Introduction

In these lectures we will study the construction of families of **black-hole** and **black-brane** solutions in general theories building on the work by Ferrara, Gibbons & Kallosh (1997).

Restricted to

STATIC

SPHERICALLY-SYMMETRIC (1)

FLAT (BRANES)

UNGAUGED (THEORIES) ($V(\phi)=0$)

ASYMPTOTICALLY FLAT

$d \geq 4$

Plan

1^{st} Lecture : $d=4$, BHs $(p=0)$

2^{nd} Lecture : $d \geq 4$, branes $(p \geq 0)$

3^{rd} Lecture : H-FGK formalism

The d=4 theories

We consider generic $d=4$ theories with

$g_{\mu\nu}(x) \longrightarrow$ spacetime metric

$\varphi^i(x) \longrightarrow$ scalar fields

$A^\Lambda{}_\mu(x) \longrightarrow$ vector fields $\quad \Lambda, \Sigma, \ldots = 1, \ldots, \bar{n}$

$\phi^i \longrightarrow \mathcal{N}_{\Lambda\Sigma}(\varphi) = \mathcal{N}_{\Sigma\Lambda}$: "period matrix"

$\phi^i \longrightarrow \mathcal{G}_{ij}(\varphi)$: metric in scalar manifold

$$A^\Lambda{}_\mu \longrightarrow F^\Lambda{}_{\mu\nu} = \partial_\mu A^\Lambda{}_\nu - \partial_\nu A^\Lambda{}_\mu \equiv 2\partial_{[\mu} A^\Lambda{}_{\nu]} ;$$

(Abelian)

Generic action $S[g_{\mu\nu}, \phi^i, A^\Lambda{}_\mu]$

$$S = \int d^4x \sqrt{|g|}\Big\{R(g) + \mathcal{G}_{ij}\partial_\mu\varphi^i\partial^\mu\varphi^j$$

$$+ 2\,\underbrace{\Im\mathrm{m}\,\mathcal{N}_{\Lambda\Sigma}}_{\text{negative-definite}}\,F^{\Lambda\mu\nu}F^{\Sigma}{}_{\mu\nu} - 2\,\Re\mathrm{e}\,\mathcal{N}_{\Lambda\Sigma}\,F^{\Lambda\mu\nu}\star F^{\Sigma}{}_{\mu\nu}\Big\}$$

No scalar potential $V(\varphi)$ | Typical of $\mathcal{N} \geq 2$

No gauged symmetries | **ungauged SUGRAS**

We want to find static, spherically-symmetric black-hole solutions $\Rightarrow$ e.o.m.?

It is convenient to start with the vectors:
S only depends on $A^\Lambda{}_\mu$ trough $F^\Lambda{}_{\mu\nu}$

$$\Rightarrow \frac{\delta S}{\delta A^\Lambda{}_\mu} = -\partial_\nu \frac{\delta S}{\delta F^\Lambda{}_{\nu\mu}} = -4\partial_\nu\left(\sqrt{|g|} * G_\Lambda{}^{\nu\mu}\right) = -4\sqrt{|g|}\, \boxed{\nabla_\nu * G_\Lambda{}^{\nu\mu}}$$

$$\frac{\delta S}{\delta F^\Lambda{}_{\nu\mu}} \equiv 4\sqrt{|g|} * G_\Lambda{}^{\nu\mu}$$

$$\frac{\delta S}{\delta F^\Lambda{}_{\nu\mu}} = 4\sqrt{|g|}\left[\Im m\, \omega_{\Lambda\Sigma} F^{\Sigma\,\nu\mu} - \Re e\, \omega_{\Lambda\Sigma} * F^{\Sigma\,\nu\mu}\right]$$

$$\Rightarrow \boxed{G_{\Lambda\,\nu\mu} = \Im m\, \omega_{\Lambda\Sigma} * F^{\Sigma\,\nu\mu} + \Re e\, \omega_{\Lambda\Sigma} F^{\Sigma\,\nu\mu}}$$

or $$\boxed{G_\Lambda{}^{+\,\nu\mu} = \overline{\omega}_{\Lambda\Sigma}\, F^{\Sigma +}}$$ linear, twisted self-duality constraint

Bianchi identities $\quad d F^\Lambda = 0 \quad \longrightarrow \quad \boxed{\nabla_\nu * F^{\Lambda\nu\mu} = 0}$

Maxwell equations $\quad d G_\Lambda = 0 \quad \longleftarrow \quad \boxed{\nabla_\nu * G_\Lambda{}^{\nu\mu} = 0}$

Define $\quad \left(\mathcal{F}^M \right) \equiv \begin{pmatrix} F^\Lambda \\ G_\Lambda \end{pmatrix} \quad \Rightarrow \quad \boxed{d \mathcal{F}^M = 0}$

Define $\quad \left(\Omega^{MN} \right) = \left(\Omega_{MN} \right) \equiv \begin{pmatrix} & \mathbb{1} \\ -\mathbb{1} & \end{pmatrix}$

Define $\quad \left(\mathcal{M}_{MN} \right) = \begin{pmatrix} I_{\Lambda\Sigma} + R_{\Lambda\Omega} I^{\Omega\Delta} R_{\Delta\Sigma} & -R_{\Lambda\rho} I^{\rho\Sigma} \\ -I^{\Lambda\rho} R_{\rho\Sigma} & I^{\Lambda\Sigma} \end{pmatrix}$

$\mathcal{N}_{\Lambda\Sigma} = R_{\Lambda\Sigma} + i I_{\Lambda\Sigma}$;

$I^{\Lambda\rho} I_{\rho\Sigma} = \delta^\Lambda{}_\Sigma$;

$$\mathcal{M}\, \Omega\, \mathcal{M} = \Omega \quad \text{(symplectic)}$$

The linear, twisted, self duality constraint is

$$\boxed{*\mathcal{F}^M = \Omega^{MP}\mathcal{M}_{PN}\mathcal{F}^N}$$

Maxwell + Bianchi invariant under

$$\mathcal{F}^M \longrightarrow \mathcal{F}'^M = S^M{}_N \mathcal{F}^N \; ; \quad S \in GL$$

The above constraint must be invariant:

$$*\mathcal{F}' = \Omega\mathcal{M}'\mathcal{F}' \; ;$$

$$S * \mathcal{F} = \Omega\mathcal{M}' S \mathcal{F} \; ; \quad *\mathcal{F} = \underbrace{S^{-1}\Omega\mathcal{M}'S}_{\Omega\mathcal{M}}\mathcal{F}$$

(EXERCISE)

$$\Rightarrow \boxed{\mathcal{M}' = \kappa S\mathcal{M}S^{-1} \; , \quad S\Omega S^{-1} = \kappa\Omega \; ; \; \kappa^2 = +1}$$

The functional form of $\mathcal{M}(\varphi)$ must not change:

$$\mathcal{M}'(\varphi) = \kappa S \mathcal{M}(\varphi) S^{-1} = \mathcal{M}(\varphi'(\varphi))$$

for some transformation $\varphi'^{i}(\varphi)$

$\varphi'^{i}(\varphi)$ must be an isometry of $G_{ij}(\varphi)$

for the other equations of motion to be invariant.

The Einstein equations are

$$\frac{\delta S}{\delta g^{\mu\nu}} = \sqrt{|g|}\Big\{ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{ij}\Big[\partial_\mu \varphi^i \partial_\nu \varphi^j - \frac{1}{2} g_{\mu\nu} (\partial\varphi)^2\Big] + 4\underbrace{I_{\Lambda\Sigma}\Big[F^\Lambda{}_\mu{}^\rho F^\Sigma{}_{\nu\rho} - \frac{1}{4} g_{\mu\nu} F^{\Lambda\rho\sigma} F^\Sigma{}_{\rho\sigma}\Big]}_{\substack{\parallel \\ \frac{1}{2}\mathcal{M}_{MN}\mathcal{F}^M{}_\mu{}^\rho \mathcal{F}^N{}_{\nu\rho} \\ \parallel \\ \frac{1}{2}\Omega_{MN}\mathcal{F}^M{}_\mu{}^\rho \star\mathcal{F}^N{}_{\nu\rho}}}\Big\}$$

(EXERCISE)

invariant if $\varphi'^i(\varphi)$ is an isometry.

$\Rightarrow$ Invariant only for $\kappa = +1$ (symplectic transf.)

Finally, the equations of motion of the scalars

$$\frac{\delta S}{\delta \varphi^j} = \sqrt{|g|}\, g_{ji} \Big\{ \nabla^2 \varphi^i + \Gamma_{jk}{}^{i}\, \partial^\mu \varphi^j \partial_\mu \varphi^k$$

$$- 2\, \partial^j G_\Lambda{}^{\mu\nu} \, {*F^\Lambda}_{\mu\nu} \Big\} \equiv \sqrt{|g|}\, g_{ji}\, \mathcal{E}^i$$

Under the combined infinitesimal transformations

$$\delta \varphi^i = k^i(\varphi) \; ; \qquad \nabla_{(i} k_{j)} = 0$$

$$\delta \mathcal{F}^M = T^M{}_N \mathcal{F}^N \; ; \qquad T^P{}_{[N} \Omega_{M]P} = 0 \text{ (symplectic)}$$

assuming

$$k^i \partial_i \mathcal{M}_{MN} = -2\, T^P{}_{(M} \mathcal{M}_{N)P} \; ;$$

$$\boxed{\delta \mathcal{E}^i = \partial_j k^i\, \mathcal{E}^j}$$ (EXERCISE)

The equations of motion of our generic theory are invariant under isometries $\varphi'^i(\varphi)$ of $g_{ij}(\varphi)$ such that

$$\boxed{\begin{gathered} \mathcal{M}(\varphi') = S\,\mathcal{M}(\varphi)\,S^{-1} \\ \text{with} \\ S\,\Omega\,S^{-1} = \Omega \end{gathered}}$$

Gaillard & Zumino 1981

$\oplus$ Isometries of scalars not coupled to vectors!

The black-hole metric

→ single black hole

A general static, spherically symmetric $d=4$ metric has only 2 independent functions of r:

$$ds^2 = f(r)\,dt^2 - g(r)\,dr^2 - r^2\,d\Omega^2_{(2)}\,;$$

A black-hole metric has only one $e^{2U(\rho)}$ (Gibbons)

$$\boxed{\begin{aligned} ds^2 &= e^{2U}\,dt^2 - e^{-2U}\Big[\underbrace{f^4(\rho)\,d\rho^2 + f^2(\rho)\,d\Omega^2_{(2)}}\Big] \\ f(\rho) &\equiv r_0/\sinh(r_0\rho) \end{aligned}}$$

radial coordinate $[L^{-1}]$

non-extremality parameter $[L]$

$$\boxed{\gamma_{\underline{mn}}\,dx^m dx^n}$$

Examples: Schwarschild

Schwarzschild: $\begin{cases} e^{2U} = e^{2M\rho} \\ r_0 = M \end{cases}$ mass

$$e^{2U} \xrightarrow[\rho \to 0]{} 1 \quad \left(\begin{array}{c}\text{asymptotic} \\ \text{flatness}\end{array}\right) \; ; \qquad e^{2U} \underset{\rho \to 0}{\sim} 1 + 2M\rho$$

$$e^{2U} \xrightarrow[\rho \to -\infty]{} 0 \quad \left(\begin{array}{c}\text{event} \\ \text{horizon}\end{array}\right)$$

missing yesterday

$$ds^2 \sim e^{2M\rho}\, dt^2 - e^{-2M\rho}\left[(2M e^{M\rho})^4 d\rho^2 + (2M e^{M\rho})^2 d\Omega^2_{(2)} \right]$$

Sin título

$$ds^2 \sim e^{2M\rho}\left(dt^2 - (2M)^4 d\rho^2\right) - (2M)^2 d\Omega^2_{(2)}$$

$$x = (2M)^2 \rho \qquad \sim \underbrace{e^{x/(2M)}\left(dt^2 - dx^2\right)}_{\text{Rindler space}} - \underbrace{(2M)^2 d\Omega^2_{(2)}}_{\text{2-sphere radius } (2M)}$$

acceleration $\dfrac{1}{4M} = \kappa$

$$T = \frac{\kappa}{2\pi} = \frac{1}{8\pi M} \,;$$

$$S = \frac{A}{4} = \frac{\cancel{4}\pi(2M)^2}{\cancel{4}}$$

Examples: Reissner-Nordstrom

Reissner-Nordström:
$$\begin{cases} e^{2U} = \left(\cosh r_0\rho - M\dfrac{\sinh r_0\rho}{r_0}\right)^{-2} \\ r_0^2 = M^2 - Q^2 \end{cases}$$

charge

mass

$$e^{2U} \xrightarrow[\rho\to 0]{} 1 \quad \left(\text{asymptotic flatness}\right) \; ; \quad e^{2U} \underset{\rho\to 0}{\sim} 1 + 2M\rho$$

$$e^{2U} \underset{\rho\to -\infty}{\sim} e^{2r_0\rho}\left(1+\frac{M}{r_0}\right)^{-2} = e^{2\log\frac{r_0}{r_+} + 2r_0\rho}$$

$$\log\frac{r_0}{r_+} = C$$

$$ds^2 \sim e^{2(C+r_0\rho)}dt^2 - e^{-2(C+r_0\rho)}\Big[\underbrace{(r_0 e^{r_0\rho})^4 d\rho^2 + (r_0 e^{r_0\rho})^2 d\Omega_{(2)}^2}_{\text{missing factors}}\Big]$$

$$ds^2 \sim e^{2(C+r_0\rho)}\left[dt^2-(2r_0)^4e^{-4C}d\rho^2\right]-e^{-2C}(2r_0)^2\,d\Omega^2_{(2)}$$

$$\sim \underbrace{e^{\frac{\cancel{2}x\,e^{2C}}{2r_0}}\left[dt^2-dx^2\right]}_{\text{Rindler space with}}-\underbrace{e^{-2C}(2r_0)^2\,d\Omega^2_{(2)}}_{\text{2-sphere radius } e^{-C}r_0}$$

$$x=(2r_0)^2e^{-2C}\left(\rho+\frac{C}{r_0}\right)$$

Rindler space with

$$a=\frac{e^{2C}}{4r_0}=\kappa$$

$$T=\frac{\kappa}{2\pi}=\frac{e^{2\log\frac{r_0}{r_+}}}{8\pi r_0}$$

$$=\frac{r_0}{8\pi(r_+)^2}\xrightarrow[r_0\to 0]{}0\ ;$$

2-sphere radius $e^{-C}r_0$

$$S=\frac{\cancel{4}\pi\left(e^{-C}2r_0\right)^2}{\cancel{4}}$$

$$=4\pi r_0^2\,e^{-2\log\frac{r_0}{r_+}}$$

$$=4\pi r_+^2\xrightarrow[r_0\to 0]{}4\pi M^2$$

$$=4\pi Q^2$$

In general, if $e^{U} \underset{\rho \to -\infty}{\sim} e^{C + r_0 \rho}$

$T = \dfrac{e^{2C}}{8\pi r_0}$; $S = 4\pi r_0^2 \, e^{-2C}$;

$\Rightarrow \boxed{2\, T S = r_0}$ If S remains finite when $r_0 \to 0$ then $T \to 0$

The same metric covers the interior of the inner horizon which lies at $\rho \to +\infty$

$$e^{2U} \underset{\rho \to +\infty}{\sim} e^{2 r_0 \rho} \left(1 - \frac{M}{r_0}\right)^{-2} = e^{2 \log \frac{r_0}{r_-} + 2 r_0 \rho}$$

$\log \frac{r_0}{r_-} \equiv C_-$

$T_- = \dfrac{e^{2C_-}}{8\pi r_0}$; $S_- = 4\pi r_0^2 \, e^{-2C_-}$; $\boxed{2\, T_\pm S_\pm = r_0}$

For the Reissner-Nordström BH

$$S_+ S_- = \left(4\pi r_+^2\right)\left(4\pi r_-^2\right) = \pi^2\left(M^2 - r_0^2\right)^2$$

$$= 16\pi^2 Q^4 = \left(S_{\text{extremal}}\right)^2$$

(Goldstein, Jejjala & Nampuri 1410.3478 in general)

Extremal limit

This is the $T \to 0$ limit $\Rightarrow r_0 = 2ST \to 0$
S finite

In that limit

$$f(\rho) = \frac{r_0}{\sinh r_0 \rho} \sim \frac{1}{\rho} \, ;$$

$$\underline{\gamma_{mn}} \, dx^m dx^n \sim \left(\frac{d\rho}{\rho^2}\right)^2 + \frac{1}{\rho^2} d\Omega^2_{(2)}$$

$\rho = -1/r \longrightarrow \quad \sim \quad dr^2 + r^2 d\Omega^2_{(2)} \, ;$
(The Euclidean metric in $\mathbb{R}^3$)

Ex

The Schwarzschild BH does not have an extremal limit:

$T_{\text{Schwarzschild}} \longrightarrow \infty$

$S_{\text{Schwarzschild}} \longrightarrow 0$

$r_0 \longrightarrow 0$

The Reissner-Nordström BH does have an extremal limit:

$$T_{RN} = \frac{r_0}{8\pi r_+^2} \longrightarrow 0$$

$$S_{RN} = 4\pi r_+^2 \longrightarrow \pi Q^2$$

harmonic function in $\mathbb{R}^3$

$$e^{2U} = \left(\cosh r_0\rho - M\frac{\sinh r_0\rho}{r_0}\right)^{-2} \longrightarrow \left(1 - \frac{M}{r_0}\rho\right)^{-2}$$

Reduction and effective action

To find solutions it is necessary to make an Ansatz for the matter fields.

(i) The metric be the static, spherically-symmetric one studied before ($g_{\mu\nu} \longrightarrow \boxed{r_0\,,\ U(\rho)\,,\ \gamma_{mn}(\rho)}$)

(ii) The scalars only depend on ρ: $\boxed{\varphi^i = \varphi^i(\rho)}$

(iii) The vector fields $A^\Lambda{}_\mu$ are indirectly given by $\boxed{A^M{}_t = \psi^M(\rho)} \Rightarrow \boxed{F^M{}_{\rho t} = \dot\psi^M}$

The complete $F^M{}_{\mu\nu}$ can be reconstructed from $F^M{}_{\rho t}$ using the self-duality constraint.

The reduction will be performed in 2 stages

1. Keeping the 3-d metric $\gamma_{\underline{mn}}$ unspecified
   $\rightarrow$ we obtain a 3-d effective action

2. Specifying $\gamma_{\underline{mn}} dx^{\underline{m}} dx^{\underline{n}} = f^4(\rho) d\rho^2 + f^2(\rho) d\Omega^2_{(2)}$

$$f(\rho) = \frac{r_0}{\sinh(r_0 \rho)}$$

First stage

$$*F^M = \Omega^{MN} M_{NP} F^P \Rightarrow F_M = M_{MN} * F^N$$

$$\longrightarrow \quad F_M \text{ of } \varphi \qquad \text{(EXERCISE)}$$

$\gamma_{mn}(g)$

$$\Rightarrow \nabla_\nu \left(* F_M{}^{\nu\mu} \right) = 0 \longrightarrow \boxed{\nabla_{m} \left(e^{-2U} M_{MN} \partial^{m} \psi^N \right) = 0}$$

compare

The 4 equations in $d=4$ become 1 in $d=3$.

The scalar equations reduce to

$$\boxed{\nabla_{m} \left(G_{ij} \partial^{m} \varphi^j \right) - \frac{1}{2} \partial_i G_{jk} \, \partial^{m} \varphi^j \partial_{m} \varphi^k - \frac{1}{2} \partial_i \left(4 e^{-2U} M_{MN} \right) \partial^{m} \psi^M \partial_{m} \psi^N = 0}$$

compare

The Einstein equations split into 3 sets

$\mu\nu = 00 \rightarrow$ $$\boxed{R + 2(\partial U)^2 - 4\nabla^2 U + g_{ij}\partial^{\underline{m}}\varphi^i\partial_{\underline{m}}\varphi^j - 4e^{-2U}M_{MN}\partial^{\underline{m}}\psi^M\partial_{\underline{m}}\psi^N = 0\,;}$$

$\mu\nu = 0m \rightarrow$ $$\boxed{\partial_{[\underline{m}}\psi^\Lambda\partial_{\underline{n}]}\psi_\Lambda = 0\,;}$$

$\mu\nu = mn \rightarrow$ $$\boxed{\begin{aligned} &G_{mn} + 2\left[\partial_m U\partial_n U - \tfrac{1}{2}\delta_{mn}(\partial U)^2\right] \\ &\qquad + g_{ij}\left[\partial_m\varphi^i\partial_n\varphi^j - \tfrac{1}{2}\delta_{mn}\partial^r\varphi^i\partial_r\varphi^j\right] \\ &+ 4e^{-2U}M_{MN}\left[\partial_m\psi^M\partial_n\psi^N - \tfrac{1}{2}\delta_{mn}\partial^r\psi^M\partial_r\psi^N\right] \\ &= 0\,; \end{aligned}}$$

All these equations (except $\partial_{[\mu}\psi^\Lambda \partial_{\nu]}\psi_\Lambda = 0$) can be derived from the effective action

$$\boxed{S[\gamma,\phi] = \int d^3x \sqrt{|\gamma|}\left[R(\gamma) + G_{AB}(\phi)\,\partial^\mu\phi^A\,\partial_\mu\phi^B\right]}$$

$$(\phi^A) = \begin{pmatrix} U \\ \varphi^i \\ \psi^M \end{pmatrix} \quad \text{and} \quad (G_{AB}) = \left(\begin{array}{c|c|c} 2 & & \\ \hline & G_{ij} & \\ \hline & & 4e^{-2U}\mathcal{M}_{MN} \end{array}\right)$$

(Gibbons, Breitenlohner & Maison 1988)

Second stage

We substitute γ_{mn} into the equations of motion (not into the action: beware of Kaluza's error!!)

Using $R_{\rho\rho} = -2r_0^2$ (only $\neq 0$) we get two equations:

$$\boxed{\begin{aligned} &\frac{d}{d\rho}\left(g_{AB}\dot{\phi}^B\right) - \frac{1}{2}\partial_A g_{BC}\dot{\phi}^B\dot{\phi}^C = 0; \\ &g_{AB}\dot{\phi}^A\dot{\phi}^B - 2r_0^2 = 0; \end{aligned}}$$

Geodesic equation

Constraint

$r_0 = 0 \rightarrow$ extremal $\rightarrow$ null

$r_0 \neq 0 \rightarrow$ non-extremal $\rightarrow$ timelike

$G_{AB}(\phi)$ does not depend on the potentials ψ^M

$\rightarrow$ conserved momenta $\left(Q^M\right) = \begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix}$ magnetic / electric

$$\frac{d}{d\rho}\left(G_{MN}\dot{\psi}^N\right) = 0 \rightarrow G_{MN}\dot{\psi}^N = 4e^{-2U}\mathcal{M}_{MN}\dot{\psi}^N = Q_M$$

Eliminating $\dot{\psi}^M$ in the e.o.m. we get

$$\boxed{\begin{aligned} &\ddot{U} + e^{2U} V_{bh} = 0 \\ &\frac{d}{d\rho}\left(G_{ij}\dot{\varphi}^j\right) - \frac{1}{2}\partial_i G_{jk}\dot{\varphi}^j\dot{\varphi}^k + e^{2U}\partial_i V_{bh} = 0, \\ &\dot{U}^2 + \frac{1}{2}G_{ij}\dot{\varphi}^i\dot{\varphi}^j + e^{2U}V_{bh} = r_0^2, \end{aligned}}$$

equations of motion

"Hamiltonian constraint"

Where

$$\boxed{-V_{bh}(Q,\varphi) \equiv -\frac{1}{2} Q^M \mathcal{M}_{MN} Q^N ;}$$

Black-hole potential (FGK)

The first two equations can be derived from the effective action

$$\boxed{S_{FGK} = \int d\rho \left\{ \dot{U}^2 + \frac{1}{2} g_{ij} \dot{\varphi}^i \dot{\varphi}^j - e^{2U} V_{bh} \right\}}$$

No explicit dependence on $\rho \Rightarrow H$ is conserved

The third equation is

$$\boxed{H = r_0^2}$$

The FGK theorems: attractor mechanism

The **FGK** system can describe all static, spherically-symmetric, charged BHs.

$\Rightarrow$ We can derive general results

1 **Extremal case** $r_0 = 0$; $S \neq 0$

In the near-horizon limit $\rho \to -\infty$

$$e^{-2U} \sim S/\pi \,\rho^2 \rightarrow ds^2 \sim \underbrace{\frac{\pi}{S}\frac{dt^2}{\rho^2} - \frac{S}{\pi}\frac{d\rho^2}{\rho^2}}_{AdS_2 \text{ with radius } \sqrt{\pi/S}} - \underbrace{\frac{S}{\pi}\, d\Omega^2_{(2)}}_{S^2 \text{ with radius } \sqrt{\pi/S}}$$

Robinson-Bertotti solution

If $S \neq 0$ the scalars will be finite over the horizon.
We assume that

$$\lim_{\rho \to -\infty} g_{ij} \dot\varphi^i \dot\varphi^j e^{2U} \rho^4 = \lim_{\rho \to -\infty} \frac{\pi}{S} g_{ij} \dot\varphi^i \dot\varphi^j \rho^2 = \xi^2 < \infty$$

$$\boxed{e^U \sim \sqrt{\frac{\pi}{S}} \frac{1}{\rho}}$$

$\left(\dot\varphi^i \sim \mathcal{O}\left(\frac{1}{\rho}\right) \text{ or faster}\right)$

$$0 = \lim_{\rho \to -\infty} \frac{S^2}{\pi} e^{2U} \rho^4 \, H =$$

$$= \lim_{\rho \to -\infty} \frac{S^2}{\pi} e^{2U} \rho^4 \left[\dot U^2 + \frac{1}{2} g_{ij} \dot\varphi^i \dot\varphi^j + e^{2U} V_{bh}\right]$$

$$S + \frac{1}{2}\frac{S^3}{\pi^2}\xi^2 + \pi V_{bh}(\varphi_h, Q) = 0$$

$$\Rightarrow \boxed{S/\pi \leq -V_{bh}(\varphi_h, Q)}$$

The finiteness of the scalars on the horizon requires $\xi = 0$

$$\Rightarrow \qquad \boxed{\frac{S}{\pi} = -V_{bh}(\varphi_h, Q)}$$

We need the constants φ^i_h to use this result.

Take the FGK e.o.m. for the scalars

$$\lim_{\rho \to -\infty} \rho^2 \left\{ \ddot{\varphi}^i + \cancel{\Gamma_{jk}{}^i \dot{\varphi}^j \dot{\varphi}^k} + e^{2U} g^{ij} \partial_j V_{bh} \right\}$$

$$\downarrow$$

$$\frac{\pi}{S} g^{ij} \partial_j V_{bh} \Big|_{\varphi_h}$$

$$\Rightarrow \varphi^i \sim \frac{\pi}{S} g^{ij} \partial_j V_{bh} \Big|_{\varphi_h} \log(-\rho) + a\rho + \mathcal{O}(1/\rho)$$

Finiteness $\Rightarrow$ $a = 0$ and $\boxed{\partial_j V_{bh} \Big|_{\varphi_h} = 0}$

The possible φ_h^i are the critical points of V_{bh}

$\Rightarrow$ just an algebraic computation.

If there are no flat directions around the critical point

$$\boxed{\varphi_h^i = \varphi_h^i(Q)}$$

Independently of φ_∞^i

$\rightarrow$ **ATTRACTOR** from $\rho = 0$ to $\rho = -\infty$

Generically, though

$$\boxed{\varphi_h^i = \varphi_h^i(\alpha, Q)}$$

$\alpha = \alpha(\varphi_\infty)$ parametrize the flat directions

ALWAYS $\frac{S}{\pi} = -V_{bh}(\varphi_h(\alpha, Q), Q)$ independent of α! (Sen)

2 Non-extremal case

At spatial infinity ($\rho \longrightarrow 0^-$)

$$U \sim M\rho \qquad \& \qquad \varphi^i \sim \varphi^i_\infty + \Sigma^i \rho$$

$$H = r_0^2$$

$$\dot{U}^2 + \frac{1}{2} G_{ij} \dot{\varphi}^i \dot{\varphi}^j - e^{2U} V_{bh} = r_0^2$$

$$\boxed{M^2 + \frac{1}{2} G_{ij}(\varphi_\infty) \Sigma^i \Sigma^j - V_{bh}(\varphi_\infty, Q) = r_0^2} \geqslant 0$$

A general Bogomol'nyi-like bound

But useless: **no-hair** $\Rightarrow \Sigma^i = \Sigma^i(\varphi_\infty, M, Q)$ unknown!

Double-extremal black holes

These are extremal black holes with constant scalars

$$\Rightarrow \quad \varphi^i_\infty = \varphi^i_h(\alpha, Q)$$

and $\Sigma^i = 0$

$$\boxed{M^2 = -V_{bh}(\varphi_\infty, Q) = -V_{bh}(\varphi_h, Q) = S/\pi}$$

Given a critical point φ^i_h, the double-$\sqrt{\text{non!}}$extremal solution is the Reissner-Nordström solution

$$e^{-U} = \cosh r_0\rho - M\,\frac{\sinh r_0\rho}{r_0}\,;$$

with

$$r_0^2 = M^2 + V_{bh}(\varphi_h, Q)\,;$$

(Any theory!)

The FGK formalism in N=2, d=4 SUGRA

vector multiplets

To apply this formalism we just need

i) Bosonic field content (besides the metric) $\begin{cases} \text{Complex scalars } Z^i \\ \text{Vector fields } A^0_\mu, A^i_\mu = A^\Lambda_\mu \end{cases}$

ii) Couplings $\left\{ \begin{array}{l} \mathcal{G}_{ij}(\varphi) \to \mathcal{G}_{i\bar{j}}(Z, \bar{Z}) \\ W_{\Lambda\Sigma}(\varphi) \to W_{\Lambda\Sigma}(Z, \bar{Z}) \end{array} \right\}$ Special geometry

Hypermultiplets do not lead to black holes and we do not consider them.

(Not in $V_{bh}(\varphi, Q)$)

$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$ is a Special-Kähler metric

$w_{\Lambda\Sigma}(z, \bar{z})$ is such that

$$-V_{bh}(z, \bar{z}, Q) = +\frac{1}{2} Q^M \mathcal{M}_{MN} Q^N$$

$$= |Z|^2 + 4 g^{i\bar{j}} \partial_i |Z| \partial_{\bar{j}} |Z|$$

$$Z(z, \bar{z}, Q) \equiv Q^M V_M(z, \bar{z})$$ central charge

$$(V^M) = \begin{pmatrix} \mathcal{L}^\Lambda(z, \bar{z}) \\ \mathcal{M}_\Lambda(z, \bar{z}) \end{pmatrix}$$

canonical, covariantly holomorphic symplectic section

(defines the theory)

It can be shown that

$$\partial_i |Z| \Big|_{\hat{z}} = 0 \Rightarrow \partial_i V_{bh} \Big|_{\hat{z}} = 0 \Rightarrow \hat{z} = z_h$$

but not the other way around!

The critical points of $|Z|$ are the SUSY critical points of V_{bh} $\Rightarrow$ "SUSY ATTRACTORS"

Unique, $z^i_h = z^i_h(Q)$ (no moduli) 2

$$\boxed{\begin{aligned} S/\pi &= |Z_h| \\ M^2 &= |Z_\infty|^2 \end{aligned}}$$

Supersymmetric BHs

('Bogomol'nyi' or BPS "bound")

However

but there are more, associated to "fake central charges" $W(z, \bar{z}, Q)$ such that

$$\boxed{-V_{bh} = |W|^2 + 4 g^{i\bar{j}} \partial_i |W| \partial_{\bar{j}} |W|}$$

There is something special about this form of V_{bh}

Flow equations

If the BH potential can be written as

$$-V_{bh} = |W|^2 + 4 g^{i\bar{j}} \partial_i |W| \partial_{\bar{j}} |W|$$

then the FGK effective action can be written in a Bogomol'nyi - like form

$$S = \int d\rho \left\{ \left(\dot{U} \pm e^U |W| \right)^2 + g_{i\bar{j}} \left(\dot{z}^i \pm 2 e^{2U} \partial^i |W| \right) \left(\quad \right)^{\bar{j}} \right\}$$

up to total derivatives.

Then, S is extremised when the "flow equations" are satisfied

$$\boxed{\begin{aligned} \dot{U} &= \mp e^{U} |W| \\ \dot{z}^{i} &= \mp 2 e^{U} \partial^{i} |W| \end{aligned}}$$

They are not easy to integrate but they give general results:

$\rho \to 0^{-}$ $\quad\boxed{M = \mp |W_{\infty}|}\quad$ etc

The FGK formalism in N>2, d=4 SUGRA