SOME RESULTS

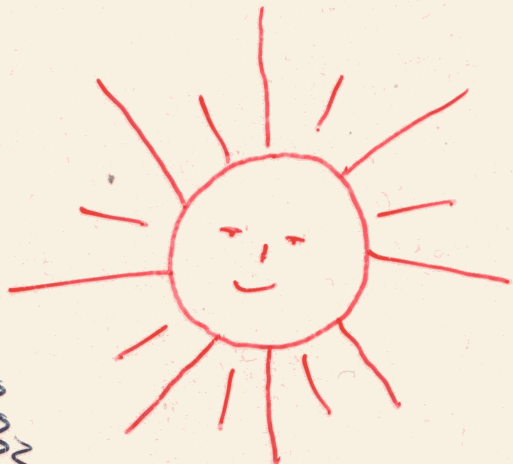
IN

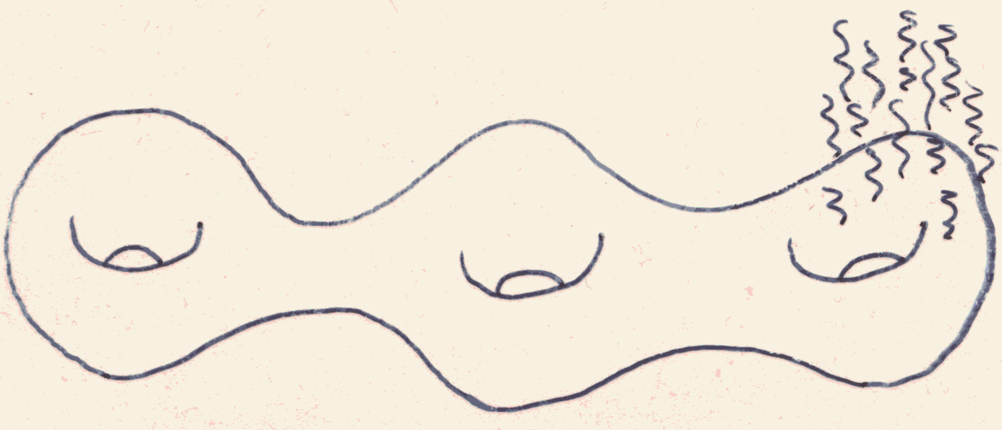
STRING PERTURBATION THEORY

AT

FINITE TEMPERATURE

* INTRODUCTION TO STRING PERTURBATION THEORY
  - BOSONIC STRINGS
  - FERMIONIC AND HETEROTIC STRINGS
  - MODULI SPACE
* FINITE TEMPERATURE
  - INTRODUCTION
  - DUALITY
  - CRITICAL TEMPERATURES





E. ALVAREZ
T. O.
M.A.R. OSORIO

RESULTS

* FOR BOSONIC AND HETEROTIC STRINGS THERE IS A CRITICAL TEMPERATURE AT WHICH THE THERMAL ENERGY DIVERGES ORDER BY ORDER IN PERTURBATION THEORY

→ PHASE TRANSITION?

→ LIMITING TEMPERATURE IN THE UNIVERSE?

(E. ALVAREZ
T. O.
M.A.R. OSORIO)

* ALSO FOR BOSONIC AND HETEROTIC STRINGS THERE IS A SYMMETRY (β-DUALITY) WHICH RELATES THERMAL PROPERTIES AT HIGH AND LOW TEMPERATURES

IN FACT, IT RELATES PROPERTIES ABOVE AND BELOW THE CRITICAL TEMPERATURE (AND ITS DUAL!)

IT IS A SYMMETRY OF THE WHOLE PERTURBATIVE SERIES.

→ IS IT ALSO A NON-PERTURBATIVE SYMMETRY?

(E. ALVAREZ
M.A.R. OSORIO)

* THERE SEEMS TO BE SOME PROBLEMS WITH THE MODULAR INVARIANCE OF MANY EXPRESSIONS FOR THE $g=2$ COSMOLOGICAL CONSTANT FOR THE HETEROTIC STRING.

→ IS THAT A MANIFESTATION OF THE FAMOUS (FOR SOME PEOPLE) AMBIGUITY?

→ ARE THERE SOME OTHER FUNDAMENTAL PROBLEMS WITH FERMIONIC STRINGS BEYOND GENUS 2?

... SEE THE FOLLOWING TRANSPARENCIES →

(TOPICS IN DIFFERENT COLOR WILL BE EXPLAINED IN DETAIL)

* INTRODUCTION TO STRING PERTURBATION THEORY

— CLOSED ORIENTED BOSONIC STRINGS

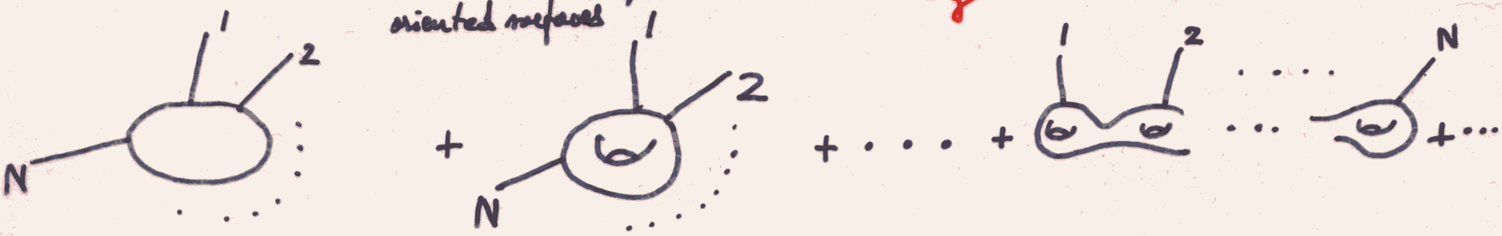
$$\chi = 2(1-g) - N$$

$$A(1,2,\dots N) \overset{DEF}{=} \int_{\substack{\text{2-dimensional oriented} \\ \text{surfaces with } N \text{ boundaries + boundary conditions}}} \mathcal{D}X^\mu \mathcal{D}g_{\alpha\beta}\, e^{-S - \ln K \frac{1}{4\pi}\int d^2\xi \sqrt{g} R^{(2)}} \overset{REDEF}{=}$$

$(D=26)$

$$\overset{REDEF}{=} K^{N-2} \sum_{g=0}^{\infty} K^{2g} \int_{\substack{\text{genus } g \\ \text{2-dimensional compact} \\ \text{oriented surfaces}}} \mathcal{D}X^\mu \mathcal{D}g_{\alpha\beta}\, e^{-S}\, V(1)\dots V(N) =$$

$$V(i) = \int_{\Sigma_g} d^2\xi_i \sqrt{g} f_i(X, g)$$



GENUS g CONTRIBUTION TO THE COSMOLOGICAL CONSTANT ($\langle VAC | VAC \rangle$)

$$Z_g = \int \mathcal{D}X^\mu \mathcal{D}g_{\alpha\beta}\, e^{-S}$$

$$S = \int d^2\xi \sqrt{g}\, g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu$$

LOCAL (GAUGE) INVARIANCES

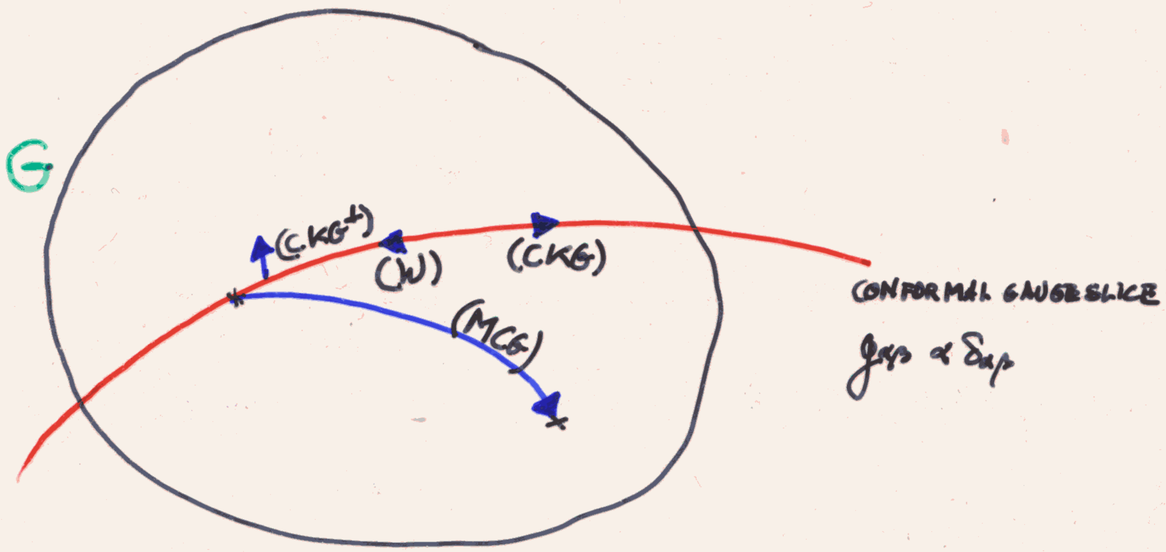
$$\Rightarrow \mathcal{D}g_{\alpha\beta} \longrightarrow \frac{\mathcal{D}g_{\alpha\beta}}{(\text{Vol Diff})(\text{Vol } W)} \quad (D=26)$$

— Diffeomorphisms (Diff)

* CONNECTED WITH THE IDENTITY (Diff_0)
  1) GENERATED BY CONFORMAL KILLING VECTORS (CKG) (RESPECT THE CONFORMAL GAUGE GLOBALLY)
  2) THEIR ORTHOGONAL COMPLEMENT ($CKG^\perp$)
* NOT CONNECTED WITH THE IDENTITY ($\text{Diff}/\text{Diff}_0$) : Mapping Class Group : (MCG)

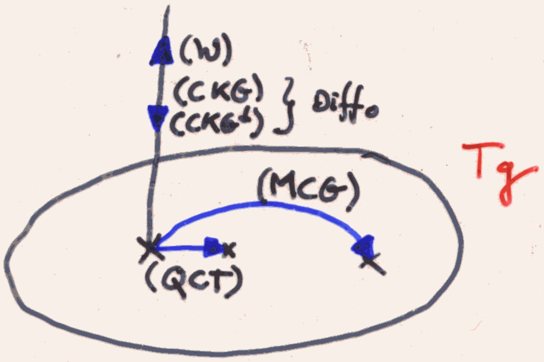
— Weyl transformations (W)

+ SPACE OF GENUS g METRICS $\mathcal{G}$



+ TEICHMÜLLER SPACE T_g

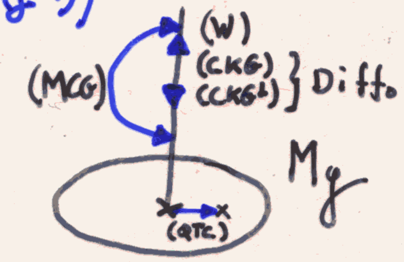
$$T_g = \frac{\mathcal{G}}{(W)\times(\mathrm{Diff}_0)}$$



QCT: QUASICONFORMAL TRANSFORMATIONS $(3(g-1))$

+ MODULAR SPACE M_g

$$M_g = T_g/(\mathrm{MCG}) = \frac{\mathcal{G}}{(W)\times(\mathrm{Diff})}$$



$\parallel$

$\{$GAUGE EQUIVALENCE CLASSES$\} \stackrel{\mathrm{DEF}}{=} \{$ RIEMANN SURFACES$\}$

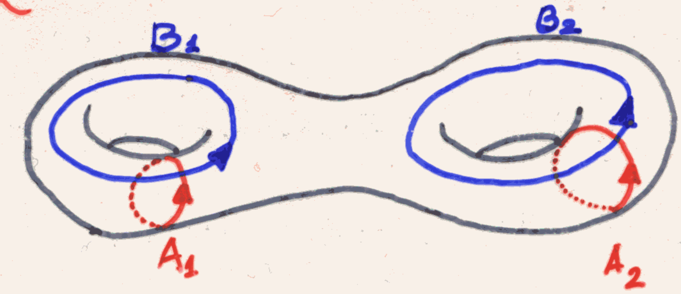
$\Rightarrow$ THE PATH INTEGRAL REDUCES TO AN ORDINARY INTEGRAL OVER THE $3(g-1)$ (COMPLEX)-DIMENSIONAL MODULI SPACE: A FUNDAMENTAL REGION OF THE (M.C.G.) IN THE TEICHMÜLLER SPACE (ALSO $3(g-1)$-DIMENSIONAL) $(g \geq 2)$

WE CAN CHOOSE ANY FUNDAMENTAL REGION, AND ALL OF THEM ARE RELATED BY (M.C.G.)-TRANSFORMATIONS (MODULAR TRANSFORMATIONS IN PARTICULAR)

$\Rightarrow$ $\boxed{\text{THE INTEGRAND MUST BE MODULAR INVARIANT}}$

* THE MODULAR GROUP

CANONICAL HOMOLOGY BASIS



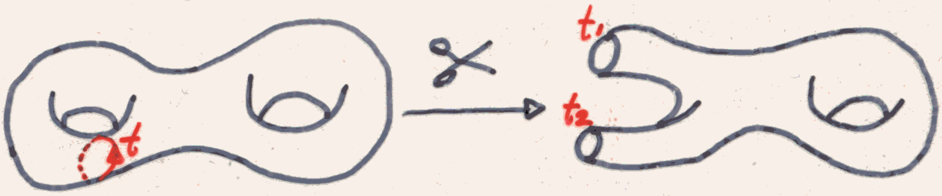
INTERSECTION MATRIX
$(2g \times 2g)$

$$\left(\begin{array}{c|c} (A_i,A_j) & (A_i,B_j) \\ \hline (B_i,A_j) & (B_i,B_j) \end{array}\right) = \left(\begin{array}{c|c} 0 & \delta_{ij} \\ \hline -\delta_{ij} & 0 \end{array}\right) = \left(\begin{array}{c|c} 0 & \mathbb{I} \\ \hline -\mathbb{I} & 0 \end{array}\right)$$

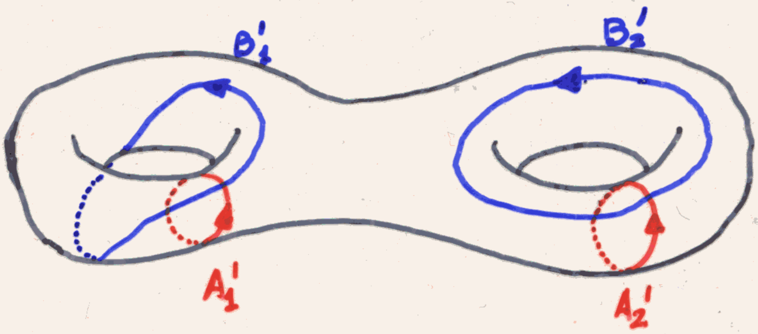
DEHN TWIST

IS A DIFFEOMORPHISM NOT CONNECTED WITH THE IDENTITY. WE CAN VISUALIZE IT AS FOLLOWS:

1) CUT THE SURFACE AROUND A CLOSED PATH (PARAMETRIZED $t = t_1 = t_2 \in [0, 2\pi]$)



2) REGLUE THE SURFACE IDENTIFYING $t_1 \sim t_2 + 2\pi$



(M.C.G.) = { DEHN TWISTS }

MODULAR GROUP = { DEHN TWISTS AROUND NONTRIVIAL HOMOLOGY CYCLES }

* MODULAR TRANSFORMATIONS PRESERVE THE INTERSECTION MATRIX AND THEY ARE LINEAR TRANSFORMATIONS OF THE HOMOLOGY BASIS } $\Rightarrow Sp(2g, \mathbb{Z})$

GENERATORS : DEHN TWISTS AROUND

$$\left.\begin{array}{lll} A_1 \cdots A_g & & \\ B_1 \cdots B_g & & \\ A_1 - A_2 & A_2 - A_3 & A_{g-1} - A_g \end{array}\right\} 3g-1$$

PERIOD MATRIX

ON EVERY GENUS g RIEMANN SURFACE THERE EXIST g ABELIAN DIFFERENTIALS (COMPLEX HOLOMORPHIC 1-FORMS) $\omega_i = \omega_i(z)\, dz \quad i=1\dots g$

NORMALIZATION: $\displaystyle\int_{A_i} \omega_j = \delta_{ij}$

$\Rightarrow \displaystyle\int_{B_i} \omega_j = \tau_{ij}$

τ_{ij} : $g \times g$ complex symmetric matrix

$\Rightarrow \frac{1}{2} g(g+1)$ independent complex components

$\mathrm{Im}\, \tau_{ij}$ positive definite $\Rightarrow \tau_{ij} \in \mathcal{H}_g$ (Siegel's upper half plane)

$\rightarrow$ PERIOD MATRIX

IF WE PERFORM A MODULAR TRANSFORMATION

$$\underbrace{\left(\begin{array}{c|c} D & C \\ \hline B & A \end{array}\right)}_{\in Sp(2g,\mathbb{Z})} \begin{pmatrix} A_1 \\ \vdots \\ A_g \\ B_1 \\ \vdots \\ B_g \end{pmatrix} = \begin{pmatrix} \\ \\ \end{pmatrix}'$$

$\Rightarrow \displaystyle\int_{A_i'} \omega_j' = \delta_{ij} \Rightarrow \omega_j' = \omega_k (C\tau + D)^{-1}_{kj}$

$$\boxed{\int_{B_i'} \omega_j' = \tau_{ij}' = (A\tau + B)_{ik} (C\tau + D)^{-1}_{kj}}$$

$* \ \frac{1}{2} g(g+1) = 3(g-1) \quad \Rightarrow \quad g = \begin{cases} 2 \\ 3 \end{cases}$

FOR GENUS TWO AND THREE WE CAN USE THE τ_{ij} $i \le j$ AS COORDINATES IN MODULI SPACE, AND A FUNDAMENTAL REGION $\mathcal{H}_g / Sp(2g,\mathbb{Z})$ AS M_g.

HOWEVER, WE KNOW IN GENERAL HOW TO RELATE THE ELEMENTS OF τ WITH SOME GOOD COORDINATES IN M_g

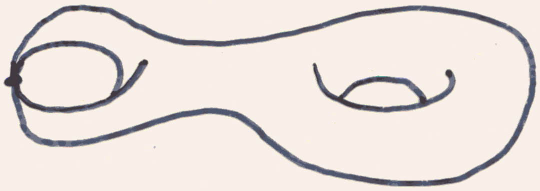
$\longrightarrow$

THE BOUNDARY OF MODULI SPACE

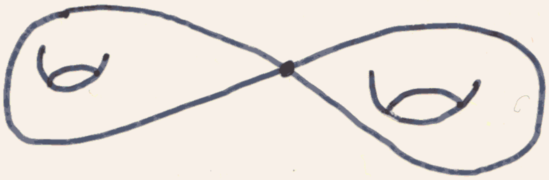
* POINTS ON THE BOUNDARY OF M_g CORRESPONDS TO DEGENERATE SURFACES (SURFACES WITH NODES) (PINCHED SURFACES).

TWO TYPES

1) A NONTRIVIAL HOMOLOGY CYCLE IS PINCHED (A_1)

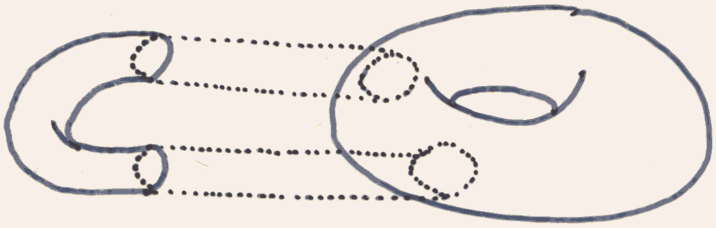


2) A TRIVIAL HOMOLOGY CYCLE IS PINCHED (0)



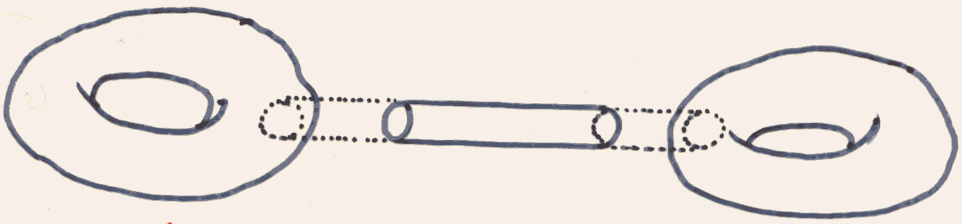
BOTH CASES CAN BE DESCRIBED BY THE SO CALLED "PLUMBING FIXTURE"
(D. J. FAY)

1)



$g-1$

2)



g_1 $\qquad$ g_2 $\qquad$ $g_1+g_2=g$

THE PLUMBING FIXTURE IS PARAMETRIZED BY THE COMPLEX PARAMETER δ, $|\delta|<1$ IN SUCH A WAY THAT WE GET THE BOUNDARY OF M_g IN $\boxed{\delta=0}$ AND (IN THE CONSTANT CURVATURE SLICE), THE CIRCUMFERENCE OF THE CYLINDER GOES TO ZERO AS

$$l = \frac{-2\pi^2}{\ln|\delta|}$$

AND ITS LENGHT TO INFINITY AS $\quad s = \pi/l = -\dfrac{\ln|\delta|}{2\pi}$

* FOUR IMPORTANT FACTS ABOUT THE FIRST CASE

a) δ IS A GOOD COORDINATE NEAR THE BOUNDARY OF M_g

b) THE ZONE OF THE PINCHING IS A DOUBLE RAMIFIED COVERING OF THE SPHERE BRANCHED AT $\pm\delta^{1/2}$.

$$y^2 = z^2 - \delta \qquad z \in \hat{\mathbb{C}}$$

y - local coordinate near the pinching

c) $\operatorname{Arg}\delta$ PARAMETRICES NATURALLY THE "TWIST" OF THE PLUMBING FIXTURE, AND, THUS, $\operatorname{Arg}\delta \in [-\pi, \pi]$.

d) THE PERIOD MATRIX HAS THE FOLLOWING δ-DEPENDENCE (WE CHOOSE THE A_1 CYCLE)

$$\tau_g = \left(\begin{array}{c|c} \frac{\ln\delta}{2\pi i} + C_1 & \vec{\tau}^{\,t} \\ \hline \vec{\tau} & \tau_{(g-1)} \end{array}\right) + O(\delta)$$

$\tau_{(g-1)}$ IS THE PERIOD MATRIX OF THE REMAIMING GENUS $g-1$ SURFACE

c) + d) $\Longrightarrow$ $\quad \operatorname{Re}\tau_{11} \in [\operatorname{Re}C_1 - 1/2, \operatorname{Re}C_1 + 1/2] + O(\delta)$

$\operatorname{Im}\tau_{11} = S$, THE LENGHT OF THE TUBE... $\xrightarrow[\delta\to 0]{} \infty$

BELAVIN-KNIZHNIK THEOREM (PHYSICS LETTERS 168B (1986) 201)

WE CAN ALWAYS WRITE THE GENUS g CONTRIBUTION TO THE COSMOLOGICAL CONSTANT IN THIS WAY: ($g \geq 2$!)

$$Z_g = \int_{M_g} (\det \mathrm{Im}\,\tau)^{-13}\, W \wedge \overline{W}$$

WHERE

$$W = W(y_1 \cdots y_{3(g-1)})\, dy_1 \wedge \cdots \wedge dy_{3(g-1)}$$

$y_i \to$ COMPLEX COORDINATES ON M_g

is A HOLOMORPHIC $3(g-1)$ FORM ON M_g WITH SECOND ORDER POLES AT THE BOUNDARY OF M_g. W is, THUS, DETERMINED UP TO A MULTIPLICATIVE CONSTANT. (MOORE: PHYSICS LETTERS 176B (1986) 369, $g=2$)

By IDENTIFYING s WITH THE PROPER TIME OF STRING MODES PROPAGATING THROUGH THE PINCHED HANDLE, AS $s \to \infty$ ONLY THE LOWEST ENERGY MODES PROPAGATE: THE TACHYON CAUSES THE POLES OF ORDER FOUR OF Z_g.

THIS CAN BE EASILY SEEN AT $g=1$ AND 2 (MOORE). (KNIZHNIK)

WE WILL HAVE A BEHAVIOR

$$Z_g \sim \int_{\delta \to 0} \frac{d\delta \wedge d\bar{\delta}}{|\delta|^4}$$

WHEN A HANDLE IS BEING PINCHED

* THE FERMIONIC STRING

(H. VERLINDE PH.D. THESIS, UTRECHT UNIV.)
($g \geq 2$!)

THE PRELIMINAR SITUATION IS THE (SUPER) ANALOGUE OF THE BOSONIC CASE (SUPER MODULI SPACE SM_g ...)

AFTER (SUPERCONFORMAL) GAUGE FIXING AND EXPLICIT INTEGRATION OVER THE ODD COORDINATES OF SM_g, WE HAVE:

$$Z_g^{(\text{FERMIONIC})} = \int \mathcal{D}x^\mu \mathcal{D}\psi^\mu \mathcal{D}b\, \mathcal{D}c\, \mathcal{D}\beta\, \mathcal{D}\gamma \Big|^2 \Big| \prod_a Y[\chi_a] \prod_i \langle \mu_i | b \rangle \Big|^2 e^{-S_0}$$

$Y[\chi_a] = \delta(\langle \chi_a, \beta \rangle) \langle \chi_a, T_F \rangle \rightarrow$ PICTURE CHANGING OPERATORS (PCOs)

$a = 1 \ldots 2(g-1)$ $\quad \chi_a \rightarrow$ GRAVITINO ZERO MODES

$i = 1 \ldots 3(g-1)$ $\quad \mu_i \rightarrow$ BELTRAMI DIFFERENTIALS

$$S_0 = -\frac{1}{\pi} \int_{\Sigma_g} \frac{1}{2} \partial X^\mu \bar{\partial} X_\mu + \Big(\frac{1}{2} \psi^\mu \bar{\partial} \psi_\mu + \underbrace{b \bar{\partial} c}_{\text{GHOSTS}} + \underbrace{\beta \bar{\partial} \gamma}_{\text{SUPERGHOSTS}} + \text{c.c} \Big)$$

Z_g IS AGAIN AN INTEGRAL OVER M_g.

* BUT, NOW, Z_g SEEMS TO DEPEND ON THE χ_a's: CHANGING $\chi_a \rightarrow \chi_a + \delta\chi_a$ THE INTEGRAND OF Z_g CHANGES IN A TOTAL DERIVATIVE IN M_g. WE DON'T KNOW IF THERE IS A CONTRIBUTION TO Z_g COMING FROM THE BOUNDARY OF M_g THROUGH THIS DERIVATIVE. IF THIS IS SO, THEN Z_g IS NOT UNAMBIGUOUSLY DEFINED.

$\Rightarrow$ IN THE LIGHT-CONE GAUGE THERE IS NO SUCH AMBIGUITY: ONLY A SET OF χ_a's CANCELS THE BACKGROUND CHARGE.

IN THE GENERAL CASE WE ONLY KNOW PARTICULAR CHOICES OF χ_a's USEFUL ONLY LOCALLY IN M_g. WE COULD NEED TO DO A CHANGE OF CHOICE TO STUDY ANOTHER REGION OF M_g.

FINALLY, IF

$$Z_g^{(\text{FERMIONIC})} = \int_{M_g} \left(\frac{\det}{\mathrm{Im}\,\tau}\right)^{-5} u \wedge \overline{u} \qquad Z_g^{(\text{BOSONIC})} = \int_{M_g} \left(\frac{\det}{\mathrm{Im}\,\tau}\right)^{-13} W \wedge \overline{W}$$

$$Z_g^{(\text{HETEROTIC})} = \int_{M_g} \left(\frac{\det}{\mathrm{Im}\,\tau}\right)^{-5} u \wedge \overline{W}^{(E_8\times E_8)}_{(SO(32)/\mathbb{Z}_2)}$$

SPIN STRUCTURES

(L. ALVAREZ-GAUMÉ, G. MOORE, C. VAFA C.M.P. 106 (1986) 1)

FERMIONS ARE SINGLE VALUED ON A DOUBLE COVERING OF THE RIEMANN SURFACE. THEN, THEY CAN BE PERIODIC OR ANTIPERIODIC AROUND EVERY NONTRIVIAL HOMOLOGY CYCLE. A SET OF PERIODICITY CONDITIONS AROUND A CANONICAL HOMOLOGY BASIS DETERMINES A SPIN STRUCTURE, AND CAN BE REPRESENTED BY A "THETA CHARACTERISTIC"

$$\begin{bmatrix} \vec{\alpha} \\ \vec{\beta} \end{bmatrix} \qquad \vec{\alpha}, \vec{\beta} \in \frac{(\mathbb{Z}_2)^g}{2} \qquad \begin{array}{l} A_i \to (-1)^{2\alpha_i + 1} \\ B_j \to (-1)^{2\beta_j + 1} \end{array}$$

α_i, β_i: $0 \to$ ANTIPERIODIC $\quad 1/2 \to$ PERIODIC

— THETA CHARACTERISTICS ARE NOT MODULAR INVARIANT.

— WE CAN NOT DEFINE SPIN STRUCTURES WITHOUT A HOMOLOGY BASIS.

$\Rightarrow$ WE CAN NOT WRITE A MODULAR INVARIANT PARTITION FUNCTION FOR SUPERSTRINGS WITH FERMIONS WITH ONLY ONE SPIN STRUCTURE. WE MUST SUM OVER A SET OF SPIN STRUCTURES CLOSED UNDER MODULAR TRANSFORMATIONS. (AT LEAST)

$$\Rightarrow \quad \mathcal{U} = \sum_s \phi_s \, \mathcal{U}_s \qquad s \to \begin{bmatrix} \vec{\alpha} \\ \vec{\beta} \end{bmatrix}$$

$\phi_s \to$ PHASES $\longrightarrow$ THE GSO PROJECTION

THEY ARE CHOSEN IN SUCH A WAY THAT $Z_g = 0$ (SUPERSYMMETRY) (TACHYONS OUT!)

EVEN SPIN STRUCTURES $\qquad e^{4\pi i \vec{\alpha}\cdot\vec{\beta}} = 1$ (NO ZERO MODES)

ODD SPIN STRUCTURES $\qquad e^{4\pi i \vec{\alpha}\cdot\vec{\beta}} = -1$ (1 ZERO MODE)

ONLY EVEN SPIN STRUCTURES CONTRIBUTE TO $\mathcal{U}$. THEY TRANSFORM BETWEEN THEMSELVES:

$$\Rightarrow \quad \mathcal{U} = \sum_{s \text{ even}} \phi_s \, \mathcal{U}_s$$

$\boxed{\text{GENUS} \quad g = 1}$

$$\mathcal{U}_{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]} = \frac{\vartheta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]^4(0/\tau)}{\eta^{12}(\tau)}\, d\tau \qquad \mathcal{U}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} = \frac{\vartheta\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]^4(0/\tau)}{\eta^{12}(\tau)}\, d\tau$$

$$\mathcal{U}_{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]} = \frac{\vartheta\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]^4(0/\tau)}{\eta^{12}(\tau)}\, d\tau$$

$$\left\{ \begin{aligned} &\vartheta\begin{bmatrix}\vec{\alpha}\\\vec{\beta}\end{bmatrix}(\vec{z}/\tau) = \sum_{\vec{n}\in\mathbb{Z}^g} \exp\left\{\pi i (\vec{n}+\vec{\alpha})^t \tau (\vec{n}+\vec{\alpha}) + 2\pi i (\vec{n}+\vec{\alpha})^t(\vec{z}+\vec{\beta})\right\} \\ &\vartheta\begin{bmatrix}\vec{\alpha}\\\vec{\beta}\end{bmatrix}(-\vec{z}/\tau) = e^{4\pi i \vec{\alpha}\vec{\beta}}\, \vartheta\begin{bmatrix}\vec{\alpha}\\\vec{\beta}\end{bmatrix}(\vec{z}/\tau) \\ &\eta(\tau)^{-1} = q^{1/24} \prod_{n=1}^{\infty} (1-q^n) \qquad q = e^{2\pi i \tau} \end{aligned} \right.$$

$S : \tau \to -1/\tau \qquad S\,\mathcal{U}\begin{bmatrix}0\\0\end{bmatrix} = \tau^{-6}\,\mathcal{U}\begin{bmatrix}0\\0\end{bmatrix} \qquad S\,\mathcal{U}\begin{bmatrix}1\\0\end{bmatrix} = \bar{\tau}^{6}\,\mathcal{U}\begin{bmatrix}0\\1\end{bmatrix}$

$T : \tau \longrightarrow \tau+1 \qquad S\,\mathcal{U}\begin{bmatrix}0\\1\end{bmatrix} = \tau^{-6}\,\mathcal{U}\begin{bmatrix}1\\0\end{bmatrix} \qquad S(\mathrm{Im}\,\tau)^{-6} = \tau^{6}\bar{\tau}^{6}(\mathrm{Im}\,\tau)^{-6}$

$$T\,\mathcal{U}\begin{bmatrix}0\\0\end{bmatrix} = -\mathcal{U}\begin{bmatrix}0\\1\end{bmatrix} \qquad T\,\mathcal{U}\begin{bmatrix}0\\1\end{bmatrix} = -\mathcal{U}\begin{bmatrix}0\\0\end{bmatrix}$$

$$T\,\mathcal{U}\begin{bmatrix}1\\0\end{bmatrix} = \mathcal{U}\begin{bmatrix}1\\0\end{bmatrix} \qquad T(\mathrm{Im}\,\tau)^{-6} = (\mathrm{Im}\,\tau)^{-6}$$

$$\Rightarrow \phi\begin{bmatrix}0\\0\end{bmatrix} = -\phi\begin{bmatrix}1\\0\end{bmatrix} = -\phi\begin{bmatrix}0\\1\end{bmatrix} = 1 \;\Big|\; \mathcal{U} \sim \eta^{-12}\left[\vartheta\begin{bmatrix}0\\0\end{bmatrix}^4 - \vartheta\begin{bmatrix}1\\0\end{bmatrix}^4 - \vartheta\begin{bmatrix}0\\1\end{bmatrix}^4\right] = 0$$

(GSO PROJECTION)

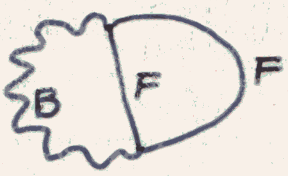
"Equatio identica satis abstrusa" (Jacobi, 1829)

FOR THE LEFT MOVERS WE CAN CHOOSE $\overline{\mathcal{U}}$, $\overline{W}$ COMPACTIFIED (IF NOT, WE DON'T HAVE MODULAR INVARIANCE) OR, ALSO, INSTEAD OF $\mathcal{U}\wedge\overline{\mathcal{U}}$

$$\sum_{\substack{s,t\\ \text{even}}} \delta_{st}\,\mathcal{U}_s \wedge \overline{\mathcal{U}}_t \qquad \text{(DIXON-HARVEY)}$$

IN THE GENUS 1 CASE, THE GSO PROJECTION CORRESPONDS TO WEIGHT WITH A DIFFERENT SIGN SPACE-TIME BOSONS AND FERMIONS

* THIS INTERPRETATION CANNOT BE POSSIBLE BEYOND $g=1$: WE CAN HAVE A FIELD WHICH CORRESPONDS TO A BOSON ON A HANDLE AND TO A FERMION ON ONE ANOTHER HANDLE. WE CAN HAVE GSO PROJECTIONS WITH THIS INTERPRETATION LOCALLY, ON A HANDLE $\Rightarrow$ THEY CANNOT BE MODULAR INVARIANT.



LET'S STUDY CLOSER THE GENUS 1 CANCELLATION.

THE GSO PROJECTION MUST ELIMINATE TACHYONS I.E. DIVERGENCIES IN THE DEGENERATION LIMIT:

$$\tau = \frac{\ln\delta}{2\pi i} + C_1 \qquad \mathrm{Re}\,\tau \overset{\text{def}}{=} \frac{\varphi}{2\pi} + \mathrm{Re}\,C_1 \;;\; \varphi = \mathrm{Arg}\,\delta$$

$$Z_1^{(\text{HETEROTIC})} = \int_{M_1} (\mathrm{Im}\,\tau)^{-5} \sum_{s\ \text{even}} \phi_s\, \mathcal{U}_s \wedge \overline{W}\begin{pmatrix} E_8\times E_8 \\ SO(32)/\mathbb{Z}_2 \end{pmatrix}$$

$\mathrm{Im}\,\tau \longrightarrow$ logarithmic divergencies.

$$\phi_{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]} \mathcal{U}_{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]} \wedge \overline{W}\begin{pmatrix} E_8\times E_8 \\ SO(32)/\mathbb{Z}_2 \end{pmatrix} \sim +\Big(|\delta|^{-5/2} e^{\pi i\,\mathrm{Re}\tau} e^{3\pi\,\mathrm{Im}C_1} + 8|\delta|^{-2} e^{2\pi i\,\mathrm{Re}\tau} e^{2\pi\,\mathrm{Im}C_1} +$$

$$+ |\delta|^{-3/2} e^{\pi\,\mathrm{Im}C_1}\left[36\, e^{3\pi i\,\mathrm{Re}\tau} + 480\, e^{-\pi i\,\mathrm{Re}\tau}\right] + \dots \Big)\, d|\delta| \wedge d\,\mathrm{Re}\tau$$

$$\phi_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} \mathcal{U}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} \wedge \overline{W}\big(\quad\big) \sim -\Big(16\,|\delta|^{-2} e^{2\pi i\,\mathrm{Re}\tau} e^{2\pi\,\mathrm{Im}C_1} + \dots\Big)\, d|\delta| \wedge d\,\mathrm{Re}\tau$$

$$\phi_{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]} \mathcal{U}_{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]} \wedge \overline{W}\big(\quad\big) \sim -\Big(|\delta|^{-5/2} e^{\pi i\,\mathrm{Re}\tau} e^{3\pi\,\mathrm{Im}C_1} - 8|\delta|^{-2} e^{2\pi i\,\mathrm{Re}\tau} e^{2\pi\,\mathrm{Im}C_1} +$$

$$+ |\delta|^{-3/2} e^{\pi\,\mathrm{Im}C_1}\left[36\, e^{3\pi i\,\mathrm{Re}\tau} + 480\, e^{-\pi i\,\mathrm{Re}\tau}\right] + \dots \Big)\, d|\delta| \wedge d\,\mathrm{Re}\tau$$

> INTEGRATION OVER $\mathrm{Re}\,\tau \in [\mathrm{Re}\,C_1 - 1/2,\ \mathrm{Re}\,C_1 + 1/2]$ PROJECTS OUT MODES WITH $e^{2\pi i m\,\mathrm{Re}\tau}$

* FINITE TEMPERATURE

Bosonic { B. McLAIN, B.D.B. ROTH — CMP 111 (1987) 539

HETEROTIC { I.J. ATICK, E. WITTEN — NPB 310 (1988) 291

THE QUATITY OF INTEREST IS THE THERMAL FREE ENERGY:

$$F(\beta) = -\frac{1}{\beta} \log \operatorname{Tr} e^{-\beta \hat{H}}$$

{ THE TRACE IDENTIFIES FINAL AND INITIAL (SPACE-TIME) STATES.
THE HAMILTONIAN $\hat{H}$ IS A TIME-TRANSLATION OPERATOR.

THEN, IN THE PATH INTEGRAL REPRESENTATION OF F( )

$$F^{(Bos)}(\beta) = \sum_{g=1}^{\infty} K^{2(g-1)} F_g(\beta) = \sum_{g=1}^{\infty} \int_{(B.C.)_\beta} \frac{\mathcal{D}g_{\alpha\beta}\, \mathcal{D}X^\mu}{\mathrm{Vol}(W)\,\mathrm{vol}(\mathrm{Diff})}\, e^{-S} K^{2(g-1)}$$

(THERE IS NO $g=0$ CONTRIBUTION)

$\Rightarrow$ WE MUST INTEGRATE OVER COMPACT SURFACES WITHOUT PUNCTURES AND WE MUST IDENTIFY

$$\boxed{X^0 \sim X^0 + \beta}$$

{ X^0 IS COMPACTIFIED ON THE 1-TORUS OF RADIUS $\beta/2\pi$.
X^0 IS A MULTIVALUED FIELD ALL OF WHOSE VALUES DIFFER BY m TIMES β.

$\Rightarrow$ THE VALUE OF A CONFIGURATION OF X^0 CAN JUMP BY AN INTEGER MULTIPLE OF β AROUND A CLOSED PATH ON A GENUS g RIEMANN SURFACE Σ_g

> THEN, WE MUST CONSIDER ALL X^0 CONFIGURATIONS, CLASSIFIED BY THEIR MONODROMY CONDITIONS AROUND A BASIS OF CLOSED PATHS ON Σ_g: $A_1, \dots, A_g, B_1, \dots, B_g$
> $(n_1, \dots, n_g, m_1, \dots, m_g)$

THE CONFIGURATIONS OF THE "SECTOR" LABELLED BY $(n_1,\dots,n_g,m_1,\dots,m_g)$ $\left((\vec{n},\vec{m})\right)$ ARE THOSE WHOSE VALUES ON A POINT $P\in\Sigma_g$, $X^0(P)$ TURN INTO $X^0(P)+n_i\beta$ (RESP. $m_j\beta$) IN GOING ONCE AROUND A_i (RESP. B_j)

THEN, THE PATH INTEGRAL SPLITS:

$$F_g(\beta)=\sum_{(\vec{n},\vec{m})\in\mathbb{Z}^{2g}}\int_{B.C.(\vec{n},\vec{m})\beta}\frac{\mathcal{D}g_{\alpha\beta}}{\mathrm{vol}(W)}\frac{\mathcal{D}X^\mu}{\mathrm{vol}(\mathrm{Diff})}\,e^{-S[X,g,\beta]}$$

✱ WE CAN OBTAIN EVERY X^0 CONFIGURATION IN A GIVEN SECTOR IN THIS WAY:

$$X^0_{(\vec{n},\vec{m})}=X^{0\,(\text{classical})}_{(\vec{n},\vec{m})}+X^{0\,(\text{quantum})}$$

$$-\partial_z\partial_{\bar z}X^{0\,(\text{classical})}_{(\vec{n},\vec{m})}=0$$ AND HAS $(\vec{n},\vec{m})$-MONODROMY PROPERTIES

THERE IS ONLY ONE REAL MULTIVALUED FUNCTION SATISFYING THESE CONDITIONS:

$$\boxed{X^{0\,(\text{classical})}_{(\vec{n},\vec{m})}{}^{(z,\bar z)}=\frac{i}{2}\beta\left\{\int^z\omega_i\,(\mathrm{Im}\,\tau_{ij})^{-1}\left(\bar\tau_{jk}n_k-m_j\right)-\text{c.c.}\right\}}$$

$$\left(\begin{array}{l}\partial_{\bar z}\,\omega_i=0 \text{ BY DEFINITION} \qquad (\partial_z\bar\omega_i=0)\\ z \text{ AROUND } A_j \Rightarrow \int^z\omega_i\rightarrow\int^z\omega_i+\oint_{A_j}\omega_i=\int^z\omega_i+\delta_{ij}\\ z \text{ AROUND } B_j \Rightarrow \int^z\omega_i\rightarrow\int^z\omega_i+\oint_{B_j}\omega_i=\int^z\omega_i+\tau_{ij}\end{array}\right)$$

ONE FUNCTION IS A "CONSTANT" IN FUNCTIONAL INTEGRATION:

$$\Rightarrow \mathcal{D}X^{0} = \mathcal{D}X^{0\,(\text{quantum})}$$

$$S[X_{(\vec{n},\vec{m})}, g, \beta] = S[X^{(\text{quantum})}, g] + S[X^{(\text{classical})}_{(\vec{n},\vec{m})}, g, \beta] =$$

$$= S[X^{(q)}, g] + \frac{\beta^2}{2\pi}[m_i \tau_{ij} - m_j](\mathrm{Im}\,\tau)^{-1}_{jk}[\bar{\tau}_{kl} m_l - m_k]$$

UNDER A MODULAR TRANSFORMATION

$$\left(\begin{array}{c|c} D & C \\ \hline B & A \end{array}\right)\begin{pmatrix} A_1 \\ \vdots \\ A_g \\ B_1 \\ \vdots \\ B_g \end{pmatrix} = \begin{pmatrix} \\ \\ \end{pmatrix}' \longrightarrow \begin{pmatrix} \vec{m}' \\ \vec{m}' \end{pmatrix} = \left(\begin{array}{c|c} D & C \\ \hline B & A \end{array}\right)\begin{pmatrix} \vec{m} \\ \vec{m} \end{pmatrix}$$

DIFFERENT SECTORS ARE INTERCHANGED BETWEEN THEM.

$\Rightarrow$ TO OBTAIN MODULAR INVARIANCE WE MUST SUM OVER ALL $(\vec{n},\vec{m})$ SECTORS. (WE KNEW IT!)

$$F_g(\beta) = \sum_{(\vec{n},\vec{m})\in\mathbb{Z}^{2g}} \int_{M_g} (\det \mathrm{Im}\,\tau)^{-13}\, W \wedge \bar{W}\, e^{-S[X^{(\text{clas})}_{(\vec{n},\vec{m})}, g, \beta]}$$

$\ast$ WE CAN WRITE THE β-DEPENDENT PART IN THIS WAY:

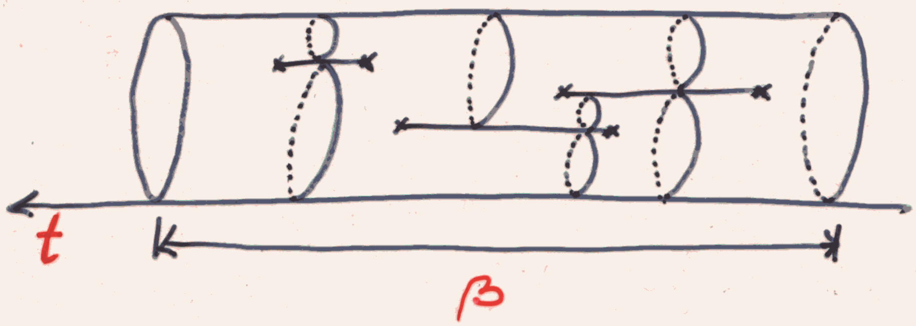
$$\vartheta\begin{bmatrix} \vec{0} \\ \vec{0} \end{bmatrix}\left(\vec{0} \middle/ \tfrac{1}{4}\Omega_g(\beta)\right) \quad ; \quad \Omega_g(\beta) = \frac{2i\beta^2}{\pi^2}\begin{pmatrix} \tau_1\tau_2^{-1}\tau_1 + \tau_2 & -\tau_1\tau_2^{-1} \\ -\tau_2^{-1}\tau_1 & \tau_2^{-1} \end{pmatrix}$$

$$\tau_1 = \mathrm{Re}\,\tau$$
$$\tau_2 = \mathrm{Im}\,\tau$$

* OBSERVATIONS

* THE NUMBERS $(m_1,\dots,m_g, m_1,\dots,m_g)$ HAVE NO TOPOLOGICAL INFORMATION. ONLY $r =$ G.C.D. $(\vec{m},\vec{m})$ IS A TOPOLOGICAL INVARIANT : $\gamma = \frac{1}{r}\sum_{i=1}^{g}(m_i A_i + m_i B_i)$ CAN BE INTENDED AS A TIME (OR SPACE) DIRECTION ON Σ_g. r COUNTS THE NUMBER OF TIMES γ IS WRAPPED AROUND X^0.

* IN LIGHT-CONE GAUGE CALCULATIONS WE HAVE SUCH A γ PATH CHOSEN FROM THE BEGINNING, AND WE HAVE ONLY ONE SECTOR LABELLED BY r. (E. ALVAREZ, M.A.R. OSORIO PHYS. REV. D36 (1987) 1175)



MANDELSTAM DIAGRAM.

WE HAVE NOT MODULAR INVARIANCE.

$\Rightarrow$ WE MUST INTEGRATE OVER A FUNDAMENTAL REGION OF THE SUBGROUP OF $Sp(2g,\mathbb{Z})$ WHICH LEAVES INVARIANT THIS WHOLE SECTOR $(\supset M_g)$. THIS RESULT CAN BE EXTENDED TO THE MODULAR INVARIANT ONE BY COSET TECHNIQUES ETC.* (E. ALVAREZ & M.A.R. OSORIO NPB304 (1988) 327 ($g=1$ AND HETEROTIC); McLAIN & ROTH (g ARBITRARY, BOSONIC))

IT IS SHOWN THAT THE TWO RESULTS ARE EQUIVALENT.

* BY "PUTTING ALL THE WINDINGS ON A HANDLE" USING $Sp(2g,\mathbb{Z})$ TRANSFORMATIONS.

– BY PERFORMING A POISSON RESUMMATION:

$$\vartheta\left[\begin{smallmatrix}\vec{0}\\ \vec{0}\end{smallmatrix}\right]\left(\vec{0}\,/\,\tfrac{1}{4}\Omega_g(\beta)\right) = \left(\frac{\sqrt{2\pi}}{\beta}\right)^{g}\left(\det \operatorname{Im}\tau\right)^{1/2}\sum_{\substack{\vec{p}_L\\ \vec{p}_R}} e^{i\pi\left(\vec{p}_L^{\,t}\,\tau\,\vec{p}_L - \vec{p}_R^{\,t}\,\bar{\tau}\,\vec{p}_R\right)}$$

$$\left.\begin{array}{l} \vec{p}_L = \dfrac{\pi}{\beta}\vec{m} + \dfrac{\beta}{2\pi}\vec{n} \\ \vec{p}_R = \dfrac{\pi}{\beta}\vec{m} - \dfrac{\beta}{2\pi}\vec{n}\end{array}\right\} \quad \begin{array}{c}\vec{n}\\ \vec{m}\end{array} \in \mathbb{Z}^{g} \qquad \begin{array}{l}\vec{n} \to \text{"windings"}\\ \vec{m} \to \text{"momenta"}\end{array}$$

(These are not the same $\vec{n}\,\vec{m}$ as in ϑ)

THE EXPONENT IS INVARIANT UNDER THE SIMULTANEOUS CHANGES:

$$\beta \longrightarrow \frac{2\pi^2}{\beta} \qquad \vec{n} \leftrightarrow \vec{m} \quad = \quad \text{windings} \leftrightarrow \text{momenta}$$

$$\Rightarrow F_g^{(B)}\left(\frac{2\pi^2}{\beta}\right) = \left(\frac{\beta^2}{2\pi^2}\right)^{g} F_g^{(B)}(\beta) \qquad F(\kappa,\beta) = \sum_{g=1}^{\infty} \kappa^{2(g-1)} F_g(\beta)$$

$$\Rightarrow \boxed{F^{(B)}\left[\frac{\kappa\pi\sqrt{2}}{\beta}, \frac{2\pi^2}{\beta}\right] = \frac{\beta^2}{2\pi^2}\, F^{(B)}(\kappa,\beta)}$$

β–DUALITY (E. ALVAREZ, M.A.R. OSORIO, PHYS. REV. D40 (1989) 1150)

FOR THE BOSONIC STRING THIS IS ONLY A FORMAL PROPERTY DUE TO DIVERGENCIES IN THE INTEGRAND AND IN THE PERTURBATION SERIES (GROSS & PERIWAL)

– DIVERGENCIES OF $F_g(\beta)$ COME FROM DIVERGENCIES IN $W\wedge\bar{W}$ WHICH ARE GIVEN BY THE BELAVIN-KNIZHNIK THEOREM

$$W\wedge\bar{W} \sim \frac{d\delta\wedge d\bar{\delta}}{|\delta|^4} \qquad \left(\tau_{ii} \sim \frac{\ln\delta}{2\pi i} + c_i + O(\delta)\right)$$

AND ARE MODULED BY β-DEPENDENT TERMS (EXCEPT THE $(\vec{n},\vec{m}) = (\vec{0},\vec{0})$ TERM → THE TACHYON) IN SUCH A WAY THAT THE LOWEST ORDER TERMS DIVERGE FOR (δ – INTEGRATION)

$$\beta < \pi\sqrt{8} \qquad \text{AND ITS DUAL} \qquad \beta \geq \pi/\sqrt{2}$$

(HAGEDORN TEMPERATURE) $\left(\beta \to \frac{2\pi^2}{\beta}\right)$

ALL THE TERMS IN THE PERTURBATIVE SERIES DIVERGE AT THE HAGEDORN TEMPERATURE: IT RECEIVES NO LOOP CORRECTIONS!!

WHAT IS THE PHYSICAL INTERPRETATION?

TWO POSSIBLE ANALOGIES:

1) COSMIC STRINGS

(NUMERICAL SIMULATIONS $\longrightarrow$ LIMITING TEMPERATURE)

2) Q.C.D. $\rightsquigarrow$ PHASE TRANSITION AT T_H

(CONFINED $\longrightarrow$ DECONFINED PHASE)

* WE COULD EXPECT THAT WHEN WE INTRODUCE INTERACTIONS, STATES ACQUIRE A WIDTH Γ.

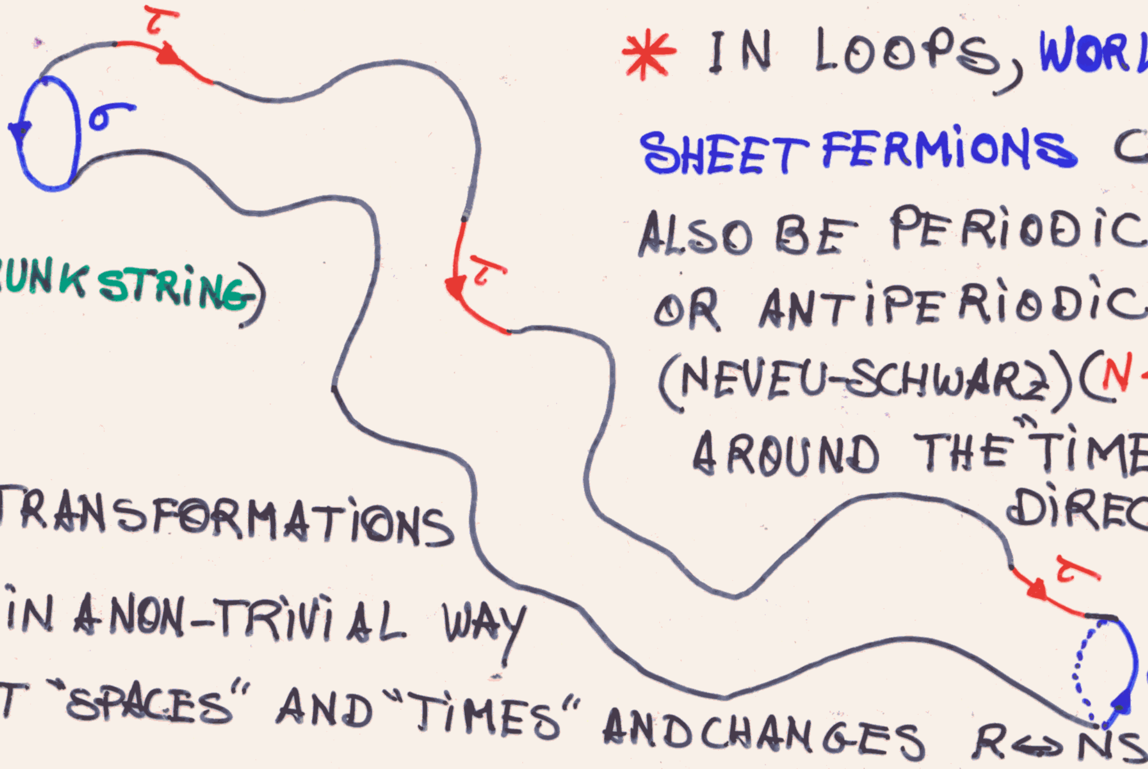
IF Γ GROWS QUICKLY ENOUGH WITH THE MASS AND WE EXCLUDE IN $F(\beta)$ STATES WHOSE MEAN LIFE IS LESSER THAN β, THEN IT HAS BEEN ARGUED (GLEISER & TAYLOR P.L. B64 (1985) 36) THAT $T_H = \infty$.

WE HAVE SEEN THAT THIS IS NOT THE CASE FOR BOSONIC STRINGS AND WE WILL SEE THAT THE SAME HAPPENS IN THE HETEROTIC CASE.

* FERMIONIC STRINGS

SPACE-TIME FERMIONS HAVE ANTIPERIODIC β-(EUCLIDEAN TIME) BOUNDARY CONDITIONS AND THEY MUST BE WEIGHTED WITH A MINUS (STATISTICAL) SIGN IN THE PARTITION FUNCTION → ("GSO")

WORLD-SHEET FERMIONS ARE SPACE TIME FERMIONS IF THEY ARE PERIODIC (RAMOND)(R) AROUND THE "SPACE" DIRECTION σ OF THE STRING →

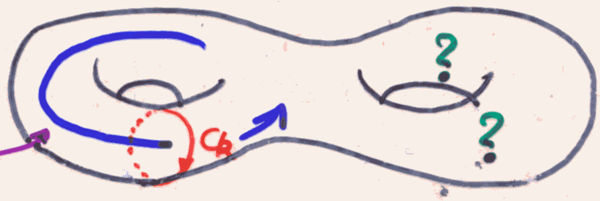


(DRUNK STRING)

* IN LOOPS, WORLD-SHEET FERMIONS CAN ALSO BE PERIODIC (R) OR ANTIPERIODIC (NEVEU-SCHWARZ)(N-S) AROUND THE "TIME" DIRECTION.

* MODULAR TRANSFORMATIONS PERMUTE IN A NON-TRIVIAL WAY WORLD-SHEET "SPACES" AND "TIMES" AND CHANGES R ⟺ NS B.C.

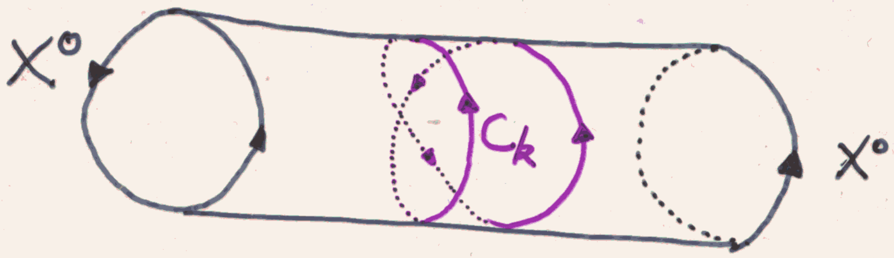
* A WORLD SHEET FERMION WITH R B.C. AROUND A HOMOLOGY CYCLE C_R, PROPAGATES AS A SPACE-TIME FERMION ALONG THE "ORTHOGONAL" DIRECTION



S-T FERMION

WHEN IT HAS THE WINDING NUMBER m_k $(\in \mathbb{Z})$, IT IS WRAPPED AROUND THE (TARGET) SPACE-TIME TIME X^0 (OF LENGTH β) m_k TIMES WHEN IT PASSES THROUGH C_k:

m_k



R - B.C. m_k WINDING NUMBER

CHANGE IN SIGN AROUND

X^0	C_k
-1	$+1$

m_k TIMES $X^0 \Rightarrow (-1)^{m_k}$

$\Longrightarrow$ WE MUST INTRODUCE A $(-1)^{m_k}$ ADITIONAL FACTOR.

N-S B.C. m_k WINDING NUMBER

CHANGE IN SIGN AROUND

X^0	C_k
$+1$	-1

m_k TIMES $X^0 \Rightarrow +1$

$\Rightarrow$ WE MUST INTRODUCE A $(+1)$ ADITIONAL FACTOR.

REMEMBER:

SPIN STRUCTURE $\begin{bmatrix} \vec{\alpha} \\ \vec{\beta} \end{bmatrix}$

R B.C. AROUND $A_k \longrightarrow \alpha_k = \frac{1}{2}$

$B_k \longrightarrow \beta_k = \frac{1}{2}$

N-S B.C. AROUND $A_k \longrightarrow \alpha_k = 0$

$B_k \longrightarrow \beta_k = 0$

THEN, IMPOSING MODULAR INVARIANCE WE GET THE ADITIONAL PHASE WE MUST INTRODUCE:

(ATICK & WITTEN, ALVAREZ & OSORIO)

$$\mathcal{U}\left(\begin{bmatrix}\vec{\alpha}\\ \vec{\beta}\end{bmatrix} \Big/ (\vec{m}, \vec{\mathfrak{m}})\right) = \exp\left\{2\pi i\left(\vec{m}\vec{\beta} + \vec{\mathfrak{m}}\vec{\alpha} + \frac{\vec{m}\vec{\mathfrak{m}}}{2}\right)\right\}$$

THE CONTRIBUTION OF ALL $(\vec{m}, \vec{\mathfrak{m}})$ "SECTORS" TO $F_g(\beta)$ FOR A GIVEN SPIN STRUCTURE $\begin{bmatrix}\vec{\alpha}\\ \vec{\beta}\end{bmatrix}$ IS:

$$\vartheta\begin{bmatrix}\vec{\alpha} & \vec{\alpha}\\ \vec{\beta} & \vec{\alpha}\end{bmatrix}(\vec{0} / \tilde{\Omega}_g) \qquad \tilde{\Omega}_g = \Omega_g + \frac{1}{2}\left(\begin{array}{c|c} 0 & \mathbb{I} \\ \hline \mathbb{I} & 0\end{array}\right)_{2g\times 2g}$$

THIS THETA FUNCTION IS MODULAR COVARIANT:

UNDER $T\left(\begin{bmatrix}\vec{\alpha}\\ \vec{\beta}\end{bmatrix}\right) = \begin{bmatrix}\vec{\alpha}'\\ \vec{\beta}'\end{bmatrix}$ $\quad T(\tau) = \tau'$ $\quad \tilde{\Omega}'_g = \tilde{\Omega}_g(\tau')$

$$(T \in Sp(2g, \mathbb{Z}))\quad T\left(\vartheta\begin{bmatrix}\vec{\alpha} & \vec{\alpha}\\ \vec{\beta} & \vec{\alpha}\end{bmatrix}(\vec{0} / \tilde{\Omega}_g)\right) = \vartheta\begin{bmatrix}\vec{\alpha}' & \vec{\alpha}'\\ \vec{\beta}' & \vec{\alpha}'\end{bmatrix}(\vec{0} / \tilde{\Omega}'_g)$$

THEN, THE THERMAL FREE ENERGY FOR THE HETEROTIC STRING $F_g^{(\text{HETEROTIC})}(\beta)$ IS:

$$F_g^{(\text{HETEROTIC})}(\beta) = \int_{M_g} (\det \operatorname{Im}\tau)^{-5}\left(\sum_{s\ \text{even}} \phi_s\, \mathcal{U}_s\, \vartheta[s](0/\tilde{\Omega}_g)\right) \wedge \overline{W}_{\left(E_8\times E_8\right) \left(\frac{SO(32)}{\mathbb{Z}_2}\right)}$$

AND IS MODULAR INVARIANT (IF $Z_g^{(\text{HETEROTIC})}$)