Exercise sheet for Advanced Mathematics Part-I, Group Theory

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- 1. Consider the group S_3 of permutations of 3 elements, $e: (abc) \to (abc)$, $\pi_1: (abc) \to (bac)$, etc
 - What is the order of the group? (more generally what is the order of the group S_n ?)
 - Write out the multiplication table.
 - Find a three dimensional faithful representation.
 - Find the conjugacy classes.
 - Find the proper subgroup H of order 3. In terms of the matrix representations what distinguishes this subgroup?
 - Is *H* normal in S_3 ? If yes what is the factor group? If the factor group is *F*. Does the group S_3 have the direct product structure $F \times H$?
- 2. Suppose G is a finite group of order 2N and H is a subgroup of order N. Show that H is a normal subgroup.
- 3. Compute the left- and right- invariant measures for the group SU(2) in the parametrization

$$U = \begin{pmatrix} \cos(\alpha)e^{i\beta} & \sin(\alpha)e^{i\gamma} \\ -\sin(\alpha)e^{-i\gamma} & \cos(\alpha)e^{-i\beta} \end{pmatrix}$$
(1)

- 4. Show that the defining two-dimensional representation of SO(2) is reducible by explicitly constructing the isomorphism $SO(2) \rightarrow D_{-1} \oplus D_{+1}$, where D_m is the U(1) representation formed by $\exp(im\phi)$ with $m \in \mathbb{Z}$.
 - Extend SO(2) to O(2) (parity!). Show that the two-dimensional representations are irreducible for $m \neq 0$ (hint: conjugation by parity). Discuss the case m = 0!
- 5. You know already the canonical homomorphism $\Phi_R : SU(2) \to SO(3)$. Based on that knowledge construct a similar homomorphism $\Phi_L :$ $SL(2;\mathbb{C}) \to SO(1,3;\mathbb{R})$ (hint: consider the matrices $X = x^0 \mathbf{1}_2 + x^i \sigma_i$ and $\bar{X} = x^0 \mathbf{1}_2 - x^i \sigma_i$).

Let V be a carrier space of the fundamental representation of $SL(2; \mathbb{C})$,

i.e. for $\psi \in V$ we have $S : \psi \to S\psi$. Let \tilde{V} be the dual space of one forms acting on V, (linear maps $\tilde{\psi} : V \to \mathbb{C}$, $\tilde{\psi}(\alpha_1\psi_1 + \alpha_2\psi_2) = \alpha_1\tilde{\psi}(\psi_1) + \alpha_2\tilde{\psi}(\psi_2)$. How does $SL(2;\mathbb{C})$ act on \tilde{V} ? Now take the complex conjugate space \bar{V} , how does S act on $\bar{\psi} \in \bar{V}$ and the corresponding one forms? Express the homomorphism Φ_L in terms of the representations on the dual spaces (hint: consider $\bar{X}.X$ and the identity $\epsilon\sigma_i\epsilon = \sigma_i^T$ with $\epsilon = i\sigma_2$. It is useful to introduce $\sigma^{\mu} = (\mathbf{1}_2, \vec{\sigma})$ and $\bar{\sigma}^{\mu} = (\mathbf{1}_2, -\vec{\sigma})$.

- 6. Take the Dynkin diagrams of B_2 , C_2 and G_2 . Write down their Cartan matrices and construct the weight diagrams corresponding of the two fundamental and the adjoint representations for each. Draw their root systems. What is the relation between B_2 and C_2 ?
- 7. Construct the (weight diagram) of the representation with highest weight (3,0) of the Lie algebra A_2 . Find an application in particle physics (hint: Ω^-)!