|_{\phi^f = 0} \left. \frac{\delta}{\delta \phi^{f^1}} \left(\delta_\varepsilon \phi^b \right) \right|_{\substack{\phi^f = 0 \\ \varepsilon \text{ Killing}}} = 0 } \quad \text{KSIs}$$

$\Rightarrow$ Certain combinations of the equations of motion of the bosonic fields are automatically satisfied.

If one looks for supersymmetric solutions among the supersymmetric configurations only a few independent e.o.m.'s to be imposed.

Other general properties of susy solutions

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- BHs : - vanishing temperature ⇒ extremal ⇒ usually one can have many in static equilibrium.
  - interpolating between maximally supersymmetric solutions
    → asymptotically (Minkowski, AdS)
    → at the event horizon ($AdS_2 \times S^3$ or similar)
    ⇓
    Supersymmetric near-horizon geometries can be determined more easily! They can be very interesting in $d > 4$!

- Strings, branes : generalisation of the above properties + possible intersections.

- Stationary gravitational waves (Hpp waves)
- Gödel-like spacetimes (⇒ supersymmetry does not exclude CTCs)

N=1,d=5 Supergravities

miércoles, 30 de noviembre de 2016 18:25

Supersymmetric theories are labeled by the number N of unconstrained supersymmetry parameters, the dimension of the spacetime d and the matter content and gauged symmetries.

N=1 means there is only one parameter: a spinor. The smallest spinor representation in d=5 (1,4) has 8 real independent parameters but we cannot construct a Lorentz-invariant theory if we use less than 8 imposing constraints on the spinor. Sometimes people refer to these theories as theories with "8 supercharges" and also as N=2, d=5 theories because 8 independent spinor components correspond to 2 minimal spinors in d=4 (Weyl or Majorana).

Why N=1? The more (super) symmetry, the more constrained theories are: N=4 is unique, but there are many models of N=1, d=5 supergravity with interesting properties (related to N=2, d=4 models also very interesting because they are the simplest that admit supersymmetric BHs)

Why d=5? Most of what we find in the N=2, d=4 models can be uplifted to N=1, d=5. But d=5 is a far reached arena because in d=5 BH uniqueness theorems and the theorems on the topology of BH event horizons are violated (with no need of cosmological constant). Thus, one finds black rings, black strings....
Moreover, in d=5 one finds AdS_3 factors in near-horizon geometries which are related to 2-d CFTs.

Spinors, gamma matrices and identities

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The structure of a supersymmetric theory is dictated by the properties of its spinors and γ matrices (Clifford algebra)

In $d=5$ there are no Weyl or Majorana spinors. The minimal spinor ψ^α is a <u>Dirac</u> 4-component complex spinor. However, there is another way of looking at it: as <u>a symplectic-Majorana spinor</u> $\psi^\alpha{}_i$ $i=1,2$. As we are going to see, only by rewriting Dirac spinors in this way we can make manifest an underlying symmetry of the theory called <u>R-symmetry</u> and which in this case corresponds to the global $SU(2) \sim Sp(1)$ group acting in the fundamental representation over the spinor indices $i=1,2$.

R-symmetry is the group of outer automorphisms of the $N=1, d=5$ supersymmetry algebra and we expect all $N=1, d=5$ theories to be invariant under them (but do not associate that superalgebra to the SUGRA as you do not associate the Poincaré algebra to GR).

To define symplectic-Majorana spinors, we first review the γ-matrices

$$\{\gamma^a, \gamma^b\} = +2\eta^{ab} \quad ; \quad (\eta^{ab}) = \mathrm{diag}(+ - - - -) \quad 4\times 4 \; (\gamma^a)^\alpha{}_\beta$$

There is no Majorana (purely real or imaginary) representation. We can take $\gamma^0 - \gamma^3$ imaginary (same as in $d=4$) and $\gamma^4 \equiv \gamma_0\gamma_1\gamma_2\gamma_3 \in \mathbb{R}$

$$\rightarrow \gamma^0\gamma^1\gamma^2\gamma^3\gamma^4 = +1$$

<u>Dirac conjugation matrix D</u>

$$D_\pm^{-1} \gamma^{a\dagger} D_\pm = \pm \gamma^a$$

$$\Gamma_s(M_{ab}) \sim \gamma_{ab}$$

$$D^{-1} \Gamma_s(M_{ab})^\dagger D = -\Gamma_s(M_{ab})$$

$$\psi^\alpha \rightarrow \bar\psi_\alpha = (\psi^\beta)^* D_{\beta\alpha}$$

($\bar\psi_\alpha$ transforms covariantly if ψ^α transforms contravariantly)

<u>Charge conjugation matrix C</u>

$$C_\pm^{-1} \gamma^{aT} C_\pm = \pm\gamma^a$$

$$C^{-1} \Gamma_s(M_{ab})^T C = -\Gamma_s(M_{ab})$$

$$\psi^c{}_\alpha = \psi^\beta C_{\beta\alpha}$$

($\psi^c{}_\alpha$ covariant)

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Majorana condition $\bar\psi = \psi_c \Rightarrow$ Majorana spinor (1/2 indep. comp.)

In $d=5$ $\quad D_{(+)} = i\gamma^0$; $C_{(+)} = i\gamma^{04}$ and the Majorana condition reads

$$\psi^\dagger D = \psi^T C \, ; \quad \left(\psi^{*T} i\gamma^0 = \psi^T i\gamma^{04}\right)\cdot i\gamma^0 \, ; \quad \psi^{*T} = -\psi^T\gamma^4 \, ;$$

$\psi^* = -\gamma^{4T}\psi$

γ^4 real, unitary and $(\gamma^4)^2 = -1$; $\gamma^{4\dagger} \overset{\text{unit}}{=} \gamma^{4-1} \overset{(\gamma^4)^2=-1}{=} -\gamma^4$; real $\rightarrow$ $\gamma^{4\dagger} = \gamma^{4T}$ $\Bigg\}$ $\boxed{\psi^* = \gamma^4\psi}$

This condition is not consistent : $\left(\psi^* = \gamma^4\psi\right)^* \Rightarrow \psi = \gamma^4\psi^* \Rightarrow \psi = -\gamma^4\psi$

The only possibility is to take a pair of spinors ψ^i $\quad i=1,2$ (or several pairs)

and suppose $\boxed{\psi^{*i} = \Omega_{ij}\gamma^4\psi^j \quad \oplus \quad \Omega_{ij}\Omega_{jk} = -\delta_{ik}}$

Symplectic-Majorana condition $\boxed{\bar\psi^i = \Omega_{ij}\psi^j{}_c} = \psi_{c\,i}$

There is a natural action of $USp(N)$ on (pairs of) symplectic-Majorana spinors that preserves their nature (the above condition):

$$\psi'^i = \Lambda^i{}_j\psi^j \, ; \quad \bar\psi^{i\prime} = (\Lambda^i{}_j)^*\bar\psi^j \, ;$$

$$(\Lambda^{*-1})^m{}_i\left[(\Lambda^i{}_j)^*\bar\psi^j = \Omega_{ik}\Lambda^k{}_l\psi^l{}_c \, ;\right] \quad \bar\psi^m = \underbrace{(\Lambda^{*-1})^m{}_i\Omega_{ik}\Lambda^k{}_l}_{\Omega_{ml}}\psi^l$$

$(\Lambda^{*-1})\,\Omega\,\Lambda = \Omega$; Solved by $\Lambda^\dagger = \Lambda^{-1} \in U(N)$
$\qquad \Lambda^T\Omega\Lambda = \Omega \quad \in Sp(N)$

The structures that occur in the supergravity theory are related to the possible spinor bilinears one can construct with just N pairs of S-M spinors $\bar\psi_i\gamma_{a_1\cdots a_n}\psi^j = \Omega_{ik}\psi^k C\gamma_{a_1\cdots a_n}\psi^j$

In the $N=1$ case we can convert the pairs of indices i,j into "adjoint indices" $M=0, x=0,1,2,3$ using the Pauli matrices $(\sigma^M)^i{}_j$ $\sigma^0 = 1_{2\times 2}$

$$\bar\psi\gamma_{a_1\cdots a_n}\sigma^M\psi = \underbrace{(\sigma^M)^j{}_i\,\Omega_{jk}}_{\substack{\text{symmetric in } ik \; M=0 \\ \text{antisymmetric in } ik \; M=x}}\bar\psi^i\left(\gamma_{a_1\cdots a_n}\right)^T C^T\psi^k$$

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$\xi = 0$ commuting
$\xi = 1$ anticommuting

$$\left(\psi^{i\,T} C \gamma_{a_1 \ldots a_m} \psi^j\right)^T = (-1)^{\xi} \psi^{j\,T} C C^{-1} \left(\gamma_{a_1 \ldots a_m}\right)^T C^T \psi^i$$

$$C^T = \left(i\gamma^{04}\right)^T = i\gamma^{4T}\gamma^{0T} = i\gamma^4\gamma^0 = -i\gamma^{04} = -C;$$

$$\left.\begin{aligned} \gamma^{0T} &= -\gamma^{0\dagger} = -\gamma^{0-1} = -\gamma^0 \\ \gamma^{4T} &= +\gamma^{4\dagger} = +\gamma^{4-1} = -\gamma^4 \end{aligned}\right\}$$

$$C^{-1}\left(\gamma_{a_1 \ldots a_m}\right)^T C = C^{-1}\gamma^T_{a_m} \cdots \gamma^T_{a_1} C = C^{-1}\gamma^T_{a_m} C C^{-1}\gamma^T_{a_{m-1}} \cdots\cdots =$$

$$= +\gamma_{a_m \ldots a_1} = (-1)^{m-1} \cdot (-1)^{m-2} \cdots (-1)\, \gamma_{a_1 \ldots a_m} = (-1)^{\left[\frac{m}{2}\right]} \gamma_{a_1 \ldots a_m}$$

$$\bar{\psi}^i \gamma_{a_1 \ldots a_m} \psi^j = \left(\quad\right)^T = (-1)^{\xi + 1 + \left[\frac{m}{2}\right]} \bar{\psi}^j \gamma_{a_1 \ldots a_m} \psi^i$$

anticommuting spinors — even : $m = 0, 1, 4$: odd — commuting spinors
odd : $m = 2, 3$: even

$$\left(\gamma^{a_1 \ldots a_m} = \frac{(-1)^{[m/2]}}{(5-m)!} \varepsilon^{a_1 \ldots a_m b_1 \ldots b_{5-m}} \gamma_{b_1 \ldots b_{5-m}}\right)$$

Anticommuting spinor bilinears

$$\boxed{\bar{\psi}_i (\sigma^x)^i{}_j \psi^j, \quad \bar{\psi}_i \gamma_a (\sigma^x)^i{}_j \psi^j \quad \bar{\psi}_i \gamma_{ab} \psi^j}$$

(Also $\bar{\psi}_{[\mu| i} \gamma_a \bar{\psi}^i{}_{|\nu]}$ → torsion etc)

Commuting spinor bilinears

$$\boxed{\begin{aligned} &\bar{\psi}_i \psi^i, \quad \bar{\psi}_i \gamma_a \psi^i; \\ &\bar{\psi}_i \gamma_{ab} \sigma^{x\,i}{}_j \psi^j \to \text{triplets of 2-forms.} \end{aligned}}$$

How about the reality? Remember $\bar{\psi}^i$ is really ψ^i_c: $\psi_{ic} = \bar{\psi}^i = \psi^{i\dagger} D$

$$\left(\bar{\psi}_i \psi^i\right)^* \to \left(\bar{\psi}^i \psi^i\right)^\dagger = (-1)^{\xi} \psi^{i\dagger} D^\dagger \psi^i = (-1)^{\xi+1} \psi^{i\dagger} D \psi^i =$$

$$= (-1)^{\xi+1} \bar{\psi}^i \psi^i \longrightarrow (-1)^{\xi+1} \bar{\psi}_i \psi^i$$

⇒ For commuting spinors
$$\left\{\begin{aligned} &i\bar{\psi}_i \psi^i \text{ is real} \\ &i\bar{\psi}_i \gamma_a \psi^i \text{ is real} \\ &\bar{\psi}_i \gamma_{ab} \sigma^{x\,i}{}_j \psi^j \text{ is real} \end{aligned}\right.$$

R-symmetry

miércoles, 30 de noviembre de 2016 18:42

For $N=1$ $USp(N) = SU(2)$. If we use a single Dirac spinor we do not see this property. If we introduce ψ^2 / $\psi^{2*} = -\gamma^4 \psi^1$ then we have an $SU(2)$ doublet. All the spinors of the theory (including the supersymmetry transformation parameters) are written in this way and all of them transform simultaneously under $SU(2)$ which is the <u>R-symmetry group</u>.

This is going to be a global symmetry of the theory even after we couple it to supersymmetric matter. This can happen in two ways:

i) The bosonic fields in the matter multiplets are R-invariant.

ii) The bosonic fields in the matter multiplets transform in some well-defined way under R-symmetry.

i) → Vector and tensor multiplets (which include scalars)

ii) → Hypermultiplets (which only include scalars)

For $N>1$ in $d=5$ ($N>2$ in $d=4$) the scalars of the theory parametrise homogeneous (symmetric, actually) spaces

$(d=4)$ $N=3,4$ $G/H \times G^{matter}/H_{matter}$ H: R-symmetry group

$N \geq 5$ G/H $H \subset$ R-symmetry group

In coset spaces there is a natural H (R-) connection ϑ whose pullback over the spacetime can couple naturally to the spinors which transform in the same way: <u>scalar-dependent R-symmetry transformations</u>. ϑ is necessary in the covariant derivative of spinors.

$$\mathfrak{g} = \{P_a, M_i\} \qquad -u^{-1} du = e^a P_a + \vartheta^i M_i$$

$$\mathfrak{h} = \{M_i\} \qquad u(\phi)\text{: } G/H \text{ coset representative}$$

$$u' = g\, u\, h^{-1}(\phi)$$

$$\psi' = \Gamma_r(h(\phi))\,\psi$$

$$\boxed{\mathcal{D}\psi = \nabla\psi + \vartheta^i\, \Gamma_r(M_i)\,\psi}$$

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Low-N theories are more flexible: the scalars do not need to live in a fixed coset space (nor in a coset space at all). Then, how is the coupling of the scalars to the spinors made consistent with R-symmetry?

→ • The scalar manifolds are required to have the R-symmetry group as a subgroup of the holonomy group.

• Furthermore a specific relation between the curvature and some structures is imposed:

$N=1, d=4$ $\left\{ \begin{array}{l} \text{R-symmetry} = U(1) \\ \text{Kähler-Hodge scalar manifolds} \left\{ \begin{array}{l} U(1) \subset \text{holonomy} \subset U(n) \\ R \text{ related to } \mathcal{J} \\ \text{(to be investigated)} \end{array} \right. \\ \text{(Ricci 2-form = Kähler 2-form)} \end{array} \right.$

$N=2, d=4$ $\left\{ \begin{array}{l} \text{R-symmetry} = U(2) = U(1) \times SU(2) \\ \text{Special Kähler scalar manifolds } (\Rightarrow \text{Kähler-Hodge}) \\ \text{Quaternionic-Kähler scalar manifolds } SU(2) \subset \text{holonomy} \\ \oplus \quad R^x = \mathcal{J}^x \text{ specific, non-vanishing, constant} \\ (SU(2) \text{ curvature = hypercomplex structure}) \end{array} \right.$

$N=1, d=5$ $\left\{ \begin{array}{l} \text{R-symmetry} = SU(2) \\ \text{Very special manifolds (R-symmetry neutral)} \\ \text{Quaternionic-Kähler manifolds with same property} \end{array} \right.$

pullbacks of the

The Kähler connection or the SU(2) component of the spin connection of the KH or QK manifolds appears in the covariant derivatives of the spinors.

Now, a global transformation of the scalar manifold (isometry) induces a scalar-dependent transformation of the spinors.

Where do the $R \sim \mathcal{J}$ conditions come from?

3

sábado, 3 de diciembre de 2016 19:48

I do not really know. Supersymmetry demands them. We can, however, see that those conditions are similar to the properties of the curvature in coset spaces. More important: they are essential for gauging the theories because they play an important rôle in the definition of momentum map.

Supermultiplets

miércoles, 30 de noviembre de 2016 18:42

All supergravities are constructed by coupling a supermultiplet that contains the metric (Vielbein $e^a{}_\mu$) and its supersymmetric partners (gravitini ψ^i_μ) plus other fields, called "supergravity multiplet", to matter supermultiplets (if they exist). The theory one constructs from the supergravity multiplet is invariant under supersymmetry transformations by itself and it is called "pure supergravity".

The matter supermultiplets may be used to construct consistent theories on a fixed background by themselves, but we will not go into this problem.

The supermultiplets of N=1, d=5 supergravity are:

Supergravity multiplet : $\{e^a{}_\mu, \psi^i_\mu, A_\mu\}$ A_μ : graviphoton

ψ^i_μ : (SM) gravitino

Vector supermultiplet : $\{A_\mu, \lambda^i, \phi\}$ λ^i: (SM) gaugino

Tensor supermultiplets : $\{B_{\mu\nu}, \lambda^i, \phi\}$ (dual to vectors)

Hypermultiplets $\{q^X, \zeta^A\}$ $X=1,\cdots,4$, $A=1,2 \rightarrow SU(2)$

We are not going to consider tensor multiplets in most of what we are going to do. Some results still have to be generalized to include them properly. In many cases they can be replaced by vector multiplets, but not always.

In general we will have more than one vector multiplet or hypermultiplet. First, we need to adapt our notation to this possibility: $\{A^x{}_\mu, \lambda^{xi}, \phi^x\}$ $x=1,\cdots,n_V$

$(A_\mu{}^I)=(A^0_\mu, A^x{}_\mu)$; $A^0_\mu=A_\mu$; $\{q^X, \zeta^A\}$ $X=1\cdots 4n_H$; $A=1\cdots 2n_H$

Second, we have to study the possible couplings of these fields.

Real Special Geometry

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In general the scalar fields of a supergravity theory parametrize some space with a non-trivial metric and their kinetic terms are a σ-model.
Furthermore, they may couple to other fields through functions with special properties: kinetic matrices etc.
For the scalar fields in the vector multiplets ϕ^x all these structures are integrated into "<u>Real Special Geometry</u>"
The n_V-dimensional space parametrised by these scalars can always be seen as a codimension-1 hypersurface in a (n_V+1)-dimensional space with coordinates h^I, $I=0,\dots,n_V$ (same labels as the vector fields) defined by an equation of the form $C_{(IJK)} h^I h^J h^K = 1$
There are linear and non-linear symmetries of this hypersurface.
The former are the linear symmetries of the symmetric tensor C_{IJK} and they are the only ones that can be symmetries of the whole theory (the vectors $A^I{}_\mu$ must transform linearly).
These symmetries (if any) act non-linearly on the physical scalar fields that solve the equation $h^I(\phi)$.
Some definitions: $h_I \equiv C_{IJK} h^J h^K$ (covariant)

$$\Rightarrow h_I h^I = 1 ;$$

$$a_{IJ} \equiv -2 C_{IJK} h^K + 3 h_I h_J ;$$

$$\Rightarrow a_{IJ} h^J = h_I ; \quad a^{IJ} h_J = h^I \quad \text{etc.}$$

a_{IJ} is the metric on the embedding space $ds^2 = h_{IJ} dh^I dh^J$
It will play the role of kinetic matrix for the vector fields and has the same symmetries as C_{IJK} (linear)
The induced metric on the hypersurface $g_{xy} = 3 a_{IJ} \partial_x h^I \partial_y h^J$ is the metric of the σ-model. $h_I \partial_x h^I = h^I \partial_x h_I = 0$

$$A^I = (h_J A^J) h^I - 3 (\partial_x h_J A^J) g^{xy} \partial_y h^I \quad \text{etc.}$$

$\Rightarrow$ h_I can be used to project onto the graviphoton sector.

$\partial_x h^I$ ($\sim h_x^I$) can be used to project onto the matter sector.

From the supersymmetry point of view, these are not A^0_μ, A^x_μ, but the combinations that occur in the gravitino and gaugino supersymmetry transformation rules.

So far we know 2 terms of the bosonic action

σ-model $\frac{1}{2} g_{xy} g^{\mu\nu} \partial_\mu \phi^x \partial_\nu \phi^y$;

vectors $-\frac{1}{4} a_{IJ} g^{\mu\nu} g^{\rho\sigma} F^I{}_{\mu\rho} F^J{}_{\nu\sigma}$;

Hypers and QK manifolds

sábado, 3 de diciembre de 2016   21:23

The scalars in the hypermultiplets parametrize a quaternionic-Kähler manifold of dimension $4n_H$. They are defined as follows:

i) They are endowed with a triplet of complex structures $J^r{}_X{}^Y$

$X, Y = 1, \ldots, 4n_H$; $r, s, t = 1, 2, 3$

$$J^r{}_X{}^Y J^r{}_Y{}^Z = -\delta_X{}^Z \qquad \left((J^r)^2 = -1\right)$$

ii) The triplet satisfies the algebra of imaginary quaternions

$$J^x J^y = -\delta^{xy} + \varepsilon^{xyz} J^z$$

iii) The Riemannian metric g_{XY} is Hermitean w.r.t. the three complex structures

$$J^{(r)}_X{}^Z J^r{}_Y{}^W g_{ZW} = g_{XY} \;; \quad r = 1,2,3$$

$\Rightarrow$ 3 Kähler 2-forms $K^r{}_{XY} = J^r{}_X{}^Z g_{ZY}$;

iv) There is an SU(2) bundle over the QK manifold with a connection 1-form $A^r{}_X\, dq^X$ and the L-C + SU(2) covariant derivative of the K^rs is zero:

$$D_X K^r{}_{YZ} = \nabla_X K^r{}_{YZ} + \varepsilon^{rst} A^s{}_X K^t{}_{YZ} = 0$$

v) The curvature of the SU(2) connection $F^r{}_{XY}$ is proportional to the Kähler 2-forms $J^r{}_{XY}$

$$K^r{}_{XY} = \kappa\, J^r{}_{XY} \;; \quad \begin{cases} \kappa = 0 \Rightarrow \text{hyper-Kähler} \\ \kappa \neq 0 \Rightarrow \text{quaternionic-Kähler} \end{cases}$$

In supergravity $N=1$, $d=5$ supergravity $\boxed{\kappa = -1}$

QK manifolds have special holonomy:

$$\underbrace{SU(2) \times Sp(2n_H)}_{2\,\times\,(4n_H^2 - 1)} \subset \underbrace{SO(4n_H)}_{2n_H(4n_H - 1) \,=\, 8n_H^2 - 2n_H}$$

The tangent-space indices can be represented by pairs $\underset{SU(2)\,\rightarrow}{i}\;\underset{\leftarrow\,Sp(2n_H)}{A}$

Vierbein $\longrightarrow$ "Quadbein" $f_X{}^{iA}$

Given a QK manifold of dimension $4n_H$, the kinetic term of the hyperscalars is the σ-model

$$\frac{1}{2} g_{XY} g^{\mu\nu} \partial_\mu q^X \partial_\nu q^Y$$

There are no more couplings of the hypers to the rest of the fields: the structure of the theory does not allow it. Only when we gauge global symmetries some couplings will appear because we will have to use vectors from the vector multiplets. If the symmetries being gauged act on the hypers, there will be couplings between these two kinds of multiplets.

The ungauged theory

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We have all the elements we need to describe the simplest $N=1$, $d=5$ supergravity theories in which no global symmetry has been gauged. We are interested in the <u>bosonic action only</u> ($\phi^f=0$) since it is enough to get the equations of motion of the bosonic fields ($\phi^f=0$ is a "consistent truncation"). We want the <u>supersymmetry transformation rules of the fermions for vanishing fermions</u> ($\phi^f=0$) because they will become the KSEs. Finally, we want the <u>complete supersymmetry transformation rules of the bosons</u> (setting $\phi^f=0$ would trivialize them) because we want to use them to find the KSIs.

The action contains the elements mentioned before plus an Einstein-Hilbert term and a Chern-Simons term for the vector fields which is very characteristic of these theories:

$$S = \int d^5x\sqrt{|g|}\,\Big\{ R + \tfrac{1}{2} g_{xy}\,\partial_\mu\phi^x\partial^\mu\phi^y + \tfrac{1}{2} g_{XY}\,\partial_\mu q^X\partial^\mu q^Y$$
$$-\tfrac{1}{4} a_{IJ}\, F^I\cdot F^J + \underbrace{\frac{1}{12\sqrt{3}}}_{\text{supersymmetry}} C_{IJK}\,\frac{\varepsilon^{\mu\nu\rho\sigma\delta}}{\sqrt{|g|}} F^I{}_{\mu\nu} F^J{}_{\rho\sigma} A^K{}_\delta \Big\}$$

The supersymmetry transformation rules of the fermions for vanishing fermions are

$$\delta_\varepsilon\psi^i_\mu = \overset{\text{Lorentz + SU(2) covariant derivative}}{D_\mu}\varepsilon^i - \frac{1}{8\sqrt{3}}\overbrace{h_I F^{I\,\alpha\beta}}^{\text{graviphoton}}(\gamma_{\mu\alpha\beta} - 4 g_{\mu\alpha}\gamma_\beta)\,\varepsilon^i\,; \quad \rightarrow \text{Differential}$$

algebraic
$$\begin{cases}\delta_\varepsilon\lambda^{ix} = \frac{1}{4}\left(\not\partial\phi^x - \frac{1}{2} h^x{}_I \not F^I\right)\varepsilon^i\,; \quad \rightarrow \text{matter vector fields} \\ \delta_\varepsilon\zeta^A = \frac{1}{2} f_X{}^{iA}\not\partial q^X\,\varepsilon_i\,; \qquad \left(\text{We could use } f^X{}_{iA}\,\zeta^A\right) \quad \text{R-symmetry}\end{cases}$$

→ constraints.

All the symmetries of the theory act covariantly in both sides of the equations.

In absence of hypers, $D_\mu \rightarrow \nabla_\mu$ (Levi-Civita only)

The supersymmetry transformation rules of the bosons are always algebraic and, furthermore, do not change when we gauge the theory:

$$\delta_\varepsilon e^a{}_\mu = \frac{i}{2}\bar{\varepsilon}_i \gamma^a \psi_\mu{}^i ;$$

$$\delta_\varepsilon A^I{}_\mu = -i\frac{\sqrt{3}}{2} h^I \bar{\varepsilon}_i \psi^i_\mu + \frac{i}{2} h^I{}_x \bar{\varepsilon}_i \gamma_\mu \lambda^{x\,i}$$

$$\delta_\varepsilon \phi^x = \frac{i}{2}\bar{\varepsilon}_i \lambda^{x\,i} ;$$

$$\delta_\varepsilon q^X = -i\, f_{iA}{}^X \bar{\varepsilon}^i \zeta^A$$

the combination that we could have used before ξ^X_i

Examples of ungauged N=1, d=5 supergravities

i) Pure supergravity $n_V = n_H = 0$ & $C_{000} = 1$ (simplicity)

$h^0 = 1$; $a_{00} = 1$;

Suppressing the index "0"

$$S = \int d^5x \sqrt{|g|}\left\{ R - \frac{1}{4}F^2 + \frac{1}{12\sqrt{3}}\frac{\varepsilon^{\mu\nu\rho\sigma\delta}}{\sqrt{g}} F_{\mu\nu} F_{\rho\sigma} A_\delta \right\}$$

→ Einstein-Maxwell + a particular Chern-Simons term.

$$\begin{cases} \delta_\varepsilon e^a{}_\mu = \frac{i}{2}\bar{\varepsilon}_i \gamma^a \psi_\mu{}^i ; \\ \delta_\varepsilon \psi_\mu{}^i = \nabla_\mu \varepsilon^i - \frac{1}{8\sqrt{3}} F^{\alpha\beta}\left(\gamma_{\mu\alpha\beta} - 4 g_{\mu\alpha}\gamma_\beta\right)\varepsilon^i ; \\ \delta_\varepsilon A_\mu = -i\frac{\sqrt{3}}{2}\bar{\varepsilon}_i \psi^i_\mu ; \end{cases}$$

ii) ST[2, n_V] models (strictly speaking, it is the N=2, d=4 models that one obtains from them by dimensional reduction that are called this way).

They have $n_H = 0$ and are defined by $C_{0xy} = \frac{1}{3!}\eta_{xy}$;

$(\eta_{xy}) = \mathrm{diag}(+ - - \cdots -)$

$$C_{IJK} h^I h^J h^K = \frac{1}{2} h^0 \eta_{xy} h^x h^y = 1 ;$$

We can solve this constraint in this way:

>> Only $SO(1,1)\times SO(1,n_V-1)$ is linearly realized, <<
though!

domingo, 4 de diciembre de 2016 18:18

The $\eta_{xy}h^x h^y$ part is invariant under $SO(1,n_{V-1})$ transformations, but, together with the conformal factor $\frac{1}{2}h^0$, the symmetry is the conformal group of the metric η_{rs} : $SO(2,n_{V-1})$. Since the dimension of the hypersurface is n_V, this model describes the coset space

$$SO(2,n_{V-1})\big/SO(1,n_{V-1}) \rightarrow \frac{(n_V+1)n_V}{2}-\frac{n_V(n_V-1)}{2}=n_V \qquad \boxed{AdS_{n_V}}$$

$$\left(\begin{aligned}&(X^0)^2+\eta_{xy}X^xX^y=1;\\&(X^0)^2+(X^1)^2=\tfrac{1}{2}h^0(h^1)^2 \quad ; \quad X^2=\left(\tfrac{h^0}{2}\right)^{1/2}h^2\dots\\&\Rightarrow \tfrac{1}{2}h^0\eta_{xy}h^xh^y=1\end{aligned}\right)$$

We can use all the parametrizations of AdS_{n_V} to find $h^I(\phi)$.
Using the definition $h_I\equiv C_{IJK}h^Jh^K$:

$$a_{IJ}\equiv -2C_{IJK}h^K+3h_Ih_J.$$

$$\Rightarrow h_0=\frac{1}{3!}\eta_{xy}h^xh^y=(3h^0)^{-1} \quad ; \quad h_x=\frac{1}{3}\eta_{xy}h^0h^y;$$

$$h_Ih^I=h_0h^0+h_xh^x=\frac{1}{3}+\frac{1}{3}h^0\eta_{xy}h^xh^y=\frac{1}{3}+\frac{2}{3}=1;$$

$$a_{00}=3h_0h_0=\frac{1}{3(h^0)^2}$$

$$a_{0x}=-2\frac{1}{3!}\eta_{xy}h^y+3h_0h_x=-\frac{1}{3}\eta_{xy}h^y+\frac{1}{h^0}\frac{1}{3}\eta_{xy}h^0h^y=0$$

$$a_{xy}=-2\frac{1}{3!}\eta_{xy}h^0+3\left(\frac{1}{3^2}\eta_{xz}h^z\eta_{yw}h^w\right)(h^0)^2=$$

$$=-\frac{1}{3}\left[\eta_{xy}h^0+(h^0)^2\eta_{xz}\eta_{yw}h^zh^w\right];$$

The σ-model metric is just that of $SO(1,1)\times SO(1,n_V-1)$

The Chern-Simons term is

$$\left(\frac{\varepsilon}{12\sqrt{3}\sqrt{g}}\right)C_{IJK}F^IF^JA^K=\frac{1}{3!}\Big(2\underbrace{F^0\eta_{rs}F^rA^s}_{2\partial(A^0\eta_{rs}F^rA^s)\,+\,A^0\eta_{rs}F^rF^s}+\eta_{rs}F^rF^sA^0\Big)$$

$$\Rightarrow \boxed{\frac{1}{24\sqrt{3}}\frac{\varepsilon^{\mu\nu\rho\sigma\delta}}{\sqrt{|g|}}\eta_{rs}F^rF^sA^0}$$

domingo, 4 de diciembre de 2016 19:20

STU model $n_H = 0$; $n_V = 2$; $C_{IJK} = \frac{1}{3!} |\varepsilon_{IJK}|$ $\boxed{h^0 h^1 h^2 = 1}$

The symmetry of this model is $SO(1,1)^3$ with each factor acting on a pair $h^0 h^1$, $h^0 h^2$, $h^1 h^2$ by rescalings with opposite weights.

$$h_I = \frac{1}{3 h^I} \quad ; \quad a_{IJ} = 3\, \delta_{I(J)}\, h_{(J)}^2 \, ;$$

A convenient parametrization with reduced metric $g_{xy} = \delta_{xy}$ is

$$h^0 = e^{\sqrt{6}\phi^1} \; ; \quad h^1 = e^{-\sqrt{\frac{3}{2}}\phi^1 - \frac{3}{\sqrt{2}}\phi^2} \; ; \quad h^2 = e^{-\sqrt{\frac{3}{2}}\phi^1 + \frac{3}{\sqrt{2}}\phi^2} \; ;$$

$$S = \int d^5x \sqrt{|g|} \Big\{ R + \frac{1}{2} \partial_\mu \phi^x \partial^\mu \phi^x - \frac{1}{12} e^{-2\sqrt{6}\phi^1} (F^0)^2 - \frac{1}{12} e^{\sqrt{6}\phi^1 + 3\sqrt{2}\phi^2} (F^1)^2$$
$$- \frac{1}{12} e^{\sqrt{6}\phi^1 - 3\sqrt{2}\phi^2} (F^2)^2 + \frac{1}{12\sqrt{3}} \frac{\varepsilon}{\sqrt{g}} F^0 F^1 A^2 \Big\}$$

Observe that the ST[2,2] model, which parametrizes $SO(1,1) \times SO(1,1)$ related to the STU model:

$$\frac{1}{2} h^0 \left[(h^1)^2 - (h^2)^2 \right] = h^0 \, \overbrace{\frac{1}{\sqrt{2}} (h^1 + h^2)}^{h^{1\prime}} \, \overbrace{\frac{1}{\sqrt{2}} (h^1 - h^2)}^{h^{2\prime}}$$

They are the same model in a different basis.

Global symmetries

miércoles, 30 de noviembre de 2016 18:43

Is it possible to construct other theories of N=1, d=5 supergravity? There are only two possibilities which can be understood as deformations of a given ungauged theory (⇒ same # of d.o.f.)

1.- Massive deformations: we can try to introduce a Stückelberg-type coupling between 2-forms and 1 forms (in general (k+1)- to k-forms)

$$F = dA \longrightarrow dA + mB \qquad A=0 \Rightarrow \text{massive 2-forms consistently with susy.}$$

$$H = dB$$

Since we are not including 2-forms in the theory, we are not going to explore this possibility.

2.- Gaugings of global symmetries of the ungauged theory: we modify the couplings so that the theory becomes invariant under the local version of a global symmetry. Since we need connection 1-forms, this can only be done when the theory already has vector multiplets that can play that rôle.

Which are the global symmetries of an N=1, d5 supergravity?

i) SU(2) R-symmetry acting only on the SM spinors.

* However, in presence of hypers they are coupled to the spinors via the pullback of the SU(2) connection $A^r_x\,\partial_\mu q^x$ that occurs in the SU(2)-covariant derivative. Thus, we must do something to the hypers to compensate for this transformation. That "something" must leave invariant the QK metric g_{xy} and so it must be an isometry. But it must leave the whole QK structure invariant as well. In particular, it must preserve the hyper-Kähler structure J^r ⇒ it must be a tri-holomorphic isometry.

Thus, in presence of hypers, R-symmetry is no longer an independent symmetry: it is part of the group of tri-holomorphic isometries of the QK manifold. We will discuss this point later.

ii) Isometries of the scalar manifolds that leave invariant the whole theory. This requires the preservation of all the geometric structures required by supersymmetry (Real Special Geometry and QK Geometry) and also the couplings to the rest of the fields. In the QK case we have already said something but we are going to discuss the question in more detail now.

Global symmetries associated to triholomorphic isometries of the QK manifold

The scheme of how they work is the following (no mathematical rigour)

① We transform the hypers $\delta q^x = \alpha^A k_A{}^x(q)$ $\qquad \alpha^A$ constants

$$[k_A, k_B] = -f_{AB}{}^C k_C ;$$

② If the $k_A{}^x(q)$ are Killing vectors of g_{xy} ; $\pounds_{k_A} g_{xy} = 0$ and the kinetic term is left invariant.

③ If $\pounds_{k_A} J^r = 0$; $r=1,2,3$, the QK structure is preserved ???

* This statement is not SU(2)-covariant. It must be replaced by $\mathbb{L}_{k_A} K^r = 0$ where $\mathbb{L}_{k_A}$ is the SO(2)-covariant Lie derivative $\mathbb{L}_{k_A} K^r{}_{xy} = \pounds_{k_A} K^r{}_{xy} - \varepsilon^{rst} W_A^s K^t{}_{xy}$

W_A^s is the compensator

$W_A{}^s \equiv k_A{}^x A^s{}_x - P_A{}^s$ $\qquad \leftarrow$ definition of the triholomorphic momentum map.

$W_A{}^s$ undergoes a compensating SU(2) transformation when we perform the scalar transformation corresponding to the isometry generated by k_A. But it does not transform covariantly; $P_A{}^s$ does (the inhomogeneous term is absorbed by $k_A{}^x A_x$).

Remember: this is similar to the Lorentz case (Kosmann derivative) where $W_A{}^{ab} = k_A{}^\mu \omega_\mu{}^{ab} - \nabla^{[a} k^{b]}{}_A \rightarrow$ "momentum map"

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$$\left.\begin{array}{l} \pounds_{k_A} K^r{}_{XY} = 0; \\ \oplus \\ D_Z K^r{}_{XY} = 0; \end{array}\right\} \quad -2\left(\nabla_{[X|} k_A{}^Z\right) K^r{}_{Z|Y]} + \varepsilon^{rst} P_A{}^s K^t{}_{XY} = 0;$$

Contracting with K^{rXY} $\rightarrow$ $K^{rXY}\nabla_X k_{AY} = -2m_H P_A{}^r$;

Acting on both sides with D_Z $\rightarrow$ $K^{rXY}\nabla_Z\nabla_X k_{AY} = -2m_H D_Z P_A{}^r$;

Using the integrability condition for Killing vectors $\rightarrow$ $K^{rXY}\left(-R^W{}_{ZXY}k_{AW}\right) = -2m_H D_Z P_A{}^r$

$$D_Z P_A{}^r = \frac{1}{2m_H} k_A{}^W \underbrace{R_{WZXY}K^{rXY}}_{-2m_H F^r{}_{WZ} = -2m_H \kappa K^r{}_{WZ}}$$

$$\Rightarrow \boxed{D_Z P_A{}^r = \kappa K^r{}_{ZW} k_A{}^W}$$

This is the usual definition of the triholomorphic momentum.

However, remember the previous equation

$$\boxed{D_Z P_A{}^r = \frac{1}{2m_H} k_A{}^W R_{WZXY} K^{rXY}}$$

because we can show how this structure occurs in more general contexts where there are no complex structures involved

$\rightarrow$ Generalized concept of momentum map.

Generalized momentum maps

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Given a Riemannian (or pseudo Riemannian, the signature plays no role here) manifold M_d admitting an isometry group generated by the Killing vectors $k_A{}^a$: $\nabla_a k_{Ab} = \nabla_{[a|} k_{A|b]}$ we define the generalized momentum map associated to the Levi-Civita connection $\omega_\mu{}^a{}_b$ and the Killing vectors $k_A{}^\mu$ by

$$\boxed{P_A{}^a{}_b \equiv \nabla^a k_{A\,b} = -\nabla_b k_A{}^a} \quad \in so(d)$$

Properties:

i) It associates an infinitesimal isometry in the base space $k_A{}^\mu$ to an infinitesimal rotation in the tangent space.

ii) The Kosmann Lie derivative on an object T transforming in the representation r of the rotation group is given by

$$\mathbb{L}_{k_A} = \pounds_{k_A} - \frac{1}{4}\left(k_A{}^\mu \omega_\mu{}^{ab} - 2\, P_A{}^{ab}\right) \Gamma_r(M_{ab})$$

iii) If M has special holonomy generated by some $(M_i)^a{}_b \in so(d)$ we can reduce

$$P_A{}^a{}_b = P_A{}^i\, \Gamma_r(M_i)^a{}_b\,; \qquad [M_i, M_j] = -f_{ij}{}^k M_k$$

$$\omega_\mu{}^a{}_b = -\vartheta_\mu{}^i\, \Gamma_r(M_i)^a{}_b\,;$$

$$R_{\mu\nu\,b}{}^a = -R_{\mu\nu}{}^i(\vartheta)\, \Gamma_r(M_i)^a{}_b\,;$$

and using the integrability condition for Killing vectors

$$\nabla_a \nabla_b k_A{}^c = k_A{}^d R_{d\,ab}{}^c \longrightarrow \nabla_a P_A{}^c{}_b = k_A{}^d R_{d\,ab}{}^c$$

and reducing it to the holonomy group becomes

$$\boxed{D_a P_A{}^i = R_{ab}{}^i(\vartheta)\, k_A{}^b} \qquad \left(D_a P_A{}^i = \partial_a P_A{}^i + f_{jk}{}^i \vartheta_a{}^j P_A{}^j\right)$$

which is the general form of the equation that defines the momentum map. $\Rightarrow$ Defined up to covariantly-constant pieces: FI terms

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iv) If M_d is a Riemannian symmetric space G/H ; $u' = g u h^{-1}$ the coset rep.

$g \equiv 1 + \alpha^A T_A$;

$h \equiv 1 - \alpha^A W_A{}^i M_i$; $\{T_A\} = \{P_a, M_i\}$ $A = a, i$.

$x^{\mu\prime} \equiv x^\mu + \alpha^A k_A{}^\mu$; $-u^{-1} du = e^a P_a + \vartheta^i M_i$;

It can be easily shown that

$$k_A{}^a = -\Gamma_{Adj}(u^{-1})^a{}_A \quad ;$$

$$W_A{}^i = -k_A{}^\mu \vartheta^i{}_\mu - P_A{}^i \; ; \text{ with } P_A{}^i = \Gamma_{Adj}(u^{-1})^i{}_A \; ;$$

P_A transforms as

$$\boxed{P_B{}' \, \Gamma_{Adj}(g)^B{}_A = \Gamma_{Adj}(h)^i{}_j \, P_A{}^j \; ;}$$

and satisfies the equation that defines the momentum map.

v) If M is a Kähler-Hodge manifold $\mathcal{R}_{\mu\nu}$ is related to $K_{\mu\nu}$

$\mathcal{R}_{\mu\nu} \to$ Ricci 2-form $\mathcal{R}_{\mu\nu} = \frac{1}{2} R_{\mu\nu\rho\sigma} \mathcal{J}^{\rho\sigma}$; where $\mathcal{J}^\rho{}_\nu$ is the complex structure

$K_{\mu\nu} \to$ Kähler 2-form $K_{\mu\nu} = g_{\mu\rho} \mathcal{J}^\rho{}_\nu$;

The part of the holonomy group which is relevant for us is the $U(1)$ of $\mathcal{R}_{\mu\nu}$. In general $P_A{}^a{}_b \notin u(1)$ and we are going to be interested only on those isometries for which $P_A{}^a{}_b \in u(1)$. These are the holomorphic isometries. We are going to show that they satisfy the equation that defines the momentum map for the $u(1)$ subgroup.

Holomorphic isometries preserve the Kähler 2-form (covariantly constant by def)

$$0 = \mathcal{L}_{k_A} K_{ab} = \cancel{k_A{}^c \nabla_c K_{ab}} + 2\nabla_{[a|} k_A{}^c K_{c|b]} =$$

$$= 2\nabla_{[a}(k_A{}^c K_{c|b]}) + \cancel{2 k_A{}^c \nabla_{[a} K_{b]c}}$$

$$\Rightarrow \; k_A{}^c K_{cb} = \partial_b f_A \quad \text{locally}$$

$$\partial_a f_A = -K_{ab} k_A{}^b \sim -\mathcal{R}_{ab} k_A{}^b = -\tfrac{1}{2} R_{abcd} \mathcal{J}^{cd} k_A{}^b$$

$$\Rightarrow \; f_A \sim P_A \quad \text{for } u(1) \text{ only}$$

relation to be investigated.

To be improved

vi) For QK manifolds we have already seen how it works.
KH & QK are essential. just K or hK does not work!

Gauging isometries

lunes, 5 de diciembre de 2016 0:21

The procedure to gauge a global symmetry of a theory which acts on the fields according to

i) Scalars $\delta_\alpha \phi^x = \alpha^A k_A{}^x(\phi)$; $[k_A, k_B] = -f_{AB}{}^C k_C$;
$k_A{}^x$ Killing vectors of the scalar manifold respecting all the structures demanded by supersymmetry.

ii) Vectors $\delta_\alpha A^I{}_\mu = \alpha^A (T_A)^I{}_J A^J{}_\mu$ where the $T_A{}^I{}_J$ are in the adjoint representation of the group that we want to gauge. Since not all the vectors transform, it is customary to write $A \to I$ and $(T_A)^K{}_J \longrightarrow f_{IJ}{}^K$: $\delta_\alpha \phi^x = \alpha^I k_I{}^x(\phi)$; $[k_I, k_J] = -f_{IJ}{}^K k_K$;
$k_A \longrightarrow k_I$ (same meaning) $\delta_\alpha A^I{}_\mu = \alpha^J f_{JK}{}^I A^J$;

iii) Spinors: we must distinguish between isometries of the RS and QK manifolds because the spinors only feel the transformations associated to the latter:

$$\begin{cases} \delta_\alpha \phi^x = \alpha^I k_I^x(\phi); & [k_I, k_J] = -f_{IJ}{}^K k_K ; \\ \delta_\alpha q^X = \alpha^I K_I{}^X(q); & [K_I, K_J] = -f_{IJ}{}^K K_K ; \\ \delta_\alpha A^I{}_\mu = \alpha^J f_{JK}{}^I A^K{}_\mu ; & \end{cases}$$

same algebras because $A^x{}_\mu$, ϕ^x are in the same multiplet and $A^I{}_\mu$, $k^I(\phi)$ must transform in the same way.

If the $K_I{}^X \neq 0$, then $\delta_\alpha \psi^i = \alpha^I \underbrace{\left(K_I{}^X A_X{}^r + P_I{}^r\right)}_{\text{compensator } W_A{}^r} \frac{i}{2}(\sigma^r)^i{}_j \psi^j$

$$D_\mu \psi^i = \partial_\mu \psi^i + A^r{}_X \partial_\mu q^X \, i(\sigma^r)_j{}^i \psi^j ;$$
$$\delta_\alpha \psi^i = \underbrace{\alpha^I W_I{}^r i(\sigma^r)_j{}^i \psi^j}_{\text{induced } SU(2) \text{ transformations.}} ;$$

only covariant under global transformations: α^I constant

goes as follows:

a) Replace derivatives by gauge-covariant derivatives

$$\partial_\mu \phi^x \longrightarrow \mathfrak{D}_\mu \phi^x = \partial_\mu \phi^x + g A^I{}_\mu k_I{}^x(\phi);$$
$$\partial_\mu q^X \longrightarrow \mathfrak{D}_\mu q^X = \partial_\mu q^X + g A^I{}_\mu K_I{}^X(q);$$
$$D_\mu \psi^i \longrightarrow \mathfrak{D}_\mu \psi^i = \underset{\text{Lorentz}}{D_\mu} \psi^i + g A^I{}_\mu P_I{}^r i(\sigma^r)_j{}^i \psi^j ;$$

$$k_I{}^X \mathcal{A}_X{}^r - W_I{}^r$$

Observe that $\mathcal{D}_\mu \psi^i = \nabla_\mu \psi^i + \left(\mathcal{A}^r{}_X \partial_\mu q^X + g A^I_\mu P_I{}^r\right) i(\sigma^r)_j{}^i \psi^j$

$$= \nabla_\mu \psi^i + \left[\mathcal{A}^r{}_X \left(\partial_\mu q^X + g A^I_\mu k_I^X\right) - g W_I{}^r\right] i(\sigma^r)_j{}^i \psi^j$$

$$= \nabla_\mu \psi^i + \left(\mathcal{A}^r{}_X \mathcal{D}_\mu q^X - g W_I{}^r\right) (\sigma^r)_j{}^i \psi^j$$

$\Rightarrow$ To obtain the gauge-covariant derivative it is not enough to replace the pullback of the SO(2) connection by the "covariant pullback" if $W_I{}^r \neq 0$ (which is usually the case)

Observe that on the RS manifold, the preservation of the RS structure by the isometries amounts $T_A{}^L{}_{(I} C_{JK)L} = 0 \Rightarrow f_{M(I}{}^L C_{JK)L} = 0$

and $\mathbb{L}_A h^I = \pounds_{k_A} h^I - (T_A)^I{}_J h^J = k_A{}^x \partial_x h^I - (T_A)^I{}_J h^J = 0$

$\Rightarrow k_A{}^x = -\sqrt{3}\, T_A{}^I{}_J h_I^x h^J$; $\boxed{k_I{}^x = -\sqrt{3}\, f_{IJ}{}^K h^J h_K{}^x}$

$\Rightarrow$ They vanish for Abelian isometry groups. They cannot be gauged alone!

b) Replace the vector field strengths by covariant field strengths

$$F^I = dA^I \longrightarrow dA^I + g f_{JK}{}^I A^J \wedge A^K$$

c) The Chern-Simons term has to be modified to make it gauge-invariant (up to total derivatives)

$$\frac{1}{12\sqrt{3}} \frac{\varepsilon}{\sqrt{|g|}} C_{IJK} F^I F^J A^K \longrightarrow \frac{1}{12\sqrt{3}} \frac{\varepsilon}{\sqrt{|g|}} C_{IJK} \Big\{ F^I F^J A^K - \frac{1}{2} g f_{LM}{}^I F^J A^K A^L A^M$$

$$+ \frac{1}{10} g^2 f_{LM}{}^I f_{NP}{}^J A^K A^L A^M A^N A^P \Big\}$$

d) The closure of the local supersymmetry algebra requires the modification of the supersymmetry transformation rules of the fermions. Apart from the replacement of derivatives and field strengths by their covariant counterparts, new terms named "fermion shifts" have to be added. They are generically related to the triholomorphic momentum maps:

$$\delta_\varepsilon \psi_\mu{}^i = \mathcal{D}_\mu \varepsilon^i - \frac{1}{8\sqrt{3}} h_I F^{I\alpha\beta}(\gamma_{\mu\alpha\beta} - 4 g_{\mu\alpha}\gamma_\beta)\varepsilon^i + \frac{1}{2\sqrt{3}} g h^I P_{Ij}{}^i \gamma_\mu \varepsilon^j ;$$

$$\delta_\varepsilon \lambda^{ix} = \frac{1}{2}\left(\not{\partial}\phi^x - \frac{1}{2} h^x{}_I \not{F}^I\right)\varepsilon^i + g h^{Ix} P_{Ij}{}^i \varepsilon^j ;$$

Different basis! $\rightarrow$ $\delta_\varepsilon \zeta^A = \frac{1}{2} f_X{}^{iA}\left(\not{\partial} q^X + \sqrt{3} g h^I K_I{}^X\right)\varepsilon_i ;$

lunes, 5 de diciembre de 2016 17:48

These new terms, in their turn, generate new terms in the variation of the action that need to be compensated by the introduction of a scalar potential:

$$V(\phi,q) = -g^2 \Big\{ \left(4h^I h^J - 2g^{xy} h_x{}^I h_y{}^J\right) P_I{}^r P_J{}^r$$
$$- \frac{3}{2} h^I h^J \left(k_I{}^x k_J{}^y g_{xy} + k_I{}^X k_J{}^Y g_{XY}\right)$$

$$k_I{}^x h^I \sim f_{IJ}{}^K h^J h_K{}^x \cancel{h^I} = 0.$$

$\Rightarrow$ The scalar potential vanishes if $n_H = 0$. ($P_I{}^r = 0$)

However, it is possible to have $P_I{}^r \neq 0$ for $n_H = 0$! (Fayet-Iliopoulos terms)

Fayet-Iliopoulos terms

miércoles, 30 de noviembre de 2016 18:43

If $n_H = 0$ one can still find P_I^r's which are constant and satisfy the basic properties needed. They are called Fayet-Iliopoulos terms and come in two flavors:

U(1) : P_I^r = constant for 1 value of I and one value of r (chosen at will because these transformations only affect the spinors (it is a U(1) subgroup of the SU(2) R-symmetry group). Using it, we gauge that $U(1) \subset SO(2)$ R-symmetry group.

SU(2) P_I^r = 3 possible vectors $\vec{e}_r$ satisfying $\vec{e}_r \times \vec{e}_s = \varepsilon_{rst} \vec{e}^t$ (I → r). In this case SU(2) acts on the coordinates of the RSG because the vectors have to transform in the adjoint of SU(2) in order to be used as gauge fields (connections)

In these cases the scalar potential becomes

U(1): $V = -g^2 \left[4(h^*)^2 - 2 g^{xy} \partial_x h^* \partial_y h^* \right]$ ← not definite

SU(2) $V = -g^2 \left[4\, h^I h^I - 2 g^{xy} \partial_x h^I \partial_y h^I \right]$ (using the simplest choice)

We can use FI terms to gauge U(1) in pure supergravity because it already has one vector (the graviphoton): $V = -4g^2 < 0 \Rightarrow$ AdS-like

This theory is called "minimal gauged d=5 supergravity".

Relations with other SUGRAs via redox

miércoles, 30 de noviembre de 2016 18:45

- $d=11$ supergravity ("M-theory"), compactified on a Calabi-Yau 3-fold with Hodge numbers $h_{(1,1)}, h_{(2,1)}$ and with triple intersection numbers C_{IJK} $I,J,K=0,\dots,h_{(1,1)}-1$, becomes an $N=1, d=5$ supergravity with $h_{(1,1)}-1=n_V$ vector multiplets and $h_{(2,1)}+1=n_H$ hypermultiplets.
- Further reduction to $d=4$ on S^1 gives a theory of $N=2, d=4$ supergravity and a truncation (equivalently, a compactification on $S^1/\mathbb{Z}_2$) gives an $N=1, d=4$ theory.
- Reversing the order of the compactifications, we find that this is equivalent to first reduce $d=11$ supergravity on S^1 to get $N=2A, d=10$ and then compactify on CY_3 to $N=2, d=4$ and then truncate ($\mathbb{Z}_2$) or first truncate $N=2A, d=10$ to $N=1, d=10$ supergravity and then to $N=1, d=4$ on CY_3). In this order you don't see $d=5$.
- Observe that not all $N=2, d=4$ supergravities can be obtained in this way: only the cubic models with prepotential $\mathcal{F} \sim \frac{C_{IJK}\mathcal{X}^I\mathcal{X}^J\mathcal{X}^K}{\mathcal{X}^0}$
- Another way to arrive to $d=5$ from higher dimensions is via $d=6$. This has not been studied in depth because there is only one $N=(0,1), d=6$ supergravity for a given matter content and the reduction to $N=1, d=5$ just gives 1 particular model (for that matter content)

  We have recently discovered that, actually there are 2 6-dimensional matter contents that give rise to exactly the same $N=1, d=5$ model which suggests the existence of an underlying duality between the 2 6-dimensional theories compactified on S^1.

  Furthermore, one of the 6-dimensional theories involved arises in the compactification of $N=2A, d=10$ on K_3 or (dually) on the compactification of $N=1, d=10$ on T^4.
- Thus, a generic $N=1, d=5$ supergravity can always be assigned an

11-dimensional origin (if the CY_3 with those Hodge numbers exist) but only a particular model can be given a 6-dimensional origin and, in some subcases, 2.

- The 6-dimensional theories can also be obtained from F-theory compactification.

How about the gauged theories? Very complicated. In general, the gaugings have to be already present in higher dimensions. They may arise in the string compactifications (self-dual radii, K_3 singularities...) but this cannot be seen at the level of the supergravity theory.

General classifications of supersymmetric solutions

miércoles, 30 de noviembre de 2016 18:29

Finding supersymmetric solutions is an interesting problem:
- First of all, they are solutions of gravity
- Some of them can play the rôle of vacua (+compactification)
- Specially interesting solutions are expected

Unbroken supersymmetry is a very strong condition, as we have stressed, and this opens up the possibility of finding all the supersymmetric solutions or, at least, finding a recipe to construct them all more or less systematically. For some reason this is usually referred to as "classifying" all the supersymmetric solutions of a theory.

In the $N=1$, $d=5$ case it would be desirable to do it for all theories simultaneously, studying the most general theory so we can later apply the result to any particular case. This is a very complicated problem, but it can be solved, giving a set of simple equations and conditions whose solutions correspond to solutions of the complete theory.

Working with the most general theory has some drawbacks because one cannot use the simplifications that appear in certain cases, "refining" the classification because some equations become simpler or trivial. Thus, (also for historical and pedagogical reasons) we are going to start by studying the simplest theories first.

There are essentially 2 methods to carry on this task:

① <u>Spinorial geometry</u>: gives a very refined classification of the solutions with the different amounts of unbroken supersymmetry they have.

② <u>Bilinear method</u>: gives the general form of all the configurations that have some unbroken supersymmetry, irrespectively of how much.

We are going to use the second method.

Tod's program

miércoles, 30 de noviembre de 2016 18:46

In 1981 Tod showed how in pure $N=2$, $d=4$ supergravity one could "classify" all the supersymmetric solutions by using the consistency conditions of the KSEs, (including the integrability conditions).
Nobody tried to repeat this analysis for other supergravity theories until Gauntlett, Gutowski, Hull, Pakis & Reall did it for pure $N=1$, $d=5$ supergravity. This was due, in fact, to the fact that Tod had used the Newman-Penrose formalism which is not well known among the supergravity / superstring community and is only valid for $d=4$. GGHPR used a different idea: transform the KSEs into equations for the spinor bilinears first and then study the consistency of these equations. This method works in any dimension.
With this modification (and another one we will introduce) the whole program consists in the following steps:

1) Assume the bosonic configuration ϕ^b ($\phi^f=0$) admits at least one Killing spinor ϵ, so the KSEs $\partial\epsilon + \phi^b\epsilon = 0$ are satisfied by it. Since these equations are linear we can consider ϵ to be a commuting spinor and operate with it in the standard way.
2) Take all the bilinears that can be constructed with ϵ. $\bar\epsilon\epsilon$, $\bar\epsilon\gamma^a\epsilon$ .. and find which equations they satisfy because ϵ satisfies the KSE: $\partial(\bar\epsilon\epsilon) = \partial\bar\epsilon\,\epsilon + \bar\epsilon\partial\epsilon = -\bar\epsilon\bar\phi^b\epsilon - \bar\epsilon\phi^b\epsilon$ etc.
The resulting equations may not hold all the information contained in the KSEs, but it seems to contain all the information relevant to the problem of finding the ϕ^b.
3) Analyze the consistency of the equations of the bilinears. In this analysis we must take into account that not all the bilinears are independent: they satisfy a non-trivial algebra that can be found by using Fierz identities.

4) Once the analysis has been completed finding all the necessary conditions that the supersymmetric configurations must satisfy we have to show that they are also sufficient to ensure the existence of a KS. In all the theories considered so far this has been the case and, at that point one has identified all the supersymmetric configurations of the theory.

5) In order to identify the supersymmetric solutions among the supersymmetric configurations we must impose the equations of motion (2nd order, KSEs: 1st order). The KSIs, expressed in terms of the spinor bilinears, tell us which ones are independent for supersymmetric configurations and need to be imposed.

(The KSIs are equivalent to certain projections of the integrability conditions of the KSEs, but computing the latter can be very hard.)

That's it!

In the $N=1$, $d=5$ case the spinor bilinears and their algebra are the same for all the theories (only depend on the spinor representation and the Clifford algebra) and we are going to compute them first.

Since the bosonic supersymmetry transformation rules are also common for all the theories (with the same matter content), the KSIs are also common to all the theories, although the e.o.m. themselves will be different.

Spinor bilinears

miércoles, 30 de noviembre de 2016 18:46

The spinor bilinears that one can construct with just one SM spinor ϵ^i were determined before:

$$\begin{cases} f = i\bar\epsilon_i\epsilon^i \,; & \text{a real function} \\ V^a = i\bar\epsilon_i\gamma^a\epsilon^i \,; & \text{a real vector field} \\ \Phi^r{}_{ab} = \bar\epsilon_i\gamma_{ab}\sigma^{r\,i}{}_j\epsilon^j \,; & \text{a triplet of real 2-forms} \end{cases}$$

How are they related? Use the <u>Fierz identities</u> $\hat r = 0, r$

$$(\bar\lambda M\varphi)(\bar\psi N\chi) = -\frac{1}{8}(\bar\lambda M\sigma^{\hat r}N\chi)(\bar\psi\sigma^{\hat r}\varphi) - \frac{1}{8}(\bar\lambda M\gamma^a\sigma^{\hat r}N\chi)(\bar\psi\gamma_a\sigma^{\hat r}\varphi) + \frac{1}{16}(\bar\lambda M\gamma^{ab}\sigma^{\hat r}N\chi)(\bar\psi\gamma_{ab}\sigma^{\hat r}\varphi)\,;$$

$\downarrow$

$(\bar\lambda_{\alpha i}\;{}_{j\beta}M_{\cdot}{}^{i\alpha}\,\varphi^{j\beta})$

$$\Rightarrow \boxed{\begin{aligned} & V^aV_a = f^2\,; \\ & V_aV_b = \eta_{ab}f^2 + \frac{1}{3}\Phi^r{}_a{}^c\Phi^r{}_{bc}\,; \\ & V^a\Phi^r{}_{ab} = 0\,; \\ & V^a(\star\Phi^r)_{abc} = -f\Phi^r{}_{bc}\,; \\ & \Phi^r{}_a{}^c\Phi^s{}_{cb} = -\delta^{rs}(\eta_{ab}f^2 - V_aV_b) - \varepsilon^{rst}f\Phi^t{}_{ab}\,; \\ & \Phi^r{}_{[ab}\Phi^s{}_{cd]} = -\frac{1}{4}f\delta^{rs}\varepsilon_{abcde}V^e\,; \end{aligned}}$$

(For higher N this algebra has not been computed)

In view of this algebra there are two different possibilities depending on the causal character of the vector field V^a (non-spacelike $V^2 = f^2 \geq 0$)

$f \neq 0 \rightarrow V^a$ timelike $\rightarrow$ timelike supersymmetric solutions.

$f = 0 \rightarrow V^a$ null $\rightarrow$ null supersymmetric solutions.

These two cases are studied separately but there are solutions with more than one KS that belong to the two classes. This formalism is not designed to identify them.

KSIs

martes, 6 de diciembre de 2016 0:40

The general form is $\left(\mathcal{E}_b \equiv \frac{\delta S}{\delta \phi^b}\Big|_{\phi^f=0} \right)$

$$\mathcal{E}_b \frac{\delta}{\delta \phi^{f'}} \delta_\varepsilon \phi^b \Big|_{\phi^f=0} = 0 \; ; \quad \underline{1 \text{ relation for each } \phi^f}$$

The boson's supersymmetry rules are, in all $N=1, d=5$ supergravities, given by

$$\left.\begin{aligned}
\delta_\varepsilon e^a{}_\mu &= \tfrac{i}{2} \bar{\varepsilon}_i \gamma^a \psi_\mu^i \; ; & &\to \mathcal{E}_a{}^\mu \\
\delta_\varepsilon A^I{}_\mu &= -i\tfrac{\sqrt{3}}{2} h^I \bar{\varepsilon}_i \psi^i_\mu + \tfrac{i}{2} h^I{}_x \bar{\varepsilon}_i \gamma_\mu \lambda^{xi} & &\to \mathcal{E}_I{}^\mu \\
\delta_\varepsilon \phi^x &= \tfrac{i}{2} \bar{\varepsilon}_i \lambda^{xi} \; ; & &\to \mathcal{E}_x \\
\delta_\varepsilon q^X &= -i f_{iA}{}^X \bar{\varepsilon}^i \zeta^A \; ; & &\to \mathcal{E}_X
\end{aligned}\right\} \text{ different for each theory.}$$

$$\boxed{\begin{aligned}
\psi_\mu^i &: \quad \tfrac{i}{2} \bar{\varepsilon}_i \gamma^a \mathcal{E}_a{}^\mu - \tfrac{i\sqrt{3}}{2} \bar{\varepsilon}_i h^I \mathcal{E}_I{}^\mu = 0 \; ; \\
\lambda^{ix} &: \quad \tfrac{i}{2} \bar{\varepsilon}_i \gamma_\mu h^I{}_x \mathcal{E}_I{}^\mu + \tfrac{i}{2} \bar{\varepsilon}_i \mathcal{E}_x = 0 \; ; \\
\zeta^A &: \quad -i \bar{\varepsilon}^i f_{iA}{}^X \mathcal{E}_X = 0 \; ;
\end{aligned}}$$

To convert these identities into tensor identities, we act with ε^j, $\gamma^a \varepsilon^j$ etc from the right

$$\left(\bar{\varepsilon}^i f_{iA}{}^X \mathcal{E}_X\right) 2i \varepsilon_j = 2i \bar{\varepsilon}^i \varepsilon_j f_{iA}{}^X \mathcal{E}_X = -f \delta^i{}_j f_{iA}{}^X \mathcal{E}_X = -f f_{jA}{}^X \mathcal{E}_X$$

$f \neq 0 \Rightarrow f_{iA}{}^X \mathcal{E}_X = 0 \Rightarrow \mathcal{E}_X = 0$

$f = 0 \Rightarrow$ nothing

$$\left(\bar{\varepsilon}^i f_{iA}{}^X \mathcal{E}_X\right) 2i \gamma^a \varepsilon_j = 2i \bar{\varepsilon}^i \gamma^a \varepsilon_j f_{iA}{}^X \mathcal{E}_X = -\delta^i{}_j V^a f_{iA}{}^X \mathcal{E}_X$$

$f \neq 0 \Rightarrow$ same conclusion

$f = 0 \Rightarrow$ introduce another null vector W_a ; $W^2 = 0$ $W_a V^a = 1$

$\Rightarrow \boxed{\mathcal{E}_X = 0}$ too

Doing the same in all the KSIs one gets

$$\boxed{\begin{aligned} & f\,\mathcal{E}_{\mu\nu} + \tfrac{\sqrt{3}}{2} h^I \mathcal{E}_{I\mu} V_\nu = 0\,; \\ & \mathcal{E}_{\mu\nu} V^\nu + \tfrac{\sqrt{3}}{2} h^I \mathcal{E}_{I\mu} = 0\,; \end{aligned} \quad \text{in the timelike case} \\ \begin{aligned} & \mathcal{E}_x V_\mu - f\, h^I{}_x \mathcal{E}_{I\mu} = 0\,; \\ & f\,\mathcal{E}_x - h_x{}^I \mathcal{E}_{I\mu} V^\mu = 0\,; \end{aligned} \quad \text{in the timelike case} \\ \mathcal{E}_X = 0\,;}$$

Observe that the derivation of these identities assumes the local existence of the vectors $A^I{}_\mu$. In practice we often know the field strengths and we have to impose the Bianchi identities $B^I{}_{\mu\nu} = 0$. In the KSEs only the field strengths occur and their integrability conditions only involve them without assuming $B^I{}_{\mu\nu} = 0$. Thus, if one derives the above identities from the KSE's integrability conditions one finds that the $B^I{}_{\mu\nu}$ occur in them. They always occur in combination with the Einstein equations in the combination

$$\boxed{\mathcal{E}_{\mu\nu} \longrightarrow \mathcal{E}_{\mu\nu} + \tfrac{\sqrt{3}}{2} h_I \star B^I{}_{\mu\nu}}$$

Now, let us see what the above identities look like in the timelike and null cases:

$\boxed{f \neq 0 \text{ (timelike)}}$ defining $e^0{}_\mu = V_\mu / f$

$$\boxed{\begin{aligned} & \mathcal{E}_{ab} + \tfrac{\sqrt{3}}{2} h^I \mathcal{E}_{I(a}\delta^0{}_{b)} = 0\,; \\ & h_I \star B^I{}_{ab} + h^I \mathcal{E}_{I[a}\delta^0{}_{b]} = 0\,; \\ & \mathcal{E}_x - h_x^I \mathcal{E}_I{}^0 = 0\,; \\ & \mathcal{E}_X = 0\,; \end{aligned}}$$

Most of the e.o.m.s are satisfied if $\mathcal{E}_I{}^0 = 0$. $\mathcal{E}_{00}$ requires $\mathcal{E}_I{}^0 = 0$ and the $h^x{}_I B^I{}_{ab} = 0$ have to be imposed directly

$\Rightarrow$ In the timelike case, the only equations that have to be checked explicitly are $\boxed{h^x{}_I \mathcal{B}^I{}_{ab} = 0 \,;\quad \mathcal{E}_I{}^a = 0;}$

$\boxed{f = 0 \text{ (null)}}$ $\quad V_a \equiv l_a \,;\quad W_a \equiv n_a \,;\quad n_a l^a = 1 \,;$

$$\boxed{\begin{aligned} &\mathcal{E}_{\mu\nu} l^\nu + \frac{\sqrt{3}}{2} h_I * \mathcal{B}^I{}_{\mu\nu} l^\nu = 0 \,; \\ &h^I \mathcal{E}_I{}^\mu = 0 \,; \\ &h_x{}^I \mathcal{E}_{I\mu} l^\mu = 0 \,; \\ &\mathcal{E}_\alpha = 0 \,; \\ &\mathcal{E}_x = 0 \,; \end{aligned}}$$

Many components to be checked. At least we do not have to check the scalar equations of motion which are typically very involved.

Ungauged pure SUGRA

miércoles, 30 de noviembre de 2016 18:47

Remember this theory only has the supergravity multiplet $\{e^a{}_\mu, \psi_\mu{}^i, A_\mu\}$ and, therefore only one fermion and only one KSE:

$$\boxed{\nabla_\mu \varepsilon^i - \frac{1}{8\sqrt{3}} F^{\alpha\beta}\left(\gamma_{\mu\alpha\beta} - 4 g_{\mu\alpha}\gamma_\beta\right)\varepsilon^i = 0;}$$

Deriving equations for the bilinears f, V^a, $\Phi^i{}_{ab}$ is straightforward:

$$df - \frac{1}{\sqrt{3}} i_V F = 0;$$

$$\nabla_{(\mu} V_{\nu)} = 0; \quad \leftarrow \text{ Killing} \Rightarrow \text{isom.}$$

$$dV + \frac{2}{\sqrt{3}} f F + \frac{1}{\sqrt{3}} * (F \wedge V) = 0;$$

$$\nabla_\alpha \Phi^i{}_{\beta\gamma} + \frac{1}{\sqrt{3}} F^{\delta\sigma}\Big[g_{\delta[\beta} * \Phi^i{}_{\gamma]\alpha\sigma} - g_{\delta\alpha} * \Phi^i{}_{\beta\gamma\sigma} - \frac{1}{2} g_{\alpha[\beta} * \Phi^i{}_{\gamma]\delta\sigma} \Big] = 0;$$

Some immediate consequences are

$$V^\mu \partial_\mu f = 0; \quad \Rightarrow \text{ invariant under the isometry generated by } V_\mu$$

$$d\Phi^i = 0; \quad \Rightarrow \text{ we'll see.}$$

To proceed, the timelike and null cases are considered separately

Timelike

Choose an adapted coordinate $V^\mu \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial t} \Rightarrow V_\mu dx^\mu = f^2(dt + \omega)$

($\omega \neq 0$ because $V \wedge dV \neq 0$ in general)

$$\Rightarrow ds^2 = f^2(dt+\omega)^2 - f^{-1} h_{\underline{m}\underline{n}}\, dx^m dx^n;$$

($d=5$ conformastationary form. $h_{mn} \to$ "base space metric", Euclidean. f, ω, h_{mn} time-independent)

$$\nabla^2 \underset{\substack{\downarrow \\ \text{time-indep}}}{\phi} = \frac{1}{\sqrt{|g|}} \partial_{\underline{m}}\left(\sqrt{|g|}\, g^{\underline{m}\underline{n}} \partial_{\underline{n}} \phi\right) = \frac{1}{f^{-1}\sqrt{|h|}} \partial_{\underline{m}}\left(\sqrt{|h|}\, \cancel{f^{-1}} \cancel{f} h^{\underline{m}\underline{n}} \partial_{\underline{n}} \phi\right) = f \nabla^2_{(4)} \phi$$

Using these coordinates, the bilinear identities $\Phi\Phi$ reduce to an identity on the base space:

$$\boxed{\Phi^{r}{}_{m}{}^{p}\,\Phi^{s}{}_{p}{}^{n} = -\delta^{rs}\,\delta_{m}{}^{n} + \varepsilon^{rst}\,\Phi^{t}{}_{m}{}^{p}} \qquad \boxed{\star_4 \Phi^{r} = -\Phi^{r}}$$

→ the algebra of hyper-complex structures! Observe that this result only depends on the Clifford algebra and on the choice of coordinates and it will hold true in the timelike cases of all $N=1, d=5$ supergravities.

The following property is specific of this case:

$$\boxed{\nabla_{(4)\,m}\,\Phi^{r}{}_{np} = 0}$$

⇒ the base space must be a 4-dimensional hyper-Kähler manifold!

The vector field strength contains two pieces: "electric" and "magnetic."

The electric part can be determined from the KSEs and is

$$-\sqrt{3}\, d\left[f(dt+\omega)\right]$$

and the magnetic is denoted by $-\sqrt{3}\,\Theta$ and it is constrained to be self-dual on the base space (⇒ instanton?)

$$\Rightarrow \quad \boxed{F = -\sqrt{3}\left\{ d\left[f(dt+\omega)\right] + \Theta \right\} \;;\; \star_4\Theta = +\Theta \;;}$$

The last condition involves ω

$$\boxed{(d\omega)^{+} = -\tfrac{3}{2} f^{-1}\,\Theta \;;}$$

These are the necessary conditions for unbroken timelike supersymmetry.

Plugging them into the KSEs we see that

$$\boxed{\varepsilon^{i} = f^{1/2}\,\varepsilon_0^{\,i} \;; \qquad \tfrac{1}{2}(1+\gamma^0)\,\varepsilon^{i}_0 = 0 \;;}$$

solves them. Thus, although we only assumed the existence of one Killing spinor, we see that there are at least 4 in every timelike supersymmetric configuration:

$$\left\{\tfrac{1}{2}(1-\gamma^0)\right\}^2 = \tfrac{1}{4}\left(1+(\gamma^0)^2 - 2\gamma^0\right) = \tfrac{1}{4}\,2(1-\gamma^0) \;\Rightarrow \text{projector.}$$

$\mathrm{Tr}\,\gamma^0 = 0$; $(\gamma^0)^2 = +1$ ⇒ $\dim \mathrm{Ker}\,(1-\gamma^0) = 2\ (\times 2 \text{ complex}) = 4$.

Now we want to determine which of these supersymmetric configurations are solutions. Imposing the Maxwell and Bianchi identities we arrive at these two equations

$$\boxed{\begin{aligned} \nabla^2_{(4)} f^{-1} - \tfrac{1}{4} \, ④ \cdot ④ &= 0 \, ; \\ d \, ④ &= 0 \, ; \end{aligned}}$$ (⇒ Abelian instanton!)

Problem solved! Many interesting features and rich structure. Solving these equations will require making ansatzs, though. The general solution is not known.

<u>Null</u> $l_\mu \equiv V_\mu$; $l^\mu \partial_\mu \equiv \partial_{\underline{v}}$; $l_\mu dx^\mu = f(du + \not{x})$ ← different f !

$f \neq 0$ in this case because $dl = -\frac{1}{\sqrt{3}} * (F \wedge l) \neq 0$

$\not{x} = 0$ in this case because $l \wedge dl = 0$ $(l^2 = 0)$

$$\boxed{ds^2 = 2f \, du \, (dv + H du + \omega) - f^{-2} \gamma_{\underline{rs}} \, dx^r dx^s \, ;}$$

$f, H, \omega, \gamma_{\underline{rs}}$ v-independent. (May depend on u and x^r)

In this case, the bilinear identities $\nabla \Phi$ lead to the conclusion

$$\Phi^r = du \wedge v^r \quad ; \quad v^r{}_{\underline{s}} \, v^s{}_{\underline{t}} = \gamma_{\underline{st}} \quad \text{(Dreibeins)}$$

and the equation $d\Phi^r = 0$ leads to $\underset{(3)}{d} v^r = 0 \Rightarrow v^r = dx^r$

$\gamma_{\underline{rs}} = \delta_{\underline{rs}}$

Finally

$$\boxed{F = \left[\tfrac{1}{\sqrt{3}} f^2 *_{(3)} (d_{(3)} \omega) \right] \wedge du + \sqrt{3} *_{(3)} d_{(3)} f^{-1}} \, ;$$

$f, H, \omega_{\underline{r}} \, dx^r$ unconstrained beyond the fact that they are v-indep.

These are the necessary conditions.

They are also sufficient: the KSEs are solved for all these backgrounds for any u-independent ε^i satisfying $\gamma^+ \varepsilon^i = 0 \Rightarrow \frac{1}{2}$ BPS.

Now, let us consider the equations of motion.

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The analysis of the equations of motion shows that they are solved for $\nabla^2_{(3)} K = 0$; $\nabla^2_{(3)} L = 0$, $\nabla^2_{(3)} N = 0$;

$$f^{-1} = K \ ; \qquad M = L/K \ ; \qquad H = -\tfrac{1}{2} LM + N$$

$$\ast_{(3)} \hat{d}\omega = \sqrt{3}\left(L\, dK - K\, dL\right) - 3K^2 \dot{\alpha} \ ; \qquad d\alpha \equiv \ast_{(3)} dK$$

$$\dot{K}\, \partial_z K = 0 \ ;$$

Ungauged SUGRA coupled to vectors and/or hypers

miércoles, 30 de noviembre de 2016 18:48

Let us consider now the coupling to vector multiplets and hypermultiplets.
The KSEs are now

$$\mathcal{D}_\mu \varepsilon^i - \frac{1}{8\sqrt{3}} h_I F^{I\alpha\beta}(\gamma_{\mu\alpha\beta} - 4 g_{\mu\alpha}\gamma_\beta)\varepsilon^i = 0;$$

$$\left(\partial\!\!\!/ \phi^x - \frac{1}{2} h^x{}_I \not{F}^I\right)\varepsilon^i = 0;$$

$$f_X{}^{iA}\, \partial\!\!\!/ q^X \varepsilon_i = 0;$$

The equations derived for the spinor bilinears in the pure supergravity case have the same form with the replacement $\nabla_\mu \to \mathcal{D}_\mu$ (SO(2)-covariant) and $F \to h_I F^I$ (new graviphoton).

$$\boxed{\begin{aligned}
& df - \frac{1}{\sqrt{3}} i_V h_I F^I = 0; \\
& \nabla_{(\mu} V_{\nu)} = 0; \qquad \leftarrow \text{ may} \Rightarrow \text{isom.} \\
& dV + \frac{2}{\sqrt{3}} f h_I F^I + \frac{1}{\sqrt{3}} \star\left(h_I F^I \wedge V\right) = 0; \\
& \mathcal{D}_\alpha \Phi^r{}_{\beta\gamma} + \frac{1}{\sqrt{3}} h_I F^{I\delta\sigma}\Big[g_{\delta[\beta} \star \Phi^r{}_{\gamma]\alpha\sigma} - g_{\delta\alpha} \star \Phi^r{}_{\beta\gamma\sigma} \\
& \qquad - \frac{1}{2} g_{\alpha[\beta} \star \Phi^r{}_{\gamma]\delta\sigma}\Big] = 0;
\end{aligned}}$$

$$\Rightarrow \quad V^\mu \partial_\mu f = 0; \qquad \mathcal{D}\Phi^r = 0;$$

But now there are algebraic equations that follow from the 2 new KSEs:

$$\boxed{\begin{aligned}
& h_I^x F^I{}_{\alpha\beta} \Phi^{r\alpha\beta} = 0; \\
& f\, d\phi^x + h_I{}^x i_V F^I = 0; \\
& \Phi^r{}_{\mu\nu} \partial^\nu \phi^x + \frac{1}{4} \varepsilon_{\mu\nu\alpha\beta\gamma} h^x_I F^{I\nu\alpha} \Phi^{r\beta\gamma} = 0; \\
& \pounds_V q^X = 0; \\
& f \partial_\mu q^X + \Phi^r{}_\mu{}^\nu \partial_\nu q^Y J^r{}_Y{}^X = 0;
\end{aligned}}$$

$$V^\mu \partial_\mu \phi^x = 0; \quad V^\mu \partial_\mu q^X = 0;$$

Let's consider separately the timelike and null cases:

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Timelike → Same coordinate choices

The metric can be written in the same conformastationary form. The $\Phi^{r}{}_{mn}$ are antiself-dual forms defined on the base space and satisfy the same algebra. The difference is that, now, instead of $\nabla_{(4)m}\Phi^{r}{}_{np}=0$ we find

$$\boxed{\mathcal{D}_{(4)m}\Phi^{r}{}_{np}=0}$$

This looks like a QK manifold. However 4d QK manifolds are defined in a different way because having $SU(2)\times SU(2)\sim SO(4)$ is not a restriction at all. often QK manifolds are required to have $\dim\geqslant 8$.

The above equation must be understood as a relation between the anti-selfdual part of the Levi-Civita connection and the pullback of the $SU(2)$ connection of the QK space of the hypers. In the frame in which the $\Phi^{r}{}_{mn}$ are constant

$$\boxed{\omega^{-}_{(4)m\,n}{}^{p}=-A^{r}{}_{X}\,\partial_m q^X\,\Phi^{r}{}_{n}{}^{p}\ ;}$$

When A^{r} is trivial ($n_H=0$) $\omega_{(4)}$ is selfdual and the base space is HK.

The equation that involves the hyperscalars can be put in the form

(Generalization of the Cauchy–Riemann equations) $\boxed{\partial_m q^X=\Phi^{r}{}_{m}{}^{n}\,\partial_n q^Y\,J^{r}{}_{Y}{}^{Z}}$ ← "quaternionic map"

$$\Downarrow$$

$$\mathcal{D}_m\partial^m q^X=0\qquad(\mathcal{E}_X=0)$$

Not many solutions of these equations are known...

Finally, the vector field strengths take the form

$$\boxed{F^I=-\sqrt{3}\left\{d\left[fh^I(dt+\omega)\right]+\Theta^I\right\}\ ;\ \star_{(4)}\Theta^I=+\Theta^I}$$

and

$$\boxed{(d\omega)^{+}=-\frac{3}{2}\frac{h_I}{f}\Theta^I}$$ (The antiselfdual part follows by demanding $d^2\omega=0$.)

These are the necessary conditions for supersymmetry. Observe that the

the only constraint of the h^I is that they have to be time-independent.
These conditions ensure the existence of Killing spinors satisfying

$$\varepsilon^i = f^{1/2}\,\varepsilon^i{}_0\,; \qquad \tfrac{1}{2}(1-\gamma_0)\,\varepsilon^i = 0\,;$$
$$\tfrac{1}{2}\left[\delta \pm \tfrac{1}{4}\,\cancel{\Phi}^{(x)}\,\sigma^{(x)}\right]_j{}^i\,\varepsilon^j = 0\,;$$

The last 3 conditions ($x=1,2,3$) are needed only if the corresponding component $\Phi^x \neq 0$. They are projectors only on the subspace that satisfies the first condition. They satisfy the SU(2) algebra so imposing 2 is equivalent to imposing the 3rd.
The meaning of the last projectors is that $\cancel{\Phi}^x$ and σ^x have the same effect on the Killing spinors and the SU(2) of the tangent space is identified with the SU(2) of R-symmetry.
Each of these conditions reduces the number of independent KS by a half:
no hypers $\Rightarrow$ 1/2 BPS
1 extra condition $\Rightarrow$ $1/2 \times 1/2 = 1/4$ BPS
2 extra conditions = 3 extra conditions $\Rightarrow$ $1/2\times1/2\times1/2 = 1/8$ (the minimum)

Finally the supersymmetric solutions must satisfy these equations:

$$\boxed{\begin{aligned} &\nabla^2_{(4)}\left(h_I/f\right) - \tfrac{1}{4}\,C_{IJK}\,\Theta^J\cdot\Theta^K = 0\,;\\ &d\,\Theta^I = 0\,; \end{aligned}}$$

Again, Abelian selfdual instantons in a HK ($n_H=0$) or QK ($n_H\neq0$ + nontrivial q^x) manifold.

Null

We make the same coordinates and we get a metric of the same form

Now, however $\Phi^{i} = du \wedge v^{2}$ $\oplus$ $\mathcal{D}\Phi^{i} = 0 \Rightarrow du \wedge \mathcal{D} v^{2} = 0$

$\Rightarrow \mathcal{D}_{(3)} v^{i} = 0$ which can be interpreted as Cartan Structure equations for the Dreibeins v^{2} and the 3-d Levi-Civita connection

$$\boxed{\omega_{(3)r}{}^{st} = 2\, \varepsilon^{stk} A^{k}{}_{X}\, \partial_{r} q^{X}}$$

(Again, a relation between the Levi-Civita connection of the base space (here transverse space or wave-front space) and the SU(2) connection of the QK space, but now it determines the LC connection completely)

The hyperscalars must satisfy now

$$\boxed{\partial_{r} q^{X} - f_{X}{}^{iA}\, \sigma^{r}{}_{i}{}^{j} = 0;}$$

The vector field strengths have the form

$$\boxed{\begin{aligned} F^{I} &= \left[\frac{1}{\sqrt{3}} f^{2} h^{I} \star_{(3)} d_{(3)} \omega - \overset{\text{new}}{\psi^{I}}\right] \wedge du + \sqrt{3} \star_{(3)} d_{(3)} \left(h_{I}/f\right) \\ h_{I} \psi^{I} &= 0; \end{aligned}}$$

These conditions are necessary for supersymmetry and they are also sufficient to ensure the existence of KS satisfying

$$\partial_{v} \varepsilon^{i} = 0\,; \qquad \gamma^{+} \varepsilon^{i} = 0\,; \qquad \text{(special gauge!)}$$

$$\frac{1}{2}\left\{1 - \gamma^{(2)} \sigma^{(2)}\right\} \varepsilon = 0$$

The additional conditions are similar to those of the timelike and the amount of unbroken supersymmetry is the same.

The analysis of the equations of motion is complicated but in the end the solutions are given in terms of functions K^{I}, L_{I}, N harmonic in the 3-d base space plus a number of horrible gauge-fixing conditions.

$$K^{I} = h^{I}/f\,; \quad L_{I} = C_{IJK} K^{J} M^{K}, \quad H = \tfrac{1}{2} C_{IJK} K^{I} M^{J} M^{K} + N \text{ etc.}$$

Pure, gauged supergravity

miércoles, 30 de noviembre de 2016 18:49

(a.k.a. minimal gauged $d=5$ supergravity)

At the bosonic level, this theory differs from the ungauged one in the cosmological-constant term $V = -4g^2 = +3\Lambda$; $\boxed{\Lambda = -\frac{4}{3}g^2 < 0}$

At the fermionic level the theory has more important differences: all the spinors (here it is just the gravitino) are charged w.r.t. the graviphoton so their covariant derivatives must contain a Maxwell connection and the supersymmetry transformation rules acquire a fermion shift term:

$$\delta_\varepsilon \psi_\mu^i = \mathfrak{D}_\mu \varepsilon^i - \frac{1}{8\sqrt{3}} F^{\alpha\beta}(\gamma_{\mu\alpha\beta} - 4 g_{\mu\alpha}\gamma_\beta)\varepsilon^i + \frac{i}{2\sqrt{3}} g P^r(\sigma^r)_j{}^i \gamma_\mu \varepsilon^j;$$

with

$$\mathfrak{D}_\mu \varepsilon^i = \nabla_\mu \varepsilon^i + i g A_\mu P^r(\sigma^r)_j{}^i \varepsilon^j,$$

Any non-vanishing value of P^r is possible (usually we normalize $P^r P^r = 1$) but it is sometimes helpful to pick $P^r = \delta^r{}_1$, for instance.

There are many things that do not change (KSIs), but we expect additional terms in the equations of the bilinears plus the replacement $\nabla_\mu \to \mathfrak{D}_\mu$. We know this derivative plays an important rôle in the identification of the base space.

The final result can be presented as follows: the timelike supersymmetric solutions are determined by $\hat{f}$, $\hat{\omega}_{\underline{m}} dx^m$, $\hat{h}_{\underline{mn}} dx^m dx^n$, the hypercomplex structure $\hat{\Phi}^r{}_{\underline{mn}}$ and the vector field $\hat{A}_{\underline{m}}$.

- $\hat{\Phi}^r$ is covariantly-constant w.r.t. the $U(1)$-covariant derivative on the base space: $\hat{\mathfrak{D}}_{\underline{m}} \hat{\Phi}^r{}_{\underline{np}} = \hat{\nabla}_{\underline{m}} \hat{\Phi}^r{}_{\underline{np}} + g\,\varepsilon^{rst} P^s \hat{A}_{\underline{m}} \hat{\Phi}^t{}_{\underline{np}} = 0$

- $(\hat{d}\hat{\omega})^+ = -\frac{3}{2}\hat{f}^{-1}\hat{\Theta}$; $\quad \Theta = -\frac{1}{\sqrt{3}}(d\hat{A})^+ \equiv -\frac{1}{\sqrt{3}}\hat{F}^+$
- $\hat{F}^- = (d\hat{A})^- = -2\hat{f}^{-1} P^r \hat{\Phi}^r$ $\quad$ ($=0$ in the ungauged case)

  $\hat{F} = \hat{F}^+ + \hat{F}^- = -\sqrt{3}\,\Theta - 2\hat{f}^{-1}P^r\hat{\Phi}^r$
- $\hat{\nabla}^2 f^{-1} - \frac{1}{6}\hat{F}\cdot\hat{\star}\hat{F} - \frac{1}{2\sqrt{3}}\hat{F}\cdot f(\hat{d}\hat{\omega})^- = 0$

Solving these equations one finds the building blocks. The physical fields are given by

$$\begin{cases} ds^2 = \hat{f}^2 (dt + \hat{\omega})^2 - \hat{f}^{-1} h_{\underline{mn}} dx^m dx^n \\ A = -\sqrt{3}\, \hat{f} (dt + \hat{\omega}) + \hat{A} ; \qquad (A_{\underline{m}} = \hat{A}_{\underline{m}} - \sqrt{3} \hat{f} \omega_{\underline{m}}); \\ F = -\sqrt{3}\, d [\hat{f} (dt + \hat{\omega})] + \hat{F} ; \end{cases}$$

One can go much further: the equation $\hat{\mathfrak{D}}_{\underline{m}} \hat{\Phi}^r{}_{\underline{np}} = 0$ implies that the SU(2) singlet $P^r \hat{\Phi}^r{}_{\underline{mn}}$ is a covariantly constant complex structure on the base space, which is a Kähler space.

Choosing $P^r = \delta^r{}_1$ $\qquad \Phi^1{}_{\underline{mn}} \equiv K_{\underline{mn}}$

$$\hat{\mathfrak{D}}_{\underline{m}} \hat{\Phi}^r{}_{\underline{np}} = 0 \Rightarrow \left. \begin{cases} \hat{\nabla}_{\underline{m}} \hat{\Phi}^1{}_{\underline{np}} = 0 \\ \hat{\nabla}_{\underline{m}} \hat{\Phi}^2{}_{\underline{np}} = g \hat{A}_{\underline{m}} \hat{\Phi}^3{}_{\underline{np}} ; \\ \hat{\nabla}_{\underline{m}} \hat{\Phi}^3{}_{\underline{np}} = -g \hat{A}_{\underline{m}} \hat{\Phi}^2{}_{\underline{np}} ; \end{cases} \right\} \to \text{integrability}$$

$$\to \boxed{ \begin{array}{ll} \hat{R}_{mn} = g \hat{F}_{mn} ; & \to \quad \hat{R} = 8 g^2 \hat{f}^{-1} \\ \hat{F}^-{}_{mn} = -2g \hat{f}^{-1} K_{mn} ; & \end{array} }$$

The null case will not be covered due to lack of time.

N=1, d=5 SEYM theories

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These are the simplest supersymmetrisation of the d=5 Einstein-Yang-Mills (EYM) theories. Supersymmetry requires the vectors to come in full vector supermultiplets, and, thus, these theories have one real scalar for each vector (minus one). No hypers are needed. On the other hand, the gauge group must be non-Abelian for the gauging to be non trivial. In these theories, the scalar potential vanishes identically because the fermion shifts also vanish identically. The spinor fields are uncharged and their covariant derivatives only have the Levi-Civita connection. The only changes in the KSEs are, therefore, the replacement of $\partial\phi^x$ by $\mathcal{D}\phi^x$ and $F^I = dA^I$ by the gauge-covariant F^I. The same will be true in the equations of the KS bilinears. The results will be the gauge-covariant version of those obtained for ungauged supergravity coupled to vector multiplets only:

The timelike supersymmetric solutions are constructed from the building blocks $\hat{A}^I{}_{\underline{m}} dx^m$, $\hat{f}_I$, $\hat{\omega}$ defined on $h_{\underline{mn}}$ satisfying the conditions and equations

$$\begin{cases} - h_{\underline{mn}}\, dx^m dx^n \text{ is hyper Kähler.} \\ - \hat{\star}\hat{F}^I(\hat{A}) = +\hat{F}^I \quad \Rightarrow \text{non-Abelian self-dual instantons on HK spaces !} \\ - \hat{\mathcal{D}}^2 \hat{f}_I - \frac{1}{6} C_{IJK}\hat{F}^J\cdot\hat{F}^K = 0 \\ - (d\hat{\omega})^+ = \sqrt{3}\,\hat{f}_I\, \hat{F}^{I+}; \end{cases}$$

Then, using these building blocks, the physical fields are obtained:

- Define $\hat{f}$ by $h_I/\hat{f} \equiv \hat{f}_I$; the h_I satisfy a constraint which is homogeneous of order 3/2 in the h_I $\;F(h.) = 1$

$\Rightarrow F(\hat{f}.) = \hat{f}^{-3/2} \quad \Rightarrow h_I = \hat{f}_I\, F^{-2/3}(\hat{f}.) \quad \Rightarrow h^I$

A definition of the scalars that always works is $\phi^x = \frac{h_x}{h_0} = \frac{\hat{f}_x}{\hat{f}_0}$

- $ds^2 = \hat{f}^2 (dt + \hat{\omega})^2 - \hat{f}^{-1} h_{\underline{mn}}\, dx^m dx^n$

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$$- A^I = -\sqrt{3}\, h^I \hat{f}(dt + \hat{\omega}) + \hat{A}^I;$$

or $$F^I = -\sqrt{3}\, \hat{\mathfrak{d}}\left[h^I \hat{f}\left(dt + \hat{\omega}\right)\right] + \hat{F}^I;$$

Again we will focus on the timelike case and leave the null case aside.

The additional isometry trick

miércoles, 30 de noviembre de 2016 18:32

One of the main problems we face in order to find general families of solutions is that we need to be able to give the general form of the base space. Otherwise we are restricted to solving the equations for a particular base space and then for another etc.

However, the general form of 4d Kähler or hyperKähler metrics is not known.

In the hyperKähler case, though, we know the general form of the metrics that admit a triholomorphic isometry: Gibbons-Hawking metrics:

$$h_{mn}\, dx^m dx^n = H^{-1}(dz+\chi)^2 + H\, d\vec{x}^{\,2}$$

$$\text{with } d\chi = \star_3 dH \quad (\Rightarrow d\star_3 dH = 0,\ \text{harmonic in } \mathbb{E}^3)$$

Using this general form we can try to solve the necessary and sufficient conditions satisfied by the timelike supersymmetric solutions of the $N=1, d=5$ supergravities with no hypers and no F-I terms.

In the Kähler case several ansatz for Kähler metrics with more than one isometry (toric, orthotoric) have been proposed. More recently we have proposed one for Kähler metrics with one isometry that is connected to the Gibbons-Hawking ansatz:

$$h_{mn}\, dx^m dx^n = H^{-1}(dz+\chi)^2 + H\left[d\varrho^2 + W^2(dx^2+dy^2)\right]$$

$$\left.\begin{aligned} 2\partial_{[x}\chi_{\varrho]} &= \partial_y H\,; \\ 2\partial_{[y}\chi_{\varrho]} &= -\partial_x H\,; \\ 2\partial_{[y}\chi_{x]} &= \partial_\varrho (W^2 H)\,; \end{aligned}\right\} \quad \partial_x\partial_x H + \partial_y\partial_y H + \partial_\varrho\partial_\varrho (W^2 H) = 0$$

$W=1 \rightarrow$ hyperKähler, but there are intermediate possibilities.

These ansatz also offer naturally the possibility of dimensional reduction of the solution to $d=4$, connecting with theories of $N=2, d=4$.

Gibbons-Hawking ansatz

miércoles, 30 de noviembre de 2016 18:51

Let's consider this ansatz in the context of $N=1, d=5$ SEYM theories. These theories always admit a $g \to 0$ limit in which the ansatz is also valid.

The building blocks of any timelike supersymmetric solution are

1.- A hyperKähler metric $\to$ Gibbons-Hawking

2.- A self-dual $\hat{F}^I$ on the HK space $\Rightarrow$ we use the following result by Kronheimer relating YM instantons in HK and YMH monopoles in $\mathbb{E}^3$

$$\text{If } \hat{\star}\hat{F}^I(\hat{A}) = +\hat{F}^I(\hat{A}) \text{ and } \begin{cases} \check{A}^I{}_{\underline{r}} \equiv \frac{1}{2\sqrt{6}}\left(-\hat{A}^I{}_{\underline{r}} + \chi_{\underline{r}}\hat{A}^I{}_{\underline{z}}\right); & r=1,2,3 \\ \Phi^I \equiv \frac{1}{2\sqrt{6}} H \hat{A}^I{}_{\underline{z}}; \end{cases}$$

then $\boxed{\check{F}^I(\check{A}) = \star_3 \check{\mathfrak{D}}\Phi^I}$ in $\mathbb{E}^3$ Bogomol'nyi equation Φ^I: Higgs

3.- The solutions f_I to the equation

$$\hat{\mathfrak{D}}^2 f_I - \frac{1}{6} C_{IJK}\hat{F}^J \cdot \hat{F}^K = 0$$

on the HK space $\to$ Knowing the GH metric and the instanton/monopole fields ($\Rightarrow$ Higgs fields) we are going to reduce this equation to a much simpler one

4.- The solution $\omega_{\underline{m}}dx^m$ to the equation $\to$ now it will be easier to find

$$(d\hat{\omega})^+ = \sqrt{3}\hat{f}_I \hat{F}^{I+}$$

In order to solve the equations for the building blocks it is convenient to introduce new variables:

$L_I \equiv \hat{f}_I - 8C_{IJK}\Phi^J\Phi^K/H;$

$\hat{\omega} \equiv \omega_5(dz+\chi) + \omega_{\underline{r}}dx^r;$

$M \equiv \omega_5 - 16\sqrt{2}\, C_{IJK}\Phi^I\Phi^J\Phi^K/H^2 + 3\sqrt{2}\, L_I\Phi^I/H;$

$\hat{A}^I = 2\sqrt{6}\left[H^{-1}\Phi^I(dz+\chi) - \check{A}^I\right];$ (Kronheimer)

Then, the whole problem is reduced to the following equations for the new variables $\underbrace{H, M, \Phi^I, L_I}_{\text{functions}}, \underbrace{\omega, \check{A}^I, \chi}_{1-\text{forms}}$ in $\mathbb{E}^3$

$$
\begin{cases}
d\star_3 dM = 0\,; \\
\left.\begin{aligned}
\star_3 dH &= d\chi\,; \\
\star_3 \breve{\mathfrak{D}}\breve{\Phi}^I &= \breve{F}^I\,;
\end{aligned}\right\} \to \text{Bogomol'nyi monopole-like eqs.} \\
\breve{\mathfrak{D}}^2 L_I = g^2 f_{IJ}{}^{L} f_{KL}{}^{M}\, \bar{\Phi}^J \bar{\Phi}^L L_M\,; \\
\underset{\uparrow \text{ ``Bubbling equations''}}{\star_3 d\omega =} H dM - M dH + 3\sqrt{2}\left(\bar{\Phi}^I \breve{\mathfrak{D}} L_I - L_I \breve{\mathfrak{D}}\bar{\Phi}^I\right);
\end{cases}
$$

We will see specific solutions of these equations. For the moment, observe that M and H are harmonic functions in $\mathbb{E}^3$ and many are known. On the other hand, all the spherically-symmetric solutions of the Bogomol'nyi equation (for $SU(2)$) were found by Protogenov long time ago. Particular examples are the 't Hooft-Polyakov monopole in the BPS limit and the $SU(2)$ Wu-Yang monopole. However, these are not the most interesting solutions because they lead to singular instantons. The most interesting ones are the "colored monopoles" that correspond to the BPST $SU(2)$ instanton and for which Ramírez has found multi-center solutions (also dyonic, involving $L_I \neq 0$).

The "bubbling equations" always admit a solution if the other equations are solved: the integrability equations seem to be satisfied. However, the harmonic functions in $\mathbb{E}^3$ have singularities and making sure the integrability conditions are satisfied there leads to important constraints in the locations of the singularities (BHs) discovered by Denef and Bates. The "Higgs" fields that appear in the Abelian Bogomol'nyi equations are always harmonic functions. This will not be the case in the non-Abelian case using Ramírez's dyonic solution.

What happens if the integrability conditions are not satisfied at the singularities? $\to$ ω has to be defined patch by patch and gluing them usually requires time to be periodic $\to$ CTCs.

Iso-Kähler ansatz

miércoles, 30 de noviembre de 2016 18:51

The ansatz made for the Kähler metric of the base space allows one to solve immediately the equations that involve the metric function $\hat{f}$ and the components $\hat{A}_m$ of the single vector field of minimal gauged N=1, d=5 supergravity:

$$\begin{cases} g\hat{A}_m = \hat{J}_m{}^n \partial_n \log W ; \\ 8g^2 \hat{f}^{-1} = \hat{\nabla}^2 \log W^2 ; \end{cases}$$

(Remember $A = -\sqrt{3}\,\hat{f}(dt+\hat{\omega}) + \hat{A}$)

However, in this theory, these fields are also constrained by other equations. Plugging these solutions in them will lead to differential equations of up to fourth order (other ansatzs give 6th order differential equations) involving W^2.

(It is important to notice that in the SEYM case one chooses first the 6th space (H) and then solves the equations in that space. Here it is different: different solutions have different K base spaces (W,H) and one needs to determine them through simple ansatzs).

It is convenient to make a change of variables similar to that of the SEYM case:

$$L = \hat{f}^{-1} - \frac{1}{12} K^2/H ;$$

(K similar to Φ but, here $gK = \partial_\varrho \log W^2$ $\Rightarrow$ L is completely determined by H, W)

$$\hat{\omega} = \omega_2 (dz + \alpha) + \omega ;$$

$$M = \omega_2 - \frac{\sqrt{3}}{4} LK/H - \frac{1}{24\sqrt{3}} K^3/H ;$$

In terms of these variables, the equations that need to be solved are

$$\star_3 d\omega = H dM - M dH + \frac{\sqrt{3}}{4}(K dL - L dK) - H\left(\omega_2 \partial_\varrho \log W^2 - 2\sqrt{3}\, g \hat{f}^{-2}\right) d\varrho ;$$

$$\bar{\nabla}^2 L = 4H(gL)^2 - \frac{2}{3}H(gK)^2 - \frac{4}{3} gL\partial_\varrho K - \frac{1}{3} gK\partial_\varrho L + \frac{4}{3} Hg\partial_\varrho M ;$$

$$\bar{\nabla}^2 M = -\sqrt{3}\, L\left(gKL + 2\partial_\varrho L\right) ;$$

coupled

These equations ensure the integrability of that of ω.

jueves, 8 de diciembre de 2016 12:20

In solving these equations we must take into account that $\hat{f}$, K, H and L are completely determined by W through

$$\begin{cases} \partial_x^2 H + \partial_y^2 H + \partial_\rho^2 (W^2 H) = 0\,; \\ g K = \partial_\rho \log W^2\,; \\ L = \dfrac{1}{8g^2 H} \left\{ \bar{\nabla}^2 \log W^2 - \dfrac{2}{3}\left(\partial_\rho \log W^2\right)^2 \right\}; \\ \hat{f}^{-1} = \dfrac{1}{8g^2}\, \hat{\nabla}^2 \log W^2\,; \end{cases}$$

$\Rightarrow$ 4th-order equations for W^2.

Some of the most remarkable supersymmetric solutions

miércoles, 30 de noviembre de 2016 18:34

The theories of $N=1$, $d=5$ supergravity are characterised by the richness of their supersymmetric solutions. In this section we are going to review some of the most interesting. This will also illustrate how one constructs complete, physically meaningful solutions from the building blocks supersymmetry indicates we must use.

Maximally supersymmetric solutions

miércoles, 30 de noviembre de 2016 18:52

As we mentioned in our general introduction to supersymmetric solutions, these are interesting because they can be seen as vacua of the theory.

To find these solutions one must use a combination of techniques, since the one we have described does not take into account the number of KS and only ensures there is one. One must study the integrability conditions of the KSEs and study the KSEs by themselves.

First of all, to find the maximally supersymmetric solutions it is enough to consider pure (or pure gauged) supergravity: the additional KSEs are always algebraic constraints on the KS that cannot be solved for non-trivial matter fields (terms of different order in γ-matrices etc.).

Let us summarize the results. The maximally supersymmetric solutions of

Ungauged	Gauged (FI)
Minkowski	AdS_5
$KG5$	
$AdS_2\times S^3 \xleftrightarrow[\text{interpolating family}]{\text{1-parameter}} AdS_3\times S^2$	
Gödel	

In the ungauged case, most of them are related to the maximally supersymmetric vacua of the 6- and 4-dimensional theories with 8 supercharges: $N=(0,1)$, $d=6$ and $N=2$, $d=4$:

$$
\begin{tikzcd}
\mathrm{Mink}_6 \arrow[d] & & AdS_3\times S^3 \arrow[dl] \arrow[dr] \arrow[d, "\xi"] \arrow[rr, "\text{Penrose}"'] & & KG6 \arrow[d] \arrow[dr] & \\
\mathrm{Mink}_5 \arrow[d] & AdS_2\times S^3 \arrow[rr, dash] \arrow[dr] & {} \arrow[d, "\xi"] & AdS_3\times S^2 \arrow[dl] \arrow[r, "\text{Penrose}"'] & KG5 \arrow[d] & G5 \\
\mathrm{Mink}_4 & & AdS_2\times S^2 \arrow[rr, "\text{Penrose}"'] & & KG4 &
\end{tikzcd}
$$

The solutions of the 1-parameter family that interpolates between $AdS_2 \times S^3$ (the near-horizon limit of extremal BHs) and $AdS_3 \times S^2$ (the near-horizon limit of extremal BS) arise as the near horizon limits of the BMPV rotating BH and the parameter ξ is related to the angular momentum. We will derive them in this way.

The KG5 is a homogeneous pp-wave (Hpp) that can be obtained as the Penrose limit of $AdS_3 \times S^2$.

The maximally supersymmetric vacuum that is not connected to the rest in an obvious way is G5, which is a homogeneous solution similar to the Gödel solution of $d=4$ (but with some differences because here we do not have cosmological constant). It is given by

$$ds^2 = (dt + \omega)^2 - d\vec{x}_{(4)}^{\,2}\,, \quad A = -\sqrt{3}\,\omega\,, \quad \omega = \frac{1}{\sqrt{3}}\lambda \underbrace{(x^1 dx^2 - x^3 dx^4)}_{\text{Kähler 1-forms of } \mathbb{R}^2}$$

$$\left(\hat{f} = 1\,, \; h_{\underline{m}\underline{n}} \to \mathbb{R}^4\right)$$

In the gauged theory there are similar (but not maximally supersymmetric) solutions.

Asymptotically-flat BHs with Abelian charges

miércoles, 30 de noviembre de 2016 18:52

... of $N=1$, $d=5$ supergravity coupled to vector supermultiplets (Observe: the presence of hypers can be seen as hair and does not allow the construction of regular black holes, at least in the ungauged theory).

As a rule supersymmetric BHs belong to the timelike class.

Which HK base space to choose? We can use the GH ansatz because it includes $\mathbb{R}^4$ ($H=1$, $\chi=0$) and also $\mathbb{R}^4 - \{0\}$ ($H=\frac{1}{|\vec{x}|}$, $\chi=\cos\theta d\varphi$) which we known to occur in simple solutions.

In the Abelian case the equations of the building blocks are

$$\begin{cases} d\star_3 dM = 0\,; \\ \star_3 dH = d\chi\,; \quad \Rightarrow \quad d\star_3 dH = 0 \\ \star_3 d\Phi^I = d\breve{A}^I\,; \quad \Rightarrow \quad d\star_3 d\Phi^I = 0 \\ d\star_3 d L_I = 0\,; \\ \star_3 d\omega = HdM - MdH + 3\sqrt{2}\left(\Phi^I dL_I - L_I d\Phi^I\right). \end{cases}$$

and, therefore, a solution is completely determined by the choice of harmonic functions in $\mathbb{E}^3$ H, M, Φ^I, L_I such that the integrability conditions of the equation for ω are satisfied everywhere on $\mathbb{R}^3$.

If we want to describe a single BH on spherically symmetric choice is most appropriate: $H=1$ ($\mathbb{R}^4$) or $H=\frac{1}{r}$ ($\mathbb{R}^4-\{0\}$)

The first choice is really not good once we have reduced the problem to $\mathbb{E}^3$ because we will get harmonic functions in $\mathbb{E}^3$ with $SO(3)$ but not $SO(4)$ symmetry. We could have done this before, but, in that case it was difficult to solve all the equations explicitly, except if we set $\hat{F}^{I+}=0 \Rightarrow$

$$\begin{cases} \hat{\nabla}^2 f_I = 0 & f_I \sim a_I + q_I/\rho^2\,; \quad \rho = |\vec{x}_{(4)}| \\ \hat{\omega} = 0 & \rightarrow ds^2 = \hat{f}^2 dt^2 - \hat{f}^{-1}\left(d\rho^2 + \rho^2 d\Omega^2_{(3)}\right); \\ F^I = d\{\hat{f}h^I\}\wedge dt \end{cases}$$

These are purely electric, static BHs. This may seem to be enough but there is much more in these theories (violating no-hair/uniqueness "ths")

If we choose $H = \frac{1}{r}$, the HK space is

$$h_{\underline{mn}}\, dx^m dx^n = r\,(dz + \cos\theta\, d\varphi)^2 + \frac{1}{r}\left(dr^2 + r^2\, d\Omega^2_{(2)}\right)$$

$$r = \frac{\rho^2}{4} \longrightarrow \quad = \frac{\rho^2}{4}\,(dz + \cos\theta\, d\varphi)^2 + \frac{4}{\rho^2}\left(\frac{\rho\, d\rho}{2}\right)^2 + \frac{\rho^2}{4}\, d\Omega^2_{(2)}$$

$$dr = \tfrac{1}{2}\rho\, d\rho \qquad = d\rho^2 + \rho^2\, \frac{1}{4}\Big[\underbrace{(dz + \cos\theta\, d\varphi)^2 + d\Omega^2_{(2)}}_{4}\Big]$$

$$= d\rho^2 + \rho^2\, d\Omega^2_{(3)} \quad \rightarrow \quad \mathbb{E}^4 - \{0\}$$

and the harmonic functions in $\mathbb{E}^3$ $a + \frac{b}{r}$ become $a + \frac{4b}{\rho^2}$ harmonic functions in $\mathbb{E}^4 - \{0\}$, so the solutions will have (in principle) $SO(4)$ symmetry.

Observe that, for harmonic functions of this kind $\star_3 d\omega = 0$ everywhere and, thus, $\omega = 0$. There is still ω_5, of course.

Summarizing: the solutions are determined by

$$H = 1/r\,; \qquad \Phi^I = c^I + p^I/r\,;$$

$$M = a + b/r\,; \qquad L_I = e_I + q_I/r\,;$$

and from them we can reconstruct the physical fields. We are particularly interested in the metric. We need $\hat{f}$ and $\hat{\omega}$

$$\hat{\omega} = \omega_5\,(dz + \cos\theta\, d\varphi)$$

$$\omega_5 = M + 16\sqrt{2}\, C_{IJK}\, \bar{\Phi}^I \bar{\Phi}^J \bar{\Phi}^K H^2 + 8\sqrt{2}\, L_I \bar{\Phi}^I / H\,;$$

If we want a static solution we must set to zero M and some of the Φs and Ls so that $\omega_5 = 0$. In general we cannot eliminate the constants in the harmonic functions if we want the solution to be asymptotically flat. If we set all of them to zero we can always adjust M to get $\omega_5 = 0$ even if all the others are $\neq 0$. Setting the constants to zero is equivalent to take the $r \to 0$ limit and we will later see what happens in that case.

As for $\hat{f}$, we have $h_I/\hat{f} = L_I + 8\, C_{IJK}\, \Phi^J \Phi^K / H$;

Let us give an example of how to extract $\hat{f}$ from these relations: if the RSG model has a scalar manifold that is a symmetric Riemannian space

$$C^{IJK} C_{J(LM} C_{NP)K} = \frac{1}{27}\, \delta^I{}_{(L} C_{MNP)} \qquad \left(C^{IJK} = C_{IJK} \right)$$

$$\Rightarrow h^I = 27\, C^{IJK} h_J h_K \Rightarrow 27\, C^{IJK} h_I h_J h_K = 1; \quad 27\, C^{IJK} \frac{h_I}{\hat{f}} \frac{h_J}{\hat{f}} \frac{h_K}{\hat{f}} = \hat{f}^{-3}$$

$$\Rightarrow \hat{f}^{-1} = \left\{ C^{IJK} \left(L_I + 8 C_I \Phi^2/H\right) \left(L_J + 8 C_J \Phi^2/H\right) \left(L_K + 8 C_K \Phi^2/H\right) \right\}^{1/3}$$

For large r $\hat{f}^{-1} \sim$ constant that can be normalize to 1 if it does not vanish

For $r \to 0$ $\hat{f}^{-1} \sim$ constant $\frac{1}{r}$

$$\Rightarrow ds^2 \sim \frac{r^2}{(\text{constant})^3}\, dt^2 - \text{constant}\, \frac{1}{r} \left[\frac{1}{r} dr^2 + r\,(dz + \psi)^2 + r\, d\Omega^2_{(2)} \right]$$

$AdS_2 \times S^3$

The constant is related to the entropy $(\text{constant})^{3/2}$ and is a function of the charges q_I and "dipole momenta p^I".

If ω_5 does not vanish and, at $r \to \infty$ $\omega_5 \sim J/r$, then the solution has angular momentum J and the near-horizon limit includes $\left[dt + \frac{J}{2}(dz + \cos\theta\, d\varphi) \right]^2$

$\Rightarrow$ 1-parameter family that interpolates between $AdS_2 \times S^3$ and $AdS_3 \times S^2$

Black rings

miércoles, 30 de noviembre de 2016 18:54

In $d=5$ BHs can have two independent angular momenta that characterize them. The supersymmetric, asymptotically flat ones only have 1. One can have 2, but then the horizon of the solution no longer has the topology of an sphere. It is a doughnut $S^1 \times S^2 \to$ "black ring".

These solutions cannot exist in $d=4$ (Hawking: the topology of an event horizon is always S^2 in $d=4$), which makes them interesting.

Constructing the simplest supersymmetric BR in pure sugra/gravity is really simple: choose the functions

$$H = \frac{1}{2} \; ; \quad L = 1 + \frac{q_0 - a^2}{4} N \; ; \quad K = -\frac{a}{2} N \; ; \quad M = \frac{3a}{4}\left(1 - \frac{R^2}{4} N\right) ;$$

with $N = \dfrac{1}{|\vec{x} - \vec{x}_1|}$; $\vec{x}_1 = (0, 0, -R^2/4)$

The values of the coefficients ensure asymptotic flatness and integrability of ω plus absence of CTCs, Chirac-Dirac strings etc.

$$\hat{f}^{-1} = 1 + q_0 N - a^2 N (1 - r N) ;$$

$$\omega = (F - \cos\theta\, G)\, d\varphi + (F - G)\, d\psi ;$$

$$F = \frac{3a}{4}\left[1 - \left(r + \frac{R^2}{4}\right) N\right]$$

$$G = \frac{a}{16} r N^2 \left[(3q_0 - a^2) + 2a^2 r N\right].$$

Then $M = \sqrt{3}\, q_0$; $J_1 = \frac{1}{2\sqrt{3}} a (3q_0 - a^2)$; $J_2 = \frac{1}{2\sqrt{3}} a (6R^2 + 3q_0 - a^2)$

Checking the regularity etc. is complicated and we will not do it here.

Solutions with non-Abelian Yang-Mills fields

miércoles, 30 de noviembre de 2016 18:55

The next step in the construction of asymptotically-flat supersymmetric BH solutions is the inclusion of non-Abelian Yang-Mills fields. Let's consider $N=1, d=5$ SEYM theories with an $SU(2)$ gauging plus, possibly, some Abelian ungauged fields. It is important to remember that we are not using $SU(2)$ FI terms to gauge R-symmetry. Those terms would give rise to a non-vanishing scalar potential. SEYM theories have $V=0$. For the sake of clarity, we are going to consider a particular model which we have already mentioned and which can be embedded in String theory $C_{0xy} = \frac{1}{3!}\eta_{xy}$; $\eta = \mathrm{diag}(+ - \cdots -)$; $x,y = 1, \ldots, n_{V5} > 3$

Let's take just $n_{V5} = 4$. The Abelian sector is $I = 0,1$ and the $SU(2)$ is $r+1 = 2,3,4$ $(r = 1,2,3)$

The equations that we need to solve are

$$d\star_3 dM = 0 \ ; \quad d\star_3 dH = 0 \ ; \quad d\star_3 d\Phi^0 = 0 \ ; \quad d\star_3 d\Phi^1 = 0 \ ;$$

$$\underbrace{d\star_3 dL_0 = 0 \ ; \quad d\star_3 dL_1 = 0 \ ;}_{\text{Abelian sector}}$$

non-Abelian sector
$$\begin{cases} \star_3 \breve{\mathfrak{D}} \Phi^{r+1} = \breve{F}^{r+1} \ ; \\ \breve{\mathfrak{D}} \star_3 \breve{\mathfrak{D}} L_{r+1} = g^2 \left[\Phi^{t+1} \Phi^{t+1} L_{r+1} - \Phi^{r+1} \Phi^{t+1} L_{t+1} \right] ; \end{cases}$$

$$\star_3 d\omega = H dM - M dH + 3\sqrt{2}\left(\Phi^0 dL_0 + \Phi^1 dL_1 - L_0 d\Phi^0 - L_1 d\Phi^1 \right)$$
$$+ 3\sqrt{2}\left(\Phi^{r+1} \breve{\mathfrak{D}} L_{r+1} - L_{r+1} \breve{\mathfrak{D}} \Phi^{r+1} \right) ;$$

and we have to take into account that, for this model

$$\hat{f}^{-1} = H^{-1}\left\{ \frac{1}{4}\left(6HL_0 + 8\eta_{xy}\Phi^x\Phi^y \right)\left[9H^2 \eta^{xy} L_x L_y + 48 H \Phi^0 L_x \Phi^x \right.\right.$$
$$\left.\left. + 64 (\Phi^0)^2 \eta_{xy} \Phi^x \Phi^y \right] \right\}^{1/3}$$

just $L_0, L_1 \neq 0$ give a regular BH $\quad \hat{f}^{-1} = \left\{ \frac{27}{2} L_0 (L_1)^2 \right\}^{1/3}$

Adding $\Phi^{r+1} \neq 0 \quad \hat{f}^{-1} = \left\{ \frac{27}{2}\left(L_0 - \frac{4}{3} \frac{\Phi^{r+1}\Phi^{r+1}}{H} \right)(L_1)^2 \right\}^{1/3}$

Keeping $M = 0 \Rightarrow \omega = 0$ and we only need to choose the solution of the $SU(2)$ Bogomol'nyi equations.

If we are after spherically-symmetric solutions we can use the "hedgehog ansatz" (all the spherically-symmetric solutions fit in it)

$$\hat{A}^{r+1}{}_{\underline{s}} = h(r)\, \varepsilon^r{}_{st}\, x^s\, dx^t \,; \quad \bar{\Phi}^{r+1} = f(r)\, x^r \,;$$

Plugging this ansatz into the Bogomol'nyi equations we arrive at a system of ODEs for f and h (g=1 for simplicity)

$$\begin{cases} r\dot{h} + 2h + f(1+r^2 h) = 0\,; \\ r(\dot{h}-\dot{f}) - r^2 h(h-f) = 0\,; \end{cases}$$

with 2 families of solutions

① $(\mu, s) \Rightarrow \begin{cases} rf = -\frac{1}{r}\left[1-\mu r \coth(\mu r+s)\right]; \\ rh = \frac{1}{r}\left[\frac{\mu r}{\sinh(\mu r+s)} - 1\right]; \end{cases}$ $s=0 \to$ BPS 't Hooft-Polyakov (the only regular)

② $\lambda \Rightarrow \begin{cases} rf = -\frac{1}{r(1+\lambda^2 r)}\,; \\ rh = rf\,; \end{cases}$ $\lambda = 0 \to$ Wu-Yang SU(2) monopole; $\lambda \neq 0$ "colored monopole"

The latter is related via Kronheimer to the BPST instanton with $H=\frac{1}{r}$;

$$\hat{A}^{r+1} = \frac{-1}{1+\frac{\lambda^2}{4}\rho^2}\, \sigma^{r+1}{}_L \,; \quad \hat{*}\hat{F} = +\hat{F}\,;$$

From the point of the metric, all we need are the "Higgs fields"

$$\bar{\Phi}^{r+1} = -\frac{1}{r(1+\lambda^2 r)}\frac{x^r}{r}\,; \quad \to \quad \boxed{\bar{\Phi}^{r+1}\bar{\Phi}^{r+1} = \frac{1}{r^2(1+\lambda^2 r)^2}}$$

When $r \to 0$ $\quad \frac{\bar{\Phi}^{r+1}\bar{\Phi}^{r+1}}{H} \sim \frac{1}{r}$ $\Rightarrow$ like an Abelian field, contributes to the entropy

When $r \to \infty$ $\quad \frac{\bar{\Phi}^{r+1}\bar{\Phi}^{r+1}}{H} \sim \frac{1}{r^3}$ $\Rightarrow$ falls much faster and should not contribute to the mass

$\left(\times \frac{1}{g^2}\right)$ $\Rightarrow$ hair!

Then, choosing $L_{0,1} = a_{0,1} + 4q_{0,1}/r$;

$$\hat{f}^{-1} = \left\{ \frac{27}{2}\left[a_0 + 4\frac{q_0}{r} - \frac{4}{3g^2}\frac{1}{r(1+\lambda^2 r)^2}\right]\left(a_1 + 4\frac{q_1}{r}\right)^2 \right\}^{1/3}$$

$$[T_r, T_s] = -\varepsilon_{rs}{}^t T_t\,; \Rightarrow F^r = dA^r - \frac{1}{2}g\,\varepsilon^r{}_{st}\, A^s \wedge A^t$$

viernes, 9 de diciembre de 2016 1:29

For $r \to \infty$ $\quad \hat{f}^{-1} = \left\{ \frac{27}{2} a_0 a_1^2 + 4\left(a_1^2 q_0 + 2 a_0 a_1 q_1\right) \frac{1}{r} + \mathcal{O}\frac{1}{r^2} \right\}^{1/3}$

$$= \underbrace{3\left(\frac{a_0 a_1^2}{2}\right)^{1/3}}_{\overset{!}{=} 1} \left\{ 1 + \underbrace{\frac{4}{3}\frac{\left(a_1^2 q_0 + 2 a_0 a_1 q_1\right)}{\frac{27}{2} a_0 a_1^2}}_{\equiv M = \frac{8}{3^4}\left(\frac{q_0}{a_0} + 2\frac{q_1}{a_1}\right)} \frac{1}{r} \right\}$$

← central charge

Independent of g !

For $r \to 0$ $\quad \hat{f}^{-1} \sim \underbrace{\left\{ \frac{27}{2} \left(4 q_0 - \frac{4}{3 g^2}\right)\left(4 q_1\right)^2 \right\}^{1/3}}_{R^2} \frac{1}{r}$

$\Delta \sim R^3 \sim \{ \quad \}^{1/2}$

↓

Entropy of the BH depends on $\frac{1}{g}$! $<$ than the Abelian one.

Asymptotically-AdS5 solutions

miércoles, 30 de noviembre de 2016 18:54

The construction of supersymmetric asymptotically-AdS$_5$ is more complicated than the asymptotically-flat because now the Kähler base space is different for each different solution while in the asymptotically-flat one one could have many different solutions with the same hyperKähler base space. This is one of the reasons why one needs to know more about 4-dimensional Kähler manifolds.

To give some idea of the complexity of the problem, we can start by reviewing how the only maximally supersymmetric solution (AdS$_5$) occurs in this setting.

Gutowski and Reall proved that AdS$_5$ always appears with $\overline{\mathbb{CP}}^2$ (a.k.a. Bergmann space, $SU(1,2)/U(2)$, universal hypermultiplet space) as base space.

Indeed, if we define AdS$_5$ as the hyperboloid $Z^0\bar{Z}^0 - Z^i\bar{Z}^i = 1$ in $\mathbb{C}^{1,2}$ with the canonical metric $ds^2 = dZ^0 d\bar{Z}^0 - dZ^i d\bar{Z}^i$, defining

$Z^0 = |Z^0| e^{it}$, $Z^i = Z^0 \xi^i$, $\Rightarrow$ $|Z^0|^{-2} = 1 - \xi^i\bar{\xi}^i$

$$ds^2 = (dt + Q)^2 - G_{i\bar{j}}\, d\xi^i d\bar{\xi}^j \; ; \quad \leftarrow \text{AdS}_5$$

$$G_{i\bar{j}} = e^{\mathcal{K}}\left[\delta_{i\bar{j}} + e^{\mathcal{K}}\bar{\xi}^i \xi^j\right] \quad \leftarrow \overline{\mathbb{CP}}^2$$

$e^{-\mathcal{K}} = 1 - \xi^i\bar{\xi}^i$ is the Kähler potential.

$Q = \frac{i}{2} e^{\mathcal{K}}\left(\bar{\xi}^i d\xi^i - \text{c.c.}\right)$ is the Kähler 1-form

$\mathcal{J}_{i\bar{j}} = -\frac{1}{2}\left(\partial_i Q_{\bar{j}} - \partial_{\bar{j}} Q_i\right) = \frac{i}{2} G_{i\bar{j}}$ is the Kähler 2-form

In order to use our results we need to write the metric of $\overline{\mathbb{CP}}^2$ in the canonical form associated to one of its isometries. There are 3 simple possibilities.

$k=1$ $\quad \xi^1 = \tanh\rho \cos\frac{\theta}{2}\, e^{-\frac{i}{2}(z+\varphi)}$; $\xi^2 = \tanh\rho \sin\frac{\theta}{2}\, e^{-\frac{i}{2}(z-\varphi)}$

$$\rightarrow ds^2_{\overline{\mathbb{CP}}^2} = d\rho^2 + \frac{1}{4}\sinh^2\rho \cosh^2\rho\, (dz + \cos\theta\, d\varphi)^2 + \frac{1}{4}\sinh^2\rho\, d\Omega^2_{(2,1)}$$

where $d\Omega^2_{(2,1)} = d\theta^2 + \sin^2\theta\, d\varphi^2$

These are, basically, the global coordinates of AdS_5: $z = \psi + 2t$

$$ds^2_{AdS_5} = ch^2\rho\, dt^2 - d\rho^2 - sh^2\rho\, d\Omega^2_{(3,1)}$$

$$d\Omega^2_{(3,1)} = \frac{1}{4}\left[(d\psi + \cos\theta\, d\varphi)^2 + d\Omega^2_{(2,1)}\right]$$

$(\lambda = sh\,\rho \Rightarrow$ usual coordinates$)$

If, before this, we define

$$x^1 = tg\frac{\theta}{2}\cos\varphi\ ;\quad \rho' = \frac{1}{4} sh^2\rho\ ;\quad x^3 = tg\frac{\theta}{2}\sin\varphi$$

$$\longrightarrow\quad ds^2_{\mathbb{CP}^2} = H^{-1}(dz + \alpha)^2 + H\left\{d\rho'^2 + W^2\left[(dx^1)^2 + (dx^3)^2\right]\right\};$$

with $(\rho' \to \rho)$

$$H^{-1} = \rho(1 + 4\rho)\ ;\quad HW^2 = \rho\, \underbrace{\frac{4}{\left\{1 + [(x^1)^2 + (x^3)^2]\right\}^2}}_{\text{conformal factor of the sphere metric } \Phi_{(1)}}$$

$$\alpha = \alpha_{(1)} = 2\,\frac{x^1 dx^3 - x^3 dx^1}{1 + [(x^1)^2 + (x^3)^2]}$$

There are another 2 simple ways of writing $\mathbb{CP}^2$ in this canonical form labeled by $k = 1, 0, -1$:

$$H^{-1} = \rho(k + 4\rho)\ ;\quad HW^2 = \rho\,\Phi_{(k)}(x^1, x^3)\ ;\quad \alpha = \alpha_{(k)}\ ;$$

$$\Phi_{(k)} = \frac{4}{\left\{1 + k[(x^1)^2 + (x^3)^2]\right\}^2}\ ;\quad \alpha_{(k)} = 2\,\frac{x^3 dx^1 - x^1 dx^3}{1 + k[(x^1)^2 + (x^3)^2]}\ ;$$

$$\Rightarrow\quad Q = 2\rho\,(dz + \alpha_{(k)})\ ;$$

Inspired by these metrics, we can consider Kähler metrics with

$$H = H(\rho)\ ;\quad W^2 = \psi(\rho)\,\Phi(x^1, x^3) \qquad \text{(factorized)}$$

$$\Rightarrow\quad \cancel{(\partial_1^2 + \partial_3^2) H} + \partial_\rho^2 (HW^2) = 0 \quad\Rightarrow\quad \partial_\rho^2(H\psi) = 0$$

$$H = \frac{(\alpha\rho + \beta)}{\psi}\ ;\qquad H = \rho^\varepsilon/\psi(\rho)\ ;\qquad \varepsilon = 0, 1 \quad \text{up to shifts of } \rho.$$

This defines a huge class of Kähler metrics.

We also assume now that $\hat{f} = \hat{f}(\rho)$ only

lunes, 19 de diciembre de 2016 5:09

Then, we consider the next equation

$$f^{-1} = \frac{1}{8g^2}\hat{\nabla}^2 \log W^2 = \frac{1}{8g^2 H}\bar{\nabla}^2(\log \psi + \log \Phi)$$

$$= \frac{1}{8g^2 H}\left[\frac{1}{W^2}\left(\partial_\rho (W^2 \partial_\rho + \partial_1^2 + \partial_3^2\right)(\log \psi + \log \Phi)\right]$$

$$= \frac{1}{8g^2 H\psi}\left[\partial_\rho\left(\psi \partial_\rho \log\psi\right) + \underbrace{\frac{(\partial_1^2+\partial_3^2)\log \Phi}{\Phi}}_{\text{constant ! } \Rightarrow \text{ Liouville's equation.}}\right]$$

$$\Rightarrow \boxed{\Phi = \Phi(z)}$$

The next equation is

$$gK = \partial_\rho \log W^2 = \partial_\rho \psi \quad \Rightarrow \text{determines } K \text{ in terms of } \psi.$$

The next is

$$\bar{\nabla}^2 M = -\sqrt{3}\, g L (g K L + 2\partial_\rho L)$$

Integrating once

$$\rightarrow M' = \frac{\alpha}{\psi} - \sqrt{3} g L^2 ;$$

and using this result to eliminate M from the next equation

$$\bar{\nabla}^2 L = 4H(gL)^2 - \frac{2}{3}L(gK)^2 - \frac{4}{3} g L K' - \frac{1}{3} g K L' + \frac{4}{\sqrt{3}} g H M'$$

we get

$$L'' + \frac{4}{3}L'\frac{\psi'}{\psi} + \frac{4}{3}L\frac{\psi''}{\psi} - \frac{2}{3}L\left(\frac{\psi'}{\psi}\right)^2 - \frac{4}{\sqrt{3}}\alpha g \frac{\rho^6}{\psi^2} = 0$$

which, together with the equation that defines L in terms of f and K, H

$$L = \frac{\psi}{8g^2\rho^6}\left\{-\frac{2k}{\psi} - \frac{2}{3}\left(\frac{\psi'}{\psi}\right)^2 + \frac{\psi''}{\psi}\right\}$$

gives the following 4th-order equation:

$$\varepsilon = 0 \rightarrow \quad -32\sqrt{3}\alpha g^3 + 4k(\psi')^2 - 8k\psi\psi'' + 3\psi^2\psi'''' = 0 ;$$

$$\varepsilon = 1 \rightarrow \quad 96\sqrt{3}\alpha g^3 \rho^4 + 4\rho(\psi')^2(-3k\rho + \psi') + 12\psi(2k\rho - \psi')(-\psi' + \rho\psi'')$$
$$+ 9\psi^2(4k - 2\psi'' + 2\rho\psi''' - \rho^2\psi'''') = 0$$

Simplest ansatz : Ψ polynomial in ϱ

For simplicity, we will only review the $\varepsilon = 1$ solutions

$$\Psi = \frac{1}{a}\left[c\varrho^3 + \varrho^2 + b\varrho + \frac{b^2}{3(1-ak)}\right]$$

$$\text{if} \quad \alpha = \frac{1 + 3ak + 3bc\left(\frac{3bc}{1-ak} - 2(1-2ak)\right)}{24\sqrt{3}\,a^3 g^3}$$

4 indep parameters a, b, c, k (one lost by normalization)

$$\Rightarrow \hat{f} = \frac{4a(g/\sqrt{3})^2 \varrho}{c\varrho + \frac{1-ak}{3}};$$

$$\boxed{\frac{4a}{c}\left(g/\sqrt{3}\right)^2 = 1 \rightarrow \text{normalization}}$$

$$H = \frac{a\varrho}{c\varrho^3 + \varrho^2 + b\varrho + \frac{b^2}{3(1-ak)}};$$

$$W^2 H = \varrho\, \Phi(k);$$

$$\Upsilon = \Upsilon(k);$$

$$\omega_2 = \underbrace{\alpha}_{=0 \text{ for asymptotically } AdS_5} + \frac{18c^2\varrho^3 + 18c(1-ak)\varrho^2 + [9bc + 3(1-ak)^2]\varrho + 2b(1-ak)}{16\sqrt{3}\,g^3 a^2 \varrho^2}$$

AdS_5 + rotating BHs with 3 different near-horizon geometries.