Non-extremal black holes and branes of N=2, d=4,5 Supergravity

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Work done in collaboration with

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Plan of the Talk:

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In this talk I will present a general ansatz and a general formalism to construct non-extremal black-hole and black-brane solutions and we will study some examples. First, we will review some general results.

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We start by reviewing the FGK formalism for black holes in $d = 4$.

Later, we will generalize it to black branes in any dimension.

2 – FGK formalism for $d = 4$ black holes

Ferrara, Gibbons and Kallosh (1997) considered the general 4-dimensional action

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\begin{aligned}
I &= \int d^4x \sqrt{|g|}\,\{R + \mathcal{G}_{ij}(\phi)\partial_\mu \phi^i \partial^\mu \phi^j \\
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They also considered the general metric for any static non-extremal black hole

$$ds^2 = e^{2U(\tau)} dt^2 - e^{-2U(\tau)} \left[\frac{r_0{}^4}{\sinh^4 r_0\tau} d\tau^2 + \frac{r_0{}^2}{\sinh^2 r_0\tau} d\Omega^2_{(2)} \right] .$$

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- ☞ What is r_0 like for more general black holes?

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To determine completely the metric of any static, regular, spherically symmetric black hole we only need to find the function $U(\tau)$.

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The latter can be integrated out so they are effectively replaced by the electric, q_Λ, and magnetic, p^Λ charges. The general system reduces to an effective mechanical system with variables $U(\tau)$, $\phi^i(\tau)$:

$$I_{\rm eff}[U,\phi^i] = \int d\tau \left\{(U')^2 + \tfrac{1}{2}\mathcal{G}_{ij}\phi^{i\,\prime}\phi^{j\,\prime} - e^{2U}V_{\rm bh} + {r_0}^2\right\} \,,$$

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where FGK defined the black-hole potential

$$-V_{\rm bh}(\phi,q,p) \equiv -\tfrac{1}{2}\left(p^\Lambda \quad q_\Lambda\right) \begin{pmatrix} (I+RI^{-1}R)_{\Lambda\Sigma} & -(RI^{-1})_\Lambda{}^\Sigma \\ -(I^{-1}R)^\Lambda{}_\Sigma & (I^{-1})^{\Lambda\Sigma} \end{pmatrix} \begin{pmatrix} p^\Sigma \\ q_\Sigma \end{pmatrix} ,$$

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Finding a black hole with charges p, q is equivalent to solving the above mechanical system for $U(\tau)$, $\phi^i(\tau)$.

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Each critical point yields a possible extremal black-hole solution and an $AdS_2 \times S^2$ geometry. One can go a long way with the attractor only, ignoring the full explicit solution.

In the general case one can prove the following extremality bound:

$$r_0{}^2 = M^2 + \tfrac{1}{2}\mathcal{G}_{ij}(\phi_\infty)\Sigma^i\Sigma^j + V_{\rm bh}(\phi_\infty, q, p)\,, \geq 0\,,$$

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We need to find the complete explicit solution in the non-extremal case.

Whenever we can write $-\left[e^{2U}V_{\rm bh} - {r_0}^2\right] = (\partial_U Y)^2 + 2\,\mathcal{G}^{ij}\partial_i Y \partial_j Y$ for some *(generalized) superpotential* $Y(U, \phi^i, p, q, r_0)$, we can rewrite the effective action as

$$I_{\rm eff}[U, \phi^i] = \int d\tau \left\{(U' - \partial_U Y)^2 + \tfrac{1}{2}\mathcal{G}_{ij}(\phi^{i\,\prime} - 2\,\mathcal{G}^{ik}\partial_k Y)(\phi^{j\,\prime} - 2\,\mathcal{G}^{jl}\partial_l Y) + 2Y'\right\} \,.$$

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The action is minimized by configurations satisfying the first-order gradient flow equations (Miller, Schalm & Weinberg (2007), Janssen, Smyth, Van Riet & Vercnocke (2008), Perz, Smyth, Van Riet & Vercnocke (2008))

$$U' = \partial_U Y\,, \qquad \phi^{i\,\prime} = 2\,\mathcal{G}^{ij}\partial_j Y\,.$$

Whenever we can write $-\left[e^{2U}V_{\rm bh} - {r_0}^2\right] = (\partial_U Y)^2 + 2\,\mathcal{G}^{ij}\partial_i Y \partial_j Y$ for some *(generalized) superpotential* $Y(U, \phi^i, p, q, r_0)$, we can rewrite the effective action as

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A generalized superpotential $Y(U, \phi^i, p, q, r_0)$ exists in all theories whose scalar manifold (after timelike dimensional reduction) is a symmetric coset space (in particular for all $N > 2$ supergravities) (Andrianopoli, D'Auria, Orazi & Trigiante (2009), Chemissany, Fré, Rosseel, Sorin, Trigiante & Van Riet (2010)).

In the extremal case $r_0 = 0$, if there is a generalized superpotential $Y(U, \phi^i, p, q)$, it factorizes

$$Y(U, \phi^i, p, q) = e^U W(\phi^i, p, q)\,,$$

where $W(\phi^i, p, q)$ is called the *superpotential*, and the flow equations take the form (Ceresole & Dall'Agata (2007))

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The stationary values of the superpotential $\partial_i W|_{\phi_{\rm h}} = 0$ give the the entropy:

$$S = \pi |W(\phi_{\rm h}, p, q)|^2\,,$$

while the mass is

$$M = |W(\phi_\infty, p, q)|\,.$$

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We are going to review the example of (ungauged) $N = 2$ Supergravity coupled to vector multiplets.

4 − $N = 2, d = 4$ ungauged SUGRA coupled to vector multiplets

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The field content

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Hypermultiplets can be ignored for black-hole solutions.

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Local $N = 2$ supersymmetry requires the Kähler-Hodge manifold to be a special Kähler manifold, so it is the base space of a $2(n_V + 1)$-dimensional vector bundle with $Sp[2(n_V + 1), \mathbb{R}]$ structure group, on which we can define the constrained symplectic section

$$\mathcal{V} = \begin{pmatrix} \mathcal{L}^\Lambda(Z, Z^*) \\ \mathcal{M}_\Lambda(Z, Z^*) \end{pmatrix}\ .$$

$\mathcal{V}$ can be thought of as just a redundant description of the physical scalars with manifest symplectic symmetry, which also acts on the electric and magnetic charges:

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The action of the bosonic fields of the ungauged theory is of the general FGK form:

$$\begin{aligned} S &= \int d^4x \sqrt{|g|}\, \big[R + 2\mathcal{G}_{ij^*}\partial_\mu Z^i \partial^\mu Z^{*\,j^*} + 2\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma} F^{\Lambda\,\mu\nu} F^{\Sigma}{}_{\mu\nu} \\ &\qquad -2\Re\mathrm{e}\mathcal{N}_{\Lambda\Sigma} F^{\Lambda\,\mu\nu} \star F^{\Sigma}{}_{\mu\nu}\big]\,, \Rightarrow\; -V_{\mathrm{bh}} = |\mathcal{Z}|^2 + \mathcal{G}^{ij^*}\mathcal{D}_i\mathcal{Z}\mathcal{D}_{j^*}\mathcal{Z}^*\,. \end{aligned}$$

In order to find static extremal black holes one could try to integrate directly the equations of motion of the FGK formalism for the black-hole potential of $N = 2$ $d = 4$ theories:

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There is a recipe to construct all the BPS ones:
(Denef (2000), Behrndt, Lüst & Sabra (1997), Meessen, O. (2006))

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2. The components of $\mathcal{I}$ are given by a symplectic vector real functions harmonic in the 3-dimensional transverse space. For single black holes :

$$\begin{pmatrix} \mathcal{I}^{\Lambda} \\ \\ \mathcal{I}_{\Lambda} \end{pmatrix} = \begin{pmatrix} H^{\Lambda}(\tau) \\ \\ H_{\Lambda}(\tau) \end{pmatrix} = \begin{pmatrix} H^{\Lambda}{}_{\infty} - \frac{1}{\sqrt{2}} p^{\Lambda}\tau \\ \\ H_{\Lambda\,\infty} - \frac{1}{\sqrt{2}} q_{\Lambda}\tau \end{pmatrix},$$

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3. $\mathcal{R}$ is to be found from $\mathcal{I}$ by solving the generalized *stabilization equations*.

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3. $\mathcal{R}$ is to be found from $\mathcal{I}$ by solving the generalized *stabilization equations*.

4. The scalars Z^i are given by the quotients $Z^i = \dfrac{\mathcal{V}^i/X}{\mathcal{V}^0/X} = \dfrac{\mathcal{R}^i + i\mathcal{I}^i}{\mathcal{I}^0 + i\mathcal{I}^0}$.

1. For some complex X, define the Kähler-neutral, real, symplectic vectors $\mathcal{R}$ and $\mathcal{I}$

$$\mathcal{R} + i\mathcal{I} \equiv \mathcal{V}/X \, .$$

2. The components of $\mathcal{I}$ are given by a symplectic vector real functions harmonic in the 3-dimensional transverse space. For single black holes :

$$\begin{pmatrix} \mathcal{I}^{\Lambda} \\ \\ \mathcal{I}_{\Lambda} \end{pmatrix} = \begin{pmatrix} H^{\Lambda}(\tau) \\ \\ H_{\Lambda}(\tau) \end{pmatrix} = \begin{pmatrix} H^{\Lambda}{}_{\infty} - \frac{1}{\sqrt{2}} p^{\Lambda}\tau \\ \\ H_{\Lambda\,\infty} - \frac{1}{\sqrt{2}} q_{\Lambda}\tau \end{pmatrix} ,$$

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5. The function $U(\tau)$ of the FGK formalism is given by

$$e^{-2U} = \langle\, \mathcal{R} \mid \mathcal{I}\,\rangle = \mathcal{I}^{\Lambda}\mathcal{R}_{\Lambda} - \mathcal{I}_{\Lambda}\mathcal{R}^{\Lambda} \, .$$

The asymptotic values of the harmonic functions, H^M_∞ satisfying the condition $N = \langle\, H_\infty \mid \mathcal{Q}\,\rangle = 0$ have the general form

$$H^M{}_\infty = \pm\sqrt{2}\,\Im\mathrm{m}\left(\mathcal{V}^M_\infty \frac{\mathcal{Z}^*_\infty}{|\mathcal{Z}_\infty|}\right)\,.$$

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In this case the complete explicit solutions do not give much more information than the algebraic approach, but they are going to be used as starting point for the construction of non-extremal solutions later on.

5 – Direct construction of solutions: non-extremal

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The following prescription to deform the extremal supersymmetric solutions of $N = 2$ $d = 4$ Supergravity theories has been given in Galli, O., Perz & Shahbazi (2011):

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If the supersymmetric solution is given by

$$U(\tau) = U_{\mathrm{e}}[H(\tau)]\,, \qquad Z^i(\tau) = Z^i_{\mathrm{e}}[H(\tau)]\,,$$

where U_{e} and Z^i_{e} depend on harmonic functions $H^M(\tau) = H^M{}_{\infty} - \frac{1}{\sqrt{2}}\mathcal{Q}^M\tau$ given by the standard prescription for supersymmetric black holes ,

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Then, the non-extremal solution is given by

$$U(\tau) = U_{\rm e}[H(\tau)] + r_0\tau\,, \qquad Z^i(\tau) = Z^i{}_{\rm e}[H(\tau)]\,,$$

where now the functions H are assumed to be of the form

$$H^M = a^M + b^M\, e^{2r_0\tau}\,,$$

and the constants a^M, b^M have to be determined by explicitly solving the e.o.m.

- We are assuming that all the black hole solutions have the same dependence on some functions $H^M(\tau)$, which are harmonic in the extremal case and something else in the non-extremal cases.

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It has been shown that it is possible to rewrite the FGK effective action using the $H^M(\tau)$ as variables that replace $U(\tau)$ and $\phi^i(\tau)$ (Mohaupt & Waite arXiv:0906.3451, Mohaupt & Vaughan arXiv:1006.3439 & arXiv:1112.2876, Meessen, O., Perz & Shahbazi arXiv:1112.3332). This confirms our hypothesis.

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More on this, later.

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Extremal, supersymmetric

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Extremal, non − supersymmetric

6 – A complete example: $\overline{\mathbb{CP}}^n$ model

This model and has n scalars Z^i that parametrize the coset space $SU(1,n)/SU(n)$. We add for convenience $Z^0 \equiv 1$, so we have

$$(Z^\Lambda) \equiv (1, Z^i)\,, \qquad (Z_\Lambda) \equiv (1, Z_i) = (1, -Z^i)\,, \qquad (\eta_{\Lambda\Sigma}) = \mathrm{diag}(+ - \cdots -)\,.$$

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The covariantly holomorphic symplectic section reads $\mathcal{V} = e^{\mathcal{K}/2}\begin{pmatrix} Z^\Lambda \\ \\ -\frac{i}{2} Z_\Lambda \end{pmatrix}$.

It is convenient to define the complex charge combinations $\Gamma_\Lambda \equiv q_\Lambda + \frac{i}{2}\eta_{\Lambda\Sigma} p^\Sigma$.

The central charge $\mathcal{Z}$, its holomorphic Kähler -covariant derivative and the black-hole potential are given by

$$
\begin{aligned}
\mathcal{Z} &= e^{\mathcal{K}/2} Z^{\Lambda} \Gamma_{\Lambda}\,, \\
\mathcal{D}_i \mathcal{Z} &= e^{3\mathcal{K}/2} Z_i^{*} Z^{\Lambda} \Gamma_{\Lambda} - e^{\mathcal{K}/2} \Gamma_i\,, \\
|\tilde{\mathcal{Z}}|^2 &\equiv \mathcal{G}^{ij^*} \mathcal{D}_i \mathcal{Z} \mathcal{D}_{j^*} \mathcal{Z}^* = e^{\mathcal{K}} |Z^{\Lambda} \Gamma_{\Lambda}|^2 - \Gamma^{*\,\Lambda} \Gamma_{\Lambda}\,, \\
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$$

Remember that in $N = 2$ theories, in the extremal case $|\mathcal{Z}|$ plays the rôle of superpotential W. In this case $|\tilde{\mathcal{Z}}|$ will play the rôle of "fake" superpotential.

In this case we can write

$$
\begin{aligned}
-\left[e^{2U}V_{\rm bh} - {r_0}^2\right] &= \Upsilon^2 + 4\,\mathcal{G}^{ij^*}\,\Psi_i\Psi^*_{j^*}\,, \\
\Upsilon &= \frac{e^U}{\sqrt{2}}\sqrt{|\mathcal{Z}|^2 + |\tilde{\mathcal{Z}}|^2 + e^{-2U}{r_0}^2 + \sqrt{\left(|\mathcal{Z}|^2 + |\tilde{\mathcal{Z}}|^2 + e^{-2U}{r_0}^2\right)^2 - 4|\mathcal{Z}|^2|\tilde{\mathcal{Z}}|^2}}\,, \\
\Psi_i &= e^{2U}\,\frac{\mathcal{Z}^*\,\mathcal{D}_i\mathcal{Z}}{\Upsilon}\,,
\end{aligned}
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$$\Upsilon = \frac{e^U}{\sqrt{2}}\sqrt{|\mathcal{Z}|^2+|\tilde{\mathcal{Z}}|^2+e^{-2U}{r_0}^2+\sqrt{\left(|\mathcal{Z}|^2+|\tilde{\mathcal{Z}}|^2+e^{-2U}{r_0}^2\right)^2-4|\mathcal{Z}|^2|\tilde{\mathcal{Z}}|^2}}\,,$$

$$\Psi_i = e^{2U}\,\frac{\mathcal{Z}^*\,\mathcal{D}_i\mathcal{Z}}{\Upsilon}\,,$$

Since

$$\partial_U\Psi_i-\partial_i\Upsilon=\partial_i\Psi_j-\partial_j\Psi_i=\partial_{i*}\Psi_j-\partial_j\Psi^*_{i^*}=0\,,$$

there exists a generalized superpotential, whose gradient generates the vector field $(\Upsilon,\Psi_i,\Psi^*_{j^*})$ and the first-order equations

$$U'=\Upsilon\,,\qquad\qquad Z^{i\,\prime}=2\,\mathcal{G}^{ij^*}\,\Psi^*_{j^*}\,.$$

but it is very difficult to find explicitly.

The extremal case

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We start by calculating the critical points of the black-hole potential:

$$\mathcal{G}^{ij^*}\partial_{j^*}V_{\mathrm{bh}} = 2\,Z^{\Lambda}\Gamma_{\Lambda}\left(\Gamma^{*\,i} - \Gamma^{*\,0}Z^{i}\right) = 0 \quad \Rightarrow \left\{ \begin{array}{l} Z^{i}{}_{\mathrm{h}} = \Gamma^{*\,i}/\Gamma^{*\,0}\,, \\ (\text{isolated, supersymmetric attractor}) \\ \\ Z^{\Lambda}{}_{\mathrm{h}}\Gamma_{\Lambda} = 0\,, \\ (\text{non} - \text{supersymmetric hypersurface}) \end{array} \right.$$

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Attractor	$e^{-\mathcal{K}_{\mathrm{h}}}$	$\lvert\mathcal{Z}_{\mathrm{h}}\rvert^2$	$\lvert\tilde{\mathcal{Z}}_{\mathrm{h}}\rvert^2$	$-V_{\mathrm{bhh}}$	M
$Z_{\mathrm{h}}^{i\,\mathrm{susy}} = \Gamma^{*\,i}/\Gamma^{*\,0}$	$\Gamma^{*\,\Lambda}\Gamma_{\Lambda} > 0$	$\Gamma^{*\,\Lambda}\Gamma_{\Lambda}$	0	$\Gamma^{*\,\Lambda}\Gamma_{\Lambda}$	$\lvert\mathcal{Z}_{\infty}\rvert$
$Z_{\mathrm{h}}^{\Lambda\,\mathrm{nsusy}}\Gamma_{\Lambda} = 0$	$-\Gamma^{*\,\Lambda}\Gamma_{\Lambda} > 0$	0	$-\Gamma^{*\,\Lambda}\Gamma_{\Lambda}$	$-\Gamma^{*\,\Lambda}\Gamma_{\Lambda}$	$\lvert\tilde{\mathcal{Z}}_{\infty}\rvert$

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Then, the solutions are completely determined by the harmonic functions $H^M(\tau) = H^M - \frac{1}{\sqrt{2}}\mathcal{Q}^M\tau$ with

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Defining, for convenience's sake

$$\mathcal{H}_\Lambda \equiv H_\Lambda + \tfrac{i}{2}\eta_{\Lambda\Sigma}H^\Sigma \equiv e^{\mathcal{K}_\infty/2}\frac{\mathcal{Z}_\infty}{|\mathcal{Z}_\infty|}Z^*_{\Lambda\,\infty} - \frac{1}{\sqrt{2}}\Gamma_\Lambda\tau$$

the metric function and the scalars are

$$e^{-2U} = 2\mathcal{H}^{*\,\Lambda}\mathcal{H}_\Lambda\,, \qquad\qquad Z^i = \frac{\mathcal{R}^i + i\mathcal{I}^i}{\mathcal{R}^0 + i\mathcal{I}^0} = \frac{\mathcal{H}^{*i}}{\mathcal{H}^{*0}}\,.$$

Non-extremal solutions

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Our Ansatz for the non-extremal solution is

$$e^{-2U} = e^{-2[U_{\rm e}(\mathcal{H})+r_0\tau]}\,, \qquad e^{-2U_{\rm e}(\mathcal{H})} = 2\mathcal{H}^{*\,\Lambda}\mathcal{H}_\Lambda\,, \qquad Z^i = Z^i{}_{\rm e}(\mathcal{H}) = \mathcal{H}^{*\,i}/\mathcal{H}^{*\,0}\,,$$

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where $\mathcal{H}^{\Lambda} \equiv A^{\Lambda} + B^{\Lambda}e^{2r_0\tau}\,, \quad \Lambda = 0, \cdots, n.$

The $2(n+1)$ complex constants A_{Λ}, B_{Λ} are found by imposing the e.o.m. $(f \equiv e^{r_0\tau})$

$$\begin{aligned}
\ddot{U}_{\mathrm{e}} - (\dot{U}_{\mathrm{e}})^2 - \mathcal{G}_{ij^*}\dot{Z}^i\dot{Z}^{*\,j^*} &= 0\,,\\
(2r_0)^2\left[f\ddot{U}_{\mathrm{e}} + \dot{U}_{\mathrm{e}}\right] + e^{2U_{\mathrm{e}}}V_{\mathrm{bh}} &= 0\,,\\
(2r_0)^2\left[f\left(\ddot{Z}^i + \mathcal{G}^{ij^*}\partial_k\mathcal{G}_{lj^*}\dot{Z}^k\dot{Z}^l\right) + \dot{Z}^i\right] + e^{2U_{\mathrm{e}}}\mathcal{G}^{ij^*}\partial_{j^*}V_{\mathrm{bh}} &= 0\,.
\end{aligned}$$

The e.o.m. are solved if the the constants satisfy the **algebraic** equations

$$
\begin{aligned}
\Im\mathrm{m}(B^{*\,\Lambda}A_\Lambda) &= 0\,,\\
A^{*\,\Lambda}A^\Sigma\xi_{\Lambda\Sigma} &= 0\,,\\
(A^{*\,\Lambda}B^\Sigma + B^{*\,\Lambda}A^\Sigma)\xi_{\Lambda\Sigma} &= 0\,,\\
B^{*\,\Lambda}B^\Sigma\xi_{\Lambda\Sigma} &= 0\,,\\
(2r_0)^2(B^*_iA^*_0 - B^*_0A^*_i)A^{*\,\Lambda}A_\Lambda + (\Gamma^*_iA^*_0 - \Gamma^*_0A^*_i)A^{*\,\Lambda}\Gamma_\Lambda &= 0\,,\\
-(2r_0)^2(B^*_iA^*_0 - B^*_0A^*_i)B^{*\,\Lambda}B_\Lambda + (\Gamma^*_iB^*_0 - \Gamma^*_0B^*_i)B^{*\,\Lambda}\Gamma_\Lambda &= 0\,,\\
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where $\xi_{\Lambda\Sigma} \equiv 2\left(\Gamma_\Lambda\Gamma^*_\Sigma + 8{r_0}^2A_\Lambda B^*_\Sigma\right) - \eta_{\Lambda\Sigma}\left(\Gamma^\Omega\Gamma^*_\Omega + 8{r_0}^2A^\Omega B^*_\Omega\right)\,.$

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(2r_0)^2(B^*_iA^*_0-B^*_0A^*_i)A^{*\,\Lambda}A_\Lambda+(\Gamma^*_iA^*_0-\Gamma^*_0A^*_i)A^{*\,\Lambda}\Gamma_\Lambda &= 0\,,\\
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No differential equations remain to be solved!

Furthermore, we need to normalize the metric at spatial infinity and relate A_Λ, B_Λ to the physical parameters:

$$
\begin{aligned}
2(A^{*\,\Lambda} + B^{*\,\Lambda})(A_\Lambda + B_\Lambda) &= 1\,, \\
4\Re\mathrm{e}[B^{*\,\Lambda}(A_\Lambda + B_\Lambda)] &= 1 - M/r_0\,, \\
\frac{A^{*\,i} + B^{*\,i}}{A^{*\,0} + B^{*\,0}} &= Z^i{}_\infty\,.
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\end{aligned}
$$

Up to a phase to be determined in the supersymmetric extremal limit the solution is

$$
\begin{aligned}
A_\Lambda &= \pm\frac{e^{\mathcal{K}_\infty/2}}{2\sqrt{2}}\left\{Z^*_{\Lambda\,\infty}\left[1 + \frac{(M^2 - e^{\mathcal{K}_\infty}|Z^{*\,\Sigma}_\infty\Gamma^*_\Sigma|^2)}{Mr_0}\right] + \frac{\Gamma_\Lambda Z^{*\,\Sigma}{}_\infty\Gamma^*_\Sigma}{Mr_0}\right\}, \\
B_\Lambda &= \pm\frac{e^{\mathcal{K}_\infty/2}}{2\sqrt{2}}\left\{Z^*_{\Lambda\,\infty}\left[1 - \frac{(M^2 - e^{\mathcal{K}_\infty}|Z^{*\,\Sigma}_\infty\Gamma^*_\Sigma|^2)}{Mr_0}\right] - \frac{\Gamma_\Lambda Z^{*\,\Sigma}_\infty\Gamma^*_\Sigma}{Mr_0}\right\},
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$$

Furthermore, we need to normalize the metric at spatial infinity and relate A_Λ, B_Λ to the physical parameters:

$$\begin{aligned} 2(A^{*\,\Lambda}+B^{*\,\Lambda})(A_\Lambda+B_\Lambda) &= 1\,, \\ 4\Re\mathrm{e}[B^{*\,\Lambda}(A_\Lambda+B_\Lambda)] &= 1-M/r_0\,, \\ \frac{A^{*\,i}+B^{*\,i}}{A^{*\,0}+B^{*\,0}} &= Z^i{}_\infty\,. \end{aligned}$$

Up to a phase to be determined in the supersymmetric extremal limit the solution is

$$\begin{aligned} A_\Lambda &= \pm\frac{e^{\mathcal{K}_\infty/2}}{2\sqrt{2}}\left\{Z^*_{\Lambda\,\infty}\left[1+\frac{(M^2-e^{\mathcal{K}_\infty}|Z^{*\,\Sigma}_\infty\Gamma^*_\Sigma|^2)}{Mr_0}\right]+\frac{\Gamma_\Lambda Z^{*\,\Sigma}{}_\infty\Gamma^*_\Sigma}{Mr_0}\right\}, \\ B_\Lambda &= \pm\frac{e^{\mathcal{K}_\infty/2}}{2\sqrt{2}}\left\{Z^*_{\Lambda\,\infty}\left[1-\frac{(M^2-e^{\mathcal{K}_\infty}|Z^{*\,\Sigma}_\infty\Gamma^*_\Sigma|^2)}{Mr_0}\right]-\frac{\Gamma_\Lambda Z^{*\,\Sigma}_\infty\Gamma^*_\Sigma}{Mr_0}\right\}, \end{aligned}$$

Here ${M}^2{r_0}^2=({M}^2-|\mathcal{Z}_\infty|^2)({M}^2-|\tilde{\mathcal{Z}}_\infty|^2)$, and one can show that the metric is regular in all the ${r_0}^2>0$ cases.

Supersymmetric and non-supersymmetric extremal limits

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$$\mathcal{H}_\Lambda \stackrel{M \to |\tilde{\mathcal{Z}}_\infty|}{\longrightarrow} \pm \frac{e^{\mathcal{K}_\infty/2}}{2\sqrt{2}} \left\{ Z^*_{\Lambda\,\infty} - \frac{1}{|\tilde{\mathcal{Z}}_\infty|} \left[-Z^*_{\Lambda\,\infty} \Gamma^{*\,\Sigma} \Gamma_\Sigma + \Gamma_\Lambda Z^{*\,\Sigma}_\infty \Gamma^*_\Sigma \right] \tau \right\} .$$

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   On the event horizon $\tau \to -\infty$ the scalars $Z^i = \mathcal{H}^{*\,i}/\mathcal{H}^{*\,0}$ take the values

$$Z^{*\,i}_{\rm h} = \frac{\Gamma^i Z^{*\,\Lambda}_\infty \Gamma^*_\Lambda - Z^{*\,i}_\infty \Gamma^{*\,\Sigma}\Gamma_\Sigma}{\Gamma^0 Z^{*\,\Gamma}_\infty \Gamma^*_\Gamma - \Gamma^{*\,\Omega}\Gamma_\Omega} ,$$

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   which depend manifestly on the asymptotic values.

There is no attractor behavior in a proper sense.

The structure of the extremal non-supersymmetric solution as function of the H^Ms is the same as in the supersymmetric case.

However, no simple *substitution recipe* could have led to it.

Physical properties of the non-extremal solutions

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One can compute the "entropies" of the inner and outer horizons (event horizon (+) and Cauchy horizon) at $\tau \to -\infty$ and $\tau \to +\infty$ resp.:

$$\frac{S_{\pm}}{\pi} = (M^2 - |\mathcal{Z}_\infty|^2) \pm (M^2 - |\tilde{\mathcal{Z}}_\infty|^2) \pm 2 M r_0 \,.$$

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They can also be written in the suggestive form

$$S_\pm = \pi \left(\sqrt{N_{\rm R}} \pm \sqrt{N_{\rm L}}\right)^2 \,,$$

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The product S_+S_- is manifestly moduli-independent for all values of r_0:

$$S_+S_-/\pi^2 = (\Gamma^{*\,\Lambda}\Gamma_\Lambda)^2\,.$$

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➾ Thus, if $\Gamma^{*\,\Lambda}\Gamma_\Lambda > 0$, which is the property that characterizes the supersymmetric attractor , then $|\mathcal{Z}_\infty| > |\tilde{\mathcal{Z}}_\infty|$ and the evaporation process will stop when $M = |\mathcal{Z}_\infty|$ (supersymmetry restoration).

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7 – H-FGK formalism for $N = 2$, $d = 4$ supergravity

Or: Where the H^Ms come from

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In the FGK formalism it is always possible to use a different set of variables (Mohaupt & Vaughan `arXiv:1112.2876`, Meessen, O., Perz & Shahbazi `arXiv:1112.3332`)

$$U(\tau), Z^i(\tau) \;\; (2n_V + 1) \longrightarrow \begin{pmatrix} H^\Lambda \\ H_\Lambda \end{pmatrix} \equiv H^M \,, \;\; (2n_V + 2)$$

plus one constraint that will appear automatically.

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$$\mathcal{R}^M + i\mathcal{I}^M \equiv \mathcal{V}^{\mathcal{M}}/X \,.$$

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$$\tilde{H}_M \equiv \mathcal{R}_M \,, \qquad\qquad H^M \equiv \mathcal{I}^M \,.$$

We define the Hessian potential $\mathsf{W}(H) \equiv \tilde{H}_M(H) H^M$, or $\mathsf{W}(H) \equiv \tilde{H}_M H^M(H)$.

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W is homogenous of second order in the H^M variables and satisfy

$$\partial_M \mathsf{W} \equiv \frac{\partial \mathsf{W}}{\partial H^M} = 2\tilde{H}_M \, , \qquad \partial^M \mathsf{W} \equiv \frac{\partial \mathsf{W}}{\partial \tilde{H}_M} = 2H^M \, ,$$

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Then, the FGK effective action can be written in the form

$$\begin{aligned} I_{\rm eff}[H] \;\; = \;\; \int d\tau \Big\{ & \tfrac{1}{2}\partial_M \partial_N \log \mathsf{W} \left(\dot{H}^M \dot{H}^N + \tfrac{1}{2}\mathcal{Q}^M \mathcal{Q}^N \right) \\ & + \left(\mathsf{W}^{-1} \dot{H}^M H_M \right)^2 + \left(\mathsf{W}^{-1} \mathcal{Q}^M H_M \right)^2 \Big\} \,, \end{aligned}$$

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All the information about the model is encoded in the Hessian potential $\mathsf{W}(H)$.

Having the $H^M(\tau)$ that solve this action, the black-hole solution is given by

$$e^{-2U(\tau)} = \mathsf{W}[H(\tau)] \,, \qquad Z^i(\tau) = \frac{\tilde{H}^i(H) + iH^i}{\tilde{H}^0(H) + iH^0} \,.$$

This shows that we can write all the static black-hole solutions of a given model $N = 2 \;\; d = 4$ supergravity exactly in the same way in terms of the functions $H^M(\tau)$.

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The equations of motion of the H^Ms are

$$
\begin{aligned}
-\tfrac{1}{2}\partial_M\partial_N \log \mathsf{W}\left(\dot{H}^M\dot{H}^N - \tfrac{1}{2}\mathcal{Q}^M\mathcal{Q}^N\right) + \left(\mathsf{W}^{-1}\dot{H}^M H_M\right)^2 - \left(\mathsf{W}^{-1}\mathcal{Q}^M H_M\right)^2 &= {r_0}^2\,, \\
\tfrac{1}{2}\partial_M \log \mathsf{W}\left(\ddot{H}^M - {r_0}^2 H^M\right) + \left(\mathsf{W}^{-1}\dot{H}^M H_M\right)^2 &= 0\,, \\
\tfrac{1}{2}\partial_P\partial_M\partial_N \log \mathsf{W}\left[\dot{H}^M\dot{H}^N - \tfrac{1}{2}\mathcal{Q}^M\mathcal{Q}^N\right] + \partial_P\partial_M \log \mathsf{W}\,\ddot{H}^M & \\
-\frac{d}{d\tau}\left(\frac{\partial\Lambda}{\partial\dot{H}^P}\right) + \frac{\partial\Lambda}{\partial H^P} &= 0\,,
\end{aligned}
$$

with

$$
\Lambda \equiv \left(\mathsf{W}^{-1}\dot{H}^M H_M\right)^2 + \left(\mathsf{W}^{-1}\mathcal{Q}^M H_M\right)^2\,.
$$

In the extremal case $r_0 = 0$ one sees immediately that $\dot{H}^P = \pm\frac{1}{\sqrt{2}}\mathcal{Q}^P$ satisfying the no-NUT condition $\dot{H}^P H_P = 0$ solve all the equations.

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Actually, there are **two**: one for black holes and another for black strings.

We first need a generalization of the FGK formalism to higher spacetime (d) and worldvolume $(p + 1)$ dimensions.

8 – FGK formalism for $d \geq 4$ and $p \geq 0$

The FGK formalism has been generalized to higher spacetime (d) and worldvolume ($p+1$) dimensions for generic actions including ($p+1$)-form potentials $A^{\Lambda}_{(p+1)}$

$$
\begin{aligned}
I &= \int d^d x \sqrt{|g|}\, \{R + \mathcal{G}_{ij}(\phi)\partial_\mu \phi^i \partial^\mu \phi^j \\
&\quad + 4\tfrac{(-1)^p}{(p+2)!}\left[I_{\Lambda\Sigma}(\phi) F^{\Lambda}_{(p+2)} \cdot F^{\Sigma}_{(p+2)} + \xi^2 R_{\Lambda\Sigma}(\phi) F^{\Lambda}_{(p+2)} \star F^{\Sigma}_{(p+2)}\right]\Big\}\,,
\end{aligned}
$$

where the last term occurs only when $p = \tilde{p} = (d-4)/2$ and

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R_{\Lambda\Sigma}(\phi) = -\xi^2 R_{\Sigma\Lambda}(\phi)\,, \qquad \xi^2 = (-1)^{\frac{d}{2}+1} = (-1)^{p+1}\,.
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We need a generalization of the ansatz that can describe single, static, regular, black p-branes with flat worldvolume in the directions $\vec{y}_{(p)} = (y_1, \cdots, y_p)$ living in a spacetime of $d = p + \tilde{p} + 4$ dimensions.

This ansatz is

$$
\begin{aligned}
ds^2_{(d)} &= e^{\frac{2}{p+1}\tilde{U}}\left[e^{\frac{2p}{p+1}r_0\rho}dt^2 - e^{-\frac{2}{p+1}r_0\rho}d\vec{y}^{\,2}_{(p)}\right] - e^{-\frac{2}{\tilde{p}+1}\tilde{U}}\gamma_{(\tilde{p}+3)\,mn}dx^m dx^n\,, \\
\gamma_{(\tilde{p}+3)\,mn}dx^m dx^n &\equiv \left[\frac{r_0}{\sinh(r_0\rho)}\right]^{\frac{2}{\tilde{p}+1}}\left[\left(\frac{r_0}{\sinh(r_0\rho)}\right)^2\frac{d\rho^2}{(\tilde{p}+1)^2} + d\Omega^2_{(\tilde{p}+2)}\right],
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$$\gamma_{(\tilde{p}+3)\,mn}dx^m dx^n \equiv \left[\frac{r_0}{\sinh(r_0\rho)}\right]^{\frac{2}{\tilde{p}+1}}\left[\left(\frac{r_0}{\sinh(r_0\rho)}\right)^2\frac{d\rho^2}{(\tilde{p}+1)^2} + d\Omega^2_{(\tilde{p}+2)}\right],$$

and only contains an independent function $\tilde{U}$ and the extremality parameter r_0.
The effective action is

$$I_{\rm eff}[\tilde{U},\phi^i] = \int d\tau\left\{(\dot{\tilde{U}})^2 + \tfrac{(p+1)(\tilde{p}+2)}{d-2}\mathcal{G}_{ij}\dot{\phi}^i\dot{\phi}^j - e^{2\tilde{U}}V_{\rm bb} + r_0{}^2\right\},$$

where we have defined the black-brane potential

$$-V_{\rm bb}(\phi,Q) \equiv -\tfrac{1}{2}Q^M Q^N \mathcal{M}_{MN}\,,\quad (\mathcal{M}_{MN}) \equiv \begin{pmatrix} (I-\xi^2 R I^{-1}R)_{\Lambda\Sigma} & \xi^2(RI^{-1})_{\Lambda}{}^{\Sigma} \\ -(I^{-1}R)^{\Lambda}{}_{\Sigma} & (I^{-1})^{\Lambda\Sigma}\end{pmatrix},$$

This ansatz is

$$ds^2_{(d)} = e^{\frac{2}{p+1}\tilde{U}}\left[e^{\frac{2p}{p+1}r_0\rho}dt^2 - e^{-\frac{2}{p+1}r_0\rho}d\vec{y}^{\,2}_{(p)}\right] - e^{-\frac{2}{\tilde{p}+1}\tilde{U}}\gamma_{(\tilde{p}+3)\,mn}dx^m dx^n\,,$$

$$\gamma_{(\tilde{p}+3)\,mn}dx^m dx^n \equiv \left[\frac{r_0}{\sinh(r_0\rho)}\right]^{\frac{2}{\tilde{p}+1}}\left[\left(\frac{r_0}{\sinh(r_0\rho)}\right)^2\frac{d\rho^2}{(\tilde{p}+1)^2} + d\Omega^2_{(\tilde{p}+2)}\right],$$

and only contains an independent function $\tilde{U}$ and the extremality parameter r_0.
The effective action is

$$I_{\rm eff}[\tilde{U},\phi^i] = \int d\tau\left\{(\dot{\tilde{U}})^2 + \tfrac{(p+1)(\tilde{p}+2)}{d-2}\mathcal{G}_{ij}\dot{\phi}^i\dot{\phi}^j - e^{2\tilde{U}}V_{\rm bb} + r_0{}^2\right\},$$

where we have defined the black-brane potential

$$-V_{\rm bb}(\phi,Q) \equiv -\tfrac{1}{2}Q^M Q^N\mathcal{M}_{MN}\,,\quad (\mathcal{M}_{MN}) \equiv \begin{pmatrix} (I-\xi^2 RI^{-1}R)_{\Lambda\Sigma} & \xi^2(RI^{-1})_{\Lambda}{}^{\Sigma} \\ -(I^{-1}R)^{\Lambda}{}_{\Sigma} & (I^{-1})^{\Lambda\Sigma}\end{pmatrix},$$

$\mathcal{M}_{MN}$ is an $O(n,n)$ (resp. $Sp(n,n)$) matrix when $\xi^2=+1$ (resp. -1).

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The non-extremal black-brane solutions seem to have the similar features for all p. An H-FGK formalism exists for theories associated to certain supergravity theories: black holes and black strings in $N = 2$, d=5 supergravity (Mohaupt & Waite `arXiv:0906.3451`, Mohaupt & Vaughan `arXiv:1006.3439` & `arXiv:1112.2876`, Meessen, O., Perz & Shahbazi `arXiv:1112.3332` and work to appear).

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If we then define the derived objects

$$h_I \equiv C_{IJK} h^J h^K\,, \qquad h^I_x \equiv -\sqrt{3}\,\frac{\partial h^I}{\partial \phi^x} \qquad \text{and} \qquad h_{Ix} \equiv \sqrt{3}\,\frac{\partial h_I}{\partial \phi^x}\,,$$

we can see that they satisfy the following relations

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$$h^I h_I = 1 \qquad \text{and} \qquad h^I h_{Ix} = h_I h^I_x = 0\,.$$

The scalar metric g_{xy}, and the vector kinetic matrix, a_{IJ}, are given by

$$g_{xy} = h_{Ix} h^I_y \qquad \text{and} \qquad a_{IJ} = 3 h_I h_J - 2 C_{IJK} h^K = h_I h_J + h_{Ix} h^x_J\,.$$

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The bosonic action for $N=2$ $d=5$ supergravity with n vector supermultiplets is

$$\mathcal{I}_5 = \int_5 \left(R \star 1 + \tfrac{1}{2} g_{xy}\, d\phi^x \wedge \star d\phi^y - \tfrac{1}{2} a_{IJ} F^I \wedge \star F^J + \tfrac{1}{3\sqrt{3}} C_{IJK} F^I \wedge F^J \wedge A^K \right).$$

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This theory admits black-hole ($p=0, \tilde{p}=1$) and black strings ($p=1, \tilde{p}=0$) solutions. The corresponding metric ansätze are particular cases of the general one

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\begin{aligned}
ds^2_{(d)} &= e^{\frac{2}{p+1}\tilde{U}}\left[e^{\frac{2p}{p+1}r_0\rho}dt^2 - e^{-\frac{2}{p+1}r_0\rho}d\vec{y}^{\,2}_{(p)}\right] - e^{-\frac{2}{\tilde{p}+1}\tilde{U}}\gamma_{(\tilde{p}+3)\,mn}dx^m dx^n\,,\\
\gamma_{(\tilde{p}+3)\,mn}dx^m dx^n &\equiv \left[\frac{r_0}{\sinh(r_0\rho)}\right]^{\frac{2}{\tilde{p}+1}}\left[\left(\frac{r_0}{\sinh(r_0\rho)}\right)^2\frac{d\rho^2}{(\tilde{p}+1)^2} + d\Omega^2_{(\tilde{p}+2)}\right],
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$$

where, in each case, we have to replace the black-brane potential $V_{\rm bb}$ by the the black-hole $V_{\rm bh}(\phi,q)$ and black-string potentials

$$
\left\{
\begin{array}{rcl}
-V_{\rm bh}(\phi,q) &\equiv& a^{IJ}q_I q_J = {\mathcal{Z}_{\rm e}}^2 + 3\,\partial_x\mathcal{Z}_{\rm e}\,\partial^x\mathcal{Z}_{\rm e}\,, \\
-V_{\rm bs}(\phi,p) &\equiv& a_{IJ}p^I p^J = {\mathcal{Z}_{\rm m}}^2 + 3\,\partial_x\mathcal{Z}_{\rm m}\,\partial^x\mathcal{Z}_{\rm m}\,,
\end{array}
\right.
$$

where we have defined the *electric and magnetic central charges* by

$$
\mathcal{Z}_{\rm e}(\phi,q) \equiv h^I q_I\,, \qquad \mathcal{Z}_{\rm m}(\phi,p) \equiv h_I p^I\,.
$$

H-variables for black holes

We introduce two new sets of variables, $\tilde{H}^I$ and H_I, related to the original ones $(\tilde{U}, \phi^x)$ by

$$\begin{aligned} e^{-\tilde{U}/2} h^I(\phi) &\equiv \tilde{H}^I\,, \\ e^{-\tilde{U}} h_I(\phi) &\equiv H_I\,, \end{aligned}$$

and the new (unconstrained) function W

$$\mathsf{W}(\tilde{H}) \equiv 2 C_{IJK} \tilde{H}^I \tilde{H}^J \tilde{H}^K\,,.$$

The homogeneity properties imply that

$$\begin{aligned} e^{-\frac{3}{2}\tilde{U}} &= \tfrac{1}{2}\mathsf{W}(H)\,, \\ h_I &= (\mathsf{W}/2)^{-2/3} H_I\,, \\ h^I &= (\mathsf{W}/2)^{-1/3} \tilde{H}^I\,. \end{aligned}$$

Changing the action to the H_I variables, it becomes

$$-\tfrac{3}{2}\mathcal{I}[H] = \int d\rho \left[\partial^I \partial^J \log \mathsf{W} \left(\dot{H}_I \dot{H}_J + q_I q_J\right) - \tfrac{3}{2} {r_0}^2\right].$$

K-variables for black strings

We introduce two new sets of variables, K^I and $\tilde{K}_I$, related to the original ones $(\tilde{U}, \phi^x)$ by

$$\begin{aligned} e^{-\tilde{U}} h^I(\phi) &\equiv K^I\,, \\ e^{-2\tilde{U}} h_I(\phi) &\equiv \tilde{K}_I\,, \end{aligned}$$

and the new (unconstrained) function V

$$\mathsf{V}(K) \equiv C_{IJK} K^I K^J K^K\,.$$

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Changing the action to the K^I variables, it becomes

$$-3\mathcal{I}[K] = \int d\rho \left[\partial_I \partial_J \log \mathsf{V} \left(\dot{K}^I \dot{K}^J + p^I p^J\right) - 3{r_0}^2\right].$$

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How useful are these new variables?

☞ In H-variables one immediately sees that, in the extremal case $r_0 = 0$

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always solves the equations of motion in all theories.

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☞ The B_Is are sometimes called *fake charges*. If we define the *fake electric central charges*

$$\mathcal{Z}_{\rm e}(\phi, B) \equiv h^I B_I\,,$$

It is immediate to see that the following *first-order flow equations*

$$\frac{de^{-\tilde{U}}}{d\rho} = \mathcal{Z}_{\rm e}(\phi, B)\,, \qquad \frac{d\phi^x}{d\rho} = -3e^{\tilde{U}}\partial^x \mathcal{Z}_{\rm e}(\phi, B)\,.$$

are satisfied.

Non-extremal black holes

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☞ The integration constants A_I can be easily determined, but, to find the B_Is, one has to solve the equations of motion, which reduce to

$$\begin{aligned} \partial^K \partial^I \partial^J \log \mathsf{W}(H) \left(B_I B_J - {r_0}^2 A_I A_J - q_I q_J \right) &= 0 , \\ \partial^I \partial^J \log \mathsf{W}(H) \left(B_I B_J - {r_0}^2 A_I A_J - q_I q_J \right) &= 0 . \end{aligned}$$

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$$\frac{de^{-\tilde{U}}}{d\hat{\rho}} = \mathcal{Z}_{\rm e}(\phi, B)\,, \qquad\qquad \frac{d\phi^x}{d\hat{\rho}} = -3e^{\tilde{U}} \partial^x \mathcal{Z}_{\rm e}(\phi, B)\,.$$

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☞ These equations are formally identical to those of the extremal case, but the B_Is are different and the range of the coordinate $\hat{\rho}$ is not enough to reach an attractor.

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  We have proven that part of our ansatz is completely general, constructing a formalism ("H-FGK") that simplifies the construction of extremal and non-extremal black hole solutions.
- ★ We have extended the FGK formalism to higher spacetime and worldvolume dimensions and the H-FGK formalism to the 5-dimensional $N = 2 \;\; d = 5$ case (black holes and also black strings) and shown the power of this approach.

We are closer to determining the general form of all single, static, black-hole and black-string solutions of $N = 2$,$d = 4, 5$ theories.