PERTURBATIVE APPROACH TO STRING THEORY

NON-PERTURBATIVE calc.

ACTION {Polyakov, Brink-Di Vecchia-Howe} + SUM over all (closed / open) smooth 2-D (super) manifolds

PERTURBATIVE series in the string coupling constant (dilaton expectation value)

Fix the order g

D = 26, 10
→ CRITICAL STRINGS
⇓
All the classical symmetries are preserved

Functional integral over genus g (super) manif.

gauge fixing

Integral over M_g, SM_g of functional (super)-determinants

Belavin – Knizhnik theorem: factorisation of functional determinants
Holomorphic anomaly

Chiral determinants

Fadeev – Popov ghosts
+
zero - modes

Correlators of conformal fields on R-S

Reformulation of the action

THE BELAVIN-KNIZHNIK THEOREM

GENUS g BOSONIC STRING COSMOLOGICAL CONSTANT

$$Z_g^{(BOS)} = \int_{M_g} \prod_{i=1}^{3g-3} dy_i \wedge d\bar{y}_i \left(\frac{\det' \Delta_{-1}}{\det N_{-1} \det N_2} \right) \times$$

(gauge fixing $\downarrow$ ghosts)

$$\times \left(\frac{\det N_0 \det N_1}{\det' \Delta_0} \right)^{13 \; (D=26)} \left(\det N_1 \right)^{-13 \; (D=26)}$$

(matter)

$\Delta_j = \rho^{j-1} \partial \rho^{-j} \bar{\partial}$ → Laplace op. acting on j-diff.

$(g_{ab}\, d\xi^a d\xi^b = \rho\, dz\, d\bar{z})$

$(N_j)_{\alpha\beta} = \int \rho^{1-j} \bar{\phi}_\alpha \phi_\beta \, dz \wedge d\bar{z}$ $\quad$ $\{\phi_\alpha\}$ basis of holomorphic j-diffs. (Δ_j - zero modes)

Riemann-Roch $\# = (2j-1)(g-1)$

$f_i \quad i = 1, \dots, 3g-3$ holomorphic 2-diffs.

$\eta_i \quad i = 1, \dots, 3g-3$ Beltrami diffs $\quad (-1,+1)$-diffs.

$\int \eta^i f_j \, dz \wedge d\bar{z} = \delta^i_j$ (dual basis)

Any arbitrary metric can be written as

$$g_{ab}\, d\xi^a d\xi^b = \tilde{\rho}\, | dz + \eta\, d\bar{z} |^2$$

$$\eta = \sum_{i=1}^{3g-3} y_i \eta_i \quad \Rightarrow y_i \text{ complex coordinates on } M_g$$

Conformal anomaly

$$\delta_\rho \log\left(\det{}'\Delta_j / \det N_j \det N_{1-j}\right) = \frac{c_j}{24\pi}\int \delta\rho\, \rho^{-1}\partial\bar\partial \log\rho\, d^2\xi$$

combination independent of the election of $\{\phi_a\}$ $\qquad c_j = 6j^2 - 6j + 1$

Holomorphic anomaly

$$\delta g = \rho\, \eta(y)\, d\bar z^2 + \rho\, \bar\eta(y)\, dz^2$$

Iff $F(y,\bar y)$ is the squared modulus of a holomorphic function on M_g, then the second variation $\delta_\eta \delta_{\bar\eta} \log F$ must vanish

$$\delta_\eta\delta_{\bar\eta} \log\left(\det{}'\Delta_j / \det N_j \det N_{1-j}\right) = \frac{c_j}{24\pi}\int \rho^{-2}\left(\bar\partial f\, \partial\bar f - f\bar f\, \partial\bar\partial \log\rho\right) d^2\xi$$

$$(f = \rho\bar\eta)$$

$\Rightarrow \prod_j \left(\frac{\det{}'\Delta_j}{\det N_j \det N_{1-j}}\right)^{m_j}$ is the squared modulus of a holomorphic function on M_g $\Leftrightarrow \sum_j m_j c_j = 0$

{The conformal anomaly cancels} $\Longleftrightarrow$ {The holomorphic anomaly cancels}

$$D = 26 \rightarrow \quad Z_g^{(BOS)} = \int \prod_{i=1}^{3g-3} \frac{dy_i \wedge d\bar y_i}{(\det N_1)^{13}} \,|W(y)|^2$$

CHIRAL DETERMINANTS

If at the end of our calculations the conformal-gravitational-holomorphic anomaly is going to cancel out, we can work with "holomorphic square roots" of $\left(\frac{\det' \Delta_j}{\det N_j \det N_{1-j}}\right)$

$\Rightarrow$ CHIRAL DETERMINANTS $\det \bar\partial_j$

$$\left|\det \bar\partial_j\right|^2 = \frac{\det' \Delta_j}{\det N_j \det N_{1-j}}$$ up to the anomaly!

IT'S EASIER TO CALCULATE CHIRAL DETERMINANTS (Faddeev-Popov)

$$S_j = \int \phi^{(1-j)} \bar\partial_j \phi^{(j)} \, dz \wedge d\bar z$$ ϕ fermionic fields

$\Rightarrow$ FORMALLY $$\det \bar\partial_j = \int \mathcal{D}\phi^{(1-j)} \mathcal{D}\phi^{(j)} e^{S_j} \quad \left(=0 \text{ DUE TO THE ZERO MODES}\right)$$

To SOAK UP THE ZERO MODES (TO NEUTRALIZE THE CHARGE AT ∞)

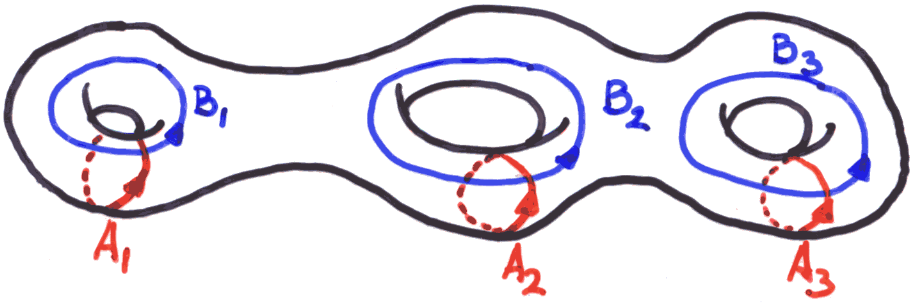
$$\longrightarrow \langle \phi^{(j)}(z_1) \cdots \phi^{(j)}(z_{(2j-1)(g-1)}) \rangle = \int \mathcal{D}\phi^{(j)} \mathcal{D}\phi^{(1-j)} \phi^{(j)}(z_1) \cdots \phi^{(j)}(z) \, e^{S_j}$$

To MAKE THE EXPRESSION INDEPENDENT OF THE POINTS $z_1 \ldots z_{(2j-1)(g-1)}$ ON THE RIEMANN SURFACE

$$\boxed{\det \bar\partial_j \stackrel{\text{def}}{=} \frac{\langle \phi^{(j)}(z_1) \cdots \phi^{(j)}(z_{(2j-1)(g-1)}) \rangle}{\det_{\alpha\beta} \phi_\alpha^{(j)}(z_\beta)}}$$ (Knizhnik)

NOW, WE ONLY HAVE TO USE FUNCTION THEORY ON A RIEMANN SURFACE Σ_g.

SOME BASICS:



$g=3$

$A_1, \dots, A_g$; $B_1, \dots, B_g$ CANONICAL HOMOLOGY BASIS

$(\, , \,)$ Intersection form $\quad (A_i, A_j) = (B_i, B_j) = 0$

$$(A_i, B_j) = -(B_j, A_i) = \delta_{ij}$$

(Real dimension $= 2g$)

HOMOLOGY $\Rightarrow$ WE CAN INTEGRATE 1-diffs ALONG THEM

$\exists \; g \left\{ \begin{matrix} \text{COMPLEX} \\ \text{HOLOMORPHIC} \end{matrix} \right\}$ 1-diffs ω_i (Abelian diffs of 1st kind)

NORMALIZATION: $\displaystyle \int_{A_i} \omega_j = \delta_{ij}$

$$\Rightarrow \boxed{\tau_{ij} = \tau_{ji} = \int_{B_i} \omega_j \qquad \text{COMPLEX PERIOD MATRIX}}$$

IT'S USEFUL TO PARAMETRIZE M_g

$D = \sum_i m_i P_i$: divisor on Σ_g (formal)

$m_i \in \mathbb{Z} \quad \forall i$

$P_i \in$ Riemann surface Σ_g

$$\deg D = \sum_i m_i$$

IF f is A λ-DIFF $(\lambda \in \mathbb{Z}/2)$ AND HAS POLES IN SOME P_i WITH MULTIPLICITY n_i AND ZEROS Q_i WITH MULTIPLICITY m_i

$$\mathrm{Div}\, f = \sum_i m_i Q_i - \sum_j n_i P_i$$

Th If f is a function on Σ_g, $\deg \mathrm{Div}\, f = 0$

$\Rightarrow$ Th If ω is a 1-diff. on Σ_g $\deg \mathrm{Div}\, \omega = g$

⋮

To ANY D, $\deg D = 0$ ($\not\exists$ f function $\mathrm{Div}\, f = D$) WE CAN ASSOCIATE A COMPLEX VECTOR IN $\mathbb{C}^g$:

→ TAKE $K \mid \partial K = D$

$$\mathrm{I}(D) = \int_K \vec{\omega} \quad ; \quad \vec{\omega} = (\omega_1, \dots, \omega_g)$$

(Jacobi map)

FOR INSTANCE $D = Q - P \Rightarrow \mathrm{I}(D) = \int_{P\,\gamma}^{Q} \vec{\omega}$

- I DEPENDS ON γ
- γ IS DEFINED UP TO A COMBINATION OF HOMOLOGY CYCLES

$\Rightarrow \mathrm{I}$ IS DEFINED UP TO AN ELEMENT OF $\mathbb{Z}^g \tau + \mathbb{Z}^g$

$$\boxed{\mathrm{I}(D) \in \mathbb{C}^g / \mathbb{Z}^g \tau + \mathbb{Z}^g \quad \text{(Jacobi variety)}}$$

NOW YOU CAN USE $\mathrm{I}(D)$ AS $\vec{z}$ ARGUMENT OF A RIEMANN THETA FUNCTION $\vartheta\left[\begin{smallmatrix}\vec{\alpha}\\ \vec{\beta}\end{smallmatrix}\right](\mathrm{I}(D) / \tau)$

ϑ IS PERIODIC ON $\mathbb{Z}^g \tau + \mathbb{Z}^g$ UP TO A τ-DEPENDENT PHASE

$\Rightarrow \vartheta\left[\begin{smallmatrix}\vec{\alpha}\\ \vec{\beta}\end{smallmatrix}\right]\left(\mathbb{I}(D)\,/\,\tau\right)$ IS A MULTIVALUED FUNCTION OF THE $P_i \in \Sigma_g$ $\left(D = \sum_i m_i P_i\right)$.

$\{$BRANCHS$\}$ $\longleftrightarrow$ $\{$HOMOLOGY CLASSES OF $K\}$

$$\gamma_2 = \gamma_1 - A_1$$

$$\int_{\gamma_2}\vec{\omega} = \int_{\gamma_1}\vec{\omega} - \int_{A_1}\vec{\omega} = \int_{\gamma_1}\vec{\omega} + (-1, 0, \dots, 0)$$

IDEA

IF WE TAKE TWO MULTIVALUED FUNCTIONS WITH THE SAME MONODROMY, AND CHOOSE A BRANCH, THEIR QUOTIENT IS A SINGLE-VALUED FUNCTION

$\mathrm{V}_{g=2}$ $z^{3/2}$, $z^{1/2}$ two-valued functions
two branches "$\pm z^{3/2}$"
"$\pm z^{1/2}$"

Going once around the origin we change of branch

$$\frac{+z^{3/2}}{-z^{1/2}} = \frac{-z^{3/2}}{+z^{1/2}} = -z$$

$$\frac{+z^{3/2}}{+z^{1/2}} = \frac{-z^{3/2}}{-z^{1/2}} = +z$$

Two possible single-valued functions → "normalization"
WITH THE SAME DIVISOR

We obtain functions on Σ_g with well-defined poles and zeros. We obtain single-valued functions defined up to a multiplicative constant.

WE CAN DO THE SAME WITH ϑ- FUNCTIONS AND OBTAIN FUNCTIONS, j-DIFFERENTIALS ETC. WELL DEFINED ON Σ_g UP TO A MULTIPLICATIVE CONSTANT THAT DEPENDS ON THE BRANCHES (= THE INTEGRATION PATHS) WE CHOOSE.

THAT'S GOOD ENOUGH TO CONSTRUCT CORRELATORS ON Σ_g

$j \in \mathbb{Z}/2$

$$\langle \prod_{i=1}^{(2j-1)(g-1)+M} \phi^{(j)}(z_i) \prod_{j=1}^{M} \phi^{(1-j)}(w_j) \rangle_e = Z_1^{-1/2}\, \vartheta[e]\Big(\sum_i z_i - \sum_j w_j - (2j-1)\Delta_g\Big) \times$$

$$\times \frac{\prod_{i<k} E(z_i, z_k) \prod_{j<\ell} E(w_j, w_\ell)}{\prod_{i,j} E(z_i, w_j)} \times \left(\frac{\prod_i \sigma(z_i)}{\prod_j \sigma(w_j)}\right)^{(2j-1)}$$

$$Z_1 = \frac{\langle \prod_{m=1}^{g} \phi^{(1)}(z_m)\, \phi^{(-1)}(w) \rangle}{\det \omega_m(z_n)}$$

(V&V = W)

BUT THE MULTIPLICATIVE CONSTANT IS A FUNCTION OF τ. WE ARE INTERESTED IN THE τ-DEPENDENCE OF $\det \bar\partial_j$. WE SHOULD FIX THE BRANCHES OF THE ϑ's, BUT THERE IS NO KNOWN GENERAL WAY TO DO IT. FOR MANY PURPOSES THE ABOVE EXPRESSIONS

ARE USELESS

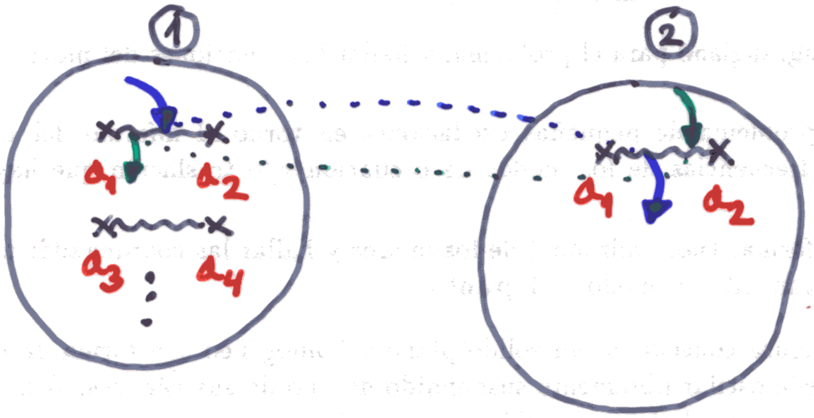
CAN WE AVOID THE USE OF ϑ-FUNCTIONS ?

→ SELBERG Z-FUNCTIONS

→ HYPERELLIPTIC FORMALISM

A genus g hyperelliptic Riemann surface is the surface that uniformizes $y = \sqrt{\prod_{i=1}^{2g+2} (z - a_i)}$ as a function of z.

- WE HAVE $2g+2$ BRANCH POINTS a_i.
- WHEN WE GO ONCE AROUND ANY OF THEM, WE CHANGE OF BRANCH → WE CHANGE OF SHEET ($\hat{\mathbb{C}}$).
- WHEN WE GO TWICE AROUND ANY a_i, WE ARRIVE AT THE SAME SHEET ($\hat{\mathbb{C}}$)



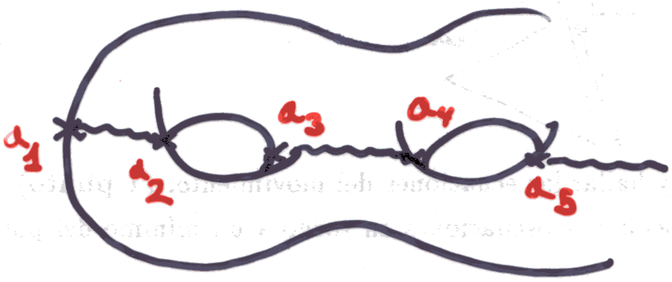
(Riemann-Hurwitz theorem)

↓

$2g+2$ branch points

↓

genus g Riemann surface



WE CAN CHOOSE THREE OF THE a_i ARBITRARILY BECAUSE OF $SL(2,\mathbb{C})$ INVARIANCE ON $\hat{\mathbb{C}}$

WE CAN PARAMETRIZE WITH THE REMAINING $2g-1$ a_i

THE MODULI SPACE OF HYPERELLIPTIC SURFACES $\subset M_g$

$$\left.\begin{array}{ll} 2g-1 = 3g-3 & \Rightarrow g=2 \\ 2g-1 = 1 & \Rightarrow g=1 \end{array}\right\}$$

Every genus 1 and 2 RS is a hyperelliptic one and the $2g-1$ "independent" a_i are good coordinates on M_g $\Rightarrow$ the independent harmonic ratios

$$\lambda(ijkl) = \frac{(a_i-a_j)(a_k-a_l)}{(a_i-a_k)(a_j-a_l)}$$

CONFORMAL FIELDS ON THE BRANCHED SPHERE

ON THE ℓ^{th} $\hat{\mathbb{C}}$-SHEET, LET'S CONSIDER $\left\{\begin{array}{l} f^{(\ell)} \to j\text{-diff} \\ \phi^{(\ell)} \to (1-j)\text{-diff} \end{array}\right|$ <u>anticomm.</u>

Action $\quad S_j^{(\ell)} = \int f^{(\ell)} \bar{\partial}_j \phi^{(\ell)} d^2 z$

Stress tensor $\quad T_j^{(\ell)} = -j f^{(\ell)} \partial \phi^{(\ell)} + (j-1)\phi^{(\ell)} \partial f^{(\ell)}$

$$f^{(\ell)}(z')\phi^{(\ell)}(z) \sim \frac{1}{z'-z} + O(1)$$

$\hat{\pi}_{a_i} \to$ operator which rotates a field 2π around a_i

$$\hat{\pi}_{a_i} f^{(\ell)}(z) = f^{(\ell+1)}(z)$$

$$\hat{\pi}_{a_i} \phi^{(\ell)}(z) = \phi^{(\ell+1)}(z)$$

CHOOSE A BASIS DIAGONALIZING $\hat{\pi}_{a_i}$

$$f_k = e^{-2\pi i \frac{(k-j)}{2}} f^{(1)} + f^{(2)} \qquad j \in \mathbb{Z}$$

$$\phi_k = e^{+2\pi i \frac{(k-j)}{2}} \phi^{(1)} + \phi^{(2)}$$

$$\hat{\pi}_{a_i} f_k = e^{\pi i (k-j)} f_k$$

$$\hat{\pi}_{a_i} \phi_k = e^{-\pi i (k-j)} \phi_k$$

THE CURRENTS $J_k = :f_k \phi_k: \; ; \; \bar\partial J_k = 0$.

ARE SINGLE VALUED AROUND ANY a_i.

$\rightsquigarrow$ IN THE VECINITY OF ANY a_i

$$* \quad J_k(z) = \frac{q_k}{z-a_i} + \mathcal{O}(1) \; ; \quad q_k = \frac{k-j}{2} = \text{“the charge of the branch point } a_i\text{”}$$

$$f(y')\phi(y) \sim \frac{1}{y'-y}$$

$$f^{(\ell)}(z')\,\phi^{(\ell)}(z) = \left(\frac{dy'}{dz'}\right)^{j}\left(\frac{dy}{dz}\right)^{1-j} f(y')\phi(y') \sim \frac{1}{2(z'-z)}\sum_{\ell=1}^{2}\left(y'/y\right)^{\ell-j}$$

$$\rightarrow f_k(z')\,\phi_m(z) \sim \delta_{k,m}\left[\frac{1}{z'-z} + \frac{q_k}{z-a_i}\right] + \mathcal{O}(1)$$

COMPARE!

$$f_k(z')\,\phi_m(z) \sim \frac{\delta_{k,m}}{z'-z} + \underbrace{:f_k\phi_m:(z)}_{k=m \;\to\; J_k} + \mathcal{O}(1) \quad \Rightarrow *$$

IF WE TRY TO BOSONIZE THE $f_k\phi_k$ THEORY $\to$ φ_k ON $\hat{\mathbb{C}}$ WITHOUT ANY MONODROMY

$$\varphi_k(z')\varphi_m(z) \sim -\delta_{km}\log(z'-z)$$

$$f_k = :e^{i\varphi_k}: \; ; \quad \phi_k = :e^{-i\varphi_k}: \; ; \quad J_k = i\partial\varphi_k$$

WE HAVE TO INCLUDE IN CORRELATION FUNCTIONS

$$V_{\vec q}(a_i) = :e^{i\vec q\,\vec\varphi(a_i)}: \; ; \quad \vec q\,\vec\varphi = \sum_k q_k\varphi_k$$

— TO COMPENSATE THE BACKGROUND CHARGE

— TO REPRODUCE $*$

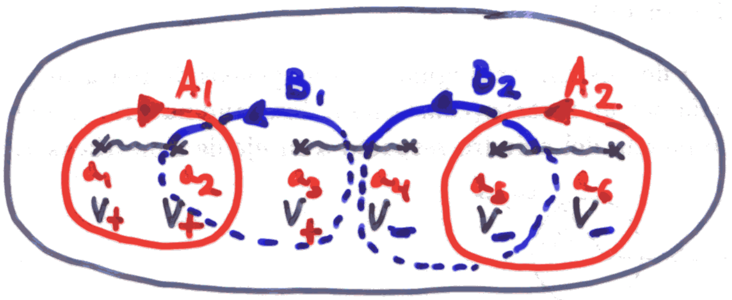
$$\boxed{\{\varphi_k\} \text{ ON } \hat{\mathbb{C}} + V_{\vec q}(a_i) = \text{CORRELATIONS OF } f,\phi \text{ ON } \Sigma_g \text{ HYPERELLIPTIC}}$$

CALCULATING IN THIS WAY THE CORRELATION FUNCTIONS WE OBTAIN UNAMBIGUOUS CHIRAL DETERMINANTS:

V_g $g=2$ $\qquad \det \bar{\partial}_2 = \prod_{k<l=1}^{6} (a_k - a_l)^{5/4}$

TO HANDLE WITH FERMIONS WE NEED TWO DIFFERENT KINDS OF BRANCH-POINT OPERATORS $\left\{ \begin{matrix} V_+(a_i) \\ V_-(a_i) \end{matrix} \right\}$ TO REPRODUCE THEIR MONODROMY PROPERTIES: THEY ARE PERIODIC AROUND TWO V_+ OR TWO V_- AND ANTIPERIODIC AROUND A PAIR $V_+ V_-$:

$g=2$



(ONE SHEET)

--- ⇒ THE OTHER SHEET

→ SPIN STRUCTURE

FOR A FERMION PERIODIC AROUND ANY A_i, B_i $\Rightarrow \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$

THE 10 EVEN SPIN STRUCTURES ON A GENUS TWO RIEMANN SURFACE ARE IN 1 TO 1 CORRESPONDENCE WITH PARTITIONS OF THE 6 BRANCH POINTS IN TWO GROUPS OF THREE:

$$s = \begin{bmatrix} \vec{\alpha} \\ \vec{\beta} \end{bmatrix} \longleftrightarrow (A_1 A_2 A_3 / B_1 B_2 B_3) \qquad \begin{matrix} A_i \\ B_i \end{matrix} \in \{a_i\}$$

$$\vartheta^4[s](\vec{0}/\tau) \propto \prod_{i<j}^{3} (A_i - A_j)(B_i - B_j) \qquad \text{(Thomae identities)}$$

ETC.

NOW THE QUESTION IS:

COULD WE USE ALL THAT MACHINERY IN ORDER TO FIND THE GENUS $g=2$ CONTRIBUTION TO THE HETEROTIC STRING COSMOLOGICAL CONSTANT?

AFTER INTEGRATING OUT THE ODD SUPERMODULI WE HAVE THE FOLLOWING SITUATION:

$$Z^{(HET)}_{g=2} = \int_{M_g} \sum_{s \text{ even}} \Lambda_s(z_1, z_2) \qquad (W)$$

$\Lambda_s \sim \langle X(z_1) X(z_2) \rangle_s$ $\quad X$: Picture-Changing Operators (PCO)

THE DEPENDENCE ON z_1 AND z_2 IS SUCH THAT

$$\sum_{s \text{ even}} \Lambda_s(z_1, z_2) = \sum_s \Lambda_s(\omega_1, \omega_2) + \underbrace{\frac{\partial}{\partial y_i} F \, dy_i}_{\text{TOTAL DERIVATIVE ON } M_2}$$

$\rightsquigarrow$ DOES IT CONTRIBUTE TO $Z^{(HET)}_{g=2}$?

IT CAN BE SHOWN THAT THE WHOLE INTEGRAND IS

$$\sum_{s \text{ even}} \Lambda_s(z_1, z_2) = \underline{\frac{\partial}{\partial y_i} (\text{SOMETHING}) \, dy_i}$$

$\rightsquigarrow$ DOES IT CONTRIBUTE . . . ?

ON THE OTHER HAND (Witten / Seiberg-Witten) WE MUST CALCULATE A Λ_s FOR A GIVEN s AND FIND THE OTHER ONES THROUGH MODULAR TRANSFORMATIONS:

$$\Lambda_{T(s)} \stackrel{def}{=} T(\Lambda_s)$$

$*$ NO ONE DOES USE THIS DEFINITION

MODULAR TRANSFORMATIONS CAN MOVE z_1 AND z_2 PRESERVING s ! DOES Λ_s CHANGE BY A TOTAL DERIVATIVE ?

HOW CAN WE OBTAIN AN EXPLICIT EXPRESSION FOR $Z^{(HET)}_{g=2}$?

FIRST → WE MUST OBTAIN AN EXPRESSION FOR $\Lambda_s(z_1,z_2)$ WITH WELL CHOSEN z_1 and z_2. Use hyperelliptic formulation !

THEN → WE MUST STABILIZE $\Lambda_s(z_1,z_2)$ UNDER THE SUBGROUP OF THE MODULAR GROUP $\Gamma_{g=2}$ THAT PRESERVES s : $\Gamma^{(s)}_{g=2}$

HOW? → BY SUMMING OVER ALL POSSIBLE

$$T\,\Lambda_s(z_1,z_2)\ ;\ T\in\Gamma^{(s)}_{g=2}$$

GIVING A DIFFERENT RESULT.

FINALLY → OBTAIN THROUGH SEIBERG-WITTEN RECIPE THE REMAINING Λ_s.

THIS PROCEDURE ENSURES EXPLICIT

MODULAR INVARIANCE

(The total derivative should arise in modular transformation)

ONE PROBLEM : WE CAN HAVE AS MANY $Z_{=}^{(HET)}$

AS PAIRS δ_1, δ_2 DIS CONNECTED BY TRANSFORMATIONS OF $\Gamma_{g=2}(8)$. BUT THE SITUATION IS BETTER. (SEE LATER)

V_{g2} KNIZHNIK'S EXPRESSION (HYPERELLIPTIC FORMALISM)

USEFUL CHOICE

$z_1 = a_1$
$z_2 = a_2$

He calculated $\Lambda_s(a_1, a_2)$ for all the even s

WE WILL OBTAIN TWO DIFFERENT EXPRESSIONS

STARTING FROM $\Lambda_s = (a_1 a_2 \cdot / \cdots)$

OR $\Lambda_{s'} = (a_1 \cdots / a_2 \cdots)$

Modular transformations are permutations of the branch points

Why?

- LET'S STABILIZE $\Lambda_{(123|456)}(a_1, a_2)$
- WE WILL OBTAIN $\Lambda_{(123|456)}(a_i, a_j)$ $\begin{cases} i,j \in \{1,2,3\} \\ \text{OR} \\ i,j \in \{4,5,6\} \end{cases}$

$$\Longrightarrow \quad \Lambda_{(\alpha_1^e \alpha_2^e \alpha_3^e | \beta_1^e \beta_2^e \beta_3^e)} = \sum_{\substack{i,j \in \{\alpha\} \\ \text{OR} \\ i,j \in \{\beta\}}} \Lambda^{(a_i, a_j)}_{(\alpha_1^e \alpha_2^e \alpha_3^e | \beta_1^e \beta_2^e \beta_3^e)}$$

- STARTING FROM $\Lambda_{(135|246)}(a_1, a_2)$

WE WILL OBTAIN $\Lambda_{(135|246)}(a_i, a_j)$ $\begin{cases} i \in \{1,3,5\} \\ j \in \{2,4,6\} \end{cases}$ OR $\begin{cases} i \in \{2,4,6\} \\ j \in \{1,3,5\} \end{cases}$

$$\Longrightarrow \quad \Lambda_{(\alpha_1^e \alpha_2^e \alpha_3^e | \beta_1^e \beta_2^e \beta_3^e)} = \sum_{\substack{\binom{i \in \{\alpha\}}{j \in \{\beta\}} \binom{i \in \{\beta\}}{j \in \{\alpha\}}}} \Lambda^{(a_i, a_j)}_{(\alpha_1^e \alpha_2^e \alpha_3^e | \beta_1^e \beta_2^e \beta_3^e)}$$

THE PROGRAM CAN BE FOLLOWED (WITH A GREAT BUT FINITE NUMBER OF STEPS!)

THE FINAL RESULTS EXHIBIT A NUMBER OF PECULIARITIES

- BOTH OF THE FINAL EXPRESSIONS VANISH BEFORE INTEGRATION, THROUGH SOME NEW "MINIMAL" THETA IDENTITIES. (EXPLANATION)
- IN ONE CASE, THE PART COMING FROM P^X VANISH BEFORE WE SUM OVER SPIN STRUCTURES (ANOMALY?)
- IN THE OTHER CASE, THE SAME HAPPENS TO THE PART COMING FROM P_e^{gh}
- THE BEHAVIOR IN THE DEGENERATION LIMIT  IS DIFFERENT. ONLY IN THE SECOND CASE IS THE SAME AS AT GENUS 1 → THE SAME GSO (EXPLANATION)

ADVANTAGES OF THESE EXPRESSIONS (COMPARING WITH PRECEDENT VANISHING EXPRESSIONS).

- THE MAIN IS: EXPLICIT MODULAR INVARIANCE → When we construct from them the thermal free energy $F_2(\beta) = \int \sum_s \Lambda_s \vartheta_s(\beta)$ we obtain a modular invariant result
- THE CANCELLATION IS "MINIMAL".

FINAL QUESTIONS:

- WE DON'T UNDERSTAND REALLY THE HIGHER GENUS SUPERSTRING CALCULATIONS.

WITH RESPECT TO THIS METHOD:

- WE DON'T UNDERSTAND THE RELATION BETWEEN SUSY AND THE THETA IDENTITIES
- WE DON'T KNOW HOW MANY DIFFEREN EXPRESSIONS CAN WE CONSTRUCT IN THIS WAY.
  (OBVIOUS GUESS: ONLY TWO!)
- WE DON'T UNDERSTAND THE ANOMALOUS CANCELLATIONS.
- WE DON'T KNOW HOW TO GO TO HIGHER GENUS.
- WE DON'T KNOW

Thanks!

THE INTEGRAND OF THE COSMOLOGICAL CONSTANT

OF THE HETEROTIC STRING

$$Z_g^{(HET)} = \int_{M_g} \prod_{i=1}^{3(g-1)} d^2 m_i \, \frac{|\det\langle \mu_i | b_k\rangle|^2}{\det \mathrm{Im}\,\tau} \left(\frac{\det'\Delta_{-1}}{\det\langle b_i|b_j\rangle}\right) \times$$

$$\times \left(\frac{\det'\Delta_0}{A \det \mathrm{Im}\,\tau}\right)^{-5} \sum_{\substack{e,f,g\\ \text{even}}} C(e,f,g) \, \overline{\left(\det \nabla^2_{\frac{1}{2},f}\right)^8} \, \overline{\left(\det\nabla^2_{\frac{1}{2},g}\right)^8} \times$$

$$\times \left(\frac{\det'\Delta_{-\frac{1}{2},e}}{\det\langle\beta_a|\beta_b\rangle_e}\right)^{-\frac{1}{2}} \frac{1}{\det\langle\chi_a|\beta_b\rangle_e} \times$$

$$\times \int \prod_{a=1}^{2(g-1)} d^2 w_a \, \det_e \chi_a(w_b) \, \langle\langle T_F(w_1)\cdots T_F(w_{2(g-1)})\rangle\rangle_e$$

Picture Changing Formalism

$$\chi_a(w) \sim \delta^{(2)}(w-w_a)$$

$$\Longrightarrow \quad X(w_a) \ \text{PCO's}$$

$$\det \mathrm{Im}\,\tau = \det\langle w_a|w_b\rangle = \det N_1$$

$$\det\langle b_i|b_j\rangle = \det N_2$$

$$\det\langle\beta_a|\beta_b\rangle_e = \det N_{\frac{3}{2},e}$$

$$T_F = \frac{1}{2}\psi^\mu \partial X^\mu + c\partial\beta + \frac{3}{2}\partial c\,\beta - \frac{1}{2}\gamma b$$

$$C(e,f,g) \begin{cases} = \phi(e) & E_8\times E_8 \\ = \phi(e)\,\delta_{fg} & SO(32)/\mathbb{Z}_2 \end{cases}$$

$\langle\langle \ \rangle\rangle \rightarrow$ normalised

KNIZHNIK'S EXPRESSION FOR $Z_2^{(HET)}$

$$Z_{g=2} = \int \sum_{\substack{e,f,g \\ \text{even}}} C(e,f,g) \prod_{i=1}^{6} \frac{d^2 a_i}{d\sigma_{\mu}^2}\, T^{-5} \prod_{k<l}^{6} \overline{(a_{kl})}^{-3} (a_{kl})^{-2} \times$$

$$\times\, \overline{Q_f}^{2}\, \overline{Q_g}^{2} \left\{ P^X Q_e + P_e^{gh} Q_e \right\}$$

$x_{ij} = x_i - x_j$

$e \longleftrightarrow (\alpha_1^e \alpha_2^e \alpha_3^e \,\|\, \beta_1^e \beta_2^e \beta_3^e)$ $\qquad \alpha, \beta \in \{1, 2, \dots 6\}$

$$\Rightarrow Q_e = \prod_{i<j}^{3} a_{\alpha_i^e \alpha_j^e}\, a_{\beta_i^e \beta_j^e} \quad \propto \vartheta^4[e](\vec{0}/\tau)$$

$$d\sigma_{\mu} = da_4\, da_5\, da_6\, (a_{45} a_{46} a_{56})^{-1} \qquad \left(\Rightarrow a_1\ a_2\ a_3\ \underline{\text{moduli}}\right)$$

$$T = \int d^2 z_1\, d^2 z_2 \left| z_{12}\, y^{-1}(z_1)\, y^{-1}(z_2) \right|^2$$

$$y^2(z) = \prod_{i=1}^{6} (z - a_i)$$

$$P^X = \frac{5}{8} a_{12}^{-1} \left\{ a_{23} a_{24} a_{25} a_{26} \frac{P_{12}}{T} + (a_1 \leftrightarrow a_2) \right\}$$

$$P_{12} = \int d^2 z_1\, d^2 z_2 \frac{(a_1 - z_1)(a_1 - z_2)}{(a_2 - z_1)(a_2 - z_2)} \left| \frac{z_{12}}{y(z_1)\, y(z_2)} \right|^2$$

$$P_e^{gh} = \frac{1}{4} a_{12}^{-1} \sum_{i=1}^{3} (a_1 - a_{\alpha_3^e})(a_1 - a_{\beta_i^e})(a_2 - a_{\beta_{i+1}^e})(a_2 - a_{\beta_{i+2}^e}) \quad \text{if } \alpha_2^e = 2$$

$$P_e^{gh} = \frac{1}{4} a_{12}^{-1} (a_1 - a_{\alpha_2^e})(a_1 - a_{\alpha_3^e})(a_2 - a_{\beta_2^e})(a_2 - a_{\beta_3^e}) \qquad \text{if } \beta_1^e = 2$$