

Exam Preparation: Practice Exercises

Group Theory for Physicists

Statements with explicit notation, and fully worked solutions
Based on *Lectures on Group Theory for Physicists*, Parts A–E

How to use this sheet. There are fifteen exercises, three for each of the five parts of the course. Difficulty is marked *Easy*, *Medium* or *Hard*: the easy ones are quick warm-ups, the medium ones are typical exam length, and the hard ones are at the level of the last problems of an exam. **Part 0** collects the notation used in the exercises, including some symbols that are only used informally in the lectures ($\langle g \rangle$, S_n , A_n , cycle notation). **Part I** contains the statements, so you can work on them first; **Part II** contains full worked solutions in which every step is spelled out and the lecture result used is named. References such as [*Lectures: Sec. 8*] refer to the section numbers of the lecture notes.

Part	Topic	Exercises	Levels
A	Finite groups	1–3	E, M, M
B	Lie groups, representations, characters	4–6	E, M, M
C	Lie algebras and $\mathfrak{su}(2)$	7–9	E, M, H
D	Structure theory and classification	10–12	E, M, H
E	Weights, classical algebras, Lorentz group	13–15	M, M, H

Part 0 — Notation used in the exercises

Groups, powers, orders, and the subgroup $\langle g \rangle$

- The product of two elements is written gh (or $g \circ h$ as in the lectures); e is the identity and g^{-1} the inverse. Powers: $g^0 = e$, $g^k = g g \cdots g$ (k factors) for $k > 0$, and $g^{-k} = (g^{-1})^k$.
- $|G|$ is the *order* of a finite group G , i.e. its number of elements. The *order of an element* g is the smallest integer $n \geq 1$ with $g^n = e$.
- **The cyclic subgroup $\langle g \rangle$.** For an element $g \in G$,

$$\langle g \rangle := \{g^k : k \in \mathbb{Z}\} = \{\dots, g^{-2}, g^{-1}, e, g, g^2, \dots\}$$

is the set of all powers of g . It is a subgroup, because $g^a g^b = g^{a+b}$ and $(g^a)^{-1} = g^{-a}$. It is called the *subgroup generated by g* , and its number of elements equals the order of g . A group G is *cyclic* if $G = \langle g \rangle$ for some g (then g is called a *generator*).

- $\mathbb{Z}_n = \{0, 1, \dots, n-1\}$ is the group of integers modulo n under *addition* mod n , with identity 0. In additive notation, “ g^k ” means $k \cdot g = g + \cdots + g$, so $\langle a \rangle = \{0, a, 2a, 3a, \dots\} \pmod{n}$. Example: in $\mathbb{Z}_{12}$, $\langle 4 \rangle = \{0, 4, 8\}$. The group $\mathbb{Z}_n$ is cyclic, generated by 1.
- $H \leq G$ means “ H is a subgroup of G ”. *Lagrange’s theorem* [Lectures: Sec. 8]: $|H|$ divides $|G|$. The *index* of H in G is $[G : H] = |G|/|H|$, the number of cosets.
- A subgroup H is *normal* (invariant), written $H \trianglelefteq G$, if $ghg^{-1} \in H$ for all $g \in G$, $h \in H$ [Lectures: Sec. 7]. For normal H the cosets gH form the *quotient group* G/H with $(gH)(g'H) = (gg')H$ [Lectures: Sec. 9].
- $G_1 \times G_2$ is the direct product, and $G_1 \rtimes G_2$ the semidirect product (with G_1 normal) [Lectures: Sec. 12]. The symbol $\cong$ means “isomorphic”.

Permutation groups: S_n , cycle notation, A_n

- S_n (the *symmetric group*) is the group of all permutations of $\{1, 2, \dots, n\}$, i.e. all bijective maps $\{1, \dots, n\} \rightarrow \{1, \dots, n\}$, with composition as product. It has $n!$ elements. The lectures meet it in Cayley’s theorem [Lectures: Sec. 5.3], and S_3 in [Lectures: Secs. 6, 7].
- **Product convention.** $\sigma\tau$ means “first apply τ , then σ ” (right factor first), so $(\sigma\tau)(k) = \sigma(\tau(k))$.
- **Cycle notation.** (abc) denotes the permutation $a \mapsto b \mapsto c \mapsto a$ that leaves all other numbers fixed; (ab) swaps a and b (a *transposition*). Thus

$$S_3 = \{e, (12), (13), (23), (123), (132)\},$$

with $(123)^{-1} = (132)$ and $(ab)^{-1} = (ab)$. Example: $(12)(13) = (132)$, because applying (13) first and then (12) gives $1 \mapsto 3 \mapsto 3$, $3 \mapsto 1 \mapsto 2$, $2 \mapsto 2 \mapsto 1$.

- **Conjugation just relabels.** For $\tau \in S_n$: $\tau(abc)\tau^{-1} = (\tau(a) \tau(b) \tau(c))$, and likewise $\tau(ab)\tau^{-1} = (\tau(a) \tau(b))$.
- A_n (the *alternating group*) is the subgroup of S_n of *even* permutations, i.e. those that can be written as a product of an even number of transpositions. It has $n!/2$ elements. We only need $n = 3$:

$$A_3 = \{e, (123), (132)\} = \langle (123) \rangle,$$

the identity and the two 3-cycles. Geometrically, S_3 is the symmetry group of an equilateral triangle: A_3 consists of the rotations (by $0^\circ, 120^\circ, 240^\circ$) and the three transpositions are the reflections.

Matrix groups, representations, characters

- $D(\varphi) = \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix}$ is the rotation by the angle φ . $SO(2) = \{D(\varphi)\}$. Further, $O(2) = \{A \in \mathbb{R}^{2 \times 2} : A^T A = \mathbb{1}\}$, and $SO(2) = \{A \in O(2) : \det A = 1\}$.
- A *representation* is a map $\Gamma : G \rightarrow GL(n, \mathbb{C})$ with $\Gamma(gh) = \Gamma(g)\Gamma(h)$; n is its dimension. It is *irreducible* if no proper subspace is mapped to itself by all $\Gamma(g)$. The *trivial representation* is $\Gamma(g) = 1$ for all g . The *character* is $\chi_\Gamma(g) = \text{tr } \Gamma(g)$; it is constant on conjugacy classes [Lectures: Sec. 22].
- **Character orthogonality and multiplicities** [Lectures: Sec. 22]: for a finite group G of order N and irreducible representations Γ_a, Γ_b , $\frac{1}{N} \sum_{g \in G} \chi_a(g) \chi_b(g)^* = \delta_{ab}$, and if $\Gamma = \bigoplus_a m_a \Gamma_a$ then $m_a = \frac{1}{N} \sum_g \chi_\Gamma(g) \chi_a(g)^*$.

Lie algebras

- $[A, B] = AB - BA$ for matrices. $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices, with $[\sigma_a, \sigma_b] = 2i\varepsilon_{abc}\sigma_c$ and $(\hat{n} \cdot \vec{\sigma})^2 = \mathbb{1}$ for a unit vector $\hat{n}$.
- $\mathfrak{su}(2)$ consists of the traceless anti-Hermitian 2×2 matrices. Two bases appear in the lectures. In Parts B and D, $\Theta_a = \frac{i}{2}\sigma_a$ (anti-Hermitian), with $[\Theta_a, \Theta_b] = -\varepsilon_{abc}\Theta_c$. In Part C (and E), the Hermitian $t_a = \frac{1}{2}\sigma_a$, with $[t_a, t_b] = i\varepsilon_{abc}t_c$, $t_\pm = t_1 \pm it_2$, hence $[t_3, t_\pm] = \pm t_\pm$ and $[t_+, t_-] = 2t_3$. In the spin- j representation, $t_3\psi_m = m\psi_m$ and $t_\pm\psi_m = \sqrt{(j \mp m)(j \pm m + 1)} \psi_{m\pm 1}$ [Lectures: Sec. 31].
- $\text{ad}(X)Y = [X, Y]$ is the adjoint action, and $K(X, Y) = \text{Tr}(\text{ad } X \text{ ad } Y)$ the Killing form [Lectures: Sec. 35]. The *derived series* of a Lie algebra $\mathfrak{g}$ is $\mathfrak{g}^{(0)} = \mathfrak{g}$, $\mathfrak{g}^{(k+1)} = [\mathfrak{g}^{(k)}, \mathfrak{g}^{(k)}]$, where $[V, W]$ is the span of all brackets $[v, w]$; $\mathfrak{g}$ is *solvable* if the series reaches $\{0\}$. An *ideal* is a subspace I with $[\mathfrak{g}, I] \subseteq I$; $\mathfrak{g}$ is *semisimple* if it has no non-zero abelian ideal, and *simple* if it is non-abelian with no ideals except $\{0\}, \mathfrak{g}$.
- **Roots.** $\langle \cdot, \cdot \rangle$ is the Euclidean inner product, e_1, e_2 the standard basis of $\mathbb{R}^2$. The Cartan matrix is $A_{jk} = 2\langle \alpha_j, \alpha_k \rangle / \langle \alpha_k, \alpha_k \rangle$ [Lectures: Sec. 40]. The *Weyl reflection* in a root α is $s_\alpha(\beta) = \beta - \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha$ [Lectures: Sec. 43].
- **Weights.** A weight of a representation is an eigenvalue vector of the Cartan generators; in a tensor product $R_1 \otimes R_2$ weights add [Lectures: Sec. 28]. For $\mathfrak{su}(3)$ the weights of the fundamental $\mathbf{3}$ are e_u, e_d, e_s with $e_u + e_d + e_s = 0$, and the weights of the conjugate $\bar{\mathbf{3}}$ are $-e_u, -e_d, -e_s$ [Lectures: Sec. 44].

Part I — Exercises

Part A: Finite groups

Exercise 1

Part A · Easy

- (a) Let G be a group of prime order p . Prove that G is cyclic, i.e. that $G = \langle g \rangle$ for some $g \in G$ (see Part 0 for $\langle g \rangle$). *Hint:* Lagrange's theorem.
- (b) Consider $\mathbb{Z}_{12} = \{0, 1, \dots, 11\}$ with addition mod 12. For every divisor d of 12, find the subgroup of order d (list its elements) and its index. Prove that there are no other subgroups.
- (c) Since $\mathbb{Z}_{12}$ is abelian, every subgroup is normal. Let $H = \langle 4 \rangle = \{0, 4, 8\}$. List the cosets $a + H$ that make up the quotient group $\mathbb{Z}_{12}/H$. Compute the order of the coset $1 + H$ in $\mathbb{Z}_{12}/H$, and decide whether $\mathbb{Z}_{12}/H$ is isomorphic to $\mathbb{Z}_4$ or to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Exercise 2

Part A · Medium

Use the notation for S_3 and A_3 from Part 0.

- (a) Compute the conjugacy classes of S_3 (the conjugacy class of σ is $\{\tau\sigma\tau^{-1} : \tau \in S_3\}$), using the relabelling rule. Check that the class sizes add up to $|S_3| = 6$.
- (b) A subgroup is normal if and only if it is a union of conjugacy classes [*Lectures: Sec. 7*]. Use this to find all normal subgroups of S_3 . In addition, show *directly* that $H = \langle (12) \rangle = \{e, (12)\}$ is not normal, by exhibiting $\tau \in S_3$ with $\tau(12)\tau^{-1} \notin H$.
- (c) Compute the cosets of A_3 in S_3 and show that S_3/A_3 has two elements, hence $S_3/A_3 \cong \mathbb{Z}_2$. Show that $S_3 \not\cong \mathbb{Z}_6$.

Exercise 3

Part A · Medium

Let $r = (123)$ and $s = (12)$ in S_3 . Recall (Part 0) that $G = G_1 \rtimes G_2$ means: G_1 is normal in G , $G_1 \cap G_2 = \{e\}$, and every $g \in G$ can be written as $g = g_1g_2$ with $g_1 \in G_1$, $g_2 \in G_2$ [*Lectures: Sec. 12.2*].

- (a) Show that r has order 3, that s has order 2, and that $srs^{-1} = r^{-1}$. Then show that $S_3 = A_3 \rtimes \langle s \rangle$ by checking the three conditions, writing all six elements of S_3 as $r^a s^b$. Conclude $S_3 \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_2$.
- (b) Explain why S_3 is *not* a direct product of A_3 and $\langle s \rangle$: which of the conditions for a direct product [*Lectures: Sec. 12.1*] fails?
- (c) Starting from $srs^{-1} = r^{-1}$, derive the multiplication rule $(r^a s^b)(r^c s^d) = r^{a+(-1)^b c} s^{b+d}$ (exponents of r taken mod 3, of s mod 2). Use it to compute $(rs)^2$ and decide whether rs is a rotation or a reflection.
- (d) Which group do you get if s acts *trivially*, i.e. if $srs^{-1} = r$ instead? Identify it with a group you know.

Part B: Lie groups, representations, characters

Exercise 4

Part B · Easy

Let $R = \text{diag}(1, -1)$ and recall $O(2)$, $SO(2)$ and $D(\varphi)$ from Part 0.

- (a) Show that $\det A = \pm 1$ for every $A \in O(2)$, that $R \in O(2)$ with $\det R = -1$, and that $O(2) = SO(2) \cup R \cdot SO(2)$ (disjoint union).
- (b) Show that $O(2)$ has exactly two connected components and that $SO(2)$ is the one containing the identity [*Lectures: Sec. 17*]. *Hint:* a path of group elements is a continuous map $\gamma : [0, 1] \rightarrow O(2)$; use that $\det \circ \gamma$ is continuous.
- (c) Show that $O(2)$ is compact, i.e. a closed and bounded subset of the space $\mathbb{R}^{2 \times 2} \cong \mathbb{R}^4$.

of matrices [Lectures: Sec. 18].

- (d) Compute $RD(\varphi)R^{-1}$ and show that $O(2) = SO(2) \rtimes \{1, R\}$ by checking the three conditions of the semidirect product.

Exercise 5

Part B · Medium

Let $\mathbb{Z}_2 = \{e, g\}$ act on $\mathbb{R}^2$ by

$$\Gamma(e) = \mathbb{1}, \quad \Gamma(g) = P := \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

- Check that Γ is a representation (check all four products). Show that it is *not* orthogonal (unitary), i.e. $\Gamma(g)^T \Gamma(g) \neq \mathbb{1}$.
- Following the proof of the unitarity theorem for finite groups [Lectures: Sec. 20], form $S = \sum_{h \in G} \Gamma(h)^T \Gamma(h)$, compute it, and verify that S is symmetric and positive definite and that $\Gamma(h)^T S \Gamma(h) = S$ for every h . What does this say about the inner product $\langle u, v \rangle_S = u^T S v$?
- Find the eigenvectors of P , show that they are orthogonal with respect to $\langle \cdot, \cdot \rangle_S$, and decompose Γ into irreducible representations. Identify them (trivial, or $g \mapsto -1$).
- Cross-check (c) using characters and the multiplicity formula of Part 0.

Exercise 6

Part B · Medium

S_3 has three conjugacy classes: $\{e\}$, $T = \{(12), (13), (23)\}$ and $C = \{(123), (132)\}$ (see Exercise 2).

- (a) The symmetry group of the equilateral triangle acts on the plane. Using Exercise 3(c) one checks that

$$\Gamma_2(r^a s^b) = D\left(\frac{2\pi}{3}\right)^a \text{diag}(1, -1)^b, \quad r = (123), \quad s = (12),$$

defines a representation of S_3 on $\mathbb{R}^2$ (you may take this for granted: the matrices obey the same relations $r^3 = s^2 = e$, $srs^{-1} = r^{-1}$). Compute its character on the classes e , T , C , and show using the multiplicity formula that Γ_2 is irreducible. Also write down the characters of the trivial representation and of the *sign representation* (+1 on e and on C , -1 on T). Verify that the three characters are orthonormal, and assemble the character table.

- Using the orthogonality of the matrix elements of irreducible representations [Lectures: Sec. 22], argue that S_3 has no irreducible representations besides these three. *Hint*: the functions $g \mapsto \sqrt{n_a/N} \Gamma_a(g)_{ij}$ are orthonormal vectors in $\mathbb{C}^N$; how many can there be?
- Let S_3 act on $\mathbb{C}^3$ by permuting the three basis vectors ($\Gamma_{\text{perm}}(\sigma)e_k = e_{\sigma(k)}$). Compute its character (number of fixed points of σ) and decompose Γ_{perm} into irreducibles.

Part C: Lie algebras and $\mathfrak{su}(2)$

Exercise 7

Part C · Easy

Let $\Theta_a = \frac{i}{2}\sigma_a$, $a = 1, 2, 3$, be the anti-Hermitian basis of $\mathfrak{su}(2)$ (Part 0).

- Verify that $[\Theta_a, \Theta_b] = -\varepsilon_{abc}\Theta_c$.
- Compute the three 3×3 matrices $\text{ad}(\Theta_a)$ in the basis $(\Theta_1, \Theta_2, \Theta_3)$.
- Compute the Killing form $K(\Theta_a, \Theta_b)$. What does this show about semisimplicity (Cartan's second criterion [Lectures: Sec. 35]), and what does the sign of K say about compactness?

Exercise 8*Part C · Medium*

Recall that the exponential of a matrix is $\exp X = \sum_{k \geq 0} X^k / k!$.

- For a unit vector $\hat{n} \in \mathbb{R}^3$ compute $\exp(\theta \hat{n} \cdot \vec{\Theta})$ explicitly as a 2×2 matrix, by splitting the series into even and odd powers.
- Show that $\theta = 2\pi$ gives $-\mathbb{1}$ and $\theta = 4\pi$ gives $\mathbb{1}$. Explain why $\exp : \mathfrak{su}(2) \rightarrow SU(2)$ is surjective but not injective. (Every $U \in SU(2)$ can be written $U = a_0 \mathbb{1} + i \vec{a} \cdot \vec{\sigma}$ with real $a_0, \vec{a}$ and $a_0^2 + |\vec{a}|^2 = 1$.)
- Explain how this is related to the double cover $SU(2) \rightarrow SO(3)$ and to the centre $\{\pm \mathbb{1}\}$ of $SU(2)$ [Lectures: Sec. 30]. Why do half-integer spins not give representations of $SO(3)$?

Exercise 9*Part C · Hard*

Use the Hermitian conventions of Part 0: in the spin- j representation $t_3 \psi_m = m \psi_m$ and $t_{\pm} \psi_m = \sqrt{(j \mp m)(j \pm m + 1)} \psi_{m \pm 1}$, with $m = -j, \dots, j$.

- For $j = \frac{3}{2}$, write down the 4×4 matrices of t_3 and $t_{\pm}$ in the basis $(\psi_{3/2}, \psi_{1/2}, \psi_{-1/2}, \psi_{-3/2})$. Verify $[t_3, t_{\pm}] = \pm t_{\pm}$ and $[t_+, t_-] = 2t_3$, and check that the Casimir $t^2 = t_1^2 + t_2^2 + t_3^2$ equals $j(j+1) = \frac{15}{4}$.
- In the tensor product $\frac{3}{2} \otimes \frac{1}{2}$ the generators act as $t_a = t_a^{(1)} + t_a^{(2)}$, so the product state $\psi_{m_1} \otimes \psi_{m_2}$ has weight $M = m_1 + m_2$. Count the states of each weight, and decompose $\frac{3}{2} \otimes \frac{1}{2}$ into irreducibles by the highest-weight (“plateau”) method [Lectures: Sec. 32]. Verify the dimension count.
- Find the highest-weight vector of the spin-1 summand explicitly, as a combination of product states, and check that it is annihilated by $t_+ = t_+^{(1)} + t_+^{(2)}$. *Hint:* it is orthogonal to the $M = 1$ state of the spin-2 multiplet, which you get by applying t_- to the top state.

Part D: Structure theory and classification**Exercise 10***Part D · Easy*

Take the simple roots $\alpha_1 = e_1 - e_2$ and $\alpha_2 = e_2$ in $\mathbb{R}^2$ (inner product, Weyl reflection and Cartan matrix as in Part 0).

- Compute $\langle \alpha_j, \alpha_k \rangle$. Generate all roots by repeatedly applying the Weyl reflections $s_{\alpha_1}, s_{\alpha_2}$ to the simple roots, and list the positive roots as combinations of α_1, α_2 with non-negative integer coefficients. Cross-check with the master formula for root strings [Lectures: Sec. 38].
- Compute the Cartan matrix and draw the Dynkin diagram, including the arrow [Lectures: Sec. 40]. Which algebra is this?

Exercise 11*Part D · Medium*

Let $\alpha_j \neq \alpha_k$ be two simple roots, with angle θ between them.

- Show that $A_{jk} A_{kj} = 4 \cos^2 \theta$.
- Use that the off-diagonal Cartan matrix entries between distinct simple roots are non-positive integers [Lectures: Secs. 38–40] to deduce $\theta \in \{90^\circ, 120^\circ, 135^\circ, 150^\circ\}$.
- For each case find the possible values of the ratio $\langle \alpha_j, \alpha_j \rangle / \langle \alpha_k, \alpha_k \rangle$, and name a rank-2 algebra realising it.

Exercise 12

Part D · Hard

For each of the following real Lie algebras decide, with proof, whether it is (i) solvable (compute the derived series), (ii) semisimple (Killing form and Cartan's criteria, or look for a non-zero abelian ideal), and (iii) simple.

- (a) The *Heisenberg algebra*: basis x, y, z with $[x, y] = z$ and $[x, z] = [y, z] = 0$.
- (b) $\mathfrak{u}(2)$: the 4-dimensional algebra of anti-Hermitian 2×2 matrices, with basis $Z = i\mathbb{1}$ and $\Theta_1, \Theta_2, \Theta_3$ as in Exercise 7. Note $[Z, \cdot] = 0$.
- (c) $\mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ [Lectures: Sec. 35, example on $\mathfrak{so}(4)$]: you may use that it has a basis $A_1, A_2, A_3, B_1, B_2, B_3$ with $[A_i, A_j] = -2\varepsilon_{ijk}A_k$, $[B_i, B_j] = -2\varepsilon_{ijk}B_k$, $[A_i, B_j] = 0$.
- (d) The Euclidean algebra of the plane $\mathfrak{e}(2)$ (rotation J and translations P_1, P_2 ; compare the Euclidean group [Lectures: Sec. 12.3]): $[J, P_1] = P_2$, $[J, P_2] = -P_1$, $[P_1, P_2] = 0$.

Part E: Weights, classical algebras, Lorentz group**Exercise 13**

Part E · Medium

Use the weights e_u, e_d, e_s of the $\mathfrak{su}(3)$ fundamental $\mathbf{3}$ (Part 0). Order the weights so that e_u is the highest, then e_d , then e_s .

- (a) List the nine weights of $\mathbf{3} \otimes \mathbf{3}$ as sums $e_i + e_j$, with the product states carrying each weight and its multiplicity. Use $e_u + e_d + e_s = 0$ to simplify those weights that appear twice.
- (b) Using only weights, show that $\mathbf{3} \otimes \mathbf{3} = \mathbf{6} \oplus \bar{\mathbf{3}}$: find the highest weight, identify the weights of the $\mathbf{6}$, and show that the remaining weights are those of $\bar{\mathbf{3}}$ [Lectures: Sec. 44].
- (c) Which summand is the symmetric part and which the antisymmetric part of the tensor product? Why are these two subspaces invariant under $\mathfrak{su}(3)$?

Exercise 14

Part E · Medium

Recall the classification of an irreducible representation Γ [Lectures: Sec. 21.1]: Γ is *real* if $\Gamma^*(g) = S\Gamma(g)S^{-1}$ for all g with S symmetric, *pseudo-real* if such an S exists but is antisymmetric, and *complex* if no S exists.

- (a) Show that $\varepsilon\sigma_a\varepsilon^{-1} = -\sigma_a^*$ for $a = 1, 2, 3$, where $\varepsilon = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Deduce that the doublet $\Gamma(U) = U$ of $SU(2)$ satisfies $\Gamma^*(U) = \varepsilon\Gamma(U)\varepsilon^{-1}$, and conclude that it is pseudo-real.
- (b) Explain why the symmetry type of S does not depend on the choice of S (Schur's lemma).
- (c) For general spin j , define $S_j\psi_m = (-1)^{j+m}\psi_{-m}$. Show that $S_j t_3 S_j^{-1} = -t_3$ and $S_j t_{\pm} S_j^{-1} = -t_{\mp}$, deduce $\Gamma_j^* = S_j\Gamma_j S_j^{-1}$ (in the Hermitian conventions, a group element is $\exp(i\vec{\theta} \cdot \vec{t})$ and the t_a are the Hermitian generators, so one needs $S_j t_a S_j^{-1} = -t_a^*$), and find which j are real and which are pseudo-real.

Exercise 15

Part E · Hard

The Lorentz algebra has generators J^i (rotations) and K^i (boosts) with

$$[J^i, J^j] = i\varepsilon^{ijk}J^k, \quad [J^i, K^j] = i\varepsilon^{ijk}K^k, \quad [K^i, K^j] = -i\varepsilon^{ijk}J^k$$

[Lectures: Sec. 47]. Define $A_i^{\pm} = \frac{1}{2}(J^i \pm iK^i)$.

- (a) Show that $[A_i^{\pm}, A_j^{\pm}] = i\varepsilon_{ijk}A_k^{\pm}$ and $[A_i^+, A_j^-] = 0$.
- (b) Explain why the finite-dimensional irreducible representations are labelled by a pair (j, j') of spins [Lectures: Sec. 47], and give their dimension. List the cases $(0, 0)$, $(\frac{1}{2}, 0)$,

$(0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2})$ with the fields they describe.

- (c) The rotation subalgebra is generated by $J = A^+ + A^-$. Use the Clebsch–Gordan decomposition [*Lectures: Sec. 32*] to find how $(\frac{1}{2}, \frac{1}{2})$ and $(1, 0) \oplus (0, 1)$ split into spin- J representations of the rotation group. Identify the pieces physically (a 4-vector V^μ , and the electromagnetic field strength $F_{\mu\nu}$ with its six components $\vec{E}, \vec{B}$).

Part II — Solutions

Part A

Solution to Exercise 1

Strategy. Part (a) is a direct application of Lagrange's theorem [Lectures: Sec. 8]. Part (b) uses Lagrange twice: once to restrict the possible orders, and once (applied inside the subgroup) to pin the subgroup down. Part (c) is a computation in the quotient group [Lectures: Sec. 9].

(a) Let $|G| = p$ with p prime. Since $p \geq 2$, there is an element $g \neq e$. The set $\langle g \rangle = \{g^k : k \in \mathbb{Z}\}$ is a subgroup of G (Part 0). By Lagrange's theorem, $|\langle g \rangle|$ divides $|G| = p$. The only positive divisors of a prime p are 1 and p . The subgroup $\langle g \rangle$ contains at least the two distinct elements e and g , so $|\langle g \rangle| \neq 1$. Hence $|\langle g \rangle| = p = |G|$. A subset of a finite set with the same number of elements is the whole set, so $\langle g \rangle = G$ and G is cyclic. (In fact every non-identity element is a generator, and in particular $G \cong \mathbb{Z}_p$.)

(b) *Step 1: candidates.* If $H \leq \mathbb{Z}_{12}$ then $|H|$ divides 12 by Lagrange, so $|H| \in \{1, 2, 3, 4, 6, 12\}$. *Step 2: existence.* For each divisor d of 12 put $m = 12/d$ and $H_d := \langle m \rangle = \{0, m, 2m, \dots, (d-1)m\}$. This has exactly d elements (the multiples km with $0 \leq k < d$ are distinct mod 12, and $dm = 12 \equiv 0$).

Step 3: uniqueness. Let H be any subgroup with $|H| = d$ and let $x \in H$. The cyclic subgroup $\langle x \rangle$ is a subgroup of H , so by Lagrange (applied inside the group H) its order divides d . Hence $d \cdot x \equiv 0 \pmod{12}$, i.e. $12 \mid dx$, which gives $m = \frac{12}{d} \mid x$. So every element of H is a multiple of m modulo 12, i.e. $H \subseteq H_d$. Since $|H| = d = |H_d|$, we get $H = H_d$. Therefore the subgroups are exactly:

order d	1	2	3	4	6	12
subgroup	$\{0\}$	$\{0, 6\}$	$\{0, 4, 8\}$	$\{0, 3, 6, 9\}$	$\{0, 2, 4, 6, 8, 10\}$	$\mathbb{Z}_{12}$
index $12/d$	12	6	4	3	2	1

There are six subgroups, one for each divisor of 12, and no others.

(c) The cosets of $H = \{0, 4, 8\}$ are

$$H = \{0, 4, 8\}, \quad 1 + H = \{1, 5, 9\}, \quad 2 + H = \{2, 6, 10\}, \quad 3 + H = \{3, 7, 11\},$$

four cosets in total, matching $[\mathbb{Z}_{12} : H] = 12/3 = 4$. The group law is $(a + H) + (b + H) = (a + b) + H$, and the identity is H itself. Now compute multiples of the coset $1 + H$:

$$\begin{aligned} 1 \cdot (1 + H) &= 1 + H \neq H, \\ 2 \cdot (1 + H) &= 2 + H \neq H \quad (2 \notin H), \\ 3 \cdot (1 + H) &= 3 + H \neq H, \\ 4 \cdot (1 + H) &= 4 + H = H \quad (4 \in H). \end{aligned}$$

So $1 + H$ has order 4. Hence $\langle 1 + H \rangle$ has four elements, i.e. it is all of $\mathbb{Z}_{12}/H$, so the quotient is cyclic of order 4, i.e. $\mathbb{Z}_{12}/H \cong \mathbb{Z}_4$. It is *not* isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, since in $\mathbb{Z}_2 \times \mathbb{Z}_2$ every element squares to the identity [Lectures: Sec. 12.1], so no element has order 4.

Cross-check with the First Isomorphism Theorem [Lectures: Sec. 10]: the map $a \mapsto a \bmod 4$ is a well-defined surjective homomorphism $\mathbb{Z}_{12} \rightarrow \mathbb{Z}_4$ (well defined because $4 \mid 12$) with kernel $\{0, 4, 8\} = H$, hence $\mathbb{Z}_{12}/H \cong \mathbb{Z}_4$.

Solution to Exercise 2

Strategy. Conjugation in S_n only relabels, so classes are labelled by cycle type. Normal subgroups are the unions of classes that are subgroups [Lectures: Sec. 7]. Cosets are listed

explicitly.

(a) *Class of (12)*. Conjugating by each $\tau \in S_3$ with the rule $\tau(12)\tau^{-1} = (\tau(1) \tau(2))$:

τ	e	(12)	(13)	(23)	(123)	(132)
$(\tau(1) \tau(2))$	(12)	(21) = (12)	(32) = (23)	(13)	(23)	(31) = (13)

(for $\tau = (123)$: $1 \mapsto 2, 2 \mapsto 3$; for $\tau = (132)$: $1 \mapsto 3, 2 \mapsto 1$). So the class is $\{(12), (23), (13)\}$.
Class of (123). Conjugation by $e, (123), (132)$ (powers of (123)) leaves it unchanged. Conjugation by a transposition reverses the cycle: $(12)(123)(12)^{-1} = (\tau(1) \tau(2) \tau(3)) = (2 \ 1 \ 3) = (1 \ 3 \ 2)$ (a cycle can be started at any entry), and in the same way (13) gives $(3 \ 2 \ 1) = (1 \ 3 \ 2)$ and (23) gives $(1 \ 3 \ 2)$. So the class is $\{(123), (132)\}$.

Class of e is $\{e\}$. Altogether

$$\begin{aligned} [e] &= \{e\}, \\ [(12)] &= \{(12), (13), (23)\}, \quad [(123)] = \{(123), (132)\}, \\ 1 + 3 + 2 &= 6 = |S_3|, \end{aligned}$$

as it must be, because the classes partition the group.

(b) A normal subgroup is a union of classes that contains e and is closed under multiplication.

- $\{e\}$ (size 1): a subgroup, normal.
- $\{e\} \cup [(12)]$ (size 4): impossible, since $4 \nmid 6$ (Lagrange). Directly: $(12)(13) = (132)$ is not in the set.
- $\{e\} \cup [(123)] = \{e, (123), (132)\} = A_3$ (size 3): closed, since $(123)^2 = (132)$, $(123)(132) = e$. So A_3 is normal.
- all of S_3 (size 6).

So the normal subgroups of S_3 are $\{e\}, A_3, S_3$. The subgroup $H = \{e, (12)\}$ contains (12) but not the other elements (13), (23) of its class, so it is not normal. Directly: for $\tau = (13)$ we get $\tau(12)\tau^{-1} = (\tau(1) \tau(2)) = (3 \ 2) = (23) \notin H$.

(c) The left cosets of A_3 : one is A_3 itself. Another is $(12)A_3 = \{(12)e, (12)(123), (12)(132)\}$. Compute (right factor first): $(12)(123)$ sends $1 \mapsto 2 \mapsto 1, 2 \mapsto 3 \mapsto 3, 3 \mapsto 1 \mapsto 2$, i.e. (23); and $(12)(132)$ sends $1 \mapsto 3 \mapsto 3, 3 \mapsto 2 \mapsto 1, 2 \mapsto 1 \mapsto 2$, i.e. (13). So $(12)A_3 = \{(12), (23), (13)\} = T$, the set of transpositions. Since $A_3 \cup T = S_3$, there are exactly two cosets, $S_3/A_3 = \{A_3, T\}$, a group of order 2. By Exercise 1(a) (prime order 2) it is cyclic, $S_3/A_3 \cong \mathbb{Z}_2$.

Finally, $\mathbb{Z}_6$ is abelian, while S_3 is not: $(12)(13) = (132)$ but $(13)(12)$ sends $1 \mapsto 2 \mapsto 2, 2 \mapsto 1 \mapsto 3, 3 \mapsto 3 \mapsto 1$, i.e. $(123) \neq (132)$. An isomorphism would preserve commutativity, so $S_3 \not\cong \mathbb{Z}_6$.

Solution to Exercise 3

Strategy. Check the three conditions of the semidirect-product definition [Lectures: Sec. 12.2] by explicit computation. For (c), the only input is the relation $srs^{-1} = r^{-1}$, applied repeatedly.

(a) *Orders.* $r = (123)$ has $r^2 = (132) \neq e$ and $r^3 = e$, so r has order 3. $s = (12)$ has $s^2 = e$ and $s \neq e$, so order 2. *Relation:* $srs^{-1} = (s(1) s(2) s(3)) = (2 \ 1 \ 3) = (1 \ 3 \ 2) = r^{-1}$.

The three conditions with $G_1 = A_3 = \langle r \rangle$, $G_2 = \langle s \rangle = \{e, s\}$:

- A_3 is normal in S_3 (Exercise 2(b)).
- $A_3 \cap \langle s \rangle = \{e, r, r^2\} \cap \{e, s\} = \{e\}$, since $s = (12)$ is not a 3-cycle.

(iii) Every element is $r^a s^b$ with $a \in \{0, 1, 2\}$, $b \in \{0, 1\}$:

$r^a s^b$	e	r	r^2	s	rs	$r^2 s$
permutation	e	(123)	(132)	(12)	(13)	(23)

Here $rs = (123)(12)$ sends $1 \mapsto 2 \mapsto 3$, $2 \mapsto 1 \mapsto 2$, $3 \mapsto 3 \mapsto 1$, i.e. (13); and $r^2 s = (132)(12)$ sends $1 \mapsto 2 \mapsto 1$, $2 \mapsto 1 \mapsto 3$, $3 \mapsto 3 \mapsto 2$, i.e. (23). These are six distinct elements, hence all of S_3 .

So $S_3 = A_3 \rtimes \langle s \rangle$. Since $A_3 = \langle r \rangle \cong \mathbb{Z}_3$ (via $r^k \mapsto k$) and $\langle s \rangle \cong \mathbb{Z}_2$, we have $S_3 \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_2$, where the generator of $\mathbb{Z}_2$ acts on $\mathbb{Z}_3$ by inversion $k \mapsto -k$.

(b) In a direct product $G_1 \times G_2$ every element of G_1 commutes with every element of G_2 (condition (i) of the direct-product theorem), and as a consequence both factors are normal. Here $sr \neq rs$: indeed $srs^{-1} = r^{-1} \neq r$ because r has order 3. Equivalently, $\langle s \rangle$ is not a normal subgroup of S_3 (Exercise 2(b)). So the commutation condition fails (only A_3 is normal), and S_3 is a genuinely semidirect, not direct, product. The non-trivial action of $\langle s \rangle$ on A_3 by conjugation is exactly the obstruction.

(c) From $srs^{-1} = r^{-1}$: conjugation by s is a homomorphism, so $sr^c s^{-1} = (srs^{-1})^c = r^{-c}$, i.e. $sr^c = r^{-c}s$. Now

$$(r^a s^b)(r^c s^d) = \begin{cases} r^{a+c} s^d & b = 0, \\ r^a (sr^c) s^d = r^a r^{-c} s s^d = r^{a-c} s^{1+d} & b = 1, \end{cases}$$

which is $r^{a+(-1)^b c} s^{b+d}$, with exponents of r mod 3 and of s mod 2 (as $s^2 = e$). In particular

$$(rs)^2 = r(srs^{-1})s^2 = r r^{-1} e = e,$$

so rs has order 2: it is a reflection (indeed $rs = (13)$). The same computation shows that all three elements $s, rs, r^2 s$ have order 2.

(d) If $srs^{-1} = r$ then r and s commute and the rule becomes $(r^a s^b)(r^c s^d) = r^{a+c} s^{b+d}$, the multiplication of the direct product $\mathbb{Z}_3 \times \mathbb{Z}_2$. The element rs then satisfies $(rs)^k = r^k s^k$, which equals e only if $3 \mid k$ and $2 \mid k$, i.e. $6 \mid k$. So rs has order 6 and generates the group: the group is cyclic of order 6, $\cong \mathbb{Z}_6$. Thus the two possible actions of $\mathbb{Z}_2$ on $\mathbb{Z}_3$ give the two groups of order 6: trivial action $\rightarrow \mathbb{Z}_6$, inversion $\rightarrow S_3$.

Part B

Solution to Exercise 4

Strategy. Use $\det(A^T A) = \det(A)^2$, the definition of connected component by continuous paths [Lectures: Sec. 17], the closed-and-bounded definition of compactness [Lectures: Sec. 18], and the definition of semidirect product [Lectures: Sec. 12.2].

(a) For $A \in O(2)$: $1 = \det(\mathbb{1}) = \det(A^T A) = \det(A^T) \det(A) = (\det A)^2$, so $\det A = \pm 1$. For $R = \text{diag}(1, -1)$: $R^T R = \mathbb{1}$ and $\det R = -1$. Disjointness is clear since determinants differ. If $A \in O(2)$ with $\det A = -1$, then $RA \in O(2)$ (a product of orthogonal matrices: $(RA)^T (RA) = A^T R^T R A = \mathbb{1}$) and $\det(RA) = (-1)(-1) = 1$, so $RA \in SO(2)$ and $A = R^{-1}(RA) = R(RA) \in R \cdot SO(2)$ (as $R^{-1} = R$). Conversely every element of $R \cdot SO(2)$ is orthogonal with determinant -1 . Hence $O(2) = SO(2) \sqcup R \cdot SO(2)$.

(b) *Each piece is connected.* For $D(\varphi) \in SO(2)$ the path $\gamma(t) = D(t\varphi)$, $t \in [0, 1]$, is continuous in $O(2)$ and joins $\mathbb{1}$ to $D(\varphi)$. Likewise $t \mapsto R D(t\varphi)$ joins R to $RD(\varphi)$ within $R \cdot SO(2)$. *No path joins the two pieces.* If $\gamma: [0, 1] \rightarrow O(2)$ is a continuous path, then $\det \circ \gamma$ is a continuous function $[0, 1] \rightarrow \{-1, +1\}$; by the intermediate value theorem a continuous function with values in $\{\pm 1\}$ is constant. So γ stays in one piece. Hence there are exactly

two connected components, $SO(2)$ (the one containing $\mathbb{1}$, the group G_0 of Sec. 17) and $R \cdot SO(2) = \{\det = -1\}$, the other being a coset of $SO(2)$ as in the corollary of Sec. 17.

(c) *Bounded*: $A^T A = \mathbb{1}$ says that the columns of A are unit vectors: $\sum_i A_{ij}^2 = 1$ for each j . So $|A_{ij}| \leq 1$ for all entries, and $O(2)$ lies in a bounded subset of $\mathbb{R}^4$. *Closed*: $O(2)$ is the preimage of the single point $\mathbb{1}$ under the continuous map $A \mapsto A^T A$; equivalently, if $A_n \rightarrow A$ with $A_n^T A_n = \mathbb{1}$ then $A^T A = \lim A_n^T A_n = \mathbb{1}$. A closed and bounded subset of $\mathbb{R}^4$ is compact (Sec. 18).

(d) With $c = \cos \varphi$, $s = \sin \varphi$ and $R^{-1} = R$:

$$R D(\varphi) R^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} c & -s \\ -s & -c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix} = D(-\varphi).$$

Now the three conditions with $G_1 = SO(2)$, $G_2 = \{\mathbb{1}, R\} \cong \mathbb{Z}_2$:

(i) $SO(2)$ is normal: conjugating $D(\varphi)$ by $D(\psi)$ gives $D(\varphi)$ (rotations commute), and by $R D(\psi)$ gives $R D(\varphi) R^{-1} = D(-\varphi) \in SO(2)$. (Alternatively: $SO(2)$ is the identity component, normal by the theorem in Sec. 17.)

(ii) $SO(2) \cap \{\mathbb{1}, R\} = \{\mathbb{1}\}$, because $\det R = -1$.

(iii) By (a), every $A \in O(2)$ is $D(\varphi)$ or $R D(\psi) = D(-\psi)R$, so $A = g_1 g_2$ with $g_1 \in SO(2)$, $g_2 \in \{\mathbb{1}, R\}$.

Hence $O(2) = SO(2) \rtimes \{\mathbb{1}, R\}$, the continuous analogue of $S_3 = A_3 \rtimes \langle s \rangle$ (Exercise 3), where R acts on $SO(2)$ by inversion.

Solution to Exercise 5

Strategy. This is the proof of the unitarity theorem of [Lectures: Sec. 20], run on a concrete example: average $\Gamma(h)^T \Gamma(h)$ over the group, use the result as a new inner product, and decompose. Part (d) re-derives the answer with the multiplicity formula of [Lectures: Sec. 22].

(a) *Representation property.* Write $P = \Gamma(g)$. Then

$$P^2 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 + 1 \cdot 0 & 1 \cdot 1 + 1 \cdot (-1) \\ 0 \cdot 1 + (-1) \cdot 0 & 0 \cdot 1 + (-1)(-1) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \Gamma(e) = \Gamma(g^2).$$

The four products $\Gamma(e)\Gamma(e) = \Gamma(e)$, $\Gamma(e)\Gamma(g) = \Gamma(g)\Gamma(e) = \Gamma(g)$, $\Gamma(g)\Gamma(g) = \Gamma(g^2) = \Gamma(e)$ all hold, so Γ is a homomorphism. *Not orthogonal*:

$$P^T P = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \neq \mathbb{1}.$$

(Equivalently the columns $(1, 0)^T$ and $(1, -1)^T$ of P are not orthonormal.)

(b) The sum over $h \in \{e, g\}$ is

$$S = \Gamma(e)^T \Gamma(e) + \Gamma(g)^T \Gamma(g) = \mathbb{1} + P^T P = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}.$$

It is symmetric, and positive definite since for $v = (x, y)^T \neq 0$, $v^T S v = 2x^2 + 2xy + 3y^2 = x^2 + (x + y)^2 + 2y^2 > 0$. *Invariance*: for $h = e$ it is trivial; for $h = g$,

$$P^T S P = P^T (\mathbb{1} + P^T P) P = P^T P + (P^T)^2 P^2 = P^T P + \mathbb{1} = S,$$

where $P^2 = \mathbb{1}$ was used. This is the step of the lectures' proof where the rearrangement theorem is used (summing over h and over $h g_0$ gives the same sum). Consequently the inner product $\langle u, v \rangle_S = u^T S v$ is invariant, $\langle \Gamma(h)u, \Gamma(h)v \rangle_S = \langle u, v \rangle_S$: with respect to it, the representation is orthogonal (unitary).

(c) *Eigenvectors.* $Pv = v$ with $v = (x, y)^T$: $(x + y, -y) = (x, y)$ gives $y = 0$, so $v_+ = (1, 0)^T$ (eigenvalue $+1$). $Pv = -v$: $(x + y, -y) = (-x, -y)$ gives $y = -2x$, so $v_- = (1, -2)^T$ (eigenvalue -1). *Orthogonality for S :* $v_+^T S v_- = (1, 0) \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = (2, 1) \cdot (1, -2) = 2 - 2 = 0$. (Note: they are *not* orthogonal for the standard inner product, $v_+ \cdot v_- = 1$, which is why Γ is not orthogonal in the standard basis.) *Decomposition:* the lines $L_\pm = \mathbb{R}v_\pm$ are invariant, since $Pv_\pm = \pm v_\pm$, and $\mathbb{R}^2 = L_+ \oplus L_-$. On L_+ the element g acts as $+1$, and on L_- as -1 . So

$$\Gamma \cong \Gamma_{\text{triv}} \oplus \Gamma_{\text{sign}}, \quad \Gamma_{\text{triv}}(g) = 1, \quad \Gamma_{\text{sign}}(g) = -1,$$

both one-dimensional, hence irreducible. In the basis (v_+, v_-) the matrix of g is $\text{diag}(1, -1)$;

with $B = \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix}$ (columns $v_\pm$) one checks $B^{-1}PB = \text{diag}(1, -1)$.

(d) $\chi(e) = \text{tr } \mathbb{1} = 2$ and $\chi(g) = \text{tr } P = 1 + (-1) = 0$. The irreducible characters of $\mathbb{Z}_2$ are $\chi_{\text{triv}} = (1, 1)$ and $\chi_{\text{sign}} = (1, -1)$ on (e, g) (check: $\frac{1}{2}(1+1) = 1$, $\frac{1}{2}(1 \cdot 1 + 1 \cdot (-1)) = 0$). With $N = 2$:

$$m_{\text{triv}} = \frac{1}{2}(\chi(e) \cdot 1 + \chi(g) \cdot 1) = \frac{1}{2}(2+0) = 1, \quad m_{\text{sign}} = \frac{1}{2}(\chi(e) \cdot 1 + \chi(g) \cdot (-1)) = \frac{1}{2}(2-0) = 1,$$

in agreement with (c).

Solution to Exercise 6

Strategy. Everything here follows from Sec. 22 of the lectures (character orthogonality, the multiplicity formula, and orthogonality of matrix elements), plus the class structure of Exercise 2. The two facts “the number of irreps equals the number of classes” and $\sum_a n_a^2 = |G|$ are *not* needed: part (b) replaces them by a direct counting argument.

(a) *The 2-dimensional representation.* Let $R_3 = D(2\pi/3) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$ and $F = \text{diag}(1, -1)$. Then $R_3^3 = D(2\pi) = \mathbb{1}$, $F^2 = \mathbb{1}$, and $FR_3F^{-1} = D(-2\pi/3) = R_3^{-1}$ (Exercise 4(d)), the same relations as for r, s ; so the matrices $\Gamma_2(r^a s^b) = R_3^a F^b$ multiply by the same rule $r^{a+(-1)^b c} s^{b+d}$ as in Exercise 3(c).

Characters. $\chi_2(e) = \text{tr } \mathbb{1} = 2$. For r^a ($a = 1, 2$), the 3-cycles, $\text{tr } D(\pm 2\pi/3) = 2 \cos(2\pi/3) = -1$. For the reflections $r^a s$: with $\theta = 2\pi a/3$,

$$D(\theta)F = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}, \quad \text{tr} = 0.$$

So $\chi_2 = (2, 0, -1)$ on the classes (e, T, C) , with class sizes $(1, 3, 2)$.

Irreducibility. $\frac{1}{N} \sum_g |\chi_2(g)|^2 = \frac{1}{6}(1 \cdot 4 + 3 \cdot 0 + 2 \cdot 1) = 1$. If $\Gamma_2 = \bigoplus_a m_a \Gamma_a$, then by orthonormality of irreducible characters $\frac{1}{N} \sum_g |\chi_2|^2 = \sum_a m_a^2$. This equals 1 only if exactly one irreducible representation appears, once. So Γ_2 is irreducible.

One-dimensional representations. The trivial one has $\chi_1 = (1, 1, 1)$. The sign representation $\Gamma_\varepsilon(\sigma) = \det \Gamma_2(\sigma)$ (which is a homomorphism since $\det$ is multiplicative; it equals $+1$ on r^a and -1 on $r^a s$) has $\chi_\varepsilon = (1, -1, 1)$. Orthonormality:

$$\langle \chi_1, \chi_\varepsilon \rangle = \frac{1}{6}(1 - 3 + 2) = 0, \quad \langle \chi_1, \chi_2 \rangle = \frac{1}{6}(2 + 0 - 2) = 0, \quad \langle \chi_\varepsilon, \chi_2 \rangle = \frac{1}{6}(2 + 0 - 2) = 0,$$

$$\langle \chi_1, \chi_1 \rangle = \frac{1}{6}(1 + 3 + 2) = 1, \quad \langle \chi_\varepsilon, \chi_\varepsilon \rangle = \frac{1}{6}(1 + 3 + 2) = 1.$$

The character table is

	e (size 1)	T (size 3)	C (size 2)
χ_1 (trivial)	1	1	1
χ_ε (sign)	1	-1	1
χ_2	2	0	-1

(b) Sec. 22 shows: for irreducible unitary representations, the n_a^2 functions $g \mapsto \sqrt{n_a/N} \Gamma_a(g)_{ij}$ (for each irrep a , $1 \leq i, j \leq n_a$) are mutually orthonormal vectors in the N -dimensional space $\mathbb{C}^N$ of functions on G . In $\mathbb{C}^N$ there can be at most N orthonormal vectors, so $\sum_a n_a^2 \leq N = 6$ for any list of inequivalent irreducible representations. The three representations found have $n = 1, 1, 2$, so $1 + 1 + 4 = 6$ already saturates the bound. Any further irreducible representation would contribute at least $n^2 \geq 1$ and violate $\sum_a n_a^2 \leq 6$. (Every representation is equivalent to a unitary one, Sec. 20.) So these three are all the irreducible representations of S_3 .

(c) For a permutation matrix, the diagonal entries are 1 exactly at the fixed points, so $\chi_{\text{perm}}(\sigma)$ is the number of fixed points: e : 3; a transposition such as (12) fixes 3: 1; a 3-cycle fixes nothing: 0. So $\chi_{\text{perm}} = (3, 1, 0)$. By the multiplicity formula, $m_a = \frac{1}{6} \sum_{\text{classes}} |\text{class}| \chi_{\text{perm}} \chi_a^*$:

$$\begin{aligned} m_1 &= \frac{1}{6}(1 \cdot 3 \cdot 1 + 3 \cdot 1 \cdot 1 + 2 \cdot 0 \cdot 1) = 1, \\ m_\varepsilon &= \frac{1}{6}(3 - 3 + 0) = 0, \\ m_2 &= \frac{1}{6}(1 \cdot 3 \cdot 2 + 3 \cdot 1 \cdot 0 + 2 \cdot 0 \cdot (-1)) = 1. \end{aligned}$$

So $\Gamma_{\text{perm}} \cong \Gamma_{\text{triv}} \oplus \Gamma_2$, with dimension check $1 + 2 = 3$. Cross-check: $\frac{1}{6} \sum |\chi_{\text{perm}}|^2 = \frac{1}{6}(9 + 3 + 0) = 2 = 1^2 + 1^2$. Concretely, the trivial summand is spanned by $(1, 1, 1)^T$ and Γ_2 is realised on the invariant plane $x_1 + x_2 + x_3 = 0$.

Part C

Solution to Exercise 7

Strategy. Compute commutators with the Pauli algebra; read off the matrix of $\text{ad}(\Theta_a)$ from the structure constants; evaluate the trace. This is the computation of the lectures' example showing that $\mathfrak{su}(2)$ is simple [Lectures: Sec. 35].

(a) Using $[\sigma_a, \sigma_b] = 2i\varepsilon_{abc}\sigma_c$,

$$[\Theta_a, \Theta_b] = \left(\frac{i}{2}\right)^2 [\sigma_a, \sigma_b] = -\frac{1}{4} \cdot 2i\varepsilon_{abc}\sigma_c = -\frac{i}{2}\varepsilon_{abc}\sigma_c = -\varepsilon_{abc}\Theta_c.$$

(b) The structure constants are $f_{ab}^c = -\varepsilon_{abc}$, and the lectures write the matrix of t_a in the adjoint representation with a lower row index and an upper column index, $(\text{ad } \Theta_a)_b^c = f_{ba}^c = \varepsilon_{abc}$ [Lectures: Sec. 29, Def. 29.1]. For $a = 1$: $(\text{ad } \Theta_1)_{23} = \varepsilon_{123} = 1$, $(\text{ad } \Theta_1)_{32} = \varepsilon_{132} = -1$, all other entries 0 (check: $[\Theta_1, \Theta_2] = -\Theta_3$ and $[\Theta_1, \Theta_3] = +\Theta_2$, so $\text{ad}(\Theta_1)$ maps $\Theta_2 \mapsto -\Theta_3$, which is minus the row-2 entry, as in the sign rule of the lectures). Similarly for $a = 2, 3$:

$$\text{ad } \Theta_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \text{ad } \Theta_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \text{ad } \Theta_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(c) $K(\Theta_a, \Theta_b) = \text{Tr}(\text{ad } \Theta_a \text{ad } \Theta_b) = \sum_{c,d} (\text{ad } \Theta_a)_c^d (\text{ad } \Theta_b)_d^c = \sum_{c,d} \varepsilon_{acd} \varepsilon_{bdc} = \sum_{c,d} \varepsilon_{adc} \varepsilon_{bcd} = -\sum_{c,d} \varepsilon_{adc} \varepsilon_{bdc} = -2\delta_{ab}$, using $\varepsilon_{bcd} = -\varepsilon_{bdc}$ and $\sum_{c,d} \varepsilon_{adc} \varepsilon_{bdc} = 2\delta_{ab}$. A direct check: $(\text{ad } \Theta_1)^2 = \text{diag}(0, -1, -1)$ has trace $-2 = K(\Theta_1, \Theta_1)$, and $\text{ad } \Theta_1 \text{ad } \Theta_2$ has only an off-diagonal entry, so $K(\Theta_1, \Theta_2) = 0$. Thus

$$K = -2 \mathbb{1}_3, \quad \det K = -8 \neq 0.$$

Semisimplicity: K is non-degenerate, so by Cartan's second criterion $\mathfrak{su}(2)$ is semisimple (in fact simple). *Compactness:* K is negative definite, which is the signature of the compact real form: $\mathfrak{su}(2)$ is the Lie algebra of the compact group $SU(2)$ (compare $\mathfrak{sl}(2, \mathbb{R})$, whose Killing form is indefinite).

Solution to Exercise 8

Strategy. Use $(\hat{n} \cdot \vec{\sigma})^2 = \mathbb{1}$ to sum the exponential series in closed form. Then relate the result to the covering $SU(2) \rightarrow SO(3)$ [Lectures: Sec. 30].

(a) We have $\theta \hat{n} \cdot \vec{\sigma} = \frac{i\theta}{2} \hat{n} \cdot \vec{\sigma}$ and $(\hat{n} \cdot \vec{\sigma})^2 = \mathbb{1}$. Hence $(\hat{n} \cdot \vec{\sigma})^{2m} = \mathbb{1}$ and $(\hat{n} \cdot \vec{\sigma})^{2m+1} = \hat{n} \cdot \vec{\sigma}$, and

$$\exp\left(\frac{i\theta}{2} \hat{n} \cdot \vec{\sigma}\right) = \sum_{m \geq 0} \frac{(i\theta/2)^{2m}}{(2m)!} \mathbb{1} + \sum_{m \geq 0} \frac{(i\theta/2)^{2m+1}}{(2m+1)!} \hat{n} \cdot \vec{\sigma} = \cos \frac{\theta}{2} \mathbb{1} + i \sin \frac{\theta}{2} \hat{n} \cdot \vec{\sigma},$$

using $(i)^{2m} = (-1)^m$ and $(i)^{2m+1} = i(-1)^m$. Written out,

$$\exp(\theta \hat{n} \cdot \vec{\sigma}) = \begin{pmatrix} \cos \frac{\theta}{2} + in_3 \sin \frac{\theta}{2} & (in_1 + n_2) \sin \frac{\theta}{2} \\ (in_1 - n_2) \sin \frac{\theta}{2} & \cos \frac{\theta}{2} - in_3 \sin \frac{\theta}{2} \end{pmatrix}.$$

(b) For $\theta = 2\pi$: $\cos \pi = -1$ and $\sin \pi = 0$, so the result is $-\mathbb{1}$ for *every* unit vector $\hat{n}$. For $\theta = 4\pi$: $\cos 2\pi = 1$, $\sin 2\pi = 0$, giving $+\mathbb{1}$. Hence all the points $2\pi \hat{n} \cdot \vec{\sigma}$, which form a whole 2-sphere in the 3-dimensional algebra, are mapped to the single group element $-\mathbb{1}$: $\exp$ is not injective (also $\exp(0) = \mathbb{1} = \exp(4\pi \hat{n} \cdot \vec{\sigma})$). *Surjectivity:* let $U = a_0 \mathbb{1} + i\vec{a} \cdot \vec{\sigma}$ with $a_0^2 + |\vec{a}|^2 = 1$. Choose $\theta \in [0, 2\pi]$ with $\cos \frac{\theta}{2} = a_0$ (possible since $a_0 \in [-1, 1]$); then $\sin \frac{\theta}{2} = \sqrt{1 - a_0^2} = |\vec{a}| \geq 0$. If $|\vec{a}| \neq 0$ put $\hat{n} = \vec{a}/|\vec{a}|$, so that $U = \cos \frac{\theta}{2} \mathbb{1} + i \sin \frac{\theta}{2} \hat{n} \cdot \vec{\sigma} = \exp(\theta \hat{n} \cdot \vec{\sigma})$. If $\vec{a} = 0$ then $U = \pm \mathbb{1}$, which is $\exp(0)$ or $\exp(2\pi \hat{n} \cdot \vec{\sigma})$. So every element is an exponential.

(c) The covering homomorphism $\pi : SU(2) \rightarrow SO(3)$ sends $\exp(\theta \hat{n} \cdot \vec{\sigma})$ to the rotation by the angle θ about $\hat{n}$; it is two-to-one, with kernel $\{\mathbb{1}, -\mathbb{1}\} = Z(SU(2)) \cong \mathbb{Z}_2$, so $SO(3) \cong SU(2)/\mathbb{Z}_2$. The path $\theta \in [0, 2\pi]$, $\theta \mapsto \exp(\theta \hat{n} \cdot \vec{\sigma})$ runs in $SU(2)$ from $\mathbb{1}$ to $-\mathbb{1}$, an *open* path; its image in $SO(3)$ is the closed loop of a full rotation by 360° (it returns to the identity rotation). This loop cannot be contracted in $SO(3)$ (it generates $\pi_1(SO(3)) = \mathbb{Z}_2$), but its lift to the simply connected cover $SU(2)$ is not closed. A representation of $SU(2)$ in which $-\mathbb{1}$ acts as -1 cannot descend to $SO(3) = SU(2)/\{\pm \mathbb{1}\}$, because the two preimages $\pm U$ of one rotation would have to act by the same matrix. In the spin- j representation $-\mathbb{1}$ acts as $(-1)^{2j}$, so half-integer spins are exactly the ones that do *not* give representations of $SO(3)$ (a spin- $\frac{1}{2}$ state picks up a factor -1 under a 2π rotation and needs 4π to return to itself).

Solution to Exercise 9

Strategy. (a) Fill in the formula for $t_\pm$ and check the algebra entry by entry. (b) Count weights and peel off multiplets from the top. (c) The spin-1 highest-weight vector is the combination of the two $M = 1$ states that is orthogonal to the spin-2 state and killed by t_+ .

(a) With $j = \frac{3}{2}$ the coefficients of $t_+ \psi_m = \sqrt{(j-m)(j+m+1)} \psi_{m+1}$ are

$$t_+ \psi_{1/2} = \sqrt{(1)(3)} \psi_{3/2} = \sqrt{3} \psi_{3/2},$$

$$t_+ \psi_{-1/2} = \sqrt{(2)(2)} \psi_{1/2} = 2 \psi_{1/2},$$

$$t_+ \psi_{-3/2} = \sqrt{(3)(1)} \psi_{-1/2} = \sqrt{3} \psi_{-1/2},$$

and $t_+\psi_{3/2} = 0$. In the ordered basis $(\psi_{3/2}, \psi_{1/2}, \psi_{-1/2}, \psi_{-3/2})$ the matrices are

$$t_3 = \text{diag}\left(\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}\right), \quad t_+ = \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad t_- = t_+^T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}.$$

$[t_3, t_\pm] = \pm t_\pm$: for any matrices, $(t_3 X - X t_3)_{ik} = (m_i - m_k)X_{ik}$ when t_3 is diagonal with entries m_i . The non-zero entries of t_+ sit at positions $(i, i+1)$, where $m_i - m_{i+1} = +1$, so $[t_3, t_+] = t_+$. The non-zero entries of t_- sit at $(i+1, i)$, where $m_{i+1} - m_i = -1$, so $[t_3, t_-] = -t_-$. $[t_+, t_-] = 2t_3$: $t_+ t_-$ is diagonal with entries $\sum_k (t_+)_{ik}^2 = (3, 4, 3, 0)$ (squares of the entries in each row of t_+), and $t_- t_+$ is diagonal with entries $\sum_k (t_+)_{ki}^2 = (0, 3, 4, 3)$ (squares of the entries in each column). So

$$[t_+, t_-] = \text{diag}(3 - 0, 4 - 3, 3 - 4, 0 - 3) = \text{diag}(3, 1, -1, -3) = 2t_3.$$

Casimir: $t^2 = t_3^2 + \frac{1}{2}(t_+ t_- + t_- t_+) = \text{diag}(\frac{9}{4} + \frac{3}{2}, \frac{1}{4} + \frac{7}{2}, \frac{1}{4} + \frac{7}{2}, \frac{9}{4} + \frac{3}{2}) = \frac{15}{4} \mathbb{1} = j(j+1) \mathbb{1}$.

(b) The weights of the product states are $M = m_1 + m_2$ with $m_1 \in \{\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}\}$ and $m_2 \in \{\frac{1}{2}, -\frac{1}{2}\}$:

M	product states $\psi_{m_1} \otimes \psi_{m_2}$	multiplicity
2	$(\frac{3}{2}, \frac{1}{2})$	1
1	$(\frac{3}{2}, -\frac{1}{2}), (\frac{1}{2}, \frac{1}{2})$	2
0	$(\frac{1}{2}, -\frac{1}{2}), (-\frac{1}{2}, \frac{1}{2})$	2
-1	$(-\frac{1}{2}, -\frac{1}{2}), (-\frac{3}{2}, \frac{1}{2})$	2
-2	$(-\frac{3}{2}, -\frac{1}{2})$	1

The highest weight is $M = 2$ with multiplicity 1, so it is the top of a spin-2 multiplet, which contributes one state at each of $M = 2, 1, 0, -1, -2$. Subtracting these leaves multiplicities 0, 1, 1, 1, 0 at $M = 2, 1, 0, -1, -2$: the highest remaining weight is $M = 1$, giving one spin-1 multiplet ($M = 1, 0, -1$), after which nothing is left. Hence

$$\frac{3}{2} \otimes \frac{1}{2} = 2 \oplus 1, \quad (2 \cdot \frac{3}{2} + 1)(2 \cdot \frac{1}{2} + 1) = 4 \cdot 2 = 8 = 5 + 3.$$

(c) Write $a = \psi_{3/2} \otimes \psi_{-1/2}$ and $b = \psi_{1/2} \otimes \psi_{1/2}$ for the two $M = 1$ states. The spin-2 state at $M = 1$ is obtained by lowering the top state with $t_- = t_-^{(1)} + t_-^{(2)}$. Using $t_- \psi_m = \sqrt{(j+m)(j-m+1)} \psi_{m-1}$: for spin $\frac{3}{2}$, $t_- \psi_{3/2} = \sqrt{3 \cdot 1} \psi_{1/2} = \sqrt{3} \psi_{1/2}$; for spin $\frac{1}{2}$, $t_- \psi_{1/2} = \sqrt{1 \cdot 1} \psi_{-1/2} = \psi_{-1/2}$. So

$$t_-(\psi_{3/2} \otimes \psi_{1/2}) = \sqrt{3} \psi_{1/2} \otimes \psi_{1/2} + \psi_{3/2} \otimes \psi_{-1/2} = \sqrt{3} b + a, \quad |\sqrt{3} b + a|^2 = 3 + 1 = 4,$$

so $|2, 1\rangle = \frac{1}{2}(a + \sqrt{3} b)$. The spin-1 highest-weight state $|1, 1\rangle$ has the same weight and lies in a different irreducible subspace; the t_a are Hermitian, so different multiplets are orthogonal, and therefore $|1, 1\rangle$ is the normalised vector orthogonal to $a + \sqrt{3} b$:

$$|1, 1\rangle = \frac{1}{2}(b - \sqrt{3} a) \quad (\text{norm}^2 = \frac{1}{4}(1 + 3) = 1, \quad \text{overlap} = \frac{1}{4}(\sqrt{3} - \sqrt{3}) = 0).$$

Check that t_+ kills it. Here $t_+ = t_+^{(1)} + t_+^{(2)}$ acts on the first factor (spin $\frac{3}{2}$) by $t_+ \psi_{1/2} = \sqrt{3} \psi_{3/2}$, $t_+ \psi_{3/2} = 0$, and on the second factor (spin $\frac{1}{2}$, coefficients $\sqrt{(j-m)(j+m+1)}$ with $j = \frac{1}{2}$) by $t_+ \psi_{1/2} = 0$, $t_+ \psi_{-1/2} = \psi_{1/2}$. Then

$$t_+ b = \sqrt{3} \psi_{3/2} \otimes \psi_{1/2} + \psi_{1/2} \otimes 0 = \sqrt{3} \psi_{3/2} \otimes \psi_{1/2}, \quad t_+ a = 0 + \psi_{3/2} \otimes \psi_{1/2} = \psi_{3/2} \otimes \psi_{1/2},$$

so $t_+(b - \sqrt{3} a) = \sqrt{3} \psi_{3/2} \otimes \psi_{1/2} - \sqrt{3} \psi_{3/2} \otimes \psi_{1/2} = 0$, as required for a highest-weight vector. The other two states of the spin-1 multiplet follow by applying t_- .

Part D

Solution to Exercise 10

Strategy. Reflections s_α map roots to roots; apply them to the simple roots until nothing new appears. Then evaluate the Cartan matrix from the definition and read the Dynkin diagram from the lecture rule.

(a) *Inner products:* $\langle \alpha_1, \alpha_1 \rangle = \langle e_1 - e_2, e_1 - e_2 \rangle = 2$, $\langle \alpha_2, \alpha_2 \rangle = 1$, $\langle \alpha_1, \alpha_2 \rangle = \langle e_1 - e_2, e_2 \rangle = -1$. So α_1 is long and α_2 short, and

$$s_{\alpha_1}(\beta) = \beta - \langle \beta, \alpha_1 \rangle \alpha_1, \quad s_{\alpha_2}(\beta) = \beta - 2\langle \beta, \alpha_2 \rangle \alpha_2.$$

Reflections:

$$\begin{aligned} s_{\alpha_1}(\alpha_1) &= -\alpha_1, & s_{\alpha_1}(\alpha_2) &= \alpha_2 - (-1)\alpha_1 = \alpha_1 + \alpha_2 = e_1, \\ s_{\alpha_2}(\alpha_2) &= -\alpha_2, & s_{\alpha_2}(\alpha_1) &= \alpha_1 - 2(-1)\alpha_2 = \alpha_1 + 2\alpha_2 = e_1 + e_2. \end{aligned}$$

Further reflections of the new roots give nothing new: $s_{\alpha_1}(e_1) = e_1 - \alpha_1 = e_2 = \alpha_2$, $s_{\alpha_1}(e_1 + e_2) = e_1 + e_2$ (as $\langle e_1 + e_2, \alpha_1 \rangle = 0$), $s_{\alpha_2}(e_1) = e_1$, $s_{\alpha_2}(e_1 + e_2) = e_1 + e_2 - 2e_2 = \alpha_1$. Together with the negatives (reflections map $-\beta$ to $-s(\beta)$) the roots are

$$\pm e_1 \pm e_2 \text{ (4 long),} \quad \pm e_1, \pm e_2 \text{ (4 short),}$$

eight in total. The four positive roots, in terms of the simple roots, are

$$\alpha_1 = e_1 - e_2, \quad \alpha_2 = e_2, \quad \alpha_1 + \alpha_2 = e_1, \quad \alpha_1 + 2\alpha_2 = e_1 + e_2.$$

Cross-check with root strings. The master formula says that the α -string through β is $\beta - p\alpha, \dots, \beta + q\alpha$ with $p - q = 2\langle \beta, \alpha \rangle / \langle \alpha, \alpha \rangle$. Take $\beta = \alpha_1$, $\alpha = \alpha_2$: the difference of two simple roots is never a root, so $p = 0$, and $-q = 2\langle \alpha_1, \alpha_2 \rangle / \langle \alpha_2, \alpha_2 \rangle = -2$ gives $q = 2$: the string $\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2$. Take $\beta = \alpha_2$, $\alpha = \alpha_1$: $p = 0$ and $q = -2\langle \alpha_2, \alpha_1 \rangle / \langle \alpha_1, \alpha_1 \rangle = 1$: the string $\alpha_2, \alpha_2 + \alpha_1$. This reproduces exactly the four positive roots. Dimension check: $\dim = \text{rank} + |\Delta| = 2 + 8 = 10 = \dim \mathfrak{so}(5)$.

(b) With $A_{jk} = 2\langle \alpha_j, \alpha_k \rangle / \langle \alpha_k, \alpha_k \rangle$:

$$A_{11} = A_{22} = 2, \quad A_{12} = \frac{2(-1)}{1} = -2, \quad A_{21} = \frac{2(-1)}{2} = -1, \quad A = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}.$$

The two nodes are joined by $A_{12}A_{21} = 2$ edges (a double line). The arrow points from the long root to the short root; the short root is the node k for which $|A_{jk}|$ is the larger off-diagonal entry, here column 2 (as $|A_{12}| = 2 > 1 = |A_{21}|$), i.e. α_2 .

$$\begin{array}{c} \circ \longrightarrow \circ \\ \alpha_1 \text{ (long)} \alpha_2 \text{ (short)} \end{array}$$

This is the Dynkin diagram B_2 , the algebra $\mathfrak{so}(5)$; by the isomorphism of rank-2 algebras it is also $C_2 = \mathfrak{usp}(4)$.

Solution to Exercise 11

Strategy. Express the Cartan entries through lengths and angle; combine “integer” (master formula), “non-positive” (simple roots) and “ < 4 ” (linear independence) to get a short list.

(a) Using $\langle \alpha_j, \alpha_k \rangle = |\alpha_j||\alpha_k| \cos \theta$:

$$A_{jk} = \frac{2\langle \alpha_j, \alpha_k \rangle}{\langle \alpha_k, \alpha_k \rangle} = \frac{2|\alpha_j| \cos \theta}{|\alpha_k|}, \quad A_{kj} = \frac{2|\alpha_k| \cos \theta}{|\alpha_j|} \implies A_{jk}A_{kj} = 4 \cos^2 \theta.$$

(b) For $j \neq k$ the entries A_{jk} and A_{kj} are non-positive integers: integers by the master formula for root strings (Sec. 38), non-positive because $\langle \alpha_j, \alpha_k \rangle \leq 0$ for distinct simple roots (Sec. 39/40; this is the statement $A_{jk} = -p_{jk}$ with $p_{jk} \geq 0$ the number of times α_k can be added to α_j). So $A_{jk}A_{kj} = 4\cos^2\theta$ is a non-negative integer. It is strictly smaller than 4: $4\cos^2\theta = 4$ would mean $\cos\theta = \pm 1$, i.e. $\alpha_k = \pm c\alpha_j$ proportional, but simple roots are linearly independent (Sec. 39). Hence $4\cos^2\theta \in \{0, 1, 2, 3\}$. With $\cos\theta \leq 0$ (non-positive inner product),

$$\begin{aligned} 4\cos^2\theta = 0 &\Rightarrow \theta = 90^\circ, \\ 1 &\Rightarrow \cos\theta = -\frac{1}{2}, \theta = 120^\circ, \\ 2 &\Rightarrow \cos\theta = -\frac{1}{\sqrt{2}}, \theta = 135^\circ, \\ 3 &\Rightarrow \cos\theta = -\frac{\sqrt{3}}{2}, \theta = 150^\circ. \end{aligned}$$

(c) If $\langle \alpha_j, \alpha_k \rangle \neq 0$, dividing the two Cartan entries gives

$$\frac{A_{jk}}{A_{kj}} = \frac{\langle \alpha_j, \alpha_j \rangle}{\langle \alpha_k, \alpha_k \rangle}.$$

We need pairs of negative integers with the given product:

θ	(A_{jk}, A_{kj})	$\langle \alpha_j, \alpha_j \rangle / \langle \alpha_k, \alpha_k \rangle$	example
90°	$(0, 0)$	not determined (roots orthogonal)	$A_1 \oplus A_1 \cong \mathfrak{so}(4)_{\mathbb{C}}$
120°	$(-1, -1)$	1 (equal length)	$A_2 = \mathfrak{su}(3)$
135°	$(-1, -2)$ or $(-2, -1)$	$\frac{1}{2}$ or 2	$B_2 = C_2$
150°	$(-1, -3)$ or $(-3, -1)$	$\frac{1}{3}$ or 3	G_2

(For 120° : the only way to have product 1 with negative integers is $(-1, -1)$, so the lengths are equal.)

Solution to Exercise 12

Strategy. For each algebra: (i) compute $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}]$, $\mathfrak{g}^{(2)}$, ...; (ii) look for a non-zero abelian ideal or compute K and use Cartan's criteria [Lectures: Sec. 35]; (iii) simple means no ideals at all except 0 and $\mathfrak{g}$.

(a) **Heisenberg.** *Derived series.* The brackets of basis pairs are $[x, y] = z$, $[x, z] = 0$, $[y, z] = 0$, so $\mathfrak{g}^{(1)} = \text{span}(z)$ and $\mathfrak{g}^{(2)} = [\text{span}(z), \text{span}(z)] = 0$: **solvable**. *Killing form.* In the basis (x, y, z) : $\text{ad } x$ sends $y \mapsto z$ and kills x, z ; $\text{ad } y$ sends $x \mapsto -z$ and kills y, z ; $\text{ad } z = 0$. Each ad maps everything into $\text{span}(z)$ and kills z , so the product of any two of them is the zero matrix and $K \equiv 0$. It is degenerate, so by Cartan's second criterion the algebra is **not semisimple**. Directly: $\text{span}(z)$ is a non-zero abelian ideal ($[\mathfrak{g}, z] = 0 \subseteq \text{span}(z)$). Not simple (it has that ideal). Consistent with Cartan's first criterion ($K(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]) = 0$).

(b) **$\mathfrak{u}(2)$.** Basis $Z = i\mathbb{1}, \Theta_1, \Theta_2, \Theta_3$. Z is a multiple of $\mathbb{1}$, so $[Z, \cdot] = 0$ and $\text{ad } Z = 0$. *Derived series.* $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}] = \text{span}(\Theta_1, \Theta_2, \Theta_3) = \mathfrak{su}(2)$, since $[\Theta_a, \Theta_b] = -\varepsilon_{abc}\Theta_c$ produces every Θ_c and Z never appears. Then $\mathfrak{g}^{(2)} = [\mathfrak{su}(2), \mathfrak{su}(2)] = \mathfrak{su}(2)$ again, and the series stops decreasing at $\mathfrak{su}(2) \neq 0$: **not solvable**. *Killing form.* $\text{ad } Z = 0$, so $K(Z, \cdot) = 0$; and $\text{ad } \Theta_a$ maps $\mathfrak{su}(2)$ into itself and kills Z , so $K(\Theta_a, \Theta_b) = -2\delta_{ab}$ as in Exercise 7. In the basis $(Z, \Theta_1, \Theta_2, \Theta_3)$: $K = \text{diag}(0, -2, -2, -2)$, which is degenerate: **not semisimple**. Directly: $\text{span}(Z)$ is a one-dimensional central, hence abelian, ideal. **Not simple** (both $\text{span}(Z)$ and $\mathfrak{su}(2)$ are proper ideals). Indeed $\mathfrak{u}(2) = \mathfrak{su}(2) \oplus \mathfrak{u}(1)$; it is neither solvable nor semisimple.

(c) **$\mathfrak{so}(4)$.** The A_i span an ideal $\mathcal{A} \cong \mathfrak{su}(2)$ and the B_i an ideal $\mathcal{B} \cong \mathfrak{su}(2)$ with $[\mathcal{A}, \mathcal{B}] = 0$. For $x = a + b$ with $a \in \mathcal{A}$, $b \in \mathcal{B}$, $\text{ad } x$ acts block-diagonally, so $K_{\mathfrak{so}(4)} = K_{\mathcal{A}} \oplus K_{\mathcal{B}}$. Rescaling $A'_i = \frac{1}{2}A_i$ gives $[A'_i, A'_j] = -\varepsilon_{ijk}A'_k$, exactly the relations of Exercise 7, so $K(A'_i, A'_j) = -2\delta_{ij}$

and $K(A_i, A_j) = -8\delta_{ij}$; the same holds for the B_i . Thus $K = \text{diag}(-8, -8, -8, -8, -8, -8)$ is non-degenerate: **semisimple** (Cartan's second criterion). **Not solvable**: $[\mathfrak{so}(4), \mathfrak{so}(4)] = \mathfrak{so}(4)$ since each $\mathfrak{su}(2)$ factor equals its own derived algebra. **Not simple**: $\mathcal{A}$ is a proper non-zero ideal. (This is the lecture example $\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$, semisimple but not simple.)
(d) $\mathfrak{e}(2)$. Derived series. $[J, P_1] = P_2$, $[J, P_2] = -P_1$, $[P_1, P_2] = 0$ give $\mathfrak{g}^{(1)} = \text{span}(P_1, P_2)$, which is abelian, so $\mathfrak{g}^{(2)} = 0$: **solvable**. **Killing form** in the basis (J, P_1, P_2) : $\text{ad } J$ rotates the translations ($P_1 \mapsto P_2$, $P_2 \mapsto -P_1$) and kills J , so $(\text{ad } J)^2 = \text{diag}(0, -1, -1)$ and $K(J, J) = -2$. $\text{ad } P_1$ maps $J \mapsto [P_1, J] = -P_2$ and kills P_1, P_2 ; $\text{ad } P_2$ maps $J \mapsto [P_2, J] = +P_1$ and kills P_1, P_2 . Hence $\text{ad } J \text{ ad } P_1$ maps $J \mapsto \text{ad } J(-P_2) = P_1$ and kills the rest (off-diagonal, trace 0), likewise for $\text{ad } J \text{ ad } P_2$, and $\text{ad } P_i \text{ ad } P_j = 0$ (the image of $\text{ad } P_j$ lies in $\text{span}(P_1, P_2)$, which $\text{ad } P_i$ kills). So

$$K = \text{diag}(-2, 0, 0),$$

degenerate: **not semisimple**. Directly: $\text{span}(P_1, P_2)$ is an abelian ideal ($[\mathfrak{g}, P_i] \subseteq \text{span}(P_1, P_2)$). Not simple. Consistency with Cartan's first criterion: $K(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]) = K(\mathfrak{g}, \text{span}(P_1, P_2)) = 0$.

Summary.

	solvable	semisimple	simple
Heisenberg	yes	no	no
$\mathfrak{u}(2)$	no	no	no
$\mathfrak{so}(4)$	no	yes	no
$\mathfrak{e}(2)$	yes	no	no

Part E

Solution to Exercise 13

Strategy. Weights of a tensor product add [Lectures: Sec. 28]. Find the highest weight, build the irreducible representation it generates using its weight diagram [Lectures: Sec. 44], and see what is left over.

(a) The product states $|i\rangle \otimes |j\rangle$ ($i, j \in \{u, d, s\}$) have weight $e_i + e_j$. There are $3 \times 3 = 9$ of them:

weight	states	mult.	using $e_u + e_d + e_s = 0$
$2e_u$	uu	1	$2e_u$
$2e_d$	dd	1	$2e_d$
$2e_s$	ss	1	$2e_s$
$e_u + e_d$	ud, du	2	$-e_s$
$e_u + e_s$	us, su	2	$-e_d$
$e_d + e_s$	ds, sd	2	$-e_u$

Total: $3 \cdot 1 + 3 \cdot 2 = 9$.

(b) With the order $e_u > e_d > e_s$, the highest weight is $2e_u$, carried by $|u\rangle \otimes |u\rangle$ (multiplicity 1). Applying the lowering operators (the root vectors for the roots $e_u - e_d$ and $e_d - e_s$, which turn a quark u into d and d into s) to uu generates the states dd, ss and the symmetric combinations $ud + du, us + su, ds + sd$: six states, forming the weight diagram

$2e_u, 2e_d, 2e_s$ (vertices of a triangle), $e_u + e_d, e_u + e_s, e_d + e_s$ (midpoints of its edges),

each with multiplicity 1. This is the irreducible representation with highest weight $2e_u$, the **6** (Dynkin label (2, 0) in Sec. 44). Subtract these six weights from the list in (a): what remains is one copy of each of $e_u + e_d, e_u + e_s, e_d + e_s$, i.e. by the last column

$$-e_s, \quad -e_d, \quad -e_u,$$

exactly the three weights of $\bar{\mathbf{3}}$ (the negatives of the weights of $\mathbf{3}$). These three states are connected by the roots $e_i - e_j$, so they form an irreducible 3-dimensional representation, namely $\bar{\mathbf{3}}$. Hence

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{6} \oplus \bar{\mathbf{3}}, \quad 9 = 6 + 3.$$

(c) The $\mathbf{6}$ consists of the *symmetric* tensors, spanned by $|i\rangle|i\rangle$ and $|i\rangle|j\rangle + |j\rangle|i\rangle$. The $\bar{\mathbf{3}}$ consists of the *antisymmetric* tensors, spanned by $|i\rangle|j\rangle - |j\rangle|i\rangle$ (with weight $e_i + e_j = -e_k$, matching ε_{ijk}). Both subspaces are invariant because the $\mathfrak{su}(3)$ generators act on $V \otimes V$ as $T \otimes \mathbb{1} + \mathbb{1} \otimes T$, which is symmetric under exchanging the two factors: it commutes with the swap operator $\text{SWAP}(v \otimes w) = w \otimes v$. Hence the ± 1 eigenspaces of the swap (symmetric and antisymmetric tensors) are invariant subspaces.

Solution to Exercise 14

Strategy. (a) Verify the identity on the three Pauli matrices and exponentiate. (b) Use Schur's Lemma II [Lectures: Sec. 21]. (c) Build S_j explicitly from the ladder formulas [Lectures: Sec. 31].

(a) We have $\varepsilon = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\varepsilon^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and $\sigma_1^* = \sigma_1$, $\sigma_2^* = -\sigma_2$, $\sigma_3^* = \sigma_3$.

Compute

$$\begin{aligned} \varepsilon\sigma_1 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3, \\ \varepsilon\sigma_1\varepsilon^{-1} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = -\sigma_1 = -\sigma_1^*, \\ \varepsilon\sigma_3 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = -\sigma_1, \\ \varepsilon\sigma_3\varepsilon^{-1} &= -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -\sigma_3 = -\sigma_3^*, \end{aligned}$$

and, since $\varepsilon \propto \sigma_2$ commutes with σ_2 , $\varepsilon\sigma_2\varepsilon^{-1} = \sigma_2 = -\sigma_2^*$. So $\varepsilon\sigma_a\varepsilon^{-1} = -\sigma_a^*$ for $a = 1, 2, 3$. *Consequence for the group.* By Exercise 8, every $U \in SU(2)$ is $U = \exp(\frac{i}{2}\vec{\theta} \cdot \vec{\sigma})$ with real $\vec{\theta}$. Then

$$\varepsilon U \varepsilon^{-1} = \exp(\frac{i}{2}\vec{\theta} \cdot \varepsilon \vec{\sigma} \varepsilon^{-1}) = \exp(-\frac{i}{2}\vec{\theta} \cdot \vec{\sigma}^*) = \left(\exp(\frac{i}{2}\vec{\theta} \cdot \vec{\sigma}) \right)^* = U^*,$$

(conjugating an exponential conjugates the argument, and $\vec{\theta}$ is real). So $\Gamma^*(U) = U^* = \varepsilon \Gamma(U) \varepsilon^{-1}$ for the doublet $\Gamma(U) = U$. The doublet is irreducible (it has dimension 2, and no line is invariant, since t_+ and t_- connect the two weight spaces $m = \pm \frac{1}{2}$). So by the definition of Sec. 21.1 it is self-conjugate with $S = \varepsilon$, and $\varepsilon^T = -\varepsilon$ is *antisymmetric*: the doublet is **pseudo-real**.

(b) Suppose S and S' both satisfy $\Gamma^* = S\Gamma S^{-1} = S'\Gamma S'^{-1}$. Then $S\Gamma(g)S^{-1} = S'\Gamma(g)S'^{-1}$ for all g ; multiplying by S'^{-1} from the left and by S from the right gives $S'^{-1}S\Gamma(g) = \Gamma(g)S'^{-1}S$, i.e. $S'^{-1}S$ commutes with every $\Gamma(g)$. By Schur's Lemma II (for the irreducible Γ over $\mathbb{C}$), $S'^{-1}S = c\mathbb{1}$, so $S = cS'$ for a scalar c . Therefore S and S' are simultaneously symmetric or antisymmetric: the type (real or pseudo-real) is well defined.

(c) In the spin- j representation Γ_j , group elements are $\exp(i\vec{\theta} \cdot \vec{t})$ with Hermitian t_a (acting by the matrices of Sec. 31, in the basis ψ_m). Then $\Gamma_j^* = \exp(-i\vec{\theta} \cdot \vec{t}^*)$ and $S\Gamma_j S^{-1} = \exp(i\vec{\theta} \cdot S\vec{t}S^{-1})$, so it suffices to find S_j with $S_j t_a S_j^{-1} = -t_a^*$. Define $S_j \psi_m = (-1)^{j+m} \psi_{-m}$, hence $S_j^{-1} \psi_{-m} = (-1)^{j+m} \psi_m$. Then:

- Apply $S_j t_3 S_j^{-1}$ to ψ_{-m} step by step. First $S_j^{-1} \psi_{-m} = (-1)^{j+m} \psi_m$; then t_3 gives $m(-1)^{j+m} \psi_m$; then $S_j \psi_m = (-1)^{j+m} \psi_{-m}$ gives $m(-1)^{2(j+m)} \psi_{-m} = m \psi_{-m}$ (as $j+m$ is an integer). On the other hand $t_3 \psi_{-m} = -m \psi_{-m}$. So $S_j t_3 S_j^{-1} \psi_{-m} = -t_3 \psi_{-m}$, i.e. $S_j t_3 S_j^{-1} = -t_3$.
- For the ladder operators, $t_+ \psi_m = c_m \psi_{m+1}$ with $c_m = \sqrt{(j-m)(j+m+1)}$ and $t_- \psi_{-m} = \sqrt{(j-m)(j+m+1)} \psi_{-m-1} = c_m \psi_{-m-1}$. Then

$$\begin{aligned} S_j t_+ S_j^{-1} \psi_{-m} &= (-1)^{j+m} S_j t_+ \psi_m \\ &= (-1)^{j+m} c_m S_j \psi_{m+1} = (-1)^{j+m} c_m (-1)^{j+m+1} \psi_{-m-1} \\ &= -c_m \psi_{-m-1} = -t_- \psi_{-m}. \end{aligned}$$

So $S_j t_+ S_j^{-1} = -t_-$, and similarly $S_j t_- S_j^{-1} = -t_+$.

Now convert to $-t_a^*$. In the basis ψ_m the matrices of $t_3, t_{\pm}$ are real, hence $t_3^* = t_3$, $t_{\pm}^* = t_{\pm}$, and $t_1 = \frac{1}{2}(t_+ + t_-)$ is real, $t_2 = \frac{1}{2i}(t_+ - t_-)$ is purely imaginary, so $t_1^* = t_1$, $t_2^* = -t_2$. Then

$$\begin{aligned} S_j t_1 S_j^{-1} &= \frac{1}{2}(-t_- - t_+) = -t_1 = -t_1^*, \\ S_j t_2 S_j^{-1} &= \frac{1}{2i}(-t_- + t_+) = t_2 = -t_2^*, \\ S_j t_3 S_j^{-1} &= -t_3 = -t_3^*. \end{aligned}$$

So $\Gamma_j^* = S_j \Gamma_j S_j^{-1}$. For $j = \frac{1}{2}$ the matrix $S_{1/2}$ in the basis $(\psi_{1/2}, \psi_{-1/2})$ has columns $S\psi_{1/2} = -\psi_{-1/2}$ and $S\psi_{-1/2} = \psi_{1/2}$, i.e. $S_{1/2} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \varepsilon$, matching (a).

Symmetry type. The matrix elements of S_j are $(S_j)_{m',m} = (-1)^{j+m} \delta_{m',-m}$. Its transpose has $(S_j^T)_{m',m} = (S_j)_{m,m'} = (-1)^{j+m'} \delta_{m,-m'} = (-1)^{j-m} \delta_{m',-m}$ (using $m' = -m$). Hence $(S_j^T)_{m',m} = (-1)^{-2m} (S_j)_{m',m}$, and since $j-m \in \mathbb{Z}$ we have $(-1)^{2m} = (-1)^{2j}$. So

$$S_j^T = (-1)^{2j} S_j.$$

For integer j the matrix S_j is symmetric: Γ_j is **real**; for half-integer j it is antisymmetric: Γ_j is **pseudo-real**. (E.g. $j = 1$, the adjoint of $SU(2)$, is real; $j = \frac{1}{2}$ is pseudo-real, as in (a).)

Solution to Exercise 15

Strategy. (a) Expand the commutators and use the antisymmetry of ε . (b) Representations of a direct sum of two $\mathfrak{su}(2)$'s are tensor products. (c) Restricting to the diagonal subalgebra is a Clebsch–Gordan problem [Lectures: Sec. 32].

(a) First note $[K^i, J^j] = -[J^j, K^i] = -i\varepsilon^{ijk} K^k = i\varepsilon^{ijk} K^k$. Then

$$\begin{aligned} [A_i^+, A_j^+] &= \frac{1}{4}([J^i, J^j] + i[J^i, K^j] + i[K^i, J^j] - [K^i, K^j]) \\ &= \frac{1}{4}(i\varepsilon^{ijk} J^k + i \cdot i\varepsilon^{ijk} K^k + i \cdot i\varepsilon^{ijk} K^k + i\varepsilon^{ijk} J^k) \\ &= \frac{1}{4}(2i\varepsilon^{ijk} J^k - 2\varepsilon^{ijk} K^k) = \frac{i}{2}\varepsilon^{ijk}(J^k + iK^k) = i\varepsilon_{ijk} A_k^+. \end{aligned}$$

(here $i \cdot i = -1$, and $-[K^i, K^j] = +i\varepsilon^{ijk} J^k$). For A^- replace $K \rightarrow -K$: the two middle terms change sign twice ($i[J, -K]$ and $i[-K, J]$ each flip sign, but the $-[K, K]$ term is unchanged, being quadratic in K)

$$[A_i^-, A_j^-] = \frac{1}{4}(i\varepsilon^{ijk} J^k + \varepsilon^{ijk} K^k + \varepsilon^{ijk} K^k + i\varepsilon^{ijk} J^k) = \frac{i}{2}\varepsilon^{ijk}(J^k - iK^k) = i\varepsilon_{ijk} A_k^-.$$

Mixed commutator:

$$\begin{aligned} [A_i^+, A_j^-] &= \frac{1}{4} \left([J^i, J^j] - i[J^i, K^j] + i[K^i, J^j] + [K^i, K^j] \right) \\ &= \frac{1}{4} \left(i\varepsilon^{ijk} J^k + \varepsilon^{ijk} K^k - \varepsilon^{ijk} K^k - i\varepsilon^{ijk} J^k \right) = 0, \end{aligned}$$

using $-i[J^i, K^j] = -i \cdot i\varepsilon^{ijk} K^k = +\varepsilon^{ijk} K^k$, $i[K^i, J^j] = i \cdot i\varepsilon^{ijk} K^k = -\varepsilon^{ijk} K^k$ and $[K^i, K^j] = -i\varepsilon^{ijk} J^k$. So the complexified Lorentz algebra is a direct sum $\mathfrak{su}(2)_{\mathbb{C}} \oplus \mathfrak{su}(2)_{\mathbb{C}}$ of two commuting copies of the $\mathfrak{su}(2)$ algebra.

(b) The A_i^+ and A_i^- each satisfy the $\mathfrak{su}(2)$ relations and commute with one another. Irreducible representations of a direct sum of two algebras are tensor products of irreducible representations of the summands: the A^+ copy acts with some spin $j \in \{0, \frac{1}{2}, 1, \dots\}$ (Sec. 31) and the A^- copy with some spin j' . So they are labelled by (j, j') , and their dimension is $(2j+1)(2j'+1)$. The simplest ones:

(j, j')	dim	field
$(0, 0)$	1	scalar ϕ
$(\frac{1}{2}, 0)$	2	left-handed Weyl spinor χ_α
$(0, \frac{1}{2})$	2	right-handed Weyl spinor $\bar{\chi}_{\dot{\alpha}}$
$(\frac{1}{2}, \frac{1}{2})$	4	vector V^μ

(c) Since $J = A^+ + A^-$, the rotation generators act on (j, j') as the sum of two angular momenta, so under rotations (j, j') decomposes by the Clebsch–Gordan rule $j \otimes j' = \bigoplus_{J=|j-j'|}^{j+j'} J$:

$$\left(\frac{1}{2}, \frac{1}{2}\right) \rightarrow \frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1 \quad (1+3=4), \quad (1, 0) \oplus (0, 1) \rightarrow 1 \oplus 1 \quad (3+3=6),$$

since $1 \otimes 0 = 1$ and $0 \otimes 1 = 1$. *Physical meaning.* The 4-vector $V^\mu = (V^0, \vec{V})$ splits into a rotation scalar ($J = 0$), the time component V^0 , and a rotation vector ($J = 1$), the space components $\vec{V}$. The antisymmetric tensor $F_{\mu\nu}$ has $4 \cdot 3/2 = 6$ independent components, which is the dimension of $(1, 0) \oplus (0, 1)$; under rotations they split into two spin-1 pieces, the electric and magnetic fields $\vec{E}, \vec{B}$, or in complex combinations $\vec{E} + i\vec{B} \in (1, 0)$ and $\vec{E} - i\vec{B} \in (0, 1)$ (the self-dual and anti-self-dual parts). The boosts $K = -i(A^+ - A^-)$ mix the $J = 0$ and $J = 1$ pieces of the vector, as they mix V^0 and $\vec{V}$.