Cobordism, K-theory and tadpoles

Part 2

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Introduction and motivation

- Mathematical formulation of quantum gravity?

- Signatures of quantum gravity in low energy EFTs?

These two questions can be addressed together!

Symmetries and currents

Consider a d-dimensional theory on X.
To a continuous $(d-n-1)$-form symmetry is a associated a (magnetic) current J_n.

If the symmetry is gauged, the current is exact, $J_n = dF_{n-1}$.
If the symmetry is broken, the current is not closed

$$0 \neq dJ_n = \delta^{(n+1)}(\Sigma)$$

Typically one uses (co)homology, for example $J_n \in H^n(X;\mathbb{Z})$.
(Co)homology behaves naturally under dimensional reduction.

Are there more general languages? Are they relevant for physics?

Cobordism as generalized homology

- (Co)Homology groups of point carry no information

$$H_n(\mathrm{pt}) = 0 \qquad (\text{if } n > 0)$$

since every cycle on pt of positive dimension is a boundary.

- Cobordism groups of point do carry information

$$\Omega_n(\mathrm{pt}) \neq 0$$

since **not every compact manifold is a boundary**.

- This information is topological and physical.

A (co)homology theory whose groups of pt are generically non-vanishing is called generalized (co)homology.
Cobordism and K-theory are examples.

To recap

- Cobordism symmetries are global and must be either broken or gauged in quantum gravity [McNamara, Vafa '19]

$$\Omega_n^{QG}(\mathrm{pt}) = 0$$

- QG-structure is not known a priori. Whitehead tower can be used as organizing principle [Andriot, Carqueville, NC '22].

- When gauging, one can combine cobordism with K-theory. Most natural for Spin/Spin^c cobordism and KO/K-theory. No coincidence but deep mathematical structure behind.

- One can recover certain string theory tadpoles (Bianchi identities) without using the effective action.

Gauging cobordism

Tadpole: integrated Bianchi identity

$$0 = \int_M dF_{n-1} = \int_M J_n$$

Goal: To reconstruct J_n without knowing string theory.

1. Add all bordism invariants (ABS orientation is just one)

$$0 = \int_M dF_{n-1} = \sum_{i \in \text{inv}} a_i \, \mu_{n,i}$$

2. Include defects classified by $K^{-n}(\mathrm{pt})$.

Thus we get a combination of bordism and K-theory

$$0 = \int_{[M]} dF_{n-1} = \sum_{i \in \text{inv}} a_i \, \mu_{n,i} + \sum_{j \in \text{def}} \int_{[M]} Q_j \, \delta^n(\Delta_{10-n,j})$$

From groups of pt to groups of X

[Blumenhagen, NC, Kneißl, Makridou '22]

- The above discussion can be generalised for pt $\to X$.

- The groups are enlarged

$$\Omega(X) = \Omega(\mathrm{pt}) \oplus \tilde{\Omega}(X), \qquad K(X) = K(\mathrm{pt}) \oplus \tilde{K}(X),$$

  so potentially more global symmetries.

- What is their interpretation?

- Notice: $X = BG$ used for anomalies of G. Instead, we take X to be a manifold, such as spheres, tori, CY.

Computing groups of X

The groups $\Omega(X)$, $K(X)$ can be computed using a spectral sequence.
It is a tool to calculate generalised (co)homology theories.

- Start from ordinary (co)homology
- Refine the approximation by means of **differentials**
- Eventually, solve an **extension problem**
  (extra information needed)

Certain differentials are physically associated to Freed-Witten anomalies. [Diaconescu, Moore, Witten '00; Bergman, Gimon, Sugimoto '01; Maldacena, Moore, Seiberg '01]

Spectral sequence

A spectral sequence consists of pages and differentials $(E^r_{p,q}, d^r)$, such that $E^{r+1} = H(E^r, d^r)$. Turning the page, $E^r \to E^{r+1}$, means getting closer to the result G_n.

However:

- Generically the explicit form of the differentials is **not** known. Only their existence is, $d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$.
- Without torsion the result is $G_n = \oplus_{p=0}^n E^\infty_{p,n-p}$. With torsion this is true up to an extension problem.

Atiyah-Hirzebruch spectral sequence:
Given $F \to E \to B$ and knowing the generalised (co)homology $G_n(F)$, one can use the spectral sequence to compute $G_n(E)$

$$E^2_{p,q} \cong H_p(B; G_q(F)) \Rightarrow G_{p+q}(E)$$

Differentials and extension problem

In this example, the second differential acts as

$$
\begin{array}{c|ccc}
3 & 0 & 0 & 0 \\
2 & E^2_{0,2} & 0 & 0 \\
1 & E^2_{0,1} & 0 & 0 \\
0 & E^2_{0,0} & 0 & E^2_{2,0} \\
\hline
 & 0 & 1 & 2
\end{array}
\quad \Rightarrow \quad
\begin{array}{c|ccc}
3 & 0 & 0 & 0 \\
2 & E^3_{0,2} \equiv B & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & E^3_{0,0} & 0 & E^3_{2,0} \equiv A \\
\hline
 & 0 & 1 & 2
\end{array}
$$

No other differential can act, thus $E^3 \cong E^\infty$.
Then we can write $G_0 = E^3_{0,0}$, while G_2 is such that

$$0 \to B \to G_2 \to A \to 0, \qquad G_2 = e(A,B) = ?$$

Without torsion $e(A,B) = A \oplus B$ but otherwise not true in general.

Example: $A = B = \mathbb{Z}_2$, then $e(A,B)$ is either $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ or $\mathbb{Z}_4$.
The AHSS cannot tell which one to choose.

(The number of extensions is given by $Ext^1(A,B)$)

Freed-Witten anomalies and the AHSS

Type I/II D-branes must wrap Spin/Spin^c manifold Y otherwise they are anomalous (assuming $B = 0$) [Freed, Witten '99].

This is encoded automatically in the AHSS. For example, for K-theory one has

$$d_3 = Sq^3(Y) \sim W_3(Y)$$

implying $d_3 = 0$ if $W_3(Y) = 0$ (so Y is Spin^c).

Non-anomalous branes are in $\ker d_3$, but those which are d_3-exact are removed from the AHSS when taking d_3 cohomology.
Branes in $\mathrm{Im}\, d_3$ have trivial K-theory charge and are unstable.
[Diaconescu, Moore, Witten '00; Bergman, Gimon, Sugimoto '01; Maldacena, Moore, Seiberg '01]

Some results

For $X = \{S^k, T^k, K3, CY_3\}$, we find ($k = dim(X)$)

$$K^{-n}(X) = \bigoplus_{m=0}^{k} b_{k-m}(X) K^{-n-m}(\mathrm{pt})$$

$$\Omega_{n+k}^{\mathrm{Spin^c}}(X) = \bigoplus_{m=0}^{k} b_m(X) \Omega_{n+k-m}^{\mathrm{Spin^c}}(\mathrm{pt})$$

- We show that they reproduce pattern of global symmetries stemming from **dimensional reduction** on X.
- They classify $(d-1-k-n)$-form charges in $D = d-k$ dimensions, arising from dimensional reduction of $d-1-n$, $d-2-n$, ..., $d-1-k-n$ form charges along the k, $k-1$, ..., 0 cycles X.

Interpretation: K-theory

$$K^{-n}(X) = \bigoplus_{m=0}^{k} b_{k-m}(X) K^{-n-m}(\mathrm{pt})$$

- They classify codimension $(n+m)$-branes wrapping $(k-m)$-cycles of X. Consistent with expectation from dimensional reduction.
- By construction, these branes do not suffer from FW anomalies, otherwise they would not survive the spectral sequence.
- All sites populated. Completeness hypothesis.
- Simlar result for KO-theory, for $X = \{S^k, T^k, K3\}$

Interpretation: Cobordism

$$\Omega^{\mathrm{Spin^c}}_{n+k}(X) = \bigoplus_{m=0}^{k} b_m(X)\Omega^{\mathrm{Spin^c}}_{n+k-m}(\mathrm{pt})$$

Each non-vanishing term in the RHS means that the $(n+k)$-manifold M is wrapped around non-trivial m-cycle of X.

Two qualitatively different cases:

- $n \geq 0$: There is associated K-theory group $K_{n+k}(X) = K^{-n}(X)$ with string interpretation.

  It reproduces expectation from dimensional reduction.

- $-k \leq n < 0$: No K-theory analogue in physics.

  Physical interpretation more speculative

Example: $X = CY_3$

$$K^0(CY_3) = K_6(CY_3) = b_6 \underbrace{K^0(\mathrm{pt})}_{\mathbb{Z}} \oplus b_4 \underbrace{K^{-2}(\mathrm{pt})}_{\mathbb{Z}} \oplus b_2 \underbrace{K^{-4}(\mathrm{pt})}_{\mathbb{Z}} \oplus b_0 \underbrace{K^{-6}(\mathrm{pt})}_{\mathbb{Z}}$$

$$\Omega_6^{\mathrm{Spin}^c}(CY_3) = b_6 \underbrace{\Omega_0^{\mathrm{Spin}^c}(\mathrm{pt})}_{\mathbb{Z}} \oplus b_4 \underbrace{\Omega_2^{\mathrm{Spin}^c}(\mathrm{pt})}_{\mathbb{Z}} \oplus b_2 \underbrace{\Omega_4^{\mathrm{Spin}^c}(\mathrm{pt})}_{\mathbb{Z}\oplus\mathbb{Z}} \oplus b_0 \underbrace{\Omega_6^{\mathrm{Spin}^c}(\mathrm{pt})}_{\mathbb{Z}\oplus\mathbb{Z}}$$

- Combining groups of pt with same $(0, 2, 4, 6)$ index, we can construct tadpoles in 4D.
- They give $b_{6,4,2,0}$ tadpoles of 4D 3-form symmetries.
- In fact, they are the dimensional reduction of tadpoles for the 10D (9,7,5,3)-form symmetries.

What about $-k \leq n < 0$?

Interpretation for $-k \leq n < 0$ less transparent.
There is no analogous K-theory group and thus not clear if we should gauge or break.

In [Blumenhagen, NC, Kneißl, Makridou '22] we propose that, for $X = CY_3$

- $\Omega_{\mathrm{ODD}}(X)$, **broken**: there is no appropriate gauge field in the theory to gauge it.
- $\Omega_{\mathrm{EVEN}}(X)$, **gauged**: contributing to tadpoles of $n \geq 0$ groups.

$\Omega_6(X)$	$b_6\Omega_0(\mathrm{pt})$	$b_4\Omega_2(\mathrm{pt})$	$b_2\Omega_4(\mathrm{pt})$	$b_0\Omega_6(\mathrm{pt})$
	C_{10}	C_8	C_6	C_4
	$O9$	$\mathrm{F}(CY_4)_{c_1(M_6)}$	$\mathrm{tr}(R \wedge R)_{\mathrm{D9,O9}}$	$\mathrm{F}(CY_4)_{c_1c_2,c_1^3(M_6)}$
$\Omega_4(X)$	$b_4\Omega_0(\mathrm{pt})$	$b_2\Omega_2(\mathrm{pt})$	$b_0\Omega_4(\mathrm{pt})$	–
	C_8	C_6	C_4	–
	$O7$	$N7_{c_1(M_4)}$	$\mathrm{tr}(R \wedge R)_{\mathrm{D7,O7}}$	–
$\Omega_2(X)$	$b_2\Omega_0(\mathrm{pt})$	$b_0\Omega_2(\mathrm{pt})$	–	–
	C_6	C_4	–	–
	$O5$	$N5_{c_1(M_2)}$	–	–
$\Omega_0(X)$	$b_0\Omega_0(\mathrm{pt})$	–	–	–
	C_4	–	–	–
	$O3$	–	–	–

- First column: localised O-planes
- $\mathrm{F}(CY_4)_x$: contribution proportional to x arising in F-theory compactified on elliptically fibered CY_4 with base M_6.
- $\mathrm{tr}(R^2)_x$: contribution proportional to $\mathrm{tr}(R^2)$ arising from CS action of x.
- New contributions?

Conclusion

- The absence of global symmetries seems to be a fact of QG. It holds true also when enlarging notion of symmetry, such as to include cobordism [McNamara, Vafa '19]
- This statement has predictive power. [Montero, Vafa '20; Dierigl, Heckmann '20; Hamada, Vafa, '21; McNamara '21; Debray, Dierigl, Heckman, Montero '21; Blumenhagen, NC, Kneißl, Makridou '22, Velàzquez, De biasio, Lüst '22. . .]
- Cobordism and K-theory can be combined in a mathematical and physical way. Their combination must be either be broken or gauged
- The generalisation pt$\to X$ (for some X) can be interpreted in terms of dimensional reduction

Future directions

- Cobordism groups with more structure (gauge fields, compact manifolds, . . .)
- Clarify origin of tadpoles from bottom-up
- Is cobordism conjecture combined with K-theory enough to reconstruct tadpoles in string theory (String Lamppost Principle)?
- Are there new objects in string theory detected by cobordism? This can happen when breaking but also when gauging.

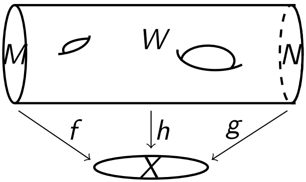
Thank you!

Extra slides

Cobordism groups of X

One can define cobordism groups for any topological space X.

$$\Omega_n^{\xi}(X) = \{\text{set of pairs } (M, f)\}/\sim$$



One generically has the splitting

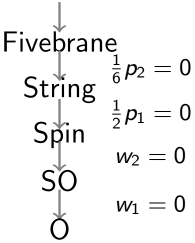
$$\Omega_n^{\xi}(X) = \Omega_n^{\xi}(\text{pt}) \oplus \tilde{\Omega}_n^{\xi}(X),$$

thus when passing from pt to X the group is enlarged.
Similarly for K-theory.

An organising principle

In general not clear how to get to QG.
Following the Whitehead tower helps [NC, Andriot, Carqueville '22].

Whitehead tower: it organises topological structures (with their obstructions) according to the degree of "connectedness".



Climbing the tower, bordism groups become smaller.

Atiyah-Bott-Shapiro orientation

Relation between cobordism and K-theory dates back to **ABS-orientation** [Atiyah, Bott, Shapiro '64]

$$\begin{aligned} \alpha_n &: \quad \Omega_n^{\mathrm{Spin}}(\mathrm{pt}) \rightarrow \mathrm{KO}^{-\mathrm{n}}(\mathrm{pt}) \\ \alpha_n^c &: \quad \Omega_n^{\mathrm{Spin}^c}(\mathrm{pt}) \rightarrow \mathrm{K}^{-\mathrm{n}}(\mathrm{pt}) \end{aligned}$$

explicitly given by the refined A-roof and Todd genus

$$\alpha_n([M]) = \begin{cases} \hat{A}(M) & n = 8k \\ \frac{1}{2}\hat{A}(M) & n = 8k+4 \\ \dim H \mod 2 & n = 8k+1 \\ \dim H^+ \mod 2 & n = 8k+2 \\ 0 & \text{otherwise} \end{cases} \qquad \alpha_n^c([M]) = \mathrm{Td}(M)$$

Starting point to prove theorem by [Hopkins, Hovey '92], see also [Conner, Floyd '66; Landweber '76; Kreck, Stolz '93].

Note: α_n, α_n^c are bordism invariants.

Example: gauging $\Omega_6^{\mathrm{Spin}^c}$

We have $\Omega_6^{\mathrm{Spin}^c} = \mathbb{Z} \oplus \mathbb{Z}$ with invariants

$$\mu_6^1 \equiv \alpha_6^c = \int td_6 = \int \frac{1}{24} c_1 c_2, \qquad \mu_6^2 = \int \frac{1}{2} c_1^3$$

- (Magnetic) 5-form global symmetry, gauged by C_4
- $K^{-6}(\mathrm{pt})$ classifies D3-branes

Combining we get

$$\int_B \sum_i Q_i\, \delta^{(6)}(\Delta_{4,i}) = \int_B \left(\tfrac{a_1}{24}\, c_2(B)\, c_1(B) + \tfrac{a_2}{2}\, c_1^3(B) \right) \equiv \frac{\chi(Y)}{24}$$

Matching with known D3-brane tadpole cancellation in F-theory for $a_1 = 12$ and $a_2 = 30$. [Sethi, Vafa, Witten '96]

Notice that c_3 cannot appear since it is **not bordism invariant**.

Example: gauging Ω_1^{Spin}

Torsion charges require care. Consider $\Omega_1^{\mathrm{Spin}} = \mathbb{Z}_2 = KO^{-1}(\mathrm{pt})$ with invariant

$$\mu_1 \equiv \alpha_1$$

and $KO^{-1}(\mathrm{pt})$ classifies $\widehat{D8}$-branes.

We get $\mathbb{Z}_2$-valued charge neutrality condition

$$\int_M \sum_i Q_i \delta^{(1)}(\Delta_{9,i}) = \mathbf{a}\,\alpha_1 \mod 2$$

- **a=even**: RHS decouples. Even number of $\widehat{D8}$-branes needed and $KO^{-1}(\mathrm{pt})$ is gauged. **New defect** needed to break Ω_1^{Spin}.
- **a=odd**: single $\widehat{D8}$-brane on S^1_p (having $\alpha_1(S^1_p) = 1$) allowed since vanishing total charge, $1 + 1 = 0$ mod 2.
  Unlikely: S^1_p valid background without $\widehat{D8}$.

Example: X = CY_3 (continued)

The cobordism invariants of each term (recall $M_6 \neq X$)

$$\mathrm{td}_0(M_6) = 1\,,$$
$$\mathrm{td}_2(M_6) = \frac{1}{2} c_1(M_6)\,,$$
$$\mathrm{td}_4(M_6) = \frac{1}{12}\left(c_2(M_6) + c_1^2(M_6)\right)\,, \qquad c_1^2(M_6)\,,$$
$$\mathrm{td}_6(M_6) = \frac{1}{24} c_2(M_6)\; c_1(M_6)\,, \qquad \frac{1}{2} c_1^3(M_6)$$

can be expanded in $H^{6-m}(X;\mathbb{Z})$ such that their Poincaré duals are in $H_m(X;\mathbb{Z})$ and counted by $b_m(X)$.

For each $m = 6, 4, 2, 0$ we combine cobordism invariants with K-theory defects and repeat the same logic as for groups of pt.

- $6 - m = 0$: 9-form symmetry in 10D, 3-form symmetry in 4D. $b_6(X) = 1$ tadpole

$$N\,\delta^{(0)}(M_6) + a^{(0)}\,\mathrm{td}_0(M_6) = 0$$

- $6 - m = 2$: 7-form symmetry in 10D, 3-form symmetry in 4D. $b^2(X) = b_4(X)$ tapdoles obtained combining $(\omega_{(2)} \in H^2(X;\mathbb{Z}))$

$$\delta^{(2)}(\mathbb{R}^{1,3}\times\Sigma_4) = \sum_{a=1}^{b_4} \delta^{(0)}(\mathbb{R}^{1,3})^{(2)a} \wedge \omega_{(2)a} \quad \text{and} \quad \mathrm{td}_2(M_6) = \sum_{a=1}^{b_4} \tilde{j}_0^{(2)a} \wedge \omega_{(2)a}$$

- $6 - m = 4$: 5-form symmetry in 10D, 3-form symmetry in 4D. $b^4(X) = b_2(X)$ tapdoles after expanding in cohomology

$$\sum_{j\in\mathrm{def}} N_j\,\delta^{(4)}(\mathbb{R}^{1,3}\times\hat{\Sigma}_{2,j}) + a_1^{(4)}\left(\frac{c_2(M_6)+c_1^2(M_6)}{12}\right) + a_2^{(4)}\,c_1^2(M_6) = 0$$

- $6 - m = 6$: 3-form symmetry in 10D, 3-form symmetry in 4D. $b^6(X) = 1$ tapdole after expanding in cohomology

$$\sum_{j\in\mathrm{def}} N_j\,\delta^{(6)}(\mathbb{R}^{1,3}\times\mathrm{pt}_j) + a_1^{(6)}\,\frac{c_2(M_6)\,c_1(M_6)}{24} + a_2^{(6)}\,\frac{c_1^3(M_6)}{2} = 0$$