Comments on dispersive bounds and the swampland

Julio Parra-Martinez

(w/ Caron-Huot, Li, Simmons-Duffin x2, w/ Helset, Cheung)

@ Back to the Swamp, IFT Madrid, September 2022

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A few questions

How much can a quantum theory of gravity, possessing a large spectral gap and satisfying minimal axioms, differ from Einstein's General Relativity?

Can we use basic principles (unitarity, causality, …) to constrain quantum gravity theories from the bottom up?

After imposing such constraints, how big is the swampland?

Outline

1. Dispersive bounds a (very) short review

2. What we can and cannot do for the swampland

(3. Bonus?)

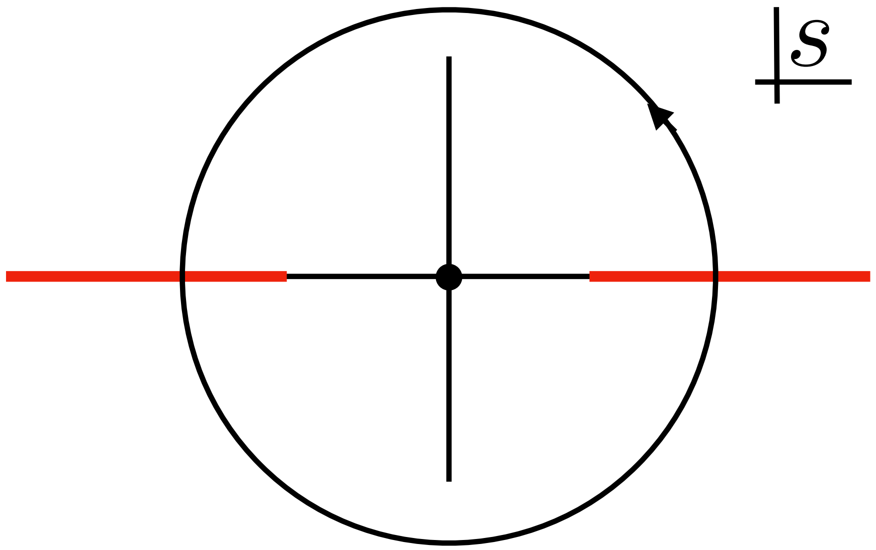
Dispersive bounds:
a (very) short review

A dispersive bound

[Pham, Truong 80s]

- Unitarity, causality, etc $\longleftrightarrow$ analytic properties of $\mathcal{M}$
- Scalar EFT with shift symmetry (e.g. dilaton) $\frac{1}{2}(\partial\phi)^2 + g_2(\partial\phi)^4 + \cdots$

$$\mathcal{M} \sim g_2(s^2+t^2+u^2) + g_3 stu + \cdots$$



A sum rule: $B(t) = \oint \frac{ds}{s^3}\mathcal{M}(s,t)$

At high energies $B(0) = \int_0^\infty \frac{ds}{s^3}\mathrm{Im}\mathcal{M}(s,0) = \int_0^\infty \frac{ds}{s^2}\sigma$

At low energies $B(0) = g_2$

$$\boxed{g_2 = \langle\sigma\rangle > 0}$$

(c.f. proof of the D=4 a-theorem)

Forward limit bounds

- Example of “old fashioned” positivity bounds
- Uses positivity in the forward limit (e.g. optical theorem)
- Can only bound sign of terms even in CoM energy squared (“s”)
- More examples of positivity:

  - Coefficient of F^4 in Maxwell EFT [Adams, Arkani-Hamed, Dubovski, Nicolis, Ratazzi]

  - Coefficient of R^4 in gravitational EFT [Cheung, Remmen]

- Forward limit dangerous when gravity is on

$$\mathcal{M}_{4\gamma} \sim \langle 12\rangle^2[34]^2 \left(8\pi G\left(\frac{s}{tu} + \underset{RF^2}{\beta^2}\frac{tu}{s}\right) + \underset{F^4}{g_2} + \cdots\right)$$

Q: Many swampland arguments are non-perturbative in nature (e.g. black holes). Why would this ever help?

A: Dispersion relations are also non-perturbative. LE part is computed in EFT, but HE part encodes non-perturbative processes. Constraints (unitarity, causality) are also general.

Gravity amplitude

- Let us focus on MHV sector

$$
\begin{aligned}
\mathcal{M}(1^+2^-3^-4^+) &= \langle 23\rangle^4[14]^4 f(s,u) \\
\mathcal{M}(1^+2^-3^+4^-) &= \langle 24\rangle^4[13]^4 f(t,s) \\
\mathcal{M}(1^+2^+3^-4^-) &= \langle 34\rangle^4[12]^4 f(t,u)
\end{aligned}
$$

- Crossing $\quad f(s,u) = f(u,s)$

- EFT expansion

$$
f = \frac{8\pi G_N}{stu} + 2\pi G_N |\hat{g}_3|^2 \frac{su}{t} + g_4 + g_5 t + g_6\, t^2 + g_6'\, su + \cdots
$$

Einstein-Hilbert $\quad R^3 \quad R^4 \quad D^2R^4 \quad D^4R^4$

Assumptions

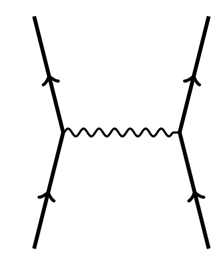
1. Low energy spectrum & EFT expansion

$$Gs^2 < GM^2 = M^2/M_{pl}^2 \ll 1$$

2. Quantum causality (= analiticity & crossing)

3. Lorentz invariance at all scales (spectrum organized by mass, spin)

4. Partial wave unitarity $0 < \rho_J^{h_i}(m^2) = |c_{J,m^2}^{h_i}|^2 < 2$



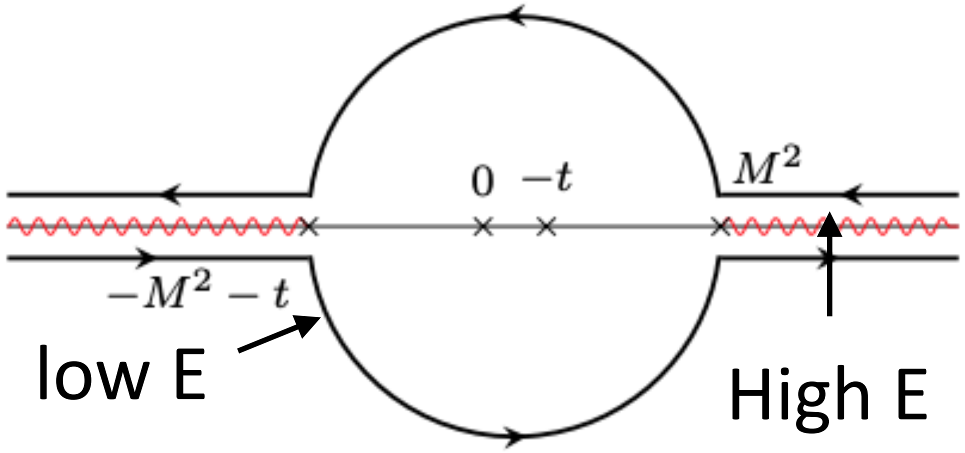
$$\mathrm{Im}\mathcal{M}(m^2,t) = \sum_J \rho_J^{h_i}(m^2) d_{h,h'}^J(1+\tfrac{2t}{m^2})$$

5. Regge boundedness $\lim_{|s|\to\infty} \mathcal{M}/|s|^2 = 0$

“Bootstrappy” Dispersive bounds

[Caron-Huot, Van Duong; Caron-Huot, Li, JPM, Simmons-Duffin]

- Sum rule $$B_k(p^2) = \oint_\infty \frac{ds}{4\pi i} \frac{(2s-p^2)}{[s(s-p^2)]^{\frac{k+2}{2}}} f(s, p^2-s) = 0$$



$$\boxed{-B_k(p^2)\Big|_{\text{low}} = B_k(p^2)\Big|_{\text{high}}}$$

$$t = -p^2$$

- If we can find a functional F such that $$F[B_k(p^2)]\Big|_{\text{high}} \geq 0$$

  then we get a bound on EFT coefficients from $$-F[B_k(p^2)]\Big|_{\text{low}} \geq 0$$

- Unitarity helps! $$\rho_J(m^2) \geq 0$$

$$F[B_k(p^2)]\Big|_{\text{high}} \sim \int dm \sum_J \rho_J(m^2) F[\mathcal{P}_J(1 - \tfrac{2p^2}{m^2})]$$

Gravity is weak (and attractive)

- Can find a functional such that $$-B_2^{(1)}(p^2)\Big|_{\rm low} = \frac{16\pi G}{p^2} + 2\pi G|\hat{g}_3|^2 p^6$$

$$F[1/p^2] = 1 \qquad F[p^6] = 0$$

and is positive on high-energy spectrum

- This implies $$G = \sum_{J,m^2} |c_{J,m^2}|^2 (\text{positive})$$

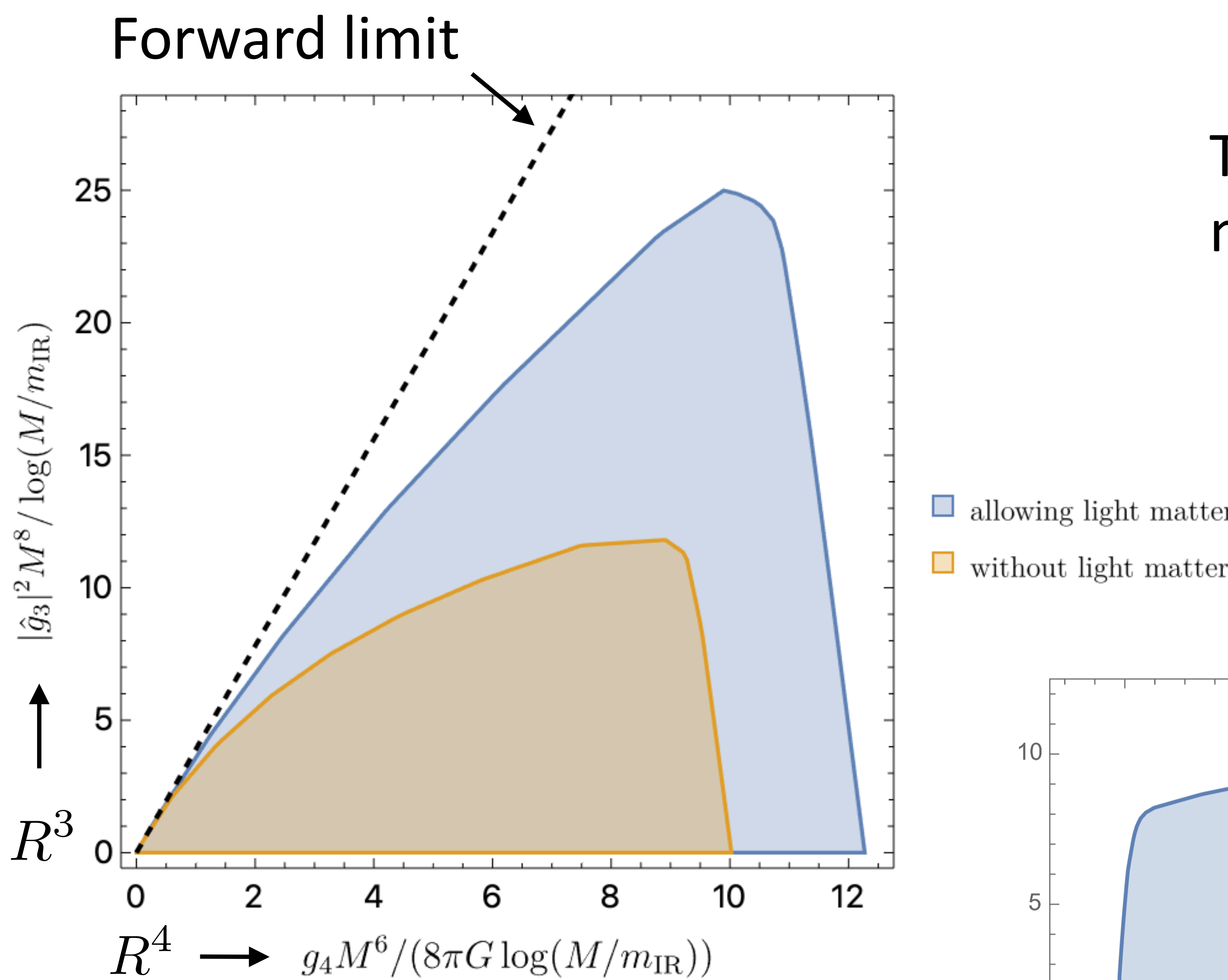
- So gravity is:
  1. Attractive (duh!)
  2. Weakly coupled at all scales M < Mpl!

- This guarantees bounds on all contacts $$M^{2i} g_i < G$$

(Sharp: dimensional analysis with O(1) coefficient fixed)

What we can and cannot do for the swampland

[Caron-Huot, Li, JPM, Simmons-Duffin]

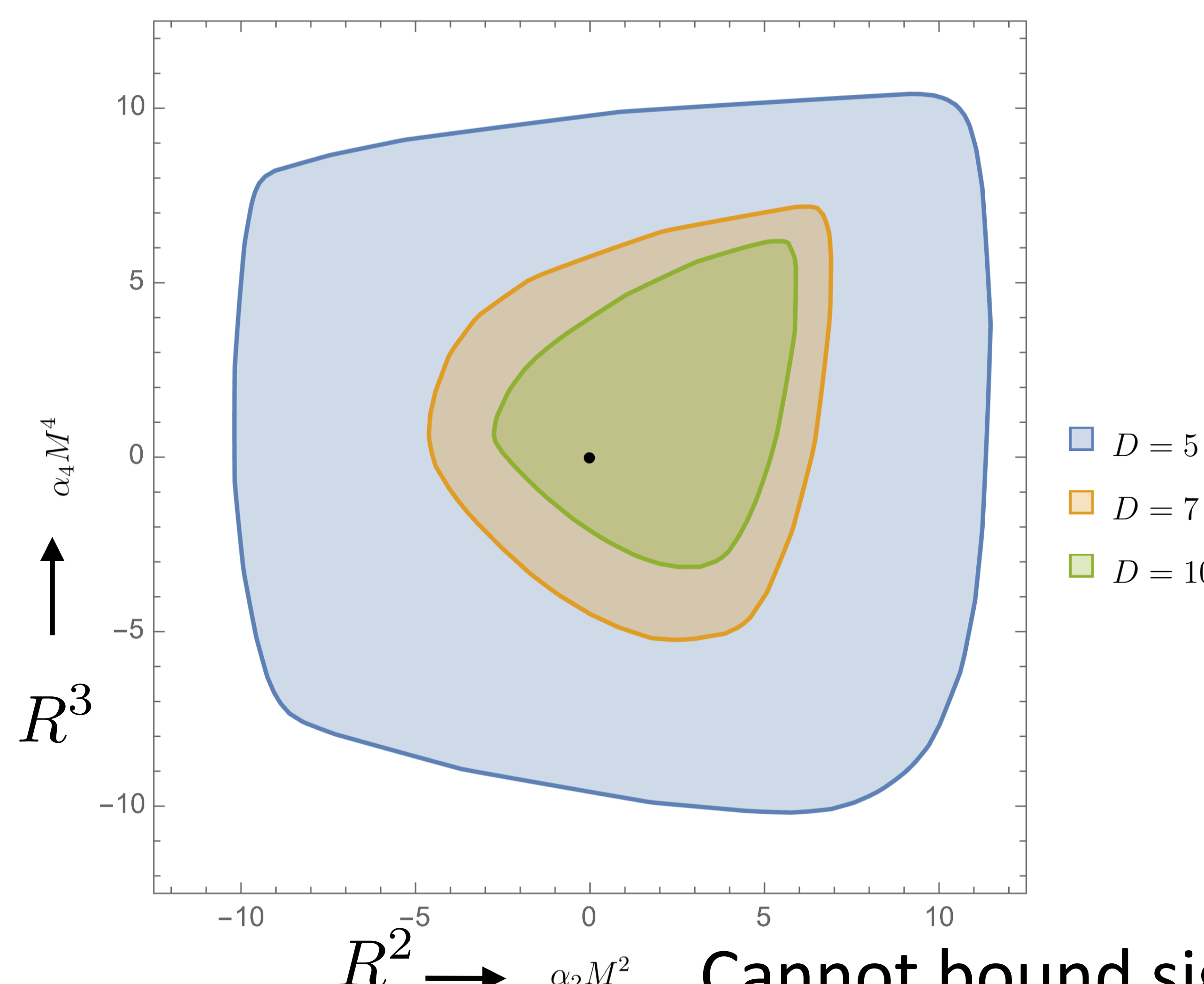


Two-sided bounds, not only positivity!

$$\left|\frac{a-c}{c}\right| \leq \frac{23}{\Delta_{\mathrm{gap}}^2} + \mathcal{O}(1/\Delta_{\mathrm{gap}}^4)$$

Provides an on-shell version of the usual species argument

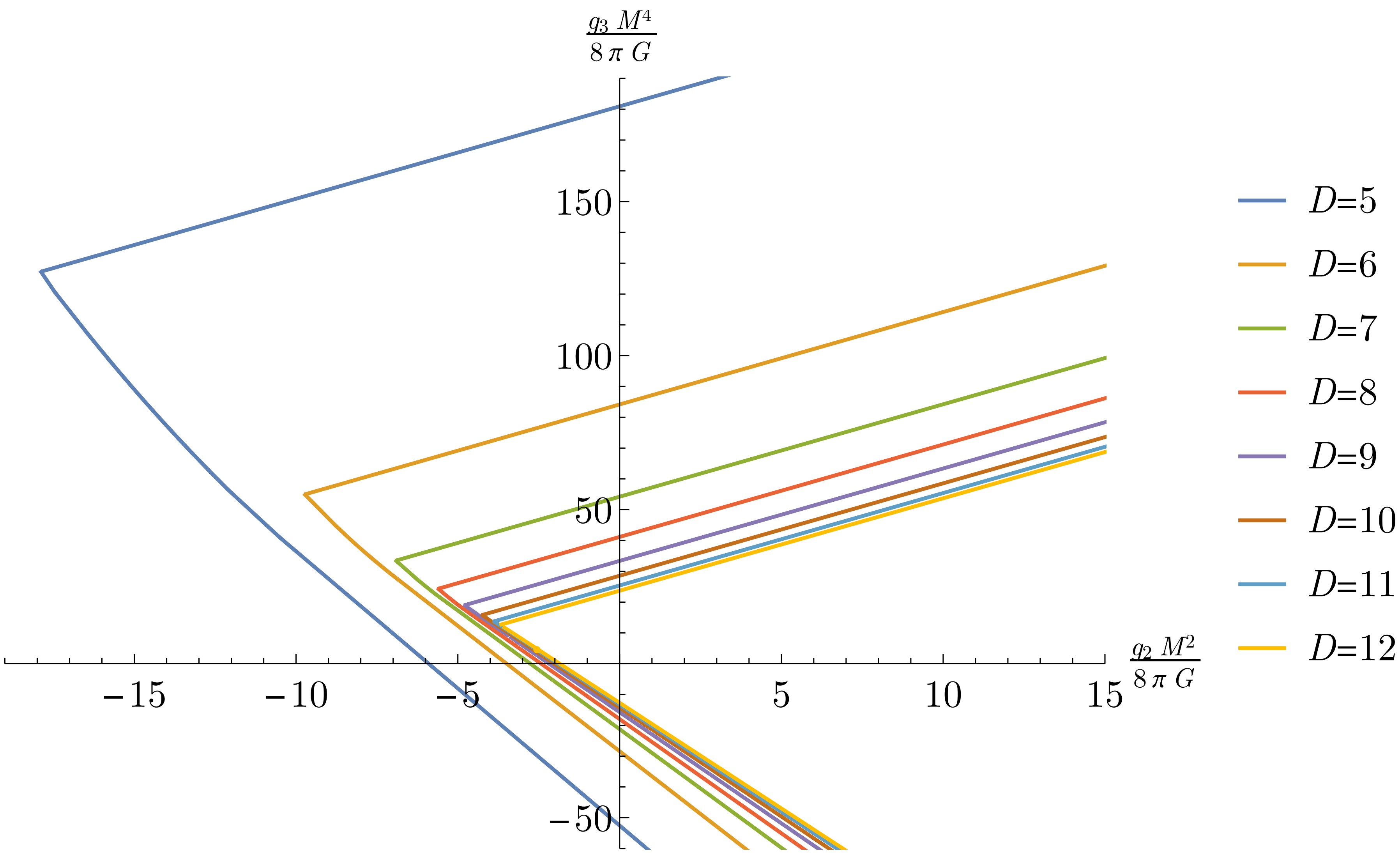
$$N < \mathcal{O}(1)\frac{M_{pl}^2}{M^2}\log\left(\mathrm{IR}\right)$$



Cannot bound sign...

Positivity is spoiled by gravity

[Caron-Huot, Mazac, Rastelli, Simmons-Duffin]



The reason is that graviton pole is attractive and dominates.
We cannot probe it beyond the EFT cutoff $t \sim M^2$

Large black holes and WGC

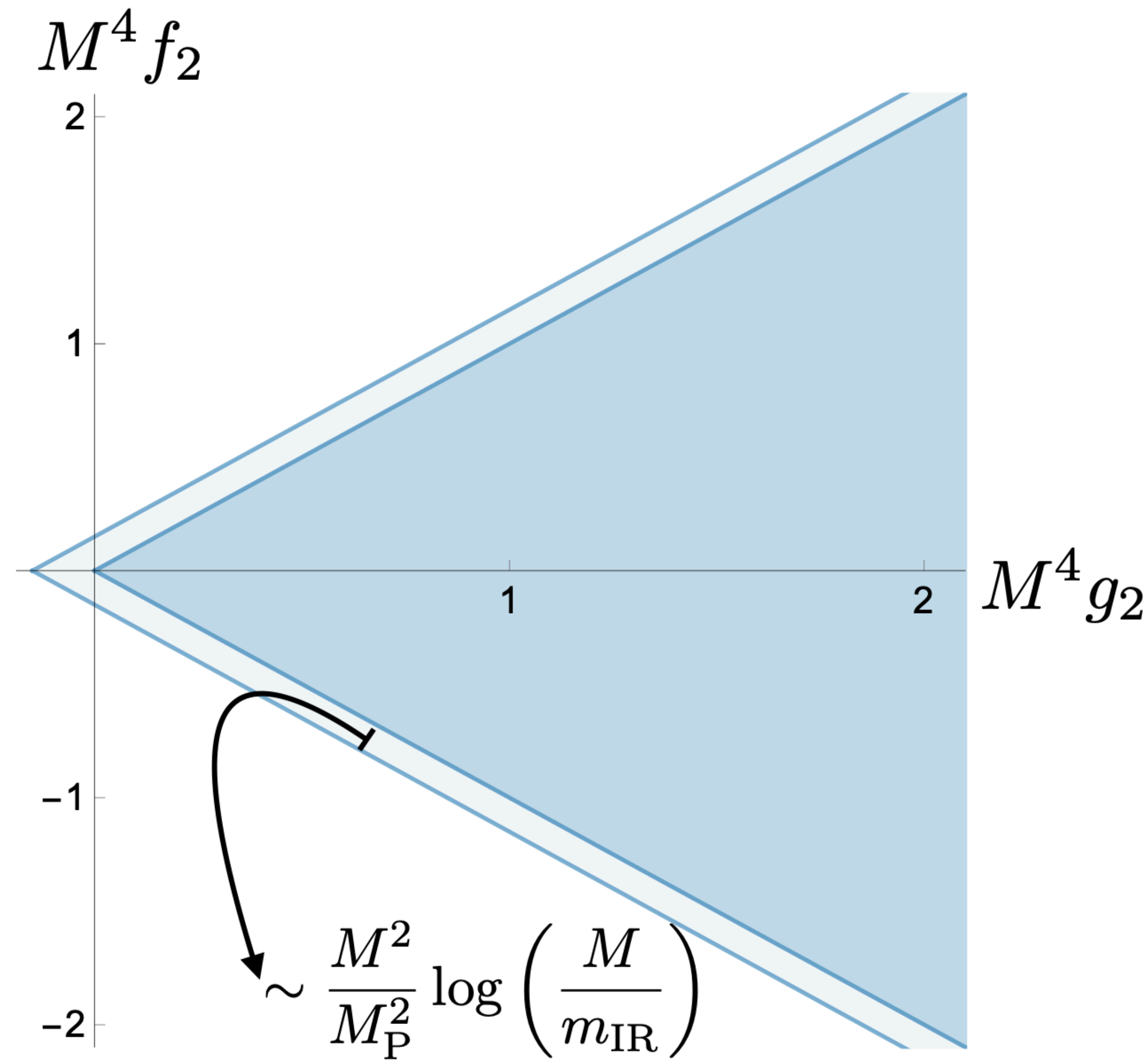
Same goes for coefficient of F^4 whose sign is correlated with shift in extremality bound

$$\text{“}\quad \delta\frac{Q}{M} \sim g_2 \quad\text{”}$$

[Kats, Motl, Padi]

In general, we cannot use dispersive bounds to conclude that large BH satisfy WGC

(See [Cheung, Liu, Remmen] and [Hamada, Noumi Shiu] for other discussions)



[Henriksson, McPeak, Russo, Vichi]

Can we quantify no global symmetries?

- We can write sum rules for coefficients that break symmetry

e.g. Leading gravitational Z2 symmetry breaking $g\phi R^2$

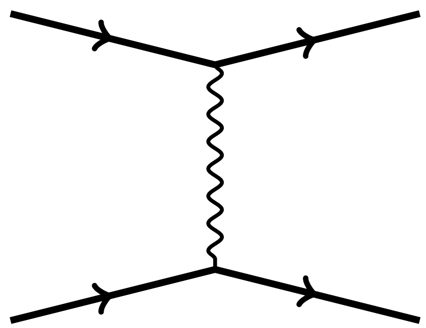
$$g^2 = \left\langle \frac{|c^{++}_{J,m^2}|^2 - |c^{+-}_{J,m^2}|^2}{m^6} \right\rangle$$

Can we prove that RHS is necessarily non-zero? BH contribution?

- Unlikely, since we cannot exclude the possibility that the symmetry is gauged by looking at EFT amplitudes

Can we quantify no global symmetries?

- What about continues symmetries, for which we have a gauge field



$$\mathcal{M} \sim \frac{s}{t}$$

- Invisible, since dispersion relations require at least two subtractions

$$\lim_{|s|\to\infty} \mathcal{M}/|s|^2 = 0 \qquad B_k = \oint \frac{ds}{s^{1+k}}\mathcal{M} = 0 \quad \forall k \geq 2$$

- We would need singly-subtracted dispersion relations

$$B_1 = \oint \frac{ds}{s^2}\mathcal{M} = c_\infty \quad ?$$

Strong form of WGC/RFC?

- There is a very special singly subtracted dispersion relation, which extracts amplitude at threshold

$$B_1' = \oint \frac{ds\mathcal{M}(s,t)}{s(s-(m_1+m_2)^2)}$$

- At low energies this is precisely the RFC combination

$$B_1'|_{\text{low}} = \frac{4m_1m_2}{M_{pl}^{D-2}}\frac{1}{t}\left(e^2q_1q_2 - \frac{D-3}{D-2}\frac{m_1m_2}{M_{pl}^{D-2}}\right) + \mathcal{O}(t^0)$$

(also works with scalars)

- However, we are free to choose any charges, which naively don't affect high-energy part of the sum rule.
- Also we can always turn of the electromagnetic coupling, so first need to attack no global symmetries

Strong form of WGC/RFC?

- Still, if we assume RFC we have an interesting sum rule

$$\frac{1}{t}\left(e^2 q_1 q_2 - \frac{D-3}{D-2}\frac{m_1 m_2}{M_{pl}^{D-2}}\right) + \cdots = \int_{M^2}^{\infty} \frac{ds\,\mathrm{Im}\mathcal{M}(s,t)}{s(s-(m_1+m_2)^2)} + c_\infty$$

with positive dominant terms for small t.

- Q for you: Is there anything universal about the boundary term in string theory?

$$c_\infty = \oint_{|s|=\infty} \frac{ds\,\mathcal{M}(s,t)}{s(s-(m_1+m_2)^2)}$$

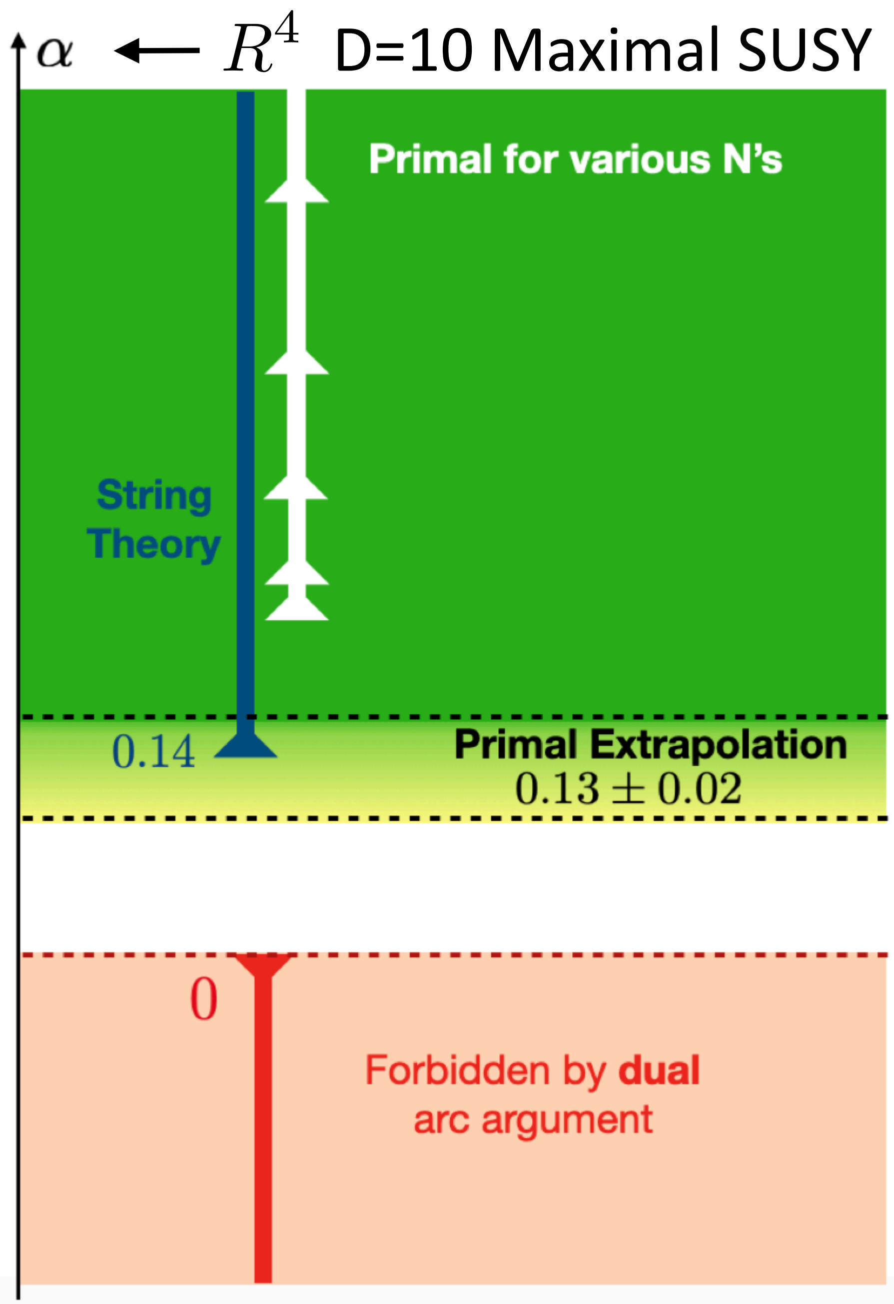
Rank conjectures? Bounds on CY data?

- Absolute upper bounds could connect to rank conjectures, bound number of CYs? e.g. $r_G \leq 26 - D$

- We can only bound ratios of Wilson coefficients

- Q for you: Can one phrase rank conjecture as upper bound on ratio of Wilson coefficients in SUGRA EFT?

- Non-rigorous absolute upper bounds (primal bootstrap) [Guerrieri, Penedones, Vieira]



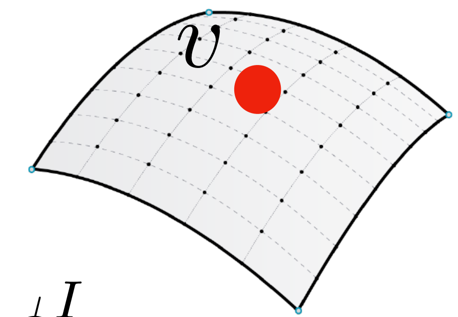
Bonus: Amplitudes and moduli spaces of vacua

Summary

- We can prove that Einstein gravity is the only option at low energies with a gap. Expectation from EFT, but with O(1) coefficients fixed.
- Some of the traditional suggestions for connecting dispersion relations and the swampland are now understood to not work in general (e.g. extremality shift)
- Understanding the gap between the “bottom up” swampland and the “top down” swampland seems a worthwhile exercise
- There still seems to be a gap between what we can do today and the goals of the swampland community. Perfect reason to continue our conversations :)

Thank you!

Geometry of amplitudes



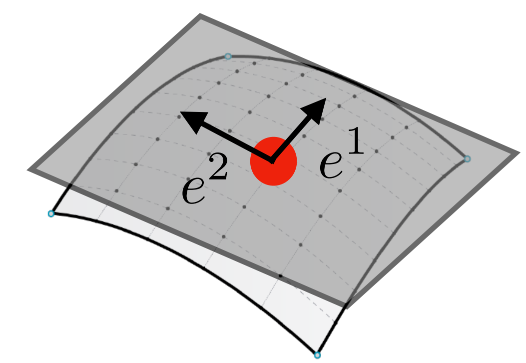
- Amplitudes defined by expanding around VEV $\Phi^I = v^I + \phi^I$

- Do not depend on field basis $\phi \to \phi + \epsilon f(\phi)$

$$S(\phi) \to S(\phi) + \frac{\delta S}{\delta \phi} \epsilon f(\phi) + \cdots$$

equations of motion

$$\propto p^2 - m^2 + \cdots$$



but on a choice of frame $\langle p^i | \phi^J(x) | 0 \rangle = e^{iJ}(v) e^{ip\cdot x}$

- States are sections of a the tangent bundle over the space of vacua. Amplitudes also live on the tangent space. [Cheung, Helset, JPM]

- Q for you: Is there any interesting behavior at infinite distance for quantities that live on the tangent space?

Some amplitudes

- Must be a function of geometric invariants! e.g. curvature of moduli space [Volkov; Dixon, Kaplunovsky, Louis]

$$R^{ijkl}(v) \qquad \nabla^m R^{ijkl}(v)$$

- Examples: [Cheung, Helset, JPM] $\frac{1}{2}g_{IJ}(\Phi)\partial_\mu\Phi^I\partial^\mu\Phi^J$

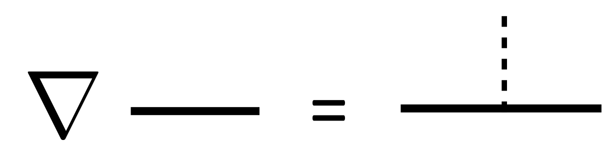
$$\begin{aligned}
A_4^{i_1i_2i_3i_4} &= R^{i_1i_3i_2i_4}s_{34} + R^{i_1i_2i_3i_4}s_{24}, \\
A_5^{i_1i_2i_3i_4i_5} &= \nabla^{i_3}R^{i_1i_4i_2i_5}s_{45} + \nabla^{i_4}R^{i_1i_3i_2i_5}s_{35} + \nabla^{i_4}R^{i_1i_2i_3i_5}s_{25} \\
&\quad + \nabla^{i_5}R^{i_1i_3i_2i_4}s_{34} + \nabla^{i_5}R^{i_1i_2i_3i_4}(s_{24}+s_{45}), \\
A_6^{i_1i_2i_3i_4i_5i_6} &= -\tfrac{1}{72}\left(R^{i_1i_3i_2j}s_{12} + R^{i_1i_2i_3j}s_{13}\right)\frac{1}{s_{123}}\left(R_j{}^{i_6i_5i_4}s_{46} + R_j{}^{i_5i_6i_4}s_{45}\right) \\
&\quad + \tfrac{1}{108}\left(R^{i_1i_3i_2j}(s_{12}-\tfrac{1}{6}s_{123}) + R^{i_1i_2i_3j}(s_{13}-\tfrac{1}{6}s_{123})\right)\left(R_j{}^{i_6i_5i_4} + R_j{}^{i_5i_6i_4}\right) \\
&\quad + \tfrac{1}{90}R^{i_1i_6i_5j}R_j{}^{i_2i_3i_4}s_{13} + \tfrac{1}{80}\nabla^{i_6}\nabla^{i_5}R^{i_1i_2i_3i_4}s_{13} + \text{perm}\,.
\end{aligned}$$

Soft scalar theorem

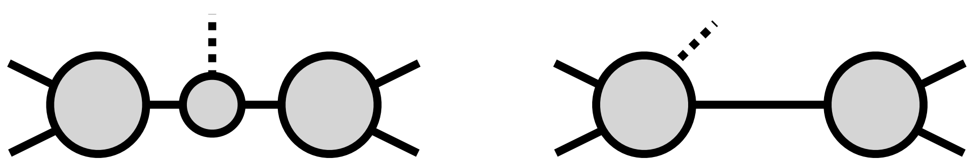
[Cheung, Helset, JPM]

- New soft theorem, intuition:

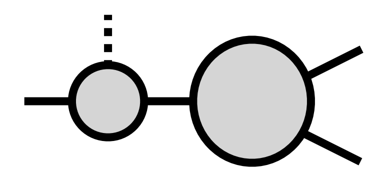
$$\lim_{q\to 0} A_{n+1} \sim \left(\nabla + \frac{\nabla m^2}{p^2 - m^2} \right) A_n$$



"Derivative w.r.t VEV"



"on-shell connection"



- Dotting the i's etc

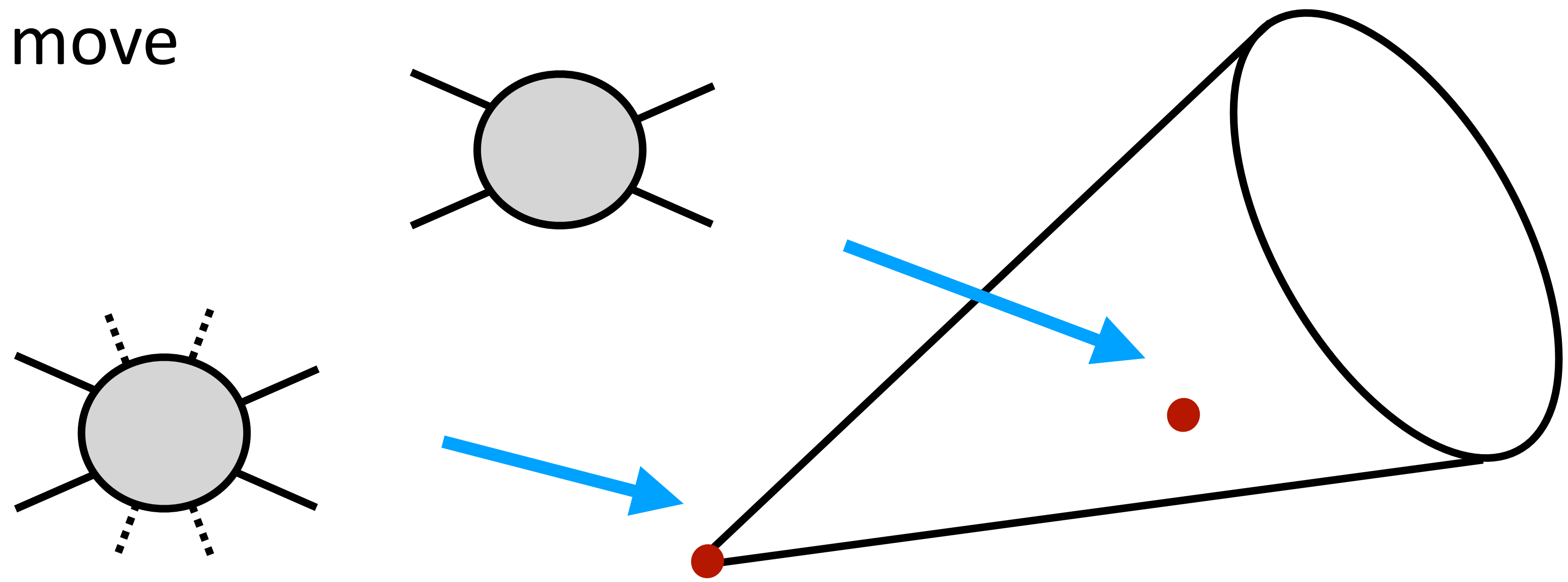
$$\lim_{q\to 0} A_{n+1}^{i_1 \cdots i_n i} = \quad \nabla^i A_n^{i_1 \cdots i_n} + \sum_{a=1}^{n} \frac{\nabla^i V^{i_a}{}_{j_a}}{(p_a + q)^2 - m_{j_a}^2} \left(1 + q^\mu \frac{\partial}{\partial p_a^\mu} \right) A_n^{i_1 \cdots j_a \cdots i_n}$$

- Works with coupling to other matter (fermions, gauge fields, …)

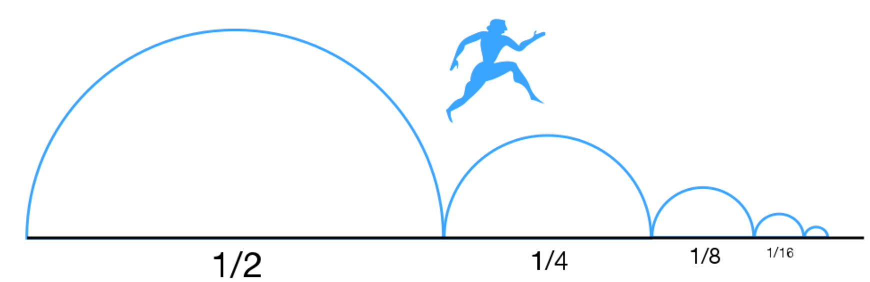
[Cheung, Derda, Helset, JPM, to appear]

Infinitesimal vs. finite vs. infinite distance

- (multi)Soft limits lets us move around moduli space



- However, even to move a finite distance we need amplitudes with infinitely many



Zeno's paradox

- Distinction between finite and infinite distance non-obvious

- Perhaps species scales can be tied to scale of breaking of unitarity

$$\mathrm{Im}\mathcal{M}_{2\to 2} \sim \sum_n |\mathcal{M}_{2\to n}|^2 < 1$$

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