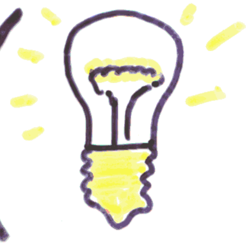
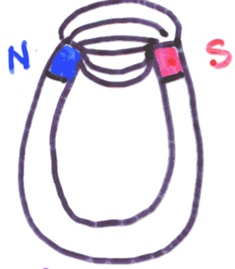
SL(2,R)

ELECTRIC – MAGNETIC

( – 

)

DUALITY

&

SUPERSYMMETRY

- Duality symmetries
- SL(2,R)
  - Origin
  - Action
    - fields
    - action
    - charges
  - vs. Supersymmetry
    - KS
    - KSI
- Target-space duality vs. Supersymmetry
- Conclusion

ELECTRIC - MAGNETIC DUALITY

Maxwell action for a 1-form A

$$S_M(A) = i\int F \wedge {}^*F \,; \qquad F = dA \Rightarrow \boxed{\begin{array}{l} dF = 0\,; \\ \\ d^*F = 0\,; \end{array}}$$

(Bianchi identity)

Equation of motion (Gauss law)

Duality
$$\begin{cases} F \to F' = i\,{}^*F \\ {}^*F \to {}^*F' = i\,F \end{cases} \qquad \begin{cases} \vec{E} \to \vec{E}' = \vec{B} \\ \vec{B} \to \vec{B}' = -\vec{E} \end{cases}$$

Bianchi id. $\rightleftarrows$ Eqs. of motion

Define A' / $F' = dA'$

Electric charge of A' = P
$\parallel$
Magnetic charge of A

$$Q \rightleftarrows P$$

<u>However</u>

$$S_M(A') = i\int F' \wedge {}^*F' = -i\int {}^*F \wedge F = -i\int F \wedge {}^*F = -S_M(A)$$

SO(2) duality group $\quad F' = ia\,{}^*F + b\,F$

Invariance of $T_{M\mu\nu} \Rightarrow a^2 + b^2 = +1$

The action behaves worse under SO(2)

GENERALIZATION TO P-FORMS

$B \rightarrow (p-1)\text{-form}$ $\qquad H = dB \qquad p\text{-form}$

$\qquad {}^*H \qquad (d-p)\text{-form}$

$$S(B) = (-1)^{pd}\, i \int H \wedge {}^*H \, ;$$

Bianchi identity $\quad \boxed{dH = 0;}$

Eq. of motion $\quad \boxed{d\,{}^*H = 0;}$

Again, there is a symmetry of the equations of motion

$$\begin{cases} H \rightarrow H' = i\,{}^*H \, ; \\ {}^*H \rightarrow {}^*H' = (-1)^{p(d-p)}\, i H \, ; \end{cases}$$

But there is no symmetry of the action

$a \quad (d-p-1)\text{-form} \,/\, H' = {}^*H = da$

$$S(a) = (-1)^{(d-p)d}\, i \int H' \wedge {}^*H' = -(-1)^{(d+p)(d-p)}\, i \int {}^*H \wedge H = (-1)^{d+1} S(B)$$

To get the dual action, the sign has to be changed by hand when d is even.

Continuous group only for some ds and ps

Electric-magnetic duality is useful as a solution-generating technique

$$S_{EM} = \int d^4x\sqrt{g}\left[-R + F^2\right] ;$$

Purely electric Reissner-Nordström BH solution:

$$\begin{cases} ds^2 = \dfrac{(r-r_+)(r-r_-)}{r^2}dt^2 - \dfrac{r^2}{(r-r_+)(r-r_-)}dr^2 - r^2 d\Omega^2 ; \\ F_{tr} = \dfrac{Q}{r^2} ; \qquad r_\pm = M \pm \sqrt{M^2 - Q^2} ; \end{cases}$$

SO(2)

$$\begin{cases} ds^2 = \dfrac{(r-r_+)(r-r_-)}{r^2}dt^2 - \dfrac{r^2}{(r-r_+)(r-r_-)}dr^2 - r^2 d\Omega^2 ; \\ F_{tr} = \dfrac{Q}{r^2} ; \quad {}^*F_{tr} = \dfrac{P}{r^2} ; \quad r_\pm = M \pm \sqrt{M^2 - (Q^2+P^2)} \end{cases}$$

The relation action $\leftrightarrow$ entropy breaks down

Which properties are preserved by this kind of transformations?

Geometric? ✓

Supersymmetric? ✓ $\qquad \delta\psi \sim F^+$

SL(2,R)

String theory effective action in $d=4$ dimensions with one Abelian vector field A

$$S_1=\int d^4x\sqrt{g}\,e^{-2\phi}\left[-R-4(\partial\phi)^2+\frac{3}{4}H^2-\frac{1}{2}F^2\right];$$

$$H=\frac{1}{3}(dB+A\wedge F);$$ Bianchi id. $\boxed{dH=0;}$

(B) Eq. of motion $\boxed{d(e^{-2\phi}\,{}^*H)=0;}$

First go to Einstein frame $g_{\mu\nu}=e^{2\phi}g_{E\mu\nu}$

$$S=\int d^4x\sqrt{g_E}\left[-R_E+2(\partial\phi)^2+\frac{3}{4}e^{-4\phi}H^2-\frac{e^{-2\phi}}{2}F^2\right];$$

now we define the (pseudo)-scalar a

$${}^*H=ie^{4\phi}da$$

$\boxed{d(e^{-4\phi}\,{}^*H)=0}$ is the Bianchi id. of a

$\boxed{dH=0}$ is the Eq. of m. of a

$$\frac{3}{4}e^{-4\phi}H^2=-\frac{1}{2}e^{4\phi}(\partial a)^2$$

The action for a is

$$\boxed{S_2=\int d^4x\sqrt{g}\left[-R+2(\partial\phi)^2+\frac{1}{2}e^{-4}(\partial a)^2-\frac{1}{2}e^{-2\phi}F^2+\frac{i}{2}aF^*F\right]}$$

- S_1 has "manifest" target-space duality which "disappears" in S_2.
- The equations of motion of S_2 have a new electric-magnetic duality "hidden" in the formulation of S_1.

$$\begin{cases} R_{\mu\nu} + 2\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}e^{4\phi}\partial_\mu a\partial_\nu a - 2e^{-2\phi}\left(F_{\mu\rho}F_\nu^{\ \rho} - \frac{1}{4}g_{\mu\nu}F^2\right) = 0; \\ \boxed{\underbrace{\nabla_\mu\left[e^{-2\phi}F^{\mu\nu} - ia\,{}^*F^{\mu\nu}\right] = 0}_{\text{Eq. of motion}}; \qquad \underbrace{\nabla_\mu {}^*F^{\mu\nu} = 0}_{\text{Bianchi id.}};} \\ \nabla^2 a + 4\nabla_\mu\phi\nabla^\mu a - ie^{-4\phi}F{}^*F = 0; \\ \nabla^2\phi - \frac{1}{2}e^{4\phi}(\partial a)^2 - \frac{1}{2}e^{-2\phi}F^2 = 0, \end{cases}$$

when $a = 0$ $\qquad \begin{cases} \phi \to -\phi \\ F \to e^{-2\phi}\,{}^*F \end{cases}$ (electric-magnetic)

Always $\qquad a \to a + a_0$

$\phi \to -\phi \qquad$ Strong $\leftrightarrow$ Weak coupling regime from the S.T. point of view.

These two operations (Montonen-Olive duality and Peccei-Quinn shift) generate the group $SL(2,R)$:

→ Introduce the $SL(2,R)$-dual of F

$$\tilde{F} \equiv e^{-2\phi}\,{}^*F - i a F\,, \qquad \lambda = a + i e^{-2\phi}\,;$$

A-Equation of motion $\quad d\tilde{F} = 0;$

A-Bianchi identity $\quad dF = 0;$

→ Play the same game

$$\begin{cases} \tilde{F} \to \tilde{F}' = \alpha \tilde{F} - i\beta F; \\ F \to F' = i\gamma \tilde{F} + \delta F; \end{cases}$$

- The definition of $\tilde{F}$ $\Rightarrow$ $\lambda \to \lambda' = \dfrac{\alpha\lambda + \beta}{\gamma\lambda + \delta}$

- Invariance of $T_{\mu\nu} \Rightarrow \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,R)$

$SL(2,R)$ is useful to generate new solutions

with the same Einstein metric

BUT

with different Stringy metric

All the (Einstein-frame) geometric properties preserved?

Action of SL(2,R) on the charges

What does "electric" and "magnetic" charge mean in this system?

Usually $d\,{*F}=0 \Rightarrow \nabla_\mu F^{\mu\nu}=0$; $F_{tr} \sim \frac{Q}{r^2}$

$\rightarrow Q$ conserved electric charge

Here

$$\left\{\begin{array}{l} d\tilde F = 0 \Rightarrow \nabla_\mu {}^*\tilde F^{\mu\nu}=0 \,;\; {}^*\tilde F_{tr} \sim \frac{\tilde Q}{r} \\ \rightarrow \tilde Q \text{ conserved electric charge} \\ dF=0 \Rightarrow \nabla_\mu {}^*F^{\mu\nu}=0 \,;\; {}^*F_{tr} \sim \frac{\upsilon P}{r^2} \\ P \text{ topologically conserved magnetic charge} \end{array}\right\}$$

SL(2,R) $\left\{\begin{array}{l} \tilde Q' = \alpha \tilde Q + \beta P \,; \\ P' = \gamma \tilde Q + \delta P \,; \end{array}\right.$

(Also) $\left\{\begin{array}{l} \tilde P' = \alpha \tilde P - \beta Q \,; \\ Q' = -\gamma \tilde P + \delta Q \,; \end{array}\right.$

$Q\tilde Q + P\tilde P = e^{-2\phi}(Q^2+P^2)$ SL(2,R)-invariant

Upon quantization SL(2,R) breaks down to SL(2,$\mathbb{Z}$)

A. Sen

R. Kallosh & T.O.

Action of SL(2,R) on the scalar charges

There are not conserved charges unless they are related to $\tilde{Q}$ and P (as in axion-dilaton BHs)

$$\lambda \sim \lambda_0 - 2e^{-2\phi_0}\frac{\Upsilon}{r} \quad ; \quad \Upsilon = \Delta + i\Sigma \; ;$$

$$\boxed{\Upsilon' = e^{i\alpha}\Upsilon} \Rightarrow \Sigma \text{ and } \Delta \text{ are rotated into each other}$$

$$F^{+} \sim \frac{\phi_0 \Gamma}{r} \qquad \boxed{\Gamma' = e^{-i\frac{\alpha}{2}}\Gamma}$$

$$\left.\begin{array}{l} |\Gamma|^2 = e^{-2\phi_0}(Q^2 + P^2) \\ |\Upsilon|^2 = \Sigma^2 + \Delta^2 \end{array}\right\} \text{ are SL(2,R) invariants}$$

Mass and angular momentum are also SL(2,R)-invariants

The parameters that can appear in a general solution are combinations of these SL(2,R)-invariant blocks.

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AXION-DILATON BLACK HOLES

{Shapere, Trivedi, Wilczek}
{R. Kallosh, T.O.}

General, $SL(2,R)$-invariant class of BH solutions

$$\begin{cases} ds^2 = \dfrac{(r-r_+)(r-r_-)}{(r^2-|\Upsilon|^2)}dt^2 - \dfrac{(r^2-|\Upsilon|^2)}{(r-r_+)(r-r_-)}dr^2 - (r^2-|\Upsilon|^2)\,d\Omega^2 ; \\ \lambda = \dfrac{\lambda_0 r + \bar{\lambda}_0 \Upsilon}{r+\Upsilon} \\ A_t^{(n)} = e^{\phi_0}\left[\Gamma^{(n)}(r+\Upsilon)+c.c.\right]/(r^2-|\Upsilon|^2) \;, \\ \tilde{A}_t^{(n)} = -e^{\phi_0}\left[\Gamma^{(n)}(\lambda_0 r+\bar{\lambda}_0\Upsilon)+c.c.\right]/(r^2-|\Upsilon|^2) ; \end{cases}$$

$$r_\pm = M \pm \underbrace{\sqrt{M^2+|\Upsilon|^2-4\sum_n|\Gamma^{(n)}|^2}}_{SL(2,R)\text{-invariant}} \;; \qquad \Upsilon = \frac{-2}{M}\sum_n \bar{\Gamma}^{(n)2}$$

When one performs an $SL(2,R)$ transformation in the general solution above, the general form of the solution doesn't change, only the boundary conditions (charges etc.) do.

The condition for absence of naked singularities is

$$\boxed{M^2+|\Upsilon|^2-4\sum_n|\Gamma^{(n)}|^2 \geq 0}$$

(Bogomolnyi bound)

It's $SL(2,R)$-invariant (A. Sen, T.O.)

EXTREME BHs & SUSY

G.W. Gibbons
C. Hull
⋮

Supersymmetry algebra ⇒ Bogomolnyi bounds on the charges of the states.

$N=1 \rightarrow \{Q, Q^*\} = M > 0$;

Saturation of the BB ⇒ Restoration of supersymmetry.

$\Downarrow$ $|M=0\rangle \rightarrow M|M=0\rangle = 0 \Rightarrow Q|M=0\rangle = 0$

Existence of Killing Spinors ε_K $\begin{cases} \delta_{\varepsilon_K} \psi_\mu \Big|_{\text{fermions}=0} = 0 \\ \vdots \end{cases}$

$\boxed{\text{Extremality of BHs}}$

Embedding a-d BHs in $N=4, d=4$ SUGRA

$$\begin{cases} \frac{1}{2}\delta_\varepsilon \psi_{\mu I} = \nabla_\mu \varepsilon_I - \frac{i}{4} e^{2\phi} \partial_\mu a\, \varepsilon_I - \frac{1}{4} \sigma^{\rho\sigma} F^+_{\rho\sigma} \alpha_{IJ} \gamma_\mu \varepsilon^J ; \\ \frac{1}{2}\delta_\varepsilon \Lambda_I = -\frac{i}{2}(e^{2\phi} \not{\partial} \lambda)\varepsilon_I + \frac{1}{2}\sigma^{\rho\sigma} F^-_{\rho\sigma} \alpha_{IJ} \varepsilon^J \end{cases}$$

Extreme d BHs admit Killing spinors,

What about a-d BHs?

$SL(2,R)$ vs. SUSY

Th: If for a bosonic configuration

$\{g_{\mu\nu}, \lambda = a + ie^{-2\phi}, F^{(n)}_{\mu\nu}\}$ $\epsilon_{\kappa I}$ is a Killing spinor

$$\begin{cases} \delta_{\epsilon_\kappa} \psi_{\mu I}(g,\lambda,F) = 0; \\ \delta_{\epsilon_\kappa} \Lambda_I (g,\lambda,F) = 0; \end{cases}$$

and $\{g'_{\mu\nu}, \lambda', F'^{(n)}_{\mu\nu}\}$ is another $SL(2,R)$-related configuration

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,R) \qquad \begin{aligned} R &= \alpha\lambda + \beta \\ S &= \gamma\lambda + \delta \end{aligned}$$

$$\begin{cases} g'_{\mu\nu} = g_{\mu\nu}; \\ F^{+\prime}_{\mu\nu} = S F^{+}_{\mu\nu}; \\ \lambda' = R/S; \end{cases}$$

Then the primed configuration admits the Killing spinor

$$\boxed{\epsilon'_{\kappa I} = e^{\frac{i}{2} \mathrm{Arg}(S)} \epsilon_{\kappa I}}$$

Good for on- or off-shell configurations

KILLING SPINOR IDENTITIES

{R. Kallosh
{T.O.

Gauge identities

In G.R. $\quad G_{\mu\nu} = T_{\mu\nu}\,;\quad \nabla^\mu G_{\mu\nu} = 0\,; \Leftarrow$ {Invariance under g. r. of S_{EH}

$S_{EH} + S_M \qquad \Downarrow$

$\nabla^\mu T_{\mu\nu} = 0\,;$

In SUGRA theories

$$S(\phi_b, \phi_f) \qquad \delta_\varepsilon S = S_{,b}\,\delta_\varepsilon \phi^b + S_{,f}\,\delta_\varepsilon \phi^f \underset{\uparrow}{=} 0\,;$$

local SUSY

This is a trivial identity if $\phi_f = 0$

So

$$(\delta_\varepsilon S)_{,f_2}\Big|_{\phi_f=0} = \Big[S_{,b,f_2}\,\delta_\varepsilon \phi^b + S_{,b}(\delta_\varepsilon \phi^b)_{,f_2} + S_{,f_1,f_2}\,\delta_\varepsilon \phi^{f_1} + S_{,f_1}(\delta_\varepsilon \phi^{f_1})_{,f_2}\Big]\Big|_{\phi_f=0} = 0\,;$$

And, if ε_K is a Killing spinor $\quad \delta_{\varepsilon_K} \phi^f\big|_{\phi_f=0} = 0\,;$

$$\boxed{S_{,b}(\delta_\varepsilon \phi^b)_{,f}\Big|_{\phi^f} = 0}$$

SL(2,R) vs. Killing Spinor Identities

$$S_{,b} \to \begin{cases} \dfrac{\delta S}{\delta g^{\mu\nu}} = J_{\mu\nu}\,; & \dfrac{\delta S}{\delta \lambda} = J\,; \\[2ex] \dfrac{\delta S}{\delta A_\mu} = J_A{}^{\mu}\,; & \dfrac{\delta S}{\delta B_\mu} = J_B{}^{\mu}\,; \end{cases}$$

All J's $= 0$ for classical solutions

KSI: $\displaystyle\sum_{\text{bosons}} J_b \left(\delta_{\epsilon_\alpha} \phi^b\right)_{,f} = 0\,;$

$$\begin{cases} \dfrac{e^{\phi}}{\sqrt{2}}\left[J_A{}^{\mu}\alpha^{IJ} + i\gamma_5 J_B{}^{\mu}\beta^{IJ}\right]\bar{\epsilon}_J\gamma_\mu + 2i\,J\,e^{-2\phi}\bar{\epsilon}^I = 0\,; \\[2ex] 2\,J^{\mu\nu}\,\bar{\epsilon}^I\gamma_\nu + \dfrac{e^{\phi}}{\sqrt{2}}\left[J_A{}^{\mu}\alpha^{IJ} + i\gamma_5 J_B{}^{\mu}\beta^{IJ}\right]\bar{\epsilon}_J = 0\,; \end{cases}$$

SL(2,R) $\quad\begin{cases} J_{\mu\nu}{}' = J_{\mu\nu}\,; \quad J_{A,B}{}'^{\mu} = \alpha\, J_{A,B}{}^{\mu}\,; \\ J' = S^2 J\,; \end{cases}$

The KSI are violated after SL(2,R)-duality in general.

This is due to the fact that J_A + SL(2,R) = magnetic currents $\Rightarrow$ one can't define A_μ which is essential for SUSY.

TARGET-SPACE DUALITY vs. SUSY

$$S = \int d^D x \, e^{-2\phi} \sqrt{-g} \left\{ -R + 4(\partial\phi)^2 - \frac{3}{4} H^2 \right\}$$

x is a redundant coordinate $\Rightarrow S$ is invariant under

$$g'_{xx} = 1/g_{xx}; \quad g'_{x\mu} = B_{x\mu}/g_{xx}; \quad g'_{\mu\nu} = g_{\mu\nu} - \frac{(g_{x\mu} g_{x\nu} - B_{x\mu} B_{x\nu})}{g_{xx}}$$

$$B'_{x\mu} = g_{x\mu}/g_{xx}; \quad B'_{\mu\nu} = B_{\mu\nu} + \frac{(g_{x\mu} B_{\nu x} - g_{x\nu} B_{\mu x})}{g_{xx}};$$

$$\phi = \phi - \frac{1}{2} \log |g_{xx}|;$$

Buscher, [illegible], Ovrut, Alvarez, Osorio ....

There is an interchange $B \leftrightarrow g \leftrightarrow \phi$

How do residual supersymmetries behave under these transformations?

This question makes sense, for instance, in $d=10$ $(N=1)$ and $d=4$ $(N=4)$

$N=1$, $d=10$:

$$\begin{cases} \delta_\varepsilon \psi_\mu = \left[\partial_\mu - \frac{1}{4} \left(\omega_\mu{}^{ab} - \frac{3}{2} H_\mu{}^{ab} \right) \gamma_{ab} \right] \varepsilon ; \\ \delta_\varepsilon \lambda = \left[\gamma^\mu \partial_\mu \phi + \frac{1}{4} H_{\mu\nu\rho} \gamma^{\mu\nu\rho} \right] \varepsilon ; \end{cases}$$

No vector fields

Hints: – g, B & ϕ are in the same supermultiplet.
– GFS and SSW are related by duality and both admit Killing spinors.

Th: In the appropriate tangent space basis, a configuration and its dual have identical Killing spinors.

no vector fields

E. Bergshoeff, R. Kallosh & T. O. M.

(Work in progress) ⇝ **WARNING!**

Observations

1) It's not clear that the Th. is true for time-like duality.

2) What about vector fields?

Th: The same happens for $N=4$, $d=4$ with one vector field (generalised duality transformations)

E. Bergshoeff, R. Kallosh & T. O. M.

CONCLUSION

- Dualities are useful tools for generating new solutions and supersymmetric configurations and for classifying them in a way consistent with SUSY

⇝ MAYOR TOOL IN THE QUEST FOR THE "MOTHER OF ALL SOLITONS"
Gibbons, Townsend

- The consistency with SUSY seems to suggest that manifestly duality-invariant theories can be built. (So far little success)

SL(2,R) : A. Sen & J. Schwarz (?)

Target-space: Is there a geometrical entity for which T-S duality is so obvious a symmetry as $g_{\mu\nu} \to g_{\nu\mu}$ in Riemannian geometry?