Cartan-classification of simple Lie-Algebras

Here we always assume a complex Lie-algebra $\tilde{\mathcal{L}}$

<u>Definition</u> Cartan subalgebra: $\mathcal{H}$

maximal Abelian subalgebra of $\tilde{\mathcal{L}}$

(maximal = every subalgebra containing $\mathcal{H}$ is non-Abelian)

<u>Definition</u> the rank of a Lie-algebra is the dimension of its Cartan subalgebra.

Let h_i be a basis of $\mathcal{H}$ then

$$[h_i, h_j] = 0 \qquad i,j = 1..l$$

also write the elements of $\hat{\mathcal{L}}$ as

$$h_1 .. h_l , a_1' , .. , a_{n-l}'$$

then $$[h_i, a_k'] = \alpha_k(h_i)\, a_k'$$

for fixed k there must be at least one $\alpha_k(h_i) \neq 0$

Note: since $[h_i, h_j] = 0$ it follows that all h_i matrices $ad(h_i)$ are diagonal

$$\forall\, h \in \mathcal{H} \qquad [h, a_k'] = \alpha_k(h)\, a_k'$$

$\alpha_k(h)$ ... linear functional on $\mathcal{H}$ $\quad \sum_{i=1}^{l} \mu_i \alpha_k(h_i)$

$\alpha_k(h)$ ... non-zero root

root subspace $\mathcal{L}_\alpha$ $[h, a_\alpha] = \alpha(h)\, a_\alpha$

$\alpha(h)$ specified by $\alpha(h_i)$ for some basis h_i

think: r - dimensional vector

Δ ... set of all root sub spaces

$\hat{\mathcal{L}}_0$ ... set of zero-roots $= \mathcal{H}$

$\hat{\mathcal{L}} = \hat{\mathcal{L}}_0 \oplus \Delta$

Let $a_\alpha \in \mathcal{L}_\alpha$ and $a_\beta \in \mathcal{L}_\beta$ then $[a_\alpha, a_\beta]$ is in $\mathcal{L}_{\alpha+\beta}$ if $\alpha+\beta \in \Delta$ otherwise it is zero

Proof: $[h, [a_\alpha, a_\beta]] = [[h, a_\alpha], a_\beta] + [a_\alpha, [h, a_\beta]] =$

$= \alpha\, [a_\alpha, a_\beta] + \beta\, [a_\alpha, a_\beta]$

$= (\alpha+\beta)\, [a_\alpha, a_\beta]$

$\rightarrow$ if a_α, a_β do not commute then $(\alpha+\beta)$ is a root otherwise they commute.

If $a_\alpha \in \mathcal{L}_\alpha$ $a_\beta \in \mathcal{L}_\beta$ and if $\alpha+\beta \neq 0$ then

$\kappa(a_\alpha, a_\beta) = 0$

$\Rightarrow$

Proof $\quad ad(a_\alpha) \cdot ad(a_\beta) \cdot a_\gamma = [a_\alpha, [a_\beta, a_\gamma]]$

this is in $\mathcal{L}_{\alpha+\beta+\gamma}$ if $\alpha+\beta+\gamma$ is a root

but since $\alpha+\beta \neq 0$ $\quad \mathcal{L}_{\alpha+\beta+\gamma} \cap \mathcal{L}_\gamma = \{0\}$

if $\alpha+\beta+\gamma$ is not a root then it is just zero

$\Rightarrow$ $ad(a_\alpha)\, ad(a_\beta)\, a_\gamma$ does not contain a_γ

so all diagonal elements are zero

$tr(ad(a_\alpha)\, ad(a_\beta)) = 0$

In particular $\kappa(h, a_\alpha) = 0$

and $\kappa(a_\alpha, a_\alpha) = 0$

$\Rightarrow$ κ is non-degenerate on $\mathcal{H}$

Proof suppos κ is degenerate, then $\exists\, h'$ such that $\kappa(h', h) = 0 \quad \forall\, h \in \mathcal{H}$ but also $\kappa(h', a_\alpha) = 0 \Rightarrow \kappa$ would be degenerate on $\mathcal{L}$. But we knew that κ is non-degenerate on $\mathcal{L} \Rightarrow$ it is non-degenerate on $\mathcal{H}$.

This means that we can chose a basis h_α such that $\kappa(h_\alpha, h) = \alpha(h)$

if $\alpha+\beta$ is a root $\quad h_{\alpha+\beta} = h_\alpha + h_\beta$

and $\quad \alpha(h_\beta) = \beta(h_\alpha) = \kappa(h_\alpha, h_\beta)$

$$\langle \alpha, \beta \rangle = \langle \beta, \alpha \rangle = \kappa(h_\alpha, h_\beta)$$

$$\Rightarrow \quad [h_\beta, a_\alpha] = \langle \alpha, \beta \rangle a_\alpha \qquad \forall\, \alpha, \beta \in \Delta$$

If $\alpha \in \Delta$ then $-\alpha \in \Delta$

Proof: suppose $\alpha \in \Delta$ but $-\alpha \notin \Delta$ then

$$\kappa(a_\alpha, a) = 0 \quad \forall\, a \in \mathcal{L}$$

$\Rightarrow$ But κ is non-degenerate

So $-\alpha \in \Delta$

Example $su(2)$ $\frac{i}{2}\sigma_1, \frac{i}{2}\sigma_2, \frac{i}{2}\sigma_3$ linearily independent over $\mathbb{C}$, good basis for $\tilde{\mathcal{L}}$ $su(2;\mathbb{C})$

Cartan subalgebra : $a_3 = \frac{i}{2}\sigma_3$ but since we

we can actually make complex linear combinations we can also take $\frac{1}{2}\sigma_3$

adjoint: $-\varepsilon_{i3j} = \begin{pmatrix} 0 & +1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\kappa(a_3, a_3) = -2$

roots: $a_1 \pm i a_2 : a_\pm$

$$[a_3, a_1 + i a_2] = -\varepsilon_{312} a_2 \mp i \varepsilon_{321} a_1 =$$
$$= -a_2 + i a_1 = i(a_1 + i a_2)$$

$$[h_1, a_{\pm\alpha}] = \pm i\, a_{\pm\alpha}$$

$\alpha_1(h_1) = i$ $\mathcal{L}_\alpha : \lambda a_+$

$\mathcal{L}_{-\alpha} : \lambda' a_-$

$$\kappa(h_\alpha, h_\alpha) = \alpha(h_\alpha)$$

$h_\alpha = \lambda h_1$ $\lambda^2(-2) = \lambda i \Rightarrow \lambda = -\frac{i}{2}$

$$\langle \alpha, \alpha \rangle = \kappa(h_\alpha, h_\alpha) = \frac{1}{2}$$

Physics convention $\frac{1}{2}\sigma_i = \tau_i$

$$[\tau_i, \tau_j] = i\,\varepsilon_{ijk}\,\tau_k$$

$$[\tau_3, \tau_1 \pm i\tau_2] = i\varepsilon_{312}\,\tau_2 \pm i\,(i\varepsilon_{321})\,\tau_1$$

$$= i\tau_2 \pm \tau_1 = \tau_1 \pm \tau_2$$

$$[\tau_3, (\tau_1 \pm i\tau_2)] = \pm\,\tau_\pm$$

roots now are $\pm 1 = \alpha$

$$h_\alpha : \quad +2\lambda^2 = \lambda \qquad \lambda = +\frac{1}{2}$$

$$\kappa(h_\alpha, h_\alpha) = \lambda^2(-2) = +\frac{1}{2}$$

$$h_\alpha = \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

More complicate example: $su(3;\mathbb{C})$

$su(3)$: 3×3 traceless Hermitian (Gell-Mann matrices)

$$\lambda_1 = \begin{pmatrix} 0&1&0\\1&0&0\\0&0&0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0&-i&0\\i&0&0\\0&0&0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1&0&0\\0&-1&0\\0&0&0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0&0&1\\0&0&0\\1&0&0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0&0&-i\\0&0&0\\i&0&0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0&0&0\\0&0&1\\0&1&0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0&0&0\\0&0&-i\\0&i&0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1&0&0\\0&1&0\\0&0&-2 \end{pmatrix}$$

$$[\lambda_p, \lambda_q] = i\,2\,f_{pqr}\,\lambda_r$$

pqr	123	147	156	246	257	345	367	458	678
f_{pqr}	1	$1/2$	$-1/2$	$1/2$	$1/2$	$1/2$	$-1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$

f_{pqr} totally antisymmetric

math conventions $a_p = i\lambda_p$

$$\kappa(a_p, a_q) = \pm 12\, \delta_{pq} \quad \binom{\text{phys}}{\text{math}}$$

Cartan subalgebra λ_3 and λ_8

λ_p are linearity independent over $\mathbb{C}$; $\dim(su(3)) = 8$

<u>roots</u>: $\lambda_1 \pm i\lambda_2$; $\lambda_6 \pm i\lambda_7$; $\lambda_4 \pm i\lambda_5$

$$[h_1, (\lambda_1 \pm i\lambda_2)] = \pm 2(\lambda_1 \pm i\lambda_2)$$
$$[h_2, (\lambda_1 \pm i\lambda_2)] = 0$$

$$[h_1, (\lambda_6 \pm i\lambda_7)] = \mp (\lambda_6 \pm i\lambda_7)$$
$$[h_2, (\lambda_6 \pm i\lambda_7)] = \pm \sqrt{3}\, (\lambda_6 \pm i\lambda_7)$$

$$[h_1, (\lambda_4 \pm i\lambda_5)] = \pm (\lambda_4 \pm i\lambda_5)$$
$$[h_2, (\lambda_4 \pm i\lambda_5)] = \pm (\lambda_4 \pm i\lambda_5)$$

<u>roots</u>: $\alpha_1 = (2, 0)$; $\alpha_2 = (-1, \sqrt{3})$; $\alpha_3 = (1, \sqrt{3})$

note $\alpha_1 + \alpha_2 = \alpha_3$

$h_{\alpha_1}, h_{\alpha_2}, h_{\alpha_3}$; $\kappa(h_1, h_1) = \kappa(h_2, h_2) = +12$

$$h_{\alpha_1} = \mu_1 h_1 + \mu_2 h_2 \qquad \kappa(h_{\alpha_1}, h_1) = \alpha_1(h_1)$$
$$\kappa(h_{\alpha_1}, h_2) = \alpha_1(h_2)$$

$$\mu_1 \kappa(h_1, h_1) + \mu_2 \kappa(h_2, h_1) = 2$$
$$\mu_1 \kappa(h_1, h_2) + \mu_2 \kappa(h_2, h_2) = 0$$

$\Rightarrow \mu_2 = 0 \quad \mu_1 = \frac{1}{6}$

$$h_{\alpha_1} = \frac{1}{6}\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix} \qquad h_{\alpha_2} = \frac{1}{6}\begin{pmatrix}0&0&0\\0&1&0\\0&0&-1\end{pmatrix}$$

$$h_{\alpha_3} = \frac{1}{6}\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}$$

$$h_{\alpha_1} + h_{\alpha_2} = h_{\alpha_3}$$

$$\langle \alpha_1, \alpha_1 \rangle = \frac{1}{3} \qquad \langle \alpha_2, \alpha_2 \rangle = \frac{1}{3}$$

$$\langle \alpha_1, \alpha_2 \rangle = -\frac{1}{6} \qquad \langle \alpha_2, \alpha_1 \rangle = -\frac{1}{6}$$

For further notice $\dfrac{2\langle \alpha_1, \alpha_2 \rangle}{\langle \alpha_1, \alpha_1 \rangle} = -1$

("canonical" normalization of Killing form: $\|\alpha\| = 2$ for long roots

$\exists$ representation

$$\mathrm{tr}(t_p, t_q) = 2h^\vee \kappa^{\text{canonical}}_{pq}$$

$h^\vee$ ... "dual Coxeter number")

Now since the Killing form κ is non-degenerate on $\mathcal{H}$ we can chose generators H_i such that, $H_i^\dagger = H_i$

$$\kappa(H_i, H_j) = \delta_{ij}$$

$$[H_i, E_\alpha] = \alpha_i E_\alpha \qquad \forall \alpha \in \Delta$$

think of roots as "vectors"

(note $H_i^\dagger = H_i$ unitary representation

$\Rightarrow \quad E_\alpha^\dagger = E_{-\alpha}$)

$h_\alpha \cdots \alpha_i H_i$

$H_\alpha = 2 \frac{\alpha_i H_i}{\alpha^2}$

$$\boxed{\begin{aligned} [H_\alpha, E_{\pm\alpha}] &= \pm 2 E_{\pm\alpha} \\ [E_\alpha, E_{-\alpha}] &= H_\alpha \end{aligned}}$$

A_1 - subalgebra for each root $\alpha \in \Delta$
(physics $su(2)$ sub algebra)

Note here $H_\alpha \simeq 2 A_3$ operator

$\Rightarrow$ eigenvalues of H_α are integers!

Proof: $tr([E_\alpha, E_{-\alpha}], h_\beta) = tr(E_{-\alpha}[h_\beta, E_\alpha]) = \langle\alpha,\beta\rangle\, tr(E_\alpha E_{-\alpha})$
$= tr(h_\alpha, h_\beta)\, tr(E_\alpha, E_{-\alpha})$

$\Rightarrow tr[([E_\alpha, E_{-\alpha}] - \underbrace{tr(E_\alpha, E_{-\alpha})}_{\text{normalise to } 2/\langle\alpha,\alpha\rangle} h_\alpha), h_\beta] = 0$

Master formula

Def: α-string of roots through β

all roots $\beta + k\alpha$ $\quad k \in \mathbb{Z}$

$\rightarrow$ correspond to an $su(2)$ representation of $(H_\alpha, E_{\mp\alpha})$; this will be characterized by some highest weight j (Note integer more since

$$[E_\alpha, E_\beta] = E_{\alpha+\beta}$$

$$[E_\alpha, \dots [E_\alpha, E_\beta]] = E_{p\alpha+\beta}$$

suppose $E_{p\alpha+\beta}$ is the highest weight

$$[H_\alpha, E_{p\alpha+\beta}] = 2p + \frac{2(\alpha\cdot\beta)}{\alpha^2} = j$$

similarly

$$[E_{-\alpha}, E_\beta] = E_{\beta-\alpha}$$

and so on until we reach $-j$

$$\exists\, q \in \mathbb{Z}: \quad -2q + \frac{2\langle\alpha,\beta\rangle}{\alpha^2} = -j$$

$$\Rightarrow \quad \boxed{p - q + \frac{2\langle\alpha,\beta\rangle}{\alpha^2} = 0}$$

Theorem $\frac{2\langle\alpha,\beta\rangle}{\alpha^2}$ can take only the values

$0, \pm 1, \pm 2, \pm 3$

$\langle \alpha, \beta \rangle$ is inner product therefore

$$|\langle \alpha, \beta \rangle|^2 \leq \langle \alpha^2 \rangle \langle \beta^2 \rangle$$ Schwartzian inequality

$$\Rightarrow \left| \frac{2\langle \alpha, \beta \rangle}{\alpha^2} \right| \left| \frac{2\langle \alpha, \beta \rangle}{\beta^2} \right| \leq 4 \quad \text{if } \alpha \neq \beta$$

$$\frac{2\langle \alpha, \beta \rangle}{\alpha^2} \; \dots \; 0, \pm 1, \pm 2, \pm 3$$

angle:

$$\frac{2\langle \alpha, \beta \rangle}{\alpha^2} = -(p-q)$$

$$\frac{2\langle \beta, \alpha \rangle}{\beta^2} = -(p'-q')$$

$$\Rightarrow \frac{\langle \alpha, \beta \rangle^2}{\alpha^2 \beta^2} = \frac{(p-q)(p'-q')}{4}$$

$$\left(\cos \Theta_{\alpha\beta} \right)^2 = \frac{(p-q)(p'-q')}{4}$$

since $(p-q)(p'-q')$ is an ~~integer~~ positive integer

- $(p-q)(p'-q') = 0 \Rightarrow \Theta_{\alpha\beta} = \pi/2 \; (3\pi/2)$ (roots are unique) (π is a $-\alpha$)
- $(p-q)(p'-q') = 1$ $\quad \pi/3, \; 2\pi/3$
- $(p-q)(p'-q') = 2$ $\quad \pi/4, \; 3\pi/4$
- $(p-q)(p'-q') = 3$ $\quad \pi/6, \; 5\pi/6$
- Θ $\quad 4$ $\quad \Theta = 0 \quad (-\alpha)$

$$p-q, \; p'-q' \in 0, 1, 2, 3$$

Furthermore: $\beta - \frac{2\langle\alpha,\beta\rangle}{\alpha^2}\alpha$ is also a root!

Note β belongs to an $su(2)$ representation of $H_\alpha, E_{\pm\alpha}$

$$[H_\alpha, E_\beta] = \frac{2\langle\alpha,\beta\rangle}{\alpha^2} E_\beta$$

therfore $-\frac{2\langle\alpha,\beta\rangle}{\alpha^2}$ must also be an eigenvalue of $ad(H_\alpha)$ (Eigenvalus: $j, j-1, \dots, -j+1, -j$)

$\Rightarrow \exists\, k \in \mathbb{Z}$ $\quad -\frac{2\langle\alpha,\beta\rangle}{\alpha^2} = \frac{2\langle\alpha,\beta\rangle}{\alpha^2} + 2k$

after applying k-times E_α (or $E_{-\alpha}$)

$$\Rightarrow \quad k = -\frac{2\langle\alpha,\beta\rangle}{\alpha^2}$$

Simple roots: Let h_{α_i} be a basis for the Cartan subalgebra such that every root can be written as

$$\beta = \sum k_i \alpha_i$$

since very $h_\beta = \sum k_i h_{\alpha_i}$

$$K(h_{\alpha_j}, h_\beta) = \sum k_i (h_{\alpha_j}, h_i)$$

β is a root, eigenvalus of a Hermitian matrix

β_j are real $\Rightarrow$ k_i are real

(one can show that they a rational)

Positive roots: choose a basis h_{α_i} $i = 1..l$

the root β is called positive if the first non-vanishing coefficients $k_1, \dots, k_l$ is positive. $(\beta > 0)$

Example $A_2 \sim su(3)$ α_1, α_2 as basis

positive roots $\alpha_1, \alpha_2, \alpha_3$

one take $\alpha_1, \alpha_1+\alpha_2$ as basis

note that now $-\alpha_2$ is a positive root

$$-\alpha_2 = \underset{>0}{\underset{\uparrow}{\alpha_1}} - (\alpha_1+\alpha_2) = -\alpha_2$$

Notion of positive roots is basis dependent!

Useful: if $\alpha > 0$ and $\beta > 0$ the $\alpha+\beta > 0$

if $\alpha > 0$ then $-\alpha < 0$

$\Rightarrow$ $\mathcal{L}$ decomposes into $\Delta_+ \oplus \mathcal{H}_e \oplus \Delta_-$

Δ_+ ... set of positive roots

also if $(\alpha-\beta) > 0$ then we $\alpha > \beta$

Definition: ~~Pos.~~ Simple root:

α is called a simple root if $\alpha \in \Delta_+$ but $\alpha \neq \beta+\gamma$ with $\beta, \gamma \in \Delta_+$

Example: A_2: α_1, α_2 simple roots

$\alpha_1+\alpha_2$

$-\alpha_1, -\alpha_2$

$-(\alpha_1+\alpha_2)$

Theorem: If α, β are simple roots then

- $\alpha - \beta$ is not a root
- $\langle \alpha, \beta \rangle \leq 0$

Proof: -) suppose $\alpha - \beta$ is a pos. root; then $\alpha = (\alpha-\beta) + \beta$ so α cannot be a simple root, but α is a simple root by assumption $\Rightarrow (\alpha - \beta) \notin \Delta_+$

If $\alpha - \beta \in \Delta_- \Rightarrow \beta - \alpha \in \Delta_+ \Rightarrow \beta$ could not be a simple root

-) let $\beta - q\alpha, \ldots, \beta + p\alpha$ be the α-string through β. but $q = 0$ since ~~$\alpha-\beta$~~ $\beta - \alpha$ is not a root

$$p = -\frac{2\langle \beta, \alpha \rangle}{\alpha^2}$$

but α^2 is positive since α_i a real

$$\Rightarrow \quad \langle \alpha, \beta \rangle \leq 0$$

This means that the angle between two simple roots can only be

$$\Theta : \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}$$

also note that

$$\frac{2\langle \alpha, \beta \rangle}{\alpha^2} = q - p \qquad \frac{2\langle \alpha, \beta \rangle}{\beta^2} = q' - p'$$

$$\Rightarrow \boxed{\frac{\alpha^2}{\beta^2} = \frac{q'-p'}{q-p}}$$

Rem: $q'-p', q-p \in \{1, 2, 3\}$

the only possible ratios of root-lengths ar $1, \sqrt{2}, \sqrt{3}$

It follows more that every root $\in \Delta_+$ can be written as

$$\beta = \sum q_i \alpha_i$$

q_i ... non-negative integer

$H_{\alpha_i}, E_{\pm\alpha_i}$: ~~decomposes~~ decomposes the algebra in representatives ~~of~~ of $su(2)^\ell$

-) $2\alpha_i$ is not a root because $H_{\alpha}, E_{\pm\alpha}$ is a spin 1-representation (adjoint)

-) If α, β are simple roots then $\alpha+\beta$ is a root only if $\langle \alpha, \beta \rangle \neq 0$

$$[E_\alpha, E_\beta] = E_{\alpha+\beta}$$

$$[H_\alpha, [E_\alpha, E_\beta]] = [[H_\alpha, E_\alpha], E_\beta] + [E_\alpha, [H_\alpha, E_\beta]]$$

$$= 2 [E_\alpha, E_\beta] + \frac{2\langle \alpha, \beta \rangle}{\alpha^2} [E_\alpha, E_\beta]$$

so if $\frac{\langle \alpha, \beta \rangle}{\alpha} = 0$ it would mean that

$H_\alpha, E_{\alpha+\beta}$ is another $su(2)$ sub algebra.

$\Rightarrow$ $\alpha+\beta$ is not a root if $\langle \alpha, \beta \rangle = 0$

alternatives $q - p = \frac{2\langle \alpha, \beta \rangle}{\beta^2}$

$\alpha, \beta \neq 0$ but $\langle \alpha, \beta \rangle = 0 \Rightarrow q = p = 0$

0 is the symmetric point of each $su(2)$ representation $+j, \dots, -j$ $q = p$
symmetric point is an element of the Cartan subalgebra (zero-root)

Definition Cartan-matrix

$l \times l$ matrix A_{jk}

$$\boxed{A_{jk} := \frac{2\langle \alpha_j, \alpha_k\rangle}{\alpha_j^2}}$$

α_j, α_k simple roots

$A_{jk} \in 0, -1, -2, -3 \quad j \neq k$

a) equal length $\quad A_{jk} = 2\cos\left(\frac{2\pi}{3}\right) = -1$

•) ratio $\frac{1}{\sqrt{2}}$: $\quad A_{jk} = 2\cos\left(\frac{3\pi}{4}\right)\frac{l_j l_k}{l_j^2} = -2\frac{1}{\sqrt{2}} \begin{cases} \sqrt{2} \\ 1/\sqrt{2} \end{cases}$

see we get -1 if j is the long root

or -2 if j is the short root

•) ratio $\frac{1}{\sqrt{3}}$ $\quad A_{jk} = -1$ if j is the long root

$A_{jk} = -3$ if j is the short root

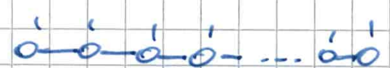
It is possible to construct all admissable Cartan-matrices $\to$ classification of all simple (complex) Lie-algebras (and therefore of the connected part of Lie-groups)

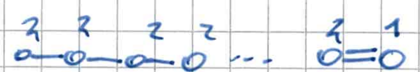
The Cartan matrix is encoded in a Dynkin diagram.

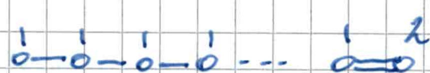
The whole Lie-algebra can be deduced from the Dynkin diagram

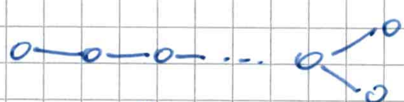
- Each simple root is a dot
- ~~dots~~ dots j and k are connected by $|A_{jk}|$ lines
- length of the roots is indicated by an index

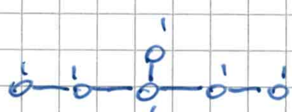
It turns out that there are 4 infinite series and 5 exceptional

A_l: 

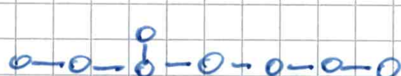
B_l: 

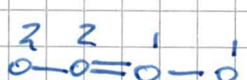
C_l: 

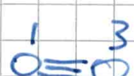
D_l: 

E_6: 

E_7: 

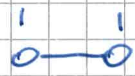
E_8: 

F_4: 

G_2: 

A, D, E are calle "simly laced"

all roots of qual length

Example: A_2 

2×2 matrix $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

A_3  $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$

B_2 :  $\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$

A_{12}

C_2 : $\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$

G_2 : $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$

Constructing the roots from the Cartan matrix:

A_3 : $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$ 3 simple roots $\alpha_1, \alpha_2, \alpha_3$

simple roots "level 1"

consider the α_2- string containing α_1

$\alpha_1 + r\alpha_2$:

$$(q-p) = \frac{2\langle\alpha_1, \alpha_2\rangle}{\alpha_2^2} = -1 = A_{21}$$

note $\alpha_1 - \alpha_2$ is not a root $\Rightarrow q = 0$

$\Rightarrow p = 1$

only $\alpha_1 + \alpha_2$

α_2 string containing α_1 is $(\alpha_1, \alpha_1 + \alpha_2)$

α_3-string containing α_1 $\quad \alpha_1 + r\alpha_3$

$$(q-p) = \frac{2(\alpha_1,\alpha_2)}{\alpha_1^2} = 0 \quad , \quad q = 0$$

$$\Rightarrow p = 0$$

α_3-string containing α_1 is α_1

α_3-string cont. α_2 : $\quad \alpha_1 + \alpha_2$

of level 2 $\quad \alpha_1 + \alpha_2$; $\alpha_2 + \alpha_3$

α_3-string through $\alpha_1 + \alpha_2$: $\quad \alpha_1 + \alpha_2 + r\alpha_3$

$$q - p = 2\frac{(\alpha_3, \alpha_1 + \alpha_2)}{\alpha_3^2} = A_{31} + A_{32} = -1$$

$\alpha_1 + \alpha_2 - \alpha_3$ is not a positive nor a negative root $\Rightarrow$ not a root $\quad q = 0 \quad p = 1$

$\Rightarrow \quad (\alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3)$ is the α_3-string containing $\alpha_1 + \alpha_2$

positives	$\alpha_1, \alpha_2, \alpha_3$	level 1
	$\alpha_1+\alpha_2 \quad \alpha_2+\alpha_3$	level 2
	$\alpha_1+\alpha_2+\alpha_3$	level 3

Roots ~~and~~ $su(3)$-diagram

for a particular root $\beta = \sum k_i \alpha_i$

$$q^i - p^i = 2\,\frac{\langle \beta, \alpha^i \rangle}{\|\alpha^i\|^2} = \sum k_j A_{ji}$$

$$[H_{\alpha_i}, E_{\alpha_j}] = 2\,\frac{\langle \alpha_i\, \alpha_j \rangle}{\|\alpha_i\|^2} E_{\alpha_j}$$

↳ Eigenvalue of H_{α_i} =

= entry in Cartan matrix

A_2 $(su(3))$:
$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \Rightarrow \begin{array}{ll} \alpha_1 = (2\ \ {-1}) & 1^{st} \text{ root} \\ \alpha_2 = (-1,\ 2) & 2^{nd} \text{ root} \end{array}$$

meaning: $(2 \quad -1)$

highest weight of α_1 - $su(2)$ subalgebra

lowest weight of α_2 - subalgebra

$$\begin{tikzcd}
 & (1,1) & \\
(2\ \ {-1}) \arrow[ur, "E_{\alpha_2}"] \arrow[dr, "E_{\alpha_1}"'] & & (-1\ \ 2) \arrow[ul, "E_{\alpha_1}"'] \arrow[dl, "E_{-\alpha_2}"] \\
 & (0,0)^2 \arrow[dl] \arrow[dr, "E_{-\alpha_1}"] & \\
(1,-2) \arrow[dr, "E_{-\alpha_1}"'] & & (-2,1) \arrow[dl, "E_{-\alpha_2}"] \\
 & (-1,-1) &
\end{tikzcd}$$

8 states =

$N^2 - 1$ for $N = 3$

root diagram.

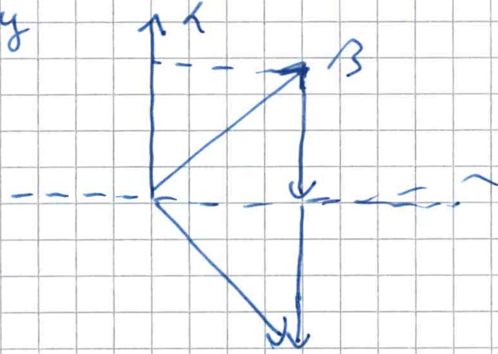
The Weyl group

Def.: $$S_\alpha(\beta) = \beta - \frac{2\langle\beta,\alpha\rangle}{\alpha^2}\alpha$$

is called a Weyl reflection on α

α, β are roots

gemetrically



reflection on the hyperplane orthogonel to α

- $S_\alpha(\alpha) = \alpha - 2\alpha = -\alpha$
- $S_\alpha(S_\alpha(\beta)) = S_\alpha\left(\beta - \frac{2\langle\beta,\alpha\rangle}{\alpha^2}\alpha\right) = \beta - \cancel{\frac{2\langle\beta,\alpha\rangle}{\alpha^2}} + \cancel{\frac{2\langle\beta,\alpha\rangle}{\alpha^2}}\alpha$

  $S_\alpha^2 = 1$
- $\langle S_\alpha(\beta), S_\alpha(\gamma)\rangle = \langle\beta,\gamma\rangle$

  $$\left\langle \beta - \frac{2\langle\beta,\alpha\rangle}{\alpha^2}\alpha, \gamma - \frac{2\langle\gamma,\alpha\rangle}{\alpha^2}\alpha \right\rangle =$$

  $$= \underline{\langle\beta,\gamma\rangle} - 2\cancel{\frac{\langle\beta,\gamma\rangle}{\alpha^2}\langle\alpha,\gamma\rangle} - \cancel{\frac{2\langle\gamma,\alpha\rangle}{\alpha^2}}\langle\beta,\alpha\rangle + \cancel{\frac{4\langle\gamma,\alpha\rangle\langle\beta,\alpha\rangle}{\alpha^4}}\alpha^2$$

Weyl group is a finite group generated by S_{α_i} where α_i are the simple roots

$$S_{\alpha_j}(\alpha_i) = \alpha_i - A_{ji}\alpha_j$$

Real Lie-algebras

Def: A Lie-algebra is called "compact" if its Killing form is negative definite.

$$B(a_p, a_q) < 0 \quad \forall\, a_p, a_q \in \mathcal{L}$$

example $su(2)$

$$\underset{a_1}{\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}, \quad \underset{a_2}{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}}, \quad \underset{a_3}{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}}$$

$$tr\left[\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right] = tr\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -2 \qquad = B(a_1 a_1)$$

$$tr\left[\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\right] = tr\begin{pmatrix} 0 & & \\ & 0 & \\ & & 0 \end{pmatrix} = 0 \qquad = B(a_1 a_2)$$

$$\vdots$$

$$\cancel{tr(ad\,a)} \quad B(a_p, a_q) = -2\,\delta_{pq}$$

→ Now it is possible to go to a basis in which

$$B(a_p, a_q) = -\delta_{pq}$$

orthonormal basis

This has the important consequence that we the structure constants are totally anti-symmetric

Proof: $B([a_p, a_q]\, a_s) = -B(a_q\, [a_p, a_s])$

$$Tr(a_p a_q a_s - a_q a_p a_s) = Tr(a_q a_s a_p - a_q a_p a_s)$$

$$c_{pq}{}^{r} B(a_r a_s) = -c_{ps}{}^{r} B(a_q a_r) =$$

$$= - c_{pqs} = + c_{psq}$$

In the orthonormal basis the structure constants

•) $c_{pqr} = - c_{qpr}$

• $c_{pqr} = - c_{prq}$

strictly speaking only in the orthonormal basis of a compact Lie-algebra

(Physics $[t_a, t_b] = i f_{abc} t_c$

assume f_{abc} totally anti-symmetric

t_a hermitian)

the adjoint representation is given by a set of anti-symmetric matrices

$$\mathrm{ad}(a_p)_{rs} = c_{rps} = - c_{rsp} = c_{srp} = - c_{spr}$$

Theorem (Weyl) A (semi-) simple Lie group is compact if and only if its corresponding Lie-algebra is compact

(Remember the Lie-algebra is always non-compact as vector space $\mathbb{R}^n$)

Sketch of proof: since the adjoint representation is given by anti-symmetric matrices it is a sub group of $SU(N)$ $\Rightarrow$ Lie algebra are (anti-symmetric = anti-hermitian) adjoint representation is faithful

Conversely: a compact Lie-group has unitary representations (all representations are equivalent to unitary ones)

$g \in G$ $\quad$ g is unitary $m \times m$ matrix

$a \in \mathcal{L}_G$ $\quad$ is antihermitian

Since G is (semi)-simple all $a \in \mathcal{L}$ can be expressed as a commutator of two other elements of $\mathcal{L}$. $\Rightarrow$ ad $(\mathcal{L})$ are traceless matrices

traceless, antihermitian matrices = subalgebra of $su(m)$, killing form of $su(m)$ is negative definite (will be shown later)

Any two compact real forms of $\mathcal{L}$ are isomorphic "the (unique) compact real form of $\mathcal{L}$.

Representations of (semi-) simple Lie-algebras

So far we have based our discussion on the adjoint representation.
Now we want to study the other representations.

Suppose $\mathcal{H}$ acts on a vector space V, $\Psi \in V$

since $[H_i, H_j] = 0$ the H_i can be diagonalized simultaneously

$$\Rightarrow \qquad H_i \Psi = \lambda_i \Psi$$

$\lambda_i \dots$ are called weights

$$H_\alpha = 2 \frac{\vec{\alpha} \cdot \vec{H}}{\alpha^2} \qquad H_\alpha \Psi = \frac{2 \langle \lambda, \alpha \rangle}{\alpha^2} \Psi$$

Since $(H_\alpha, E_{\pm\alpha})$ form an $su(2)$ subalgebra it follows that

$$2 \frac{\langle \lambda, \alpha \rangle}{\alpha^2} \in \mathbb{Z}$$

in particular the smallest possible representation (non-trivial) is spin $\frac{1}{2}$ $(1, -1)$

Definition Fundamental weight

Suppose $\vec{\alpha}_i$ are simple roots
the $\vec{\lambda}_j$ is called a fundamental weight if

$$\boxed{2 \frac{\langle \vec{\lambda}_i, \alpha_j \rangle}{\alpha_j^2} = \delta_{ij}}$$

Let λ be a weight and $\lambda + k\alpha$ be in the α-string containing λ then there exist two positive integers p, q such that

$$\boxed{2\,\frac{\vec{\alpha}\cdot\vec{\lambda}}{\alpha^2} = q - p}$$

that $-q \leq k \leq p$

<u>Definition</u> "highest weight", suppose α_i are simple roots of a simple Lie algebra, λ is called a highest weight if $\lambda + \alpha_j$ is not a weight

φ_λ is the state corresponding to λ

$$\Rightarrow \quad \boxed{E_{\alpha_j}\,\varphi_\lambda = 0} \quad \forall \text{ simple roots } \alpha_j$$

highest weight state φ_λ

<u>Definition</u> Dynkin label ℓ^i

$$2\,\frac{\langle \alpha_i^\#, \lambda\rangle}{\alpha_i^2} = \ell^i \qquad \text{(integer)}$$

In terms of fundamental weights

$$\boxed{\lambda = \sum_{i=1}^{r} \ell^i \lambda^f_{(i)}}$$

the weights of the adjoint representation are just the roots.

The rows in the Cartan matrix are the Dynkin coefficients of the simple roots.

$$H_{\alpha_1} = \frac{1}{2}\lambda_3 + \frac{\sqrt{3}}{2}\lambda_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

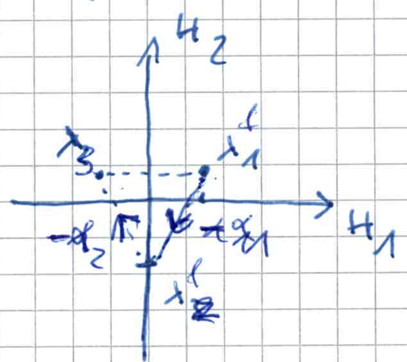
$$H_{\alpha_2} = \frac{1}{2}\lambda_3 - \frac{\sqrt{3}}{2}\lambda_8 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$H_{\alpha_3} = \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

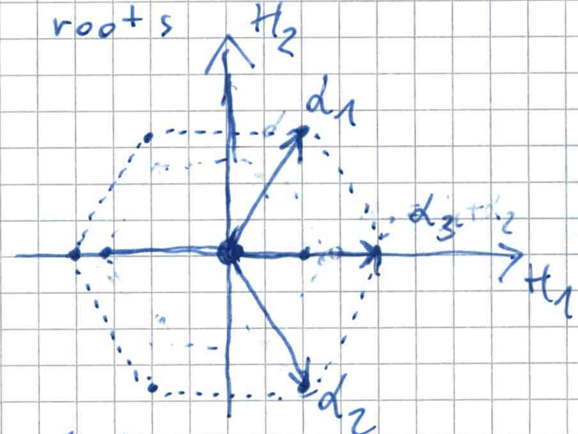
$$E_{\alpha_1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad E_{\alpha_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad E_{\alpha_3} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ highest weight $\quad E_{\alpha_1}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = E_{\alpha_2}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = E_{\alpha_3}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0$

weights



roots



$\lambda_1 = (1, \frac{1}{\sqrt{3}})$, $\lambda_2 = (0, -2/\sqrt{3})$; $\lambda_3 = (-1, \frac{1}{\sqrt{3}})$

root diagram:

eigenvalues under H_α's

$$\alpha_1 = (2, -1) \qquad \alpha_2 = (-1, 2)$$

$$\alpha_3 = \alpha_1 + \alpha_2 = (1, 1)$$

$$\begin{array}{ccc} & (1,1) & \\ \swarrow & & \searrow \\ (-1,2) & & (2,-1) \\ \searrow & & \swarrow \\ & (0,0)\,(0,0) & \\ \swarrow & & \searrow \\ (-2,1) & & (1,-2) \\ \searrow & & \swarrow \\ & (-1,-1) & \end{array}$$

We can do the same for the representation

fundamental: $(1,0)$ (Dynkin label)

$$\begin{array}{c} (1,0) \\ \downarrow {\scriptstyle -\alpha_1} \\ (-1,1) \\ \downarrow {\scriptstyle -\alpha_2} \\ (0,-1) \end{array}$$

(anti-) fundamental $(0,1)$

$$\begin{array}{c} (0,1) \\ \downarrow \\ (1,-1) \\ \downarrow \\ (-1,0) \end{array}$$

<u>Note:</u> if λ is a weight of the fundamental $(1,0)$ representation $-\lambda$ is a weight of the $(0,1)$ representation

"Complex representation"

compact real form $t_p^{\mathbb{R}} = i\, a_p^{\mathbb{R}}$

$$[t_p, t_q] = i\, c_{pqr}\, t_r$$

$$[t_p^*, t_q^*] = -i\, c_{pqr}\, t_r^*$$

and $$[-t_p^*, -t_q^*] = i\, c_{pqr}\, (-t_r^*)$$

$\Rightarrow$ $-t_p^*$ is another representation of the compact real form

We know already that the trace of the generators in the adjoint representation vanishes.

Now the trace of the generators vanishes in all representations

Note: $(H_\alpha, E_{\pm\alpha})$ form a su(2) subalgebra, but all representations of su(2) are traceless since the spins run from j to $-j$

$$\mathrm{tr}\left([E_\alpha, E_{-\alpha}]\right) = \mathrm{tr}\left([E_{-\alpha}, E_\alpha]\right) = \mathrm{tr}(H_\alpha) = 0$$

$$\mathrm{tr}\left([H_\alpha, E_{\pm\alpha}]\right) = \pm 2 E_{\pm\alpha} \Rightarrow \mathrm{tr}(E_{\pm\alpha}) = 0$$

Example $SU(3) = A_2$

su(3): traceless, (anti-) hermitian matrices

generated by the Gell-Mann matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} ; \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} ; \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} ; \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} ; \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} ; \quad \lambda_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \frac{1}{\sqrt{3}}$$

In physics $T_a = \frac{1}{2} \lambda_a$

$$\mathrm{tr}(T_a, T_b) = \frac{1}{2} \delta_{ab}$$

$$S = \frac{1}{2} \int \mathrm{tr}(F^a_{\mu\nu} F^{\mu\nu})$$

orthonormal

Carter-generators λ_3 and λ_8

$$H_1 = \frac{1}{\sqrt{2}} \lambda_3 \qquad H_2 = \frac{1}{\sqrt{2}} \lambda_8$$

Eigen systems

$$H_1: \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \right\}, \quad \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, -\frac{1}{\sqrt{2}} \right\}$$

$$H_2: \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{6}} \right\}, \quad \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{6}} \right\}; \quad \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, -\sqrt{\frac{2}{3}} \right\}$$

$$\lambda_1 = \left(\frac{1}{\sqrt{2}} \; \frac{1}{\sqrt{6}} \right); \quad \lambda_2 = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}} \right) \quad \lambda_3 = \left(0, -\frac{\sqrt{2}}{\sqrt{3}} \right)$$

$$\lambda_2 = \lambda_1 - \alpha_1 \qquad \alpha_1 = \lambda_1 - \lambda_2 = (\sqrt{2}, 0)$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{E_{-\alpha_1}} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad E_{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad E_{+1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\tilde{H}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad [\tilde{H}_1, E_1] = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 2E_1$$

$$\lambda_1 - \lambda_3 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}} + \frac{2}{\sqrt{6}} \right) = \left(\frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{\sqrt{2}} \right)$$

$$\lambda_2 - \lambda_3 = \left(-\frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{\sqrt{2}} \right)$$

positive roots

$$\underset{\alpha_1}{\left(\frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{\sqrt{2}} \right)}, \quad \underset{\alpha_2}{\left(\frac{1}{\sqrt{2}}, -\frac{\sqrt{3}}{\sqrt{2}} \right)}, \quad \underset{\alpha_3}{\left(\sqrt{2}, 0 \right)}$$

$$\alpha_1 + \alpha_2 = \alpha_3 \qquad \alpha_1 \cdot \alpha_2 = \frac{1}{2} - \frac{3}{2} = -1$$

$$\alpha_1^2 = \alpha_2^2 = \alpha_3^2 = 2 \qquad \frac{\alpha_1 \cdot \alpha_2}{\alpha_1^2} = -\frac{1}{2}$$

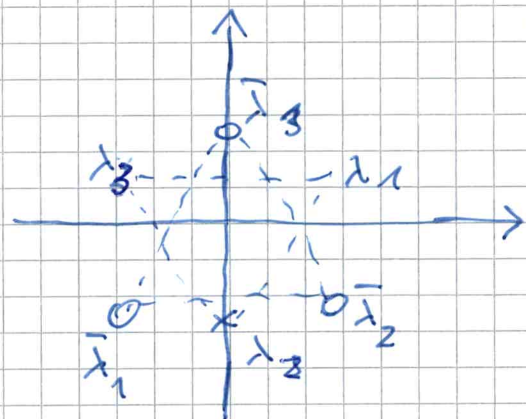
Since the Cartan-generators are diagonal and real (hence hermitian) it follows that if

$$H_i \, \Psi = \lambda_i \, \Psi$$

$$-H_i \, \Psi^* = (-\lambda_i) \, \Psi^* \qquad H_i^* = H_i$$

$\Rightarrow$ if λ is a weight of the representation R, $-\lambda$ is a weight of the representation R^*

su(3)



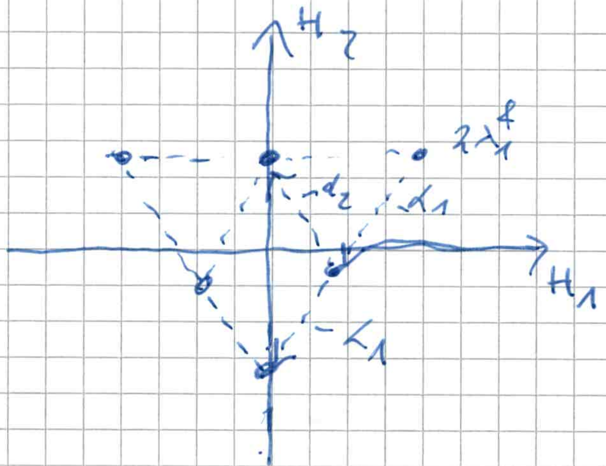
__Note:__ the adjoint representation is "real": it always contains the positive and negative roots.

All representations of the form (n,n) are real (for $SU(3)$)

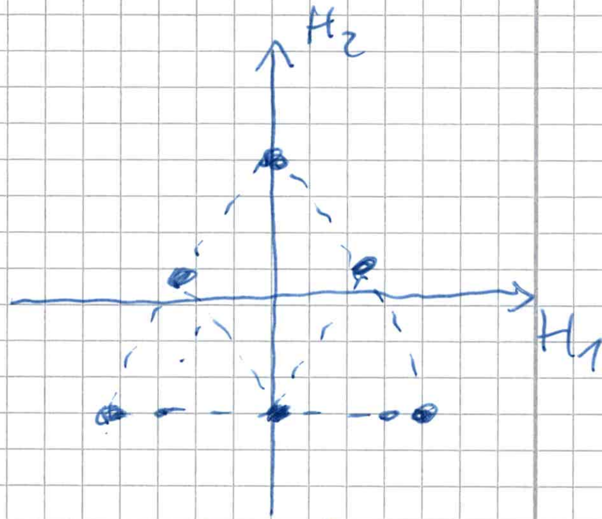
Some other representation

$(2\ 0)$	$(0\ 2)$
↙	↘
$(0,1)$	$(1,0)$
↙ ↘	↙ ↘
$(-2,2)$ $(1,-1)$	$(-1,1)$ $(2,-2)$
↘ ↙	↘ ↙
$(-1,0)$	$(0,-1)$
↘	↙
$(0\ -2)$	$(-2,0)$
6	$\bar 6$

weight diagrams



(2, 0)



(0, 2)

Problem (3,0) representation = Decuplet

find an application in particle physics

The correspondence of the classical groups to the Dynkin diagrams

SU(N) group $g = e^{a}$ $\det g = 1 \Rightarrow \operatorname{tr} a = 0$; $a^\dagger = -a$

defining property: $N \times N$ anti-hermitian traceless matrices

Basis of matrices e.g. $\left.\begin{array}{l} (e_{\ell m})_{ij} = (\delta_{\ell j}\delta_{mi} - \delta_{\ell i}\delta_{mj}) \\ (\hat{e}_{\ell m})_{ij} = i(\delta_{\ell j}\delta_{mi} + \delta_{\ell i}\delta_{mj}) \end{array}\right\}$ $\ell \neq m$

diagonal

$$H_1 = \frac{1}{\sqrt{2}}(1, -1, 0, \dots, 0)$$

$$H_2 = \frac{1}{\sqrt{6}}(1, 1, -2, \dots, 0)$$

$$H_3 = \frac{1}{\sqrt{N(N+1)}}(1, 1, 1, -3, 0 \dots, 0)$$

$$\vdots$$

$$H_{N-1} = \frac{1}{\sqrt{N(N-1)}}(1, 1, \dots, -N+1)$$

$\operatorname{tr}(H_i H_j) = \delta_{ij}$ orthonormal basis

eigen vectors $\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix}, \dots \begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix}$

$$w^{(1)} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{12}}, \dots, \frac{1}{\sqrt{N(N-1)}}\right) \quad (N-1)\text{-vector}$$

$$w^{(2)} = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{12}}, \dots, \frac{1}{\sqrt{N(N-1)}}\right)$$

$$w^{(3)} = \left(0, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{12}}, \dots, \frac{1}{\sqrt{N(N-1)}}\right)$$

$$\vdots$$

$$w^{(N)} = \left(0, \dots\dots, \frac{-N+1}{\sqrt{N(N-1)}}\right)$$

In total these are $\frac{1}{2}N(N-1) + \frac{1}{2}N(N+1) - N + N - 1 = \underline{N^2 - 1}$ matrices. $N \times N$ − 1 constraint $\operatorname{tr}(a) = 0$

Still after normalization the length of the weight vectors is

$$w^{(i)\,2} = \frac{N-1}{N}$$

easy: $w^{(N)\,2} = \frac{N-1}{N}$

Sum: $\sum_{n=1}^{N-1} = \frac{N-1}{N}$

Proof: induction $N \to N+1$ $\quad \frac{N-1}{N} + \frac{1}{N(N+1)} = \frac{N}{N+1}$ ✓

<u>roots</u> $\quad w^{i} - w^{i+1} = \alpha^{(i)}$

$$\alpha^{(1)} = (\sqrt{2}, 0, 0, 0 \dots)$$

$$\alpha^{(2)} = \left(-\tfrac{1}{\sqrt{2}}, \tfrac{\sqrt{3}}{\sqrt{2}}, 0, \dots 0\right)$$

we can declare them to be positive if we count from behind

etc...

$\alpha^{(1)\,2} = 2$; $\alpha^{(2)\,2} = \frac{1}{2} + \frac{3}{2} = 2$

$$\alpha^{(1)} \cdot \alpha^{(2)} = -1$$

how is this in general?

$$\omega^{(N)} = \left(0, \dots, 0, \frac{-N+1}{\sqrt{N(N-1)}}\right)$$

$$\omega^{(N-1)} = \left(0, \dots, 0, \frac{-N+2}{\sqrt{(N-1)(N-2)}}; \frac{1}{\sqrt{N(N-1)}}\right)$$

$$\alpha^{(N-1)} = \left(\underbrace{0,0,\ 0}_{N-2}, \frac{-\sqrt{N-2}}{\sqrt{N-1}}, \frac{+\sqrt{N}}{\sqrt{N-1}}\right)$$ "last root"

$$\alpha^{(N-1)2} = \frac{N-2}{N-1} + \frac{N}{N-1} = \frac{2(N-1)}{N-1} = 2$$

In general:

$$\omega^{(i)} = \left(\underbrace{0,0,0}_{i-2}, \frac{-\sqrt{i-1}}{\sqrt{i}}, \frac{1}{\sqrt{i(i+1)}}, \dots\right)$$

$$\omega^{(i+1)} = \left(\underbrace{0, \dots, 0}_{i-1}, \frac{-\sqrt{i}}{\sqrt{i+1}}, \dots\right)$$

$$\alpha^{(i)} = \left(\underbrace{0, \dots, 0}_{i-2}, -\frac{\sqrt{i-1}}{\sqrt{i}}, \frac{1+i}{\sqrt{i(i+1)}}, \dots\right) =$$

$$\alpha^{(i)} = \left(\underbrace{0, \dots, 0}_{i-2}, -\frac{\sqrt{i-1}}{\sqrt{i}}, \frac{\sqrt{i+1}}{\sqrt{i}}, 0, \dots\right)$$

$$\alpha^{(i)2} = \frac{i-1}{i} + \frac{i+1}{i} = 2$$

$$\alpha^{(i)} \cdot \alpha^{(j)} = 0 \qquad |i-j| > 1$$

$$\alpha^{(i)} \cdot \alpha^{(i+1)} = -\frac{\sqrt{i}}{\sqrt{i+1}} \frac{\sqrt{i+1}}{\sqrt{i}} = -1 \quad ✓$$

$\Rightarrow$ Cartan - matrix for $su(N)$: A_{N-1}

$$A_{ij} = \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & \\ & & & \ddots \end{pmatrix}$$

$$\boxed{\text{The Dynkin diagrams } A_{N-1} \text{ corresponds to } su(N)}$$

$\underline{SO(2n)}$: group $O^T = O^{-1} \Rightarrow a^T = -a$

how many matrices are there $\frac{1}{2}(2n)(2n-1) = n(2n-1)$

Carten basis $\begin{pmatrix} \sigma_2 & 0 & \\ 0 & 0 & \\ & & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & \\ 0 & \sigma_2 & 0 \\ & & \ddots \end{pmatrix}$, $\begin{pmatrix} 0 & & & \\ & 0 & & \\ & & \sigma_2 & \\ & & & \sigma_2 \end{pmatrix}$

block of 2×2 matrices $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

Eigenvectors of σ_2 are

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ \pm i \end{pmatrix} = \pm 1 \begin{pmatrix} 1 \\ \pm i \end{pmatrix}$$

$$(H_m)_{jl} = -i\left(\delta_{j,2m-1}\,\delta_{l,2m} - \delta_{j,2m}\,\delta_{l,2m-1}\right)$$

$(\pm e^{k})_j = \delta_{j,2k-1} \pm i\,\delta_{j,2k}$ $\qquad$ k-th pair of eigen vectors

m, k run from $1..n$; j, l run for $1..2n$

Remark on Chevalley basis

e.g. SU(3)

$$H_1 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} \quad ; \quad H_2 = \begin{pmatrix} 0 & 0 & \\ 0 & 1 & 0 \\ & 0 & -1 \end{pmatrix}$$

but now: $\operatorname{tr}(H_i H_j) \neq \delta_{ij} = g_{ij} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

so in order to define an inner product on root (weight) space we need the inverse metric

$$[H_i, E_\alpha] = \alpha_i E_\alpha \qquad \text{"lower" index}$$

$\langle \alpha, \beta \rangle = ?$ need to define $\langle \alpha, \beta \rangle = \alpha_i (g^{-1})^{ij} \beta_j$

$$g^{-1} = \frac{1}{3} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

weights are
$$w^{(1)} = (1,0)$$
$$w^{(2)} = (-1,1)$$
$$w^{(3)} = (0,-1)$$

$$\alpha^1 = w^1 - w^2 = (2,-1)$$
$$\alpha^2 = w^2 - w^3 = (-1,2)$$

$$(\alpha^1)^2 = (2,-1)\frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \frac{1}{3}(3,0)\cdot\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \underline{\underline{2}}$$

$$(\alpha^1 \cdot \alpha^2) = (2,-1)\frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \frac{1}{3}(3,0)\begin{pmatrix} -1 \\ 2 \end{pmatrix} = -1$$

It is also possible to emped the root system in a space of 1 higher dimension

$$\alpha^1 = \hat{e}^1 - \hat{e}^2 \qquad (\hat{e}^i, \hat{e}^j) = \delta^{ij}$$
$$\alpha^2 = \hat{e}^2 - \hat{e}^3$$
$$(\alpha^i)^2 = 2 \quad ; \quad (\alpha^i, \alpha^j) = -1$$
$$\vdots$$

all these are orthogonal to the vector $\sum_{i=1}^{N} \hat{e}^{(i)}$

$\Rightarrow$ roots lie in the hyperplane orthogonal to $\sum_{i=1}^{N} \vec{e}^{(i)}$

$$(H_m)_{ij}(\pm e^k)_j = i\left(\delta_{i,2m-1}\delta_{j,2m} - \delta_{i,2m}\delta_{j,2m-1}\right)\left(\delta_{j,2k-1} \pm i\,\delta_{j,2k}\right)$$

$$= i\left(\delta_{i,2m-1}\delta_{2m,2k-1} - \delta_{i,2m}\delta_{m,k} \pm i\,\delta_{i,2m-1}\delta_{k,m} \mp i\,\delta_{i,2m}\delta_{2m-1,2k}\right)$$

$\delta_{2m,2k-1} = 0$ (even - odd)

$$= \pm\left(\delta_{k,m}\delta_{i,2m-1} + i\,\delta_{k,m}\delta_{i,2m}\right) =$$

$$= \pm\,\delta_{k,m}(e^k)_i$$

⇒ the weight vectors are $w^k_\pm = \pm\delta_{km}$

<u>roots</u>: $w^k_\pm - w^l_\pm$ $\quad k \neq l$

$w^k_\pm + w$ $\quad$ etc

denote by e^j a unitvector in n-dimensions

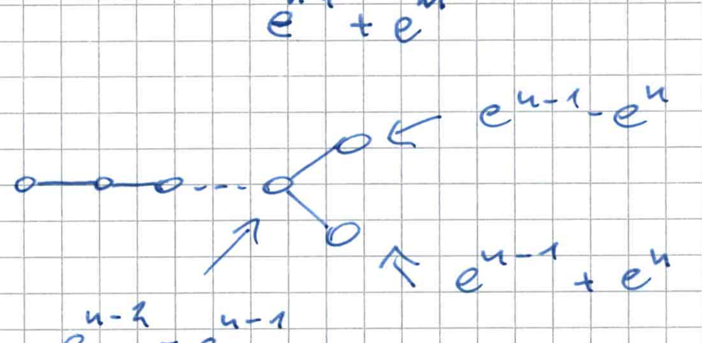
weights $\pm e^j$ $\quad j = 1..n$ $\quad (2n)$

roots $\pm e^j \pm e^k$ $\quad j \neq k$

positive roots $e^j \pm e^k$ $\quad j < k$

simple roots $e^j - e^{j+1}$ $\quad j = 1 .. n-1$

$e^{n-1} + e^n$



$$A = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & \ddots & & & \\ & & \ddots & & & \\ & & & 2 & -1 & -1 \\ & & & -1 & 2 & 0 \\ & & & -1 & 0 & 2 \end{pmatrix}$$

$\boxed{\text{The Dynkin diagram } D_n \text{ corresponds to the } SO(2n) \text{ group.}}$

SO(2n+1) rotations in odd dimensions

$$\begin{pmatrix} \sigma_2 & & \\ & 0 & \\ & & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & & 0 \\ 0 & \sigma_2 & \\ & & 0 \end{pmatrix}, \quad \ldots \quad \left(\begin{array}{c|c} \ddots \ \sigma_2 & \vdots \\ \hline \cdots & 0 \end{array}\right)$$

$\Rightarrow$ weights are $\pm \vec{e}^{\,k}$ $\quad k = 1..n$

and an additional zero weight!

roots: $\quad \pm e^k \pm e^l$ $\quad$ as before

but also $\quad \pm e^k$ $\quad$ because of the zero weight

simple roots $\quad e^j - e^{j+1}$ $\quad j = 1..n-1$

and e^n

this gives the Dynkin diagr

$$\underset{e^1-e^2}{\circ} - \circ - \circ - \circ = \underset{e^n}{\circ}$$

$$A = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & \ddots & & \\ & & & 2 & -1 \\ & & & -2 & 2 \end{pmatrix}$$

$|e^n| = 1$

"short root"

Symplectic groups

$$J = \begin{pmatrix} 0 & \mathbb{1}_N \\ -\mathbb{1}_N & 0 \end{pmatrix}$$

$$g^T J g = J$$

$USp(2M)$: $\quad g^+ = g^{-1}$

on the level of Lie-algebra

$$g = 1 + a$$

$$(1 + a^T) J (1 + a) = J$$

$$\boxed{a^T J + J a = 0}$$

and for $USp(2M)$ $\quad a^+ = -a$ antihermitian

$$a = \begin{pmatrix} P & W \\ -W^+ & Q \end{pmatrix} \qquad P^+ = -P \,;\, Q^+ = -Q$$

$$\begin{pmatrix} P^T & -W^* \\ W^T & Q^T \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} P & W \\ -W^+ & Q \end{pmatrix}$$

$$= \begin{pmatrix} +W^* & P^T \\ -Q^T & W^T \end{pmatrix} = \begin{pmatrix} W^+ & -Q^T \\ +P & W \end{pmatrix}$$

$\Rightarrow \quad W = W^T \qquad P^T = -Q$

$$\frac{n}{2}(n+1) + \frac{n^2}{2} - \left(n^2 + \frac{n}{2}\right) \text{ complex parameter}$$

$\Rightarrow$ in total $\quad n(2n+1)$ parametres

the algebra is generated by

$$A \otimes \mathbb{1}_2 \,,\; iS_1 \otimes \sigma_1 \,;\; iS_2 \otimes \sigma_2 \,;\; iS_3 \otimes \sigma_3$$

A .. antisymmetric, $S_{1,2,3}$ real symmetric parameters $\frac{n}{2}(n-1) + 3\frac{n}{2}(n+1) =$

$$= \frac{n^2}{2} - \frac{n}{2} + \frac{3n^2}{2} + \frac{3n}{2} = n(2n+1) \checkmark$$

Check that these indeed generate a ~~gr~~ algebra ←

Su 2×2 block notation

$$\begin{pmatrix} A + iS_3 & iS_1 + S_2 \\ iS_1 - S_2 & A - iS_3 \end{pmatrix}$$

Cartan-subalgebra: diagonal $\begin{pmatrix} u & 0 \\ 0 & -u^T \end{pmatrix}$

u is a $u(n)$ matrix! (not $su(n)$!)

So the Cartan subalgebra is

$$\left(\begin{array}{c|c} \begin{matrix} 1 & & \\ & 0 & \\ & & 0 \ddots \end{matrix} & \\ \hline & \begin{matrix} -1 & & \\ & 0 & \\ & & \ddots \end{matrix} \end{array}\right), \left(\begin{array}{c|c} \begin{matrix} 0 & & \\ & 1 & \\ & & 0 \ddots \end{matrix} & \\ \hline & \begin{matrix} 0 & & \\ & -1 & \\ & & 0 \ddots \end{matrix} \end{array}\right), \text{etc...}$$

weights: $\omega^{(1)} = \underbrace{(1, 0, 0 \ldots)}_{n}$

$\omega^{(n+1)} = (-1, 0 \ldots, 0)$

$\vdots$

$\omega^{(2n)} = (0 \ldots, -1)$

$2n$ weights in n-dimensional space

Note: $tr(H_i H_j) = \delta_{ij}$ already orthonormal basis

<u>roots:</u> $\hat{e}^i - \hat{e}^j$! $\quad i = 1..n-1$ like $su(n)$

and $\hat{e}^{(1)} - \hat{e}^{(n+1)} = 2\hat{e}^1, \; 2\hat{e}^2, \; \dots$

$\Rightarrow$ n-simple roots are $\hat{e}^i - \hat{e}^{i+1}$ $\quad i = 1..n-1$

$2\hat{e}^n$

<u>note:</u> $(\hat{e}^i - \hat{e}^{i+1})^2 = 2$ $\leftarrow$ short roots

$(2\hat{e}^n)^2 = 4$ $\leftarrow$ long root

Dynkin diagram $\circ - \circ - \circ \cdots \circ \Leftarrow \circ$

Cartan matrix $\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & & & \\ & & \ddots & & \\ & & & 2 & -2 \\ & & & -1 & 2 \end{pmatrix}$ " C_n "

$$\boxed{\text{The Dynkin diagram } C_n \text{ correspond to the } USp(2n) \text{ groups}}$$

Classical groups as "rotations"

Obviosly: $SO(2n)$ and $SO(2n+1)$ preserve the real norm $v_i w_i$ of two vectors in a real vector spas

Similarly: Let w_i, z_i be complex vectors and take the norm $\bar{w}_i^* z_i$ that is obviously preserved by unitary transformations

$$w^+ z \to w^+ U^+ U z \Rightarrow U^+ U = \mathbb{1}$$

So what about $USp(2n)$?

real $\to$ complex $\to$ quaternions

$$j_a : \quad j_a j_b = -\delta_{ab} + i\,\varepsilon_{abc}\, j_c$$

$$Q = A + \vec{B}\vec{j} \quad \text{"quaternion"}$$

obviously: $j_a = i\sigma_a$

quaternionic vector : vector of 2x2 matrices

$$Q^i = A^i \mathbb{1}_2 + i B^i_a \sigma_a \qquad i = 1..n$$

$$\bar{Q} = A^i \mathbb{1}_2 - i B^i_a \sigma_a$$

$$\bar{Q} Q = (A - i B^a \sigma_a)(A + i B^a \sigma_a) =$$

$$= A^2 - i \cancel{B^a \sigma_a A} + i \cancel{A B^a \sigma_a} + B^a B^b (\sigma_a \sigma_b + \cancel{i \varepsilon_{abc} \sigma_c})$$

$$= A^2 + (B^1)^2 + (B^2)^2 + (B^3)^2$$

$USp(2n)$ preserves $\bar{Q}^i Q^i$ = rotation in a quaternionic vector space.

Lie-algebra $so(1,3)$

$$g = e^{i\alpha M}$$

$$g^T \eta g = \eta$$

$$\Rightarrow \quad i\alpha M^T \eta - i\alpha \eta M = 0$$

$$M^T \eta = \eta M$$

$\Rightarrow$ ηM has to be antisymmetric

if A is antisymmetric then $M = \eta A$ is a generator of the Lorentz group

$$(A^{\mu\nu})_{\rho\sigma} = \delta^\mu_\rho \delta^\nu_\sigma - \delta^\mu_\sigma \delta^\nu_\rho$$

here $\mu\nu$ ... number the generators

$\rho\sigma$ ... are the metric indices

$$(M^{\mu\nu})^\rho{}_\sigma = \eta^{\rho\lambda} A_{\lambda\sigma} = \eta^{\rho\mu}\delta^\nu_\sigma - \eta^{\rho\nu}\delta^\mu_\sigma$$

$$[M^{\mu\nu}, M^{\lambda\kappa}] = (\eta^{\mu\rho}\delta^\nu_\sigma - \eta^{\nu\rho}\delta^\mu_\sigma)(\eta^{\lambda\sigma}\delta^\kappa_\alpha - \eta^{\kappa\sigma}\delta^\lambda_\alpha) -$$

$$-(\mu\nu \leftrightarrow \lambda\kappa)$$

$$= \eta^{\mu\rho}\eta^{\nu\lambda}\delta^\kappa_\alpha - \eta^{\mu\rho}\eta^{\nu\kappa}\delta^\lambda_\alpha - \eta^{\nu\rho}\eta^{\lambda\mu}\delta^\kappa_\lambda +$$

$$+ \eta^{\nu\rho}\eta^{\kappa\mu}\delta^\lambda_\alpha - (\dots)$$

$$\boxed{[M^{\mu\nu}, M^{\lambda\kappa}] = \eta^{\nu\lambda} M^{\mu\kappa} - \eta^{\nu\kappa} M^{\mu\lambda} - \eta^{\mu\lambda} M^{\nu\kappa} + \eta^{\mu\kappa} M^{\nu\lambda}}$$

a better way of organizing is

$$K^i = M^{0i} \qquad J^i = \frac{1}{2}\varepsilon^{ijk} M^{jk}$$

$$[K_i, K_j] = -i\varepsilon_{ijk} J_k$$
$$[J_i, J_j] = i\varepsilon_{ijk} J_k$$
$$[J_i, K_j] = i\varepsilon_{ijk} K_k$$

→ two boosts commute onto a rotation
(Physical effect: Thomas precession)
→ boost lies in Spin 1 representation
→ J_i ... generate an $SO(3)$ subgroup

In the complexified algebra define

$$A_i^{\pm} = J_i \pm i K_i$$

then $\quad [A_i^{\pm}, A_j^{\pm}] = i\varepsilon_{ijk} A_k^{\pm} \qquad [A_i^{+}, A_j^{-}] = 0$

$$so(1,3)^{\mathbb{C}} \simeq su(2) \otimes su(2)$$

representations are labelled by two copies of spins $(j, m \,; j' m')$

List the representations

•) $j = j' = 0$ trivial $\vec{J} = 0 \quad \vec{K} = 0$
scalar field $\phi(t, x)$
this does not mean that the Lorentz group does not act on ϕ it on means that

$$\phi'(x') = \phi(x)$$

$$\phi'(x) = \phi(\Lambda^{-1} x)$$

•) $j = \frac{1}{2}$, $j' = 0$ $\quad (\frac{1}{2}, 0)$

$$A^+_i = \frac{i}{2}\sigma_i \qquad A^- = 0$$

$$\frac{1}{2}(J_i + i K_i) = \frac{i}{2}\sigma_i \qquad J_i - i K_i = 0$$

$$J_i = i K_i$$

$$\Rightarrow \quad J_i = \frac{i}{2}\sigma_i \qquad K_i = \frac{1}{2}\sigma_i$$

group element $g_1 = e^{i(\varphi^i \sigma_i) + \theta^i \sigma_i} = e^{i(\vec{\varphi} - i\vec{\theta})\vec{\sigma}/2}$

$\vec{\varphi}$ .. rotation $\qquad \vec{\theta}$ .. boost $\qquad$ "left handed" spinor

•) $j = 0$; $j' = \frac{1}{2}$ $\quad (0, \frac{1}{2})$

by the same logic $\quad g_2 = e^{i(\vec{\varphi} + i\vec{\theta})\vec{\sigma}/2}$

$$g_2 = (g_1^{-1})^+$$

right-hande spinor
inverse adjoint repr.

Parity: $\vec{x} \to -\vec{x}$ $\Rightarrow$ $\vec{K} \to -\vec{K}$

So parity exchanges right- and left handed spinors.

In physics a field theory with an asymmetry between left- and right handed spinors violates parity

define 1-form $\quad \phi^\alpha \psi_\alpha \in \mathbb{C}$

$$\phi^{\beta'} (g^{-1})_{\beta'}{}^{\alpha} g_\alpha{}^\beta \psi_\beta =$$

ϕ^β transforms as $(g^{-1})^T$

Herlitz

4 objects ψ_α ; $\bar{\psi}^{\dot\alpha}$, ϕ^α , $\phi_{\dot\alpha}$

convention $\psi_\alpha \phi^\alpha$, $\bar{\psi}^{\dot\alpha} \bar{\phi}_{\dot\alpha}$

$$\phi^\alpha = \varepsilon^{\alpha\beta} \phi_\beta \qquad \bar{\phi}_{\dot\alpha} = \varepsilon_{\dot\alpha\dot\beta} \bar{\phi}^{\dot\alpha}$$

here → insert van de Waerden notation

⋮ from SUSY - lectures

Then Spinors in arbitrary dimensions