Introduction

The FGK equations are still hard to solve.
Even the flow equations are hard to solve.
In SUGRA theories, however, supersymmetric solutions are easy construct.
Can we use this fact in more general cases?

SUSY suggests us to use a new set of variables transforming linearly under duality (unlike U, φ^i)

Black holes from unbroken supersymmetry

A bosonic ($\phi^f = 0$) solution of a SUSY theory is supersymmetric if

$$\left.\begin{aligned} \delta_\varepsilon \phi^b &\sim \cancel{\bar{\varepsilon}\phi^f} = 0 \\ \delta_\varepsilon \phi^f &\sim \underbrace{\partial\varepsilon + \phi^b \varepsilon = 0}_{\text{Killing spinor equations KSE}} \end{aligned}\right\} \text{ for some } \varepsilon \neq 0$$

automatic

Supersymmetric solutions enjoy remarkable properties.
They can be identified, characterized, classified and constructed by analizing the KSE under the assumption that a solution $\varepsilon \neq 0$ exists. (Tod)

Ungauged N=2,d=4 SUGRA

All the supersymmetric solutions that can correspond to static BHs (necessarily extremal!) can be constructed with this recipe:

1. Let $\mathcal{V}^M(z,\bar{z})$ be the canonical symplectic section defining the theory.
   Introduce $X(z,\bar{z})$ with the same Kähler weight so $\mathcal{V}^M/X$ is Kähler-invariant (only Sp)

Define
$$\boxed{R^M + i I^M \equiv \mathcal{V}^M/X}$$

② R^M and I^M are not independent

$\Rightarrow$ Solve $\boxed{R^M = R^M(I)}$ Stabilisation eqs. Freudenthal duality eqs

(Hard and model-dependent)

$$I^M \xrightarrow{\text{Freudenthal duality}} R^M(I) \equiv \tilde{I}^M$$

$$\rightarrow \tilde{\tilde{I}}^M = -I^M \quad \text{(anti-involution)}$$

Define $\boxed{W(I) \equiv \tilde{I}_M(I)\, I^M}$ (Hesse potential) Homogeneous of 2^{nd} degree on I^M

$$\Downarrow$$

$$\tilde{I}_M = \frac{1}{2}\frac{\partial W}{\partial I^M}$$

All we need to know about the theory is $W(I)$

③ In the SUSY solutions related to BHs the components of I^M are functions harmonic on $\mathbb{E}^3$:

$$\boxed{I^M = H^M(x)\,; \quad \partial_{\underline{m}}\partial_{\underline{m}} H^M = 0\,; \quad m = 1,2,3}$$

For static BHs

$$\boxed{H_M \partial_{\underline{m}} H^M = 0\,;}$$

A choice of $H^M(x)$ determines completely a solution!

Now we just have to construct the physical fields in terms of the H-variables:

4 The metric is $ds^2 = e^{2U}dt^2 - e^{-2U}d\vec{x}^{\,2}$

with $\boxed{e^{-2U} = W(H)}$ (Difficult to guess from FGK eqs)

5 The scalars are given by

$$\boxed{Z^i = \frac{\tilde{H}^i + iH^i}{\tilde{H}^0 + iH^0}};$$

6 The vector field strengths are given by

$$\boxed{F^M = -\frac{1}{\sqrt{2}}\, d\left(\tilde{H}^M e^{2U}\right)\wedge dt - \frac{1}{\sqrt{2}}\, e^{2U} \star \left(dt \wedge dH^M\right)}$$

Observe that, under Freudenthal duality

$$H^M \longrightarrow \tilde{H}^M$$

$$\tilde{H}^M \longrightarrow \tilde{\tilde{H}}^M = -H^M$$

$$W(H) = \tilde{H}_M H^M \longrightarrow -H_M \tilde{H}^M = +\tilde{H}_M H^M = W(H)$$

$$z^i = \frac{\tilde{H}^i + iH^i}{\tilde{H}^0 + iH^0} \longrightarrow \frac{-H^i + i\tilde{H}^i}{-\tilde{H}^0 + iH^0} = \frac{i(\tilde{H}^i + iH^i)}{i(\tilde{H}^0 + iH^0)} = z^i$$

But $\mathcal{F}^M$ is not invariant

$$Q^M \longrightarrow \tilde{Q}^M$$

$\Rightarrow$ not a susy solution.

Still a solution?

For single BHs $\boxed{H^M = A^M + \frac{B^M}{r}}$

and $\boxed{B^M = Q^M/\sqrt{2}}$

In the $r \longrightarrow 0$ limit

$$e^{-2U} = W(H) \sim W\left(\frac{Q/\sqrt{2}}{r}\right) \sim \frac{1}{2r^2} W(Q)$$

$$e^{-2U} r^2 d\Omega^2_{(2)} \sim \frac{W(Q)}{2} d\Omega^2_{(2)} \qquad \frac{W(Q)}{2} = S/\pi$$

These H-variables (linear under duality) look very useful.

Can they be used in ~~SUSY~~ cases?

New supersymmetry-inspired variables

We would like to replace the $2n+1$ variables U, Z^i of the FGK formalism by the $2n+2$ H-variables H^M: H-FGK formalism.

The resulting theory must have 1 local symmetry. We are going to see that it is related to

"Freudenthal duality"

The H-FGK formalism for the black holes of N=2,d=4 SUGRA

The details of the change of variables are complicated.

Some helpful formulae

$$e^{-2U} = W(H);$$

$$\partial_M W \equiv \frac{\partial W}{\partial H^M} = 2\tilde{H}_M;$$

$$\partial^M W \equiv \frac{\partial W}{\partial \tilde{H}_M} = 2H^M;$$

$$\mathcal{M}_{MN}(\mathcal{N}) = \frac{1}{2} W \partial_M \partial_N W + 2 W^{-1} H_M H_N;$$

In the end, the FGK effective action becomes

$$\boxed{\begin{aligned} S_{H\text{-}FGK}[H] &= \int d\rho \left\{ \tfrac{1}{2} g_{MN} \dot{H}^M \dot{H}^N - V(H) \right\} \\ g_{MN}(H) &\equiv \partial_M \partial_N \log W - 2\frac{H_M H_N}{W^2}\,; \\ V(H) &= \left\{ -\tfrac{1}{4}\partial_M \partial_N \log W + \frac{H_M H_N}{W^2} \right\} Q^M Q^N \\ &= -V_{bh}/W\,; \end{aligned}}$$

And the Hamiltonian constraint

$$\boxed{\tfrac{1}{2} g_{MN} \dot{H}^M \dot{H}^N + V = -r_0^2}$$

The equations of motion (after massage) are

$$\boxed{\tilde{H}_M \left(\ddot{H}^M - r_0^2 H^M\right) + \frac{\left(\dot{H}^M H_M\right)^2}{W} = 0}$$

Simple SUSY-like solutions are given by

$$\boxed{\begin{aligned} &\dot{H}^M H_M = 0\,; \\ &\ddot{H}^M - r_0^2 H^M = 0\,; \end{aligned}}$$

$\xrightarrow{r_0 = 0}$ harmonic $H^M = A^M - B^M \rho$

$\xrightarrow{r_0 \neq 0}$ hyperbolic $H^M = A^M \cosh r_0 \rho - B^M \dfrac{\sinh r_0 \rho}{r_0}$

("Conventional")

In some theories there are solutions which require $\dot{H}^M H_M \neq 0$.

The FGK theorems can be recast in this language. In particular at the horizon of an extremal BH

$$\boxed{\partial_M V_{bh}(H)\Big|_{H^M_h} = 0}$$

Attractor in H-variables

$$z^i_h = \frac{\tilde{H}^i_h + i H^i_h}{\tilde{H}^0_h + i H_h} \; ; \qquad \boxed{\frac{S}{\pi} = \frac{W(H_h)}{2}}$$

For SUSY BHs $\quad H^M_h = -Q^M/\sqrt{2}\,;$

$$\longrightarrow z^i_h = \frac{\tilde{Q}^i + i p^i}{\tilde{Q}^0 + i p^0}\,;$$

Freudenthal duality

This duality appears in two ways:

1. If B^M is an attractor $\partial_M V_{bh}\big|_B = 0$
then $\tilde{B}^M$ is also an attractor $\partial_M V_{bh}\big|_{\tilde{B}} = 0$

$$\boxed{S(B) = \frac{\pi}{2} W(B) = \frac{\pi}{2} W(\tilde{B}) = S(\tilde{B})}$$

2. The metric $g_{MN}(H)$ is not invertible:
it admits a null eigenvector: $\tilde{H}^M$

$$\tilde{H}^M g_{MN} = 0 \,;$$

More: $\boxed{\tilde{H}^M \frac{\delta S_{FGK}}{\delta H^M} = 0}$ off-shell!!

This is a gauge or Noether identity associated to a local symmetry:
Calling $f(\rho)$ the local parameter

$$\delta_f S_{HFGK} = \int d\rho\, \delta_f H^M \frac{\delta S_{HFGK}}{\delta H^M} = 0,$$

where

$$\delta_f H^M \equiv f \tilde{H}^M;$$

The finite transformations are

$$\left(\tilde{H}^M + i H^M\right)' = e^{if(\rho)}\left(\tilde{H}^M + i H^M\right)$$

$$\Rightarrow \left(\mathcal{V}^M/X\right)' = e^{if(\rho)}\, \mathcal{V}^M/X \Rightarrow \boxed{X' = e^{if(\rho)} X}$$

$$f(\beta) = \pi/2 \Rightarrow \begin{cases} H^{M'} = \tilde{H}^M ; \\ \tilde{H}^{M'} = -H^M ; \end{cases}$$

Discrete Freudenthal duality

The H-FGK action is Freudenthal invariant.

Meaning?