

Outline MAV

Advanced Mathematics

- A) Group Theory
 - B) Differential Geometry
 - C) Statistics
- } K.L.
- } Jose del Peso

Part A Group Theory:

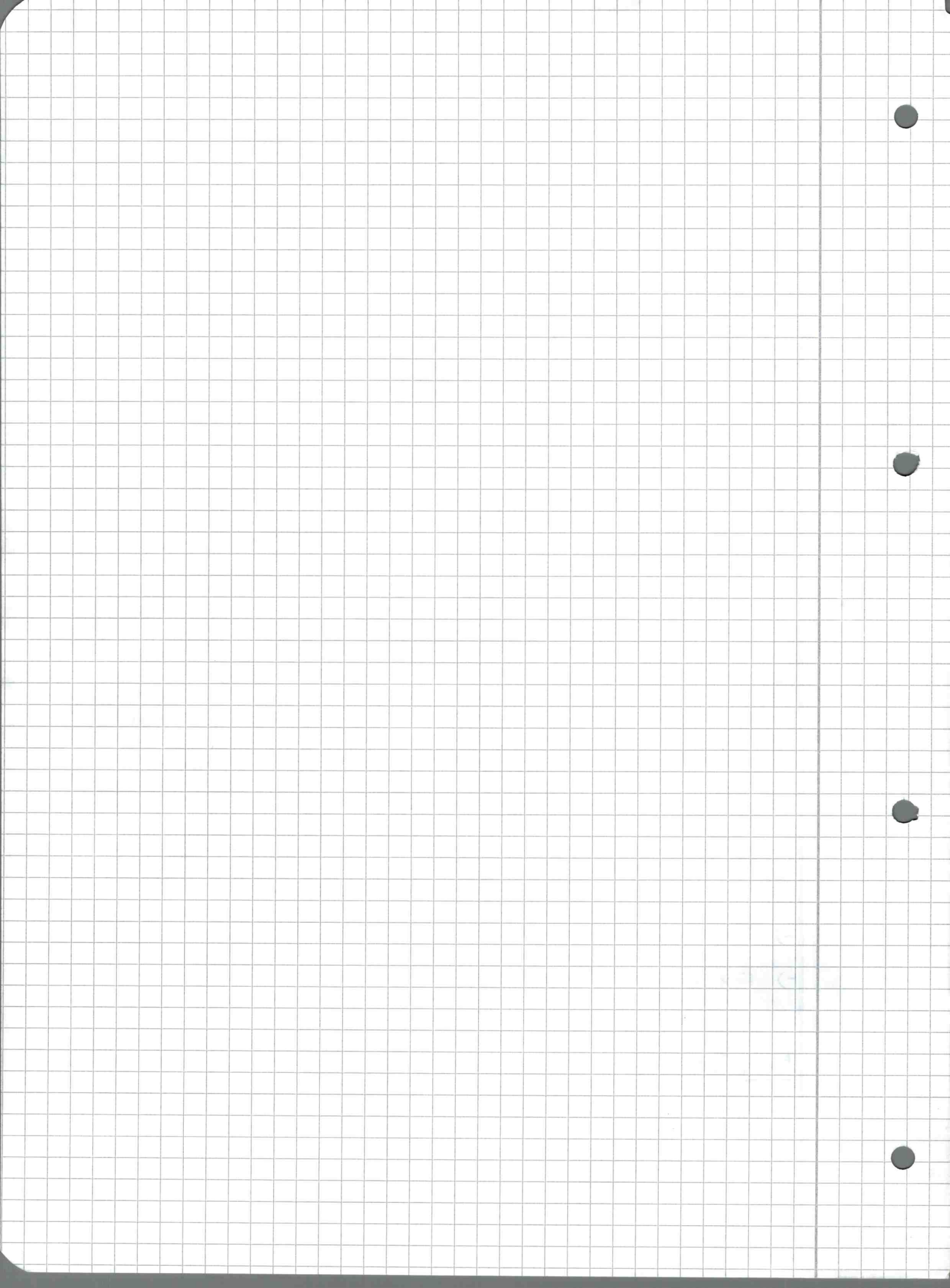
- > Introduction, concepts, structure
- > Lie-groups
- > Representations
- > Lie-Algebras
- > Classification
- > Lorentz-group, Spinors, ...

- Books:
- > H. Georgi - Lie Algebras in Physics
 - > A. Zee - Group Theory in a Nutshell
 - > Jones - Group, Representations and Physics
 - > Cornwell - Group Theory in Physics

Vol 1, 2, 3 and bridged Introduction

- > Exercise sheet
- > Exam Nov. 23

- Part B:
- > Nakahara - Geometry, Topology and Physics
 - > Nash, Sen - Topology and Geometry for Physicists



Lectures on Group Theory

In physics group theory is the theory of transformations of physical objects.

Example: rotations (multiplication, non-commutativity, topology = belt trick)

Mathematics group is an "algebraic structure" (a set with certain rules to manipulate elements of the set get other elements of the set)

Definition Group

A set G is called a "group" if the following axioms hold

- (a) Multiplication $\forall g_1, g_2 \in G \quad g_1 \circ g_2 \in G$
- (b) Associativity $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$
- (c) \exists identity $e \in G: e \circ g = g \circ e = g$
- (d) \exists inverse $g^{-1} \in G \quad g^{-1} \circ g = g \circ g^{-1} = e$

(only $e \circ g = g$ and $g^{-1} \circ g = e$ necessary!)

Examples:

(a) real numbers $r \in \mathbb{R}^+$ $r_1 \cdot r_2 = r_3 \quad r_3 \in \mathbb{R}$
 (\mathbb{R}, \cdot)
 $(r_1 \cdot r_2) \cdot r_3 = r_1 \cdot (r_2 \cdot r_3)$
 $1 = e$
 $r^{-1} = \frac{1}{r}$

(b) $(\mathbb{Z}, +)$ u_1, u_2 $u_1 + u_2 = u_3$ Integer
 $0 = e$
 $-u_1 = u_1^{-1}$ inverse

(c) $(\mathbb{Z}, -)$ is not a group \Rightarrow
 $u_1 - (u_2 - u_3) \neq (u_1 - u_2) - u_3 !$

Expl: $SO(2)$ $O^T O = \mathbb{1}_2$ det

$\Rightarrow \det(\theta) = \pm 1$

$\theta = \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix}, \quad \theta_1 \cdot \theta_2 = \theta_2 \cdot \theta_1$

"continuous" group

discrete group: $\mathbb{Z}_2: \{1, -1\}$

$\mathbb{Z}_N = \left\{ 1, e^{i\frac{2\pi}{N}}, e^{i\frac{4\pi}{N}}, \dots, e^{i\frac{(N-1)2\pi}{N}} \right\}$

all these groups are Abelian $g_1 \circ g_2 = g_2 \circ g_1$

Non-abelian discrete group: {

$\pi_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\pi_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\pi_3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

$\pi_4 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ $\pi_5 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\pi_6 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$\pi_7 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\pi_8 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

$\pi_5 \cdot \pi_7 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \pi_4$

Exercise

$\pi_2 \cdot \pi_5 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \pi_2$

check group properties

Def order = #elements

continuous non-abelian groups: expl $SO(3), SU(N)$

$SU(N)$: $\det(U) = 1$ $U^\dagger U = \mathbb{1}$ $N \times N$ matrices

1) $u_1 u_2$: $\det(u_1) \det(u_2) = 1$ \checkmark

2) $(u_1 u_2)^\dagger = u_2^\dagger u_1^\dagger \Rightarrow (u_1 u_2)^\dagger u_1 u_2 = \mathbb{1}$ \checkmark

3) $\mathbb{1}_N \in SU(N)$ \checkmark

4) $u^{-1} = u^\dagger$ \checkmark

5) associativity of matrix-multiplication \checkmark

Additional \rightarrow

Proofs $e \circ g \circ e \circ g = g \Rightarrow g \circ e = g$

\bar{g}^{-1} exists $g_1 \circ g_2 = g_3 \stackrel{?}{=} \bar{g}_3^{-1}$

$\bar{g}_3^{-1} = \bar{g}_2^{-1} \circ \bar{g}_1^{-1}$

$(e \circ g)^{-1} = e \circ \bar{g}^{-1}$

take $\bar{g}^{-1} = h$ some group element

Since $g \circ \bar{g}^{-1} = e \Rightarrow \bar{g}^{-1} \circ g = e \quad (\bar{g}^{-1} \circ \bar{g}^{-1} = e \quad \bar{g}^{-1} = \bar{g}^{-1} \circ e)$

Expl: Multiplication Table

$G = \{e, a, b\}$ $a \circ a = g$ b $a \circ b = b \circ a = e$ $b \circ b = a$

	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

$e = 1$

$e = 1$

$a = e^{i\frac{2\pi}{3}}$

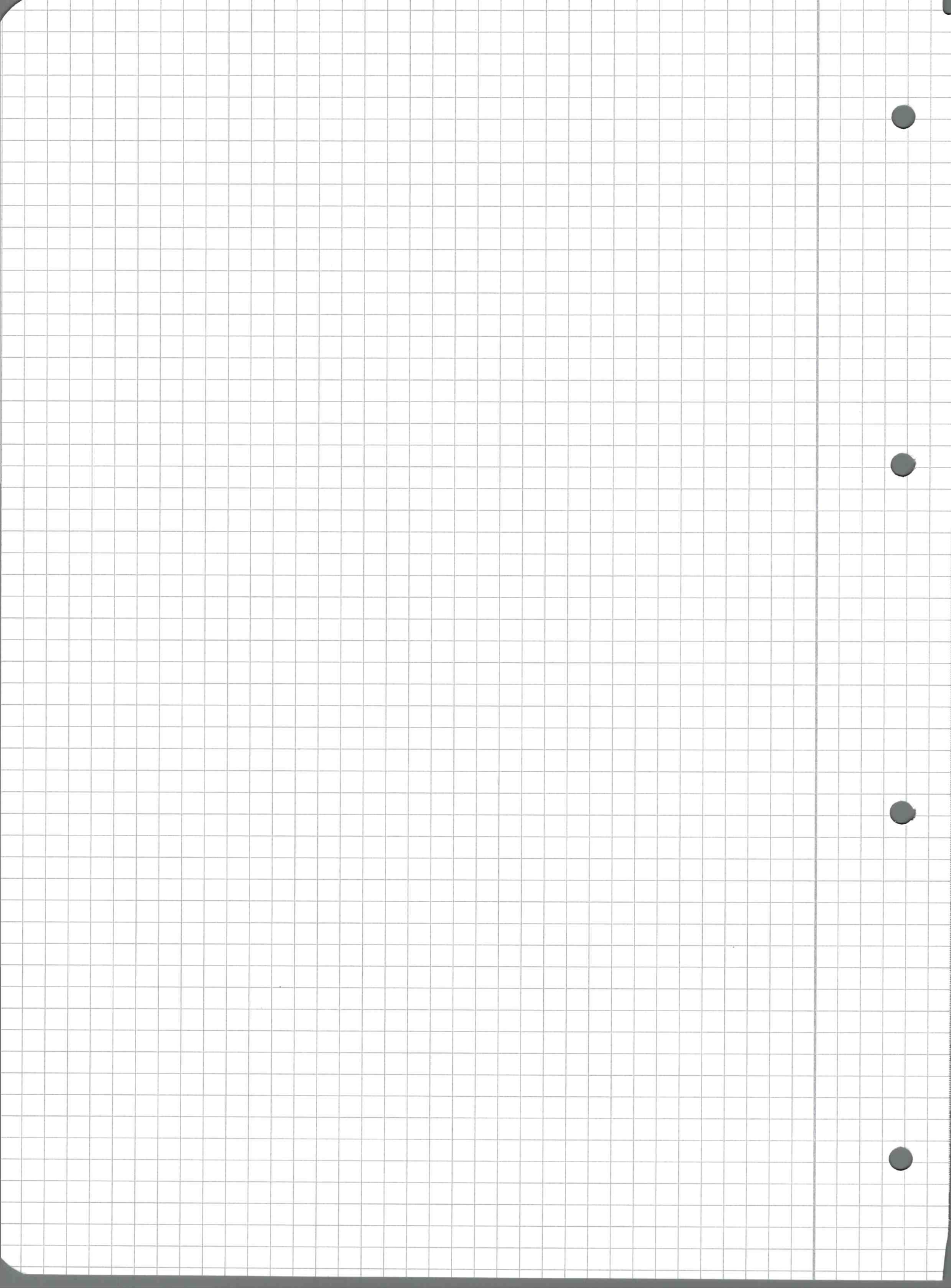
$a = e^{i\frac{4\pi}{3}}$

$b = e^{i\frac{4\pi}{3}}$

$b = e^{i\frac{2\pi}{3}}$

$a = b = e = 1$

(representations)



(3)

Def. Homomorphism: is a map from group G_1 to group G_2 which preserves the group structure

$$\phi: (G, \circ) \rightarrow (H, *)$$

$$\phi(g_1 \circ g_2) = \phi(g_1) * \phi(g_2)$$

If the homomorphism is one-to-one it is called an isomorphism

Exp(1) $(\mathbb{R}, +) \rightarrow (\mathbb{R}/\{0\}, \cdot)$

$$\begin{array}{ccc} & \downarrow & \\ x+y & e^x & \\ & \rightarrow & \\ x+y & \rightarrow & e^{x+y} = e^x \cdot e^y \quad \checkmark \end{array}$$

But not bijective: negative numbers are not in the image of e^x

$$[(\mathbb{R}, +) \rightarrow (\mathbb{R}_+, \cdot) \text{ is bijective}]$$

$$2.) (\mathbb{R}, +) \rightarrow (S^1, \cdot) \\ e^{ix}$$

Note $(G, \circ) \rightarrow \{E\}$ is a trivial homomorphism

$\{e\}$ - trivial group

Definition Subgroup S : is a subset which by itself is a group under the group multiplication law

Def: "trivial" subgroups: $G, \{E\}$

Sub groups which are not trivial are called "proper"

Theorem: Let S be a subset of G . If $s's^{-1} \in S$ for all $s', s \in S$ then S is a subgroup

Proof: (c) $s' = s \Rightarrow s s^{-1} = E \Rightarrow E \in S$

(d) $s' = E \Rightarrow E s^{-1} = s^{-1} \Rightarrow s^{-1} \in S$

(a) $s^{-1} \in S \Rightarrow (s^{-1})^{-1} \in S \Rightarrow s' (s^{-1})^{-1} = s' s \in S$

Rearrangement Theorem: For every fixed element $g' \in G$ the sets $g'o g$ (and $g o g'$) $g \in G$ contain all elements of G precisely once

Proof: take $g'' \in G$ and $g = (g')^{-1} o g'' \Rightarrow g'o g = g''$
 \Rightarrow all $g'' \in G \in g'o g \checkmark$

unique - uniquely assume $g'o g_1 = g'o g_2 \quad g_1 \neq g_2$
 impossible $(g')^{-1} o \Rightarrow g_1 = g_2$

Multiplication Table

	g_1	g_2	g_3
g_1	g_3	g_1	g_2
g_2	g_2	\dots	\dots
g_3	g_1	\dots	\dots

\leftarrow every element once

\rightarrow Permutation group & Cayley's theorem

Definition conjugate elements: if $\exists x \in G$ for $g, g' \in G$ such that $g' = x \circ g \circ x^{-1}$

g' conjugate to g (mutual)

If g conjugate to g' and g' to $g'' \Rightarrow g$ conj to g''

Definition Conjugacy class of h all elements g such that $g = x \circ h \circ x^{-1} \quad \forall x \in G$

- examples
- $\rightarrow h = E$; conjugacy class is E itself
 - \rightarrow Abelian group every element is a conjugacy class by itself

- Theorem
- $\rightarrow E$ is a class by itself
 - \rightarrow every element is member of some conjugacy class
 - \rightarrow No element can be member of two different classes

Proof: \rightarrow consider $h = g \circ h \circ g^{-1}$ with $g = h$

$$h \circ h \circ h^{-1} = h \Rightarrow h \in [h]$$

$\rightarrow g \in x \circ h_1 \circ x^{-1} \quad g \in x \circ h_2 \circ x^{-1}$

$$g \in [h] \Rightarrow h_1 \in [g] \quad h_2 \in [g]$$

$$[h_1] = [h_2]$$

Definition: Invariant subgroup S : subgroup

$\forall s \in S$ and $\forall g \in G$: $g s g^{-1} \in S$
also "normal subgroup"

Universal normal subgroups: G itself and $\{E\}$

Theorem: a subgroup S is normal if it consists of entirely of classes

→ if S is normal $\Rightarrow S$ is made up of classes

→ suppose S is a set consisting entirely of classes $\Rightarrow \forall s \in S$ $x s x^{-1} \in S$
 $\Rightarrow S$ is invariant

Dff.: G is "simple" if its only normal subgroups are $\{e\}$ and G

Definition: Coset : pick an element $g \in G$ and a subgroup H

$$gH := \{goh \mid \forall h \in H\}$$

left coset of H with respect to g

$$Hg = \{hog \mid \forall h \in H\}$$

right coset of H with respect to g

Theorems a) If $g \in H$ then $Hh = hH = H$ (Rearrangement)

b) If $g \notin H$ then gH is not a subgroup

- If gH is a subgroup it must contain the identity: $e = gh^{-1} \Rightarrow \bar{g}^{-1} = h \Rightarrow \bar{g}^{-1} \in H \Rightarrow g \in H \Rightarrow$ if $g \notin H$ gH is not a subgroup

c) Every $g \in G$ is member of some right coset

$$H \text{ contains } E \Rightarrow g = g \circ E \in gH$$

d) $gh_1 \neq gh_2$ different elements

If H is of finite order s gH contains s different elements

e) right cosets are either identical or contain no common elements

-) suppose $g_1 h_1 = g_2 h_2$ $h_1, h_2 \in H$

$$\Rightarrow g_2^{-1} g_1 = h_2 h_1^{-1} = h' \in H$$

$$\Rightarrow \bar{g}_2^{-1} g_1 H = H$$

$$\Rightarrow g_2 H = g_2 \bar{g}_2^{-1} g_1 H = g_1 H \quad \text{qed}$$

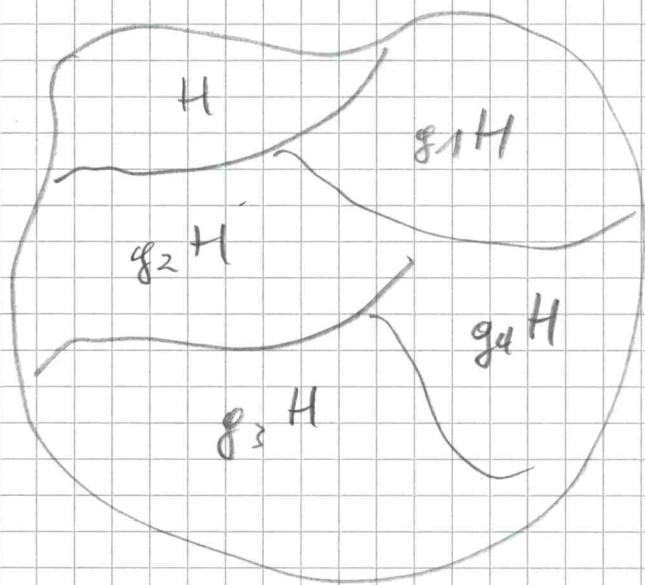
f) If $g' \in gH$ then $g'H = gH$

-) since $g'H = (gH)H = gH$

$$g' \in gH \quad g'h' = gh'h' = gh'' \Rightarrow g'H = gH$$

g) If G, H are finite of order r, s then the number of discrete cosets is r/s

Cosets form a partition of G



Note: if n is prime: no proper subgroups: \mathbb{Z}_3

Theorem: left and right cosets are identical
iff H is normal

$$gH = Hg$$

$$gh_1 = h_2g \iff goh_1og^{-1} = h_2oH$$

Definition: Quotient Group (Factor Group) G/H

If H is a normal subgroup the product on coset space

$$g_1H \circ g_2H = (g_1og_2)H$$

defines a group structure

Suppose g'_1 is a different representative of g_1H

$$\exists h'_1: g'_1h'_1 = g_1h_1, \quad g'_2 \text{ from } g_2H$$

$$\left. \begin{aligned} g'_1 \circ g'_2 & \quad g_1h'_1og_2oh_2 = g_1og_2oh_1oh_2 \\ & \quad g_1h'_1og'_2oh'_2 = g'_1og'_2oh'_1oh'_2 \end{aligned} \right\} (g_1og_2)H$$

Multiplication ✓

Associative: $g_1 H \circ (g_2 H \circ g_3 H) = (g_1 H \circ g_2 H) \circ g_3 H$ ✓

Identity: $E = H$ $g H \circ H = g H$ $H \circ g \cdot H = g H H$

Inverse: $g^{-1} H \circ g H = (g^{-1} \circ g) H = H$ ✓

Example 1) $(\mathbb{R}^N, +) = \vec{v} \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix}$ sub $\vec{v} + \vec{w}$

subgroup $H: (\mathbb{R}^P, +) = \begin{pmatrix} v_1 \\ \vdots \\ v_P \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$(\mathbb{R}^N, +) / (\mathbb{R}^P, +) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ v_{P+1} \\ \vdots \\ v_N \end{pmatrix} = \mathbb{R}^{N-P}$

($G/H \times H = G$ in this case)

Z4 $Z_4 = \{e, g, g^2, g^3\}$ $g^4 = e$ $g = e^{i\frac{\pi}{2}}$

Normal subgroup $\{1, g^2\} = Z_2$

Cosets: $[e] = \{e, g^2\}$ $[e] \cdot [e] = [e]$
 $[g] = \{g, g^3\}$ $[e] \cdot [g] = [g]$
 $[g] \cdot [g] = [e]$ } Z_2

($G/H \times H = Z_2 \times Z_2 \neq Z_4$)

\downarrow
 $\left. \begin{matrix} (e, e) & (e, g) \\ (g, e) & (g, g) \end{matrix} \right\} g^2 = 1 \text{ for } Z_2 \times Z_2, \text{ but not for } Z_4$

$G/H \times H = G$ iff $G/H \cong P$ and P is normal in G

REM: Homomorphism $\phi: (G_1, \circ) \rightarrow (G_2, *)$

$$\phi(g_1) * \phi(g_2) = \phi(g_1 \circ g_2)$$

Definition: Representation of a group: Homomorphism from G onto non-singular $d \times d$ matrices with matrix multiplication as product

Example $\mathbb{Z}_2 \{ \pm 1 \} \rightarrow (1, -1)$ 1-dimensional rep

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{2-dimensional repr.}$$

If ϕ is bijective it is called "isomorphism" and the representation is "faithful"

o) Isomorphic groups have identical representations

$$G_1 \cong G_2 \quad \phi, \phi^{-1} \quad G_1 \leftrightarrow G_2$$

$\chi(G_2)$ is a repr. of G_2 then $\chi(\phi(G_1))$ is a representation of G_1

$\chi(G_1)$ is a repr. of G_1 then $\chi(\phi^{-1}(G_2))$ is a repr. of G_2

Definition Kernel: Let ϕ be a homomorphism map from G_1 to G_2 then

$$\ker(\phi) = \{ g_1 \in G_1 \mid \phi(g) = e_2 \} \quad \text{is called}$$

Kernel of ϕ

Definition Image of ϕ

$$\text{Im}(\phi) = \{ g_2 \in G_2 \mid \exists g_1 \in G_1 \text{ st. } \phi(g_1) = g_2 \}$$

First Homomorphism Theorem:

Let ϕ be a homomorphism mapping $G \rightarrow G'$ with kernel $\ker(\phi) = K$

- o) K is a normal subgroup of G
- o) every element of the right coset gK maps onto the same element in G'
- o) define $\theta(gK) = \phi(g)$, θ is an isomorphism $G/K \cong G' \cong \text{Im}(\phi)$

$\rightarrow K$ is a subgroup $x_1, x_2 \in K \quad \phi(x_1 \cdot x_2) = \phi(x_1) \cdot \phi(x_2) = e'$

$\rightarrow x_1 \cdot x_2 \in K$

suppose $x \in K \quad x^{-1}$: $\phi(x x^{-1}) = \phi(e) = e' = \phi(x) \phi(x^{-1}) = e' \cdot \phi(x^{-1}) = e'$

$\Rightarrow \phi(x^{-1}) = e' \Rightarrow x^{-1} \in K$

$\phi(e x) = \phi(e) \phi(x) = \phi(x) = e' = \phi(e) e' \Rightarrow \phi(e) = e'$

$\Rightarrow e \in K$

$\forall g \in G \quad \forall x \in K \quad \phi(g x g^{-1}) = \phi(g) \phi(x) \phi(g^{-1}) = \phi(g) \phi(x) = \phi(g) e' = \phi(g)$

$\Rightarrow g x g^{-1} \in K \Rightarrow K$ is normal

$\rightarrow \phi(g x) = \phi(g) \phi(x) = \phi(g) \quad \phi(g K) = \phi(g)$

$\rightarrow \theta(g_1 K) \theta(g_2 K) = \phi(g_1) \phi(g_2) = \phi(g_1 g_2) = \theta(g_1 g_2 K) = \theta(g_1 K \cdot g_2 K)$

$G/K \cong G' = \text{Im}(\phi)$

Corollary: ϕ is an isomorphism if $\ker(\phi) = e$

Definition Automorphism $\phi: G \rightarrow G$ isomorphic

inner automorphism $x \in G$

$$\phi_x: g \rightarrow x \circ g \circ x^{-1}$$

otherwise "outer automorphism"

Definition Center $Z(G) = \{g \in G \mid g \circ h = h \circ g \ \forall h \in G\}$

obviously $Z(G)$ is a normal subgroup

eg. $SU(N)$: $e^{i\frac{2\pi}{N}}$ U_N $\det = 1$

Definition Direct Product $G_1 \otimes G_2$

$$\forall g_1 \in G_1, g_2 \in G_2 \text{ define } (g_1, g_2) \circ (g'_1, g'_2) = (g_1 g'_1, g_2 g'_2)$$

this provides a group structure

$$\text{obviously } (g_1, e_2) \cong G_1 \quad (e_1, g_2) \cong G_2$$

$$(g_1, e_2) \text{ commutes with } (e_1, g_2)$$

common element (e, e)

exmpl. $Z_2 \times Z_2$

Definition If $G \cong G_1 \otimes G_2$ G is called a direct product group

G_1 and G_2 are normal subgroups

Theorem If G possesses two subgroups G_1, G_2 such that

-) elements of G_1 commute with elements of G_2
 -) G_1 and G_2 have only the identity in common
 -) $\forall g \in G \exists g_1 \in G_1 \& g_2 \in G_2$ such that $g = g_1 \circ g_2$
- then G is a direct product group
 $G \cong G_1 \otimes G_2$

Example $O(3) \quad O^T O = \mathbb{1}_3$

$G_1 = SO(3) \quad \det \theta = +1$
 $G_2 = (\mathbb{1}_3 - \mathbb{1}_3) \quad \mathbb{Z}_2 \quad O(3) = \mathbb{Z}_2 \otimes SO(3)$
 Note $U(N) \neq U(N) \times SU(N)$

Definition Semi-direct product

G has two subgroups G_1, G_2 such that

-) G_1 is a normal subgroup
-) G_1 and G_2 have only the identity in common
-) $\forall g \in G \exists g_1 \in G_1, g_2 \in G_2 \quad g = g_1 \circ g_2$

$G \cong G_1 \ltimes G_2$

Expl: Euclidean group in \mathbb{R}^3

$\vec{x}' \rightarrow \theta_1 \cdot \vec{x} + \vec{t}_1$ rotation + translation

$x'' = \theta_2 \circ \theta_1 \vec{x} + \theta_2 \vec{t}_1 + \vec{t}_2$

$(\theta_2, \vec{t}_2) \circ (\theta_1, \vec{t}_1) = (\theta_2 \theta_1, \theta_2 \vec{t}_1 + \vec{t}_2) \checkmark$

identity $(\mathbb{1}_3, 0)$

inverse $(\theta^T, -\theta^T t) \circ (0, t) = (\theta^T 0, \theta^T t - \theta^T t) = E$

$$(0, t) \circ (\theta^T, -\theta^T t) = (0\theta^T, -0\theta^T t + t) = e$$

subgroups $\mathcal{G}_1 = (\mathbb{1}, t)$ translations

$\mathcal{G}_2 = (R, 0)$ rotations

but only \mathcal{G}_1 is invariant

$$\begin{aligned} (\theta, t) \circ (\mathbb{1}, \vec{a}) \circ (\theta^T, -\theta^T t) &= (\theta, \theta a + t) \circ (\theta^T, -\theta^T t) = \\ &= (0\theta^T, -\cancel{t} + \theta a + t) = (\mathbb{1}, \theta \vec{a}) \end{aligned}$$

translation

$$\begin{aligned} (\theta, t) \circ (R, 0) \circ (\theta^T, -\theta^T t) &= (R\theta, t) \circ (\theta^T, -\theta^T t) = \\ &= (R\theta^T, -R\theta^T t + t) \end{aligned}$$

↳ rotation prevents cancellation

REM: In terms of Matrices

$$g = \begin{pmatrix} \theta & \vec{t} \\ 0 & 1 \end{pmatrix}$$

$$e = \begin{pmatrix} \mathbb{1} & \vec{0} \\ 0 & 1 \end{pmatrix}$$

$$g^{-1} = \begin{pmatrix} \theta^T & -\theta^T t \\ 0 & 1 \end{pmatrix}$$

Lie groups

Lie groups are of highest importance in physics
(more specifically: "linear" Lie groups)

Aspects: - topological space
 - analytic manifold } later here: algebraic

Key feature: group elements depend on ~~some~~ number of continuous parameters ξ_1, \dots, ξ_n

Lie-group

Definition: A (linear) Lie group of dimension n is a group satisfying the following 4 conditions

1) It possesses at least one faithful finite dimensional representation ($d \times d$ matrices)

define Norm $\|g - g'\| = \left[\sum_{j,k=1}^d |\Gamma_{jk}(g) - \Gamma_{jk}(g')|^2 \right]^{1/2}$

d^2 -Complex norm (Topology)

sphere centered at the identity, M_δ
 $\|g - E\| < \delta$

2) In M_δ \exists n -parametrization $g(\xi_1, \dots, \xi_n)$
every $\vec{\xi}$ corresponds to one element, $E: \vec{\xi} = 0$

3) $\exists \eta > 0$ $\sqrt{\sum_{i=1}^n \xi_i^2} < \eta^2$ in \mathbb{R}_η every point in \mathbb{R}_η corresponds to an element in M_δ

4) Each $\Gamma_{jk}(\vec{\xi})$ is an analytic function of ξ_1, \dots, ξ_n

(analytic = expandable in a power series)

Define $(a_p)_{j\bar{k}} = \frac{\partial \Gamma_{jk}}{\partial \xi^p} \Big|_{\vec{\xi}=0}$

these are $n \times n$ matrices. They form the basis of a real linear vector space

Proof: $\Gamma_{j\bar{k}} = A_{ij} + i B_{ij} \quad \{A_{11}, A_{21}, \dots, B_{11}, B_{21}, \dots\}$

analyticity in n parameters: basis $C_1 - C_n$ and the others A 's and B 's are functions of $C_1(\vec{\xi}) - C_n(\vec{\xi})$

then form the vectors $\frac{\partial C_j}{\partial \xi^p}$

because of one-to-one correspondence

$$\det \left(\frac{\partial C_j}{\partial \xi^p} \right) \neq 0$$

$$\Rightarrow \sum \lambda_p \left(\frac{\partial C_j}{\partial \xi^p} \right) = 0 \quad \text{has only } \lambda_p = 0$$

as solutions

$$\Rightarrow \frac{\partial \Gamma}{\partial \xi^p} = a_p \quad \text{span a real linear vector space}$$

Note a_p can be complex of course

a_p span the "Lie-algebra"

in M_8 : inf $\vec{\xi} \quad \Gamma = \sum \frac{i \bar{\xi}^p \xi^q}{p \bar{q}} a_p \quad \xi$ real

condition - 1-to-one $\mathbb{R}_7 \rightarrow M_8$

$$\Rightarrow \sum \xi^p a_p = 0 \quad \text{only if } \xi^p = 0$$

(Tangent space) $a_p \dots$ Lie-algebra

Group multiplication

$$\begin{array}{ccc}
 g \cdot g' = g'' \\
 \downarrow \quad \downarrow \quad \downarrow \\
 \xi \quad \xi' \quad \xi''
 \end{array}$$

It follows from the above that

$$\xi'' = f_i(\xi_1, \dots, \xi_n, \xi'_1, \dots, \xi'_n)$$

and f_i are analytic and obey

$$\begin{aligned}
 f_i(0, \dots, 0, \xi'_1, \dots, \xi'_n) &= \xi'_i \\
 f_i(\xi_{n+1}, \xi_n, 0, \dots, 0) &= \xi_i
 \end{aligned}$$

$$(g \cdot E = g \quad ; \quad E \cdot g' = g')$$

\Rightarrow If g lies in \mathbb{R}_g then also g^{-1} lies in \mathbb{R}_g

$$g g^{-1} = e$$

$$f_i(\xi_{n+1}, \xi_n, \xi'_1, \dots, \xi'_n) = 0$$

since f_i is analytic this has a unique solution "implicit function theorem"

for infinitesimal ξ :

$$\Gamma = 1 + \xi_i \theta_i \quad \Gamma^{-1} = 1 - \xi_i \theta_i + \mathcal{O}(\xi^2)$$

Examples

1) $(\mathbb{R} \rightarrow \{\xi\}, \cdot)$ real numbers, identity is the number 1
faithful representation $\Gamma(r) = r$ 1-dimensional

norm $d(r, r') = |r - r'|$

$M_\xi: \xi = \frac{\pi}{2} \quad \frac{1}{2} < r < \frac{3}{2}$

analytic parametrization $r = e^\xi$

obviously $\xi = 0 \quad r = 1 = e$

$\mathbb{R}_\xi: \text{e.g. } \xi^2 < \left(\log \frac{3}{2}\right)^2 \quad \frac{2}{3} < e^\xi < \frac{3}{2}$

basis of the Lie algebra $\left. \frac{\partial e^\xi}{\partial \xi} \right|_{\xi=0} = 1$

Rem: $\mathbb{R}_+ = e^\xi \quad -\infty < \xi < \infty$

every group element can be written as either

$$e^\xi \quad \xi \in \mathbb{R}$$

$$\text{or } (-1) \cdot e^\xi \quad \xi \in \mathbb{R}$$

2) $O(2): O^+O = \mathbb{A}_2$ subgroup $SO(2)$ $\det(O) = +1$

parametrization $\begin{pmatrix} \cos \xi & \sin \xi \\ -\sin \xi & \cos \xi \end{pmatrix}$

$$\xi = 0 \Rightarrow \mathbb{1}_2$$

choose e.g. $\xi^2 < \left(\frac{\pi}{3}\right)^2$ \mathbb{R}_{map} clearly one-to-one
in some M_ξ around \mathbb{E}

Lie-algebra $\left. \begin{pmatrix} -\sin \xi & \cos \xi \\ -\cos \xi & -\sin \xi \end{pmatrix} \right|_{\xi=0} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

1-dimensional

O(2): Note that ξ -parameterization extends to all over SO(2)

REM O(2) $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow a^2 + b^2 = 1 = c^2 + d^2 \quad \& \quad ac + bd = 0$

$a = \cos \xi \quad b = \sin \xi$

$c = \sin \omega \quad d = \cos \omega$

$\cos \xi \sin \omega + \sin \xi \cos \omega = 0$

$\sin(\xi + \omega) = 0$

$\rightarrow \xi = -\omega \quad \begin{pmatrix} \cos \xi & \sin \xi \\ -\sin \xi & \cos \xi \end{pmatrix} \rightarrow \det(\theta) = +1$

$\rightarrow \xi = -\omega + \pi \quad \begin{pmatrix} \cos \xi & \sin \xi \\ \sin \xi & -\cos \xi \end{pmatrix} \rightarrow \det(\theta) = -1$

O(2): $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos \xi & \sin \xi \\ -\sin \xi & \cos \xi \end{pmatrix}$

Every element of O(2) can be written as an element of SO(2) multiplied by the matrix

$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Example 3

$SU(2)$: 2×2 matrices $\det(u) = +1$

and $u^\dagger u = \mathbb{1}_2$

$$u = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \quad u^\dagger = \begin{pmatrix} \alpha^* & -\beta \\ \beta^* & \alpha \end{pmatrix}$$

$$u u^\dagger = \begin{pmatrix} |\alpha|^2 + |\beta|^2 & 0 \\ 0 & |\alpha|^2 + |\beta|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow |\alpha|^2 + |\beta|^2 = 1$$

topologically S^3

$$\alpha = \alpha_1 + i\alpha_2$$

$$\beta = \beta_1 + i\beta_2$$

$$\alpha_2 = \frac{1}{2} \xi_3$$

$$\beta_1 = \frac{1}{2} \xi_2$$

$$\beta_2 = \frac{1}{2} \xi_1$$

$$\alpha_1 = \sqrt{1 - \frac{1}{4}(\xi_1^2 + \xi_2^2 + \xi_3^2)}$$

$$d(u, e) = 2 \left[1 - \sqrt{1 - \frac{1}{4}(\xi_1^2 + \xi_2^2 + \xi_3^2)} \right]^{\frac{1}{2}}$$

check that $d(u, e) < \delta$ if $\xi_1^2 + \xi_2^2 + \xi_3^2 < 2\delta^2 - \frac{1}{4}\delta^4$

Lie-algebra

$$a_1 = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$a_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$a_3 = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

Note: these are almost the Pauli-matrices:

Mathematics convention, Lie-algebra spanned by anti-hermitian matrices

Physics convention: hermitian matrices

$$a_1 = \frac{i}{2} \sigma_1 ; \quad a_2 = \frac{i}{2} \sigma_2 ; \quad a_3 = \frac{i}{2} \sigma_3$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} ; \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Recap of Lecture 1

• Def group $g_1 \circ g_2 \in G$, $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$
 $e \circ g = g$; $g^{-1} \circ g = e$

• Representation: $\Phi: G \rightarrow GL(N, \mathbb{C})$ $N \times N$ matrices
 $\Phi(g_1 \circ g_2) = \Phi(g_1) \cdot \Phi(g_2)$

• Normal subgroup $g^{-1} H g = H \quad \forall g \in G$

• Lie-groups: $g(\xi_1, \dots, \xi_n)$ analytic function
 faithful representation $\Gamma_{ij}(g(\xi_1, \dots, \xi_n))$

$\frac{\partial \Gamma}{\partial \xi_p} = a_p$ spans a real linear vector space

expl. $SU(2): \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \quad |\alpha|^2 + |\beta|^2 = 1$

$$\alpha_1 = \sqrt{1 - \frac{1}{4}(\xi_1^2 + \xi_2^2)}; \quad \alpha_2 = \frac{1}{2}\xi_3; \quad \beta_1 = \frac{1}{2}\xi_1; \quad \beta_2 = \frac{1}{2}\xi_2$$

$$a_2 = \frac{1}{2} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}; \quad a_1 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \quad a_3 = \frac{i}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$a_p = i \sigma_p$ σ_p - Pauli matrices

math-connection $a_p^\dagger = -a_p$ for $SU(N)$

$$e^{i \frac{\eta}{2} \sigma_3} = \mathbb{1} \cos \frac{\eta}{2} + i \sigma_3 \sin \frac{\eta}{2}$$

$$\Gamma(g_1 g_2) = (\Gamma(g_1) \Gamma(g_2))^T$$

$$\Gamma(g)^* \rightarrow \Gamma(g)^+ \quad \bar{\Psi} \Gamma(g)^+ \rightarrow$$

$$(\Gamma(g_2 g_1))^+ = (\Gamma(g_2) \Gamma(g_1))^+ = \bar{\Psi} \Gamma(g_1)^+ \Gamma(g_2)^+$$

$U^\dagger \sim U^*$; not in general $\Gamma^\dagger = \Gamma^+$



Def: Lie subgroup is a subgroup of a Lie group which by itself is a Lie group

Def: Connected component:

A maximal set of elements $g \in G$ that can be obtained by continuously varying the components of the faithful representation T_{ij}

Ex: $(\mathbb{R} - \{0\}, \cdot)$ $r > 0$ is a connected component

$r < 0$ is another connected component

Note $r = 0$ is not in $(\mathbb{R} - \{0\}, \cdot)$

Q(2): $SO(2)$ is a connected component

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} c & s \\ -s & c \end{pmatrix}$ is another

disconnected $\det(SO) = +1$ $\det(A \cdot SO) = -1$

Theorem: The connected component that contains the identity of a Lie group is a normal subgroup.

Proof: S be the connected component containing e

Let $s \in S$ then $s' s^{-1} \forall s' \in S$ is a connected component of G , but also $s' = s \in S$ so E is in this connected component $\Rightarrow S s^{-1}$ is a subgroup

It is normal since with $X \in G$ $X s X^{-1}$ (varying s)

is a connected component containing E

$\Rightarrow X s X^{-1} = S$

Def: A Lie group does not contain a connected normal subgroup

Covollery: Each connected component is a right coset of the connected subgroup S

The connected component is always a Lie group

Def: Connected Lie-group: consist of only one connected component $U(1)$

→ Simple: connected non-Abelian Lie group

Theorem: A connected Lie group allows a parametrization $y_1 \dots y_n$ that covers the whole group continuously but not necessarily analytically $U(1) = e^{iy_1}$ $y_1 \in [0, 2\pi)$

e.g. $SU(2)$ connected

$$|\alpha|^2 + |\beta|^2 = 1$$

$$\alpha = \cos y_1 e^{iy_2}$$

$$\beta = \sin y_1 e^{iy_3}$$

$$\text{but } 0 \leq y_1 \leq \frac{\pi}{2}, \quad 0 \leq y_2 \leq 2\pi, \quad 0 \leq y_3 \leq 2\pi$$

but $y_1 = 0$ $y_2 = 0$ and y_3 arbitrary maps to E

$$\Rightarrow \frac{\partial \Gamma}{\partial y_3} \Big|_{y=E} = 0$$

Expl: Lorentz-transformations 2D

$$\Lambda^T \eta \Lambda = \eta \quad \eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -c \\ b & -d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= a^2 - c^2 = 1 = b^2 - d^2 \quad ; \quad ab - cd = ab - dc = 0$$

$$a = \cosh x$$

$$b = \sinh y$$

$$\cosh x \sinh y - \sinh x \cosh y = 0$$

$$c = \sinh x$$

$$d = \sinh y$$

$$\sinh(x-y)$$

$$\Rightarrow x = y$$

$$\begin{pmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{pmatrix} \checkmark$$

$$\cosh x > 0 \quad \det = +1$$

$$\begin{pmatrix} -\cosh x & -\sinh x \\ -\sinh x & \cosh x \end{pmatrix}$$

$$-\cosh^2 x + \sinh^2 x \Rightarrow \det = -1$$

(y = -x)

$$\begin{pmatrix} \cosh x & \sinh x \\ -\sinh x & -\cosh x \end{pmatrix}$$

$$-\cosh^2 x + \sinh^2 x \Rightarrow \det = -1$$

$$\begin{pmatrix} -\cosh x & -\sinh x \\ -\sinh x & -\cosh x \end{pmatrix}$$

$$\cosh^2 x - \sinh^2 x \Rightarrow \det = +1$$

component containing the identity $1^0_0 > 0$

"orthochronous" Lorentz group

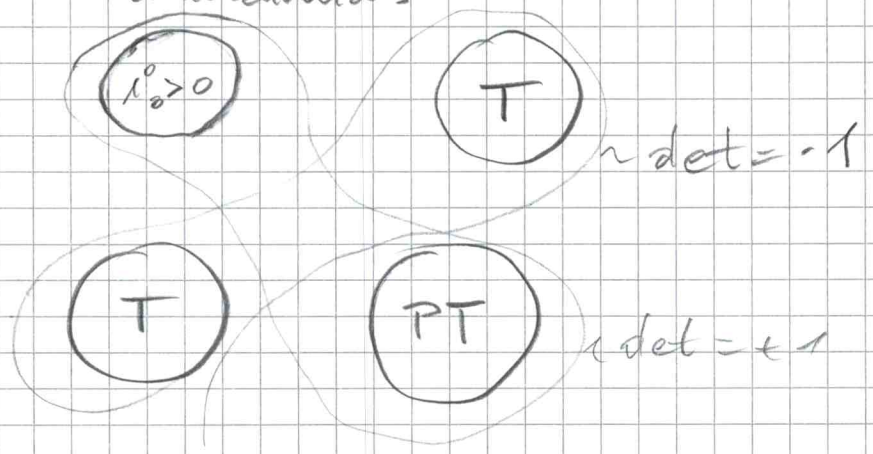
$$P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sinh x \rightarrow -\sinh x \quad \text{by } x \rightarrow -x$$

similar in 4D!

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{pmatrix} = \begin{pmatrix} \cosh x & \sinh x \\ -\sinh x & -\cosh x \end{pmatrix}$$

orthochronous



Representations of groups

Def: Representation is a Homomorphism ϕ from the group G to a group of $d \times d$ non-singular matrices under which the group multiplication maps to matrix multiplication

example: $(\mathbb{R}, +)$ $x+y$
 $\phi: e^x \cdot e^y = e^{x+y} \checkmark$
 $(\mathbb{R}, +) \rightarrow (\mathbb{R}^+, \cdot)$

Every group has the trivial representation $G \rightarrow \{1\}$

Def: Equivalent representations if \exists matrix S such that $\Gamma' = S \Gamma S^{-1}$ for two representations Γ, Γ' then Γ and Γ' are called equivalent
ex: $\mathbb{Z}_2 \langle \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Def: A representation is unitary if $\Gamma(g)$ are unitary matrices $\forall g \in G$

Theorem: A subset of points in Euclidean space is compact if and only if it is closed and bounded

Theorem: A Lie group is compact if the parameters g_1, \dots, g_n covering the connected component range over finite intervals.

Addendum: finite groups

This serves to prove the theorem that: Every representation of a finite group are equivalent to a unitary rep.

Proof: $\Gamma(g)$ be a matrix representation

$$\text{consider } S = \sum_g \Gamma^\dagger(g) \Gamma(g) \quad (\text{matrix multiplication})$$

Note: $\|\Gamma(\psi)\|^2 = \sum_g \Gamma^\dagger \Gamma \psi \geq 0$

\Rightarrow S is positive definite and Hermitian
by unitary S can be diagonalized

$$S = U^\dagger d U$$

$$d = \begin{pmatrix} d_1 & 0 & \dots \\ 0 & \ddots & \\ \vdots & & d_n \end{pmatrix} \quad d_i > 0$$

Note $d \neq 0$ because g includes e and
necessarily $\Gamma(e) = \mathbb{1}$

$$\Psi S \Psi = \sum_g \|\Gamma(g)\Psi\|^2 \quad \text{is a sum of positive terms and it includes } \|\Psi\|^2 \quad \Gamma(e) = \mathbb{1}$$

$$\Rightarrow d_i > 0$$

$$X = U^\dagger \sqrt{d} U \quad \text{and } X^{-1} \text{ exists}$$
$$X^{-1} = U \begin{pmatrix} 1/\sqrt{d_1} & & \\ & \ddots & \\ & & 1/\sqrt{d_n} \end{pmatrix} U$$

$$\Gamma'(g) = X \Gamma(g) X^{-1}$$

Now $(\Gamma'_g)^+ (\Gamma'_g) =$

$$= X^+ = U^+ \sqrt{d}^+ U = X^+ = X$$

$$(\Gamma'_g)^+ = X^{-1} \Gamma(g)^+ X$$

$$\Rightarrow (\Gamma'_g)^+ \Gamma'_g = X^{-1} \Gamma(g)^+ X X \Gamma(g) X^{-1} =$$

$$= X^{-1} \Gamma(g)^+ S \Gamma(g) X^{-1} =$$

$$= X^{-1} \Gamma(g)^+ \sum_h \Gamma(h)^+ \Gamma(h) \Gamma(g) =$$

$$= X^{-1} \sum_h \Gamma(hg)^+ \Gamma(hg) X$$

$$\underbrace{\sum_g \Gamma(g)^+ \Gamma(g)}_{= S \text{ by rearrangement theorem}}$$

$$X^{-1} S X^{-1} =$$

$$= U^+ \frac{1}{\sqrt{d}} U U^+ d U U^+ \frac{1}{\sqrt{d}} U = \mathbb{1} \checkmark \text{ qed!}$$

so the new representation is unitary!

Rem: for Lie-groups this proof "works" if the group is compact and thus allows to define a measure with finite volume

Schur's Lemma

These are actually two theorems

I) Let Γ and Γ' be two irreducible representations of a group G of dimensions d and d' and suppose that there is a $d \times d'$ matrix A such that

$$\Gamma'(g)A = A\Gamma(g) \quad \forall g \in G$$

Then either $A=0$ or $d=d'$ and $\det A \neq 0$

Proof A maps $V \rightarrow W$ let Γ' act on $W = \varphi \Gamma$ on V

$$\Gamma' A \varphi = A \Gamma \varphi \text{ subspace spanned by } \varphi$$

$\Rightarrow A\varphi$ is invariant subspace of W ($\text{Im}(A)$), but Γ' is irred.

(suppose $A\varphi = 0 \Rightarrow \Gamma' A \varphi = 0 = A \Gamma \varphi \Rightarrow \text{Ker}(A)$ is an invariant subspace of $V \Rightarrow A=0$ or $A = d \times d$ and invertible $\det(A) \neq 0$)

such that Γ' and Γ are equivalent

II) If Γ is $d \times d$ irreducible representation and

$$\Gamma(g)B = B\Gamma(g) \quad \forall g \in G$$

then $B = \lambda I$

Proof: let $A = B - \lambda I$ such that $\det A = 0$

this means λ is an eigenvalue of B

then $A\Gamma = \Gamma A$ but $\det A = 0 \Rightarrow \exists \varphi$ inv. subspace
 \downarrow
 $A\varphi = 0$

annihilating A do exist, but since Γ is

irreducible the only solution is $A=0$!

$$\Rightarrow \underline{B = \lambda I}$$

Theorem: Every irreducible representation of an Abelian group is one-dimensional

Proof: Let g, g' be two elements then

$$\Gamma(g) \cdot \Gamma(g') = \Gamma(g') \Gamma(g)$$

but by Schur's lemma $\Gamma(g') = \lambda(g') \mathbb{1}$

This is irreducible only if $\lambda = 1$

$$\mathbb{Z}_2: \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \rightarrow \mathbb{1} \oplus \{1, -1\}; \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix}$$

$g = 1, -1$

Complex representations

If $\Gamma(g)$ is a representation then $\Gamma^*(g)$ is also a representation $\Gamma^*(g) = (\Gamma(g))^*$

•) real representations $\exists S \quad S \Gamma S^{-1} = \Gamma'$
and $(\Gamma')^* = \Gamma'$

•) pseudo-real representation $\exists S : S \Gamma S^{-1} = \Gamma^*$

•) $\nexists S \Gamma S^{-1} = \Gamma^*$ complex representation

Theorem If G is finite or a compact Lie group then every representation of G is equivalent to a unitary representation.

How with proof? Do proof here!

Theorem If G is a non-compact simple Lie group then G has no finite dimensional unitary representation except for the trivial one.

Theorem: If Γ and Γ' are equivalent unitary reps of a group G i.e. $\Gamma' = S\Gamma S^{-1}$ then S can be chosen unitary as well. (unitarily equivalent)

Definition: If $\exists S$ such that the representation Γ is equivalent to

$$\Gamma' = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ 0 & \Gamma_{22} \end{pmatrix}$$

Γ is called reducible

more formal: it exists an invariant

subspace $\Gamma(g)W \subset W$ $W \sim \begin{pmatrix} W \\ 0 \end{pmatrix}$

$$\Gamma W = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ 0 & \Gamma_{22} \end{pmatrix} \begin{pmatrix} W \\ 0 \end{pmatrix} = \begin{pmatrix} \Gamma_{11}W \\ 0 \end{pmatrix} \checkmark$$

$$\forall g \in G$$

W .. vector space "carrier space" of the representation

"Module" $(\rho(\Gamma), W)$ as distinct operators $\psi_i = \langle i | \rho(\Gamma) | i \rangle$

Definition: Irreducible representation

if it is not reducible, i.e. does not have an invariant subspace

Definition: A representation is called completely reducible if it is ^{equivalent} of block diagonal form

$$\Gamma = \begin{pmatrix} \Gamma_{11} & & \\ & \Gamma_{22} & \\ & & \ddots \end{pmatrix}$$

all the blocks are irreducible

representations: if P projects onto subspace, $(1-P)$ also

Theorem: If G is finite or a compact Lie group then every reducible representation is completely reducible.

Proof as well!

Counter-examples: $(\mathbb{R}^2, \cdot) \sim e^x$

$$\Gamma = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \quad \Gamma(x) \cdot \Gamma(y) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x+y \\ 0 & 1 \end{pmatrix} \checkmark$$

Sim: Euclidean group

$$\begin{pmatrix} \theta & 1 & t \\ -\frac{1}{\theta} & 1 & - \\ 0 & 0 & 1 \end{pmatrix}$$

non-compact Lie groups

(b) on the other hand O^{ab}

$$\text{tr}(\chi_{ee}^{ab}) = \sum_g \text{tr} \left(\left[\Gamma_a^+(g) \Psi_e^a \overline{\Gamma}_b^-(g) \Gamma_a(g) \right] \right) \delta^{ab} =$$

$$= \sum_g \left(\Psi_e^a \overline{\Psi}_e^a \underbrace{\Gamma_a(g) \Gamma_a^+(g)}_{\delta_{ij}} \right) = \delta_{ee} \cdot N$$

\downarrow
 $\delta_{ij} = \delta_{ij}$

if we choose Ψ_e^a to span an orthonormal basis

$$N = \Theta(G)$$

$$\Rightarrow \delta^{ab} n_a \chi_{ee}^a = \delta^{ab} N \delta_{ee} \quad (\chi_e)_i := \delta_{ei} \text{ etc.}$$

$$\chi_{ee}^a = \frac{N}{n_a} \delta_{ee}$$

$$\Rightarrow \left\langle \sum_g \frac{n_a}{N} (\Gamma_a(g))^+_{je} (\Gamma_b(g))_{km} \right\rangle = \delta_{ab} \delta_{ee} \delta_{jm}$$

"Orthogonality relation"

Especially useful for "characters"

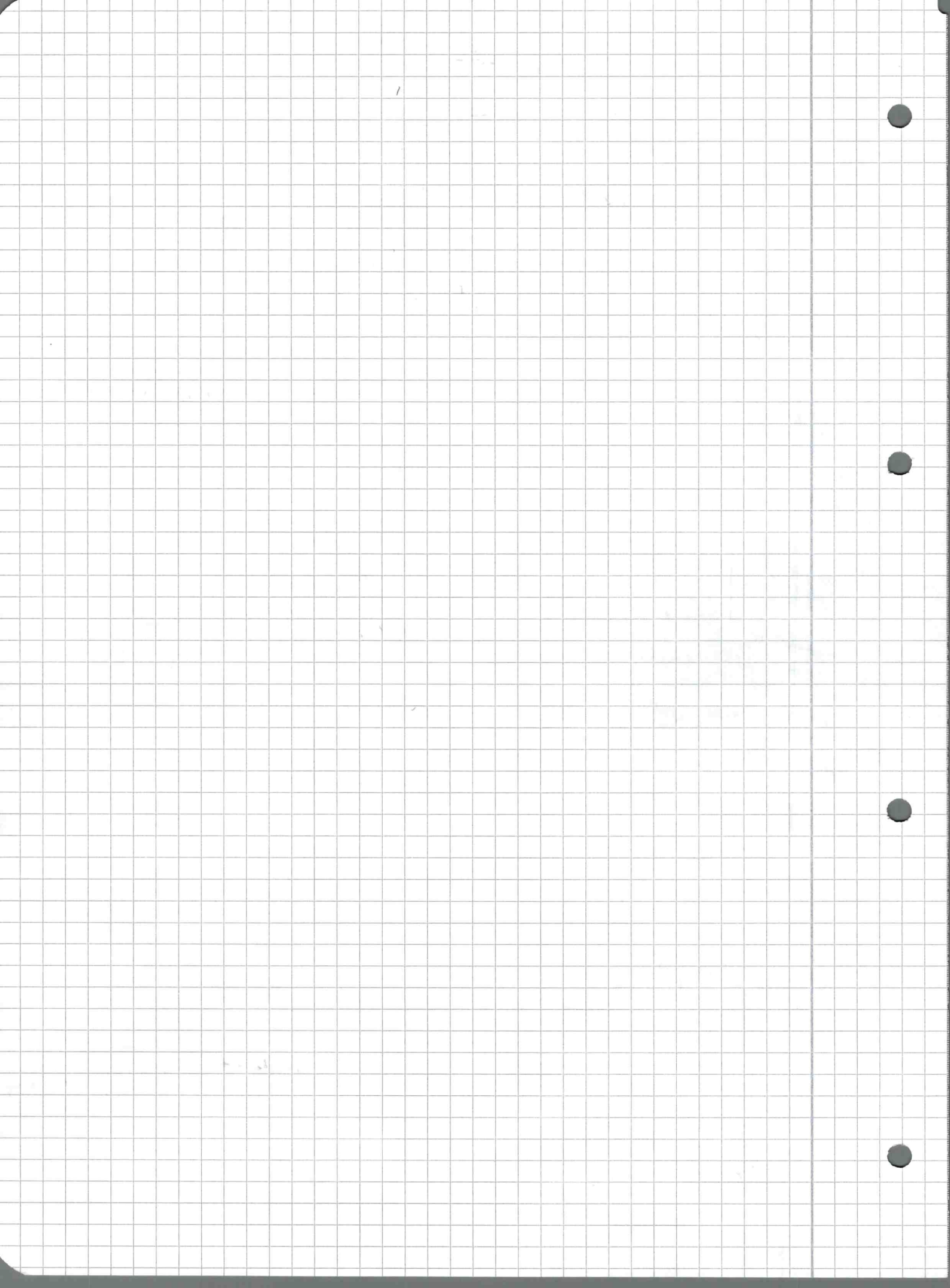
$$\chi_a(g) = \text{tr}(\Gamma_a(g)) \quad \text{invariant under similarity transformations!}$$

$$\sum_g \frac{n_a}{N} \chi_a(g)^* \chi_b(g) = \delta_{ab} n_a$$

$$\frac{1}{N} \sum_g \chi_a(g)^* \chi_b(g) = \delta_{ab}$$

Applications: Fourier: $e^{im\varphi}$, $e^{-im\varphi}$ repr. labelled by m

$$\frac{1}{N} \sum_g \rightarrow \frac{1}{2\pi} \int_0^{2\pi} d\varphi e^{-im\varphi} e^{im\varphi} = \delta_{m,m} \leftarrow \text{justify later}$$



Theorem: Every representation of a finite group is completely reducible

Proof: reducible means that there is an invariant subspace. Let $P = P^2$ be the projector onto that subspace ($P^\dagger = P$)

$$\forall g \in G: P \Gamma(g) P = \Gamma(g) P \quad \text{reducibility}$$

$$\text{since } \Gamma(g)^\dagger = \Gamma(g^{-1}) \quad \text{unitary}$$

$$P \Gamma(g^{-1}) P = P \Gamma(g^{-1}) \quad \forall g$$

$$\text{but that means } P \Gamma(g) P = P \Gamma(g) = \forall g \in G$$

then the projector onto the complement projects also to an invariant subspace

$$\begin{aligned} (1-P) \Gamma(g) (1-P) &= \Gamma(g) - \Gamma P - \cancel{P \Gamma} + \cancel{P \Gamma P} \\ &= \Gamma(g) (1-P) \quad \text{q.e.d.} \end{aligned}$$

Orthogonality relation

Let Ψ_j^a be an (orthonormal) basis of the vector space on which the representation $\Gamma_a(g)$ acts.

Consider $\sigma: V_b \rightarrow W_a$ linear

$$A_{\ell k}^{ab} = \sum_g \Gamma_a^+(g) \Psi_\ell^a \overline{\Psi}_k^b \Gamma_b(g)$$

$$\Gamma_a(g') A_{\ell k}^{ab} = \sum_g \Gamma_a(g') \Gamma_a(g^{-1}) \Psi_\ell^a \overline{\Psi}_k^b \Gamma_b(g)$$

$$\Gamma_a(g'g^{-1}) = \Gamma_a((gg^{-1})')$$

$$h = g'g^{-1} \quad g = hg'$$

$$\Gamma_a(g') A_{\ell k}^{ab} = \sum_h \Gamma_a(h) \Psi_\ell^a \overline{\Psi}_k^b \Gamma_b(hg')$$

$\Gamma_b(h) \cdot \Gamma_b(g')$

$$\Rightarrow \Gamma_a(g') A_{\ell k}^{ab} = A_{\ell k}^{ab} \Gamma_b(g')$$

so by Schur's lemma the $A^{ab} = 0$ if $a \neq b$ and otherwise proportional to $\mathbb{1}$

$$\Rightarrow A_{\ell k}^{ab} = \delta^{ab} \lambda_{\ell k}^a \mathbb{1}$$

To compute $\lambda_{\ell k}^a$ we take the trace

$$(19) \quad \text{tr}(A_{\ell k}^{ab}) = \delta^{ab} \lambda_{\ell k}^a n_a$$

$\hookrightarrow \dim \text{repr } "a"$

Now compute $\frac{d\Gamma(s)}{ds} \Big|_{s=0} = \sum_P \frac{\partial \Gamma}{\partial \alpha_P} \Big|_{s=0} \frac{\partial \alpha_P}{\partial s} = \sum_{P=1}^n \alpha_P \frac{\partial \alpha_P}{\partial s}$

Since we know that the matrices α_P spans a linear vector space we call $\frac{\partial \Gamma}{\partial s}$ the tangent vector to the curve $\Gamma(s)$ at $s=0$.

Conversally every vector $\sum \alpha_P \lambda_P$ is tangent vector to some curve.

Theorem: If a and b are tangent vectors to the curves $A(s)$ and $B(s)$ then $[a, b]$ is tangent vector to the curve $C(s^2) = A(s) B(s) A(s)^{-1} B(s)^{-1}$

$A(s) = \mathbb{1} + s a + \frac{s^2}{2} a^2$ $B(s) = \mathbb{1} + s b + \frac{s^2}{2} b^2$

$A(s)^{-1} = \mathbb{1} - s a + s^2 (a^2 - \frac{1}{2} a^2)$ $B(s)^{-1} = \mathbb{1} - s b + s^2 (b^2 - \frac{1}{2} b^2)$

$\Rightarrow C(s^2) = \mathbb{1} + s^2 [a, b] + \dots$

$s' = s^2$ $C(s) = \mathbb{1} + s' [a, b] + \dots$

\Rightarrow to every Lie-group we can associate a Lie-algebra in a natural way.

Note: the commutator is an element of the vector space spanned by the α_P 's since it appears as tangent vector to a curve in G .

$\alpha_P(t_0)$ are called the generators of the Lie-group

for $SU(2)$ we have already computed the a_p 's

$$a_1 = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad a_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad a_3 = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

commutators

$$[a_1, a_2] = \frac{1}{4} \left[\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} - \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \right] = \frac{1}{2} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -a_3$$

$$[a_2, a_3] = \frac{1}{4} \left[\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} - \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right] = \frac{1}{2} \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = -a_1$$

$$[a_1, a_3] = \frac{1}{4} \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right] = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -a_2$$

(Note the "-" signs in the math equations)

$$[\sigma_i, \sigma_j] = 2i \epsilon_{ijk} \sigma_k$$

$$\sigma_i \sigma_j = \delta_{ij} + i \epsilon_{ijk} \sigma_k$$

Theorems: 1) $A(t) = \exp(ta)$ mit $a \in \mathfrak{L}$ is a one-parameter subgroup $(-\infty < t < \infty)$

2) Every element of \mathfrak{g} in a small neighborhood of E can be written as

$$g = \exp(x_1 a_1 + x_2 a_2 + \dots + x_p a_p)$$

3) If \mathfrak{g} is a compact Lie-algebra every element of the connected component can be written in this form

Lie algebras

Lie group: \rightarrow global aspects e.g. connectedness etc.
compactness

\rightarrow local aspects \rightarrow Lie algebra
(covers some global aspects)

We know already that $a_p = \frac{\partial \Gamma}{\partial g_p}$ forms a basis of a real linear vector-space.

Now we will show that the forms an algebra \rightarrow Lie algebra and we will eventually allow to form complex linear combinations ("complexify" the algebra)
Allow for a complete classification of Lie-algebras and their representations

Def: Lie-algebra (over the real numbers) \mathfrak{L} is a (real) linear vector space equipped with a Lie-product

- $a * b \in \mathfrak{L} \quad \forall a, b \in \mathfrak{L}$ +
- $(\alpha a + \beta b) * c = \alpha a * c + \beta b * c \quad \alpha, \beta \in \mathbb{R}$
- $a * b = -b * a$
- $a * (b * c) + c * (a * b) + b * (c * a) = 0$
(Jacobi-identity)

Matrices $*$: commutator $a * b = [a, b]$

(Theorem: every (finite dimensional) Lie algebra is isomorphic to a matrix algebra with $*$ being the matrix-commutator)

Therefore we always think now of commutators

Jacobi identity

$$\begin{aligned} [a, [b, c]] &= a \cancel{bc} - a \cancel{cb} - b \cancel{ca} + c \cancel{ba} + \\ + \text{cyclic} & \quad + c \cancel{ab} - b \cancel{ca} - a \cancel{bc} + a \cancel{cb} \\ + \text{cyclic} & \quad + b \cancel{ca} - c \cancel{ba} - c \cancel{ab} + b \cancel{ac} \end{aligned}$$

✓

Note: the Jacobi identity can be written as a Leibnitz rule (= derivative)

$$[a, [b, c]] = [[a, b], c] + [b, [a, c]]$$

Since all commutators are in \mathfrak{X} and \mathfrak{X} is a linear vector space it follows

$$[a_p, a_q] = c_{pq}^r a_r \quad \text{"Structure constants"}$$

(Physics $a_p = i t_p$)

$$[t_a, t_b] = i f_{ab}^c t_c \quad f_{ab}^c = -c_{ab}^c$$

Def: Abelian Lie-algebra $[a, b] = 0, \quad f_{ab}^c = 0$

Lie groups and Lie-algebras

Take a one-parameter curve in $\vec{\xi}(s)$

where ξ_1, \dots, ξ_n parametrize a neighborhood around E in a Lie group G .

This defines a one-parameter curve in G and necessarily a subgroup. (2)

Let $\vec{\xi}(0) = 0$

This provides a direct way of characterizing some Lie-algebras

Ex: $\mathfrak{g} = \mathfrak{SL}(N, \mathbb{C}) \Rightarrow \det(g) = 1 = \det(e^a) = e^{\text{tr}(a)}$

$\Rightarrow \text{tr}(a) = 0$ defines the Lie-algebra $\mathfrak{sl}(N)$; $N=2$: 6 dimensional vector space

Notation $SU(N) \rightarrow \mathfrak{su}(N)$
 group algebra

$u^{-1} = u^\dagger \quad e^{-a} = e^{a^\dagger} \Rightarrow a = -a^\dagger \quad (t^\dagger = t \text{ in physics})$

$\Rightarrow \mathfrak{su}(N)$ traceless (anti-)hermitian matrices

$\mathfrak{g} = \mathfrak{SO}(N) \quad O^T = O^{-1} \quad a^T = -a$

note $\det(e^a) = 1$ automatically

\rightarrow anti-symmetric matrices

- it will turn out that even and odd N are very different

more generally

$SO(p, q): \quad \theta \eta = \eta \theta \quad \eta = \text{diag}(\underbrace{+1 \dots +1}_p, \underbrace{-1 \dots -1}_q)$

left- and right-invariant integration measures

Suppose we have a canonical parametrization of a compact Lie-group $\exp(x_1, \dots, x_n)$ (" x_j " coords)

$$I = \int f(A(x_1, \dots, x_n)) \sigma_L(x_1, \dots, x_n) dx_1 \dots dx_n$$

$$I = \int f(A' \cdot A(x_1, \dots, x_n)) \sigma_L(x_1, \dots, x_n) dx_1 \dots dx_n$$

we say $\sigma_L(x_1, \dots, x_n)$ is a left-invariant measure

consider $A_0^{-1} \frac{\partial A}{\partial x_k} \Big|_{x_2=x_0}$ $A(x_0) = A_0$

this is an element of \mathcal{L} since $B(x) = A_0^{-1} A(x)$

is such that at $x=x_0$ $B(x) = \mathbb{1}$

$$A^{-1} \frac{\partial A}{\partial x_k} \Big|_A = \sum \mu_{kj}^e(x_1, \dots, x_n) a_j$$

define $\sigma_L = \det \mu_{kj}^e$

$$\int f(A' A(x)) |\mu(x)| dx$$

$$A' \cdot A(x) = A(x')$$

$$A(x')^{-1} \frac{\partial A}{\partial x'_k} = \sum \mu_{kj}^e(x') a_j$$

$$A(x) \underbrace{A(x')^{-1}}_{A^{-1} \cdot A'} \frac{\partial A}{\partial x_k} \frac{\partial x^l}{\partial x'_k} = \sum \mu_{kj}^e(x') a_j$$

$$\sum \mu_{kj}^e \frac{\partial x^l}{\partial x'_k} a_j = \sum \mu_{kj}^e(x') a_j$$

$$\sigma^l = |\mu|$$

$$\sigma^l(x') = \sigma^l(x) \left| \frac{\partial x}{\partial x'} \right|$$

$$\text{or } \sigma^l(x) = \sigma^l(x') \left| \frac{\partial x'}{\partial x} \right|$$

$$\begin{aligned} \rightarrow \int f(A' \cdot A(x)) &= \int f(A(x')) \sigma^l(x') \left| \frac{\partial x'}{\partial x} \right| dx' = \\ &= \int f(A(x')) \sigma^l(x') dx' = \int f(A(x)) \sigma^l(x) dx \quad \text{q.e.d.} \end{aligned}$$

right-invariant measure

$$\frac{\partial A}{\partial x_k} A^{-1} = \sum_l \mu_{kl}(x) e_l$$

If the group consists of several components one has to sum over the components

Theorem: For compact groups the left and right invariant measures coincide "unimodular" groups and the integral is finite

Remark: In most applications it is sufficient to know that this measure exists

Exercise: compute the right and left-invariant measures of $SU(2)$

$$u = \begin{pmatrix} \cos \varphi_1 e^{i\varphi_2} & \sin \varphi_1 e^{i\varphi_3} \\ -\sin \varphi_1 e^{-i\varphi_2} & \cos \varphi_1 e^{i\varphi_3} \end{pmatrix}$$



More on the relationship of Lie-Groups
and Lie-algebras

Definition \rightarrow Subalgebra \mathfrak{X}' of \mathfrak{X} : \mathfrak{X}' is called a subalgebra of \mathfrak{X} if $\forall a \in \mathfrak{X}'; \forall b \in \mathfrak{X}' [a, b] \in \mathfrak{X}'$ $\mathfrak{X}' \subset \mathfrak{X}$

\rightarrow It is called an invariant subalgebra if $\forall a \in \mathfrak{X}$ and $b \in \mathfrak{X}' [a, b] \in \mathfrak{X}'$
"Ideal"

Theorem: \rightarrow If G and G' are Lie groups and \mathfrak{X} and \mathfrak{X}' their Lie-algebras and G' is a subgroup of G then \mathfrak{X}' is a subalgebra of \mathfrak{X}

\rightarrow If G' is an invariant subgroup (normal) then \mathfrak{X}' is an invariant subalgebra of \mathfrak{X}

\rightarrow If \mathfrak{X} is a real Lie algebra of G then each subalgebra of \mathfrak{X} is the Lie group of a subgroup of G

Theorem: If $\mathfrak{X}', \mathfrak{X}''$ are invariant subalgebras then $[\mathfrak{X}', \mathfrak{X}'']$ is an invariant subalgebra

Proof

$\forall a \in \mathfrak{X}$

$[a, [b, c]] = \underbrace{[[a, b], c]}_{b' \in \mathfrak{X}'} + \underbrace{[b, [a, c]]}_{c' \in \mathfrak{X}''}$ Jacobi-identity
 $[b', c] + [b, c'] \in [\mathfrak{X}', \mathfrak{X}'']$

Homomorphism $\Psi: \mathcal{X} \rightarrow \mathcal{X}'$

$$\text{a) } \Psi(\alpha a + \beta b) = \alpha \Psi(a) + \beta \Psi(b)$$

$$\text{b) } \Psi([a, b]) = [\Psi(a), \Psi(b)]$$

Isomorphism: one-to-one

\mathcal{X} & \mathcal{G} be an (analytic) group homomorphism $\mathcal{G} \rightarrow \mathcal{G}'$
then there is an induced \mathcal{X} -algebra homomorphism

$$\Psi(a) = \left[\frac{d}{dt} (\Phi e^{ta}) \right] \Big|_{t=0}$$

Expl $\Phi: U(1) \rightarrow SO(2)$ $\Phi: e^{ix} \rightarrow \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix}$

$$\downarrow$$
$$a = i \quad \frac{d}{dt} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \Big|_{t=0} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\Psi: i \rightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Isomorphism

Def: discrete subgroup: a subgroup \mathcal{K} of a Lie-group \mathcal{G}
is called discrete if

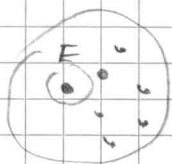
a) it is a finite group

b) it consists of countably infinite elements

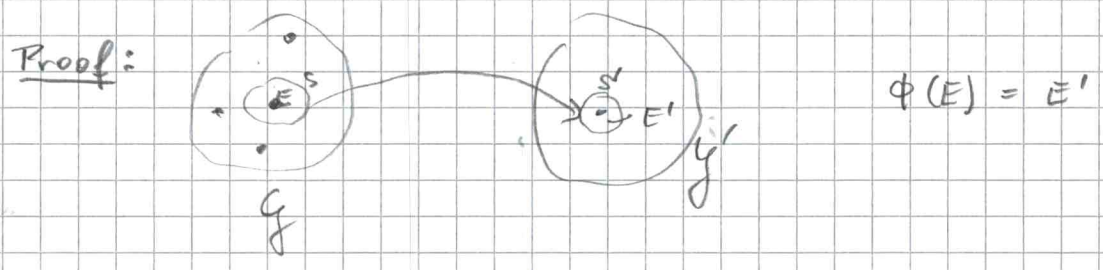
but there is neighborhood of the identity E

in which there is no element of \mathcal{K} except

for E itself



Theorem Let \mathcal{K}_ϕ be the kernel of a homomorphic mapping $\phi: \mathfrak{g} \rightarrow \mathfrak{g}'$, be discrete. then the induced homomorphism Ψ on the Lie algebras $\Psi: \mathfrak{X} \rightarrow \mathfrak{X}'$ is an isomorphism!



$\Rightarrow \phi$ is one-to-one on $S \rightarrow S'$, the small neighborhood heads of E, E'

$\Rightarrow g \sim 1 + d\alpha_i \rightarrow 1 + d\alpha'_i$

$\mathfrak{X}, \mathfrak{X}'$ is one-to-one

Significance: Lie-algebra determines the Lie-group only "locally" in S^0 connected component to the identity

Expl: (\mathbb{R}_{+}°) and $SO(2)$

$$\phi: e^x \mapsto \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix}$$

Kernel $\mathcal{K} \quad x_{\mathbb{R}} = 2\pi k \quad k \in \mathbb{Z}$

discrete and countably infinite Lie-algebra

$$\alpha_1 = 1 \quad \alpha_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(\mathbb{R}_{+}°) is non-compact
 $SO(2)$ is compact

Expl analytic homomorphism $SU(2) \rightarrow SO(3)$

$$\Phi(u) = \frac{1}{2} \operatorname{tr}(\sigma_j u \sigma_k u^\dagger) \quad j, k = 1, 2, 3$$

$$\vec{x} \in \mathbb{R}^3 \quad X = \vec{x} \cdot \vec{\sigma}$$

$$\det(X) = \begin{vmatrix} z & x-iy \\ x+iy & -z \end{vmatrix} = -z^2 - x^2 - y^2 = -r^2$$

$$\det(X) = \det(u X u^\dagger) \quad \det(u) = 1, \quad X^\dagger = X$$

$$\vec{x} \rightarrow \vec{x}' \quad \text{but} \quad \vec{x}' \cdot \vec{x}' = \vec{x} \cdot \vec{x}$$

$\rightarrow X'$ defines a vector x' with equal length as $x \Rightarrow \vec{x}, \vec{x}'$ are related by a rotation

$$\vec{x}' = R \cdot \vec{x}$$

$$\operatorname{tr}(\sigma_i \sigma_j) = 2 \delta_{ij}$$

$$\vec{x} = \frac{1}{2} \operatorname{tr}(X \cdot \vec{\sigma})$$

$$\vec{x}' = \frac{1}{2} \operatorname{tr}(X' \cdot \vec{\sigma})$$

$$\Rightarrow R_j(u) = \frac{1}{2} \operatorname{tr}(\sigma_j u \sigma_k u^\dagger)$$

$$R(u_1) \cdot R(u_2) = R(u_1 u_2)$$

$$X' = u_2 X u_2^\dagger \Rightarrow \vec{x}' = R(u_2) \vec{x}$$

$$\begin{aligned} X'' &= u_1 X' u_1^\dagger \Rightarrow X'' = R(u_1) \vec{x}' \\ &= u_1 u_2 X u_2^\dagger u_1^\dagger \\ &= R(u_1) \cdot R(u_2) \vec{x} \end{aligned}$$

$$\Rightarrow \vec{x}'' = R(u_1 u_2) \vec{x} \Rightarrow R(u_1 u_2) = R(u_1) \cdot R(u_2)$$

$$\Rightarrow R(u_1 u_2) = R(u_1) \cdot R(u_2)$$

$$\mathcal{K}: \text{tr}(u \sigma_i u^\dagger \sigma_j) = \delta_{ij}$$

$$\Rightarrow \mathcal{K} = \{\mathbb{1}_2, -\mathbb{1}_2\} \quad \mathbb{Z}_2 \quad \text{discrete kernel}$$

\Rightarrow Isomorphism in Lie algebra

$$u = 1 + d_i a_i \quad u^\dagger = 1 + d_i a_i^\dagger = 1 - d_i a_i$$

(math - convention here)

$$\frac{1}{2} \frac{d}{da_i} \text{tr} \left[(1 + d_i a_i) \sigma_i (1 - d_i a_i) \sigma_j \right] =$$

$$= \frac{1}{2} \text{tr} \left[a_i \sigma_i \sigma_j - \sigma_i a_i \sigma_j \right] = \frac{1}{2} \text{tr} \left([a_i, \sigma_i] \sigma_j \right)$$

$$a_e = \frac{i}{2} \sigma_e \Rightarrow \frac{i}{4} \text{tr} \left([\sigma_e, \sigma_i] \sigma_j \right) =$$

$$= -\frac{1}{2} \epsilon_{eij} \text{tr}(\sigma_e \sigma_j) = -\epsilon_{ijk} \delta_{ij}$$

$$= \epsilon_{ij}$$

$$a_e \rightarrow \hat{a}_{ij} = \epsilon_{ij} \quad \text{Isomorphism}$$

$$\Psi(a_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad \Psi(a_2) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \Psi(a_3) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Note: the structure constants themselves interpreted as matrices give a representation

In general: "Adjoint" representation

Representations of Lie-algebras

operator acting on a vector space

basis $\varphi^{(e)}$

$$\Gamma \varphi^{(e)} = \phi^{(e)}$$

orthonormal basis $(e^{(e)}, e^{(m)}) = \delta^{em}$

$$(e^{(e)}, \Gamma e^{(e)}) = \Gamma_{ee}$$

$$v = v_e e^{(e)}$$

$$(e_e, \Gamma v) = (e_e, \Gamma e_e) v_e = \Gamma_{ee} v_e$$

V ... carrier space (Module)

physics language the vector $\vec{v} = v_e$ transform

in representation Γ_{ee}

equivalent repr. $\Gamma' = S \Gamma S^{-1}$ $\det(S) \neq 0$

o) reducible representation

$$\Gamma \text{ equivalent to } \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ 0 & \Gamma_{22} \end{pmatrix}$$

o) irreducible representation

o) completely reducible representation

$$\begin{pmatrix} \Gamma_{11} & & & \\ & \Gamma_{22} & & \\ & & \ddots & \\ & & & \Gamma_{33} \end{pmatrix}$$

Schur lemmas hold o) $\Gamma A = A \Gamma'$

$$\Rightarrow \bullet A=0 \text{ or } A \text{ dxd } \det(A) = 0$$

o) $\nexists \Gamma = \Gamma'$ irreducible

$$A = \lambda \mathbb{1}$$

Note: Not every representation of \mathfrak{g} gives a representation of G by exponentiation

e.g. $so(3) \cong su(2)$ isomorphic one-to-one

but $(t_{\mathbf{x}})_{ij} = \epsilon_{ijk}$ exponentiates to $SO(3)$

while $\frac{i}{2}(\sigma_{ij})_{ij}$ exponentiates to $SU(2)$

Math: $SU(2)$ is not a representation of $SO(3)$
"spinor representation of $SO(3)$ "
(projective)

but: as statement about the algebra this is o.k.
since $so(3)$ and $su(2)$ are isomorphic
as Lie-algebras

Direct product representation

Supposes Lie-group ^{rep.} is of the form $\Gamma_{\mathfrak{g}} \otimes \Gamma_{\mathfrak{g}'}$

$$E = \mathbb{1} \otimes \mathbb{1}$$

then there exists a basis in $V \otimes V'$ such that

$$\Gamma_{\mathfrak{g}} = \Gamma_{\mathfrak{g}}(a) \otimes \mathbb{1} + \mathbb{1} \otimes \Gamma_{\mathfrak{g}'}(a')$$

$$\frac{\partial}{\partial t} (\mathbb{1} + t a) \otimes (\mathbb{1} + t a') \Big|_{t=0} \sim a \otimes \mathbb{1} + \mathbb{1} \otimes a'$$

'direct product representation of a Lie-algebra'

In particular if $\Gamma = \Gamma'$

$$\text{then } \psi^{(e)} \otimes \psi^{(e)} \rightarrow (\psi^{(e)} \otimes \psi^{(e)} + \psi^{(e)} \otimes \psi^{(e)}) + (\psi^{(e)} \otimes \psi^{(e)} + \psi^{(e)} \otimes \psi^{(e)})$$

which decomposes the direct product into $\frac{1}{2}d(d+1)$ symmetric subspace and $\frac{1}{2}d(d-1)$ antisymmetric subspace

$$\rightarrow \text{tensor } v_{ij} = v_{ji} \quad ; \quad v_{ij} = -v_{ji}$$

$$\Gamma \otimes \Gamma = (\Gamma \otimes \Gamma)_{\text{sym}} \oplus (\Gamma \otimes \Gamma)_{\text{asym}}$$

The adjoint representation

Jacobi-identity

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$$

base a_p, a_q, a_r

$$[a_p, [a_q, a_r]] = [a_p, c_{qr}^s a_s] = c_{qr}^s c_{ps}^t a_t + c_{rp}^s c_{qs}^t a_t + c_{pq}^s c_{rs}^t a_t = 0$$

since a_t are linearly independent

$$\Rightarrow c_{qr}^s c_{ps}^t + c_{rp}^s c_{qs}^t + c_{pq}^s c_{rs}^t = 0$$

$$(C_r)_q^s = c_{qr}^s$$

$$- (C_r C_p)_q^t + (C_p \cdot C_r)_q^t - C_{rp}^s c_{sq}^t$$

$$\boxed{[C_p, C_r] = C_{pp}^s (C_s)_q^t}$$

⇒ For a general Lie-algebra the matrices

$$(C_{\alpha})_{pq} = c_{\alpha pq}$$

form a representation of the Lie-algebra.

"Adjoint representation $ad(\alpha)$ "

$$ad(\alpha): [a, b]_{\mathfrak{g}} \subset \mathfrak{g}, b \in \mathfrak{g}$$

Carrier space is the Lie algebra itself.

$$ad(\alpha_p)_q = c_{\alpha pq}$$

physics $[t_a, t_b] = i f_{ab}^c t_c$

$$(T_a^{ad})_b^c = i f_{ba}^c$$

The corresponding adjoint representation of Lie-group

Let $A \in G, \alpha_i \in \mathfrak{g}$ then

$$Ad(A) := A \alpha_p A^{-1} = [Ad(A)]_p^q \alpha_q$$

n-dimensional representation

$$\exp(t ad(\alpha)) = Ad(\exp(t\alpha))$$

Representation of G :

$$A_1 A_2 \alpha_p A_2^{-1} A_1^{-1} = Ad(A_1)_{pq} \cdot Ad(A_2)_{qe} \alpha_e$$

$$A_1 \underbrace{Ad(A_2)_{pe}}_{\alpha'_p} \alpha_e A_1^{-1} = Ad(A_1)_{pq} \alpha'_q = Ad(A_1)_{pq} Ad(A_2)_{qe} \alpha_e \checkmark$$

So it is indeed a representation of G

The universal covering group

Suppose x_1, \dots, x_n is a parametrization of a Lie group connected

take n -function $f_i(t) = x_i$

$$g(f_1, \dots, f_n) = g(t) \in G \quad t \in \mathbb{R}_{[0,1]}$$

$g(t)$ is a path

G is connected $\Rightarrow \exists$ paths between each two group elements

if $g(1) = g(0)$ the path is called a loop

Two paths are called homotopic if there exists a continuous transformation $f_i(t) \rightarrow \tilde{f}_i(t)$

$$F(t, s) \rightarrow \begin{aligned} F_i(t, 0) &= f_i(t) \\ F_i(t, 1) &= \tilde{f}_i(t) \end{aligned}$$

if $f(t)$ is homotopic to the constant path it is called "contractible".

simply connected

Def: A connected Lie group is called contractible if all loops are contractible

Example $SU(2) : u = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}$

with $|\alpha|^2 + |\beta|^2 = 1$

$\alpha = x_1 + ix_2$ $\beta = x_3 + ix_4$

$\Rightarrow x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$

topologically a 3-sphere S^3
→ every loop is contractible

$SU(2)$ is simply connected
generally $SU(N)$ is simply connected

Expt 2 $SO(3)$ let $SO(3)$ be represented by

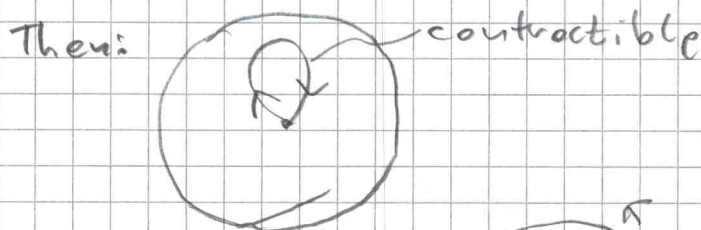
$e^{i\varphi \hat{n} \cdot \vec{\sigma}}$ \hat{n} -- direction, φ angle

$-\pi < \varphi < \pi$

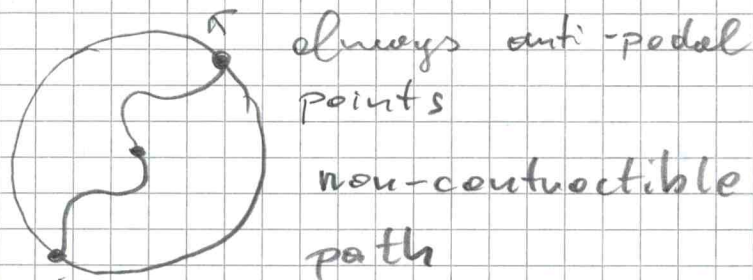


$\mathbb{1} = \varphi = 0$

But $\varphi = \pi$ is identified with $\varphi = -\pi$



But:



$SO(N)$ is not simply connected

Expl: $SO(2)$ is not simply connected

consider $\begin{pmatrix} \cos 2\pi t & \sin 2\pi t \\ -\sin 2\pi t & \cos 2\pi t \end{pmatrix} \quad t \in [0, 1]$

$F[t, s]$ continuously connected to $F[0, 1] = 0$

But $F[1, 0] = 1 \rightarrow$ continuity $F(1, s) = 1$

obviously this can not change continuously

to $F(1, 1) = 0$

$\rightarrow SO(2)$ is not simply connected.

Def: Center \mathcal{C} of a group G : subgroup consisting of all elements that commute with all elements of G

Example: $SU(N) = \left\{ \begin{pmatrix} e^{i\frac{2\pi u}{N}} & & \\ & \ddots & \\ & & e^{-i\frac{2\pi u}{N}} \end{pmatrix} \mid u = 0, \dots, N-1 \right\}$

$$SU(N) = e^{i\frac{2\pi u}{N}} \mathbb{1} \quad u = 0, \dots, N-1$$

center \mathbb{Z}_{N-1}

Theorem: If G is a connected Lie group then there exists a simply connected Lie group \tilde{G}

1) G is isomorphic to \tilde{G}/\mathcal{K} where \mathcal{K} is a discrete subgroup of the center

2) If G is simply connected G is isomorphic to \tilde{G}

3) The real Lie algebras are isomorphic

4) Every rep of the real Lie algebra of \tilde{G}

is associated with a representation of $\tilde{\mathfrak{g}}$

$\tilde{\mathfrak{g}}$ is called the universal covering group.

In particular for every Lie algebra \mathfrak{L} there exists a unique (up to isomorphism) simply connected Lie-group $\tilde{\mathfrak{g}}$ such that every Lie-group G with Lie algebra \mathfrak{L} is isomorphic to

$\tilde{\mathfrak{g}}/K$ where K is a ^{discrete} subgroup of the center of $\tilde{\mathfrak{g}}$

Every representation of \mathfrak{L} gives upon exponentiation a representation of $\tilde{\mathfrak{g}}$

(REM: $SO(3)$ is a representation of $SU(2)$)

$$SO(3) \cong SU(2)/\mathbb{Z}_2$$

This boils down the classification of connected Lie groups to the classification of Lie-algebras!

Classification of the irreducible representations of the Lie-algebra $su(2) \cong so(3)$

By exponentiation this is also a classification of the irreducible representations of $SU(2)$

a) Lie-algebra $[t_a, t_b] = i \varepsilon_{abc} t_c$

Note physics notation

$$(\phi, t\psi) = (t\phi, \psi) \quad \text{unitary representation}$$

$$(e^{it})^\dagger = e^{-it} \quad su(2) \quad u^\dagger = u^{-1}$$

consider the complexification: allow linear combinations of the t_a

$$t_\pm = t_1 \pm it_2, \quad t_3$$

$$[t_3, t_\pm] = \pm t_\pm$$

$$[t_+, t_-] = 2t_3$$

"Casimir" $t^2 = t_1^2 + t_2^2 + t_3^2$

$$t_+ t_- = t_1^2 - t_2^2 + t_3^2 = t_1^2 + t_2^2 + t_3^2$$

$$t_- t_+ = t_1^2 + t_2^2 - t_3^2 = t_1^2 + t_2^2 - t_3^2$$

$$t_+^2 = t_+ t_- + t_3^2 - t_3$$

$$t_-^2 = t_- t_+ + t_3^2 + t_3$$

$$t_+ t_- = t^2 - t_3^2 + t_3$$

$$t_- t_+ = t^2 - t_3^2 - t_3$$

Note $t^2 \neq \lambda$ but $[t^2, t_a] = 0$

$\Rightarrow t^2 \propto \mathbb{1}$ in every irreducible representation

Ex 1) $[t_a^2, t_a] = 0$, sufficient to check it for $a=1$

$$[t_1^2 + t_2^2 + t_3^2, t_1] = [t_1^2, t_1] + [t_2^2, t_1] + [t_3^2, t_1] = 0$$

$$= t_2^2 t_1 - t_1^2 t_2 + t_3^2 t_1 - t_1 t_3^2 = 0$$

$$= t_2^2 t_1 - t_1^2 t_2 + t_3^2 t_1 - t_1 t_3^2 = 0$$

$$t_2 [t_1, t_1] + [t_1, t_1] t_2 + t_3 [t_1, t_1] + [t_1, t_1] t_3 = 0$$

$$-i t_2 t_3 - i t_3 t_2 + i t_3 t_2 + i t_2 t_3 = 0$$

$t^2 \propto \mathbb{1}$; we can in addition assume that t_3 is diagonal = we assume a basis in which t_3 is diagonal

"Highest weight" ψ^j $t_3 \psi^j = j \psi^j$

$$\text{and } t_+ \psi^j = 0$$

$$t_3 t_- \psi^j = ([t_3, t_-] + t_- t_3) \psi^j = (-t_- + t_- t_3) \psi^j = (j-1) t_- \psi^j$$

\Rightarrow the vector $t_- \psi^j$ is eigenvector of t_3 with eigenvalue $j-1$!

$$t^2 \psi^j = (t_- t_+ + t_3^2 + t_3) \psi^j = (j^2 + j) \psi^j = j(j+1) \psi^j$$

Since t^2 commutes with all t_a all vectors $t_a \psi^j$ have the same eigenvalue $j(j+1)$!

\rightarrow characterizes the representation

$$t_3 (t_-)^{k_2} \psi^j = (j - k_2) \psi^j$$

suppose $(t_-)^{k_2+1} \psi^j = 0$ finite dimensional representation

$$t_+ t_- (t_-)^{k_2} \psi^j = 0 = \\ = (t^2 - t_3^2 + t_3) (t_-)^{k_2} \psi^j = (j(j+1) - (j-k_2)^2 + (j-k_2)) = 0$$

$$\Rightarrow \cancel{j(j+1)} - \cancel{j^2} + j - \cancel{j^2} + 2jk_2 - k_2^2 + j - k_2 = 0 \\ 2jk_2 + 2jk_2 - k_2^2 = k_2 = 0$$

$$\cancel{j^2} + j - \cancel{j^2} + 2jk_2 - k_2^2 + j - k_2 = 0$$

$$\Rightarrow \boxed{j = \frac{1}{2} k_2}$$

$$k_2 \in \mathbb{N}_0$$

$$m: k_2, \dots, -k_2$$

Normalization: $\psi_m^j = \frac{1}{\sqrt{m!}} \psi^j$

and suppose ψ_m^j is normalized

$$\begin{aligned} (t_- \psi_m^j, t_- \psi_m^j) &= (\psi_m^j, t_+ t_- \psi_m^j) = \\ &= (\psi_m^j, (t^2 - t_3^2 + t_3) \psi_m^j) = \\ &= j(j+1) - m^2 + m \end{aligned}$$

$$(t_-)^j = \psi_j^j, \psi_{j-1}^j, \psi_{j-2}^j, \dots, \psi_{-j}^j$$

$$m = j, j-1, \dots, -j \quad \psi_m^j \quad t_3 \psi_m^j = m \psi_m^j$$

$$\|t_- \psi_m^j\| = (t_- \psi_m^j, t_- \psi_m^j) = (\psi_m^j, t_+ t_- \psi_m^j) =$$

$$= (\psi_m^j, (t^2 - t_3^2 + t_3) \psi_m^j) = [j(j+1) - m^2 + m] =$$

$$= (j+m)(j-m+1) = j(j+1) - jm + jm - m^2 + m \quad \checkmark$$

$$\Rightarrow \begin{cases} t_- \psi_m^j = [(j+m)(j-m+1)]^{\frac{1}{2}} \psi_{m-1}^j \\ t_+ \psi_m^j = [(j+m+1)(j-m)]^{\frac{1}{2}} \psi_{m+1}^j \end{cases}$$

From this we can construct the matrices

D_{mm}^j for a $(2j+1) \times (2j+1)$ dimensional representation

$$D^j(t_1) = \frac{1}{2} \left\{ \delta_{m+1, m'} [(j-m)(j+m+1)]^{\frac{1}{2}} + \delta_{m-1, m'} [(j+m)(j-m+1)]^{\frac{1}{2}} \right\}$$

$$D^j(t_2) = \frac{i}{2} \left\{ \delta_{m+1, m'} [(j-m)(j+m+1)]^{\frac{1}{2}} - \delta_{m-1, m'} [(j+m)(j-m+1)]^{\frac{1}{2}} \right\}$$

$$D^j(t_3) = m \delta_{m, m'}$$

these are completely general results.

Valid e.g. for function spaces

variation of action on scalar functions

$$\bar{\Psi}'(\bar{x}') = \Psi(R\bar{x}) \Rightarrow \Psi'(\bar{x}) = \Psi(R^{-1}\bar{x})$$

to compute the generators on the function space

$$R \sim 1 - i\omega^a t_a \bar{x}$$

$t_a =$ generators of $SO(3) = -i\varepsilon_{ijk} =$ antisymmetric 3×3 matrices

$$\delta\Psi = \Psi'(x) - \Psi(x) = \Psi(1 - \omega^a \varepsilon_{ijk} x^j) - \Psi(x) =$$

$$= (i\omega^a t_a \Psi) = -\omega^a \varepsilon_{ijk} x^j$$

$$\Rightarrow t_a = i\varepsilon_{ajk} x_j \partial_k$$

in polar coords

$$t_1 = -i \sin\theta \frac{\partial}{\partial\theta} - \cot\theta \cos\theta \frac{\partial}{\partial\phi}$$

$$t_2 = i \cos\theta \frac{\partial}{\partial\theta} - \cot\theta \sin\theta \frac{\partial}{\partial\phi}$$

$$t_3 = i \frac{\partial}{\partial\phi}$$

$$\pm^2: \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right] \Psi_m^j(\theta, \phi) = j(j+1) \Psi_m^j$$

$$i \frac{\partial}{\partial\phi} \Psi_m^j(\theta, \phi) = m \Psi_m^j$$

$$\Psi_m^j = Y_{jm}(\theta, \phi) = (-1)^m \left[\frac{(2j+1)(j-m)!}{4\pi (j+m)!} \right]^{\frac{1}{2}} e^{im\phi} P_m^j(\cos\theta)$$

spherical harmonics

$$t_a X_{jm}^j = D_{mm'}^j X_{m'}^j$$

Clebsch-Gordan decomposition

Decomposition of a direct product representation into a sum of irreducible representations

SU(2) repr. D^{j_1} and D^{j_2}

$$D^{j_1} \otimes D^{j_2} = \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} D^j$$

$$D^{j_1+j_2} \oplus D^{j_1+j_2-1} \oplus \dots \oplus D^{j_1-j_2}$$

convention $j_1 \geq j_2$

Sanity check: can this be true? How many states are there?

direct product $(2j_1+1)(2j_2+1)$ states

$$\sum_{j=|j_1-j_2|}^{j_1+j_2} (2j+1) = \underbrace{2j_1+2j_2+1 + 2j_1+2j_2 + \dots + 2j_1-2j_2+1}_{4j_1+2}$$

and there are $\frac{1}{2} [(j_1+j_2) - (j_1-j_2) + 1]$ such pairs = $2j_2+1$

$$\Rightarrow \frac{1}{2} (2j_2+1)(4j_1+2) = (2j_1+1)(2j_2+1) \checkmark$$

of states matches

operators $t_3^{j_1}, t_{\pm}^{j_1}$ and $t_3^{j_2}, t_{\pm}^{j_2}$

$$t_3^{j_1} \otimes t_3^{j_2} = t_3^{j_1} \otimes \mathbb{1} \oplus \mathbb{1} \otimes t_3^{j_2} = t_3^j$$

$$t_3^j (\psi_{m_1}^{j_1} \otimes \psi_{m_2}^{j_2}) = m_1 + m_2 = m$$

$$\psi_{m_1}^{j_1} \otimes \psi_{m_2}^{j_2} \in \bigoplus_m^j$$

$$\bigoplus_m^j = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \underbrace{(j_1 j_2 j m | j_1 m_1 j_2 m_2)}_{\text{Clebsch-Gordan coefficients}} \psi_{m_1}^{j_1} \otimes \psi_{m_2}^{j_2}$$

Clebsch-Gordan coefficients

by assumption $\psi_{m_1}^{j_1} \otimes \psi_{m_2}^{j_2}$ is an orthonormal basis as well as Θ_m^j
 \Rightarrow matrix of Clebsch Gordan coefficients is unitary.

~~Condition~~ $(j_1 j_2 j m | j_1 m_1 j_2 m_2) = 0$ if $m \neq m_1 + m_2$

highest possible value $m_1 = j_1$; $m_2 = j_2$

there is only 1 such state

$$\Rightarrow \psi_{j_1}^{j_1} \otimes \psi_{j_2}^{j_2} = e^{i\varphi} \Theta_{j_1+j_2}^{j_1+j_2}$$

up to some arbitrary phase ($\varphi=0$) convention chooses the Clebsch-Gordan coefficients are unique. These $(j_1 j_2 j m | j_1 m_1 j_2 m_2)$ real and positive

for $m = m_1 + m_2 - 1$ there are 2 such states

$$\psi_{j_1-1}^{j_1} \otimes \psi_{j_2}^{j_2} \quad , \quad \psi_{j_1}^{j_1} \otimes \psi_{j_2-1}^{j_2}$$

- o) one can be obtained by applying $t_-^{j_1+j_2}$ to $\Theta_{j_1+j_2}^{j_1+j_2}$
- o) the other one can be obtained by either looking for the linear combination orthogonal to $t_-^{j_1+j_2} \Theta_{j_1+j_2}^{j_1+j_2}$ or by solving

$$t_+^{j_1+j_2} (\alpha \psi_{j_1-1}^{j_1} \otimes \psi_{j_2}^{j_2} + \beta \psi_{j_1}^{j_1} \otimes \psi_{j_2-1}^{j_2}) = 0$$

and normalizing.

This new state is obviously the highest weight of the $j = j_1 + j_2 - 1$ representation

at level l_k with $l_k \leq 2j_2$ the values of (m_1, m_2) are $(j_1 - l_k, j_2)$; $(j_1 - l_k + 1, j_2 - 1) \dots (j_1, j_2 - l_k)$

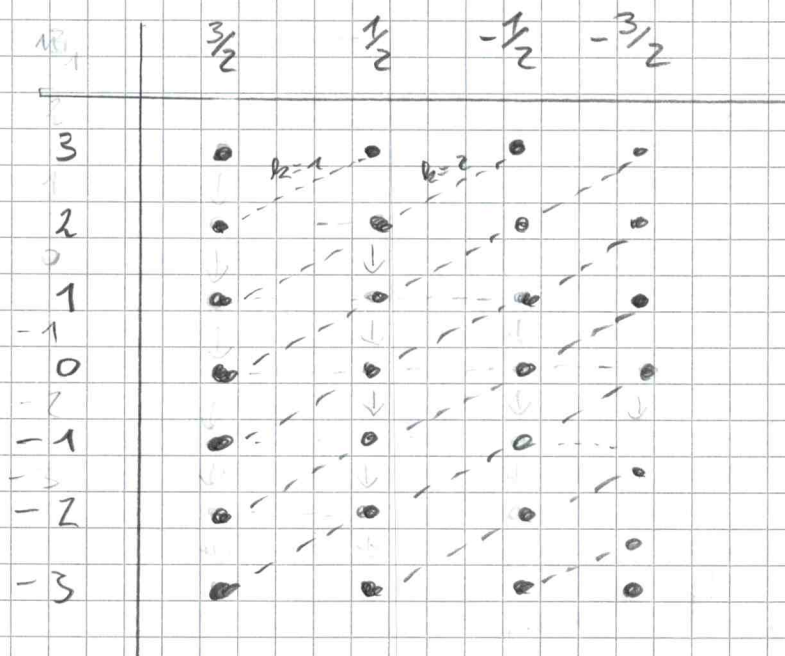
l_{k+1} such states up to $l_k = 2j_2$ where there are $2j_2 + 1$ states. l_k of these states can be obtained from the states of level $(l_k - 1)$ and the additional state can be obtained as before.

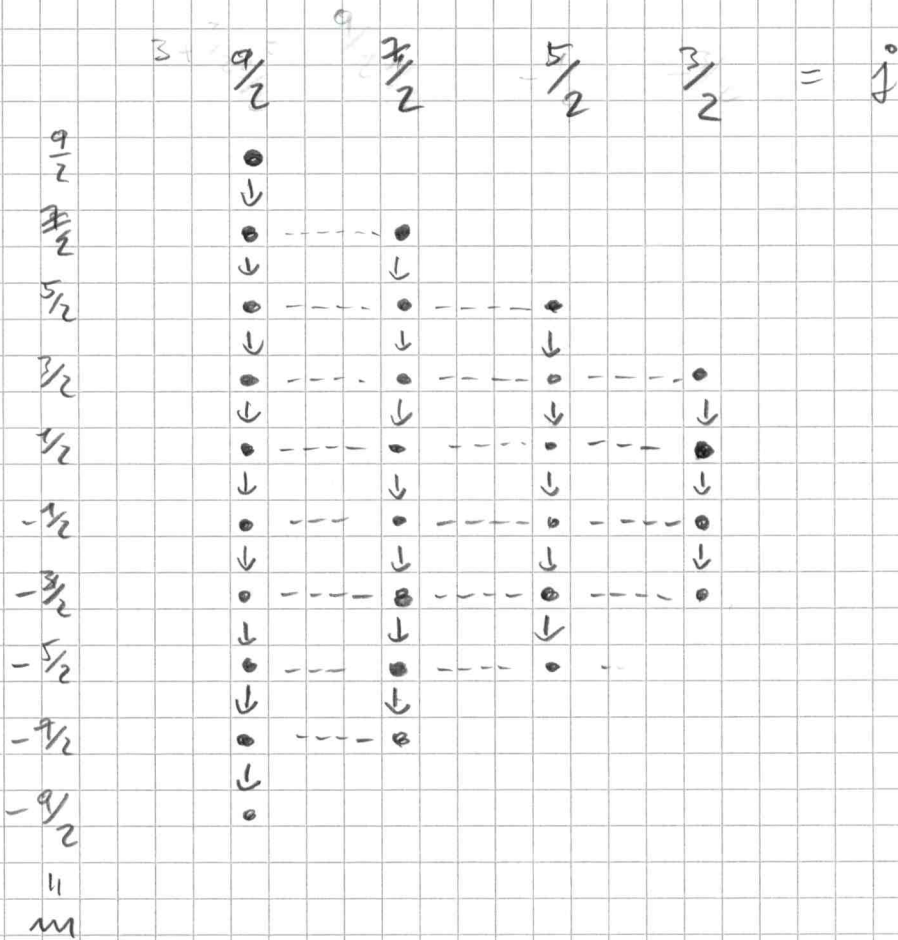
From then on all the rest of the states can be obtained by applying the lowering operator $t_{-}^{j_1 + j_2}$.

Schematic diagram:

$j_1 = 3$

$j_2 = 3/2$





Example

$$j_1 = \frac{1}{2} \otimes j_2 = \frac{1}{2}$$

physics notation $\psi_{\frac{1}{2}, \frac{1}{2}}^{\frac{1}{2}} \otimes \psi_{\frac{1}{2}, \frac{1}{2}}^{\frac{1}{2}} \sim \left| \frac{1}{2}, \frac{1}{2} \right\rangle$

$$t_{-}^{\frac{1}{2} + \frac{1}{2}} = t_{-}^{\frac{1}{2}} \otimes \mathbb{1} + \mathbb{1} \otimes t_{-}^{\frac{1}{2}}$$

$$t_{-} \psi_{m}^j = \sqrt{(j+m)(j-m+1)} \psi_{m-1}^j$$

$$t_{-}^{\frac{1}{2} + \frac{1}{2}} \psi_{\frac{1}{2}}^{\frac{1}{2}} = \sqrt{1} \left| -\frac{1}{2}, \frac{1}{2} \right\rangle + \left| \frac{1}{2}, -\frac{1}{2} \right\rangle$$

$$t_{-}^1 \theta_1^1 = \sqrt{2} \theta_0^1$$

$$\Rightarrow \theta_0^1 = |10\rangle = \frac{1}{\sqrt{2}} \left(\left| -\frac{1}{2}, \frac{1}{2} \right\rangle + \left| \frac{1}{2}, -\frac{1}{2} \right\rangle \right)$$

$$t_+ \otimes \frac{1}{2} : t_+ \left(\alpha \left| -\frac{1}{2}, \frac{1}{2} \right\rangle + \beta \left| \frac{1}{2}, -\frac{1}{2} \right\rangle \right) =$$

$$= (\alpha + \beta) \left| \frac{1}{2}, \frac{1}{2} \right\rangle \Rightarrow \alpha = -\beta$$

normalization $\alpha = \frac{1}{\sqrt{2}}$

$$\Rightarrow |0,0\rangle = \frac{1}{\sqrt{2}} \left(\left| -\frac{1}{2}, \frac{1}{2} \right\rangle - \left| \frac{1}{2}, -\frac{1}{2} \right\rangle \right)$$

$$t_- \left[\frac{1}{\sqrt{2}} \left(\left| -\frac{1}{2}, \frac{1}{2} \right\rangle + \left| \frac{1}{2}, -\frac{1}{2} \right\rangle \right) \right] \rightarrow \left| -\frac{1}{2}, -\frac{1}{2} \right\rangle = \left| -1, -1 \right\rangle$$

$$|1,0\rangle = \left| \frac{1}{2}, \frac{1}{2} \right\rangle$$

$$|1,0\rangle = \frac{1}{\sqrt{2}} \left(\left| -\frac{1}{2}, \frac{1}{2} \right\rangle + \left| \frac{1}{2}, -\frac{1}{2} \right\rangle \right)$$

$$|1,-1\rangle = \left| -\frac{1}{2}, -\frac{1}{2} \right\rangle$$

$$|0,0\rangle = \frac{1}{\sqrt{2}} \left(\left| -\frac{1}{2}, \frac{1}{2} \right\rangle - \left| \frac{1}{2}, -\frac{1}{2} \right\rangle \right)$$

explicitly: $\left(\frac{1}{2}, \frac{1}{2}; 11 \left| \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right\rangle \right) = 1$

$$\left(\frac{1}{2}, \frac{1}{2}; 10 \left| \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \right\rangle \right) = \left(\frac{1}{2}, \frac{1}{2}, 10 \left| \frac{1}{2}, \frac{1}{2}; \frac{1}{2}, -\frac{1}{2} \right\rangle \right) = \frac{1}{\sqrt{2}}$$

$$\left(\frac{1}{2}, \frac{1}{2}; 00 \left| \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \right\rangle \right) = \frac{1}{\sqrt{2}} \quad \left(\frac{1}{2}, \frac{1}{2}, 00 \left| \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \right\rangle \right) = -\frac{1}{\sqrt{2}}$$

$$\left(\frac{1}{2}, \frac{1}{2}; 1-1 \left| \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \right\rangle \right) = 1$$

Table

		$\frac{1}{2}$	$\frac{1}{2}$	
		m_1	m_2	
m_1	m_2			
1	2			

$$\frac{1}{2} \otimes \frac{1}{2} :$$

		1		
		1	0	
$\frac{1}{2}$	$\frac{1}{2}$	1	0	
		$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
		$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	-1
		$-\frac{1}{2}$	$-\frac{1}{2}$	1

Listed like that in the

PV

The structure of semi-simple Lie-algebras

Def: 1) A Lie-algebra is simple if it does not possess an invariant subalgebra is not Abelian

2) It is semi-simple if does not possess an Abelian invariant subalgebra

$su(2)$... simple

$su(2) \otimes su(3)$ semi-simple

Lie-groups: 1) Lie group is simple if it does not possess an invariant continuous subgroup

\Rightarrow its Lie-algebra is simple.

2) Lie group is semi-simple if its Lie-algebra is semi-simple

Killing-form: $\kappa: a, b \in \mathfrak{L} \rightarrow \mathbb{R}$ (for a real algebra)

$$\kappa(a, b) = \text{tr}(\text{ad}(a), \text{ad}(b))$$

Trace in the adjoint representation

Expls: $so(3)$:

$$a_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad a_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad a_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(a_i)_{ad} = \varepsilon_{ijk} a_k$$

$$a_1 \circ a_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{tr}(a_1 \circ a_2) = 0$$

$$a_1^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{tr}(a_1^2) = -2 \quad \text{etc.}$$

$$\Rightarrow \kappa_{pq} = -2\delta_{pq} \quad \mathfrak{so}(3)$$

$$\rightarrow \mathfrak{sl}(2, \mathbb{R}) \quad \det S = 1 \quad S_{ij} \text{ real}$$

$$\rightarrow \text{tr}(b) = 0$$

$$b_1 = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad b_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad b_3 = \frac{1}{2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$[b_1, b_2] = b_3 \quad [b_2, b_3] = b_1 \quad [b_3, b_1] = -b_2$$

$$\text{ad}(b_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{ad}(b_2) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \text{ad}(b_3) = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\kappa_{pq} = \begin{pmatrix} 2 & & \\ & -2 & \\ & & 2 \end{pmatrix}$$

The Killing form is "invariant"

$$\kappa([a, b], c) + \kappa(b, [a, c]) = 0$$

Proof: $\text{tr}(abc - bac + bac - bca) = 0$

Theorem The Killing form of a semi-simple Lie-algebra is non-degenerate

Suppose \mathfrak{L} contains an Abelian invariant subalgebra \mathfrak{L}'
 to $a \in \mathfrak{L}'$ $b \in \mathfrak{L}$, basis $e_1, \dots, e_{n'}$ $e_{n'+1}, \dots, e_n$

$$\text{ad}(a) = \begin{pmatrix} 0 & & \\ & 0 & \\ & & \ddots \end{pmatrix}$$

$$\text{ad}(b) = \begin{pmatrix} \dots & & \\ & \dots & \\ 0 & & \dots \end{pmatrix}$$

Abelian

invariant

other elements

invariant

$$\text{tr}(\text{ad}(a) \cdot \text{ad}(b)) \sim \begin{pmatrix} 0 & & \\ & \ddots & \\ 0 & & \ddots \end{pmatrix} \quad \det(\kappa) = 0$$

some more definitions (useful)

define $\bullet) \mathfrak{X}^{(k)} = [\mathfrak{X}, \mathfrak{X}^{(k-1)}]$

$\mathfrak{X}^{(k)}$ is called trivial if it contains only $\{0\}$

$\bullet) \mathfrak{X}^{(k)}$ is an invariant subalgebra
remember if $\mathfrak{X}', \mathfrak{X}''$ are invariant subalgebra
then $[\mathfrak{X}', \mathfrak{X}'']$ is invariant

$$\begin{aligned} [\mathfrak{X}, [\mathfrak{X}', \mathfrak{X}']] &= [[\mathfrak{X}, \mathfrak{X}'], \mathfrak{X}'] + [\mathfrak{X}', [\mathfrak{X}, \mathfrak{X}']] \\ &= [\mathfrak{X}', \mathfrak{X}'] + [\mathfrak{X}', \mathfrak{X}'] \end{aligned}$$

$\bullet) [\mathfrak{X}, \mathfrak{X}] = \mathfrak{X}^{(1)} \quad [\mathfrak{X}^{(1)}, \mathfrak{X}^{(1)}]$ etc. ~

$\bullet) \mathfrak{X}$ is called solvable if $\mathfrak{X}^{(k)}$ is trivial
for some k

\rightarrow A Lie algebra is semi-simple if it
does not contain an solvable invariant
subalgebra.

Theorem (Cartan's first criterium)

A Lie algebra \mathfrak{X} is solvable if and
only if $\kappa(a, b) = 0 \quad \forall a, b \in \mathfrak{X}^{(1)}$

Theorem: (Cartan's second criterium)

A Lie algebra is semi-simple if and
only if its Killing form is
non-degenerate $\det(\kappa) \neq 0$

th: Every semi-simple lie-algebra is either simple or the direct sum of simple Lie algebras

$$\mathfrak{L}_{ss} = \mathfrak{L}_s^1 \oplus \dots \oplus \mathfrak{L}_s^n$$

Theorem: If \mathfrak{L} is semi-simple its adjoint representation is faithful

Proof: suppose $ad(a) = ad(b)$ but $a \neq b$
then $ad(a-b) = 0$ and thus
 $tr(ad(a-b), ad(c)) = 0 \quad \forall c \in \mathfrak{L}$
 $\Rightarrow \det K = 0 \Rightarrow$ it cannot be semi-simple by Cartan's first criterion.

Theorem If \mathfrak{L} is simple then its adjoint representation is irreducible.

Proof: The linear space V is the Lie algebra itself $V = \mathfrak{L}$ $ad(a) \cdot b = [a, b]$
since \mathfrak{L} is simple it does not have an invariant subspace (= subalgebra)
 $\Rightarrow ad(a)$ is irreducible.

Example $su(2) : K_{pq} = -2\delta_{pq}$

so by Cartan's 2nd criterion it is semi-simple, but it also must be simple since otherwise $su(2) \cong \mathfrak{X}_1 \oplus \mathfrak{X}_2$

but $\dim(\mathfrak{X}_i) \geq 2$ since otherwise the algebra would be Abelian

$\Rightarrow su(2)$ is simple.

By construction of all simple Lie-algebras we will show that also $su(N)$ are simple

Example: show that $SO(4)$ is semi-simple but not simple by explicitly decomposing it as $SO(4) \cong so(3) \times so(3)$

Example: $U(N)$: the generators are the anti-hermitian $N \times N$ matrices.

$$\text{e.g. } su(2) \quad a_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad a_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad a_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \\ a_4 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$$

Lie algebra is $\mathfrak{U} \oplus su(2) \ni$ Abelian invariant subalgebra \Rightarrow it is not simple nor semi-simple

Examples: Euclidean group $\vec{x} \rightarrow O\vec{x} + \vec{t}$

Translations are (3) invariant Abelian subgroups

Similarly: Poincare group $\Lambda y \Lambda^T = y$

$$y = \text{diag}(1, -1, -1, 1) \quad x \rightarrow \Lambda x + a$$

Complexification of a real Lie algebra

We know the generators are linearly independent over the real numbers.

e.g. $su(2)$ $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$; $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$; $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$

are independent even over \mathbb{C}

but $sl(2, \mathbb{C})$ as a real Lie algebra has 6 generators

$$a_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad a_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad a_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$a_4 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad a_5 = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} \quad a_6 = \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix}$$

$$a_1 = -i a_4 \quad \text{etc.}$$

$sl(2, \mathbb{C})$ has only 3 generators as complexified Lie algebra

Formally: form pairs of generators

$$(a, b)$$

$$\text{and define } (\lambda + i\mu)(a, b) = (\lambda a - \mu b, \lambda b + \mu a)$$

$$(a, b) = a + ib$$

generators have to be linearly independent over the complex numbers.

$sl(2, \mathbb{C})$ is of dimension 3 as a complex Lie algebra

$$[a + ib, c + id] = [a, c] - [b, d] + i([b, c] + [a, d])$$

$$\begin{matrix} \downarrow & & \downarrow \\ (& &) \end{matrix}$$

$$[(a, b), (c, d)] = ([a, c] - [b, d], [b, c] + [a, d])$$

In particular we can choose a basis in the complexified Lie-algebra

$$(a_p, 0)$$

$$\text{the } [(a_p, 0), (a_q, 0)] = c_{pq} (a_r, 0)$$

a_p is a basis for the real Lie algebra with the same structure constants.

$$\Rightarrow \mathfrak{sl}(2, \mathbb{C}) : \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$\text{or } \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

they are equivalent as complex Lie algebras but not as real Lie-algebras

$$\mathfrak{sl}(2, \mathbb{R}) \neq \mathfrak{su}(2, \mathbb{R})$$

$$\text{but } \mathfrak{sl}(2, \mathbb{C}) \cong \mathfrak{su}(2, \mathbb{C})$$

If the complex Lie-algebra $\tilde{\mathfrak{L}}$ is simple the real form \mathfrak{L} is also simple.

Note the converse is not true!

A physically relevant example $SO(3,1)$

$$\Lambda^T \eta \Lambda = \eta \quad \eta = (-1, -1, -1, +1)$$

$$\text{generators: } \begin{matrix} \mathfrak{a}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \mathfrak{a}_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \mathfrak{a}_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{matrix}$$

$$\mathfrak{a}_4 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \\ -1 & & & 0 \end{pmatrix} \quad \mathfrak{a}_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & \cdot & \cdot & \cdot \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad \mathfrak{a}_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

algebra $P_{1,2,3} \dots 1,2,3$

$$[a_p, a_q] = -\epsilon_{pqr} a_r$$

$$[a_p, a_{q+3}] = -\epsilon_{pqr} a_{r+3}$$

$$[a_{p+3}, a_{q+3}] = \epsilon_{pqr} a_r$$

a_p --- rotations

a_{p+3} --- Boosts

→ No invariant subalgebra as real Lie-algebra but as complex algebra we are allowed to form the linear combinations

$$b_p = a_p + i a_{p+3} \quad \tilde{b}_p = a_p - i a_{p+3}$$

$$[b_p, b_q] = -\epsilon_{pqr} b_r$$

$$[b_p, \tilde{b}_q] = 0$$

$$[\tilde{b}_p, \tilde{b}_q] = -\epsilon_{pqr} \tilde{b}_r$$

but this is $su(2) \oplus su(2)$!

Theorem if \mathfrak{K} is simple but its complexification $\tilde{\mathfrak{K}}$ is not then $\tilde{\mathfrak{K}}$ is the direct sum of two isomorphic simple Lie-algebras

Theorem: A representation of $\tilde{\mathfrak{K}}$ provides a representation of \mathfrak{K} and vice versa

$$\Gamma_{\tilde{\mathfrak{K}}}(a_p, 0) = \Gamma_{\mathfrak{K}}(a_p)$$

