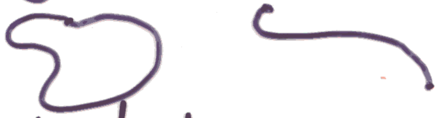
Black Holes & Other Supersymmetric Extended Objects in String Theory

INTRODUCTION: CLASSICAL SOLUTIONS OF THE LOW-ENERGY STRING EFFECTIVE ACTION

String theory is (or was) a theory of elementary 1-dimensional objects:  $\sim \alpha' \sim 1/T$

whose vibration modes are interpreted as particles. There is an infinite tower of massive modes $m \propto M_{\text{PLANCK}}$.

In the low-energy limit $\alpha' \to 0$ ($T \to \infty$) the size of the string becomes irrelevant and the massive modes can be integrated out and one gets an effective field theory for the massless modes:

$$S = \int d^{10}x \sqrt{-g}\, e^{-2\phi} \Big[\overbrace{-R + 4(\partial\phi)^2 - \tfrac{4}{3} H^2}^{\text{(common) NS-NS sector}} + \overbrace{\begin{pmatrix}\text{other} \\ \text{fields}\end{pmatrix}}^{\text{RR sector}}$$

$$H_{\mu\nu\rho} = \partial_{[\mu} B_{\nu\rho]} \qquad + \mathcal{O}(\alpha') \text{ corrections} \Big]$$

At lowest order in α' $\longrightarrow$ SUPERGRAVITY THEORY

CLASSICAL SOLUTIONS : P-BRANES

In any dimension a classical solution of the string effective action is a solution of gravity coupled to several matter fields.

p-brane solutions: solutions whose metric has p (commuting) space-like translational isometries.

⇒ If they have singularities (~ sources), they are p-dimensional ⇒ they represent the far-away field of a p-dimensional (extended) object:

- 0-brane solution ⟶ "black hole" (point-like source)
- 1-brane solution ⟶ "cosmic string" (string-like source)
- 2-brane solution ⟶ "membrane" (domain wall)
- ⋮
- (-1-brane solution ⟶ "instanton")

Simple example: M-P solutions (Majumdar-Papapetrou)

$$\begin{cases} ds^2 = V^{-2} dt^2 - V^2 d\vec{x}^{\,2} ; \\ A_t = V^{-1} ; \\ \partial_i \partial_i V = 0 ; \end{cases} \qquad S = \int d^4x \sqrt{g} \{ R + F^2 \}$$

- If V depends on x, y, z: $\quad V = 1 + \sum_{i=1}^{N} \frac{m_i}{|\vec{x}-\vec{x}_i|}$

  $N=1 \Rightarrow V = 1 + \frac{m}{r}$ extreme Reissner-Nordström BHs

  $N>1 \Rightarrow N$ extreme R-N BHs in equilibrium (Hartle Hawking)

  (BH = "0-brane")

- If V depends on x, y only

  $$V = 1 + \sum_{i=1}^{N} \frac{T_i}{2} \ln\left[(x-x_i)^2 + (y-y_i)^2\right]$$

  $N= \Rightarrow V = 1 + T \ln \rho$ infinite cosmic string parallel to the z axis

  (V is singular at $x=y=0 \; \forall z$)

  $N>1 \Rightarrow N$ parallel strings in equilibrium.

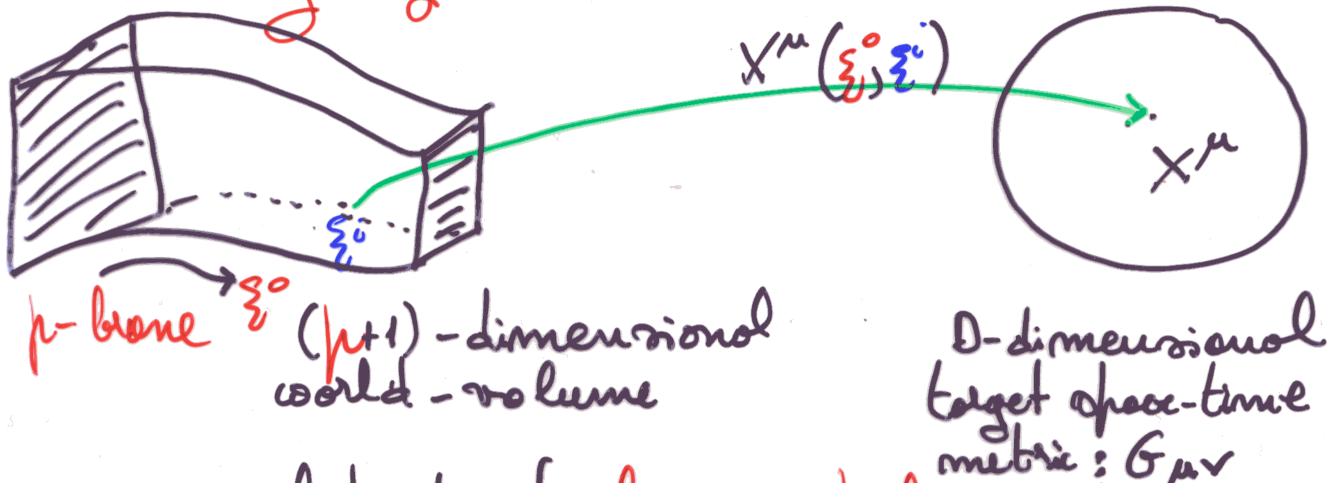
- ETC.

There can also be intersecting and overlapping p-brane solutions.

P-branes come in two flavors: electric and magnetic.

FUNDAMENTAL P-BRANES

One can construct (classical) theories of p-branes as elementary objects:



There are several kinds of bosonic p-brane actions

i) $S[X^\mu] = T\int d^{p+1}\xi \sqrt{\det g_{ij}}$ (Nambu-Goto)

$$g_{ij} = \partial_i X^\mu \partial_j X^\nu G_{\mu\nu} : \text{ induced metric}$$

ii) $S[X^\mu, \gamma_{ij}] = \frac{T}{2}\int d^{p+1}\xi \sqrt{-\gamma}\left[\gamma^{ij} g_{ij} - (p-1)\right]$ (Brink, Howe, di Vecchia, Polyakov ($p=1$))

iii) $S[X^\mu, V_i] = \int d^{p+1}\xi \frac{1}{V} \det(g_{ij} + F_{ij})$ (Born-Infeld) (Bergshoeff, London, Townsend)

ii) + SUSY $\longrightarrow$ Green-Schwarz-type actions:

$$S[X^\mu, \theta^\alpha, \gamma_{ij}] = \frac{T}{2}\int d^{p+1}\xi \left\{\sqrt{-\gamma}\left[\gamma^{ij}\Pi_i^\mu \Pi_j^\nu G_{\mu\nu} - (p-1)\right] + \underbrace{\varepsilon^{i_1 \dots i_{p+1}} B_{i_1 \dots i_{p+1}}}_{\text{Required for } \kappa\text{-symmetry}}\right\}$$

The moral of this story is that in "consistent p-brane theories", the p-brane couples not only to the space-time metric $G_{\mu\nu}$ but to a $(p+1)$-form $B_{\mu_1 \cdots \mu_{p+1}}$:

$$\left(B_{i_1 \cdots i_{p+1}} = \partial_{i_1} X^{\mu_1} \cdots \partial_{i_{p+1}} X^{\mu_{p+1}} B_{\mu_1 \cdots \mu_{p+1}} \right)$$

Thus, if we want to see our p-brane solutions as the field of a p-brane source, we must consider the joint action: $S = S_{eff} + S_{p\text{-brane}}$ and the string effective action must contain a $(p+1)$-form.

- It always contains a 2-form $B_{\mu\nu}$: it can couple to fundamental strings $(p=1)$
- Type IIA $(D=10)$ also contains a 1-form A_μ and a 3-form $C_{\mu\nu\rho}$: it can couple to 0-branes and 2-branes (membranes).
- Type IIB $(D=10)$ also contains a self-dual 4-form $D^{+}_{\mu\nu\rho\sigma}$ $\rightarrow$ 3-branes.

ELECTRIC-MAGNETIC DUALITY

Maxwell's equations:
$$\begin{cases} dF = 0 ; & \text{(Bianchi identity) } F = dA \\ d{}^*F = {}^*J ; & \text{(Eq. of motion)} \end{cases}$$

$$\begin{cases} {}^*F_{\mu\nu} = \frac{1}{2}\, \varepsilon_{\mu\nu}{}^{\rho\sigma} F_{\rho\sigma} ; \\ {}^*J_{\mu\nu\rho} = \varepsilon_{\mu\nu\rho}{}^{\sigma} J_{\sigma} ; \end{cases}$$
(Hodge dual) (same degrees of freedom)

$$d{}^*J = 0$$

We can make symmetric both equations by changing $F = dA$ to $F = dA + \omega$; $d\omega = \tilde{J}$; $\Rightarrow d\tilde{J} = 0$

$$\Rightarrow \boxed{\begin{cases} dF = \tilde{J} ; \\ d{}^*F = {}^*J \end{cases}}$$

For an electric charge ${}^*J_{123} = q\, \delta^{(3)}(\vec{x})$

For a magnetic charge $\tilde{J}_{123} = \mu\, \delta^{(3)}(\vec{x})$

$$q = \int_{M^3} {}^*J = \int_{M^3} d{}^*F = \int_{S^2} {}^*F \qquad \begin{cases}\text{dynamically} \\ \text{conserved}\end{cases}$$

$$\mu = \int_{M^3} \tilde{J} = \int_{M^3} dF = \int_{S^2} F \qquad \begin{cases}\text{topologically} \\ \text{conserved}\end{cases}$$

$$\boxed{q\,\mu = 2\pi n}$$ (Dirac's quantization condition)

NOW:

i) The equations of motion are invariant if we replace F by ${}^*F$ and $\tilde{J}$ by ${}^*J$ $\Rightarrow$ q by p

$\Rightarrow$ electric-magnetic duality $\quad \vec{E} \rightleftarrows B$

(In general $\quad F' = aF + b\,{}^*F \ldots \rightarrow U(1)$)

ii) We can rewrite the theory using, instead of A as variable, $\tilde{A}$ / ${}^*F = d\tilde{A} + \tilde{\omega}$

$\rightarrow$ dual variables (no naive substitution!)

SOME OF THESE PROPERTIES CAN BE GENERALIZED TO $(p+1)$-FORMS IN d DIMENSIONS:

$(p+2)$-form field strength $\quad F_{p+2} = dB_{p+1} \quad (+\ldots)$

$(d-p-2)$-form Hodge dual $\quad \tilde{F}_{d-p-2} = {}^*(F_{p+2})$

$$\tilde{F}_{(d-p-2)\mu_1 \ldots \mu_{d-p-2}} = \frac{1}{(d-p-2)!\,\sqrt{-g}}\,\varepsilon_{\mu_1 \ldots \mu_{d-p-2}}{}^{\nu_1 \ldots \nu_{p+2}} F_{(p+2)\nu_1 \ldots \nu_{p+2}}$$

Both $F_{(p+2)}$ and $\tilde{F}_{d-p-2}$ describe the same degrees of freedom and the theory can be rewritten in terms of $\tilde{F}_{d-p-2}$ (no naive substitution!)

The lesson to learn here is that the string effective actions can be rewritten using different p-forms:

D=10

Heterotic B_2 $(p=1)$; $H_3 = dB_2 \rightarrow K_7 = {}^*H_3$;

$K_7 = dL_6 \Rightarrow \boxed{p=5}$ 5-branes

IIA A_1 $(p=0)$; $F_2 = dA_1 \Rightarrow M_8 = {}^*F_2$;

$M_8 = dN_7 \Rightarrow \boxed{p=6}$ 6-branes

C_3 $(p=2)$; $G_4 = dC_3 \Rightarrow L_6 = {}^*G_4$;

$L_6 = dS_5 \Rightarrow \boxed{p=4}$ 4-branes

IIB D_4^+ $(p=3)$; $I_5^+ = dD_4 \Rightarrow \tilde{I}_5^+ = {}^*I_5$

$\Rightarrow$ self-dual 3-branes

ℓ_0 $(p=-1)$; $J_1 = d\ell_0 \Rightarrow \tilde{R}_9 = {}^*J_1$

$\tilde{R}_9 = dT_8$; $\rightarrow \boxed{p=7}$ 7-branes

Then, string theories ~~can~~ do couple to many kinds of p-branes. One can also define electric and magnetic charges:

p-brane $\rightarrow$ the asymptotically flat region is an S^{d-p-2} sphere (transverse space -2)

p couples to B_{p+1}, with field strength $F_{p+2} = dB_{p+1}$

$$\Rightarrow \boxed{q = \int_{S^{d-p-2}} {}^{*}(F_{p+2})}$$

In general, a p-brane can only carry electric charge (but, in the dual theory it can be interpreted as magnetic charge)

When $\boxed{d = d-p-2}$ both q & p are possible and F and $\tilde{F}$ are forms of the same degree and then one can have electric-magnetic duality

$d=8$, $p=2$

$d=6$, $p=1$

$d=4$, $p=0$ (BH) * (point-particles)

UNBROKEN SUPERSYMMETRY

p-branes are solutions of SUGRA theories with all fermions vanishing. In general, these solutions are not invariant under local SUSY transformations

$$\begin{cases} \delta\phi^f \sim \nabla\varepsilon + \varepsilon\phi^b + \cancel{\varepsilon\phi^f\phi^f} \\ \delta\phi^b \sim \cancel{\varepsilon\phi^f} = 0 \end{cases}$$

We are interested in solutions which are still invariant under a SUSY transformation, generated by some $\varepsilon \rightarrow$ KILLING SPINOR.

$\rightarrow$ "supersymmetric solution"

$$\boxed{\exists\,\varepsilon \text{ Killing} \Rightarrow \exists\, Q \text{ (SUSY charge)} \,/\, Q|\psi\rangle = 0}$$

In the rest-frame $\quad Q^2 \sim m - |Z| \quad \Rightarrow \boxed{m = |Z|}$

(in general $m \geq |Z|$) $\qquad$ Bogomol'nyi bound

Supersymmetric solutions saturate a B-bound

$\Rightarrow$ lowest possible mass $\Rightarrow$ STABILITY (ground state etc.)