Exercise sheet for Advanced Mathematics Part-II, Differential Geometry and Topology

Lecturer: Karl Landsteiner

1. Which of the following sets are closed or open? What is their closure and their boundary?

   - The integers in $\mathbb{R}$
   - Points $(x, y) \in \mathbb{R}^2$ with $xy \neq 0$.

2. Let X be the set$\{a, b, c, d, e\}$.

   - Show that $\mathbb{T} = \{\{\}, X, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$ is a topology on X.
   - List the closed subsets. Give an example of a subset that is neither closed nor open. Give examples of open and of closed sets.
   - Compute the closure of $\{a\}$, $\{a, c\}$, $\{b, d\}$.
   - Let $Y = \{b, c, d\}$. What is the interior of Y?

3. Write down the action of a scalar field in form language. Do the same for the Maxwell field. Derive the equations of motion in form language.

4. Consider a scalar field in Minkowski space. Compute the vector fields whose flows generate Poincare transformations. Take a basis for these vector fields and show that their Lie brackets generate the Poincare algebra.

5. The Chern-Simons action for a gauge field in three dimensions is

$$CS[A] = \int \mathrm{tr}\left(AdA + \frac{2}{3}A \wedge A \wedge A\right) .$$

   Compute how the Chern-Simons action behaves under a gauge transformation $CS[g^{-1}Ag + g^{-1}dg]$. What is the equation of motion?

6. Extra fun challenge: Can you write an action for the Dirac field on a curved manifold in form language?