The Tameness of QFTs and CFTs

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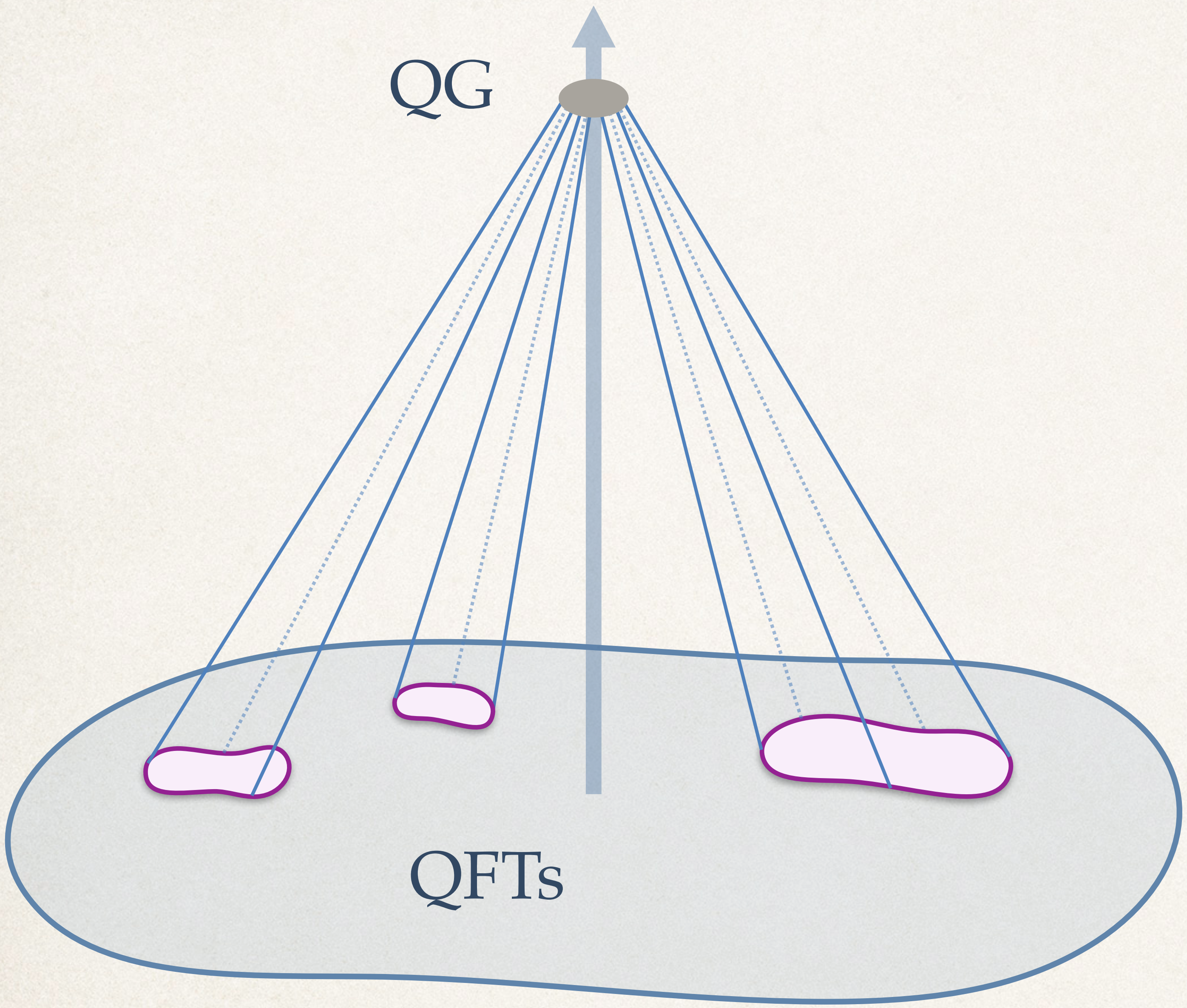
Based on:

2209.nnnn with Mike Douglas, Lorenz Schlechter

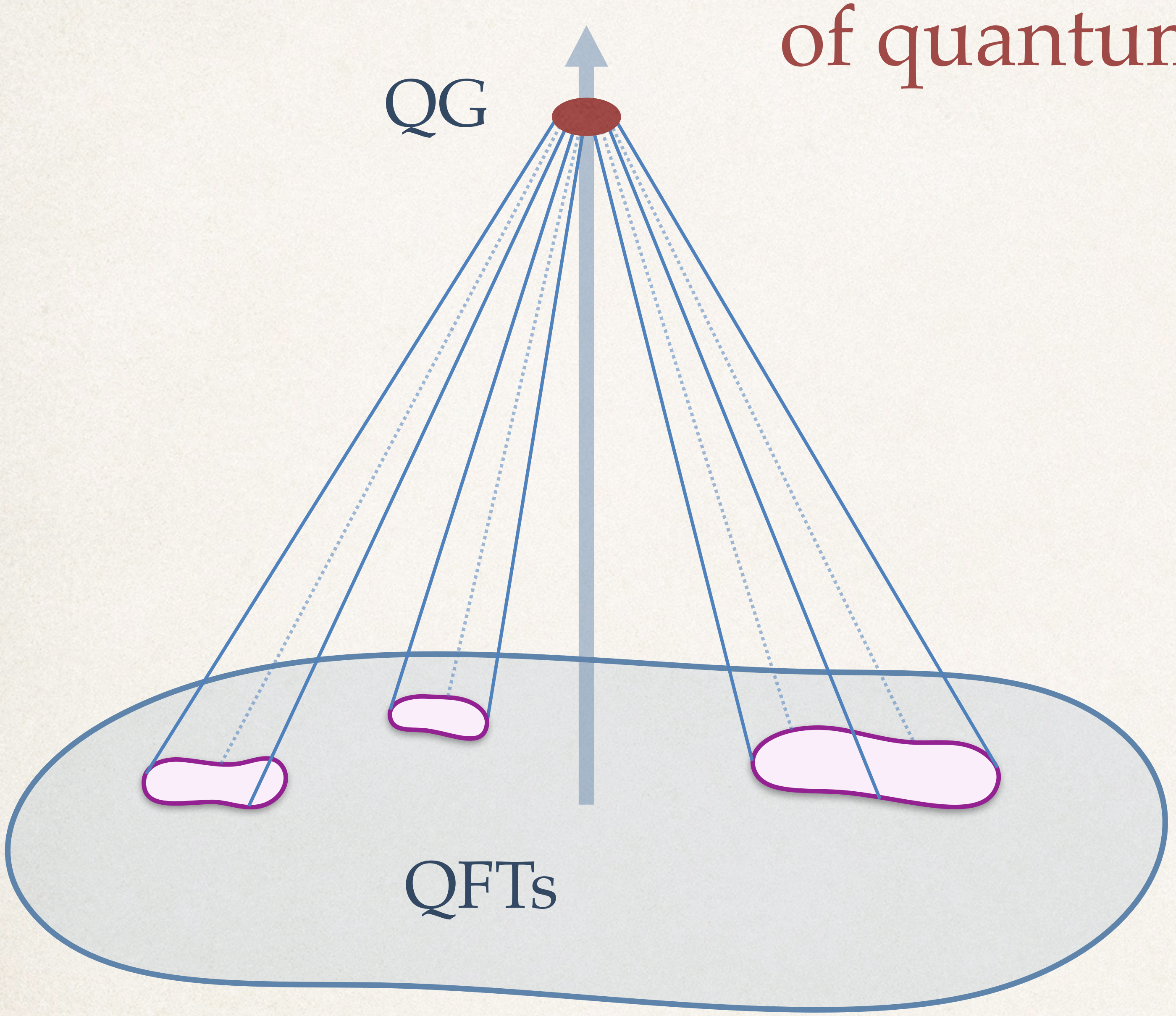
2112.08383 Tameness conjecture



Is tameness general property of landscape of effective theories compatible with QG?

Tameness conjecture [TG '21]

Is tameness a general property of quantum gravity?



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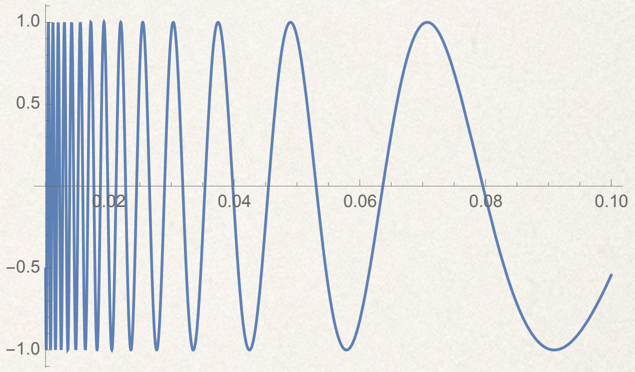

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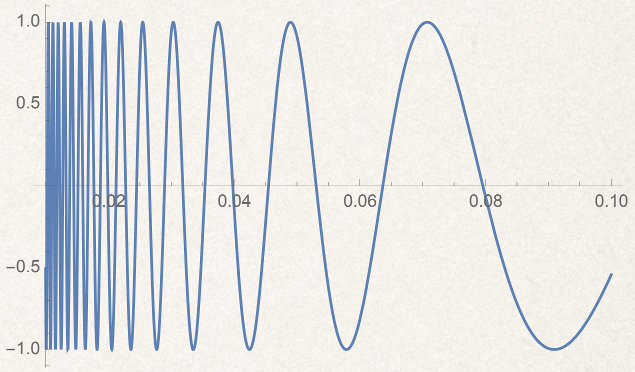

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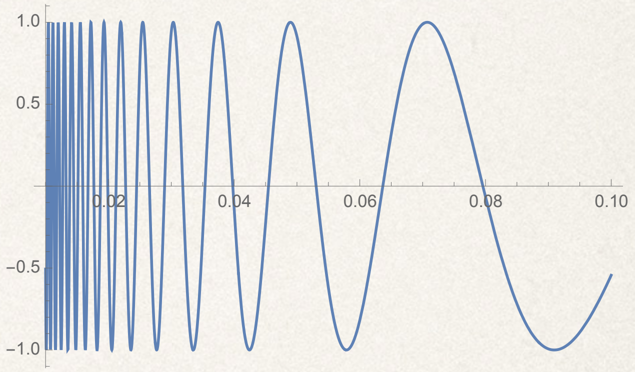

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- Grothendieck's dream to develop math. framework for geometry:
  → tame topology [Esquisse d'un programme]

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- tame/o-minimal structure $\mathcal{S}$: only subsets of $\mathbb{R}$ that are in $\mathcal{S}$ are finite unions of points and intervals

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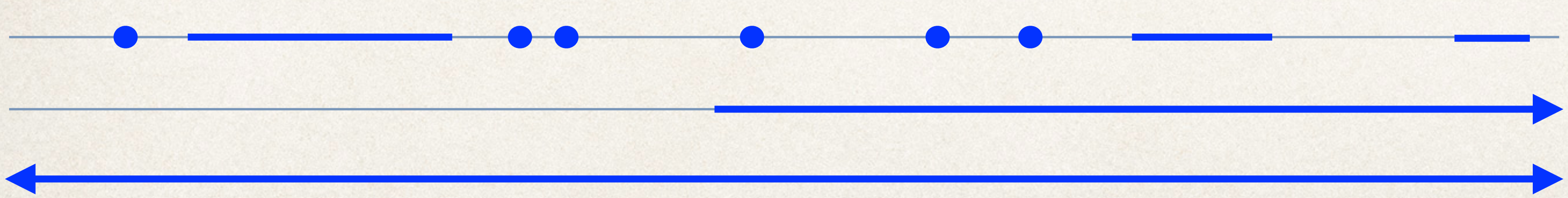

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infinitely long intervals

whole real line

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sets in o-minimal structure $\mathcal{S}$: tame sets

functions with graph being a tame set: tame functions

$\rightarrow$ tame manifold, tame bundles… a tame geometry

- ta

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Some evidence from string theory

- Type IIB/F-theory pure flux vacuum landscape is tame

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- Coupling functions in effective actions are tame functions of moduli

[TG][TG,van Vliet] in progress

- Relation between tameness, distance, and axionic string conjecture

[TG,Lanza,Li]

[Lanza,Marchesano,Martucci,Valenzuela]

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Conjecture: All correlation functions are tame functions.
Set of all CFTs is tame if d.o.f. are bounded (and gap in 2d).

Tameness in perturbative QFTs

Perturbative QFTs

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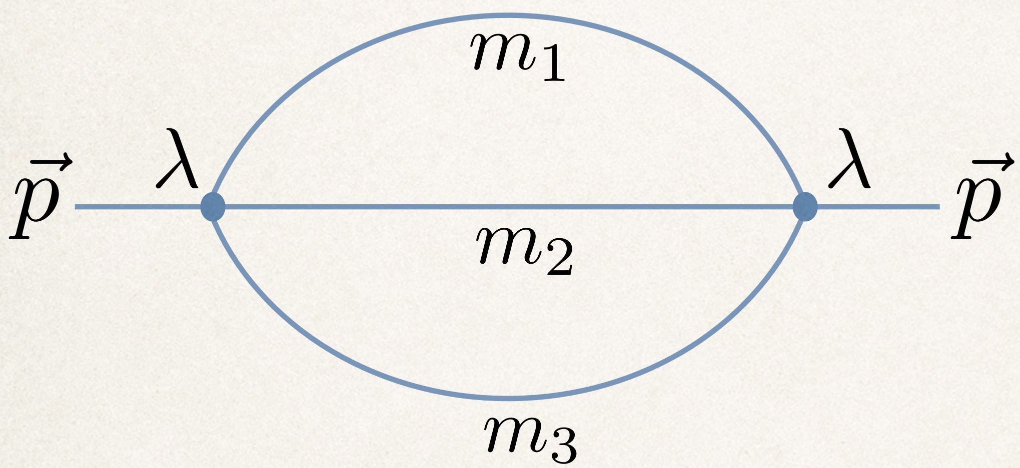

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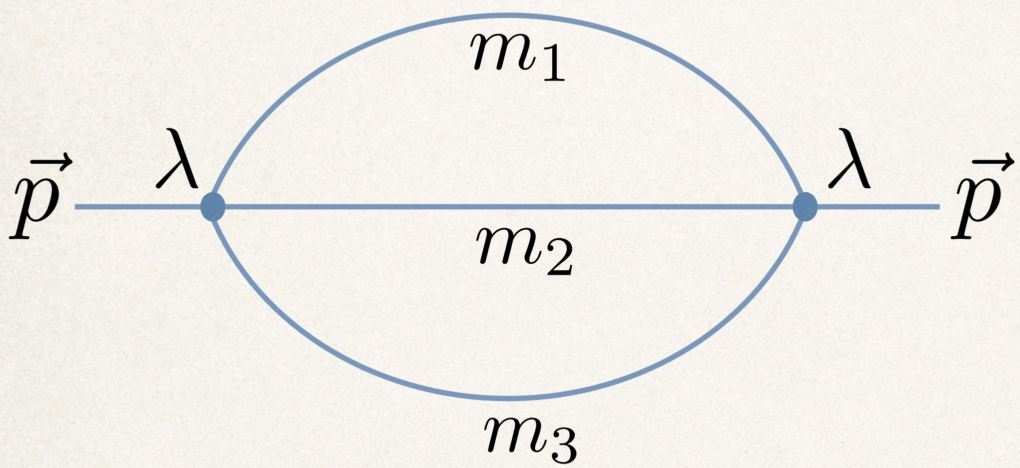

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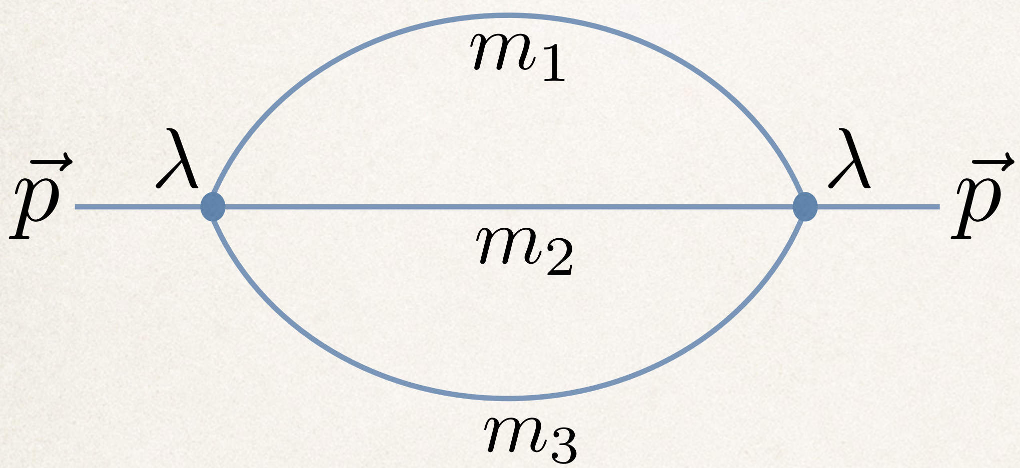

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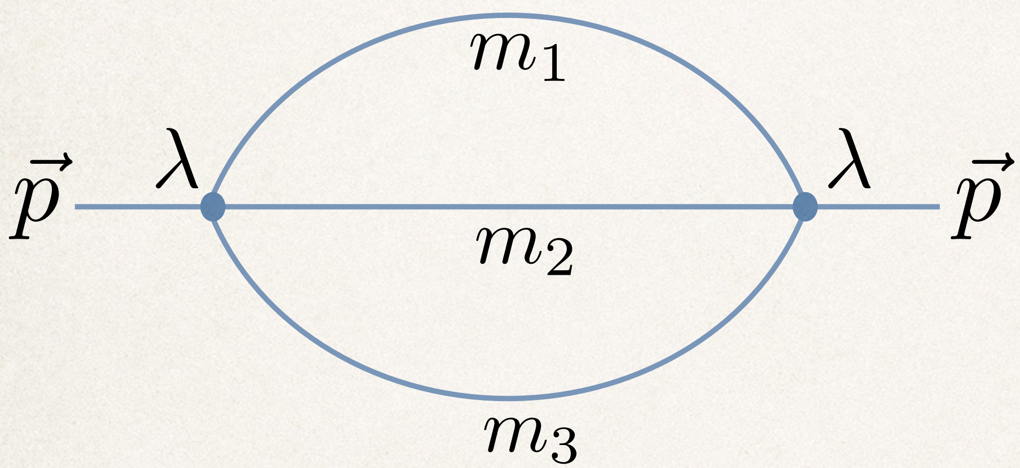

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- tameness is useful already without gravity

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$$I(p,m) = \int \left(\prod_{r=1}^{L} \frac{\mathrm{d}^d k}{i\pi^{d/2}}\right)\left(\prod_{j=1}^{n} \frac{1}{D_j^{v_j}}\right) \longrightarrow I(z) = \int_\gamma \Omega$$

review book by [Weinzierl]

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- Key point: renormalizable theories have only finitely many counterterms $\rightarrow$ tameness preserved by finite composition

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  - 2d theories: (2,2) GLSMs appearing in Type II CY compactifications

    $Z_{S^2} = e^{-K} = \bar{\Pi}\eta\Pi$ tame due to relation to periods

Challenges for tameness

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- General question: What are the precise conditions on Lagrangians in d=0,1 such that Z is tame?

  proposal in [Douglas,TG,Schlechter] - should be answerable!

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Fancy: $W_\xi = Y P_\xi(X_1, \ldots, X_k)^2 + \sum_a Z_a (\sin 2\pi i X_a)^2$ [Tachikawa]

(susy or not susy is undecidable)

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  Integrate out heavy ϕ_1: $m_1 > \Lambda$

  $$\mathcal{M}_{\text{vac}} = \left\{ \frac{\partial V}{\partial \phi_1} = 0 \right\} \cap \mathcal{M}$$ $\rightarrow$ intersection of tame spaces

  $$V(\phi_1, \phi_2) \rightarrow V(\langle \phi_1 \rangle, \phi_2)$$ $\rightarrow$ projection of tame function

Tameness of CFTs

Conformal field theories

- CFTs are axiomatically well-defined: theory set $\mathcal{T}$
  assume: CFT is unitary and local

Conformal field theories

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  $\rightarrow$ finiteness of CFTs,
  non-zero radius of convergence in CFT perturbation theory, …

CFT Tameness Conjecture

[Douglas,TG,Schlechter]

(1) Parameter space $\mathcal{P}$ labelling all CFTs in *d>1* is tame set if
- appropriate central charge (measure of d.o.f) is bounded by $\hat{c}$
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Call the o-minimal structure: $\mathbb{R}_{\mathrm{CFT}d}$

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- Construct a structure:
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start with $P(x_1, ..., x_n) = 0$

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- Note: - well-known o-minimal structures are $\mathbb{R}_{\mathrm{exp}}, \mathbb{R}_{\mathrm{an}}, \mathbb{R}_{\mathrm{an,exp}}$
  - recently: $\mathbb{R}_{\mathcal{G}^*,\mathrm{exp}}$ defines $\Gamma(x)|_{(0,\infty)}$ and $\zeta(x)|_{(0,1)}$ [Rolin etal '22]

Tameness in Quantum Gravity

Tameness for Quantum gravity

[Douglas,TG,Schlechter]

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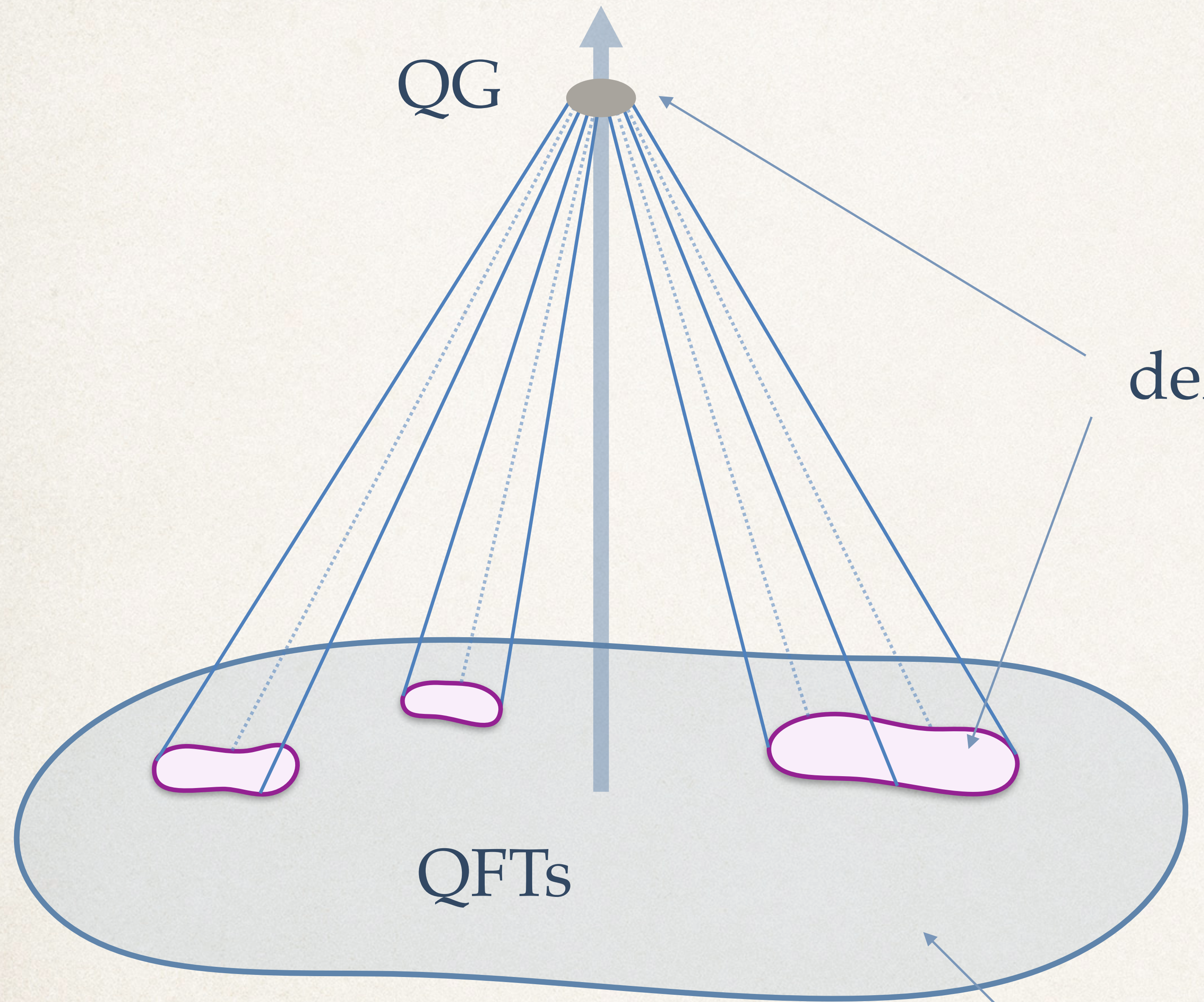
- AdS/CFT correspondence: quantum gravity in AdS_{d+1} space is defined in the o-minimal structure $\mathbb{R}_{\mathrm{CFT}d}$



define an o-minimal / tame structures

define a structures

Thanks!