Exercise sheet for Advanced Mathematics
Part-I, Group Theory

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1. Consider the group S_3 of permutations of 3 elements, $e:\ (abc) \to (abc)$, $\pi_1:\ (abc) \to (bac)$, etc ....
   - What is the order of the group? (more generally what is the order of the group S_n?)
   - Write out the multiplication table.
   - Find a three dimensional faithful representation.
   - Find the conjugacy classes.
   - Find the proper subgroup H of order 3. In terms of the matrix representations what distinguishes this subgroup?
   - Is H normal in S_3? If yes what is the factor group? If the factor group is F. Does the group S_3 have the direct product structure $G/H \times H$?

2. Suppose G is a finite group of order $2N$ and H is a subgroup of order N. Show that H is a normal subgroup.

3. Compute the left- and right- invariant measures for the group $SU(2)$ in the parametrization

$$U = \begin{pmatrix} \cos(\alpha)e^{i\beta} & \sin(\alpha)e^{i\gamma} \\ -\sin(\alpha)e^{-i\gamma} & \cos(\alpha)e^{-i\beta} \end{pmatrix} \tag{1}$$

4.
   - Show that all irreducible representations of an Abelian group are one-dimensional (use Schur's lemma).
   - Show that the defining two-dimensional representation of $SO(2)$ is reducible by explicitly constructing the isomorphism $SO(2) \to U(1)_{-1} \otimes U(1)_{+1}$, where $U(1)_m$ is the group formed by $\exp(im\phi)$ with $m \in \mathbb{Z}$.
   - Extend $SO(2)$ to $O(2)$ (parity!). Show that the two-dimensional representations are irreducible for $m \neq 0$ (hint: conjugation by parity). Discuss the case $m = 0$!

5. Show that a rotation around an axis given by a unit vector $\hat{n}$ by an angle φ is

$$R_{ij} = \delta_{ij}\cos\varphi + \hat{n}_i\hat{n}_j(1-\cos\varphi) + \sin\varphi\hat{n}_k\epsilon_{ikj} \tag{2}$$

   by

- using the homomorphism $SU(2) \to SO(3)$ and first expanding $\exp(\varphi \hat{n} \vec{a})$ where a_i are the generators of $su(2)$.
- exponentiating the defining representation of $SO(3)$ (=adjoint representation of $SU(2)$).

6. Show that the real Lie-algebra $so(4)$ is semi-simple by not simple. What are the representations of $so(4)$?

7. Take the Dynkin diagrams of B_2 (•⇒•), C_2 (•⇐•) and G_2 (•⇛•). Write down their Cartan matrices and construct the weight diagrams corresponding of the two fundamental and the adjoint representations for each. Draw their root systems. What is the relation between B_2 and C_2?