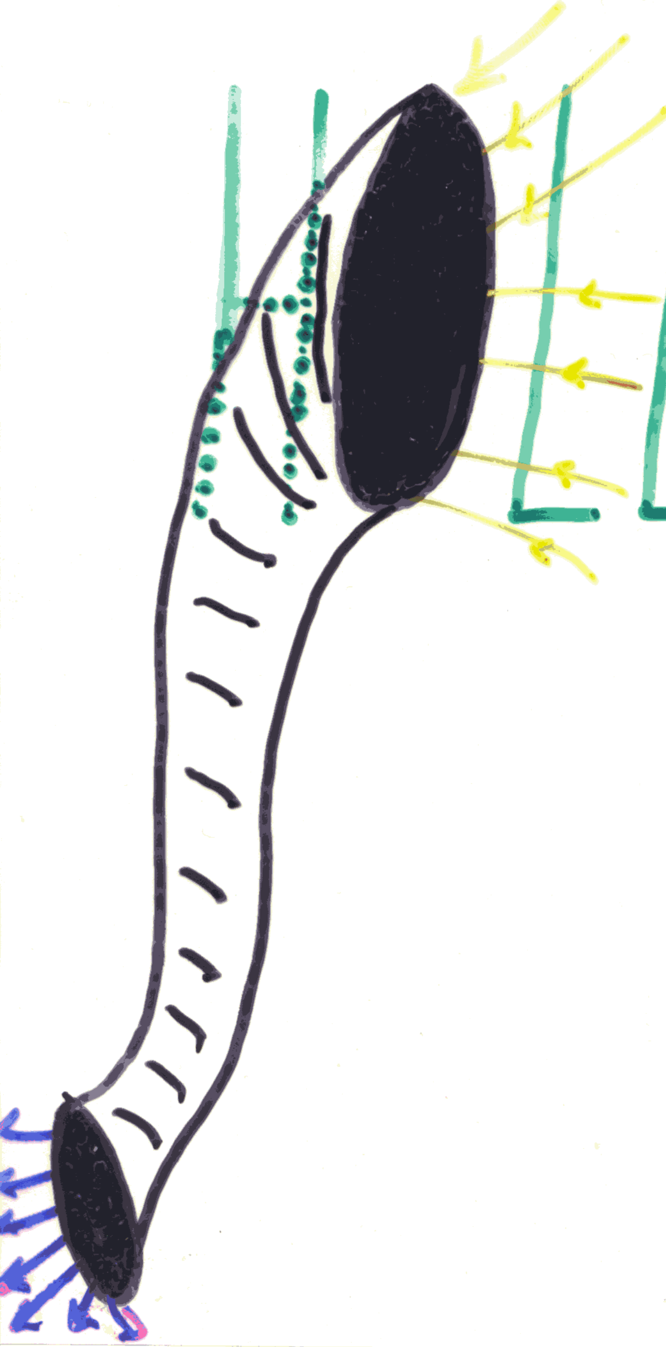
SUPERSYMMETRIC BLACK HOLES



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OVERVIEW OF
- UNBROKEN SUPERSYMMETRY
- BLACK HOLES
- DUALITY

GOAL: To show how supersymmetry can help in the study and classification of black-hole solutions (even in the absence of fermions!) together with duality symmetries.

INTRODUCTION / MOTIVATION

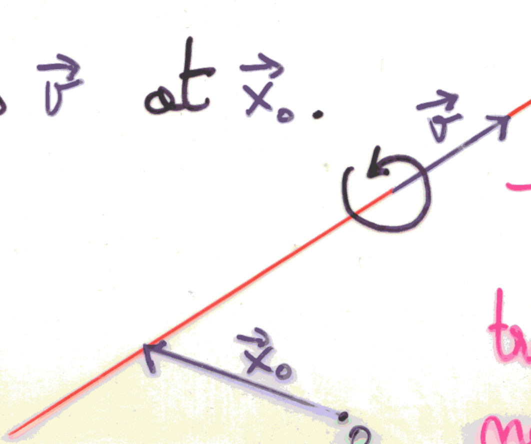
Central concept: "Residual symmetry"

> The solutions of the equations of motion of a theory do not, in general, have the symmetries of the theory. When they have some of the symmetries of the theory, we say that they have residual symmetries. The remaining symmetries of the theory become solution-generating transformations.

Simplest example: the Lagrangian of a classical point particle $L = \frac{1}{2} m \dot{\vec{x}}^2$ is invariant under the whole Galileo group of transformations $\begin{pmatrix} \mathbb{R}^3 \text{ trans.} \\ SO(3) \text{ rot.} \end{pmatrix}$

However, a typical solution $\vec{x}(t) = \vec{v}\, t + \vec{x}_0$ is only invariant under translations in the direction $\vec{v}$ and rotations with axis $\vec{v}$ at $\vec{x}_0$.

(If $\vec{v} = \vec{0} \rightarrow SO(3)$ but no translations)



The remaining transformations generate new solutions

Another example: The idea of spontaneous symmetry breaking is based on the expansion of a Lagrangian around a classical solution (vacuum) that does not have all the original (gauge) symmetry

Ex.:

$$L = -\frac{1}{4}F^2 + |D\phi|^2 - \lambda\left(|\phi|^2 - \frac{a^2}{2}\right); \quad (U(1))$$

L is invariant under $\begin{cases} A_\mu \to A_\mu - \frac{1}{e}\partial_\mu \alpha(x); \\ \phi \to e^{-i\alpha(x)}\phi; \end{cases}$

There is a whole family of solutions for which ϕ minimizes $V(\phi)$: $\boxed{\phi = e^{i\beta}\frac{a}{\sqrt{2}}; \; A_\mu = 0;}$

None of them is gauge invariant.

Performing gauge transformations on them generates new solutions

$$\phi = e^{i(\beta - e\alpha(x))}\frac{a}{\sqrt{2}}; \quad A_\mu = -\frac{1}{e}\partial_\mu \alpha(x).$$

If $\alpha(x) \xrightarrow[x\to\infty]{} 0$ these solutions are gauge equivalent

Otherwise they are physically inequivalent $\to$ multiplets

Expanding $\phi = \frac{a}{\sqrt{2}} + \eta$; $\eta \in \mathbb{R}$ (unitary gauge)

$$L(\eta, A) = -\frac{1}{4}F^2 + \frac{e^2 a^2}{4}A^2 + \frac{1}{2}(\partial\eta)^2 + 2\lambda a^2\eta^2$$
$$+ e^2\eta\left(a\sqrt{2} + \frac{\eta}{2}\right)A^2 + 2\sqrt{2}\,a\lambda\eta^3 + \lambda\eta^4$$

A closer example: The Einstein equations are covariant under arbitrary coordinate transformations. Particular solutions are only invariant under a few: ISOMETRIES

e.g. Schwarzschild $ds^2 = \left(1-\frac{2m}{r}\right)dt^2 - \left(1-\frac{2m}{r}\right)^{-1}dr^2 - r^2 d\Omega^2$;

is invariant under $\begin{cases} t \quad t+\Lambda; \\ \vec{x} \to M\vec{x}; \quad M \in SO(3) \end{cases}$;

Isometries are associated to KILLING VECTORS k^μ

$\mathcal{L}_k g_{\mu\nu} \sim \nabla_{(\mu}k_{\nu)} = 0$; (Killing equation)

Isometries are the residual symmetries in this case.

Any solution satisfies certain identities just because of the full symmetry of the theory:

$$\delta_\varepsilon \int d^4x\sqrt{-g}\,R = \int d^4x\sqrt{-g}\,\delta_\varepsilon g_{\mu\nu}\,G^{\mu\nu} =$$

$$= \int d^4x\sqrt{-g}\,\nabla_{(\mu}\varepsilon_{\nu)}\,G^{\mu\nu} = 0; \quad \forall\,\varepsilon$$

$\Rightarrow \boxed{\nabla_\mu G^{\mu\nu} = 0}$ (Bianchi identity)

If there is matter $\Rightarrow \boxed{\nabla_\mu T^{\mu\nu} = 0}$ (e-m conservation)

RESIDUAL SUPERSYMMETRIES

Theories with local (global) supersymmetry are invariant under arbitrary local (global) superreparametrizations in superspace.

In ordinary space this is equivalent to local (global) Poincaré transformations **PLUS** local (global) supersymmetry transformations of the fields that relate bosonic (ϕ^b) and fermionic (ϕ^f) fields

$$\left\{\begin{array}{l} \delta_\varepsilon \phi^b \sim \varepsilon \phi^f ; \\ \delta_\varepsilon \phi^f \sim \varepsilon \phi^b ; \end{array}\right. \qquad \left.\begin{array}{l} \varepsilon(x) \\ \varepsilon \end{array}\right\} \begin{array}{l} \text{(local)} \\ \text{(global)} \end{array}$$

We are interested in solutions of supergravity theories which have residual supersymmetries: they are invariant under some supersymmetry transformations generated by KILLING SPINORS.

We will consider only purely bosonic solutions $\boxed{\phi^f = 0}$

We call these solutions **SUPERSYMMETRIC.**

It is enough to consider the bosonic part of the action and the SUSY transformation rules of the fermions

$$\delta_\varepsilon \phi^b \sim \phi^f = 0 \quad \text{automatically}$$

$$* \quad \boxed{\delta_\varepsilon \phi^f = F(\varepsilon, \phi^b) = 0 \rightarrow \text{Killing spinor equation}}$$

The bosonic part of the SUGRA theories is not exotic:

$N=1$ $\{g_{\mu\nu}\}$

$$\begin{cases} S = \int d^4x \sqrt{-g}\, R\,; & \text{(Einstein-Hilbert)} \\ \delta_\varepsilon \psi_\mu = \nabla_\mu \varepsilon\,; \end{cases}$$

$N=2$ $\{g_{\mu\nu}, A_\mu\}$

$$\begin{cases} S = \int d^4x \sqrt{-g}\left(R + F^2\right); & \text{(Einstein-Maxwell)} \\ \delta_\varepsilon \psi_\mu = \nabla_\mu \varepsilon + \cancel{F}\gamma_\mu \gamma_5 \varepsilon\,; \end{cases}$$

graviphotons

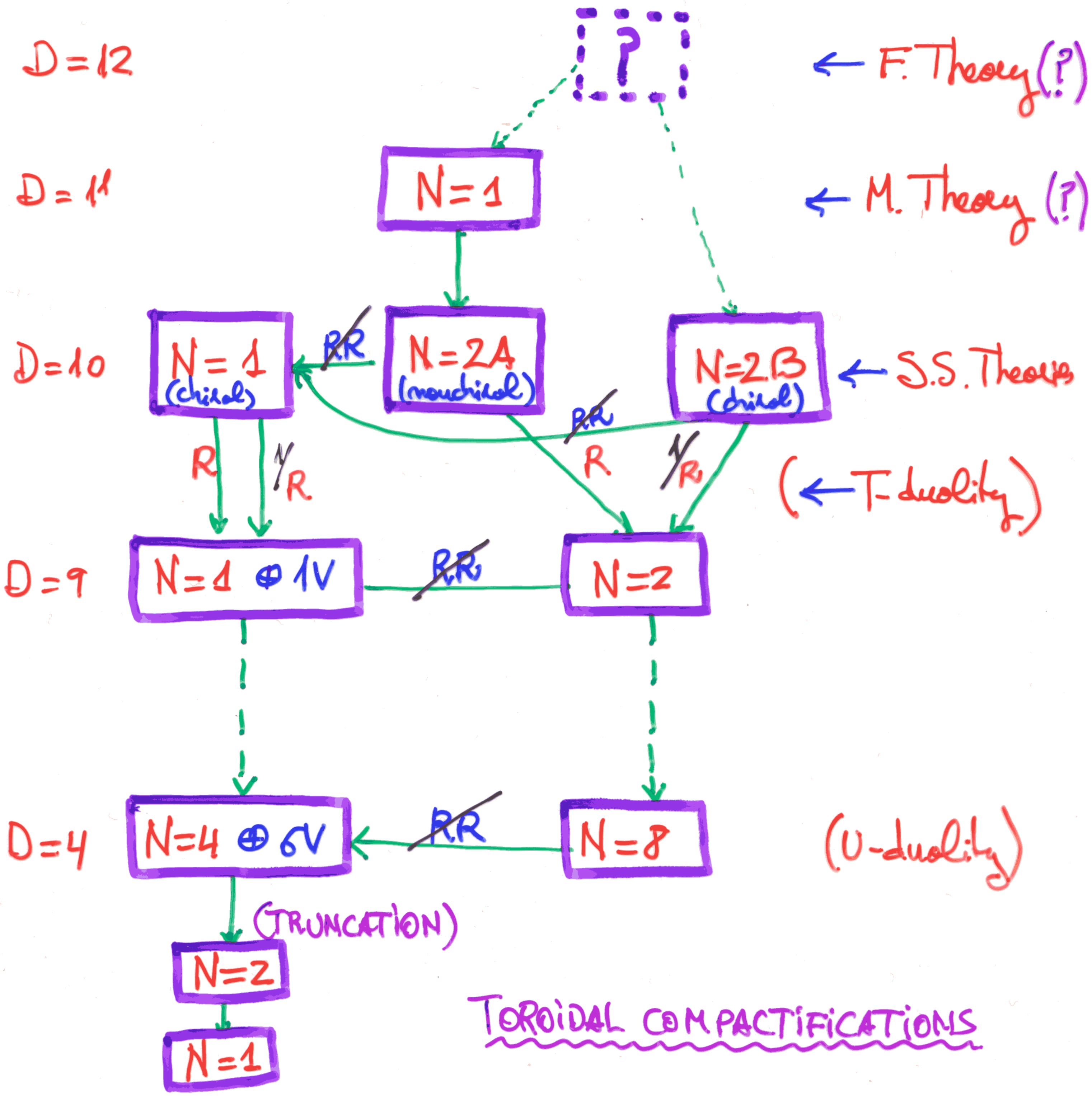
$N=4$ $\{g_{\mu\nu}, A^I_\mu, \overbrace{\phi, a}^{\lambda}\}$

$$\begin{cases} S = \int d^4x \sqrt{-g}\left(R + \frac{1}{2}\frac{\partial\lambda\,\partial\bar{\lambda}}{(\Im m\,\lambda)^2} + \left(i\lambda F^{+2} + c.c.\right)\right) \\ \delta \psi^I_\mu = \nabla_\mu \varepsilon^I + \cdots \\ \delta \Lambda^I = \cdots \end{cases}$$

→ All bosonic solutions of supergravity are solutions of the Einstein equations with special forms of matter.

SUPERGRAVITIES/SUPERSTRINGS

DIRECTORY (Simplified)



TOROIDAL COMPACTIFICATIONS

WHY?

i) Embedding a solution in SUGRA allows one to use the SUGRA machinery to derive generic properties:

- the "SUPER-BIANCHI IDENTITIES"
- positivity bounds for the mass ⇒ NESTER construction
  (Gibbons, Chester, Israel ...)

ii) When the solution is supersymmetric (has Killing spinors)

- the super-Bianchi identities simplify and become "KILLING SPINOR IDENTITIES"
- the positivity bounds are saturated and new phenomena occur:
  - equilibrium of forces ⇒ multicenter solutions
  - non-renormalization theorems.
  - alternative vacua (lowest possible energy)

  ⋮

- supersymmetric solutions are MUCH SIMPLER

e.g. $\boxed{\text{Killing spinor } \epsilon \longrightarrow \bar{\epsilon}\gamma^{\mu}\epsilon \text{ Killing vector} \quad \text{less independent fields}}$

POSITIVITY BOUNDS

If we think of a solution as representing a STATE of the SUGRA theory, it must be in a representation of the SUSY algebra (N-extended).

$$\{Q^i_\alpha, \bar{Q}^j_\beta\} = (\gamma^\mu C^{-1})_{\alpha\beta} P_\mu \ ; \ \{Q^i_\alpha, Q^j_\beta\} = \epsilon_{\alpha\beta} Z^{ij} \ ;$$

$Z^{ij} = -Z^{ji}$ CENTRAL CHARGE MATRIX $i,j = 1, \dots, N$

For massive representations, in the rest frame $(m, \vec{0})$ there is always a basis in which

$$\{S^n_{\alpha(\pm)}, \bar{S}^m_{\beta(\pm)}\} = \delta_{\alpha\beta}\, \delta^{mn} (m \pm |z_m|) \qquad m,n = 1 \dots [N/2]$$

$z_m \to$ skew eigenvalues of Z^{ij}

$$\Rightarrow (m \pm |z_m|) \geq 0 \implies \boxed{m \geq |z_m|}$$

Bogomol'nyi bound

When $m = |z_m|$ the state is annihilated by a SUSY charge $\Rightarrow$ the associated solution admits a Killing spinor and is supersymmetric

{Witten, Olive} {Ferrara, Savoy, Zumino}

What are the Z_m's ?

In $N=2$ SUGRA $Z = Q + iP \rightarrow m \geq \sqrt{Q^2+P^2}$

$Q \rightarrow$ electric charge $\Big\}$ with respect to the
$P \rightarrow$ magnetic charge $\Big\}$ graviphoton

For higher N, the Z_m's are more complicated combinations of the electric Q^I and magnetic P^I charges of the graviphotons and of the asymptotic values of the scalars "MODULI".

When all bounds are saturated

$$m = |Z_1| = \dots = |Z_{[N/2]}| ;$$

the state has the lowest possible mass for the given charges $\Rightarrow$ GROUND STATE (VACUUM?)

STABILITY

SOLITONIC NATURE

no quantum corrections

$\Rightarrow$ **BPS STATE** (Elementary state?)

The natural testing ground: **BLACK HOLES**

BLACK HOLES FOR PEDESTRIANS

For us, a (static) **BH** is a spherically symmetric asymptotically flat object with an event horizon: null hypersurface from whose interior light signals cannot reach the outer region.
(Sometimes we won't care about the horizon)

Prototype: $(R_{\mu\nu}=0)$

Schwarzschild's: $ds^2=\left(1-\frac{2m}{r}\right)dt^2-\left(1-\frac{2m}{r}\right)^{-1}dr^2-r^2d\Omega^2,$ $\quad m \to$ ADM mass

Singularity at $r=0$;
Horizon at $r=2m$ → covers the singularity (If $m>0$)

Temperature

near the horizon $(r=2m)$; $\quad r' \equiv r-2m$

$$ds^2 \sim \frac{r'}{2m}dt^2-\frac{2m}{r'}dr'^2 \quad (+\text{ angular part})$$

Wick rotation $\quad t\to i\tau \quad ; \quad \rho = 2\sqrt{2mr'}$

$$\Rightarrow ds^2 \sim d\rho^2+\frac{1}{16m^2}\rho^2 d\tau^2 \sim \text{polar coordinates}$$

$$\frac{\tau}{4m}\in[0,2\pi] \quad ; \quad \tau\in[0,\underbrace{8\pi m}_{\beta}]$$

$$\boxed{T=\frac{1}{8\pi m}}$$ → Hawking temperature

The area A of the horizon is associated to the "entropy" of the BH (Bekenstein)

$$\boxed{S = \frac{A}{4} = \frac{4\pi r_{hor}^2}{4} = 4\pi m^2}$$

The thermodynamical relation $T = \left(\frac{\partial S}{\partial E}\right)^{-1}$ is obeyed with $\boxed{E = m}$ (but $\frac{\partial m}{\partial T} < 0$!)

The area of a BH never decreases (classically) (just like the entropy). Quantum mechanically they can radiate (Hawking) but the total entropy ($\frac{A}{4}$ + radiation) does not decrease.

S can also be calculated via Euclidean PI.

Recently, it has been calculated by statistical mechanics methods using D-brane technology in certain simple cases: SUPERSYMMETRIC BHs.

T, S, m, (J), are, otherwise, purely geometrical quantities.

SUPERSYMMETRY & BHs

N=1 The B-bound is $\boxed{m \geq 0} \Rightarrow$ flat space.

Supersymmetry implies the positivity of mass.

Unbroken supersymmetry implies $T=0$ $(m=0)$ $(\cancel{x})$
$S=0$ (vacuum)

N=2 (pure → only Einstein-Maxwell) $\Rightarrow \boxed{m \geq \sqrt{q^2+p^2}}$

The only BH solutions are the Reissner-Nordström BHs

$$\boxed{\begin{array}{l} ds^2 = \frac{(r-r_+)(r-r_-)}{r^2}\, dt^2 - \frac{r^2}{(r-r_+)(r-r_-)}\, dr^2 - r^2 d\Omega^2\,; \\[2ex] A_t = \frac{q}{r}\;; \quad \tilde{A}_t = \frac{p}{r}\;; \qquad r_\pm = m \pm \sqrt{m^2-(q^2+p^2)}\;; \end{array}}$$

$r_+ \to$ event horizon

$r_- \to$ Cauchy horizon

$m^2 = q^2+p^2 \Rightarrow r_+ = r_- \Rightarrow$

$\Rightarrow$ EXTREMAL LIMIT

$m^2 < q^2+p^2$ $\left\{ \begin{array}{l} \text{no horizons} \\ \text{NAKED SINGULARITY} \end{array} \right.$

If the RN solution represents a state of N=2 SUGRA

SUSY $\Rightarrow$ NO NAKED SINGULARITIES ("COSMIC CENSORSHIP")

Extreme RN $\Rightarrow$ B-bound saturated $\Rightarrow$ $\left\{ \begin{array}{l} \text{unbroken SUSY} \\ \text{Killing spinors} \end{array} \right.$

$T = \frac{1}{4\pi} \frac{r_+ - r_-}{r_+^2}$; $\quad S = \pi r_+^2$;

$\left.\begin{cases} T \to 0 \\ S \to \pi m^2 \neq 0 \end{cases}\right\}$ in the extremal limit (supersymmetric)

SUSY is usually associated to $T=0$, but in this context there is no theorem.

$S \neq 0$ seems against an elementary state interpretation.

If we have several objects with $m_i = q_i$ (SUSY) the force between any two of them vanishes:

$\Rightarrow$ $F_{ij} = -\frac{m_i m_j}{r_{ij}^2} + \frac{q_i q_j}{r_{ij}^2} = 0 \Rightarrow$ EQUILIBRIUM

$\Rightarrow$ It makes sense to look for STATIC solutions describing several supersymmetric BHs in equilibrium.

TRICK: $m = q \Rightarrow \frac{(r - r_+)(r - r_-)}{r^2} = \left(1 - \frac{m}{r}\right)^2$

Shift $\rho = r - m$

$$\Longrightarrow \boxed{\begin{array}{l} ds^2 = V^{-2} dt^2 - V^2 d\vec{x}^2 \ , \quad V = 1 + \frac{m}{r} \\ A_t = V^{-1} ; \end{array}}$$

Using this as ansatz $\Rightarrow$ $\partial_i \partial_i V = 0 \Rightarrow$ harmonic function

Choosing the harmonic function $\boxed{V = 1 + \sum_{i=1}^{N} \frac{m_i}{|\vec{x} - \vec{x}_i|}}$

we get a metric (Majumdar Papapetrou)

that describes N extreme RN BHs with $m_i = q_i$

in equilibrium. (Hartle Hawking)

This family of solutions admits Killing spinors for ANY V ($\frac{1}{2}$ of $N=2$) "BPS STATES"

(Simple!!)

ARE THERE MORE SOLUTIONS OF $N=2$ SUGRA WITH RESIDUAL SUPERSYMMETRY?

STATIC → NO

STATIONARY → YES ⇒ IWP SOLUTIONS } t-like Killing vector (Gibbons Hull Tod)

OTHER → pp-WAVES

BUT NO MORE SUPERSYMMETRIC BH SOLUTIONS

(Till recently, the KERR-NEWMANN family was the only family of BHs known, and are solutions of $N=2$ SUGRA. It is time to move to higher N ...)

THE SUPERSYMMETRIC BHs OF $N=4$ SUGRA

$N=4$: $\{ g_{\mu\nu}, A^{(a)}_\mu, B^{(a)}_\mu, \phi, a \}$ $\lambda = a + i e^{-2\phi}$
$a = 1,2,3$ $SL(2,\mathbb{R})$ coset

$$S = \int d^4x \sqrt{g} \left\{ -R + \frac{1}{2} \frac{\partial_\mu \lambda \partial^\mu \bar{\lambda}}{(\mathrm{Im}\,\lambda)^2} + 2 \mathrm{Re}\left(i\lambda \sum_{a=1}^{3} \left((F^{(a)+})^2 + (G^{(a)+})^2 \right) \right) \right\}$$

$$\begin{cases} \frac{1}{2}\delta_\epsilon \psi_{\mu I} = \nabla_\mu \epsilon_I - \frac{i}{4} e^{2\phi} (\partial_\mu a)\, \epsilon_I - \frac{1}{2\sqrt{2}} e^{-\phi} \left(\not{F}^{(a)} \alpha^{(a)}_{IJ} + i \not{G}^{(a)} \beta^{(a)}_{IJ} \right) \gamma_\mu \epsilon^J ; \\ \frac{1}{2}\delta_\epsilon \Lambda_I = -\frac{i}{2} e^{2\phi} (\not{\partial}\lambda)\, \epsilon_I + \frac{1}{\sqrt{2}} e^{-\phi} \left(\not{F}^{(a)} \alpha^{(a)}_{IJ} + i \not{G}^{(a)} \beta^{(a)}_{IJ} \right) \epsilon^J ; \end{cases}$$

The scalars couple to the vector fields:

$$\underset{\substack{\uparrow \\ 1/g^2}}{e^{-2\phi}} F^2 + i \underset{\substack{\uparrow \\ \frac{\theta}{2\pi}}}{a} F^*F \quad \Rightarrow \left. \begin{matrix} e^{-2\phi} \\ a \end{matrix} \right\} \text{"CONSTANTS"}$$

COUPLING "CONSTANTS"

$N=4$, $d=4$ SUGRA can be obtained by dimensional reduction of $N=1$, $d=10$ SUGRA, the effective field theory of the heterotic string. Then ϕ is the DILATON and $g = e^{\phi}$ the string coupling constant (counting string loops).

a is related to the AXION 2-FORM $B_{\mu\nu}$ by

$$\boxed{\partial^\mu a \sim \varepsilon^{\mu\nu\rho\sigma} \partial_\nu B_{\rho\sigma}}$$

Finally, the metric is related to the σ-model (string) metric $G_{\mu\nu}$ by $\boxed{g_{\mu\nu} = e^{-2\phi} G_{\mu\nu}}$

$\Rightarrow$ What may look singular in $g_{\mu\nu}$ may not be singular for strings ($G_{\mu\nu}$).

WHAT SHOUD WE EXPECT FROM SUSY?

Z^{IJ}, $I,J = 1,\dots,4$ has two complex skew eigenvalues z_1, z_2

$\Rightarrow$ two B-bounds $\boxed{\begin{cases} m \geqslant |z_1| \\ m \geqslant |z_2| \end{cases}}$

When $|z_1| \neq |z_2|$ and one bound is saturated

$m = |z_1|$, $m > |z_2| \neq |z_1| \rightarrow$ $\boxed{\text{1/4 residual SUSY}}$

When $|z_1| = |z_2|$ both bounds can be simultaneously saturated $m = |z_1| = |z_2| \rightarrow$ $\boxed{\text{1/2 residual SUSY}}$

In general $m^2 \geq |z_{1,2}|^2$

$$\Rightarrow \frac{(m^2-|z_1|^2)(m^2-|z_2|^2)}{m^2} \geq 0$$

$$\boxed{m^2 + \frac{|z_1|^2|z_2|^2}{m^2} - (|z_1|^2+|z_2|^2) \geq 0;}$$

We will see that

scalar "charge" $\longrightarrow$

$$\begin{cases} |z_1|^2+|z_2|^2 = e^{2\phi_0}\sum_i (q_i^2+p_i^2); \\ \dfrac{|z_1|^2|z_2|^2}{m^2} = \Sigma^2+\Delta^2; \end{cases}$$

BH SOLUTIONS

Truncation: $S = \int d^4x\sqrt{-g}\left\{-R+2(\partial\phi)^2 - e^{-2\phi}F^2\right\}$

$\{g_{\mu\nu}, A_\mu, \phi\}$ (o.k. if $F^*F=0$)

The important point is to realize that $\phi=0 \Rightarrow F=0$

$$\boxed{\nabla^2\phi \sim e^{-2\phi}F^2}$$

(F non-null)

$(a=1)$ charged dilaton BH solution (Gibbons, Garfinkle, Horowitz, Strominger)

$$\boxed{\begin{cases} ds^2 = \left[\dfrac{(r-r_+)(r-r_-)}{R^2}\right]dt^2 - \left[\quad\right]^{-1}dr^2 - R^2 d\Omega^2; \\ e^{-2\phi} = e^{-2\phi_0}\dfrac{r-\Sigma}{r+\Sigma}; \qquad r_\pm = m\pm\sqrt{m^2+\Sigma^2-e^{2\phi_0}q^2}; \\ F_{tr} = \dfrac{e^{\phi_0}q}{(r-\Sigma)^2}; \qquad \Sigma = \dfrac{-e^{-2\phi_0}q^2}{2m}; \quad R^2 = r^2-\Sigma^2; \end{cases}}$$

The structure is very similar to RN's:

Event horizon: $r = r_+$

Cauchy horizon: $r = r_-$

Singularity: $r = |\Sigma|$

$r_+ = r_- \Rightarrow m^2 + \Sigma^2 = e^{-2\phi_0} q^2 \rightarrow m = |q| = |\Sigma|$

$\rightarrow$ extremal limit

$m^2 + \Sigma^2 \lneq e^{-2\phi_0} q^2$

$\rightarrow$ naked singularity

(The geometry of the extremal limit is not always singular in the string metric. When $m^2 + \Sigma^2 = e^{-2\phi_0} q^2$ we can talk of a "singular horizon".)

$$T = \frac{1}{4\pi} \frac{r_+ - r_-}{r_+^2 - \Sigma^2} \xrightarrow{r_+ \to r_-} \frac{1}{8\pi m} \neq 0 \, ! \text{ (Schwarzschild's)}$$

$$S = \pi (r_+^2 - \Sigma^2) \xrightarrow{r_+ \to r_-} 0;$$

However, if one calculates directly T, one finds that there is no need to compactify τ

$\Rightarrow$ the temperature is $T = 0$ or, better, is not well defined. (Similar to Minkowski in $N=1$)

Extreme dilaton black holes of this kind could correspond to (elementary / ground) states.

With only q $|Z_1| = |Z_2| \Rightarrow$ both B-bounds are saturated simultaneously. To have $|Z_1| \neq |Z_2|$ we need 2 vector fields

The metric has the same form but with

$$\begin{cases} F_{tr} = \dfrac{q}{(r-\Sigma)^2} \\ {}^*G_{tr} = \dfrac{i\,p}{(r-\Sigma)^2} \quad \text{(magnetic)} \end{cases}$$

$$\boxed{\Sigma = \frac{p^2 - q^2}{2m}; \quad m^2 + \Sigma^2 - e^{2\phi_0}(q^2+p^2)}$$

Now there are more ways of reaching the extremal limit apart from $|Z_1| = |Z_2| = m$.

When $|Z_1| \neq |Z_2|$ we get $1/4$ of the supersymmetries unbroken. When $|Z_1| = |Z_2|$ we get $1/2$. ($\to$ diagrams)

Something special happens in the "corners" of the figures: all extreme BHs have $1/4$ of $N=4$ residual supersymmetry but those in the corners have $1/2$ RESIDUAL SUPERSYMMETRY

$\boxed{\text{Non-renormalisation theorems etc.} \Rightarrow \text{NO quantum corrections to the masses or charges of these states} \to \textbf{BPS}}$

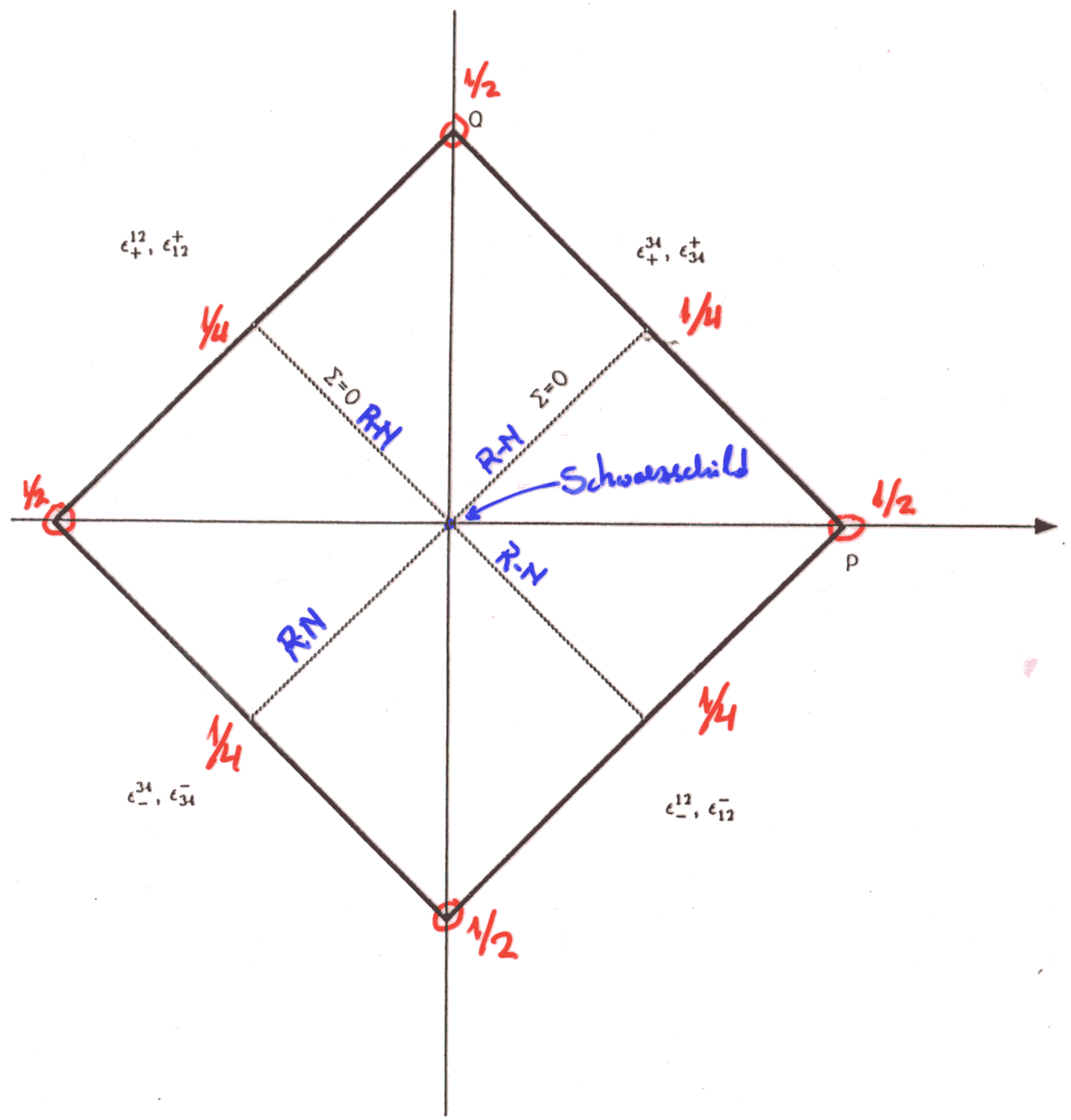


FIG. 1. The space of electrically and magnetically charged dilaton black holes with charges P and Q and a fixed mass M. $Q_{\max} = P_{\max} = \sqrt{2}M$. Every point inside the square corresponds to a regular black hole. The points outside the square (which are forbidden by supersymmetry) correspond to metrics with naked singularities. The points *on* the square correspond to extreme black holes. The unbroken $N = 1$ supersymmetries for the extreme black holes on each of the four sides (I, II, III, IV) of the square are shown. In the corners, we have unbroken $N = 2$ supersymmetry.

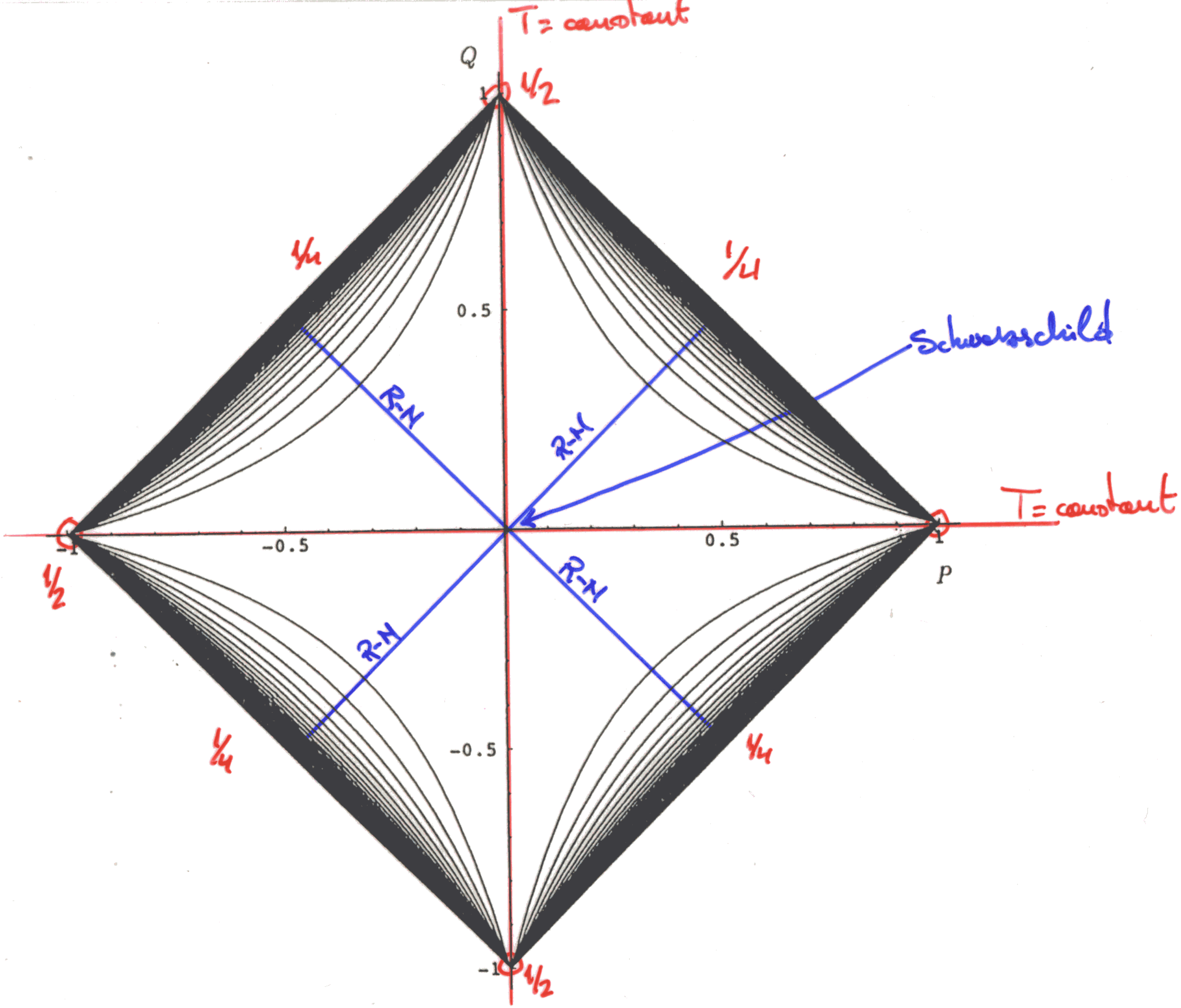


FIG. 2. Isothermals in the space of charged dilaton black holes of constant mass. The interval of temperature between two contiguous isothermals is $\frac{1}{50}T_{\max}$, where $T_{\max} = \frac{1}{8\pi M}$ is a temperature of the Schwarzschild black hole with a mass M. The two axes of coordinates are isothermals corresponding to $T = \frac{1}{8\pi M}$. The four sides of the square (excluding the corners) are isothermals corresponding to $T = 0$. The corners are very special: All the isothermals (for all the different allowed temperatures) converge to the corners. This can be better seen in Fig. 3.

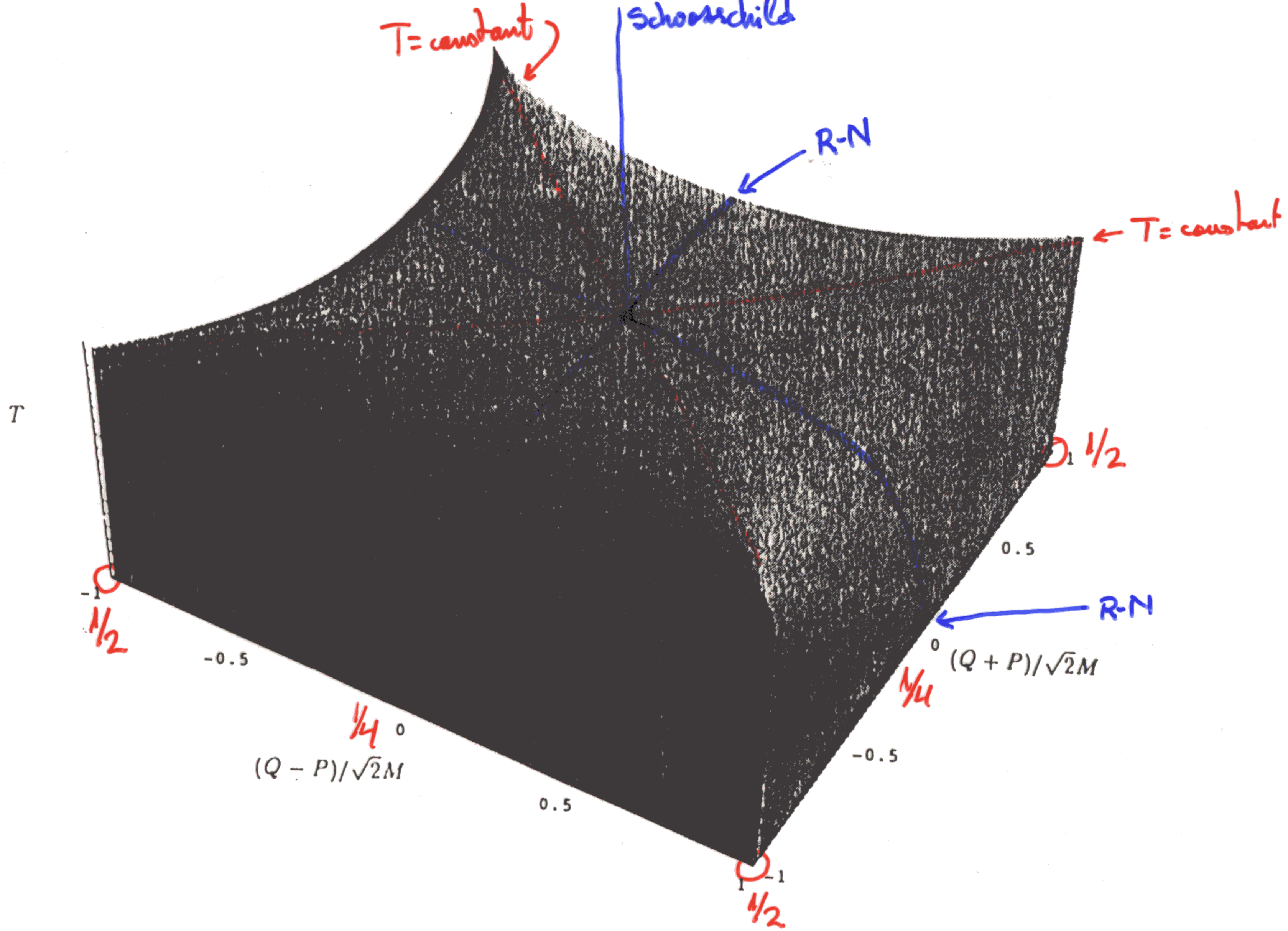


FIG. 3. The temperature of charged dilaton black holes of a given mass M as a function of $z_1/M = (Q-P)/\sqrt{2}M$ and $z_2/M = (Q+P)/\sqrt{2}M$. The extrème black holes correspond to the sides $|z_1|/M = 1$ and $|z_2|/M = 1$ of the square.

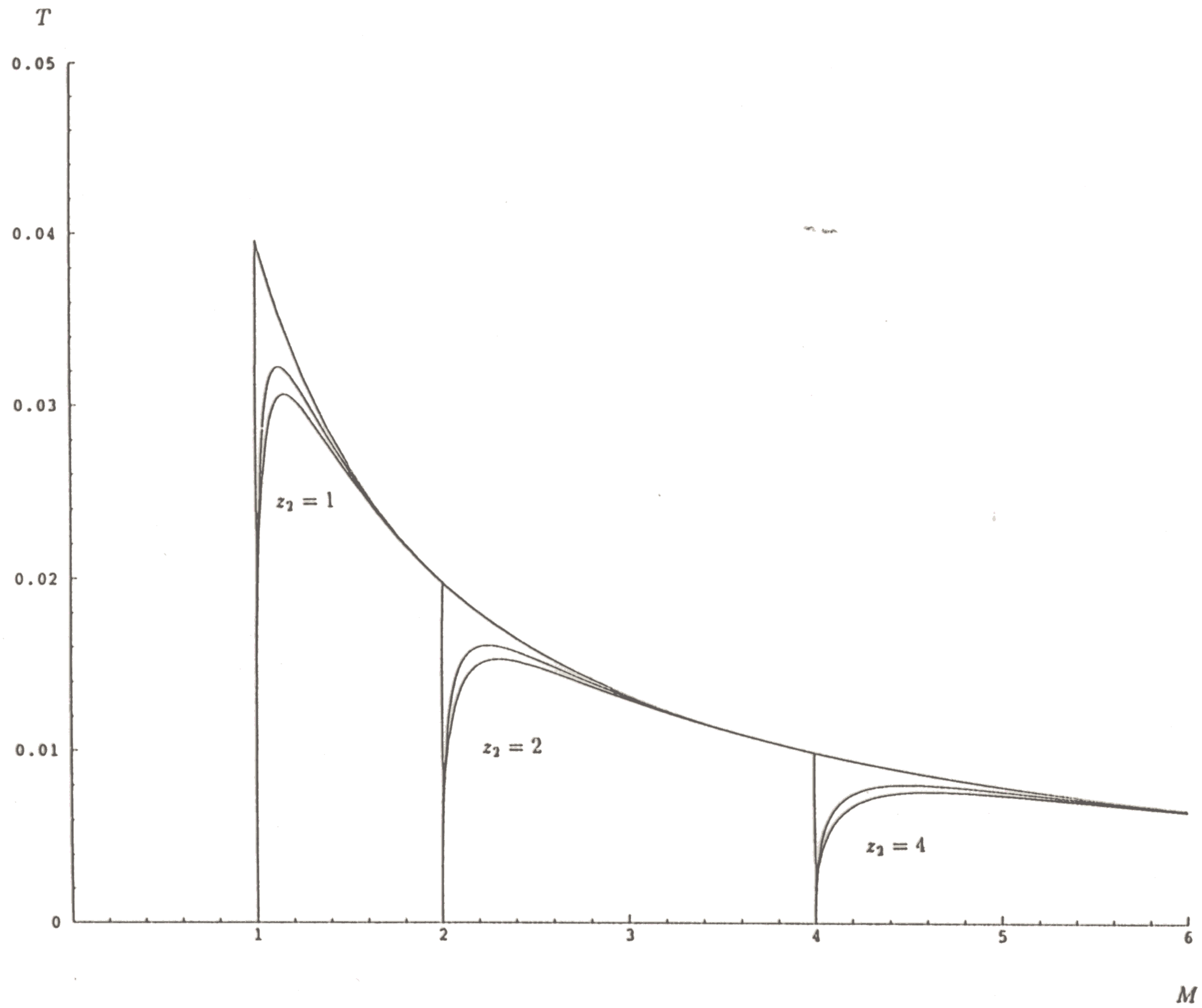


FIG. 4. The temperature vs the mass for different electric and magnetic charges. For definiteness, we take $Q > P > 0$. The black hole evaporates until its mass approaches the limiting value $M_{\rm extr} = z_2 = (Q+P)/\sqrt{2}$. The three families of curves correspond to $z_2 = 1$, $z_2 = 2$, and $z_2 = 4$. For each of these values of $z_2 = (Q+P)/\sqrt{2}$ we choose three different P/Q ratios: $P/Q = 1, \frac{1}{4}, 0$. The smoothest curves are the ones with $P = Q$ (classical Reissner-Nordström). The sharpest correspond to the limit $P/Q \to 0$, which reproduces purely electric dilaton black holes. There is always a maximum for the temperature (a point where the specific heat diverges and reverses sign), and always the temperature falls sharply to zero in the vicinity of the bound. This implies the breakdown of the thermal description when we approach extremality *for all values of P and Q*.

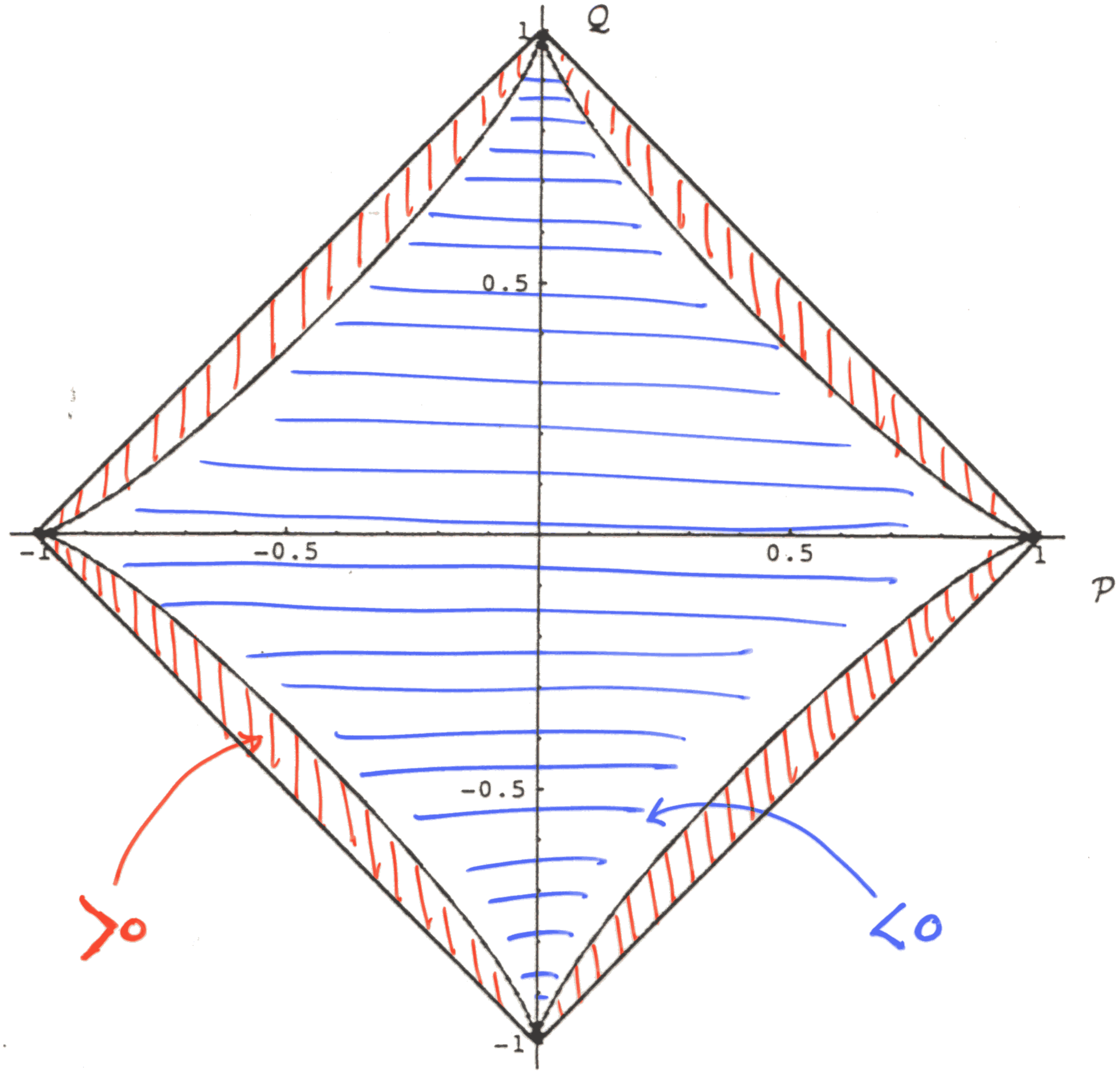


FIG. 7. The locus of points in the $(\mathcal{P}, \mathcal{Q})$ plane where the specific heat of a fixed-mass charged dilaton black hole diverges. Inside the curve, the specific heat is negative; outside, it is positive.

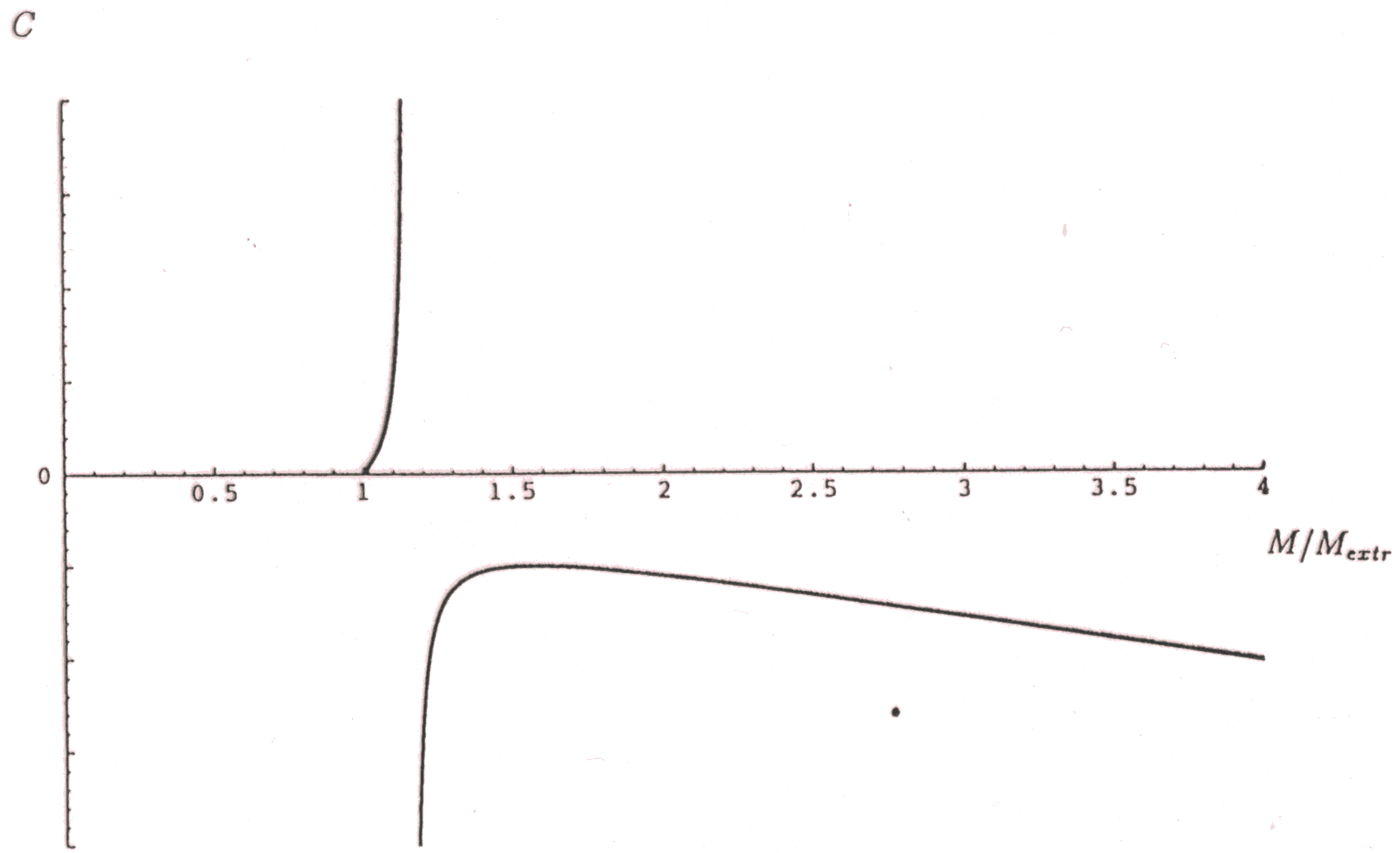


FIG. 6. The specific heat of a Reissner-Nordström black hole with $P = Q > 0$ as a function of $M/M_{\mathrm{extr}} = \sqrt{2}M/(Q+P)$.

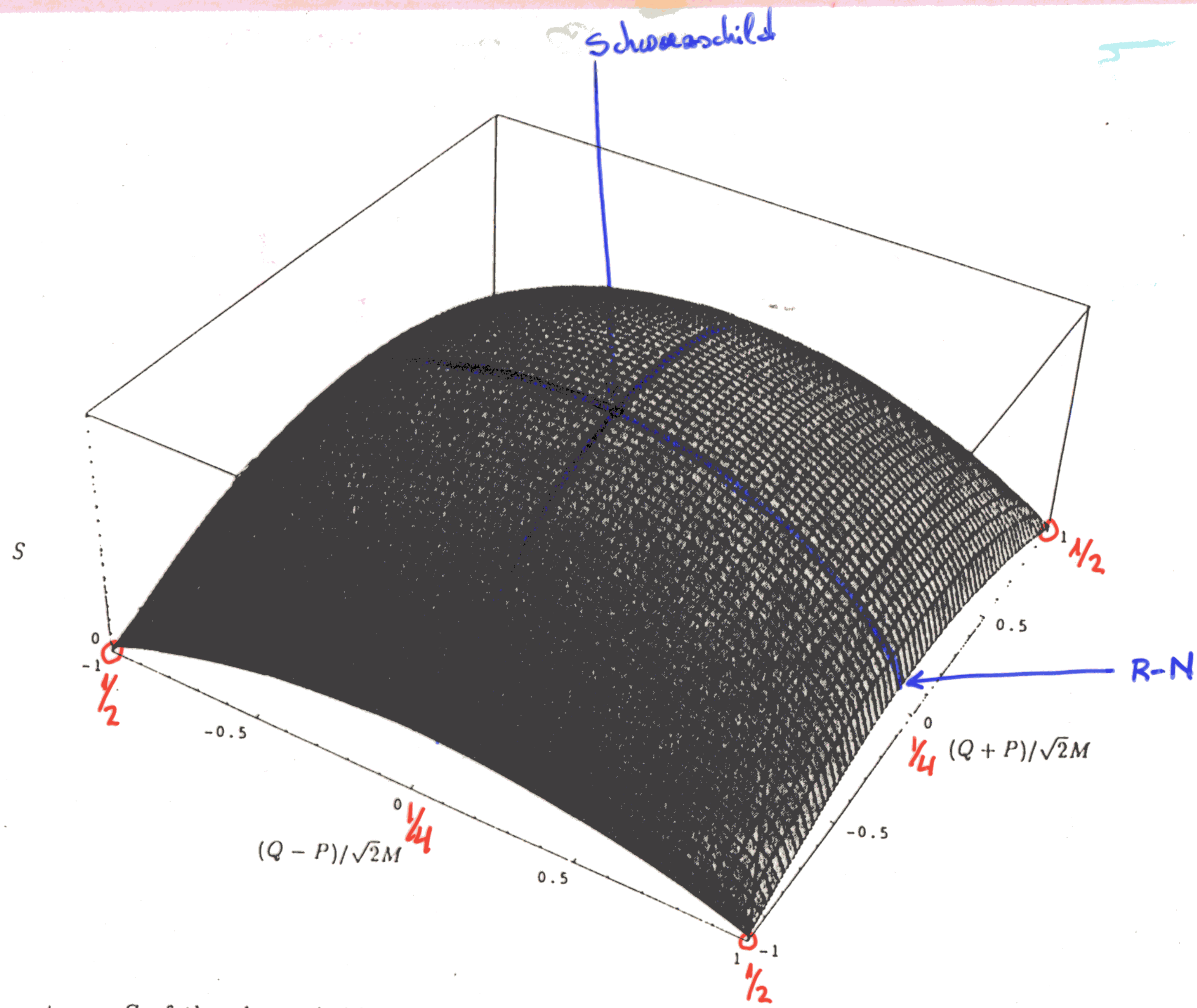


FIG. 5. The entropy S of the charged dilaton black holes as a function of z_1/M and z_2/M. It has a maximum for the Schwarzschild black hole, which corresponds to the origin of coordinates ($P = Q = \Sigma = 0$). For purely electric or magnetic extreme dilaton black holes (in the corners) it is zero. On the sides of the square the temperature vanishes, but the total entropy (Euclidean action) remains nonzero, $S = 2\pi|PQ|$.

- Again, there are multi-BH solutions in the extremal limit (now in terms of 2 real harmonic functions V, W) due to the supersymmetric equilibrium of forces:

$$\frac{m_i m_j + \Sigma_i \Sigma_j - e^{-2\phi_0} q_i q_j - e^{-2\phi_0} p_i p_j}{r_{ij}^2} = 0$$

- Observe that $S = 0$ (more precisely $A_{hor} = 0$) is associated to $1/2$ of $N=4$ residual supersymmetry. For STATIC BHs, this seems to be the rule.

ARE THERE MORE SUPERSYMMETRIC (BH) SOLUTIONS OF $N=4$ SUGRA?

$\longrightarrow$ DUALITY

DUALITY SYMMETRIES (SUGRA point of view)

Duality symmetries are SYMMETRIES OF THE EQUATIONS OF MOTION: local or global field redefinitions. They were called "HIDDEN SYMMETRIES"

Three types

i) Electric-magnetic duality (S-duality)

Minkowski vacuum Maxwell field equations

$$\partial_\mu F^{\mu\nu} = 0 \;; \quad \partial_\mu {}^*F^{\mu\nu} = 0 \longleftarrow \left(\begin{array}{l}\text{Bianchi identity} \\ \Rightarrow \exists A_\mu \text{ locally}\end{array}\right)$$

Invariant under $F \rightleftarrows {}^*F$ $\quad (\vec{E} \rightleftarrows \vec{B})$

This is a non-local transformation of A_μ

$$A'_\mu \;/\; F'_{\mu\nu} = {}^*F_{\mu\nu} = 2\,\partial_{[\mu} A'_{\nu]} \;;$$

$$\Rightarrow \boxed{A'_\mu = \int_0^1 d\lambda\, {}^*F_{\mu\nu}(\lambda x)\, \lambda x^\nu \;;}$$

More general transformations are possible:

$$F' = a F + i b\, {}^*F \;; \qquad a, b \in \mathbb{R}, \text{ arbitrary}$$

Invariance of the energy-momentum tensor

$$\Rightarrow \quad a = \cos\theta \;; \quad b = \sin\theta \;; \rightarrow \boxed{U(1) \text{ duality group}}$$

- The Lagrangian is not invariant, even under the discrete subgroup:

$$\mathcal{L} \sim \vec{E}^2 - \vec{B}^2$$

$$\left.\begin{array}{l} \vec{E} \to \vec{B} \\ \vec{B} \to -\vec{E} \end{array}\right\} \Rightarrow \mathcal{L} \to -\mathcal{L}$$

- This symmetry can be extended to curved space-time (the metric is invariant)

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu} ;$$

$$\underset{(d^*F = 0)}{\nabla_\mu F^{\mu\nu} = 0} ; \quad \underset{(dF = 0)}{\nabla_\mu {}^*F^{\mu\nu} = 0} ;$$

- But it cannot be extended to non-Abelian YM:

$$D_\mu(A) F^{\mu\nu} = 0 ; \quad D_\mu(A) {}^*F^{\mu\nu} = 0$$

- Quantum-mechanically $U(1)$ is broken to $\mathbb{Z}_2$:

The Dirac quantisation condition has to be preserved: $q\,p \in \mathbb{Z}$

(DSZ condition: $q_1 p_2 - q_2 p_1 \in \mathbb{Z}$)

ii) Field interchanges ("T-dualities") (vector fields)

$$\mathcal{L} = \frac{1}{4}\sum_{m}^{N} \left(F^{(m)}\right)^2 \longrightarrow \text{N Abelian vector fields}$$

$O(N)$ rotations of the vector fields:

$$\vec{A}_\mu{}' = M\vec{A}_\mu \; ; \quad \vec{A}_\mu = \begin{pmatrix} A_\mu^{(1)} \\ A_\mu^{(N)} \end{pmatrix}$$

Quantum-mechanically $\rightarrow O(N,\mathbb{Z})$

iii) σ-model isometries

$$\mathcal{L} = \frac{1}{2} g_{ij}(\phi)\,\partial_\mu\phi^i\partial^\mu\phi^j \; ;$$

$g_{ij}(\phi)$ is invariant under $\delta\phi^i = \varepsilon^a k_a{}^i(\phi)$ (isometries of the target space) $[k_a, k_b] = C_{ab}{}^c k_c$

SUGRA has all the three ingredients:

$$\mathcal{L} = \sqrt{-g}\Big\{ -R - \frac{1}{2} g_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j + \frac{1}{4} M_{IJ}(\phi) F^I F^J + \frac{i}{4} N_{IJ}(\phi) F^I {*F^J} \Big\}$$

and all the three symmetries. (Gaillard Zumino)

The three transformations are not independent. S and T dualities imply transformations of the scalars. Often, S and T are subgroups of a single (U) duality group. (Hull, Townsend)

Prototype: $\boxed{N=4 \quad SUGRA}$

$SL(2,\mathbb{R})$ σ-model: $\frac{1}{2}\frac{\partial\lambda\partial\bar{\lambda}}{(\mathrm{Im}\,\lambda)^2}$; $\lambda' = \frac{a\lambda + b}{c\lambda + d}$;

$ad - bc = +1$;

6 Abelian vector fields: $e^{-2\phi}\sum_{n=1}^{6}(F^{(n)})^2 - ia\sum_{n=1}^{6}F^{(n)*}F^{(n)}$

$\Longrightarrow$ $SO(6)$ invariance

Eqs. of motion: $\nabla_\mu {}^*\tilde{F}^{I\,\mu\nu} = 0$, $\tilde{F} = e^{-2\phi}\,{}^*F - ia F$

Bianchi identities: $\nabla_\mu {}^*F^{I\,\mu\nu} = 0$;

$\Rightarrow$ Electric-magnetic duality rotations: $\begin{pmatrix} F^I \\ \tilde{F}^I \end{pmatrix}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} F^I \\ \tilde{F}^I \end{pmatrix}$

$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$ and λ also transforms

$\boxed{SL(2,\mathbb{R}) \times SO(6,\mathbb{R}) \to SL(2,\mathbb{Z}) \times SO(6,\mathbb{Z}) \text{ duality}}$

COMMENT: There was no explanation for the existence of dualities in SUGRA theories. However, when they are seen as low-energy string theories, these symmetries can be interpreted as symmetries of string theory

S-duality includes $\lambda \to -1/\lambda$

$a = 0 \Rightarrow g_{string} \to \frac{1}{g_{string}}$

STRONG ~ WEAK COUPLING

T-duality includes

$$\begin{array}{ccc} A^{(1)}_\mu & \leftrightarrow & A^{(2)}_\mu \\ \downarrow & & \downarrow \\ g_{5\mu} & \leftrightarrow & B_{5\mu} \end{array}$$

→ STRING T-duality

$g_{55} \to 1/g_{55}$

$R \to 1/R$

LARGE ~ SMALL RADIUS

Duality transformations can be used to generate new solutions out of known solutions respecting (in general) the residual (super)-symmetries.

Applying repeatedly duality transformations to the known extreme BH solutions one arrives to the most general solution (so far):

SWIP $(N=4)$ (Tod) (Bergshoeff, Kallosh, O.)

$$
\begin{cases}
ds^2 = e^{2U}(dt + \vec{\omega}\, d\vec{x})^2 - e^{-2U} d\vec{x}^{\,2}\,, \\
A_t^{(i)} = 2\, e^{2U}\, \mathrm{Re}\left(k^{(i)} \mathcal{H}_2\right); \\
\tilde{A}_t^{(i)} = -2\, e^{2U}\, \mathrm{Re}\left(k^{(i)} \mathcal{H}_1\right); \\
\lambda = \mathcal{H}_1 / \mathcal{H}_2\,; \qquad e^{-2U} = 2\, \mathrm{Im}\left(\mathcal{H}_1 \overline{\mathcal{H}}_2\right) \\
\partial_{[i}\omega_{j]} = \varepsilon_{ijk}\, \mathrm{Re}\left(\mathcal{H}_1 \partial_k \overline{\mathcal{H}}_2 - \overline{\mathcal{H}}_2 \partial_k \mathcal{H}_1\right);
\end{cases}
$$

$\mathcal{H}_1, \mathcal{H}_2$ complex harmonic functions

T acts on $\vec{k} = \begin{pmatrix} k^{(1)} \\ \vdots \\ k^{(6)} \end{pmatrix}$ (vector)

S acts on $\vec{\mathcal{H}} = \begin{pmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{pmatrix}$ (doublet)

Point-like SWIP Solutions

$$\left.\begin{aligned}\mathcal{H}_1 &= \chi_0 + \frac{\chi_1}{r_1}\\ \mathcal{H}_2 &= \psi_0 + \frac{\psi_1}{r_2}\end{aligned}\right\} \qquad r_1^2 = r_2^2 = x^2 + y^2 + (z - i\alpha)^2$$

$$\boxed{\begin{aligned} ds^2 = {} & \frac{\Delta - \alpha^2 \sin^2\theta}{\Sigma}\left(dt - \omega\, d\varphi\right)^2 - \\ & - \Sigma\left(\frac{d\rho^2}{\Delta} + d\theta^2 + \frac{\Delta \sin^2\theta\, d\varphi^2}{\Delta - \alpha^2 \sin^2\theta}\right)\end{aligned}}$$

$$\left\{\begin{aligned} \omega &= \frac{2}{\Delta - \alpha^2 \sin^2\theta}\left\{ l\, \Delta \cos\theta + \alpha \sin^2\theta \left[m\left(\rho - m - |\Upsilon|\right) + \frac{1}{2}\left(|M|^2 - |\Upsilon|^2\right)\right]\right\} \\ \Delta &= \rho\left[\rho - 2\left(m + |\Upsilon|\right)\right] + \alpha^2 + \left(m + |\Upsilon|\right)^2 \\ \Sigma &= \rho\left(\rho - 2|\Upsilon|\right) + \left(\alpha \cos\theta + l\right)^2 \end{aligned}\right.$$

$M = m + i l$ ← NUT charge

(m ← ADM mass)

$\alpha = J/m$ → angular momentum per unit mass

$\Upsilon = -2 \sum_n \Gamma^{(n)2} / M$ → complex scalar charge

$\Gamma^{(n)} = q^{(n)} + i p^{(n)}$ electric-magnetic charge

Always: $\boxed{|M|^2 + |\Upsilon|^2 - 4 \sum_n |\Gamma^{(n)}|^2 = 0}$

This is the duality-invariant form of the Bogomol'nyi bound!

$$\left(|M|^2-|Z_1|^2\right)\left(|M|^2-|Z_2|^2\right)=0$$

interchanged by duality

$$\begin{cases} Z_1=\sqrt{2}\left(\Gamma^{(1)}+i\Gamma^{(3)}\right) \\ Z_2=\sqrt{2}\left(\Gamma^{(1)}-i\Gamma^{(3)}\right) \end{cases}$$

* Observe that the angular momentum does not appear in the bound and, for $J\neq 0$, the metric has a naked singularity

$\Rightarrow$ SUSY $\neq$ cosmic censor

(This is different in higher dimensions)

When $J=0$ we can calculate the area of the horizon

$$A=4\pi\left(|M|^2-|\Upsilon|^2\right)$$, independent of λ_0

On the horizon

$$\lambda_{horizon}=\frac{\lambda_0 M+\bar{\lambda}_0\Upsilon}{M+\Upsilon}\qquad (M\neq-\Upsilon)$$

Observe that the area formula can be rewritten

$$A = 4\pi \Big| |Z_1|^2 - |Z_2|^2 \Big| ; \qquad \text{(duality-invariant)}$$

in terms of the conserved charges $q^{(n)}, p^{(n)}$

$$\left\{ \begin{array}{l} {}^*F_{tr}^{(n)} \sim i\, \dfrac{p^{(n)}}{r^2} \\[2ex] \left(e^{-2\phi} F - i a\, {}^*F\right)_{tr}^{(n)} \sim \dfrac{q^{(n)}}{r^2} \end{array} \right. \qquad \left(\nabla_\mu (\;)^{\mu\nu} = 0 \right)$$

$$A = 8\pi \sqrt{(\vec{q}\cdot\vec{q})(\vec{p}\cdot\vec{p}) - (\vec{q}\cdot\vec{p})^2} \qquad \text{(manifestly moduli independent)}$$

$$= 8\pi \sqrt{\det\left[\begin{pmatrix} \vec{p}^{\,t} \\ \vec{q}^{\,t} \end{pmatrix} \begin{pmatrix} \vec{p} & \vec{q} \end{pmatrix} \right]} \qquad \text{(duality-invariant)}$$

$$\begin{pmatrix} \vec{p}\,' \\ \vec{q}\,' \end{pmatrix} = \underset{\substack{SO(6) \\ (T}}{R} \otimes \underset{\substack{SL(2,R) \\ ,\ S)}}{S} \begin{pmatrix} \vec{p} \\ \vec{q} \end{pmatrix}$$

The temperature is always zero.
Then, we have two cases:

i) $|\mathcal{M}| = |Z_{\substack{1\\2}}| \neq |Z_{\substack{2\\1}}|$.

- $1/4$ of the supersymmetries unbroken.
- Area of the (regular) horizon $\neq 0$.
- Scalars finite on the horizon.

ii) $|\mathcal{M}| = |Z_1| = |Z_2|$.

$\Rightarrow |\mathcal{M}| = |\Upsilon|$.

- $1/2$ of the supersymmetries unbroken.
- Singular horizon (zero Area).
- Sometimes the scalars blow up on the horizon.

CONCLUSION

- Local SUSY methods can be very useful even in a pure GR context.
- The existence of unbroken supersymmetry is related to very special properties:
  - zero temperature (1 SUSY) } no general
  - zero entropy (2 SUSIES) } proofs
  - equilibrium of forces
  - simplicity of solutions
- BHs are the natural testing ground for all these ideas.
- SUSY is compatible with duality symmetries (no coincidence) and it is possible to generate very general supersymmetric solutions which eventually could be used to find the non-supersymmetric ones.
- J problem ...