Non-extremal black holes of N=2, d=4,5 Supergravity

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Work done in collaboration with

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Plan of the Talk:

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In this talk I will present a general ansatz to construct non-extremal black-hole solutions and,as an example, we will study a family of solutions obtained with it. First, we will review the formalism.

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Algebraic approach
- Ferrara, Gibbons & Kallosh, (1997) (general formalism)
- Ceresole & Dall'Agata (2007) ("fake" superpotentials)

Explicit solutions

Supersymmetric (*i.e.* extremal) :
- Tod (1983) (pure $N = 2$)
- Behrndt, Luest & Sabra (1997)($N = 2$ + Vs.)
- Caldarelli & Klemm (2003) (pure gauged $N = 2$)
- Huebscher, Meessen, O. & Vaula (2007), Meessen, (2008) ($N = 2$ + Vs non − Abelian − gauged)
- Cacciatori, Klemm, Mansi & Zorzan (2008) ($N = 2$ + Vs Abelian − gaug
- Meessen, O. & Vaula (2010) (all $N \geq 2$)

Non − extremal :
- Cvetic & Youm (1996)
- O. (1996)
- Kastor & Win (1996)
- Mohaupt & Vaughan (2010) (general Ansatz d = 5)
- Galli, O., Perz & Shahbazi (2011) (general Ansatz d = 4)

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We start by reviewing the FGK formalism.

2 – FGK formalism

Ferrara, Gibbons and Kallosh (1997) considered the general 4-dimensional action

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I &= \int d^4x\sqrt{|g|}\,\{R + \mathcal{G}_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j \\
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They also considered the general metric for any static non-extremal black hole

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ds^2=e^{2U(\tau)}dt^2-e^{-2U(\tau)}\left[\frac{r_0{}^4}{\sinh^4 r_0\tau}d\tau^2+\frac{r_0{}^2}{\sinh^2 r_0\tau}d\Omega^2_{(2)}\right]\ .
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- ☞ What is r_0 like for more general black holes?

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To determine completely the metric of any static, regular, spherically symmetric black hole we only need to find the function $U(\tau)$.

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The latter can be integrated out so they are effectively replaced by the electric, q_Λ, and magnetic, p^Λ charges. The general system reduces to an effective mechanical system with variables $U(\tau), \phi^i(\tau)$:

$$I_{\text{eff}}[U, \phi^i] = \int d\tau \left\{ (U')^2 + \frac{1}{2}\mathcal{G}_{ij}\phi^{i\,\prime}\phi^{j\,\prime} - e^{2U}V_{\text{bh}} + r_0{}^2 \right\} ,$$

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where FGK defined the black-hole potential

$$-V_{\rm bh}(\phi,q,p) \equiv -\frac{1}{2}(p^\Lambda \quad q_\Lambda) \begin{pmatrix} (\mathfrak{I}+\mathfrak{R}\mathfrak{I}^{-1}\mathfrak{R})_{\Lambda\Sigma} & -(\mathfrak{R}\mathfrak{I}^{-1})_\Lambda{}^\Sigma \\ -(\mathfrak{I}^{-1}\mathfrak{R})^\Lambda{}_\Sigma & (\mathfrak{I}^{-1})^{\Lambda\Sigma} \end{pmatrix} \begin{pmatrix} p^\Sigma \\ q_\Sigma \end{pmatrix} ,$$

where

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Finding a black hole with charges p, q is equivalent to solving the above mechanical system for $U(\tau)$, $\phi^i(\tau)$.

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Each critical point yields a possible extremal black-hole solution and an $AdS_2\times S^2$ geometry. One can go a long way with the attractor only, ignoring the full explicit solution.

In the general case one can prove the following extremality bound:

$$r_0{}^2 = M^2 + \tfrac{1}{2}\mathcal{G}_{ij}(\phi_\infty)\Sigma^i\Sigma^j + V_{\rm bh}(\phi_\infty, q, p)\,, \geq 0\,,$$

where

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We need to find the complete explicit solution in the non-extremal case.

Whenever we can write $-\left[e^{2U}V_{\mathrm{bh}} - {r_0}^2\right] = (\partial_U Y)^2 + 2\,\mathcal{G}^{ij}\partial_i Y\partial_j Y$ for some *(generalized) superpotential* $Y(U, \phi^i, p, q, r_0)$, we can rewrite the effective action as

$$I_{\mathrm{eff}}[U, \phi^i] = \int d\tau \left\{(U' - \partial_U Y)^2 + \tfrac{1}{2}\mathcal{G}_{ij}(\phi^{i\,\prime} - 2\,\mathcal{G}^{ik}\partial_k Y)(\phi^{j\,\prime} - 2\,\mathcal{G}^{jl}\partial_l Y) + 2Y'\right\}\,.$$

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The action is minimized by configurations satisfying the first-order gradient flow equations (Miller, Schalm & Weinberg (2007), Janssen, Smyth, Van Riet & Vercnocke (2008), Perz, Smyth, Van Riet & Vercnocke (2008))

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A generalized superpotential $Y(U, \phi^i, p, q, r_0)$ exists in all theories whose scalar manifold (after timelike dimensional reduction) is a symmetric coset space (in particular for all $N > 2$ supergravities) (Andrianopoli, D'Auria, Orazi & Trigiante (2009), Chemissany, Fré, Rosseel, Sorin, Trigiante & Van Riet (2010)).

In the extremal case $r_0 = 0$, if there is a generalized superpotential $Y(U, \phi^i, p, q)$, it factorizes

$$Y(U, \phi^i, p, q) = e^U W(\phi^i, p, q)\,,$$

where $W(\phi^i, p, q)$ is called the *superpotential*, and the flow equations take the form (Ceresole & Dall'Agata (2007))

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The stationary values of the superpotential $\partial_i W|_{\phi_{\rm h}} = 0$ give the the entropy:

$$S = \pi |W(\phi_{\rm h}, p, q)|^2\,,$$

while the mass is

$$M = |W(\phi_\infty, p, q)|\,.$$

3 – Direct construction of solutions: extremal supersymmetric

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We are going to review the example of (ungauged) $N = 2$ Supergravity coupled to vector multiplets.

$4 - N = 2, d = 4$ ungauged SUGRA coupled to vector multiplets

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Hypermultiplets can be ignored for black-hole solutions.

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Local $N=2$ supersymmetry requires the Kähler-Hodge manifold to be a special Kähler manifold, so it is the base space of a $2(n_V+1)$-dimensional vector bundle with $Sp[2(n_V+1),\mathbb{R}]$ structure group, on which we can define the constrained symplectic section

$$\mathcal{V} = \begin{pmatrix} \mathcal{L}^\Lambda(Z,Z^*) \\ \mathcal{M}_\Lambda(Z,Z^*) \end{pmatrix}.$$

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The action of the bosonic fields of the ungauged theory is of the general FGK form:

$$\begin{aligned} S &= \int d^4x \sqrt{|g|}\, \left[R + 2\mathcal{G}_{ij^*}\partial_\mu Z^i \partial^\mu Z^{*\,j^*} + 2\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma} F^{\Lambda\,\mu\nu} F^{\Sigma}{}_{\mu\nu} \right. \\ & \left. -2\Re\mathrm{e}\mathcal{N}_{\Lambda\Sigma} F^{\Lambda\,\mu\nu} \star F^{\Sigma}{}_{\mu\nu}\right] , \Rightarrow\ -V_{\mathrm{bh}} = |\mathcal{Z}|^2 + \mathcal{G}^{ij^*}\mathcal{D}_i\mathcal{Z}\mathcal{D}_{j^*}\mathcal{Z}^* \,. \end{aligned}$$

In order to find static extremal black holes one could try to integrate directly the equations of motion of the FGK formalism for the black-hole potential of $N = 2$ $d = 4$ theories:

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There is a recipe to construct all the BPS ones:
(Denef (2000), Behrndt, Lüst & Sabra (1997), Meessen, O. (2006))

1. For some complex X, define the Kähler-neutral, real, symplectic vectors $\mathcal{R}$ and $\mathcal{I}$

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2. The components of $\mathcal{I}$ are given by a symplectic vector real functions harmonic in the 3-dimensional transverse space. For single black holes :

$$\begin{pmatrix} \mathcal{I}^{\Lambda} \\ \\ \mathcal{I}_{\Lambda} \end{pmatrix} = \begin{pmatrix} H^{\Lambda}(\tau) \\ \\ H_{\Lambda}(\tau) \end{pmatrix} = \begin{pmatrix} H^{\Lambda}{}_{\infty} - \frac{1}{\sqrt{2}} p^{\Lambda}\tau \\ \\ H_{\Lambda\,\infty} - \frac{1}{\sqrt{2}} q_{\Lambda}\tau \end{pmatrix},$$

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3. $\mathcal{R}$ is to be found from $\mathcal{I}$ by solving the generalized *stabilization equations*.

1. For some complex X, define the Kähler-neutral, real, symplectic vectors $\mathcal{R}$ and $\mathcal{I}$

$$\mathcal{R} + i\mathcal{I} \equiv \mathcal{V}/X\,.$$

2. The components of $\mathcal{I}$ are given by a symplectic vector real functions harmonic in the 3-dimensional transverse space. For single black holes :

$$\begin{pmatrix} \mathcal{I}^\Lambda \\ \\ \mathcal{I}_\Lambda \end{pmatrix} = \begin{pmatrix} H^\Lambda(\tau) \\ \\ H_\Lambda(\tau) \end{pmatrix} = \begin{pmatrix} H^\Lambda{}_\infty - \frac{1}{\sqrt{2}} p^\Lambda \tau \\ \\ H_{\Lambda\,\infty} - \frac{1}{\sqrt{2}} q_\Lambda \tau \end{pmatrix},$$

with no sources of NUT charge, *i.e.* $\langle\, H_\infty \mid \mathcal{Q}\,\rangle = H^\Lambda{}_\infty q_\Lambda - H_{\Lambda\infty} p^\Lambda = 0$

3. $\mathcal{R}$ is to be found from $\mathcal{I}$ by solving the generalized *stabilization equations*.

4. The scalars Z^i are given by the quotients $Z^i = \dfrac{\mathcal{V}^i/X}{\mathcal{V}^0/X} = \dfrac{\mathcal{R}^i + i\mathcal{I}^i}{\mathcal{I}^0 + i\mathcal{I}^0}$.

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5. The function $U(\tau)$ of the FGK formalism is given by

$$e^{-2U} = \langle\, \mathcal{R} \mid \mathcal{I}\,\rangle = \mathcal{I}^{\Lambda}\mathcal{R}_{\Lambda} - \mathcal{I}_{\Lambda}\mathcal{R}^{\Lambda}\,.$$

The asymptotic values of the harmonic functions, H^M_∞ satisfying the condition $N = \langle\, H_\infty \mid \mathcal{Q}\,\rangle = 0$ have the general form

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In this case the complete explicit solutions do not give much more information than the algebraic approach, but they are going to be used as starting point for the construction of non-extremal solutions later on.

5 – Direct construction of solutions: non-extremal

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Based on the study of several examples, the following prescription to deform the extremal supersymmetric solutions of $N = 2 \;\; d = 4$ Supergravity theories has been given (Galli, O., Perz & Shahbazi (2011)):

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If the supersymmetric solution is given by

$$U(\tau) = U_{\rm e}[H(\tau)]\,, \qquad\qquad Z^i(\tau) = Z^i_{\rm e}[H(\tau)]\,,$$

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Then, the non-extremal solution is given by

$$U(\tau) = U_{\rm e}[H(\tau)] + r_0\tau\,, \qquad\qquad Z^i(\tau) = Z^i{}_{\rm e}[H(\tau)]\,,$$

where now the functions H are assumed to be of the form

$$H^M = a^M + b^M\, e^{2r_0\tau}\,,$$

and the constants a^M, b^M have to be determined by explicitly solving the e.o.m.

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- Furthermore, preliminary results indicate that

It may be possible to prove this hypothesis in general.
work in progress.

We are going to give an explicit example, showing that one can recover both the extremal supersymmetric and non-supersymmetric black holes of a model from the general non-extremal solution found with this prescription.

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$\longrightarrow$

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Extremal, supersymmetric

Non – extremal, supersymmetric

Extremal, non – supersymmetric

6 – A complete example: $\overline{\mathbb{CP}}^n$ model

This model and has n scalars Z^i that parametrize the coset space $SU(1,n)/SU(n)$. We add for convenience $Z^0 \equiv 1$, so we have

$$(Z^\Lambda) \equiv (1, Z^i)\,, \qquad (Z_\Lambda) \equiv (1, Z_i) = (1, -Z^i)\,, \qquad (\eta_{\Lambda\Sigma}) = \mathrm{diag}(+-\cdots-)\,.$$

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It is convenient to define the complex charge combinations $\Gamma_\Lambda \equiv q_\Lambda + \frac{i}{2}\eta_{\Lambda\Sigma} p^\Sigma$.

The central charge $\mathcal{Z}$, its holomorphic Kähler -covariant derivative and the black-hole potential are given by

$$
\begin{aligned}
\mathcal{Z} &= e^{\mathcal{K}/2} Z^{\Lambda} \Gamma_{\Lambda} \,, \\
\mathcal{D}_i \mathcal{Z} &= e^{3\mathcal{K}/2} Z_i^{*} Z^{\Lambda} \Gamma_{\Lambda} - e^{\mathcal{K}/2} \Gamma_i \,, \\
|\tilde{\mathcal{Z}}|^2 &\equiv \mathcal{G}^{ij^*} \mathcal{D}_i \mathcal{Z} \mathcal{D}_{j^*} \mathcal{Z}^* = e^{\mathcal{K}} |Z^{\Lambda} \Gamma_{\Lambda}|^2 - \Gamma^{*\,\Lambda} \Gamma_{\Lambda} \,, \\
-V_{\rm bh} &= |\mathcal{Z}|^2 + |\tilde{\mathcal{Z}}|^2 \,.
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\end{aligned}
$$

Remember that in $N=2$ theories, in the extremal case $|\mathcal{Z}|$ plays the rôle of superpotential W. In this case $|\tilde{\mathcal{Z}}|$ will play the rôle of “fake” superpotential.

In this case we can write

$$-\left[e^{2U}V_{\rm bh} - {r_0}^2\right] = \Upsilon^2 + 4\,\mathcal{G}^{ij^*}\,\Psi_i\Psi^*_{j^*}\,,$$

where

$$\Upsilon = \frac{e^U}{\sqrt{2}}\sqrt{|\mathcal{Z}|^2 + |\tilde{\mathcal{Z}}|^2 + e^{-2U}{r_0}^2 + \sqrt{\left(|\mathcal{Z}|^2 + |\tilde{\mathcal{Z}}|^2 + e^{-2U}{r_0}^2\right)^2 - 4|\mathcal{Z}|^2|\tilde{\mathcal{Z}}|^2}}\,,$$

$$\Psi_i = e^{2U}\,\frac{\mathcal{Z}^*\,\mathcal{D}_i\mathcal{Z}}{\Upsilon}\,,$$

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$$\Psi_i = e^{2U}\frac{\mathcal{Z}^*\,\mathcal{D}_i\mathcal{Z}}{\Upsilon}\,,$$

Since

$$\partial_U\Psi_i-\partial_i\Upsilon=\partial_i\Psi_j-\partial_j\Psi_i=\partial_{i*}\Psi_j-\partial_j\Psi^*_{i^*}=0\,,$$

there exists a generalized superpotential, whose gradient generates the vector field $(\Upsilon,\Psi_i,\Psi^*_{j^*})$ and the first-order equations

$$U'=\Upsilon\,,\qquad\qquad Z^{i\,\prime}=2\,\mathcal{G}^{ij^*}\,\Psi^*_{j^*}\,.$$

although it is very difficult to find explicitly.

The extremal case

The extremal case

We start by calculating the critical points of the black-hole potential:

$$\mathcal{G}^{ij^*}\partial_{j^*}V_{\rm bh} = 2\,Z^\Lambda\Gamma_\Lambda\left(\Gamma^{*\,i} - \Gamma^{*\,0}Z^i\right) = 0 \quad \Rightarrow \left\{ \begin{array}{l} Z^i{}_{\rm h} = \Gamma^{*\,i}/\Gamma^{*\,0}\,, \\ \text{(isolated, supersymmetric attractor)} \\ \\ Z^\Lambda{}_{\rm h}\Gamma_\Lambda = 0\,, \\ \text{(non} - \text{supersymmetric hypersurface)} \end{array} \right.$$

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Attractor	$e^{-\mathcal{K}_{\rm h}}$	$\lvert\mathcal{Z}_{\rm h}\rvert^2$	$\lvert\tilde{\mathcal{Z}}_{\rm h}\rvert^2$	$-V_{\rm bhh}$	M
$Z_{\rm h}^{i\,\rm susy} = \Gamma^{*\,i}/\Gamma^{*\,0}$	$\Gamma^{*\,\Lambda}\Gamma_\Lambda > 0$	$\Gamma^{*\,\Lambda}\Gamma_\Lambda$	0	$\Gamma^{*\,\Lambda}\Gamma_\Lambda$	$\lvert\mathcal{Z}_\infty\rvert$
$Z_{\rm h}^{\Lambda\,\rm nsusy}\Gamma_\Lambda = 0$	$-\Gamma^{*\,\Lambda}\Gamma_\Lambda > 0$	0	$-\Gamma^{*\,\Lambda}\Gamma_\Lambda$	$-\Gamma^{*\,\Lambda}\Gamma_\Lambda$	$\lvert\tilde{\mathcal{Z}}_\infty\rvert$

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Then, the solutions are completely determined by the harmonic functions $H^M(\tau) = H^M - \frac{1}{\sqrt{2}}\mathcal{Q}^M\tau$ with

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Defining, for convenience's sake

$$\mathcal{H}_\Lambda \equiv H_\Lambda + \tfrac{i}{2}\eta_{\Lambda\Sigma}H^\Sigma \equiv e^{\mathcal{K}_\infty/2}\frac{\mathcal{Z}_\infty}{|\mathcal{Z}_\infty|}Z^*_{\Lambda\,\infty} - \frac{1}{\sqrt{2}}\Gamma_\Lambda\tau$$

the metric function and the scalars are

$$e^{-2U} = 2\mathcal{H}^{*\,\Lambda}\mathcal{H}_\Lambda\,, \qquad\qquad Z^i = \frac{\mathcal{R}^i + i\mathcal{I}^i}{\mathcal{R}^0 + i\mathcal{I}^0} = \frac{\mathcal{H}^{*i}}{\mathcal{H}^{*0}}\,.$$

Non-extremal solutions

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Our Ansatz for the non-extremal solution is

$$e^{-2U} = e^{-2[U_{\mathrm{e}}(\mathcal{H})+r_0\tau]}\,, \qquad e^{-2U_{\mathrm{e}}(\mathcal{H})} = 2\mathcal{H}^{*\,\Lambda}\mathcal{H}_{\Lambda}\,, \qquad Z^i = Z^i{}_{\mathrm{e}}(\mathcal{H}) = \mathcal{H}^{*\,i}/\mathcal{H}^{*\,0}\,,$$

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where $\mathcal{H}^\Lambda \equiv A^\Lambda + B^\Lambda e^{2r_0\tau}\,, \quad \Lambda = 0, \cdots, n.$

The $2(n+1)$ complex constants A_Λ, B_Λ are found by imposing the e.o.m. $(f \equiv e^{r_0\tau})$

$$\begin{aligned}
\ddot{U}_{\rm e} - (\dot{U}_{\rm e})^2 - \mathcal{G}_{ij^*}\dot{Z}^i\dot{Z}^{*\,j^*} &= 0\,,\\
(2r_0)^2\left[f\ddot{U}_{\rm e} + \dot{U}_{\rm e}\right] + e^{2U_{\rm e}}V_{\rm bh} &= 0\,,\\
(2r_0)^2\left[f\left(\ddot{Z}^i + \mathcal{G}^{ij^*}\partial_k\mathcal{G}_{lj^*}\dot{Z}^k\dot{Z}^l\right) + \dot{Z}^i\right] + e^{2U_{\rm e}}\mathcal{G}^{ij^*}\partial_{j^*}V_{\rm bh} &= 0\,.
\end{aligned}$$

The e.o.m. are solved if the the constants satisfy the **algebraic** equations

$$\begin{aligned}
\Im\mathrm{m}(B^{*\,\Lambda}A_\Lambda) &= 0\,,\\
A^{*\,\Lambda}A^{\Sigma}\xi_{\Lambda\Sigma} &= 0\,,\\
(A^{*\,\Lambda}B^{\Sigma}+B^{*\,\Lambda}A^{\Sigma})\xi_{\Lambda\Sigma} &= 0\,,\\
B^{*\,\Lambda}B^{\Sigma}\xi_{\Lambda\Sigma} &= 0\,,\\
(2r_0)^2(B^*_i A^*_0 - B^*_0 A^*_i)A^{*\,\Lambda}A_\Lambda + (\Gamma^*_i A^*_0 - \Gamma^*_0 A^*_i)A^{*\,\Lambda}\Gamma_\Lambda &= 0\,,\\
-(2r_0)^2(B^*_i A^*_0 - B^*_0 A^*_i)B^{*\,\Lambda}B_\Lambda + (\Gamma^*_i B^*_0 - \Gamma^*_0 B^*_i)B^{*\,\Lambda}\Gamma_\Lambda &= 0\,,\\
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where $\xi_{\Lambda\Sigma} \equiv 2\left(\Gamma_\Lambda\Gamma^*_\Sigma + 8{r_0}^2 A_\Lambda B^*_\Sigma\right) - \eta_{\Lambda\Sigma}\left(\Gamma^\Omega\Gamma^*_\Omega + 8{r_0}^2 A^\Omega B^*_\Omega\right)\,.$

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B^{*\,\Lambda}B^\Sigma\xi_{\Lambda\Sigma} &= 0\,,\\
(2r_0)^2(B_i^*A_0^* - B_0^*A_i^*)A^{*\,\Lambda}A_\Lambda + (\Gamma_i^*A_0^* - \Gamma_0^*A_i^*)A^{*\,\Lambda}\Gamma_\Lambda &= 0\,,\\
-(2r_0)^2(B_i^*A_0^* - B_0^*A_i^*)B^{*\,\Lambda}B_\Lambda + (\Gamma_i^*B_0^* - \Gamma_0^*B_i^*)B^{*\,\Lambda}\Gamma_\Lambda &= 0\,,\\
(\Gamma_i^*A_0^* - \Gamma_0^*A_i^*)A^{*\,\Lambda}\Gamma_\Lambda + (\Gamma_i^*B_0^* - \Gamma_0^*B_i^*)B^{*\,\Lambda}\Gamma_\Lambda &= 0\,,
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No differential equations remain to be solved!

Furthermore, we need to normalize the metric at spatial infinity and relate A_Λ, B_Λ to the physical parameters:

$$
\begin{aligned}
2(A^{*\,\Lambda} + B^{*\,\Lambda})(A_\Lambda + B_\Lambda) &= 1\,, \\
4\Re\mathrm{e}[B^{*\,\Lambda}(A_\Lambda + B_\Lambda)] &= 1 - M/r_0\,, \\
\frac{A^{*\,i} + B^{*\,i}}{A^{*\,0} + B^{*\,0}} &= Z^i{}_\infty\,.
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Up to a phase to be determined in the supersymmetric extremal limit the solution is

$$
\begin{aligned}
A_\Lambda &= \pm\frac{e^{\mathcal{K}_\infty/2}}{2\sqrt{2}}\left\{Z^*_{\Lambda\,\infty}\left[1 + \frac{(M^2 - e^{\mathcal{K}_\infty}|Z^{*\,\Sigma}_\infty\Gamma^*_\Sigma|^2)}{Mr_0}\right] + \frac{\Gamma_\Lambda Z^{*\,\Sigma}{}_\infty\Gamma^*_\Sigma}{Mr_0}\right\}, \\
B_\Lambda &= \pm\frac{e^{\mathcal{K}_\infty/2}}{2\sqrt{2}}\left\{Z^*_{\Lambda\,\infty}\left[1 - \frac{(M^2 - e^{\mathcal{K}_\infty}|Z^{*\,\Sigma}_\infty\Gamma^*_\Sigma|^2)}{Mr_0}\right] - \frac{\Gamma_\Lambda Z^{*\,\Sigma}_\infty\Gamma^*_\Sigma}{Mr_0}\right\},
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$$\begin{aligned} 2(A^{*\,\Lambda}+B^{*\,\Lambda})(A_\Lambda+B_\Lambda) &= 1\,, \\ 4\Re\mathrm{e}[B^{*\,\Lambda}(A_\Lambda+B_\Lambda)] &= 1-M/r_0\,, \\ \frac{A^{*\,i}+B^{*\,i}}{A^{*\,0}+B^{*\,0}} &= Z^i{}_\infty\,. \end{aligned}$$

Up to a phase to be determined in the supersymmetric extremal limit the solution is

$$\begin{aligned} A_\Lambda &= \pm\frac{e^{\mathcal{K}_\infty/2}}{2\sqrt{2}}\left\{Z^*_{\Lambda\,\infty}\left[1+\frac{(M^2-e^{\mathcal{K}_\infty}|Z^{*\,\Sigma}_\infty\Gamma^*_\Sigma|^2)}{Mr_0}\right]+\frac{\Gamma_\Lambda Z^{*\,\Sigma}{}_\infty\Gamma^*_\Sigma}{Mr_0}\right\}, \\ B_\Lambda &= \pm\frac{e^{\mathcal{K}_\infty/2}}{2\sqrt{2}}\left\{Z^*_{\Lambda\,\infty}\left[1-\frac{(M^2-e^{\mathcal{K}_\infty}|Z^{*\,\Sigma}_\infty\Gamma^*_\Sigma|^2)}{Mr_0}\right]-\frac{\Gamma_\Lambda Z^{*\,\Sigma}_\infty\Gamma^*_\Sigma}{Mr_0}\right\}, \end{aligned}$$

Here $M^2{r_0}^2=(M^2-|\mathcal{Z}_\infty|^2)(M^2-|\tilde{\mathcal{Z}}_\infty|^2)$, and one can show that the metric is regular in all the ${r_0}^2>0$ cases.

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$$\mathcal{H}_\Lambda \stackrel{M \to |\tilde{\mathcal{Z}}_\infty|}{\longrightarrow} \pm \frac{e^{\mathcal{K}_\infty/2}}{2\sqrt{2}} \left\{ Z^*_{\Lambda\,\infty} - \frac{1}{|\tilde{\mathcal{Z}}_\infty|} \left[-Z^*_{\Lambda\,\infty}\Gamma^{*\,\Sigma}\Gamma_\Sigma + \Gamma_\Lambda Z^{*\,\Sigma}_\infty \Gamma^*_\Sigma \right] \tau \right\} .$$

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On the event horizon $\tau \to -\infty$ the scalars $Z^i = \mathcal{H}^{*\,i}/\mathcal{H}^{*\,0}$ take the values

$$Z^{*\,i}_{\rm h} = \frac{\Gamma^i Z^{*\,\Lambda}_\infty \Gamma^*_\Lambda - Z^{*\,i}_\infty \Gamma^{*\,\Sigma}\Gamma_\Sigma}{\Gamma^0 Z^{*\,\Gamma}_\infty \Gamma^*_\Gamma - \Gamma^{*\,\Omega}\Gamma_\Omega} ,$$

which depend manifestly on the asymptotic values (so there is no attractor behavior in this case).

The structure of the extremal non-supersymmetric solution as function of the H^Ms is the same as in the supersymmetric case.

However, no simple *substitution recipe* could have led to it.

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One can compute the "entropies" of the inner and outer horizons (event horizon (+) and Cauchy horizon) at $\tau \to -\infty$ and $\tau \to +\infty$ resp.:

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They can also be written in the suggestive form

$$S_\pm = \pi\left(\sqrt{N_\mathrm{R}} \pm \sqrt{N_\mathrm{L}}\right)^2\,,$$

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The product $S_+ S_-$ is manifestly moduli-independent for all values of r_0:

$$S_+ S_- / \pi^2 = (\Gamma^{*\,\Lambda} \Gamma_\Lambda)^2 \,.$$

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There is an attractor behavior in the evaporation process.

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Now, the extremality parameter is $\mathcal{B}$ and the event horizon is at $\rho \to +\infty$ ($\rho = -\tau$ in $d = 4$). In general the inner horizon is not covered by the metric. One arrives to the effective mechanical system

$$\mathcal{I}[U, \phi^i] = \int d\rho \left\{(\dot{U})^2 + \tfrac{(d-3)}{(d-2)}\, \mathcal{G}_{ij}\dot{\phi}^i\dot{\phi}^j - e^{2U} V_{\text{bh}} + \mathcal{B}^2\right\} ,$$

where the black-hole potential is given by (only electric charges)

$$V_{\text{bh}} = \alpha^2\, \tfrac{2(d-3)}{(d-2)}\, \Im^{\Lambda\Sigma} q_\Lambda q_\Sigma .$$

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In Meessen & O. `arXiv:1107.5454` we showed, by direct integration of the equations of motion of the effective mechanical system, that the deformation procedure used in $d = 4$ dimensions also works in simple examples of $N = 2 \;\; d = 5$ Supergravity coupled to vector supermultiplets.

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★ We are currently working on generalizations to non-static solutions and to $p \neq 0$ black branes.

We may be close to determining the general form of all single, static, black-hole solutions of $N = 2$,$d = 4, 5$ theories.