Outline MAV

Advanced Mathematics

A) Group Theory } K.L.

B) Differential Geometry

C) Statistics } Jose del Peso

Part A: Group Theory:

-) Introduction, concepts, structure

-) Lie-groups

-) Representations

-) Lie-Algebras

-) Classification

-) Lorentz-group, Spinors, ...

Books:

-) H. Georgi - Lie Algebras in Physics

-) A. Zee - Group theory in a nutshell

-) Jones - Group, Representations and physics

-) Cornwell - Group Theory in Physics (Vol 1, 2, 3 and abridged Introduction)

-) Exercise sheet

-) Exam Nov. 23

Part B:

-) Nakahara - Geometry, Topology and Physics

-) Nash, Sen - Topology and Geometry for Physicists

Lectures on Group Theory

In physics group theory is the theory of transformations of physical objects.

Example: rotations (multiplication, non-commutativity, topology = belt trick)

Mathematics group is an "algebraic structure" (a set with certain rules to manipulate elements of the set get other elements of the set)

Definition Group

A set $\mathcal{G}$ is called a "group" if the following axioms hold

(a) Multiplication $\forall g_1, g_2 \in \mathcal{G}$ $\quad g_1 \circ g_2 \in \mathcal{G}$

(b) Associativity $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$

(c) $\exists$ identity $e \in \mathcal{G}$: $e \circ g = g \circ e = g$

(d) $\exists$ inverse $g^{-1} \in \mathcal{G}$ $\quad g^{-1} \circ g = g \circ g^{-1} = E$

(only $e \circ g = g$ and $g^{-1} \circ g = e$ necessary!)

Examples:

(a) real numbers $r \in \mathbb{R}^+$ $\quad r_1 \cdot r_2 = r_3 \quad r_3 \in \mathbb{R}$

$(\mathbb{R}, \cdot)$

$(r_1 r_2) r_3 = r_1 (r_2 r_3)$

$1 = E$

$r^{-1} = \frac{1}{r}$

(b) $(\mathbb{Z}, +)$ n_1, n_2 $\quad n_1 + n_2 = n_3$ Integer

$0 = E$

$-n_1 = "n_1^{-1}"$ inverse

(c) $(\mathbb{Z}, -)$ is not a group

$n_1 - (n_2 - n_3) \neq (n_1 - n_2) - n_3$!

Expl: $SO(2)$ $\qquad O^T O = \mathbb{1}_2$

$$\Rightarrow \det(\theta) = +1$$

$$\theta = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}, \qquad \theta_1 \cdot \theta_2 = \theta_2 \theta_1$$

"continous" group

discrete group: Z_2: $\{1, -1\}$

$$Z_N = \{1, e^{i\frac{2\pi}{N}}, e^{i\frac{4\pi}{N}}, \dots, e^{i\frac{(n-1)2\pi}{N}}\}$$

all these groups are Abelian $\quad g_1 \circ g_2 = g_2 \circ g_1$

Non-abelian discrete group: (

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad M_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad M_3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$M_4 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad M_5 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad M_6 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$M_7 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad M_8 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

$$M_5 \cdot M_7 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = M_4$$

Exercise

$$M_7 \cdot M_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = M_2$$

check group properties

Def order = #(elements)

continous non-Abelian groups: expl $SO(3)$, $SU(N)$

<u>$SU(N)$</u>: $\det(U) = 1 \qquad U^\dagger U = \mathbb{1} \qquad N \times N$ matrices

- $U_1 U_2$: $\det(U_1)\det(U_2) = 1$ ✓
  $(U_1 U_2)^\dagger = U_2^\dagger U_1^\dagger \Rightarrow (U_1 U_2)^\dagger U_1 U_2 = \mathbb{1}$ ✓
- $\mathbb{1}_N \in SU(N)$ ✓
- $U^{-1} = U^\dagger$ ✓
- associativity of matrix-multiplication ✓

Additional

Proofs $e\circ g = g \Rightarrow g\circ e = g$

g^{-1} exists $\quad g_1\circ g_2 = g_3 \quad ? \quad g_3^{-1}$

$$g_3^{-1} = g_2^{-1}\circ g_1^{-1}$$

$$(e\circ g)^{-1} = e\circ g^{-1}$$

take $g^{-1} = h$ some group element

Sim.: $g\circ g^{-1} = e \Rightarrow g^{-1}\circ g = e \quad ((g^{-1})^{-1}\circ g^{-1} = e \quad g^{-1} = g' \in$

Expl.: Multiplication Table

$G = \{e, a, b\}$ $\quad a\circ a = b \quad a\circ b = b\circ a = e \quad b\circ b = a$

	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

$e = 1$ $\qquad$ $e = 1$

$a = e^{i2\pi/3}$ or $a = e^{i4\pi/3}$

$b = e^{i4\pi/3}$ $\qquad$ $b = e^{i2\pi/3}$

$a = b = e = 1$

(representations)

<u>Def.</u> Homomorphism: is a map from group $\mathcal{G}_1$ to group $\mathcal{G}_2$ which preserves the group structure

$$\phi : (\mathcal{G}, \circ) \to (H, *)$$

$$\phi(g_1 \circ g_2) = \phi(g_1) * \varphi(g_2)$$

If the homomorphism is one-to-one it is celld an isomorphism

<u>Expl.</u>: 1.) $(\mathbb{R}, +) \to (\mathbb{R}/\{0\}, \cdot)$

$$\downarrow$$
$$e^x$$

$$x + y \to e^{x+y} = e^x \cdot e^y \quad \checkmark$$

But not bijective: negative numbers are not in the Image of e^x

[$(\mathbb{R}, +) \to (\mathbb{R}_+, \cdot)$ is bijective]

2.) $(\mathbb{R}, +) \to (S^1, \cdot)$

$$e^{i\alpha x}$$

<u>Note</u> $(\mathcal{G}, \circ) \to \{E\}$ is a trivial homomorphism

$\{e\}$ .. trivial group

Definition Subgroup S : is a subset which by itself is a group under the group multiplication law

Def: "trivial" subgroups: $\mathcal{G}$, $\{E\}$

Subgroups which are not trivial are called "proper"

Theorem: Let S be a subset of $\mathcal{G}$. If $s' s^{-1} \in S$ for all $s', s \in S$ then S is a subgroup

Proof:
(c) $s' = s \Rightarrow s s^{-1} = E \Rightarrow E \in S$
(d) $s' = E \Rightarrow E s^{-1} = s^{-1} \Rightarrow s^{-1} \in S$
(a) $s^{-1} \in S \Rightarrow (s^{-1})^{-1} \in S \Rightarrow s'(s^{-1})^{-1} = s's \in S$

Rearrangement Theorem: For every fixed element $g' \in \mathcal{G}$ the sets $g' \circ g$ (and $g \circ g'$) $g \in \mathcal{G}$ contain all elements of $\mathcal{G}$ precisely once

Proof: • take $g'' \in \mathcal{G}$ and $g = (g')^{-1} \circ g'' \Rightarrow g' \circ g = g''$
$\Rightarrow$ all $g'' \in \mathcal{G} \in g' \circ g$ ✓

unique - uniqueness assume $g' \circ g_1 = g' \circ g_2$ $g_1 \neq g_2$
impossible $(g')^{-1} \circ \Rightarrow g_1 = g_2$

Multiplication Table

	g_1	g_2	g_3
g_1	g_3	g_1	g_2
g_2	g_2		
g_3	g_1		

← every element once

↑

→ Permutation group & Cayley's theorem

<u>Definition</u> conjugate elements: if $\exists\, x \in G$ for $g, g' \in G$ such that

$$g' = x \circ g \circ x^{-1}$$

g' conjugate to g (mutual)

If g conjugate to g' and g' to $g'' \Rightarrow g$ conj to g''

<u>Definition</u> Conjugacy class of h all elements g such that $g = x \circ h \circ x^{-1} \quad \forall x \in G$

<u>examples</u>
- $h = E$; conjugacy class is E itself
- Abelian group every element is a conjugacy class by itself

<u>Theorem</u>
- E is a class by itself
- every element is member of some conjugacy class
- No element can be member of two different classes

<u>Proof:</u>
- consider h $\quad g \circ h \circ g^{-1}$ with $g = h \Rightarrow$

$$h \circ h \circ h^{-1} = h \Rightarrow h \in [h]$$

- $g \in x \circ h_1 \circ x^{-1} \qquad g \in x \circ h_2 \circ x^{-1}$

$$g \in [h] \Rightarrow h_1 \in [g] \qquad h_2 \in [g]$$

$$[h_1] = [h_2]$$

Definition: Invariant subgroup S

$\forall s \in S$ and $\forall g \in G$: $g \circ s \circ g^{-1} \in S$

also "normal subrgroup"

Universal normal subgroups: G itself and $\{E\}$

Theorem: a subgroup S is normal if ~~it~~ consists of entierely of classes

-) if S is normal $\Rightarrow$ S is made up of classes

-) suppose S is a set consisting entirely of classes $\Rightarrow$ $\forall s \in S$ $\quad x \circ s \circ x^{-1} \in S$

$\Rightarrow$ S is invariant

Def. G is "simple" if its only normal subgroups are $\{e\}$ and G

Definition: Coset : pick an element $g \in G$ and a subgroup H

$gH := \{ g \circ h \mid \forall h \in H \}$

left coset of H with respect to g

$Hg := \{ h \circ g \mid \forall h \in H \}$

right coset of H with respect to g

Theorems a) If $g \in H$ then $Hh = hH = H$ (Rearrangement

b) If $g \notin H$ then gH is not a subgroup

- If gH is a subgroup it must contain the identity $e = gh \Rightarrow g^{-1} = h \Rightarrow g^{-1} \in H$ $\Rightarrow g \in H \Rightarrow$ if $g \notin H$ gH is not a subgroup

c) Every $g \in G$ is member of some right coset

H contains $E \Rightarrow g = g \circ E \in gH$

d) $gh_1 \neq gh_2$ different elements
If H is of finite order s gH contains s different elements

•) right cosets are either identical or contain no common elements

-) suppose $g_1 h_1 = g_2 h_2$ $h_1, h_2 \in H$

$\Rightarrow g_2^{-1} g_1 = h_2 h_1^{-1} = h' \in H$

$\Rightarrow g_2^{-1} g_1 H = H$

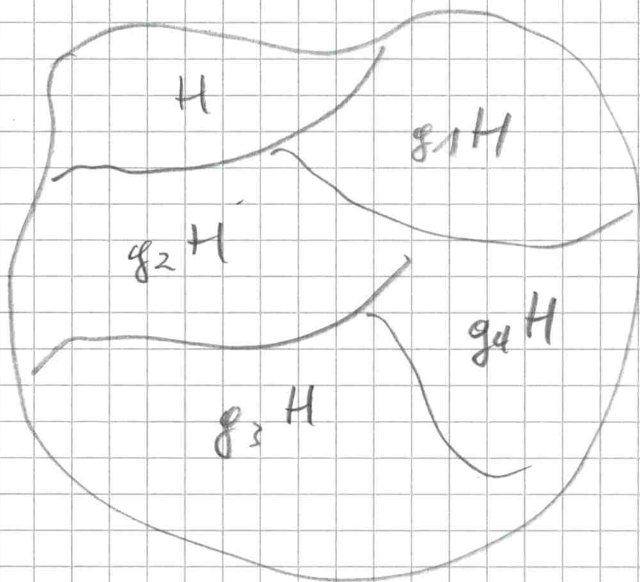
$\Rightarrow g_2 H = g_2 g_2^{-1} g_1 H = g_1 H$ qed

•) If $g' \in gH$ then $g'H = gH$

-) since $g'H = (gH)H = gH$
$g' = gh$ $g'h' = ghh' = gh'' \Rightarrow g'H = gH$

•) If G, H are finite of order r, s then the number of discrete cosets is r/s

Cosets form a pertition of G



Note: if r is prime: no proper subgroups ; $\mathbb{Z}_3$

Theorem: left and right cosets are identical iff H is normal

$$gH = Hg$$

$$gh_1 = h_2 g \iff g \circ h_1 \circ g^{-1} = h_2 \in H$$

Definition: Quotient Group (Foctor Group) G/H
If H is a normal subgroup the product on coset space

$$g_1 H \circ g_2 H = (g_1 \circ g_2) H$$

defines a group structure

Suppose g_1' is a diffent reprentative of $g_1 H$
$\exists h'$, $g_1' h' = g_1 h$, g_2' fcr $g_2 H$

$g_1' \circ g_2'$

$$\left.\begin{array}{l} g_1 h_1 \circ g_2 \circ h_2 = g_1 \circ g_2 \circ h_1 \circ h_2 \\ g_1' h_1' \circ g_2' \circ h_2' = g_1' \circ g_2' \circ h_1' \circ h_2 \end{array}\right\} (g_1 \circ g_2) H$$

Multiplication ✓

Associative: $g_1H\circ(g_2H\circ g_3H) = (g_1H\circ g_2H)\circ g_3H$ ✓

Identity: $E = H \qquad gH\circ H = gH \qquad H\circ g\circ H = gHH$

Inverse: $g^{-1}H\circ gH = (g^{-1}\circ g)H = H$ ✓

Example 1, $(\mathbb{R}^N,+)$: $\vec{v}\begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix}$ sub $\vec{v}+\vec{w}$

subgroup H: $(\mathbb{R}^P,+)$: $\begin{pmatrix} v_1 \\ \vdots \\ v_P \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$$(\mathbb{R}^N,+)/(\mathbb{R}^P,+) : \begin{pmatrix} 0 \\ 0 \\ 0 \\ v_{P+1} \\ \vdots \\ v_N \end{pmatrix} = \mathbb{R}^{N-P}$$

($G/H\times H = G$ in this case)

2, Z_4 : $\{e, g, g^2, g^3\}$ $g^4 = e \qquad g = e^{i\frac{\pi}{2}}$

Normal subgroup $\{1, g^2\}$ Z_2

Cosets: $[e] : \{e, g^2\}$ $\qquad [e]\cdot[e] = [e]$

$[g] : \{g, g^3\}$ $\qquad [e]\cdot[g] = [g]$

$[g]\cdot[g] = [e]$ $\Big\}\ Z_2$

($G/H\times H = Z_2\times Z_2 \neq Z_4$)

$\downarrow$

$(e,e)\ (e,g)$

$(g,e)\ (g,g)$ $\Big\}$ $g^2 = 1$ for $Z_2\times Z_2$ but not for Z_4

$G/H\times H = G$ iff $G/H \simeq P$ and P is normal in G

Herlitz

REM: Homomorphism $\phi : (G_1, \circ) \to (G_2, *)$

$$\phi(g_1) * \phi(g_2) = \phi(g_1 \circ g_2)$$

Définition: Representation of a group: Homomorphism from G onto ~~non~~-singular $d \times d$ matrices with matrix multiplication as product

Example $Z_2 \{e, g\} \to (1, -1)$ 1-dimensional rep

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ 2-dimension repr.

If ϕ is bijective it is called "isomorphism" and the representation is "faithful"

∘) Isomorphic groups have identical representation

$G_1 \simeq G_2$ ϕ, ϕ^{-1} $G_1 \leftrightarrow G_2$

$\Psi(G_2)$ is a repr. of G_2 then $\Psi(\phi(G_1))$ is a representation of G_1

$\chi(G_1)$ is a repr. of G_1 then $\chi(\phi^{-1}(G_2))$ is a repr of G_2

Definition Kernel: Let ϕ be a homomorphic map from G_1 to G_2 then

$\ker(\phi) = \{ g_1 \in G_1 \,/\, \phi(g) = e_2 \}$ is called

Kernel of ϕ

Definition Image of ϕ

$$\mathrm{Im}(\phi) = \{ g_2 \in G_2 \mid \exists g_1 \in G_1 \text{ s.t. } \phi(g_1) = g_2 \}$$

First Homomorphism Theorem:

Let ϕ be a homomorphic mapping $G \to G'$ with kernel $\ker(\phi) = K$

- K is a normal subgroup of G
- every element of the right coset gK maps onto the same element in G'
- define $\Theta(gK) = \phi(g)$, Θ is an isomorphism

$$G/K \simeq G' \sim \mathrm{Im}(\phi)$$

- K is a subgroup $k_1, k_2 \in K$ $\phi(k_1 \cdot k_2) = \phi(k_1) \cdot \phi(k_2) = e'$

$\to k_1 \cdot k_2 \in K$

suppose $k \in K$ k^{-1}: $\phi(k k^{-1}) = \phi(e) = e' =$

$= \phi(k)\phi(k^{-1}) = e' \cdot \phi(k^{-1}) = e'$

$\Rightarrow \phi(k^{-1}) = e' \Rightarrow k^{-1} \in K$

$\phi(e k) = \phi(e)\phi(k) = \phi(k) = e' = \phi(e) e' \Rightarrow \phi(e) = e'$

$\Rightarrow e \in K$

$\forall g \in G$, $k \in K$ $\phi(g \circ k \circ g^{-1}) = \phi(g)\phi(k)\phi(g^{-1}) = \phi(g)\phi(g^{-1}) = e'$

$\Rightarrow g \circ k \circ g^{-1} \in K \Rightarrow K$ is normal

- $\phi(g k) = \phi(g)\phi(k) = \phi(g)$ $\phi(gK) = \phi(g)$

- $\Theta(g_1 K)\, \Theta(g_2 K) = \phi(g_1)\phi(g_2) = \phi(g_1 g_2) = \Theta(g_1 g_2 K) =$

$= \Theta(g_1 K \cdot g_2 K)$

$$G/K \simeq G' = \mathrm{Im}(\phi)$$

Corollary: ϕ is an isomorphism if $K(\phi) = e$

Definition Automorphism $\phi: \mathcal{G} \to \mathcal{G}$ isomorphic

inner automorphism $x \in \mathcal{G}$

$$\phi_x : g \to x \circ g \circ x^{-1}$$

otherwise "outer automorphism"

Definition Center $Z(\mathcal{G}) = \{ g \in \mathcal{G} \mid g \circ h = h \circ g \ \forall h \in \mathcal{G} \}$

obviously $Z(\mathcal{G})$ is a normal subgroup

e.g. $SU(N)$: $e^{i\frac{2\pi}{N}n} \mathbb{1}_N$ $\det = 1$

Definition Direct Product $\mathcal{G}_1 \otimes \mathcal{G}_2$

$\forall g_1 \in \mathcal{G}_1, g_2 \in \mathcal{G}_2$ define $(g_1, g_2) \circ (g_1', g_2') = (g_1 g_1', g_2 g_2')$

this provides a group structure

obviously $(g_1, e_2) \simeq \mathcal{G}_1$ $(e_1, g_2) \simeq \mathcal{G}_2$

(g_1, e_2) commutes with (e_1, g_2)

common element (e, e)

exmpl. $Z_2 \times Z_2$

Definition If $\mathcal{G} \simeq \mathcal{G}_1 \otimes \mathcal{G}_2$ $\mathcal{G}$ is called a direct product group

$\mathcal{G}_1$ and $\mathcal{G}_2$ are normal subgroups

Theorem If $\mathcal{G}$ possesses two subgroups $\mathcal{G}_1, \mathcal{G}_2$ such that

- elements of $\mathcal{G}_1$ commute with elements of $\mathcal{G}_2$
- $\mathcal{G}_1$ and $\mathcal{G}_2$ have only the identity in common
- $\forall g \in \mathcal{G} \;\; \exists g_1 \in \mathcal{G}_1$ & $g_2 \in \mathcal{G}_2$ such that $g = g_1 \circ g_2$

then $\mathcal{G}$ is a direct product group

$$\mathcal{G} \simeq \mathcal{G}_1 \otimes \mathcal{G}_2$$

Example $O(3)$ $\quad O^T O = \mathbb{1}_3$

$$\mathcal{G}_1 = SO(3) \qquad \det O = +1$$

$$\mathcal{G}_2 = (\mathbb{1}_3, -\mathbb{1}_3) \quad \mathbb{Z}_2 \qquad\qquad O(3) = \mathbb{Z}_2 \otimes SO(3)$$

Note $U(N) \neq U(1) \times SU(N)$

Definition Semi-direct product

$\mathcal{G}$ has two subgroups $\mathcal{G}_1, \mathcal{G}_2$ such that

- $\mathcal{G}_1$ is a normal subgroup
- $\mathcal{G}_1$ and $\mathcal{G}_2$ have only the identity in common
- $\forall g \in \mathcal{G} \;\; \exists g_1 \in \mathcal{G}_1, g_2 \in \mathcal{G}_2 \qquad g = g_1 \circ g_2$

$$\mathcal{G} \simeq \mathcal{G}_1 \ltimes \mathcal{G}_2$$

Expl: Euclidean group in $\mathbb{R}^3$

$$\vec{x}' \to O_1 \cdot \vec{x} + \vec{t}_1 \qquad \text{rotation + translation}$$

$$x'' = O_2 x' + t_2 \Rightarrow O_2 O_1 \vec{x} + O_2 t_1 + t_2$$

$$(O_2, t_2) \circ (O_1, t_1) = (O_2 O_1, O_2 t_1 + t_2) \quad ✓$$

identity $(\mathbb{1}_3, 0)$

inverse $\quad (O^T, -O^T t) \circ (O, t) = (O^T O, O^T t - O^T t) = e$

$$(O, t) \circ (O^T, -O^T t) = (O O^T, -O O^T t + t) = e$$

subgroups $\quad \mathcal{G}_1 = (\mathbb{1}, t) \quad$ translations

$\qquad\qquad\quad \mathcal{G}_2 = (O, 0) \quad$ rotations

but only $\mathcal{G}_1$ is invariant

$$(O, t) \circ (\mathbb{1}, \vec{a}) \circ (O^T, -O^T t) = (O, Oa + t) \circ (O^T, -O^T t) =$$

$$= (O O^T, -\cancel{t} + Oa + \cancel{t}) = (\mathbb{1}, O\vec{a})$$

translation

$$(O, t) \circ (R, 0) \circ (O^T, -O^T t) = (OR, t) \circ (O^T, -O^T t) =$$

$$= (ORO^T, -ORO^T t + t)$$

↳ rotation prevents cancellation

<u>REM:</u> In terms of Matrices

$$g = \begin{pmatrix} O & \vec{t} \\ 0 & 1 \end{pmatrix}$$

$$e = \begin{pmatrix} \mathbb{1} & \vec{0} \\ 0 & 1 \end{pmatrix}$$

$$g^{-1} = \begin{pmatrix} O^T & -O^T t \\ 0 & 1 \end{pmatrix}$$

Lie groups

Lie groups are of highest importance in physics
(more specifically: "linear" Lie groups)

Aspects:
- topological space
- analytic manifold

} later here: algebraic

Key feature: group elements depend on some number of continuous parameters $\xi_1 \dots \xi_n$

Definition: A (linear) Lie-group of dimension n is a group satisfying the following 4 conditions

1| It possesses at least one faithful finite dimensional representation ($d \times d$ matrices)

define Norm $\| g - g' \| = \left[\sum_{j,k=1}^{d} \left| \Gamma_{jk}(g) - \Gamma_{jk}(g') \right|^2 \right]^{1/2}$

d^2 - Complex norm (Topology)

sphere centered at the identity M_δ

$\| g - E \| < \delta$

2| In M_δ $\exists$ n-parametrization $g(\xi_1, \dots, \xi_n)$
every $\vec{\xi}$ corresponds to one element, E: $\vec{\xi} = 0$

3| $\exists$ $\eta > 0$ $\sqrt{\sum_{i=1}^{n} \xi_i^2} < \eta^2$ in $\mathbb{R}_\eta$ every point in $\mathbb{R}_\eta$ corresponds to an element in M_δ

4| Each $\Gamma_{jk}(\vec{\xi})$ is an analytic function of $\xi_1, \dots, \xi_n$

(analytic = expandable in a power series)

Define $\qquad (a_p)_{jk} = \left.\frac{\partial \Gamma_{jk}}{\partial \xi_p}\right|_{\vec{\xi}=0}$

these are n $d \times d$ matrices. They form the basis of a real linear vector space

Proof: $\Gamma_{jk} = A_{ij} + i B_{ij} \qquad \{A_{11}, A_{12}, \ldots, B_{11}, B_{12}, \ldots\}$

analyticity in n parameters: basis $C_1 \ldots C_n$ and the others A's and B's are functions of $C_1(\vec{\xi}) \ldots C_n(\vec{\xi})$

then form the vectors $\frac{\partial C_j}{\partial \xi_p}$

because of one-to-one correspondence

$$\det\left(\frac{\partial C_j}{\partial \xi_p}\right) \neq 0$$

$$\Rightarrow \quad \sum \lambda_p \left(\frac{\partial C_j}{\partial \xi_p}\right) = 0 \quad \text{has only } \lambda_p = 0$$

as solution

$\Rightarrow \quad \frac{\partial \Gamma}{\partial \xi_p} = a_p$ span a real linear vector space

Note a_p can be complex of course

a_p span the "Lie-algebra"

in M_δ inf. $\vec{\xi}$ $\qquad \Gamma = \mathbb{1} + \sum_p \xi_p a_p \qquad \xi$ real

condition 1-to-one $\mathbb{R}_n \to M_\delta$

$\Rightarrow \quad \sum \xi_p a_p = 0$ only if $\xi_p = 0$

(Tangent space) $a_p \ldots$ Lie-algebra

Group multiplication

$$\begin{array}{ccc} g & \cdot\, g' = & g'' \\ \downarrow & \downarrow & \downarrow \\ \vec{\xi} & \vec{\xi}\,' & \vec{\xi}\,'' \end{array}$$

It follows from the above that

$$\xi_i'' = f_i(\xi_1 \dots \xi_n, \xi_1' \dots \xi_n')$$

and f_i are analytic and obey

$$f_i(0, \dots, 0, \xi_1' \dots \xi_n') = \xi_i'$$
$$f_i(\xi_1, \dots, \xi_n, 0 \dots, 0) = \xi_i$$

$$(g \cdot E = g \quad ; \quad E \cdot g' = g')$$

$\Rightarrow$ If g lies in $\mathbb{R}_\eta$ then also g^{-1} lies in $\mathbb{R}_\eta$

$$g\, g^{-1} = e$$

$$f_i(\xi_1, \dots, \xi_n ; \xi_1' \dots \xi_n') = 0$$

since f_i is analytic this has a unique solution
"Implicit function theorem"

for infinitesimal ξ:

$$\Gamma = 1 + \xi_i a_i \qquad \Gamma^{-1} = 1 - \xi_i a_i + O(\xi^2)$$

Examples

1) $(\mathbb{R}-\{0\}, \cdot)$ real numbers, identity is the number 1

faithful representation $\Gamma(r) = r$ 1-dimensional

metric $d(r,r') = |r-r'|$

M_δ : $\delta = \frac{1}{2}$ $\frac{1}{2} < r < \frac{3}{2}$

analytic parametrization $r = e^{\xi}$

obviously $\xi = 0$ $r = 1 = e$

$\mathbb{R}_\eta$: e.g. $\xi^2 < (\log \frac{3}{2})^2$ $\frac{2}{3} < e^{\xi} < \frac{3}{2}$

basis of the Lie algebra $\frac{\partial e^{\xi}}{\partial \xi}\Big|_{\xi=0} = 1$

Rem: $\mathbb{R}_+ : e^{\xi}$ $-\infty < \xi < \infty$

every group element can be written as either

e^{ξ} $\xi \in \mathbb{R}$

or $(-1) \cdot e^{\xi}$ $\xi \in \mathbb{R}$

2) $O(2) : O^T O = \mathbb{1}_2$ subgroup $SO(2)$ $\det(O) = +1$

parametrization $\begin{pmatrix} \cos\xi & \sin\xi \\ -\sin\xi & \cos\xi \end{pmatrix}$

$\xi = 0 \Rightarrow \mathbb{1}_2$

chose e.g. $\xi^2 < (\frac{\pi}{3})^2$ $\mathbb{R}_\eta$ clearly one-to-one

in some M_δ around E

Lie-algebra $\begin{pmatrix} -\sin\xi & \cos\xi \\ -\cos\xi & -\sin\xi \end{pmatrix}\Big|_{\xi=0} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

1-dimensional

O(2): Note that ξ-parametrization extends to all over SO(2)

REM O(2) $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow \quad a^2 + b^2 = 1 = c^2 + d^2 \quad \& \quad ac + bd = 0$

$a = \cos\xi \qquad b = \sin\xi$

$c = \sin\omega \qquad d = \cos\omega$

$\cos\xi \sin\omega + \sin\xi \cos\omega = 0$

$\sin(\xi + \omega) = 0$

$\rightarrow \xi = -\omega \qquad \begin{pmatrix} \cos\xi & \sin\xi \\ -\sin\xi & \cos\xi \end{pmatrix} \longrightarrow \det(O) = +1$

$\rightarrow \xi = -\omega + \pi \qquad \begin{pmatrix} \cos\xi & \sin\xi \\ \sin\xi & -\cos\xi \end{pmatrix} \rightarrow \det(O) = -1$

O(2): $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\xi & \sin\xi \\ -\sin\xi & \cos\xi \end{pmatrix}$

Every element of O(2) can be written as an element of SO(2) multiplied by the matrix

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Example 3 $SU(2)$: 2×2 matrices $\det(u) = +1$

and $u^+ u = 1\!\!1_2$

$$u = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \qquad u^+ = \begin{pmatrix} \alpha^* & -\beta \\ \beta^* & \alpha \end{pmatrix}$$

$$u u^+ = \begin{pmatrix} |\alpha|^2 + |\beta|^2 & 0 \\ 0 & |\alpha|^2 + |\beta|^2 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2 = 1$$

topologically S^3!

$\alpha = \alpha_1 + i\alpha_2 \qquad \beta = \beta_1 + i\beta_2$

$\alpha_2 = \frac{1}{2}\xi_3 \qquad \beta_1 = \frac{1}{2}\xi_2 \qquad \beta_2 = \frac{1}{2}\xi_1 \qquad \alpha_1 = \sqrt{1 - \frac{1}{4}(\xi_1^2 + \xi_2^2 + \xi_3^2)}$

$$d(u, e) = 2\left[1 - \sqrt{1 - \tfrac{1}{4}\textstyle\sum \xi_i^2}\right]^{\frac{1}{2}}$$

check that $d(u,e) < \delta$ if $\xi_1^2 + \xi_2^2 + \xi_3^2 < 2\delta^2 - \frac{1}{4}\delta^4$

Lie-algebra $\quad a_1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad a_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad a_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$

Note: these are almost the Pauli-matrices!

Mathematics convention, Lie-algebra spanned by anti-hermitian matrices

Physics convention: hermitian matrices

$$a_1 = \frac{i}{2}\sigma_1 \; ; \quad a_2 = \frac{i}{2}\sigma_2 \; ; \quad a_3 = \frac{i}{2}\sigma_3$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} ; \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Recap of Lecture 1

- Def group $\quad g_1 \circ g_2 \in \mathcal{G}$, $(g_1 \circ g_2) g_3 = g_1(g_2 \circ g_3)$

$$e \circ g = g \quad ; \quad g^{-1} \circ g = e$$

- Representation: $\Phi : \mathcal{G} \to GL(N, \mathbb{C})$ $\quad N \times N$ matrices

$$\Phi(g_1 \circ g_2) = \Phi(g_1) \cdot \Phi(g_2)$$

- Normal subgroup $\quad g^{-1} H g = H \quad \forall g \in \mathcal{G}$

- Lie-groups: $g(\xi_1, \dots, \xi_n)$ analytic function

faithful representation $\quad \Gamma_{ij}(g(\xi_1, \dots, \xi_n))$

$$\frac{\partial \Gamma}{\partial \xi_p} = a_p$$ spans a real linear vector space

expl. $\quad SU(2)$: $\begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}$ $\quad |\alpha|^2 + |\beta|^2 = 1$

$$\alpha_1 = \sqrt{1 - \tfrac{1}{4}(\xi \cdot \xi)} \; ; \; \alpha_2 = \tfrac{1}{2}\xi_3 \; ; \; \beta_1 = \tfrac{1}{2}\xi_1 \; , \; \beta_2 = \tfrac{1}{2}\xi_1$$

$$a_2 = \frac{1}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} ; \quad a_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \quad a_3 = \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$a_p = i\,\sigma_p \qquad \sigma_p \dots \text{ Pauli matrices}$$

math-convention $\quad a_p^+ = -a_p \quad$ for $SU(N)$

$$e^{i\frac{\varphi}{2}\sigma_3} = \mathbb{1}\cos\frac{\varphi}{2} + i\sigma_3 \sin\frac{\varphi}{2}$$

$$\Gamma(g_1 g_2) = (\Gamma(g_1)\Gamma(g_2))^+$$

$$\Gamma(g)^* \to \Gamma(g)^+ \qquad \xrightarrow{\bar{\Phi}\ \Gamma(g)^+}$$

$$(\Gamma(g_2 g_1))^+ = (\Gamma(g_2)\Gamma(g_1))^+ = \bar{\Phi}\ \Gamma(g_1)^+ \Gamma(g_2)^+$$

$U^+ \sim U^*$; not in general $\Gamma^+ = \Gamma^{-1}$

Def: Lie subgroup is a subgroup of a Lie group which by itself is a Lie group

Def: Connected component:
A maximal set of elements $g \in \mathcal{G}$ that can be obtained by continously varying the components of the faithful representation T_{ij}

Example: $(\mathbb{R}-\{0\}; \cdot)$ $r>0$ is a connected component
$r<0$ is another connected component
Note $r=0$ is not in $(\mathbb{R}-\{0\}, \cdot)$

$O(2)$: $SO(2)$ is a connected component
$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} c\xi & s\xi \\ -s\xi & c\xi \end{pmatrix}$ is another

disconnected $\det(SO) = +1$ $\det(1 \cdot SO) = -1$

Theorem: The connected component that contains the identity of a Lie group $\mathcal{G}$ is a normal subgroup.

Proof: S be the connected component containing e
Let $s \in S$ then $s' s^{-1}$ $\forall s' \in S$ is a connected component of $\mathcal{G}$, but also $s' = s \in S$ so E is in this connected component $\Rightarrow S$ is a subgroup
It is normal since with $X \in \mathcal{G}$ $x s x^{-1}$ (varying s) is a connected component containing E
$\Rightarrow x S x^{-1} = S$

Corollary: Each connected component is a right coset of the connected supgroup S

The connected compment is always a Lie graup

Def: Connected Lie-group: cunist of only one connected component $U(1)$

→ Simple: connected non-Abelian Lie group
no connected proper subgroups $SU(N)$

Theorem: A connected Lie group allows a parametrization $y_1 \cdots y_n$ that covers the whol group continously but not necessarily analytically $U(1) = e^{iy2\pi}$
$y \in [0,1]$

e.g. $SU(2)$ connected

$$|\alpha|^2 + |\beta|^2 = 1 \qquad \alpha = \cos y_1 \, e^{iy_2}$$

$$\beta = \sin y_1 \, e^{iy_3}$$

$$0 \le y_1 \le \frac{\pi}{2} \; ; \; 0 \le y_2 \le 2\pi \; , \; 0 \le y_3 \le 2\pi$$

but $y_1 = 0$ $y_2 = 0$ and y_3 arbitrary maps to E

$$\Rightarrow \frac{\partial \Gamma}{\partial y_3}\Big|_{g=E} = 0$$

Expl: Lorentz-transformations 2D

$$\Lambda^T \eta \Lambda = \eta \qquad \eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -c \\ b & -d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= a^2 - c^2 = 1 = b^2 - d^2 \; ; \; ab - cd = ab - dc = 0$$

$a = \operatorname{ch} x$ $\quad b = \operatorname{sh} y$ $\quad \operatorname{ch} x \operatorname{sh} y - \operatorname{sh} x \operatorname{ch} y = 0$

$c = \operatorname{sh} x$ $\quad d = \operatorname{sh} y$ $\quad \operatorname{sh}(x-y)$

$\Rightarrow x = y$

$\begin{pmatrix} \mathrm{ch}\,x & \mathrm{sh}\,x \\ \mathrm{sh}\,x & \mathrm{ch}\,x \end{pmatrix}$ ✓ $\qquad \mathrm{ch}\,x > 0 \qquad \det = +1$

$\begin{pmatrix} -\mathrm{ch}\,x & -\mathrm{sh}\,x \\ \mathrm{sh}\,x & \mathrm{ch}\,x \end{pmatrix} \qquad -\mathrm{ch}^2x + \mathrm{sh}^2x \Rightarrow \det = -1$

$(y = -x)$

$\begin{pmatrix} \mathrm{ch}\,x & \mathrm{sh}\,x \\ -\mathrm{sh}\,x & -\mathrm{ch}\,x \end{pmatrix} \qquad -\mathrm{ch}^2x + \mathrm{sh}^2x \Rightarrow \det = -1$

$\begin{pmatrix} -\mathrm{ch}\,x & -\mathrm{sh}\,x \\ -\mathrm{sh}\,x & -\mathrm{ch}\,x \end{pmatrix} \qquad \mathrm{ch}^2x - \mathrm{sh}^2x \Rightarrow \det = +1$

component containing the identity $\Lambda^0{}_0 > 0$

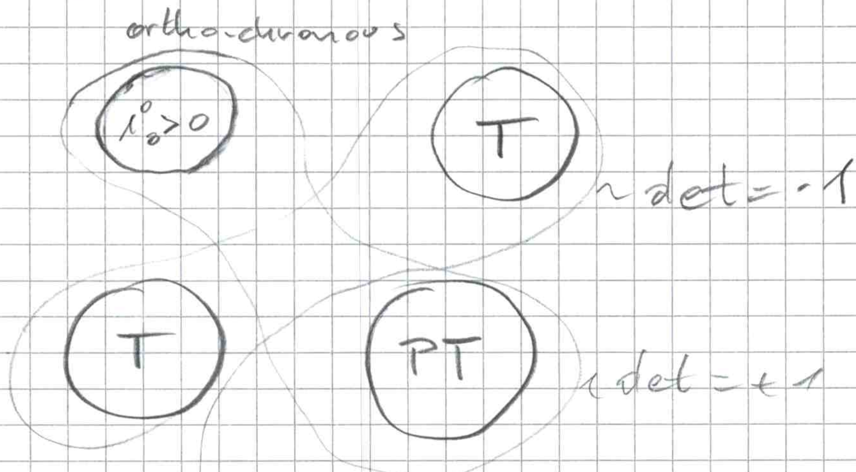
"orthochronous" Lorentz group

$P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$\mathrm{sh}\,x \to -\mathrm{sh}\,x$ by $x \to -x$

similar in 4D!

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \mathrm{ch}\,x & \mathrm{sh}\,x \\ \mathrm{sh}\,x & \mathrm{ch}\,x \end{pmatrix} = \begin{pmatrix} \mathrm{ch}\,x & \mathrm{sh}\,x \\ -\mathrm{sh}\,x & -\mathrm{ch}\,x \end{pmatrix}$



Representations of groups

Def: Representation is a Homomorphism ϕ from the group $\mathcal{G}$ to a group of $d \times d$ non-singular matrices under which the group multiplication maps to matrix multiplication

example: $(\mathbb{R}, +)$ $\quad x+y$

$$\phi: e^x \qquad e^x \cdot e^y = e^{x+y} \checkmark$$

$$(\mathbb{R}, +) \to (\mathbb{R}^+, \cdot)$$

Every group has the trivial representation $\mathcal{G} \to \{1\}$

Def: Equivalent representation if $\exists$ matrix S such that $\Gamma' = S \Gamma S^{-1}$ for two representations Γ, Γ' then Γ, and Γ' are called equivalent

example: $\mathbb{Z}_2: \begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}0&1\\1&0\end{pmatrix} \to \begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}1&0\\0&-1\end{pmatrix}$

Def: A representation is unitary if $\Gamma(g)$ are unitary matrices $\forall g \in \mathcal{G}$

(~~**Theorem:** A subset of points in Euclidean space is compact if and only if it is closed and bounded~~)

Theorem: A Lie group is compact if the parameters $g_1 \dots g_n$ covering the connected component range over finite intervals.

<u>Theorem</u> If G is finite or a compact Lie group then every representation of G is equivalent to a unitary representation.

Now with proof! No proof here!

<u>Theorem</u> If G is a non-compact simple Lie group then G has no finite dimensional unitary representation except for the trivial one.

<u>Theorem:</u> If Γ and Γ' are equivalent unitary reps of a group G i.e. $\Gamma' = S\Gamma S^{-1}$ then S can be chosen unitary as well.
(unitarily equivalent)

<u>Definition:</u> If $\exists S$ such that the representation Γ is equivalent to

$$\Gamma' = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ 0 & \Gamma_{22} \end{pmatrix}$$

Γ is called reducible

more formal: it exists an invariant subspac $\Gamma(g) W \subset W$ $\quad W \sim \begin{pmatrix} w \\ 0 \end{pmatrix}$

$$\Gamma W = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ 0 & \Gamma_{22} \end{pmatrix} \begin{pmatrix} w \\ 0 \end{pmatrix} = \begin{pmatrix} \Gamma_{11} w \\ 0 \end{pmatrix} \checkmark$$

$\forall g \in G$

W .. vector space "carrier space" of the representation

"Module" $(\Phi(\Gamma), W)$ as abstract operators
think of Q.M. $\quad O|\psi\rangle \quad O_{ij} = \langle i|O|j\rangle$

__Definition:__ Irreducible representation

If it is not reducible, i.e. does not have an invariant subspace

__Definition:__ A representation is called completely reducible if it is equivalent of block diagonal form

$$\Gamma = \begin{pmatrix} \Gamma_{11} & & & \\ & \Gamma_{22} & & \\ & & \Gamma_{33} & \\ & & & \ddots \end{pmatrix}$$

all the blocks are irreducible representations; If P projects into in-subspace, $(1-P)$ else

__Theorem:__ If G is finite or a compact Lie group then every reducible representation is completely reducible.

Proof aswell!

__Counter-examples:__ $(\mathbb{R}^+, \cdot) \sim e^{x}$

$$\Gamma = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \qquad \Gamma(x)\cdot\Gamma(y) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & x+y \\ 0 & 1 \end{pmatrix} \checkmark$$

__Sim:__ Euclidean group

$$\left(\begin{array}{c|c} O & \vec{t} \\ \hline 0 & 1 \end{array}\right)$$ non-compact Lie groups

Addendum: finite groups

This serves to prove the theorem that: Every rep. of a finite group are equivalent to a unitary rep.

Proof: $\Gamma(g)$ be a matrix representation

consider $S = \sum_g \Gamma^+(g).\Gamma(g)$ (matrix multiplication

Note: $\| \Gamma . \psi \| = \bar{\Psi} \sum_g \Gamma^+ . \Gamma \psi \geq 0$

$\Rightarrow$ S is positive definite and Hermitian

by a unitary S can be diagonalized

$$S = U^+ d\, U$$

$$d = \begin{pmatrix} d_1 & 0 & \cdots \\ 0 & \ddots & \\ \vdots & & d_N \end{pmatrix} \qquad d_i \geq 0$$

Note $d \neq 0$ because g includes e and necessarily $\Gamma(e) = \mathbb{1}$

$$\bar{\Psi} S \psi = \sum_g \| \Gamma(g) \psi \|^2$$

is a sum over positive terms and it includes $\| \psi \|^2$ $\Gamma(e) = \mathbb{1}$

$\Rightarrow$ $d_i > 0$

$$X = U^+ \sqrt{d}\, U \qquad \text{and } X^{-1} \text{ exists}$$

$$X^{-1} = U^+ \begin{pmatrix} 1/\sqrt{d_1} & & \\ & \ddots & \\ & & 1/\sqrt{d_N} \end{pmatrix} U$$

$$\Gamma'(g) = X\, \Gamma(g)\, X^{-1}$$

Now $\quad (\Gamma'(g))^\dagger (\Gamma'(g)) =$

$$= \quad X^\dagger = u^\dagger \sqrt{d}^{\,\dagger} u \qquad X^\dagger = X$$

$$(\Gamma'(g))^\dagger = X^{-1} \Gamma(g)^\dagger X$$

$$\Rightarrow \quad (\Gamma'(g))^\dagger \Gamma'(g) = X^{-1} \Gamma(g)^\dagger X X \Gamma(g) X^{-1} =$$

$$= X^{-1} \Gamma(g)^\dagger S \Gamma(g) X^{-1} =$$

$$= X^{-1} \Gamma(g)^\dagger \sum_h \Gamma(h)^\dagger \Gamma(h) \Gamma(g) =$$

$$= X^{-1} \underbrace{\sum_h \Gamma(hg)^\dagger \Gamma(hg)}_{\sum_g \Gamma(g)^\dagger \Gamma(g) = S} X$$

by rearangment theorem

$$X^{-1} S X^{-1} =$$

$$= u^\dagger \frac{1}{\sqrt{d}} u u^\dagger d u u^\dagger \frac{1}{\sqrt{d}} u = \mathbb{1} \checkmark$$ qed!

so the new representation is unitary!

<u>Rem:</u> for Lie-groups this proof "works" if the group is compact and thus allows to define a measure with finite volume

Theorem: Every representation of a finite group is completely reducible

Proof: reducible means that there is an invariant subspace. Let $P = P^2$ be the projector onto that subspace ($P^\dagger = P$)

$\forall g \in G: P\Gamma(g)P = \Gamma(g)P$ reducibility

since $\Gamma(g)^\dagger = = \Gamma(g^{-1})$ unitary

$$P\Gamma(g^{-1})P = P\Gamma(g^{-1}) \quad \forall g$$

but that means $P\Gamma(g)P = P\Gamma(g) \quad \forall g \in G$

then the projector onto the complement projects also to an invariant subspace

$$(1-P)\Gamma(g)(1-P) = \Gamma(g) - \Gamma P - \cancel{P\Gamma} + \cancel{P\Gamma}P$$
$$= \Gamma(g)(1-P) \qquad \text{q.e.d}$$

Herlitz

Schur's lemma

These are actually two theorems

I) Let Γ and Γ' be two irreducible representations of a group $\mathcal{G}$ of dimensions d and d' and suppose that there is a $d \times d'$ matrix A such that

$$\Gamma'(g) A = A \Gamma(g) \qquad \forall g \in \mathcal{G}$$

Then either $A = 0$ or $d = d'$ and $\det A \neq 0$

Proof A maps $V \to W$, Γ' acts on W, Γ on V

$$\Gamma' A \psi = A \Gamma \psi$$

$\Rightarrow$ $A\psi$ is invariant subspace of W $(\mathrm{Im}(A))$, but Γ' is irred.

(suppose $A\psi = 0$ $\Rightarrow$ $\Gamma' A \psi = 0 = A \Gamma \psi$ $\Rightarrow$ $\mathrm{Ker}(A)$ is an invariant subspace of V $\Rightarrow$ $A = 0$ or $A = d \times d$ and invertible)

$\det(A) \neq 0$

such that Γ' and Γ are equivalent

II) If Γ is $d \times d$ irreducible representation and

$$\Gamma(g) B = B \Gamma(g) \qquad \forall g \in \mathcal{G}$$

then $B = \lambda \mathbb{1}$

Proof: let $A = B - \beta \mathbb{1}$ such that $\det A = 0$

this means β is an eigenvalue of B

then $A \Gamma = \Gamma A$ but $\det A = 0 \Rightarrow \exists \psi$ inv. subspace $A\psi = 0$

anihilating A do exist, ~~but~~ since Γ is irreducible the only solution is $A = 0$!

$\Rightarrow$ $\underline{B = \beta \mathbb{1}}$

theorem: Every irreducible representation of an Abelian group is one-dimensional

Proof: let g, g' be two elements then

$$\Gamma(g) \cdot \Gamma(g') = \Gamma(g') \Gamma(g)$$

but by Shurs lemma $\Gamma(g') = \lambda(g') \mathbb{1}$

This is irreducible only if $\mathbb{1} = 1$

$$\mathbb{Z}_2 : \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \to \mathbb{1} \oplus \{1, -1\} \; ; \left\{ \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix} \right\}$$

$g = 1, -1$

Complex representations

If $\Gamma(g)$ is a representation then $\Gamma^*(g)$ is also a representations $\Gamma^*(g) = (\Gamma(g))^*$

- real representations $\exists \; S \quad S \Gamma S^{-1} = \Gamma'$ and $(\Gamma')^* = \Gamma'$
- pseudo-real representation $\exists \; S : S \Gamma S^{-1} = \Gamma^*$
- $\nexists \; S \Gamma S^{-1} = \Gamma^*$ complex representation

Orthogonality relation

let φ_j^a be an (orthonormal basis) of the vector space on which the representation $\Gamma_a(g)$ acts.

element

$\theta^{ab}: V_b \to W_a$ linear

Consider

$$A^{ab}_{\ell k} = \sum_g \Gamma_a^+(g) \underbrace{\varphi_\ell^a \bar{\varphi}_k^b}_{\theta^{ab}} \Gamma_b(g)$$

$$\Gamma_a(g') A^{ab}_{\ell k} = \sum_g \underbrace{\Gamma_a(g') \Gamma_a(g^{-1})}_{} \varphi_\ell^a \bar{\varphi}_k^b \Gamma_b(g)$$

$$\Gamma_a(g' g^{-1}) = \Gamma_a((g g'^{-1})^{-1})$$

$$h = g g'^{-1} \qquad g = h g'$$

$$\Gamma_a(g') A^{ab}_{\ell k} = \sum_h \Gamma_a^+(h) \varphi_\ell^a \bar{\varphi}_k^b \underbrace{\Gamma_b(h g')}_{\Gamma_b(h) \circ \Gamma_b(g')}$$

$$\Rightarrow \quad \Gamma_a(g') A^{ab}_{\ell k} = A^{ab}_{\ell k} \Gamma_b(g')$$

so by Schur's lemma the $A^{ab} = 0$ if $a \neq b$ and otherwise proportional to $\mathbb{1}$

$$\Rightarrow \quad A^{ab}_{\ell k} = \delta^{ab} \lambda^a_{\ell k} \mathbb{1}$$

To compute $\lambda^a_{\ell k}$ we take the trace

$$(*) \qquad \mathrm{tr}(A^{ab}_{\ell k}) = \delta^{ab} \lambda^a_{\ell k} n_a$$

↳ dim Repr "a"

(b) on the other hand O^{ab}

$$\mathrm{tr}(\Lambda^{ab}_{ek}) = \sum_g \mathrm{tr}\left[\overbrace{\left(\Gamma_a^+(g)\, \Psi^a_e\right)\left(\overline{\Psi}^a_k\, \Gamma_a(g)\right)}\right] \delta^{ab} =$$

$$= \sum_g \mathrm{tr}\Big(\underset{\delta_{ei}\,\delta_{kj}}{\underset{\perp}{\Psi^a_e \overline{\Psi}^a_k}}\, \underbrace{\Gamma_a(g)\, \Gamma_a^+(g)}_{\delta_{ij}}\Big) = \delta_{ke} \cdot N$$

if we chose Ψ^a_k to span an orthonormal basis $(\Psi_e)_i = \delta_{ei}$ etc...

$$N = O(G)$$

$$\Rightarrow \quad \delta^{ab}\, n_a\, \lambda^a_{ke} = \delta^{ab}\, N\, \delta_{ke}$$

$$\lambda^a_{ke} = \frac{N}{n_a}\, \delta_{ke}$$

$$\Rightarrow \quad \boxed{\sum_g \frac{n_a}{N} \left(\Gamma_a(g)^+\right)_{je} \left(\Gamma_b(g)\right)_{km} = \delta_{ab}\, \delta_{ek}\, \delta_{jm}}$$

"Orthogonality relation"

Especially useful for "Characters"

$$\chi_a(g) = \mathrm{tr}\left(\Gamma_a(g)\right)$$

invariant under similarity transformations!

$$\sum_g \frac{n_a}{N}\, \chi_a(g)^* \chi_b(g) = \delta_{ab}\, n_a$$

$$\boxed{\frac{1}{N} \sum_g \chi_a(g)^* \chi_b(g) = \delta_{ab}}$$

Application Fourier: $e^{im\varphi}$, $e^{in\varphi}$ repr. labelled by m, n

$$\frac{1}{N}\sum_g \to \frac{1}{2\pi}\int_0^{2\pi} d\varphi\; e^{-im\varphi}\, e^{in\varphi} = \delta_{n,m}$$

← justify later

Lie algebras

Lie group: -) global aspects e.g. connectedness etc.
compactness

-) local aspects → Lie algebra
(even some global aspects)

We know already that $a_p = \frac{\partial \Gamma}{\partial \xi_p}$ form a basis of a real linear vector-space.
Now we will show that the form an algebra → Lie algeb
And we will eventally allow to form complex linear combination ("complexify" the algebra)
Allow for a complete classification of Lie-algebras and their representations

Def: Lie-algebra (over the real numbers) $\mathcal{L}$ is a (real) linear vector space equipped with a Lie-product

- $a * b \in \mathcal{L} \quad \forall\, a, b \in \mathcal{L}$
- $(\alpha a + \beta b) * c = \alpha\, a * c + \beta\, b * c \qquad \alpha, \beta \in \mathbb{R}$
- $a * b = - b * a$
- $a * (b * c) + c * (a * b) + b * (c * a) = 0$
  (Jacobi-identity)

Matrices $*$: commutator $a * b = [a, b]$

(Theorem: every (finite dimensional) Lie algebra is isomorphic to a matrix algebra with $*$ being the matrix-commutator)

Therefore we always think now of commutators

Jacobi identity

$$[a,[b,c]] = a\cancel{b}c - a\cancel{c}b - b\cancel{c}a + c\cancel{b}a +$$
$$+ \text{cyclic} \qquad + c\cancel{a}b - b\cancel{a}c - a\cancel{b}c + a\cancel{c}b$$
$$+ \text{cyclic} \qquad + b\cancel{c}a - c\cancel{b}a - c\cancel{a}b + b\cancel{a}c$$

✓

<u>Note:</u> the Jacobi identity can be written as a <u>Leibnitz rule</u> (= derivative)

$$[a,[b,c]] = [[a,b],c] + [b,[a,c]]$$

Since all commutators are in $\mathcal{X}$ and $\mathcal{X}$ is a linear vector space it follows

$$[a_p, a_q] = c_{pq}{}^{r}\, a_r \qquad \text{"Structure constants"}$$

(Physics $a_p = i\, t_p$

$$[t_a, t_b] = i f_{ab}{}^{c}\, t_c \qquad f_{ab}{}^{c} = -c_{ab}{}^{c}$$

<u>Def:</u> Abelian Lie-algebra $[a,b] = 0$, $f_{ab}{}^{c} = 0$

<u>Lie groups and Lie-algebras</u>

Take a one-parameter curve in $\vec{\xi}(s)$
where $\xi_1, \dots, \xi_n$ parametrize a neighbourhood
around E in a Lie group $\mathcal{G}$.
This defines a one-parameter curve in $\mathcal{G}$
not necessarily a subgroup.
let $\vec{\xi}(0) = 0$

Now compute $\left.\frac{d\Gamma(s)}{ds}\right|_{s=0} = \sum_p \left.\frac{\partial \Gamma}{\partial \xi_p}\right|_{\xi=0} \frac{\partial \xi_p}{\partial s} = \sum_{p=1}^{n} a_p \frac{\partial \xi_p}{\partial s}$

Since we know that the matrices a_p span a linear vector space we call $\frac{\partial \Gamma}{\partial s}$ the tangent vector to the curve $\Gamma(s)$ at $s=0$.

Conversally every vector $\sum a_p \lambda_p$ is tangent vector to some curve.

Theorem: If a and b are tangent vectors to the curves $A(s)$ and $B(s)$ then $[a,b]$ is tangent vector to the curve $C(s^2) = A(s)\,B(s)\,A(s)^{-1}\,B(s)^{-1}$

$A(s) = \mathbb{1} + sa + \frac{s^2}{2} a_2$ $\qquad B(s) = \mathbb{1} + sb + \frac{s^2}{2} b_2$

$A(s)^{-1} = 1 - sa + s^2\left(a^2 - \frac{1}{2} a_2\right)$ $\qquad B(s)^{-1} = \mathbb{1} - sb + s^2\left(b^2 - \frac{1}{2} b_2\right)$

$\Rightarrow \quad C(s^2) = \mathbb{1} + s^2 [a,b] + \dots$

$s' = s^2 \qquad C(s) \doteq \mathbb{1} + s'[a,b] + \dots$

$\Rightarrow$ to every Lie-group we can associate a Lie-algebra in a natural way.

Note: the commutator is an element of the vector space spanned by the a_p's since it appears as tangent vector to a curve in $\mathcal{G}$.

a_p (t_a) are called the generators of the Lie-group for $SU(2)$ we have already computed the a_p's

$$a_1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad a_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad a_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

commutators

$$[a_1, a_2] = \frac{1}{4}\left[\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} - \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -a_3$$

$$[a_2, a_3] = \frac{1}{4}\left[\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} - \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = -a_1$$

$$[a_1, a_3] = \frac{1}{4}\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -a_2$$

(Note the "-" signs in the math equations

$$\boxed{[\sigma_i, \sigma_j] = 2i\,\varepsilon_{ijk}\,\sigma_k} \qquad \sigma_i \sigma_j = \delta_{ij} + i\varepsilon_{ijk}\sigma_k$$

<u>Theorems:</u> •) $A(t) = \exp(ta)$ mit $a \in \mathcal{L}$ is a one-parameter subgroup $(-\infty < t < \infty)$

•) Every element of G in a small neighborhood of E can be written as

$$g = \exp(x_1 a_1 + x_2 a_2 + \dots + x_p a_p)$$

•) If G is a compact Lie-group every element of the connected component can be written in this form

This provides a direct way of characterizing some Lie-algebras

<u>Expl:</u> •) $G = Sl(N,\mathbb{C})$: $\det(g) \overset{!}{=} 1 = \det(e^{a}) = e^{tr(a)}$

$\Rightarrow \quad tr(a) = 0$ defines the Lie-algebra $sl(N)$; $N=2$: 6 dimensional vectorspace

<u>Notation</u> $SU(N) \rightarrow su(N)$
group algebra

$u^{-1} = u^{+} \quad e^{-a} = e^{a^{+}} \quad \Rightarrow \quad a = -a^{+} \quad (t^{+} = t$ in phys$)$

$\Rightarrow$ $su(N)$ traceless (anti-) hermitian matrices

•) $G = SO(N) \quad O^{T} = O^{-1} \quad a^{T} = -a$
note $\det(e^{a}) = 1$ automatically
$\rightarrow$ anti-symmetric matrices
- it will turn out that even and odd N are very different

more generally
$SO(p,q)$: $\quad \theta\eta = \eta\theta \quad \eta = \mathrm{diag}(\underbrace{+1 .. +1}_{p}, \underbrace{-1 .. -1}_{q})$

left- and right-invariant integration measures

Suppose we have a canonical parametrization of a compact Lie-group ("g" coords)

$$I = \int f\left(A(x_1, \ldots, x_n)\right) \sigma_\ell(x_1 \ldots x_n)\, dx_1 \ldots dx_n$$

$$I = \int f\left(A' \cdot A(x_1 \ldots x_n)\right) \sigma_\ell(x_1 \ldots x_n)\, dx_1 \ldots dx_n$$

we say $\sigma_\ell(x_1 \ldots x_n)$ is a left-invariant measure

consider $\quad A_0^{-1} \frac{\partial A}{\partial x_k}\Big|_{x_k = x^0} \qquad A(x^0) = A_0$

this is an element of $\mathcal{L}$ since $B(x) = A_0^{-1} A(x)$ is such that at $x = x^0$ $B(x) = \mathbb{1}$

$$A^{-1} \frac{\partial A}{\partial x_k}\Big|_A = \sum \mu^\ell_{kj}(x_1 \ldots x_n)\, a_j$$

define $\sigma_\ell = \det \mu_{kj}$

$$\int f\left(A' A(x)\right) |\mu(x)|\, d^n x$$

$$A' \cdot A(x) = A(x')$$

$$A^{-1}(x') \frac{\partial A}{\partial x'_k} = \sum \mu_{kj}(x')\, a_j$$

$$A^{-1}(x) \underbrace{A'^{-1} \cdot A'}_{\mathbb{1}} \frac{\partial A}{\partial x_\ell} \frac{\partial x_\ell}{\partial x'_k} = \sum \mu_{kj}(x')\, a_j$$

$$\sum \mu_{\ell j} \frac{\partial x_\ell}{\partial x'_k}\, a_j = \sum \mu_{kj}(x)\, a_j$$

$$\sigma^{\ell} = |\mu|$$

$$\sigma^{\ell}(x') = \sigma^{\ell}(x) \left|\frac{\partial x}{\partial x'}\right|$$

or $$\sigma^{\ell}(x) = \sigma^{\ell}(x') \left|\frac{\partial x'}{\partial x}\right|$$

$$\rightarrow \int f(A' \cdot A(x)) = \int f(A(x'))\, \sigma^{\ell}(x') \left|\frac{\partial x'}{\partial x}\right| dx^4 =$$

$$= \int f(A(x'))\, \sigma^{\ell}(x')\, dx'^4 = \int f(A(x))\, \sigma^{\ell}(x)\, dx^4 \quad \text{q.e.d}$$

right-invariant measure

$$\frac{\partial A}{\partial x_k} A^{-1} = \sum_{\ell} \mu_{k\ell}(x)\, a_{\ell}$$

If the group consists of several components one has to sum over the components

Theorem: For compact groups the left and right invariant measures coinside "unimodular" group and the integral is finite

Rem: In most applications it is sufficient to know that this measure exists

Exercise: compute the right and left invariant measures of $SU(2)$

$$u = \begin{pmatrix} \cos y_1 e^{iy_2} & \sin y_1 e^{iy_3} \\ -\sin y_1 e^{-iy_3} & \cos y_1 e^{-iy_2} \end{pmatrix}$$

More on the relationship of Lie-Groups and Lie-algebras

Definition: -) Subalgebra $\mathcal{X}'$ of $\mathcal{X}$: $\mathcal{X}'$ is called a subalgebra of $\mathcal{X}$ if $\forall a \in \mathcal{X}'$; $\forall b \in \mathcal{X}'$ $[a, b] \in \mathcal{X}'$ $\mathcal{X}' \subset \mathcal{X}$

-) It is called an invariant subalgebra if $\forall a \in \mathcal{X}$ and $b \in \mathcal{X}'$ $[a, b] \in \mathcal{X}'$
"Ideal"

Theorems: -) If $\mathcal{G}$ and $\mathcal{G}'$ are Lie groups and $\mathcal{X}$ and $\mathcal{X}'$ their Lie-algebras and $\mathcal{G}'$ is a subgroup of $\mathcal{G}$ then $\mathcal{X}'$ is a subalgebra of $\mathcal{X}$

-) If $\mathcal{G}'$ is an invarient subgroup (normal) then $\mathcal{X}'$ is an invariant subalgebra of $\mathcal{X}$

-) If $\mathcal{X}$ is a real Lie algebra of $\mathcal{G}$ then each subalgebra of $\mathcal{X}$ is the Lie group of a subgroup of $\mathcal{G}$

Theorem: If $\mathcal{X}'$, $\mathcal{X}''$ are invariant subalgebras then $[\mathcal{X}', \mathcal{X}'']$ is an invariant subalgebra

Proof: $\forall a \in \mathcal{X}$

$$[a, [b, c]] = [\underbrace{[a, b]}_{b' \in \mathcal{X}'}, c] + [b, \underbrace{[b, c]}_{c' \in \mathcal{X}''}]$$ Jacobi-identity

$$[b', c] + [b, c'] \in [\mathcal{X}', \mathcal{X}'']$$

Homomorphism $\quad \psi: \mathcal{X} \to \mathcal{X}'$

-) $\psi(\alpha a + \beta b) = \alpha \psi(a) + \beta \psi(b)$

-) $\psi([a,b]) = [\psi(a), \psi(b)]$

Isomorphism: one-to-one

Le ϕ be an (analytic) group homomorphism $\mathcal{G} \to \mathcal{G}'$ then there is an induced Lie-algebra homomorphism

$$\psi(a) = \left[\frac{d}{dt}\left(\phi\, e^{ta}\right)\right]\Big/_{t=0}$$

Expl $\phi \quad U(1) \to SO(2) \quad \phi: e^{ix} \Rightarrow \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix}$

$\downarrow$

$a = i \qquad \frac{d}{dt}\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\Big/_{t=0} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$\psi: \ 1 \to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Isomorphism

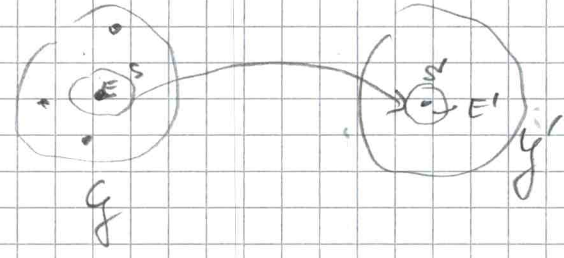
Def: discrete subgroup: a subgroup $\mathcal{H}$ of a Lie-group $\mathcal{G}$ is calld discrete if

-) it is a finite group

-) it consits of countably infinite elements but their is neighborhood of the identity E in which there is no element of $\mathcal{H}$ except for E itself

Theorem Let $\mathcal{K}$, the kernel of a homomorphic mapping $\phi : \mathcal{G} \to \mathcal{G}'$, be discrete. Then the induced homomorphism ψ on the Lie algebras $\psi : \mathcal{X} \to \mathcal{X}'$ is an isomorphism!

Proof:



$\phi(E) = E'$

$\Rightarrow$ ϕ is one-to-one on $S \to S'$, the small neighbor hoods of E, E'

$\Rightarrow$ $g \sim 1 + \alpha a_i \longrightarrow 1 + \alpha a_i'$

$\mathcal{X}, \mathcal{X}'$ is one-to-one

Significance: Lie-algebra determines the Lie-group only "locally" in S! component connected to the identity

Expl: $(\mathbb{R}_+, \cdot)$ and $SO(2)$

$$\downarrow$$

$$\phi : e^x \longrightarrow \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix}$$

Kernal $\mathcal{K}$ $x_k = 2\pi k \quad k \in \mathbb{Z}$

discrete and countably infinite

Lie-algebra

$$a_1 = 1 \qquad \hat{a}_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$(\mathbb{R}_+, \cdot)$ is non-compact

$SO(2)$ is compact

Expl   analytic homomorphism $SU(2) \to SO(3)$

$$\Phi(u) = \frac{1}{2} tr\left(\sigma_j u \sigma_k u^+\right) \qquad j,k = 1..3$$

$\vec{x} \in \mathbb{R}^3 \qquad \underline{X} = \vec{x} \cdot \vec{\sigma}$

$$\det(X) = \begin{vmatrix} z & x-iy \\ x+iy & -z \end{vmatrix} = -z^2 - x^2 - y^2 = -r^2$$

$$\det(X) = \det(u X u^+) \qquad \det(u) = 1 \,;\, X^+ = X$$

$$\vec{x} \to \vec{x}' \quad \text{but} \quad \vec{x}' \cdot \vec{x}' = \vec{x} \cdot \vec{x}$$

$\to \quad X'$ defines a vector x' with equal length as $x \Rightarrow \vec{x}, \vec{x}'$ are relate by a rotation

$$\vec{x}' = R \cdot \vec{x}$$

$$tr(\sigma_i \sigma_j) = 2\delta_{ij} \qquad \vec{x} = \frac{1}{2} tr(X \cdot \vec{\sigma})$$
$$\vec{x}' = \frac{1}{2} tr(X' \vec{\sigma})$$

$$\Rightarrow \quad R_{ij}(u) = \frac{1}{2} tr(\sigma_i u \sigma_k u^+)$$

$$R(u_1) \cdot R(u_2) = R(u_1 u_2)$$

$$X' = u_2 X u_2^+ \quad \Rightarrow \quad \vec{x}' = R(u_2) x$$

$$X'' = u_1 X' u_1 \quad \Rightarrow \quad X'' = R(u_1) \vec{x}'$$
$$= R(u_1) \cdot R(u_2) \vec{x}$$

$$= u_1 u_2 X u_2^+ u_1^+$$

$$\Rightarrow \vec{x}'' = R(u_1 u_2) \vec{x}$$

$$\Rightarrow \quad R(u_1 u_2) = R(u_1) \cdot R(u_2)$$

$\mathcal{K}$: $\operatorname{tr}(u\sigma_i u^\dagger \sigma_j) = \delta_{ij}$

$\Rightarrow \mathcal{K} = \{\mathbb{1}_2, -\mathbb{1}_2\}$ $\quad \mathbb{Z}_2$ discrete kernel

$\Rightarrow$ Isomorphism on Lie-algebra

$u = 1 + \alpha_i a_i \qquad u^\dagger = 1 + \alpha_i a_i^\dagger = 1 + \alpha_i a_i$

(math-convention here)

$$\frac{1}{2}\frac{d}{d\alpha_l}\operatorname{tr}\left[(1+\alpha_p a_p)\sigma_i(1-\alpha_q a_q)\sigma_j\right] =$$

$$= \frac{1}{2}\operatorname{tr}\left[a_l\sigma_i\sigma_j - \sigma_i a_l \sigma_j\right] = \frac{1}{2}\operatorname{tr}\left([a_l,\sigma_i]\sigma_j\right)$$

$$a_l = \frac{i}{2}\sigma_l \Rightarrow \frac{i}{4}\operatorname{tr}\left([\sigma_l,\sigma_i]\sigma_j\right) =$$

$$= -\frac{1}{2}\varepsilon_{lik}\operatorname{tr}(\sigma_k\sigma_j) = -\varepsilon_{lik}\delta_{kj}$$

$$= \varepsilon_{ilj}$$

$a_l \to \hat{a}_{ij} = \varepsilon_{ilj}$ Isomorphism

$$\Psi(a_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad \Psi(a_2) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \Psi(a_3) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

<u>Note:</u> the structure constants themselves interpreted as Matrices give a representation

In general: "Adjoint" representation

Representations of Lie-algebras

operator acting on a vector space

basis $\varphi^{(\ell)}$.

$$\Gamma \varphi^{(\ell)} = \phi^{(\ell)}$$

orthonormal basis $(e^{(\ell)}, e^{(m)}) = \delta^{\ell m}$

$$(e_{(\ell)}, \Gamma e_{(k)}) = \Gamma_{\ell k}$$

$V = v_\ell^\ell e_{(\ell)}$

$$(e_k, \Gamma v) = (e_k, \Gamma e_\ell) v_\ell \doteq \Gamma_{k\ell} v_\ell$$

V... carrier space (Module)

physics language the vector $\vec{v} \hat{=} v_\ell$ transform in representation $\Gamma_{k\ell}$

equivalent repr. $\Gamma' = S \Gamma S^{-1}$ $\det(S) \neq 0$

- reducible representation

$$\Gamma \text{ equivalent to } \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ 0 & \Gamma_{22} \end{pmatrix}$$

- irreducible representation

- completly reducible representation

$$\begin{pmatrix} \Gamma_{11} & & \\ & \Gamma_{22} & \\ & & \ddots \\ & & & \Gamma_{33} \end{pmatrix}$$

Schur lemmas hold •) $\Gamma A = A \Gamma'$

$\Rightarrow$ • $A = 0$ or A $d \times d$ $\det(A) = 0$

•) if $\Gamma = \Gamma'$ irreducible

$$A = \lambda \mathbb{1}$$

Note: Not every representation of $\mathcal{L}$ gives a representation of $\mathcal{G}$ by exponentiation

e.g. $so(3) \simeq su(2)$ isomorphic one-to-one

but $(t_\ell)_{ij} = \varepsilon_{i\ell j}$ exponentiates to $SO(3)$

while $\frac{i}{2}(\sigma_\ell)_{ij}$ exponentiates to $SU(2)$

Math: $SU(2)$ is not a representation of $SO(3)$

"spinor representation of $SO(3)$"
(projective)

but: as statement about the algebra this is o.k.
since $so(3)$ and $su(2)$ are isomorphic
as Lie-algebras

Direct product representation

Suppose Lie-group repr. is of the form $\Gamma_g \otimes \Gamma'_g$

$E = \mathbb{1} \otimes \mathbb{1}$

then there exists a basis in $V \otimes V'$ such that

$$\Gamma_x = \Gamma_x(a) \otimes \mathbb{1} + \mathbb{1} \otimes \Gamma'_x(a)$$

$$\frac{\partial}{\partial t} (\mathbb{1} + ta) \otimes (\mathbb{1} + ta')\Big|_{t=0} \sim a \otimes \mathbb{1} + \mathbb{1} \otimes a'$$

'direct product representation of a Lie-algebra'

In particular if $\Gamma = \Gamma'$

then $\psi^{(e)} \otimes \psi^{(k)} \to \left(\psi^{(e)} \otimes \psi^{(k)} + \psi^{(k)} \otimes \psi^{(e)}\right) +$
$+ \left(\psi^{(e)} \otimes \psi^{(k)} - \psi^{(e)} \otimes \psi^{(k)}\right)$

which decomposes the direct product into
$\frac{1}{2} d(d+1)$ symmetric subspace and
$\frac{1}{2} d(d-1)$ antisymmetric subspace

$\to$ tensor $v_{ij} = v_{ji}$; $v_{ij} = -v_{ji}$

$$\Gamma \otimes \Gamma = (\Gamma \otimes \Gamma)_{sym} \oplus (\Gamma \otimes \Gamma)_{asym}$$

The adjoint representation

Jacobi-identity

$$[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0$$

base a_p, a_q, a_r

$$[a_p,[a_q,a_r]] = [a_p, c_{qr}{}^s a_s] = c_{qr}{}^s c_{ps}{}^t a_t$$
$$+ c_{rp}{}^s c_{qs}{}^t a_t$$
$$+ c_{pq}{}^s c_{rs}{}^t a_t = 0$$

since a_t are linear independent

$$\Rightarrow \quad c_{qr}{}^s c_{ps}{}^t + c_{rp}{}^s c_{qs}{}^t + c_{pq}{}^s c_{rs}{}^t = 0$$

$$(C_r)_q{}^s = c_{qr}{}^s$$

$$-(C_r C_p)_q{}^t + (C_p \cdot C_r)_q{}^t - c_{rp}{}^s c_{sq}{}^t$$

$$\boxed{[C_p, C_r] = c_{pr}{}^s (C_s)_q{}^t}$$

$\Rightarrow$ For a general Lie-algebra the matrice

$$(C_r)_q{}^s = c_{qr}{}^s$$

form a representation of the Lie-algebra.
"Adjoint representation $ad(a)$

$$ad(a): [a, b]_p \quad \forall\, b_i \in \mathcal{L}$$

Carrier space is the Lie algebra itself.

$$ad(a_p)_q{}^t = c_{qp}{}^t$$

physics $\quad [t_a, t_b] = i f_{ab}{}^c t_c$

$$(T^{ad}_a)_b{}^c = i f_{ba}{}^c$$

The correponding adjoint representation of Lie-group
Let $A \in \mathcal{G}$ $\quad a_i \in \mathcal{L}$ the

$$Ad(A) := A a_p A^{-1} = [Ad(A)]_p{}^q a_q$$

n-dimensinal representation

$$\exp(t\, ad(a)) = Ad(\exp(ta))$$

Representation of $\mathcal{G}$:

$$A_1 A_2 a_p A_2^{-1} A_1^{-1} = Ad(A_1)_{pq} \circ Ad(A_2)_{qe} a_e$$

$$A_1 \underbrace{Ad(A_2)_{pe} a_e}_{a'_p} A^{-1} = Ad(A_1)_{pq}\, a'_q = Ad(A_1)_{pq}\, Ad(A_2)_{qe}\, a_e \checkmark$$

So it is indeed a representation of $\mathcal{G}$

~~the universal covering group~~

Suppose $x_1, .., x_n$ is a parametrization of a Lie Group connected

take n- function $f_i(t) = x_i$

$$g(f_1, .., f_n) = g(t) \in G \;\forall\, t \in [0,1]$$

$g(t)$ is a path

G is connected $\Rightarrow \exists$ paths between each two group elements

If $g(1) = g(0)$ the path is called a loop

Two paths are called homotopic if there exists a continuous transformation $f_i(t) \to \tilde{f}_i(t)$

$$F(t,s) \to \begin{array}{l} F_i(t,0) = f_i(t) \\ F_i(t,1) = \tilde{f}_i(t) \end{array}$$

If $f(t)$ is homotopic to the constant path it is called "contractible".

Def: A connected Lie group is called ~~contractible~~ simply connected if all loops are contractable

<u>Example</u> $SU(2)$: $u = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}$

with $|\alpha|^2 + |\beta^2| = 1$

$\alpha = x_1 + i x_2 \qquad \beta = x_3 + i x_4$

$\Rightarrow \quad x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$

topologically a 3-sphere S^3

→ every loop is contractible

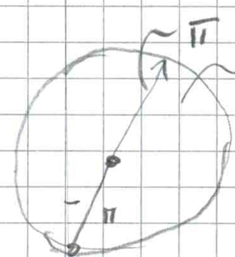
$SU(2)$ is simply connected

generally $SU(N)$ is simply connected

<u>Expl 2</u> $SO(3)$ let $SO(3)$ be represented by

$e^{i\varphi \hat{n}\vec{\sigma}}$ $\hat{n}$ -- direction, φ angle

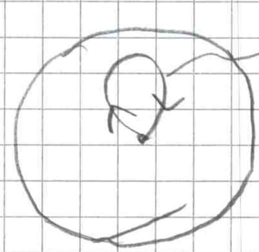
$-\pi < \varphi < \pi$



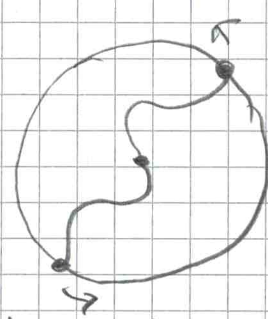
$\mathbb{1} = \varphi = 0$

But $\varphi = \pi$ is identified with $\varphi = -\pi$

Then:



<u>But:</u>



$SO(N)$ is not simply connected

Expl: $SO(2)$ is not simply connected

consider $\begin{pmatrix} \cos 2\pi t & \sin 2\pi t \\ -\sin 2\pi t & \cos 2\pi t \end{pmatrix}$ $t \in [0,1]$

$F[t,s]$ continuously connected to $F[t,1] = 0$

But $F[1,0] = 1$ → continuity $F(1,s) = 1$

obviously this can not change continiously to $F(1,1) = 0$

→ $SO(2)$ is not simply connected.

Def: Center $\mathcal{C}$ of a group $\mathcal{G}$: subgroup consisting of all elements that commute with all elements of $\mathcal{G}$

Example: $SU(N)$: $\begin{pmatrix} e^{\frac{2\pi n}{N}} & \\ & \ddots \\ & & e^{\frac{(..)n}{N}} \end{pmatrix}$ + permutation

$SU(N)$: $e^{i\frac{2\pi n}{N}} \mathbb{1}$ $n = 0 .. N-1$

center Z_{N}

theorem: If $\mathcal{G}$ is a connected Lie group then there exists a simply connected Lie group $\tilde{\mathcal{G}}$

- $\mathcal{G}$ is isomorphic to $\tilde{\mathcal{G}}/\mathcal{K}$ where $\mathcal{K}$ is a discrete subgroup of the center
- If $\mathcal{G}$ is simply connected $\mathcal{G}$ is isomorphic to $\tilde{\mathcal{G}}$
- The real Lie algebras are isomorphic
- Every rep of the real Lie algebra of $\tilde{\mathcal{G}}$

is associated with a representation of $\tilde{G}$

$\tilde{G}$ is called the universal covering group.

In particular for every Lie algebra $\mathcal{L}$ there exists a unique (up to isomorphism) simply connected Lie-group $\tilde{G}$ such that every Lie-group G with Lie algebra $\mathcal{L}$ is isomorphic to $\tilde{G}/K$ where K is a discrete subgroup of the center of $\tilde{G}$

Every representation of $\mathcal{L}$ gives upon exponentiation a representation of $\tilde{G}$

(REM: $SO(3)$ is a representation of $SU(2)$

$SO(3) \cong SU(2)/Z_2$)

This boils down the classification of connected Lie groups to the classification of Lie-algebras!

Classification of the irreducible representations of the Lie-Algebra $su(2) \simeq so(3)$

By exponentiation this is also a classification of the irreducible representations of $SU(2)$

•) Lie-algebra $[t_a, t_b] = i\varepsilon_{abc} t_c$

Note physics notation

$$(\phi, t\psi) = (t\phi, \psi) \qquad \text{unitary representation}$$

$$(e^{it})^{+} = e^{-it} \qquad SU(2) \quad u^{+} = u^{-1}$$

consider the complexification: allow linear combinations of the t_a

$$t_{\pm} = t_1 \pm t_2 \qquad , \quad t_3$$

$$[t_3, t_{\pm}] = \pm t_{\pm}$$
$$[t_+, t_-] = 2t_3$$

"Casimir" $\quad t^2 = t_1^2 + t_2^2 + t_3^2$

$$t_+ t_- = t_1^2 - i(t_1 t_2 - t_2 t_1) + t_2^2 = t_1^2 + t_2^2 + t_3$$
$$t_- t_+ = t_1^2 + i(t_1 t_2 - t_2 t_1) + t_2^2 = t_1^2 + t_2^2 - t_3$$

$$t^2 = t_+ t_- + t_3^2 - t_3$$
$$t^2 = t_- t_+ + t_3^2 + t_3$$

$$t_+ t_- = t^2 - t_3^2 + t_3$$
$$t_- t_+ = t^2 - t_3^2 - t_3$$

Note $t^2 \notin \mathcal{L}$ but $[t^2, t_a] = 0$

$\Rightarrow t^2 \propto \mathbb{1}$ in every irreducible representation

~~$[t^2, t_a]$~~ $[t^2, t_a] = 0$, sufficient to show it for $a=1$

$$[t_1^2 + t_2^2 + t_3^2, t_1] = [t_2^2, t_1] + [t_3^2, t_1] =$$

$$= t_2^2 t_1 - t_1 t_2^2 + t_3^2 t_1 - t_1 t_3^2 =$$

$$= t_2^2 t_1 + t_2 t_1 t_2 - t_2 t_1 t_2 - t_1 t_2^2 + (2 \to 3)$$

$$t_2 [t_2, t_1] + [t_2, t_1] t_2 + t_3 [t_3, t_1] + [t_3, t_1] t_3$$

$$-i\, t_2 t_3 - i\, t_3 t_2 + i\, t_3 t_2 + i\, t_2 t_3 = 0$$

$t^2 \propto \mathbb{1}$; we can in addition assume that t_3 is diagonal = we assume a basis in which t_3 is diagonal

"Highest Weight" ψ^j $\qquad t_3 \psi^j = j \psi^j$

and $t_+ \psi^j = 0$

$$t_3 t_- \psi^j = ([t_3, t_-] + t_- t_3)\psi^j = (-t_- + t_- t_3)\psi^j =$$
$$= (j-1)\, t_- \psi^j$$

$\Rightarrow$ the vector $t_- \psi^j$ is eigenvector of t_3 with eigenvalue $j-1$!

$$t^2 \psi^j = (t_- t_+ + t_3^2 + t_3)\psi^j = (j^2 + j)\psi^j = j(j+1)$$

Since t^2 commutes with all t_a ~~all~~ vectors $t_a \psi^j$ have the same eigenvalue $j(j+1)$!

$\rightarrow$ characterizes the representation

$$t_3 (t_-)^k \varphi^j = (j-k)\,\varphi^j$$

suppose $(t_-)^{k+1} \varphi^j = 0$ finite dimensional representation

$$t_+ t_- (t_-)^k \varphi^j = 0 =$$

$$= (t^2 - t_3^2 + t_3)(t_-)^k \varphi^j = \left(j(j+1) - (j-k)^2 + (j-k)\right) = 0$$

$$\Rightarrow \cancel{j(j+1)} - \cancel{j^2} + j - \cancel{j^2} + 2jk - k^2 + j - k = 0$$

$$2j + 2jk - k^2 - k = 0$$

$$\cancel{j^2} + j - \cancel{j^2} + 2jk - k^2 + j - k = 0$$

$$\Rightarrow \boxed{j = \frac{1}{2} k}$$

$k \in \mathbb{N}_0$

$m: k, \dots, -k$

Normalization: $t_- \varphi^j_m$ $\cancel{= (t_-)^{m+1} \varphi^j}$

~~and suppose φ^j_m is normalized~~

$$(t_- \varphi^j_m, t_- \varphi^j_m) = (\varphi^j_m, t_+ t_- \varphi^j_m) =$$

$$= (\varphi^j_m, (t^2 - t_3^2 + t_3)\varphi^j_m) =$$

$$= j(j+1) - m^2 + m$$

$$(t_-)^q : \quad \psi^j_j, \ \psi^j_{j-1}, \ \psi^j_{j-2}, \dots, \ \psi^j_{-j}$$

$$m = +j, \dots, -j \qquad \psi^j_m \qquad t_3 \psi^j_m = m \psi^j_m$$

$$\| t_- \psi^j_m \| = (t_- \psi^j_m, t_- \psi^j_m) = (\psi^j_m, t_+ t_- \psi^j_m) =$$

$$= (\psi^j_m, (t^2 - t_3^2 + t_3) \psi^j_m) = [j(j+1) - m^2 + m] =$$

$$= (j+m)(j-m+1) \qquad = j(j+1) - jm + jm - m^2 + m \quad \checkmark$$

$$\Rightarrow \boxed{\begin{aligned} t_- \psi^j_m &= [(j+m)(j-m+1)]^{\frac{1}{2}} \psi^j_{m-1} \\ t_+ \psi^j_m &= [(j+m+1)(j-m)]^{\frac{1}{2}} \psi^j_{m+1} \end{aligned}}$$

From this we can construct the matrices $D^j_{mm'}$ for a $(2j+1) \times (2j+1)$ dimensional representation

$$D^j(t_1) = \frac{1}{2} \Big\{ \delta_{m+1,m'} [(j-m)(j+m+1)]^{\frac{1}{2}} + \\ + \delta_{m-1,m'} [(j+m)(j-m+1)]^{\frac{1}{2}} \Big\}$$

$$D^j(t_2) = \frac{-i}{2} \Big\{ \delta_{m+1,m'} [(j-m)(j+m+1)]^{\frac{1}{2}} - \\ - \delta_{m-1,m'} [(j-m)(j+m+1)]^{\frac{1}{2}} \Big\}$$

$$D^j(t_3) = m \, \delta_{m,m'}$$

these are completely general results.

Valid e.g. for function spaces

rotations acting on scalar functions

$$\vec{\psi}'(\vec{x}') = \psi(R\vec{x}) \Rightarrow \psi'(\vec{x}) = \psi(R^{-1}\vec{x})$$

to compute the generators on the function space

$R \sim 1 - i\omega^a t_a \vec{x}$

t_a = generators of $SO(3)$ = $-i\varepsilon_{iaj}$ = antisymmetric 3x3 matrices

$$\delta\psi = \psi'(x) - \psi(x) = \psi(1 - \omega^a \varepsilon_{iaj} x^j) - \psi(x) =$$

$$= i\omega^a t_a \psi = -\omega^a \varepsilon_{iaj} x^j$$

$$\Rightarrow \boxed{t_a = i\varepsilon_{aij} x_i \partial_j}$$

in polar coords

$$t_1 = -i\sin\varphi \frac{\partial}{\partial\theta} - \cot\theta\cos\varphi \frac{\partial}{\partial\varphi}$$

$$t_2 = i\cos\varphi \frac{\partial}{\partial\theta} - \cot\theta\sin\varphi \frac{\partial}{\partial\varphi}$$

$$t_3 = i\frac{\partial}{\partial\varphi}$$

$$t^2: \left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial}{\partial\varphi^2}\right]\psi^j_m(\theta,\varphi) = j(j+1)\psi^j_m$$

$$i\frac{\partial}{\partial\varphi}\psi^j_m(\theta,\varphi) = m\,\psi^j_m$$

$$\psi^j_m = Y_{jm}(\theta,\varphi) = (-)^m\left[\frac{(2j+1)(j-m)!}{4\pi(j+m)!}\right]^{\frac{1}{2}} e^{im\varphi} P^j_m(\cos\theta)$$

spherical harmonics

$$\boxed{t_a Y_{jm} = D^j_{mm'} Y^j_{m'}}$$

Clebsch - Gordon decomposition

Decomposition of a direct product representation into a sum of irreducible representations

SU(2) repr. D^{j_1} and D^{j_2}

$$\boxed{D^{j_1} \otimes D^{j_2} = \bigoplus_{j=j_1-j_2}^{j_1+j_2} D^j}$$

$$D^{j_1+j_2} \oplus D^{j_1+j_2-1} \oplus \dots \oplus D^{j_1-j_2}$$

convention $j_1 \geq j_2$

Sanity check: can this be true? How many states are there?

direct product $(2j_1+1)(2j_2+1)$ states

$$\sum_{j=j_1-j_2}^{j_1+j_2} (2j+1) = 2j_1+2j_2+1 + \underbrace{2j_1+2j_2 + \dots + 2j_1-2j_2+1}_{4j_1+2}$$

and there are $\frac{1}{2}\left[(j_1+j_2)-(j_1-j_2)+1\right]$ such pairs $= 2j_2+1$

$$\Rightarrow \quad \frac{1}{2}(2j_2+1)(4j_1+2) = (2j_1+1)(2j_2+1) \quad ✓$$

\# of states matches

operators $t_3^{j_1}, t_\pm^{j_1}$ and $t_3^{j_2}, t_\pm^{j_2}$

$$t_a^{j_1} \otimes t_a^{j_2} = t_3^{j_1} \otimes \mathbb{1} \oplus \mathbb{1} \otimes t_3^{j_2} = t_3^j$$

$$t_3^j \left(\psi_{m_1}^{j_1} \otimes \psi_{m_2}^{j_2}\right) = m_1 + m_2 = m$$

$$\psi_{m_1}^{j_1} \otimes \psi_{m_2}^{j_2} \quad ; \quad \Theta_m^j$$

$$\Theta_m^j = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \underbrace{(j_1 j_2 j m \mid j_1, m_1 ; j_2 m_2)}_{\text{Clebsch - Gordon coefficients}} \psi_m^{j_1} \otimes \psi_m^{j_2}$$

by assumption $\varphi^{j_1}_{m_1} \otimes \varphi^{j_2}_{m_2}$ is an orthonormal basis as well as Θ^j_m

$\Rightarrow$ matrix of Clebsch Gordan coefficients is unitary.

~~Construction of~~ $(j_1 j_2 j m \mid j_1 m_1 ; j_2 m_2) = 0$ if $m \neq m_1 + m_2$

highest possible value $m_1 = j_1$; $m_2 = j_2$

there is only 1 such state

$$\Rightarrow \quad \varphi^{j_1}_{j_1} \otimes \varphi^{j_2}_{j_2} = e^{i\varphi}\, \Theta^{j_1+j_2}_{j_1+j_2}$$

up to some arbitrary phase ($\varphi = 0$) convention choices the Clebsh-Gordan coefficients are unique. Chose $(j_1 j_2 j m \mid j_1 m_1 ; j_2 m_2)$ real and positive

for $m = m_1 + m_2 - 1$ there are 2 such states

$$\varphi^{j_1}_{j_1-1} \otimes \varphi^{j_2}_{j_2} \, , \quad \varphi^{j_1}_{j_1} \otimes \varphi^{j_1}_{j_1-1}$$

- one can be obtained by applying $t_-^{j_1+j_2}$ to $\Theta^{j_1+j_2}_{j_1+j_2}$
- the other one can be obtained by either looking for the linear combination orthogonal to $t_-^{j_1+j_2}\Theta^{j_1+j_2}_{j_1+j_2}$ or by solving

$$t_+^{j_1+j_2}\left(\alpha\, \varphi^{j_1}_{j_1-1} \otimes \varphi^{j_2}_{j_2} + \beta\, \varphi^{j_1}_{j_1} \otimes \varphi^{j_2}_{j_2-1}\right) = 0$$

and normalizing.

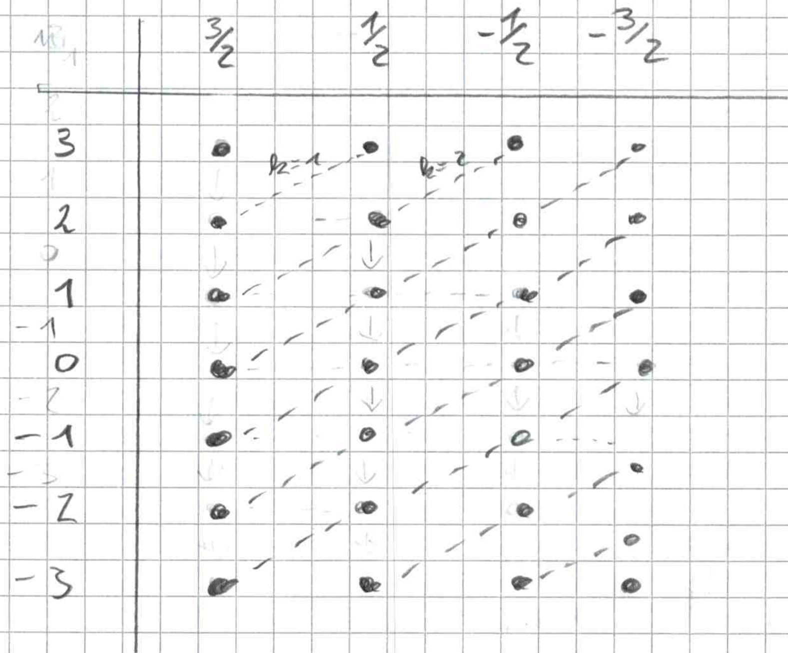
This new state is obviously the highest weight of the $j = j_1 + j_2 - 1$ representation of level k with $k \le 2j_2$ the values of (m_1, m_2) are (j_1-k, j_2) ; $(j_1-k+1, j_2-1) \dots (j_1, j_2-k)$

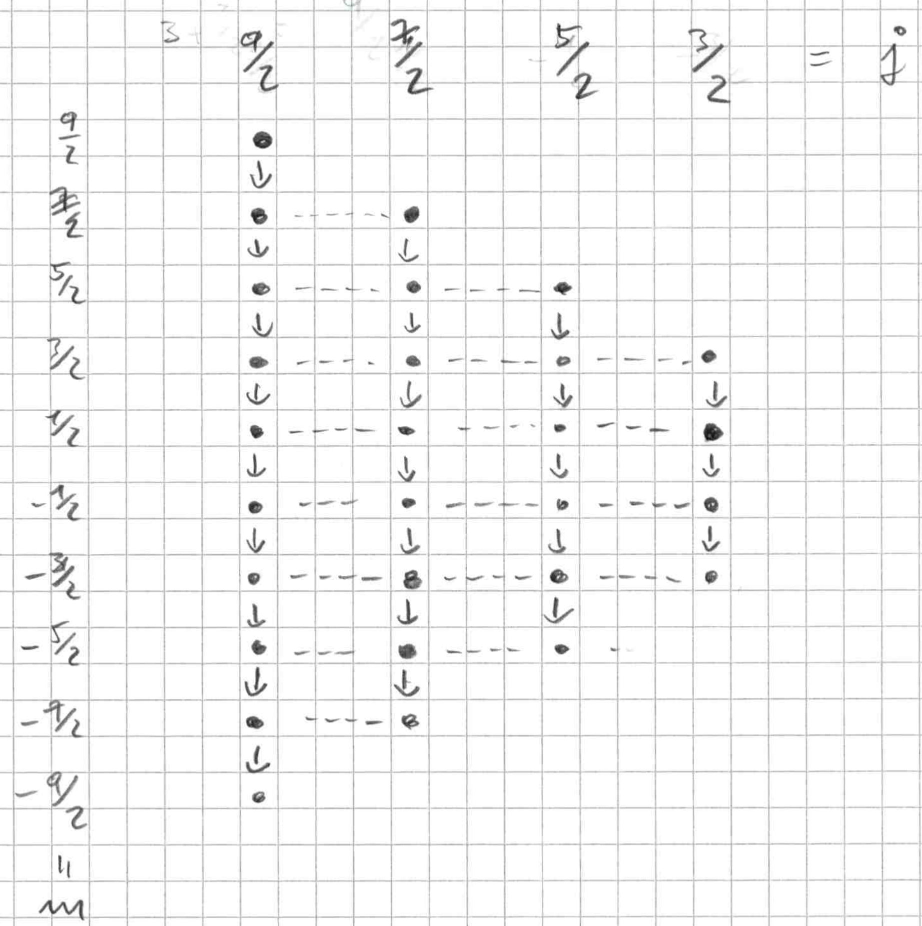
$k+1$ such states up to $k = 2j_2$ where there are $2j_2+1$ states. k of these states can be obtained from the states of level $(k-1)$ and the additional state can be obtained as before.

From then on all the rest of the states can be obtained by applying the lowering operator $t_-^{j_1+j_2}$.

Schematic diagram:

$j_1 = 3$ $\quad j_2 = 3/2$





Example $j_1 = \frac{1}{2} \otimes j_2 = \frac{1}{2}$

physics notat. $\psi^{\frac{1}{2}}_{\frac{1}{2}} \otimes \psi^{\frac{1}{2}}_{\frac{1}{2}} \sim |\frac{1}{2};\frac{1}{2}\rangle$

$$t_-^{\frac{1}{2}+\frac{1}{2}} = t_-^{\frac{1}{2}} \otimes \mathbb{1} + \mathbb{1} \otimes t_-^{\frac{1}{2}}$$

$$t_- \psi^j_m = \sqrt{(j+m)(j-m+1)}\,\psi^j_{m-1}$$

$$t_-^{\frac{1}{2}\otimes\frac{1}{2}} \psi^{\frac{1}{2}}_{\frac{1}{2}} = |-\tfrac{1}{2},\tfrac{1}{2}\rangle + |\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$t_-^1\, \Theta^1_1 = \sqrt{2}\;\Theta^1_0$$

$$\Rightarrow \Theta^1_0 = |1 0\rangle = \frac{1}{\sqrt{2}}\left(|-\tfrac{1}{2},\tfrac{1}{2}\rangle + |\tfrac{1}{2},-\tfrac{1}{2}\rangle\right)$$

$$t_+^{\frac{1}{2}\otimes\frac{1}{2}} : \; t_+\left(\alpha\,|-\tfrac{1}{2},\tfrac{1}{2}\rangle + \beta\,|\tfrac{1}{2},-\tfrac{1}{2}\rangle\right) =$$

$$= (\alpha+\beta)\,|\tfrac{1}{2},\tfrac{1}{2}\rangle \Rightarrow \alpha = -\beta$$

normalization $\quad \alpha = \frac{1}{\sqrt{2}}$

$$\Rightarrow |0,0\rangle = \frac{1}{\sqrt{2}}\left(|-\tfrac{1}{2},\tfrac{1}{2}\rangle - |\tfrac{1}{2},-\tfrac{1}{2}\rangle\right)$$

$$t_-\left[\frac{1}{\sqrt{2}}\left(|-\tfrac{1}{2},\tfrac{1}{2}\rangle + |\tfrac{1}{2},-\tfrac{1}{2}\rangle\right)\right. \longrightarrow |-\tfrac{1}{2},-\tfrac{1}{2}\rangle = |-1,-1\rangle$$

$$|1,1\rangle = |\tfrac{1}{2},\tfrac{1}{2}\rangle$$
$$|1,0\rangle = \frac{1}{\sqrt{2}}\left(|-\tfrac{1}{2},\tfrac{1}{2}\rangle + |\tfrac{1}{2},-\tfrac{1}{2}\rangle\right)$$
$$|1,-1\rangle = |-\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$|0,0\rangle = \frac{1}{\sqrt{2}}\left(|-\tfrac{1}{2},\tfrac{1}{2}\rangle - |\tfrac{1}{2},-\tfrac{1}{2}\rangle\right)$$

explicitly:

$$(\tfrac{1}{2},\tfrac{1}{2};11|\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}) = 1$$

$$(\tfrac{1}{2}\tfrac{1}{2};10|\tfrac{1}{2}\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2}) = (\tfrac{1}{2}\tfrac{1}{2},10|\tfrac{1}{2}\tfrac{1}{2};\tfrac{1}{2},-\tfrac{1}{2}) = \frac{1}{\sqrt{2}}$$

$$(\tfrac{1}{2}\tfrac{1}{2};00|\tfrac{1}{2}\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2}) = \frac{1}{\sqrt{2}} \qquad (\tfrac{1}{2}\tfrac{1}{2},00|\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}-\tfrac{1}{2}) = -\frac{1}{\sqrt{2}}$$

$$(\tfrac{1}{2}\tfrac{1}{2}\;1-1|\tfrac{1}{2}\tfrac{1}{2}\;-\tfrac{1}{2}-\tfrac{1}{2}) = 1$$

Table

		j / m	
m_1	m_2	$\cdot$	:

$\frac{1}{2}\otimes\frac{1}{2}$:

		1 / 1	1 / 0	0 / 0	1 / -1
$\frac{1}{2}$	$\frac{1}{2}$	1			
$\frac{1}{2}$	$-\frac{1}{2}$		$1/\sqrt{2}$	$1/\sqrt{2}$	
$-\frac{1}{2}$	$\frac{1}{2}$		$1/\sqrt{2}$	$-1/\sqrt{2}$	
$-\frac{1}{2}$	$-\frac{1}{2}$				1

Listed like that in the

PR

The structure of semi-simple Lie-algebras

Def: ◦) A Lie-algebra is simple if it does not possess an invariant subalgebra is not Abelian

◦) It is semi-simple if does not possess an Abelian invariant subalgebra

$su(2)$ ... simple

$su(2) \otimes su(3)$ semi-simple

Lie-groups: ◦) Lie group is simple if it does not possess an invariant continuous subgroup

$\Rightarrow$ ~~its~~ its Lie-algebra is simple.

◦) Lie group is semi-simple if its Lie-algebra is semi-simple

Killing-form: $\kappa: a, b \in \mathcal{L} \to \mathbb{R}$ (for a real algebra)

$$\kappa(a,b) = tr\left(ad(a), ad(b)\right)$$

Trace in the adjoint representation

Expls: $so(3)$:

$$a_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad a_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad a_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(a_p)_{qr} = \varepsilon_{qpr}$$

$$a_1 \cdot a_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad tr(a_1 \cdot a_2) = 0$$

$$a_1^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad tr(a_1^2) = -2 \qquad \text{etc.}$$

$$\Rightarrow \quad \kappa_{pq} = -2\delta_{pq} \qquad SO(3)$$

-) $sl(2,\mathbb{R})$ $\det S = 1$ S_{ij} real

$\rightarrow tr(b) = 0$

$$b_1 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \qquad b_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad b_3 = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$[b_1, b_2] = b_3 \qquad [b_2, b_3] = b_1 \qquad [b_3, b_1] = -b_2$$

$$ad(b_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad ad(b_2) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad ad(b_3) = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\kappa_{pq} = \begin{pmatrix} 2 & & 0 \\ & -2 & \\ 0 & & 2 \end{pmatrix}$$

The Killing form is "invariant"

$$\kappa([a,b],c) + \kappa(b,[a,c]) = 0$$

<u>Proof</u>: $tr(abc - bac + bac - bca) = 0$

<u>Theorem</u> The Killing form of a semi-simple Lie-algebra is non-degenerate

suppose $\mathcal{L}$ contains an Abelian invariant subalgebra $\mathcal{L}'$

take $a \in \mathcal{L}'$ $b \in \mathcal{L}$, basis $a_1 \dots a_{n'}$ $a_{n'+1} \dots a_n$

$$ad(a) = \begin{pmatrix} 0 & \cdots \\ 0 & 0 \end{pmatrix} \qquad ad(b) = \begin{pmatrix} \cdots & \cdots \\ 0 & \cdots \end{pmatrix}$$

$\rightarrow$ abelian $\qquad$ $\rightarrow$ invariance

$\rightarrow$ invariance

$\rightarrow$ other elements

$$tr(ad(a)\,ad(b)) \sim \begin{array}{c} \\ n' \\ n-n' \\ \\ \end{array}\begin{array}{c} n' \\ \begin{pmatrix} 0 & \cdots \\ 0 & \cdots \end{pmatrix} \\ n' \end{array} \qquad \det(\kappa) = 0$$

some more definitons (useful)

define •) $\mathcal{X}^{(k)} = [\mathcal{X}, \mathcal{X}^{(k-1)}]$

$\mathcal{X}^{(k)}$ is called trivial if it contains only $\{0\}$

•) $\mathcal{X}^{(k)}$ is an invariant subalgebra

remember if $\mathcal{X}', \mathcal{X}''$ are invariant subalgebra then $[\mathcal{X}', \mathcal{X}'']$ is invariant

$$[\mathcal{X}, [\mathcal{X}', \mathcal{X}'']] = [[\mathcal{X}, \mathcal{X}'], \mathcal{X}''] + [\mathcal{X}', [\mathcal{X}, \mathcal{X}'']]$$
$$= [\mathcal{X}', \mathcal{X}''] + [\mathcal{X}', \mathcal{X}'']$$

•) $[\mathcal{X}, \mathcal{X}] = \mathcal{X}^{(1)}$ $\quad [\mathcal{X}^{(1)}, \mathcal{X}^{(1)}]$ etc...

•) $\mathcal{X}$ is called solvable if $\mathcal{X}^{(k)}$ is trivial for some k

→) A Lie algebra is semi-simple if it does not contain an solvable invariant subalgebra

Theorem (Cartan's first criterium)
A Lie algebra $\mathcal{X}$ is solvable if and only if $\kappa(a,b) = 0 \quad \forall a, b \in \mathcal{X}^{(1)}$

Theorem: (Carton's second criterium)
A Lie algebra is semi-simple if and only if its Killing form is non-degenerate $\quad \det(\kappa) \neq 0$

Th: Every semi-simple Lie-algebra is either simple or the direct sum of simple Lie algebras

$$\mathcal{L}_{ss} = \mathcal{L}_s^1 \oplus \dots \oplus \mathcal{L}_s^n$$

Theorem: If $\mathcal{L}$ is semi-simple its adjoint representation is faithful

Proof: suppose $ad(a) = ad(b)$ but $a \neq b$
then $ad(a-b) = 0$ and thus
$$tr(ad(a-b), ad(c)) = 0 \quad \forall c \in \mathcal{L}$$
$\Rightarrow$ $\det K = 0$ $\Rightarrow$ it cannot be semi-simple by Cartan's first criterium.

Theorem If $\mathcal{L}$ is simple then its adjoint representation is irreducible.

Proof: the linear space V is the Lie algebra itself $V \simeq \mathcal{L}$ $\quad ad(a) \circ b = [a, b]$
since $\mathcal{L}$ is simple it does not have an invariant subspace (= subalgebra)
$\Rightarrow$ $ad(a)$ is irreducible.

Example $su(2)$: $\kappa_{pq} = -2\delta_{qq}$

so by Cartan 2^{nd} criterium it is semi-simple. But it also must be simple since otherwise $su(2) = \mathcal{L}_1 \oplus \mathcal{L}_2$

but $\dim(\mathcal{L}_i) \geq 2$ since otherwise the algebra would be Abelian

$\Rightarrow$ $su(2)$ is simple.

By construction of all simple Lie-algebras we will show that also $su(N)$ are simple

Example: show that $SO(4)$ is semi-simple but not simple by explicitly decomposing it as $SO(4) \simeq so(3) \times so(3)$

Example: $u(N)$; the generators are the anti-hermitian $N \times N$ matrices.

e.g. $su(2)$ $a_1 = \begin{pmatrix} 0 & i \\ +i & 0 \end{pmatrix}$ $a_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $a_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$

$a_4 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$

Lie algebra is $\mathbb{1} \oplus su(2)$ $\exists$ Abelian invariant subalgebra $\Rightarrow$ it is not simple nor semi-simple

Examples: Euclidean group $\vec{x} \to O\vec{x} + \vec{t}$

Translation are (3) invariant Abelian subgroups

Similarly: Poincare group $\Lambda \eta \Lambda^T = \eta$

$\eta = \mathrm{diag}(1,-1,-1,-1)$ $x \to \Lambda x + a$

Complexification of a real Lie-algebra

We know the generators are linearly independent over the real numbers.

e.g. $su(2)$ $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$; $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$; $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$

are independent even over $\mathbb{C}$

but $sl(2,\mathbb{C})$ as a real Lie algebra has 6 generators

$a_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $a_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $a_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$a_4 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ $a_3 = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}$ $a_4 = \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix}$

$a_1 = -i a_4$ etc...

$sl(2,\mathbb{C})$ has only 3 generators as complexified Lie algebra

Formally: form pairs of generators

(a, b)

and define $(\lambda + i\mu)(a,b) = (\lambda a - \mu b, \lambda b + \mu a)$

$(a, b) = a + ib$

generators have to be linearily independent over the complex numbers.

$sl(2,\mathbb{C})$ is of dimension 3 as a complex Lie-algebra

$[a+ib, c+id] = [a,c] - [b,d] + i([b,c] + [a,d])$

$\downarrow \qquad \downarrow$

$(\qquad , \qquad)$

$[(a,b),(c,d)] = ([a,c]-[b,d], [b,c]+[a,d])$

In particular we can chose a basis in the complexified Lie-algebra

$$(a_p, 0)$$

the $[(a_p, 0), (a_q, 0)] = c_{pq}{}^{r} (a_r, 0)$

a_p is a basis for the real Lie algebra with the same structure constants.

a) $sl(2, \mathbb{C})$: $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$

or $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}; \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

they are equivalent as complex Lie algebras but not as real Lie-algebras

$$sl(2, \mathbb{R}) \not\simeq su(2, \mathbb{R})$$

$$\text{but } sl(2, \mathbb{C}) \simeq su(2, \mathbb{C})$$

If the complex Lie-algebra $\tilde{\mathcal{L}}$ is simple the a real form $\mathcal{L}$ is also simple.

Note the converse is not true!

A physically relevant example $SO(3,1)$

$$\Lambda^T \eta \Lambda = \eta \qquad \eta = (-1, -1, -1, +1)$$

generators : $a_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $a_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $a_3 = \begin{pmatrix} 0 & 1 & \\ -1 & 0 & 0 \\ & 0 & 0 \end{pmatrix}$

$a_4 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & & \cdots & \\ 0 & & \cdots & \\ -1 & & & 0 \end{pmatrix}$ $a_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & & \cdots & \\ 0 & 0 & -1 & 0 \end{pmatrix}$ $a_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$

algebra p, q ... 1 ... 3

$$[a_p, a_q] = -\varepsilon_{pqr} a_r$$
$$[a_p, a_{q+3}] = -\varepsilon_{pqr} a_{r+3}$$
$$[a_{p+3}, a_{q+3}] = \varepsilon_{pqr} a_r$$

a_p ... rotations
a_{p+3} ... Boosts

→ No invariant subalgebra as real Lie-algebra but as complex algebra we are allowed t form the linear combinations

$$b_p = a_p + i a_{p+3} \qquad \tilde{b}_p = a_p - i a_{p+3}$$

$$[b_p, b_q] = -\varepsilon_{pqr} b_r$$
$$[b_p, \tilde{b}_q] = 0$$
$$[\tilde{b}_p, \tilde{b}_q] = -\varepsilon_{pqr} \tilde{b}_r$$

but this is $su(2) \oplus su(2)$!

<u>Theorem</u> if $\mathcal{L}$ is simple but its complexification $\tilde{\mathcal{L}}$ is not then $\tilde{\mathcal{L}}$ is the direct sum of two isomorphic simple Lie-algebras

<u>Theorem</u>: A representation of $\tilde{\mathcal{L}}$ provides a representation of $\mathcal{L}$ and vice versa

$$\Gamma_{\tilde{\mathcal{L}}}(a_p, 0) = \Gamma_{\mathcal{L}}(a_p)$$