Lectures on Differential Geometry and Topology

In modern physics the notions of topology and (differential) geometry are extremely important.

- Differential geometry : General Relativity
- Topology: condensed matter physics (top. Insulators)
- topological phases (Aharonov Bohm effect)

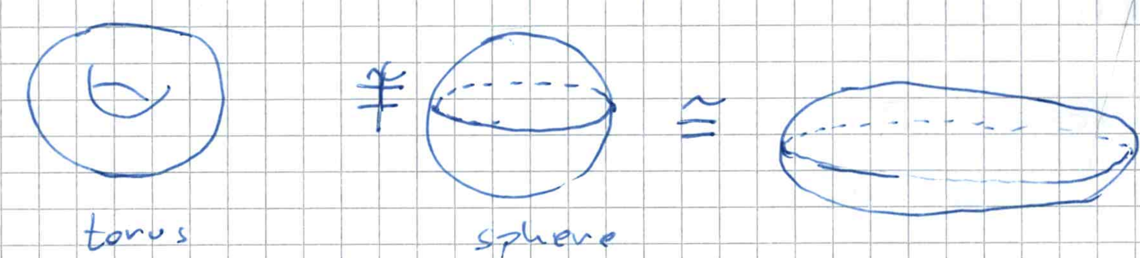
⋮

- String theory

+ Topology
- Manifolds
- differentiation on manifolds
- homotopy, homology, co-homology
  classify topology
- Fibre bundles (physics application : Gauge theories)

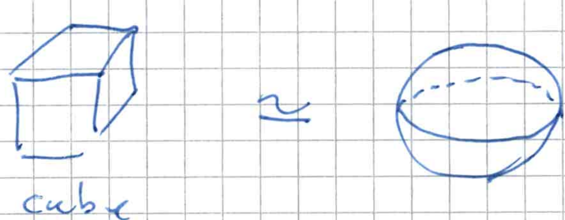
Books Nakahara, Sen & Nash "Topology, Geometry, Physics"

I. Topology :



topology formalizes the notion of "things" being continuously deformable

(Balloon, ...



Definition Topological space

Let X be any set and $Y = \{Y_\alpha\}$ be a collection of subsets (finite or infinit) of Y. X, Y form a topological space if

- $\emptyset$ (empty set) and $X \in Y$
- ~~Any~~ subcollection $\{Z_\alpha\}$ of Y has the property $\cup Z_\alpha \in Y$
- any finite subcollection $\{Z_{\alpha_1}, \dots, Z_{\alpha_n}\}$ is such that $\cap Z_{\alpha_i} \in Y$

X is a topological space and X_α are called open sets

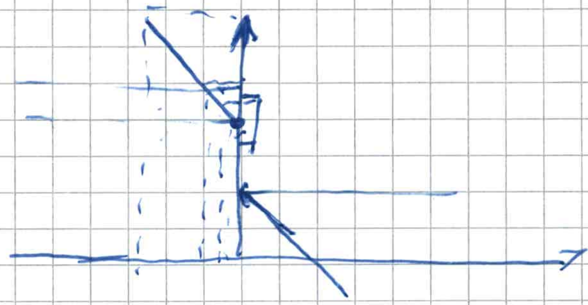
Examples

(a) X and Y is the collection of all possible subsets "discrete" topology

(b) X and $Y = \{\emptyset, X\}$ "trivial" topology

(c) $X = \mathbb{R}$; $\forall x \in \mathbb{R}$ take $a < x < b$ and the open iterval (a, b) "usual" topology

Definition: A function $f: X \to Y$ with X, Y being topological spaces is continous if the inverse image of all open sets Y are open sets X



$$f(x)\begin{cases} 2-x & x \le 0 \\ 1-x & x > 0 \end{cases}$$

$f: \mathbb{R} \to \mathbb{R}$

$$f^{-1}(3,4) = (-2,-1)$$

$$f^{-1}(2-\varepsilon, 2+\varepsilon) = (-\varepsilon, 0]$$

↳ not open!

Note: f does not open sets to oppen sets

Note: in the definition of topological space only finite intersections are allowed

otherwise with open sets on $\mathbb{R}$ we could do the following

$$X_\alpha = \left(x - \tfrac{1}{\alpha}, x + \tfrac{1}{\alpha}\right) \qquad \alpha = 1, 2, 3.$$

$$\bigcap_{\alpha=1}^{\infty} X_\alpha = x$$

So each point would count as open set = discrete topology

But then all maps would be continous as well.

For two given topologyis $T_1 = \{X_\alpha\}$, $\{X'_\alpha\} = T_2$

if $T_1 \supset T_2$ T_1 is larger than T_2 ; T_2 smaller T_1

finer coarser

Definition Neighbourhood

N is a neighbourhood of a point $x \in X$ if $x \in N$ and N contains an open set X_λ (N is not necessarily open itself)

Examples (a) $\mathbb{R}$ $[-1, 2]$ is a neighbourhood of 1 (and 0, and 0.5, ...)

but it is not an open set

Definition closed set:

Let T be a topology on X. a subset U of X is closed if its complement $U - X$ is open.

(Curiously X itself and $\emptyset$ are both open and closed)

Example $\mathbb{R}$: $(-\infty, a) \cup (b, \infty) = U \Rightarrow [a, b] = \mathbb{R} - U$ is closed ; U is open

Definition Closure of a set: take a set U and let F_α be a closed set containing U the family of such sets is $\{F_\alpha\}$ then

$$\bar{U} = \bigcap_\alpha F_\alpha$$

Example $\mathbb{R}$ $U = (a, b) \to \bar{U} = [a, b]$

"the smallest closed set containing U"

(Note $\bar{\bar{U}} = \bar{U}$)

Definition Interior of a set U: Let O_α be the collection of all open subsets of U then the interior of U is $\bigcup_\alpha O_\alpha = U^\circ$

Definition: Boundary : $\bar{U} - U^\circ$ closure of U minus the interior

example: $\mathbb{R}^2$: $U = x^2+y^2 \leq 1$

Interior U° is $x^2+y^2 < 1$

∂U boundary is $x^2+y^2 = 1$

but $U = x^2+y^2 \leq 1$

$U^\circ = x^2+y^2 < 1 = U$

$\partial U = \bar{U} - U^\circ = x^2+y^2 = 1$

[We want eventually integrate on open intervals
$U=(a,b)$ $\int_U f' = f|_{\partial U} = f(b) - f(a)$]

therefore we define the boundary through the closure of U

$U \cap \partial U = \emptyset \Rightarrow U$ open

$U \cap \partial U = \partial U \Rightarrow U$ closed

Fun example $\mathbb{R}$ usual topology $U = \frac{p}{q}$ rational points ; U° is the empty set , none of the subsets of U is open

closure of U is $\mathbb{R}$

$\partial U = \bar{U} - U^\circ = \mathbb{R} - \emptyset$!

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Definition Compactness :

A collection of sets $\{F_\alpha\}$ is a cover of U if $\bigcup_\alpha F_\alpha$ contains U

U is called compact if fore every open cover $\{F_\alpha\}$ there exists a finite sub cover.

example (a) $\mathbb{R}^2$ open disc B^2: $x^2 + y^2 < 1$ non-compact!

cover F_α: $x^2+y^2 < 1 - \frac{1}{\alpha}$ $\alpha = 2, \dots \infty$

$$\bigcup_\alpha F_\alpha = B^2$$

but there is no finite subcover!

(b) $\mathbb{R}^2$ closed disc $x^2+y^2 \leq 1$

and take $F_\alpha = x^2+y^2 < 1 - \frac{1}{\alpha} + \varepsilon$

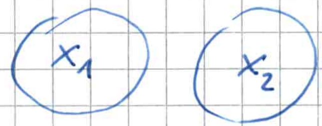
has a finite subcover: integer α for which $\frac{1}{\alpha_\varepsilon} < \varepsilon$

then $\{F_1, \dots, F_{\alpha_\varepsilon}\}$ is a finite subcover

closed disc is compact

Definition Connectedness

X is connected if $X \neq X_1 \cup X_2$ with $X_1 \cap X_2 = \emptyset$
where X_1, X_2 are open sets

$X =$ 

$X = X_1 \cup X_2$

Definition: Homeomorphism

two topological spaces are homeomorphic iff there exists a continous map $\alpha: X_1 \to X_2$ such α^{-1} is also continous.

"X_1 and X_2 are topologically equivalent"

Topological invariant: does not change under homeomorphism

Examples:
- Dimension
- Connectedness
- Compactness

Compactness $f: X \to Y$ X compact f homeomorphism
then let $\{F_\lambda\}$ be an open cover of Y, since f is homeomorphism $\{f^{-1}(F_\lambda)\}$ is an open cover of X. But X is compact so it contains a finite subcover $\{f^{-1}(F_{i_1}), \dots, f^{-1}(F_{i_n})\}$ and again since f is a homeomorphism this maps to a finite subcover of Y

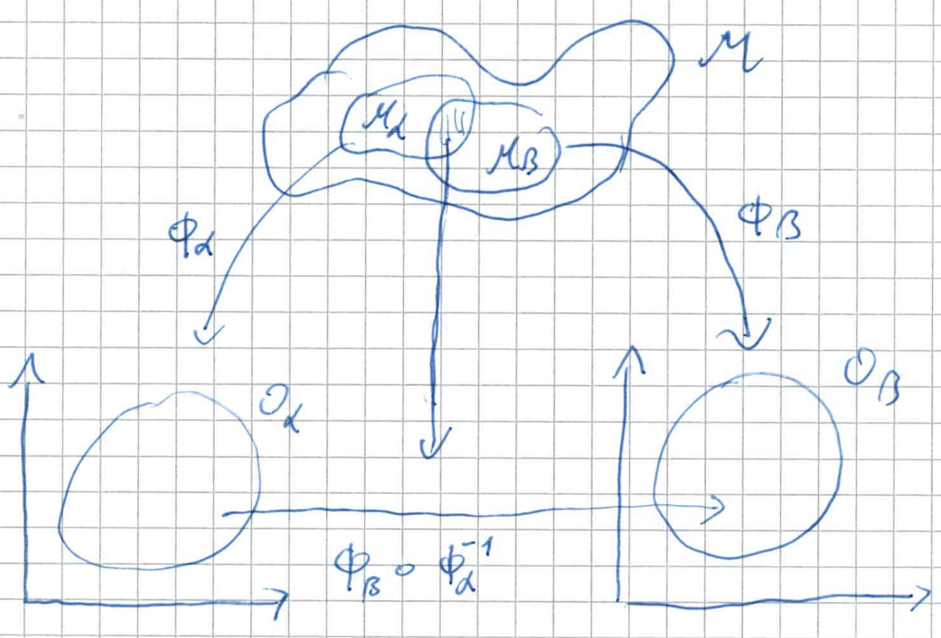
II. Differential Geometry

continous $\rightarrow$ differentiable

Definition Manifold: M is a differentiable manifold if

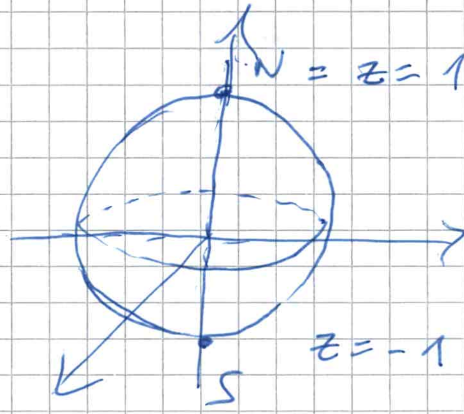
- M is a topological space
- M is provided with a set of pairs $\{(M_\alpha, \phi_\alpha)\}$
  $\{M_\alpha\}$ is an open cover of M $\quad \bigcup_\alpha M_\alpha = M$
  ϕ_α are homeomorphisms from M_α to an open subset of $\mathbb{R}^n$
- On the overlap $M_\alpha \cap M_\beta$ the map
  $\phi_\beta \circ \phi_\alpha^{-1}$ is (infinitly) differentiable (smooth).
  C^∞

The pair (M_α, ϕ_α) is called a chart.
The collection (M_α, ϕ_α) is called an atlas.



$\phi_\beta \circ \phi_\alpha^{-1}$ ... transition function
change of local coordinates

Example S^2:



$$x^2+y^2+z^2=1 \qquad u_\pm = \left(\frac{x}{1\pm z}, \frac{y}{1\pm z}\right)$$

maps from $S^2 \to \mathbb{R}^2$ except for the points $z=\pm 1$

$$x_+^2 + y_+^2 = \frac{1-z}{1+z} \qquad x_-^2 + y_-^2 = \frac{1+z}{1-z}$$

$$x_+ = \frac{x}{1+z} = \frac{x}{1-z}\,\frac{1-z}{1+z} = x_-\,(x_+^2+y_+^2)$$

$$\boxed{\left(\frac{x_+}{x_+^2+y_+^2}, \frac{y_+}{x_+^2+y_+^2}\right) = (x_-, y_-)} \qquad (\phi_- \circ \phi_+^{-1})$$

the map is well defined and C^∞ except for the two point $z=\pm 1$

Example projective space $\mathbb{RP}^n$: subset of $\mathbb{R}^{n+1}/\{0\}$ with the identification

$$(x_1, \dots, x_n) \cong \lambda(x_1, \dots, x_n)$$

charts: $x_1 \neq 0$; $x_2 \neq 0$; ...

$$\phi_\alpha = \left(\frac{x_1}{x_\alpha}, \frac{x_2}{x_\alpha}, \dots, \frac{x_{\alpha-1}}{x_\alpha}, \frac{x_{\alpha+1}}{x_\alpha}, \dots, \frac{x_{n+1}}{x_\alpha}\right)$$

$\Uparrow$ α.. left out

$\alpha = 1 .. n+1$

Def: Hausdorff space: topological space such that for any two point p, q there exist open sets X_p, X_q such that $X_p \cap X_q = \emptyset$

trivial example: $\mathbb{R}^n$ with usual topology

$$\delta = d(p,q)$$

$$X_p : d(x,p) < \frac{\delta}{2} - \varepsilon$$

$$X_q : d(x,q) < \frac{\delta}{2} - \varepsilon$$

Non-Hausdorff space: trivial topology $\{X, \emptyset\}$

Definition Metric space: topological space in which the open sets are define with a distance function $d(x,y)$

Distance (norm)
- $d(x,y) = d(y,x)$
- $d(x,y) \geq 0 \iff d(x,y) = 0 \quad x = y$
- $d(x,z) \leq d(x,y) + d(y,z)$

(triangle inequality)

open sets are "spheres" centered at a point

$$S_x : \{ y , d(x,y) < \varepsilon \}$$

- Obviously $\mathbb{R}^n$ with Euclidean norm

$$d(x,y) = [(x_1 - y_1)^2 + \dots + (x_n - y_n)^2]^{\frac{1}{2}}$$

Riemannian geometry

$$L_{pq} = \int_p^q \sqrt{g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}} \, dt \quad ; \quad d = \inf L_{ab}$$

Orientability: A manifold is orientable if the Jacobian of the local change of coordinates is positive on all overlaps

$\vec{x} = \phi_\alpha \; ; \; \vec{y} = \phi_\beta \qquad \phi_\beta \circ \phi_\alpha^{-1} \quad \det\left(\frac{\partial y^i}{\partial x^j}\right) > 0$

Calculus on manifolds

Suppose we have a Manifold M and a function f on M $\quad f: M \to \mathbb{R}$

Then we also define a curve as a 1-parameter family of points $p(t) \in M \quad p: \mathbb{R} \to \mathcal{M}$

We are interested in the derivative of f along the curve $p(t)$

$$\frac{d}{dt} f(p(t))$$

In local coordinates $\phi_\alpha : \vec{x} \qquad p: \vec{x}(t)$

$$\frac{d}{dt} \tilde{f}(p(t)) = \frac{d}{dt} \tilde{f}(\phi_\alpha^{-1}(x(t))) = \frac{d}{dt} f(x(t))$$

$$= \frac{dx^i}{dt} \frac{\partial f}{\partial x^i} = \underline{X} \cdot f$$

$\underline{X}$ ... is a differential operator acting on functions $f: \mathcal{M} \to \mathbb{R}$

obviously $\frac{dx^i}{dt}$ is the vector tangent to $x^i(t)$

obviously every such differential operator is tangent to some curve

$$X = \xi^i(p) \frac{\partial}{\partial x^i}$$

the $\quad x^i(t) = x^i(p) + \xi^i(p) \cdot t$

is a curve passing through p with tangent vector $\xi^i(p)$.

<u>Def</u>: Tangent space: space of all possible tangent vectors at the point $p \in M$

$$T_p(M)$$

"tangent space to M at the point p"

(also: all curves passing through p)

A basis is given by the differentials $\frac{\partial}{\partial x^i}$ in a particular chart.

$$y = y(x): \quad \frac{\partial}{\partial x^i} = \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j}$$

old notation "contra-variant" vectors

<u>Note</u>: the commutator of two vector fields is again a vector field

$$[X, Y] = [\xi^i \frac{\partial}{\partial x^i}, \eta^j \frac{\partial}{\partial x^j}] = \xi^i \partial_i \eta^j \partial_j + \xi^i \eta^j \partial_i \partial_j$$

$$- \eta^i \partial_i \xi^j \partial_j - \xi^i \eta^j \partial_i \partial_j$$

$$= (\xi^i \partial_i \eta^j - \eta^i \partial_i \xi^j) \partial_j$$

"Push forward" suppose we have a map from
$M \to N$ $\quad M, N$ manifolds

$$f_* : T_p(M) \longrightarrow T_{f(p)}(N)$$

let g be a function $g : N \to \mathbb{R}$

$$(f_* X)(g) := X(g \circ f)$$

$T_p(M)$: $\xi^i \frac{\partial}{\partial x^i}$ $\qquad f: y^k = y^k(x^i)$

$T_{f(p)}(M)$: $\xi^i \frac{\partial y^k}{\partial x^i} \frac{\partial}{\partial y^k} = \eta^i \frac{\partial}{\partial y^i}$

[Note: $\phi_\beta \circ f \circ \phi_\alpha^{-1}$; in physics $y^k(x^i)$]

"Pull back":

Dual space: linear map from a vector-space
$$\lambda : W \to \mathbb{C} \ (\mathbb{R})$$
the space of all such linear maps is the "dual" vector space W^*

suppose you have a map from $f_* : V \to W$
This allows to define a 1-form on V from a 1-form on W via
$$\lambda(f_*(v)) \to \mathbb{C} \qquad v \in V$$
$$f^*(\lambda(v)) = \lambda(f_*(v)) \in V^*$$
$$f^*(\lambda) = \mu \in V^*$$

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push-forward $\quad M \xrightarrow{f} N$

$$T_p(M) \xrightarrow{f_*} T_{f(p)}(M)$$

$$T_p^*(M) \xleftarrow{f^*} T^*_{f(p)}(M)$$

what are these

Differential forms: definition $\langle df, \underline{X} \rangle = \underline{X} \cdot f$

df ... differential of function f

In local coordinates

$$df = \frac{\partial f}{\partial x^i} dx^i$$

$$\left\langle \frac{\partial f}{\partial x^i} dx^i ; \xi^j \partial/_{\partial x^j} \right\rangle = \xi^i \frac{\partial f}{\partial x^i}$$

$$\Rightarrow \qquad \left\langle dx^i , \frac{\partial}{\partial x^j} \right\rangle = \delta^i_j$$

dx^i ... basis for $T_p^*(M)$ co-tangent space to M at point p

more generally $\quad a_i dx^i \qquad a_i$ ... covariant vector

pull back

$$y^k = y^k(x^i)$$

$$dy^k = \frac{\partial y^k}{\partial x^i} dx^i \qquad \text{"differential"}$$

$$\in T^*_{f(p)}(N) \qquad \in T_p^*(M)$$

(push-forward $\quad \frac{\partial}{\partial x^i} = \frac{\partial y^k}{\partial x^i} \frac{\partial}{\partial y^k}$)

Now we can define tensors

$$T^a_b \in T_p(M) \otimes \dots \otimes T_p(M) \otimes T^*_p(M) \otimes \dots \otimes T^*_p(M)$$

in a coordinate basis

$$T^{i_1 \dots i_a}{}_{j_1 \dots j_b} \frac{\partial}{\partial x^{i_1}} \dots \frac{\partial}{\partial x^{i_a}} dx^{j_1} \dots dx^{j_b}$$

example: metric $\quad ds^2 = g_{ij}\, dx^i dx^j$

Of utmost importance is the space of anti-symmetric n-forms $\quad \omega_p \in T^*_p(M) \otimes \dots \otimes T^*_p(M)$

$$\omega = \frac{1}{n!} \omega_{i_1 \dots i_n} dx^{i_1} \wedge \dots \wedge dx^{i_n}$$

"wedge" product $\quad dx \wedge dy = - dy \wedge dx$

"exterior" algebra: the wedge product of an n-form with an m-form is an $(n+m)$ form

$$\omega \wedge \nu = (n+m) \text{ form}$$

"exterior" derivative "d"

$$d: \Lambda^n T^*_p(M) \rightarrow \Lambda^{n+1} T^*_p(M)$$

$$\Omega^n(M) \rightarrow \Omega^{n+1}(M)$$

$\Omega^n(M)$ space of n-forms on M

$$\left.\begin{array}{l} \cdot)\ d^2 = 0 \qquad \cdot)\ d(\omega \wedge \nu) = d\omega \wedge \nu + (-)^p \omega \wedge d\nu \\ \text{if } f \text{ is a function; } f \in \Lambda^0(M) \quad \langle df, X \rangle = X(f) \end{array}\right\} \text{d is uniquely determined}$$

$$\boxed{d\omega = \frac{1}{n!} \frac{\partial}{\partial x^i} \omega_{i_1 \dots i_n} \, dx^i \wedge dx^{i_1} \wedge \dots \wedge dx^{i_n}}$$

functions = 0-forms

covariant vectors = 1-forms

anti-symmetric 2-tensor = 2-forms etc..

obviously $\boxed{d^2 = 0}$

since it always involves $\frac{\partial^2}{\partial x^i \partial x^j} dx^i \wedge dy^j = 0$

<u>Integration of differential forms</u>

take $\mathbb{R}^n$: and the form $\omega = dx^1 \wedge \dots \wedge dx^n$

"top-form" since on an n-dimensional space all $n+i$ forms vanish $i \geq 1$

make a change of coordinates

$$\omega = \frac{\partial x^1}{\partial y^{i_1}} dy^{i_1} \wedge \frac{\partial x^2}{\partial y^{i_2}} dy^{i_2} \wedge \dots \wedge \frac{\partial x^n}{\partial y^{i_n}} dy^{i_n}$$

$$= \frac{\partial x^1}{\partial y^{i_1}} \dots \frac{\partial x^n}{\partial y^{i_n}} \varepsilon(i_1 \dots i_n) \, dy^1 \wedge \dots \wedge dy^n$$

$\hookrightarrow$ ε-symbol

$$= \det\left(\frac{\partial x}{\partial y}\right) dy^1 \wedge \dots \wedge dy^n$$

$\hookrightarrow$ Jacobian

the n-form naturally picks up the Jacobian

this we can use to integrate a function f on M:

on each patch U_α: $\int_U f \, dx^1 \wedge \ldots dx^n$

define a partition of unity

- $0 < e_\alpha(x) < 1$
- $e_\alpha(x) = 0$ if $x \notin U_\alpha$
- $e_1 + e_2 \ldots + e_\ell = 1$

then obviously $\quad f = \sum_\alpha f e_\alpha$

$$\int f \omega = \sum_\alpha \int_{U_\alpha} e_\alpha f \, dx_\alpha^1 \wedge \ldots dx_\alpha^n$$

on a Riemannian manifold we can define the volume form

$$\omega_V = \sqrt{g} \, dx^1 \wedge \ldots \wedge dx^n$$

Here we also might introduce the ε-tensor defined as

$$\varepsilon_{i_1 \ldots i_n} = \sqrt{g} \, \underbrace{\varepsilon(i_1 \ldots i_n)}_{\pm 1, 0}$$

$$\omega_V = \frac{1}{n!} \varepsilon_{i_1 \ldots i_n} dx^{i_1} \wedge \ldots \wedge dx^{i_n}$$

Examples:

$\mathbb{R}^3$: 0-form f $\qquad df = \frac{\partial f}{\partial x^i} dx^i = (\mathrm{grad} f)_i \, dx^i$

1-form $\qquad A = A_x dx + A_y dy + A_z dz$

$\qquad B = B_x dx + B_y dy + B_z dz$

$$
\begin{aligned}
A \wedge B = \; & \underline{A_x B_y} \, dx \wedge dy + \underline{A_x B_z} \, dx \wedge dz \; + \\
& + A_y \underline{B_x} \, dy \wedge dx + A_y B_z \, dy \wedge dz \; + \\
& + A_z \underline{B_x} \, dz \wedge dx + A_z B_y \, dz \wedge dy \; = \\
= \; & (A_x B_y - A_y B_x) \, dx \wedge dy \; + \\
& + (A_x B_z - A_z B_x) \, dx \wedge dz \; + \\
& + (A_y B_z - A_z B_y) \, dy \wedge dz
\end{aligned}
$$

$\rightarrow$ cross - product

$$
\left(
\begin{aligned}
& v_i = \varepsilon_{ijk} A_j B_k \\
& v_x = A_y B_z - A_z B_y \\
& v_y = -A_x B_z + A_z B_x \\
& v_z = A_x B_y - A_y B_x
\end{aligned}
\right)
$$

$$
A \wedge B \wedge C = \underbrace{\begin{vmatrix} A_x & B_x & C_x \\ A_y & B_y & C_y \\ A_z & B_z & C_z \end{vmatrix}}_{\det} dx \wedge dy \wedge dz
$$

$$dA = \frac{\partial A_x}{\partial y} dy \wedge dx + \frac{\partial A_x}{\partial z} dz \wedge dx + \frac{\partial A_y}{\partial x} dx \wedge dy +$$
$$+ \frac{\partial A_y}{\partial z} dz \wedge dy + \frac{\partial A_z}{\partial x} dx \wedge dz + \frac{\partial A_z}{\partial y} dy \wedge dz =$$

$$= \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) dx \wedge dy + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) dz \wedge dx$$
$$\left(\frac{\partial A_y}{\partial z} - \frac{\partial A_z}{\partial y}\right) dy \wedge dz$$

$\rightarrow (\vec{\nabla} \times \vec{A})$ $\quad dA$ encodes the curl of "a vector field"

Further properties of the wedge product

- $\omega_n \wedge \omega_m = (-1)^{n.m} \omega_m \wedge \omega_n$
- $f^*(\omega_n \wedge \omega_m) = (f^*\omega_n) \wedge (f^*\omega_m)$

Integration

differential 1-form $\int df$ on $\mathbb{R}$

$$\int_a^b df = \int_a^b \frac{\partial f}{\partial x} dx = f(b) - f(a)$$

2-form $\omega = da$

$$\int \omega = \int \frac{\partial a_i}{\partial x^j} dx^j \wedge dx^i = \int \left(\frac{\partial a_1}{\partial x^2} - \frac{\partial a_2}{\partial x^1}\right) dx^1 \wedge dx^2$$
$$= \oint a_i dx^i$$

Generally: Stokes theorem:

$$\boxed{\int_M d\omega = \int_{\partial M} \omega}$$

Diffentiable structure:

manifold M and atlas $\{X_\alpha, \phi_\alpha\}$ be C^∞

class of functions $f \circ \phi_\alpha^{-1} = f(x)$ be C^∞

suppose there is another atlas $\{X_\beta, \psi_\beta\}$ which is also C^∞

question is the union $\{X_\alpha, \phi_\alpha\} \cup \{Y_\beta, \psi_\beta\}$ also C^∞?

→ equivalence classes of "differentiable structures"

Fun fact: •) S^7 has 28 inequivalent differentiable structures

•) $\mathbb{R}^4$ has a continuum of ~~of~~ inequivalent differentiable structures
(Donaldson Theory, topological SYM, Seiberg-Witten theory ...)

•) Diffeomorphism $f: M \to N$ $f^{-1}: N \to M$

$$\phi_\alpha^{-1} \circ f \circ \phi_\beta \in C^\infty$$

and $\phi_\beta^{-1} \circ f^{-1} \circ \phi_\alpha$ is C^∞

then M, N are called diffeomorphic

3 Definitions

Def: Immersion: $f: \mathcal{M} \to \mathcal{N}$ is called "Immersion" if $f_*\ T_p(\mathcal{M}) \to T_{f(p)}(\mathcal{N})$ is injective

Def: Embedding: $f: \mathcal{M} \to \mathcal{N}$ is called "Embedding" if f itself is injective and an immersion.

Def: Let f be an embedding $\mathcal{M} \to \mathcal{N}$.
$f(\mathcal{M})$ is called a submanifold of $\mathcal{N}$

(Fun-fact: the Klein bottle: there does not exist an embedding to $\mathbb{R}^3$ only immersion
= always a self-intersection
Embedding exists in $\mathbb{R}^4$!)

Example $f: \mathbb{R} \to \mathbb{R}^2$ $(x,y) = (t^3, t^2)$

not an immersion since

$$\underbrace{\frac{\partial}{\partial t}}_{\text{basis for } T_p(\mathbb{R})} = \frac{\partial x}{\partial t}\frac{\partial}{\partial x} + \frac{\partial y}{\partial t}\frac{\partial}{\partial y} = \underbrace{3t^2}_{=0 \text{ for } t=0}\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial y}$$

(basis for a vector space: $\det(v_a{}^i) \neq 0$)

f is one-to-one but not an embedding

better f_ε $(x,y) = (t^3 + \varepsilon t, t^2)$ $\varepsilon > 0$

f_ε is an embedding

$\varepsilon < 0$?

$f_-: \quad (x, y) = (t^3 - t,\ t^2)$

2-points $\quad t = \pm 1 \quad \to \quad (0, 1)$

$\frac{\partial}{\partial t}: \quad \frac{\partial x}{\partial t}\frac{\partial}{\partial x} + \frac{\partial y}{\partial t}\frac{\partial}{\partial y} = \left(3t^2 - 1,\ 2t\right)\begin{pmatrix} \partial/\partial x \\ \partial/\partial y \end{pmatrix}$

$= (3t^2 - 1)\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial y}$

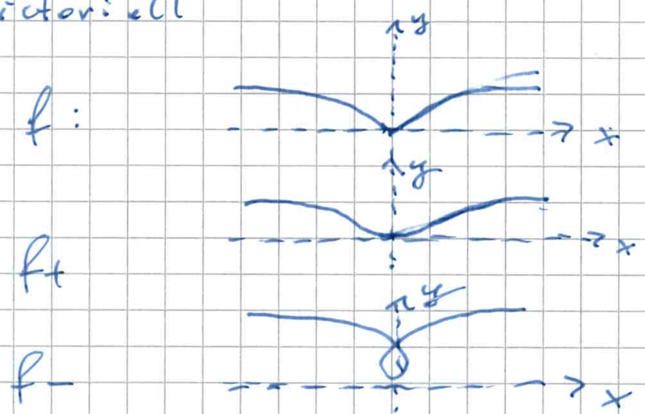
$t = 0$ is o.k. $\quad \frac{\partial}{\partial t}\Big|_{t=0} \to -\frac{\partial}{\partial x}$ ✓

$t = \pm 1 \qquad \frac{\partial}{\partial t} \to \left(2\frac{\partial}{\partial x} + 2\frac{\partial}{\partial y}\right)$ ✓

$\left(2\frac{\partial}{\partial x} - 2\frac{\partial}{\partial y}\right)$ ✓

f_- is not an embedding

pictoriell



(More on) The Lie-derivative

We have already defined a vector-field

Vector: equivalence class of curves $[p(t)]$ passing through p_0 such that

$$\frac{d}{dt} f(p(t)) \Big|_{t=0} = X \cdot f$$

local coordinates $X = \dot{x}^\mu(t) \frac{\partial}{\partial x^\mu}$

Def.: Flow of a vector-field: map $\sigma: \mathbb{R} \times M \to M$

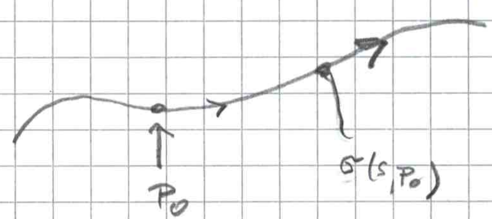
M ... manifold

$$\frac{d}{dt} \sigma(t, p_0) = X(\sigma(t,p))$$

$\mathcal{X}$ ... set of vector fields on $\mathcal{M}$

A flow fulfills

$$\boxed{\sigma(t, \sigma(s, p_0)) = \sigma(t+s, p_0)}$$



Proof: $\sigma(t)$ is the solution to a differential equation with initial condition $\sigma(0) = p_0$ the solution is unique.

Since the solution is unique we can chose any point on the curve as initial condition and recover the curve

Example: $X(x,y) = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$

$$\frac{d}{dt}x(t) = -y(t) \qquad \ddot{x} = -x$$

$$\frac{d}{dt}y(t) = x(t) \qquad \ddot{y} = -y$$

$$\Rightarrow \quad x = a\cos t + b\sin t$$

$$y = c\cos t + d\sin t$$

$$x(t=0) = x_0 \qquad a = x_0$$

$$y(t=0) = y_0 \qquad c = y_0$$

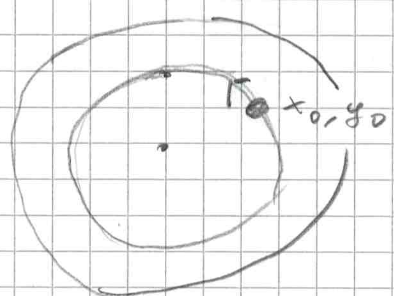
$$\dot{x} = -y \quad \Rightarrow \quad -y_0 = b$$

$$\dot{y} = x \quad \Rightarrow \quad x_0 = d$$

flow: $\sigma_0 \begin{pmatrix} \sigma^x \\ \sigma^y \end{pmatrix} = \begin{pmatrix} x_0\cos t - y_0\sin t \\ y_0\cos t + x_0\sin t \end{pmatrix}$

$\forall \; x_0, y_0 \in \mathbb{R}^2$

$$\sigma^{x^2} + \sigma^{y^2} = x_0^2 c^2 t - 2x_0 y_0 \, st\, ct + y_0^2 s^2 t + $$
$$+ y_0^2 c^2 t + 2x_0 y_0 \, st\, ct + x_0^2 s^2 t = x_0^2 + y_0^2$$



flow are concentric circles around the origin

Integral-curve

A flow generates a diffeomorphism $M \to M$
for t fixed σ_t : diff (M)
abelian group
- $\sigma_t \circ \sigma_s = \sigma_{s+t}$
- σ_0 = identity
- σ_{-t} is inverse

In our example σ_t $t = \theta$ is a rotation by θ

$$\sigma_{t=\theta} : \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

"one parameter group of transformations"
the vector field X is called the generator of the flow
Indeed in local coordinates p_0: x_0^μ

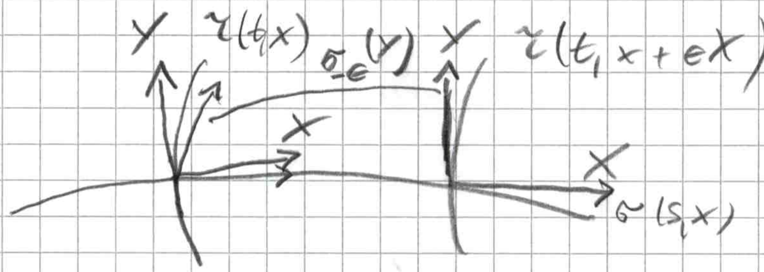
$$\sigma_\epsilon^\mu = x_0^\mu + \epsilon X^\mu$$

Lie-derivative of a vector field Y along the flow generated by X

Let σ be the flow generated by X, how does the vector field Y change along the flow of X?
$Y(\sigma_\epsilon(x))$ $\sigma_\epsilon : x^\mu \to x^\mu + \epsilon X^\mu$

but $Y(\sigma_\epsilon(x))$ cannot be compared to $Y(x)$
solution: use the push-forward in the backwards direction

$$\boxed{\mathcal{L}_X(Y) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left[(\sigma_{-\epsilon})_*(Y(\sigma_\epsilon(x)) - Y(x)\right]}$$



equivalently

$$\mathcal{L}_X(Y) = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[Y(x) - \sigma_{\epsilon,*}\left(Y(\sigma_{-\epsilon}(x))\right)\right]$$

$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[Y(\sigma_{\epsilon}(x)) - \sigma_{\epsilon,*}Y(x)\right]$$

In coordinates

$$Y|_{\sigma_\epsilon} = Y^\mu(x^\nu+\epsilon X^\nu)\frac{\partial}{\partial\underbrace{(x^\mu+\epsilon X^\mu)}_{y^\mu}} =$$

$$Y^\mu(x^\nu+\epsilon X^\nu)\frac{\partial x^\lambda}{\partial y^\mu}\frac{\partial}{\partial x^\lambda}$$

$$y^\mu = x^\mu + \epsilon X^\mu \qquad x^\mu = x^\mu - \epsilon X^\mu$$

$$\frac{\partial x^\lambda}{\partial y^\mu} = \delta^\lambda_\mu - \epsilon\frac{\partial X^\lambda}{\partial x^\mu}$$

$$(\sigma_{-\epsilon})_* Y = \left[Y^\mu(x) + \epsilon X^\nu\frac{\partial Y^\mu}{\partial x^\nu}\right]\left(\delta^\lambda_\mu - \epsilon\frac{\partial X^\lambda}{\partial x^\mu}\right)\frac{\partial}{\partial x^\lambda}$$

$$\mathcal{L}_X(Y) = \frac{1}{\epsilon}\Big[\Big(\cancel{Y^\lambda(x)} + \epsilon X^\nu\frac{\partial Y^\lambda}{\partial x^\nu} - \epsilon Y^\mu\frac{\partial X^\lambda}{\partial x^\mu} + O(\epsilon^2)\Big)\frac{\partial}{\partial x^\lambda}$$

$$\cancel{- Y^\lambda\frac{\partial}{\partial x^\lambda}}$$

$$\Rightarrow \mathcal{L}_X(Y) = \left(X^\nu\frac{\partial Y^\lambda}{\partial x^\nu} - Y^\nu\frac{\partial X^\lambda}{\partial x^\nu}\right)\frac{\partial}{\partial x^\lambda} =$$

$$= [X,Y] \; !$$ Lie-bracket

Now we have a geometrical interpretation of the Lie-derivative!

[physics:

$$Y^{\mu\prime}(x') = \frac{\partial y^\mu}{\partial x^\nu} X^\nu(x) \qquad y = x' = x + \epsilon$$

$$Y^{\mu\prime}(x) = \frac{\partial y^\mu}{\partial x^\nu} X^\nu(x - \epsilon^\lambda)$$

$$Y^{\mu\prime}(x) - Y^\mu(x) = (\delta^\mu_\nu + \partial_\nu \epsilon^\mu)(X^\nu - \varepsilon^\lambda \partial_\lambda X^\nu) - Y^\mu =$$

$$= \cancel{X^\mu} + \partial_\nu \epsilon^\mu Y^\nu - \varepsilon^\nu \partial_\nu Y^\mu - \cancel{Y^\mu}$$

$$= -[\varepsilon, Y] \quad]$$

Important properties of the Lie-bracket

- $[X, Y] = -[Y, X]$
- $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$
- for $f: M \to N$

$$f_*([X, Y]) = [f_* X, f_* Y] \quad \leftarrow \text{Exercise!}$$

We can also define the Lie-derivative of a 1-form along X

$$\mathcal{L}_X(\omega) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left[\sigma_\epsilon^*(\omega(\sigma_\epsilon(x))) - \omega(x)\right]$$

Show that in component

$$\boxed{\mathcal{L}_X(\omega_\mu) = (X^\lambda \partial_\lambda \omega_\mu + \partial_\mu X^\lambda \omega_\lambda)\, dx^\mu}$$

"covariant vector"

Def: inner derivative $i_X : \Omega^p \to \Omega^{p-1}$

$$i_X(\omega) = \omega(X, x_1, .. x_n)$$

$$\omega = \frac{1}{p!} \omega_{\mu_1 .. \mu_p} dx^{\mu_1} \wedge .. \wedge dx^{\mu_p}$$

$$\boxed{i_X \omega = \frac{1}{(p-1)!} X^\mu \omega_{\mu \mu_1 .. \mu_{p-1}} dx^{\mu_1} \wedge .. \wedge dx^{\mu_p}}$$

Properties

- nilpotent $(i_X)^2 = 0$
- $i_X(\omega \wedge \eta) = i_X\omega \wedge \eta - (-)^p \omega \wedge (i_X \eta)$

  if ω is a p-form

Show $\mathcal{L}_X \omega$ ω 1-form can be written as

$$\boxed{\mathcal{L}_X = d i_X + i_X d}$$

this is valid for arbitrary p-forms!

e.g. $\mathcal{L}_X i_X \omega = (d i_X + i_X d) i_X \omega =$

$= i_X d i_X \omega = i_X (d i_X + i_X d) \omega$

$= i_X \mathcal{L}_X \omega$

$$\Rightarrow \quad \mathcal{L}_X i_X \omega = i_X \mathcal{L}_X \omega$$

de Rahm cohomology

Preliminary: Poincaré's lemma

Theorem: If $\mathcal{M}$ is contractible to a point then all closed forms on $\mathcal{M}$ are exact

$$d\omega = 0 \Rightarrow \omega = d\eta$$

Proof: contractible: smooth map $I \times \mathcal{M} \to \mathcal{M}$

$I = [0,1]$ such that

$$\alpha(0,x) = x$$

$$\alpha(1,x) = x_0 \quad \text{one point in } \mathcal{M}$$

Consider a p-form on $I \times \mathcal{M}$

$$\omega = f_{i_1 \dots i_{p-1}}(t,x)\, dt \wedge dx^{i_1} \dots dx^{i_{p-1}} +$$
$$+ g_{j_1 \dots j_p}(t,x)\, dx^{j_1} \wedge \dots \wedge dx^{j_p}$$

Obviously ω can be integrated over I

$$P\omega = \int_0^1 dt\, \omega = \int_0^1 dt\, f_{i_1 \dots i_{p-1}}(t,x)\, dx^{i_1} \wedge \dots \wedge dx^{i_{p-1}}$$

$P\omega$ is a $p-1$ form on $\mathcal{M}$

Define also the family of maps $\beta_t : \mathcal{M} \to I \times \mathcal{M}$
$$x \to (t,x)$$

β_t^* ... pullback of a form on $I \times \mathcal{M}$ to $\mathcal{M}$

$$\beta_t^* \omega = g_{j_1 \dots j_p}^{(t,x)}\, dx^{j_1} \wedge \dots \wedge dx^{j_p} \qquad (\text{"}t = \text{const"})$$

then $$d P\omega + P d\omega = \beta_1^* \omega - \beta_0^* \omega$$

$$dP\omega = \int_0^1 dt' \frac{\partial f_{i_1 \dots i_{p-1}}(t', x)}{\partial x^{i_p}} dx^{i_p} \wedge dx^{i_1} \wedge \dots dx^{i_{p-1}}$$

$$Pd\omega = \int dt \Big(\frac{\partial f_{i_1 \dots i_{p-1}}}{\partial x^{i_p}} dx^{i_p} \wedge dt \wedge dx^{i_1} \wedge \dots dx^{i_{p-1}}$$

$$+ \frac{\partial g_{j_1 \dots j_p}}{\partial t} dt dx^{j_1} \dots dx^{j_p} +$$

$$+ \frac{\partial g_{j_1 \dots j_p}}{\partial x^i} dx^i \wedge dx^{j_1} \dots dx^{j_p} \Big)$$

$$Pd\omega = \int_0^1 dt' \Big(- \frac{\partial f_{i_1 \dots i_{p-1}}}{\partial x^{i_p}} dx^{i_p} \wedge \dots dx^{i_{p-1}} +$$

$$+ \frac{\partial g_{j_1 \dots j_p}}{\partial t} dx^{j_1} \wedge \dots \wedge dx^{j_p}$$

$$\Rightarrow \quad dP\omega + Pd\omega = g_{j_1 \dots j_p}(t=1, x) dx^{j_1} \dots dx^{j_p} - g_{j_1 \dots j_p}(0, x) dx^{j_1} \dots dx^{j_p}$$

$$= \beta_1^* \omega - \beta_0^* \omega$$

apply this to the form $\alpha^* \omega$ p-form on $I \times M$

$$dP\alpha^*\omega + Pd\alpha^*\omega = \beta_1^* \alpha^* \omega - \beta_0^* \alpha^* \omega$$

$$= (\alpha \beta_1)^* \omega - (\alpha \beta_0)^* \omega$$

$\alpha \beta_0$ --- identy

$(\alpha \beta_1)^* = 0 \qquad x = x_0$ pull backs all forms to the point x_0

$$d\alpha^* \omega = \alpha^* d\omega = 0 \Rightarrow \qquad \omega = -d(P\alpha^* \omega)$$

example: $\mathbb{R}^2 - \{0\}$; $\omega = \frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy$

$$d\omega = -\frac{1}{x^2+y^2} dy \wedge dx + \frac{y}{(x^2+y^2)^2} 2y \, dy \, dx +$$

$$+ \frac{1}{x^2+y^2} dx \wedge dy - \frac{x}{(x^2+y^2)^2} 2x \, dx \, dy =$$

$$= \frac{2}{x^2+y^2} dx \wedge dy - 2 \frac{y^2+x^2}{(x^2+y^2)^2} dx \wedge dy = 0$$

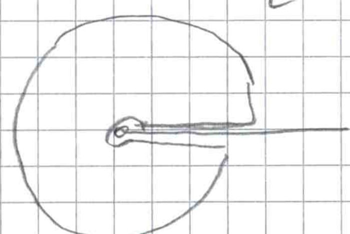
obviously $\mathbb{R} - \{0\}$ is not contractible

although $\eta = \arctan\left(\frac{y}{x}\right)$

$$d\eta = \frac{dy/x}{1+y^2/x^2} - \frac{y\,dx}{x^2} \frac{1}{1+y^2/x^2} = -\frac{y\,dx}{x^2+y^2} + \frac{x\,dy}{x^2+y^2} \checkmark$$

but η is not defined at $x = 0$

or $\frac{y}{x} = \tan\theta$ $\quad \eta = \theta$



θ is well-defined on $\mathbb{R}^2 - \mathbb{R}^0_+$
(or any half-line starting at 0 and going to infinity)

since $d\eta = \omega$ $\quad \eta = \eta + \lambda$ $\quad d\lambda = 0$

λ 0-form $\lambda = \text{const}$

all $\eta + c$ work on $\mathbb{R}^2 - \mathbb{R}^0_+$

co-homology:

p-chain $\qquad C = a_1 \lambda_1 + a_2 \lambda_2 + \dots$

$\lambda_i : \Delta_p \to M \qquad C^\infty$ map

Δ_p is a p-simplex in $\mathbb{R}^p$

let ω be a p-form
then we can define

$$\boxed{\int_C \omega = \sum_i a_i \int_{\Delta p} \lambda_i^* \omega}$$

duality relation: a p-form ω defines a linear map from the set of p-chains in M into $\mathbb{R}$: ω is a co-chain

$$\omega : C_p \to \mathbb{R}$$
$$C \to \langle \omega, C \rangle = \int_C \omega$$

This means that we can map homology to co-homology by Stokes theorem

$$\int_{\partial C} \omega = \int_C d\omega$$

$$\langle \omega, \partial C \rangle = \langle d\omega, C \rangle$$

d and ∂ are adjoint operators

Z^p .. spaces of closed forms $d\omega = 0$
B^p ... exact forms

$$H^p(M) := Z^p(M) / B^p(M)$$

$H^p(M)$ is dual to $H_p(M)$

Z_p : closed p-chains $\partial C_p = 0$
B_p : boundaries of p+1 chains $\sim \partial C_{p+1}$

$$([\omega], [c]) = \int_c \omega$$

equivalence class e ... representatives

$$\int_{C_p + \partial C_{p+1}} \omega = \int_{C_p} \omega_p + \int_{C_{p+1}} d\omega_p = \int_{C_p} \omega_p \quad \checkmark$$

$$\int_{C_p} \omega + d\eta = \int_{C_p} \omega_p + \int_{\partial C_p} \eta = \int_{C_p} \omega_p$$

$\Rightarrow$ $H^p(M)$ is dual to $H_p(M)$

Homology "complex" ; de-Rahm complex

$$\xleftarrow{\partial_{p-1}} C_{p-1} \xleftarrow{\partial_p} C_p \xleftarrow{\partial_{p+1}} C_{p+1} \xleftarrow{\partial_{p+2}}$$

$$\mathrm{Im}(\partial_{p+1}) \subset \mathrm{Ker}(\partial_p) : \qquad H_p(M) = \mathrm{Ker}(\partial_p) / \mathrm{Im}(\partial_{p+1})$$

cohomologies

$$\omega_{p-1} \xrightarrow{d_{p-1}} \omega_p \xrightarrow{d_{p+1}} \omega_{p+1} \xrightarrow{d_{p+2}} \dots \qquad H^p(M) = \mathrm{Ker}(d_{p+1}) / \mathrm{Im}(d_p)$$

some examples:

•) $H^p(\mathbb{R}^n)$: consider the map $\phi: x \to (1-t)x$

contracts $\mathbb{R}^n$ to the origin

$\Rightarrow$ all closed forms are exact

$\ker(d_{p+1}) = \operatorname{Im}(d_p) \qquad p \geq 1$

$$H^p(\mathbb{R}^n) = 0 \qquad p = 1 \ldots n$$

However $H^0(\mathbb{R}^n) = \mathbb{R}$

there are just the constant functions

Note there are simply no "-1" forms

$$df = 0 \Rightarrow f = \text{const}$$
$$\left(\frac{\partial f}{\partial x_i} = 0\right)$$

•) If M is simply connected (every curve can be contracted to a point) then

$$H^1(M) = 0 \qquad (H_1(M) = 0)$$

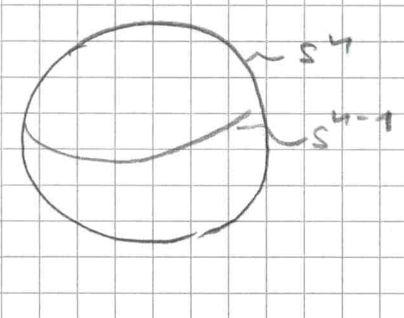
•) $H^1(S^1, \mathbb{R}) = \mathbb{R}$ "volume" form $\int_{S^1} \omega = L$ length

and $d\omega = 0$ since S^1 1-dimensional

•) $\dim H^1(M, \mathbb{R}) = \#(\text{holes in } M)$

•) spheres: $H^p(S^n, \mathbb{R}) = H^{p-1}(S^{n-1}, \mathbb{R})$

idea



on northpole $\omega = d\eta_+$

on the southpole $\omega = d\eta_-$

on equator $\eta_+ - \eta_-$ closed

Combining these result gives

$$H^p(S^n, \mathbb{R}) = 0 \qquad p > n$$

$$H^p(S^n, \mathbb{R}) = \mathbb{R} \qquad p = n$$

$$H^p(S^n, \mathbb{R}) = 0 \qquad 1 \leq p \leq n$$

$$H^0(S^n, \mathbb{R}) = \mathbb{R}$$

c) Sim: $\qquad H^p(T^n) = \mathbb{R}^\alpha \qquad \alpha = \binom{n}{p}$

$$T^n = S^1 \times S^1 \times \dots S^1 \qquad n\text{-times}$$

$H^1(S^1) = \mathbb{R}$ pick p-cycles S^1

$\dfrac{n!}{p!(n-p)!}$ ways of picking S^1's

Def: Betti-numbers

$$b^p(M) = \dim\left(H^p(M)\right)$$

Euler charcteristic (topological inveriant)

$$\chi(M) = \sum_{p=1}^{m} (-1)^p \, b^p(M)$$

Hodge - duality and harmonic forms

Hodge - Theory

Question: What is the dim $\Omega^p(M)$ when $\dim(M) = n$?

$$\#(dx^{i_1} \wedge \dots \wedge dx^{i_p}) = \binom{n}{p}$$

$$\Rightarrow \quad \dim \Omega^p(M) = \dim \Omega^{n-p}(M)$$

Is there a (natural) isomorphism $\Omega^p \leftrightarrow \Omega^{n-p}$?

Yes!, if we are dealing with a metric space

$T^{(2,0)}$ Tensor $g = $ " ds^2 " $= g_{\mu\nu}\, dx^\mu \otimes dx^\nu$

for $v, w \in T_p(M)$

- $g(v,w) = g(w,v)$
- $g(v,v) \geq 0$ (Riemanin) ; $g(v,v) = 0 \Rightarrow v = 0$
- $g(v,w) + g(w,z) \geq g(v,z)$

We define the ε - symbol $\varepsilon(i_1 \dots i_n) = \pm 1$

$$\varepsilon(i_1 \dots i_n) = \begin{cases} +1 & \text{if } i_1 \dots i_n \text{ is even permutation of } (1 \dots n) \\ -1 & \text{if } i_1 \dots i_n \text{ is odd premutation } (1 \dots n) \end{cases}$$

we also define the ε - tensor

$$\varepsilon\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right) = \sqrt{g}$$

$$\varepsilon : \frac{1}{n!}\, \varepsilon_{i_1 \dots i_n}\, dx^{i_1} \wedge \dots \wedge dx^{i_n}$$

in components: $\varepsilon_{i_1 \dots i_n} = \sqrt{g}\, \varepsilon(i_1 \dots i_n)$

now we define the Hodge dual

$$\boxed{*\omega = \frac{1}{p!(n-p)!} \varepsilon^{i_1 \dots i_p}{}_{j_{p+1} \dots j_n} \omega_{i_1 \dots i_p} dx^{j_{p+1}} \wedge \dots \wedge dx^{j_n}}$$

Rem Mathematicians like to use the ε-symbol

$$*\omega = \frac{\sqrt{g}}{p!(n-p)!} g^{i_1 j_1} \dots g^{i_p j_p} \omega_{j_1 \dots j_p} \varepsilon(i_1 \dots i_p\, i_{p+1} \dots i_n) dx^{i_{p+1}} \wedge \dots \wedge dx^{i_n}$$

(Physicist "Tensor" is" $T_{\mu_1 \dots \mu_n}{}^{\nu_1 \dots \nu_k}$)

$*$: p-form $\longrightarrow$ n-p form

Now we can write the volume form

$$*1 = \frac{1}{n!} \varepsilon_{i_1 \dots i_n} dx^{i_1} \wedge \dots \wedge dx^{i_n} = \sqrt{g}\, dx^1 \wedge \dots \wedge dx^n$$

under a coordinate change (= change of chart)

$$\underbrace{g'_{\mu\nu} \frac{\partial y^\mu}{\partial x^\lambda} \frac{\partial y^\nu}{\partial x^\kappa}}_{g} dx^\lambda \wedge dx^\kappa \qquad S =$$

$$\det g = \det g' (\det S)^2$$

$$dx^1 \wedge \dots \wedge dx^n = \frac{\partial x^1}{\partial y^{i_1}} \dots \frac{\partial x^n}{\partial y^{i_n}} dy^{i_1} \wedge \dots \wedge dx^{i_n} =$$

$$= (\det S)^{-1} dy^1 \wedge \dots \wedge dy^n$$

$$\Rightarrow \sqrt{g}\, dx^1 \wedge \dots \wedge dx^n = \sqrt{g'} \underbrace{|\det S| (\det S)^{-1}}_{1 \text{ if orientation is preserved}} dy^1 \wedge \dots \wedge dy^n$$

→ "invariant volume element"

$$**\omega = \frac{1}{p!}\frac{1}{(n-p)!p!}\,\omega_{i_1\dots i_p}\,\varepsilon^{i_1\dots i_p}{}_{j_{p+1}\dots j_n}\,\varepsilon^{j_{p+1}\dots j_n}{}_{k_1\dots k_p}\,dx^{k_1}\wedge\dots\wedge dx^{k_p}$$

$$\varepsilon^{i_1\dots i_p}{}_{j_{p+1}\dots j_n}\,\varepsilon^{k_{p+1}\dots j_n}{}_{k_1\dots k_p} = (-)^{p(n-p)}\,\varepsilon_{j_{p+1}\dots j_n}{}^{i_1\dots i_p}\,\varepsilon^{j_{p+1}\dots j_n}{}_{k_1\dots k_p} =$$

$$= (-)^{n(n-p)}(n-p)!\left(\delta^{i_1}_{k_1}\delta^{i_2}_{k_2}\dots \pm \dots\right) dx^{k_1}\wedge\dots\wedge dx^{k_p} =$$

$$= (-)^{n(n-p)}(n-p)!\,p!\,dx^{i_1}\wedge\dots\wedge dx^{i_p}$$

$$\Rightarrow **\omega = \frac{1}{p!(n-p)!p!}\,\omega_{i_1\dots i_p}\,p!(n-p)!\,dx^{i_1}\wedge\dots\wedge dx^{i_p} =$$

$$= (-)^{n(n-p)}\,\omega$$

$$\boxed{*^2 = (-)^{n(n-p)}}$$

<u>REM</u>: in Lorentzian spaces $\sqrt{g} \to \sqrt{|g|}$

then $\varepsilon^{i_1\dots i_n} = -\frac{1}{\sqrt{|g|}}\,\varepsilon(i_1\dots i_n)$

so we get an additional minus

$$\boxed{(*)^2 = (-)^{p(n-p)+1}}$$

Now we can define an inner product of p-forms: Let $\omega, \eta \in \Omega^p(M)$

$$\langle \omega, \eta\rangle : \int_M \omega\wedge *\eta = \int_M \eta\wedge *\omega$$

$\omega\wedge *\eta$ is a n-form, can be integrated over the n-dimensinal manifold

$\langle\omega,\omega\rangle > 0$ if g Rimannian

Application : Actions in Physics

e.g. scalar action: ϕ be a 0-form $\Omega^0(M)$

$d\phi$ 1-form

$$S[\phi] = \frac{1}{2}\int_M d\phi \wedge * d\phi$$

$$d\phi = \partial_\mu \phi \, dx^\mu$$

$$* d\phi = \frac{1}{(n-1)!} \varepsilon^{\mu}{}_{\nu_1 \dots \nu_2 \nu_3} \partial_\mu \phi \, dx^{\nu_1} \wedge \dots \wedge dx^{\nu_3}$$

$$\underline{d\phi \wedge * d\phi} = \partial_\lambda \phi \frac{1}{(n-1)!} \varepsilon^{\mu}{}_{\varrho\kappa\sigma} \partial_\mu \phi \, dx^\lambda \wedge dx^\varrho \wedge dx^\kappa \wedge dx^\sigma$$

$$\varepsilon(\lambda \varrho \kappa \sigma) \, dx^1 \wedge \dots \wedge dx^n$$

$$\sqrt{g}\, \varepsilon^{\lambda \varrho \kappa \sigma} \quad dx^1 \wedge \dots \wedge dx^n$$

$$= \partial_\lambda \phi \frac{1}{(n-1)!} (n-1)! (\delta^\mu_\kappa) g^{\kappa\lambda} \partial_\mu \phi \, dx^1 \wedge \dots dx^n \,, =$$

$$= \underline{\sqrt{g}\, g^{\mu\lambda} \partial_\mu \phi \, \partial_\lambda \phi}$$

equation of motion

$$\delta S = \frac{1}{2}\int \left[d\delta\phi \wedge * d\phi + \underbrace{d\phi \wedge * d(\delta\phi)}_{d(\delta\phi) \wedge * d\phi} \right]$$

$$\delta S = \int_M d(\delta\phi) \wedge * d\phi = \underbrace{\int_M d(\delta\phi \wedge * d\phi)}_{\int_{\partial M} \delta\phi \wedge * d\phi} - \int_M \delta\phi \, d * d\phi$$

$$\int_{\partial M} \delta\phi \wedge * d\phi - \int_M \delta\phi \wedge (d * d\phi)$$

boundary condition $\delta\phi\big|_{\partial M} = 0$

$$\Rightarrow \quad \delta S = 0 \Rightarrow -\int_M \delta\phi \wedge (d * d\phi) = 0$$

$$\left((-)^{n(n-1)} ** = 1 \right)$$

$$\left(= \int \delta\phi \wedge * \left((-)^{n(n-1)} * d * d\phi \right) \right.$$

$$\left. \int (-)^{n(n-1)} * d * d\phi \wedge * \delta\phi = 0 \right)$$

e.q. motion $\boxed{d * d\phi = 0}$

co-derivative: ω be $(p-1)$ form, η a p-form

$$\langle d\omega, \eta \rangle = \langle \omega, d^{+}\eta \rangle = \langle \omega, \delta\eta \rangle$$

$$\int d\omega \wedge *\eta = \int_M d(\omega \wedge *\eta) - \int_M (-)^{p+1} \omega \wedge d*\eta =$$

$= d*\eta$ is $n-p+1$ form

$$** = (-)^{(n-p+1)(n-n+p-1)} = (-)^{(n-p+1)(p-1)}$$

Inserting the signs $\cancel{p+1} + np - p - (p-1)^2 =$

$$= np - n + p \cancel{+1} - p^2 + \cancel{2p - 1}$$

$$p(p-1) - 1$$

$$\int d\omega \wedge *\eta = \int \omega \wedge * d^{+}\eta$$

$$\Rightarrow \quad \boxed{d^{+}\eta = (-)^{np-n+1} * d * \eta}$$

Herlitz

Now we can define the Laplacian on forms as

$$\Delta = (d + d^\dagger)^2 = dd^\dagger + d^\dagger d$$

$$d: \Omega^p(M) \to \Omega^{p+1}(M)$$
$$\delta: \Omega^p(M) \to \Omega^{p-1}(M)$$

scalar field: e.o.m. $\quad * d * d\phi = 0$

$$\delta d \phi = 0$$

but note that $\delta\phi \propto * d \underbrace{* \phi}_{\text{top-form}} = 0$

$\Rightarrow$ e.o.m scalar is $\Delta\phi = 0$

Exercise gauge field $\quad F = \frac{1}{2} F_{\mu\nu}\, dx^\mu dx^\nu$

$$F = dA$$

- write the Maxwell action in form language
- show that the e.o.m is $\Delta A = 0$ in the gauge $d^\dagger A = 0$ (write $d^\dagger A$ in components)

Def.: Harmonic forms: $\quad \Delta\omega = 0 \quad \mathrm{Harm}^p(M)$

Theorem: Hodge-decomposition

$$\Omega^p(M) = d\Omega^{p-1}(M) \oplus d^\dagger \Omega^{p+1}(M) \oplus \mathrm{Harm}^p(M)$$

$$\omega = d\alpha + d^\dagger\beta + \gamma \qquad \text{globally well defined}$$

α -- $p-1$
β -- $p+1$
γ -- $\Delta\gamma = 0$

Note that $(\omega,\omega) > 0$

$$(\omega,\omega) = (\gamma,\gamma) + (d\alpha, d\alpha) + (d^{\dagger}\beta, d^{\dagger}\beta) +$$
$$+ 2(\gamma, d\alpha) + 2(d^{\dagger}\beta, d\alpha) + 2(\gamma, d^{\dagger}\beta)$$

the first line is pos. definite

the second line is not $\Rightarrow$

$$(\gamma, d\alpha) = (d^{\dagger}\beta, d\alpha) = (\gamma, d^{\dagger}\beta) = 0 \;!$$

<u>Hodge theorem</u>: On an compact Riemannian manifold (M,g): $H^p(M) \cong \mathrm{Harm}^p(M)$

$\to$ every exact form has a unique harmonic representative.

Lie - Groups as manifolds

Def: A Lie group is a differentiable manifold G such that

- ·) $G \times G \to G \qquad g_1 \circ g_2 = g_3 \qquad g_{1,2,3} \in G$
- ·) $\exists\ g^{-1} \quad G \to G \qquad g^{-1} \circ g = e$

Note: g is mere a point in G but it also induces a diffeomorphism

(This is equivalent to previous definition)

left - translation $\qquad L_a g := a \circ g \qquad a \in G$ fixed

right - translation $\qquad R_a g := g \circ a$

$$L_a, R_a : \quad G \to G$$

As a manifold G comes equipped with a tangent-space $T(G)$

L_a is a map $G \to G \Rightarrow L_a^*$ is a map $T_g(G) \to T_{ag}(G)$

Def: Let $X \in \mathcal{X}(G)$ be a vector field on G.
X is called left-invariant if

$$L_a^*(X)\big|_g = X\big|_{ag}$$

let x^i be local coordinates $(x^i = x^i(g))$ and $\xi^i \frac{\partial}{\partial x^i} = X$

$$L_a^*(X) = \xi^i(g) \frac{\partial \cancel{x}x^j(ag)}{\partial x^i} \frac{\partial}{\partial x^j(ag)} = \xi^i(ag) \frac{\partial}{\partial x^i(ag)}$$

$$\text{"} \ \xi^i(x) \frac{\partial y^k}{\partial x^i} = \xi^k(y) \ \text{"}$$

Any vector $V \in T_e(G)$ defines a unique Left-invariant vector field by

$$X_V\big|_g = L_g^*(V)$$

proof: $X_V\big|_{ag} = L_{ag}^* V = (L_a^* L_g^*) V = L_a^*(X_V|_g)$

The inverse is of course also true: a left-invariant vector field defines a unique vector in $T_e(G)$.

$\Rightarrow$ L_g^* is an isomorphism of $T_e(G) \to T_g(G)$

the space of left-invariant vector fields is isomorphic to the space of vectors of $T_e(G)$ as a vector space

But even more: since

$$L_a^*([X,Y])\big|_g = [L_a^*(X)|_g, L_a^*(Y)|_g] = [X,Y]\big|_g$$

$\Rightarrow$ The space of left-invariant vector fields is isomorphic to the Lie-algebra $T_e(G)$ with the product being the Lie-derivative

example $GL(N)$: coordinates x^{ij} matrix entries

e: $x^{ij} = \delta^{ij}$

$$L_a g = ag = \sum_{j=1}^{N} x^{ij}(a)\, x^{jk}(g)$$

$$X_V\big|_g = L_g^*(V) \qquad V = \sum v^{ij} \frac{\partial}{\partial x^{ij}_{(e)}}$$

$$L_g^*(V) = v^{ij} \frac{\partial}{\partial x^{ij}_{(e)}} \left(x^{kl}(g)\, x^{lm}(e) \right) \frac{\partial}{\partial x^{km}_{(g)}} =$$

$$= v^{ij}\, \delta^l_i \delta^m_j\, x^{kl}(g) \frac{\partial}{\partial x^{km}(g)} = x^{ki} v^{im} \frac{\partial}{\partial x^{km}} = (gv)^{km} \frac{\partial}{\partial x^{km}}$$

Check also that the commutator gives

$$[X_v, X_w]|_g = (g\,[v,w])^{ij} \frac{\partial}{\partial x^{ij}(g)}$$

with $[v,w] = v^{ik} w^{kj} - w^{ik} v^{kj}$

the matrix-commutator.

Def: Let G be a Lie-group. The set of left invariant vector fields $\mathfrak{g} \in \mathfrak{X}$ forms the Lie-algebra with the product being the Lie-bracket.

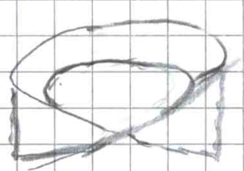
$$[X_a, X_b] = c_{ab}{}^{c} X_c$$

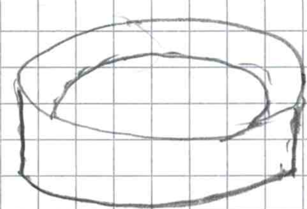
Fibre - bundles

Def: A fibre bundle is

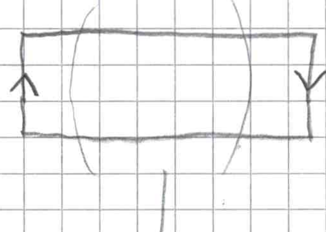
- a topologigal space E called total space
- equipped with a projection $\pi(E) = X$ wher X is a topological space called base
- a topological spac F called fibre
- a Group G of homemorphisms of $F \to F$
- a set of open sets U_α covering X and maps $\phi_\alpha : \phi_\alpha : \pi^{-1}(U_\alpha) \to X \otimes F$ "Local trivializotion"

Example: Moebius strip vs. cylinder

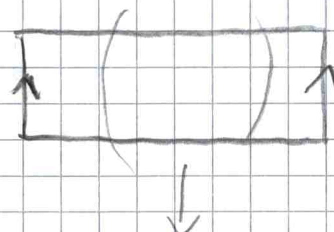




cut open



U_α



U_α

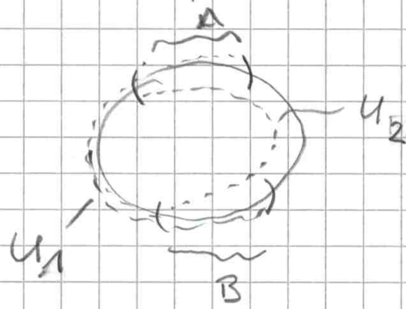
locally $\simeq (a,b) \times I \qquad I = [0,1]$

BUT: Moebius strip is not globally $S^1 \times I$

Base: S^1, Fibre: closed interval

after cutting open, how do we "glue" together the Moebius strip?

S^1 :



$$U_1 \cap U_2 = A \cup B$$

local trivialization on U_1 : (x_1, f_1) $\quad x_1 \in S^1$, $f_1 \in I$

U_2 : (x_2, f_2)

x be a point on S^1, x_i are its local coords in U_i

on A: $\quad x \times F \quad (x, f) \quad x_2 = x_2(x_1)$

$$(x_1, f_1) \equiv (x_2(x_1), f_2)$$

on B: $\quad x \times F \quad (x, f)$

$$(x_1, f_1) \equiv (x_2(x_1), -f_2)$$

Structure group $G = \{1, -1\}$ $\mathbb{Z}_2$

more generally on overlaps $U_\alpha \cap U_\beta$

$$U_\alpha : E \to U_\alpha \times F \sim (x, f)$$

x point in U_α, f point F

on $U_\alpha \cap U_\beta$ $\quad (x_\alpha, f_\alpha) \equiv (x_\beta, g_{\alpha\beta}(x) \cdot f_\beta)$

$g_{\alpha\beta}(x) \in G$ "transition function"

Examples of Fibre bundles

- Tangent bundle $T(M)$: tangent vector $V = v^i \frac{\partial}{\partial x^i}\Big|_p$

  local trivialization $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$

  $V \to (p, v^i)$

  p ... point on M

  structure group : on $U_\alpha \cap U_\beta$: $v^i_\alpha = \frac{\partial x^i_\alpha}{\partial x^j_\beta} v^j_\alpha$

  x^i_α, x^j_β local coordinates on U_α and U_β

  $J_{\alpha\beta} = \frac{\partial x^i_\alpha}{\partial x^j_\beta}$ ... matrix $GL(N, \mathbb{R})$ $= J_{\alpha\beta}$ .. Jacobian

  $\Rightarrow$ structure group is $GL(N, \mathbb{R})$

- Co-tangent bundle $\omega = \omega_i dx^i$

  $\omega_{i,\alpha} = \omega_{j,\beta} \frac{\partial x^j_\beta}{\partial x^i_\alpha} = \left((J^{-1}_{\alpha\beta})^T \cdot \omega_\beta\right)_i$

  $\rightarrow$ structure group is $GL(N, \mathbb{R})$ but if $J_{\alpha\beta}$ is a transition function on $T(M)$ then $(J^{-1}_{\alpha\beta})^T$ is the corresponding transition function on $T^*(M)$

- Tensor bundles $T \sim T_{i_1 \dots i_p}{}^{j_1 \dots j_q} dx^{i_1} \otimes \dots \otimes dx^{i_p} \frac{\partial}{\partial x^{j_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{j_q}}$

  transition function

  $$\bigotimes_p (J^{-1}_{\alpha\beta})^T \bigotimes_q J_{\alpha\beta}$$

•) Frame bundle : take a basis for $T_p(M)$: F_a
$a = 1 \dots N \qquad N = \dim(M)$

The matrix of basis vectors $F_1, F_2, \dots, F_N$
is non-degenerate $\to GL(N, \mathbb{R})$
a change of basis $F'_a = g_a{}^b F_b$
means that $g_a{}^b \in GL(N, \mathbb{R})$

$$\Rightarrow \left.\begin{array}{ll} F_a{}^i & \in GL(N, \mathbb{R}) \\ g & \in GL(N, \mathbb{R}) \end{array}\right\} \text{"principle bundle"!}$$

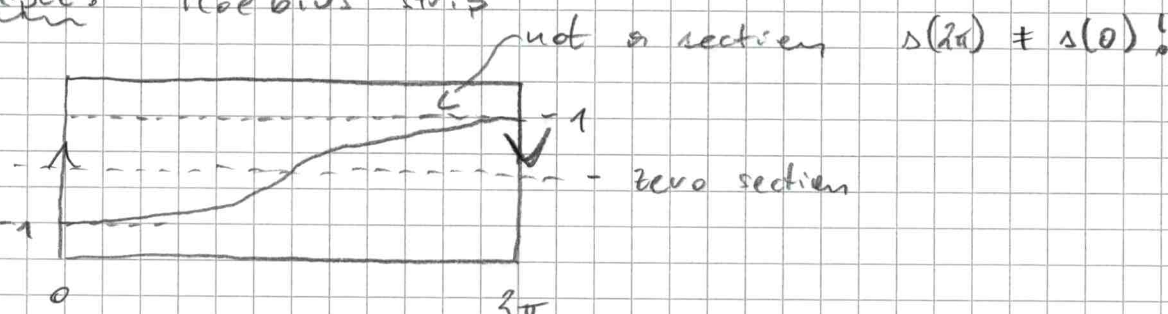
Note by a choice of coordinates we can chose
$F_a{}^i = \delta_a^i$!

•) Co-fram bundle : $f^a : \langle f^a, F_b \rangle = \delta^a_b$
structure group $GL(N, \mathbb{R})$

<u>Def</u>: Section : a section of a bundle $E \xrightarrow{\pi} M$
is a continous map $s : M \to E$ with $\pi \circ s = id_M$

(s: assign one point in the fibre to each
point $x \in M$)

<u>Example</u>: Moebius strip

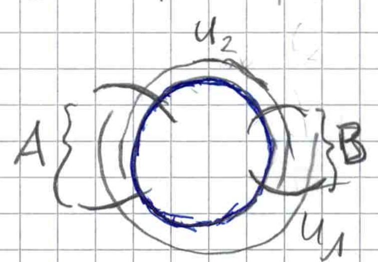


Def: A bundle is called trivial if it is globally a product $M \times F$ M .. base, F .. fibre

example: Cylinder

Def: Principle bundle: given a bundle (E, Π, F, G, M) with transition functions $\{g_{\alpha\beta}\}$ the principle bundle is defined by $(P(E), \Pi, G, G, M)$ with the same transition functions $\{g_{\alpha\beta}\}$

Example: principle bundle of the Moebius strip



transition function on A:

$(1, -1) \to (1, -1)$

transition function on B:

$(1, -1) \to (-1, 1)$

fibre $= \{-1, 1\}$ = structure group

Question: when is a bundle trivial?

Theorem: A bundle E is trivial if the principle bundle $P(E)$ has a section.

Note: the only continuous assignments $s: S^1 \to \{1, -1\}$ would be $s(\theta) = +1$ or $s(\theta) = -1$

So if $s(2\pi) = -1 = -s(0)$ this is not a section.

Proof: a) if $P(E)$ is trivial then $\exists$ section. This is clear since the constant map $s(x) = g_0$ with $g_0 \in \mathcal{G}$ $x \in M$ is a section.
(All transition functions can be chosen to be the identity $g_{\alpha\beta} = e$ $\forall \alpha, \beta$ }

b) Suppose $s(x)$ is a section on $P(E)$. Then an arbitrary point on $P(E)$ can be written as $p = s(x) \cdot a$ $p \in P(E)$; $a \in G$; $x \in M$

Take the map

$$\phi : p = s(x) \cdot a \rightarrow (x, a)$$

ϕ is a homeomorphism

$P(E) \rightarrow M \times G$

Remark: since E and $P(E)$ share the same transition functions E is trivial as well.

The pullback bundle:

Suppose (E, Y, π, F) is a bundle with fibre F

$$E \xrightarrow{\pi} Y$$

and f is a map from a manifold X to Y

$f: X \to Y$

then we can construct the pullback bundle f^*E by

- let V_α be a covering of Y, define a covering of X by $U_\alpha = f^{-1}(V_\alpha)$
- as transition function on X chose
$$t^*_{\alpha\beta} = t^*_{\alpha\beta}(f(x))$$
- as fibre chose the same fibre F

$$\begin{tikzcd}
f^*E \arrow[r, "f_*"] \arrow[d, "\pi_f"'] & E \arrow[d, "\pi"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

Theorem: Let E be a bundle over Y and let f, g be maps $f: X \to Y$; $g: X \to Y$ and consider the pullback bundles f^*E, g^*E.
If the maps f and g are homotopic then the bundles f^*E, g^*E are equivalent.

Homotopic: f can be continuously deformed int g

$\exists\ \alpha(t)$ $t \in [0,1]$ $\quad \alpha(0) = f$

$\alpha(1) = g$

$\alpha(t)$ continous

Corollary: bundles over a contractable space are trivial

Contractable: $X \to x_0$ a point

$\mathbb{R}^n$: $x \in \mathbb{R}^n$ $f = x$ $g = 0$

$$\alpha(t) = (1-t)\,x$$

if the base of a bundle is a point then all transition functions are $t_{ij} = e$ $\Rightarrow$ trivial bundle.

Non-uniqueness of transition functions:

let E be a bundle with (ϕ_α, U_α) $\phi_\alpha : E \to U_\alpha \times F$

E' be a bundle over the same base and

$\psi_\alpha : E' \to U_\alpha \times F$

and let $\phi_\alpha \circ \psi_\alpha^{-1}$ be a diffeomorphism of the fiber induced by an element of the structure group λ_α.

Then the transition functions are related by

$$t_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1} = \phi_\alpha \circ \psi_\alpha^{-1} \circ \psi_\alpha \circ \psi_\beta^{-1} \circ \psi_\beta \circ \phi_\beta^{-1} =$$

$$= \lambda_\alpha \circ t'_{\alpha\beta} \circ \lambda_\beta^{-1}$$

Since λ_α is a diffeomorphism E, E' are diffeomorphic and thus E, E' are equivalent

$\Rightarrow$ For a trivial bundle the transition functions have the form

$$t_{\alpha\beta} = \lambda_\alpha \circ \lambda_\beta^{-1}$$

Further possible contractions of bundles:

e.g. if the base is a "cylinder"

$$X = S^n \times \mathbb{R}^m$$

$\exists$ equivalent bundle with base S^n

Not only is it possible to reduce the base space but also the fibre can be reduced.

Theorem: If the fibre is contractible the bundle E always has a section.

$\Rightarrow$ if the structure group is contractable the bundle is trivial.

$\Rightarrow$ We can also contract the structure group of the principle bundle

Theorem: If G is a connected Lie group then

$$G = H \times D$$

where H is the maximal compact subgroup and D is contractible.

Example frame-bundle $G = Gl(N, \mathbb{R})$

but

$$Gl(N, \mathbb{R}) = O(N) \times C$$

C... symmetric positive matrices

O(N)... maximal compact subgroup.

⇒ We can contract the structure group of the frame (and co-frame) bundle to $O(N)$

That means without loss of generality we can assume a basis of the co-frame bundle e^a such that

$$\delta_{ab}\, e^a e^b \quad \text{is invariant}$$

in a coordinate basis $e^a = e^a_\mu\, dx^\mu$

$$\delta_{ab}\, e^a e^b = e^a_\mu\, e^b_\nu\, dx^\mu \otimes dx^\nu$$

is a symmetric, positive definite tensor of rank two = Riemannian metric.

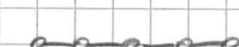
→ Of greatest importance in physics, it allows to define spinors on curved manifolds!

e^a ... co-frame $\qquad (e^a, E_b) = \delta^a_b$

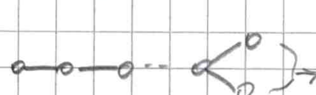
E_b ... orthonormal frame

$E_b^{\ \mu} \frac{\partial}{\partial x^\mu}$,

Physics: e^a_μ ... vielbein, "tetrad"

Why: $GL(N,\mathbb{R}) \to A_n$ series 

has no spinor representation

$SO(2n,\mathbb{R}) \to$  → Spinor reps

$SO(2n+1,\mathbb{R})$  → spinor rep

Also: Pseudo-Riemannian $\qquad \eta_{ab}\, e^a e^b = ds^2$

$\eta = (-1, +1, \dots, +1)$

Connections on bundles

Consider principle bundle $P \xrightarrow{\pi} M$

The transition functions act on the fibre from the left

$$g_\alpha = t_{\alpha\beta}\, g_\beta \qquad \phi_\alpha : P \to U_\alpha \times G$$

Consider the right-action of G on P. $u \in P$

$a \in G: \qquad u \to u \cdot a$

Obviously the right action commuts with the left action

$$g_\alpha \cdot a = t_{\alpha\beta} \cdot g_\beta \cdot a$$

$\to$ right action is independent of the local trivializations (ϕ_α, U_α) and maps out the whole fibre over a point $x \in M$;

$\pi(p) = x$ then $(x, g) \to (x, g a)$ maps each the whole fibre G over x

right action is free and transitive

- free: fixed point only for $a = e$
- transitive all $g_1 g_2 \in G$ $\exists$ a such that $g_2 = g_1 a$

$\Rightarrow$ the right action moves us along the fibre over a point $x \in M$.

define a basis of the vertical subspace $V_u(P)$

by $\displaystyle A^{\#} f(u) = \frac{d}{dt}\left(u \cdot e^{At}\right)$

with $A \in \mathcal{L}(G)$

since this is defined by the right action it characterises $V_u(P)$ independently of the local trivializations.

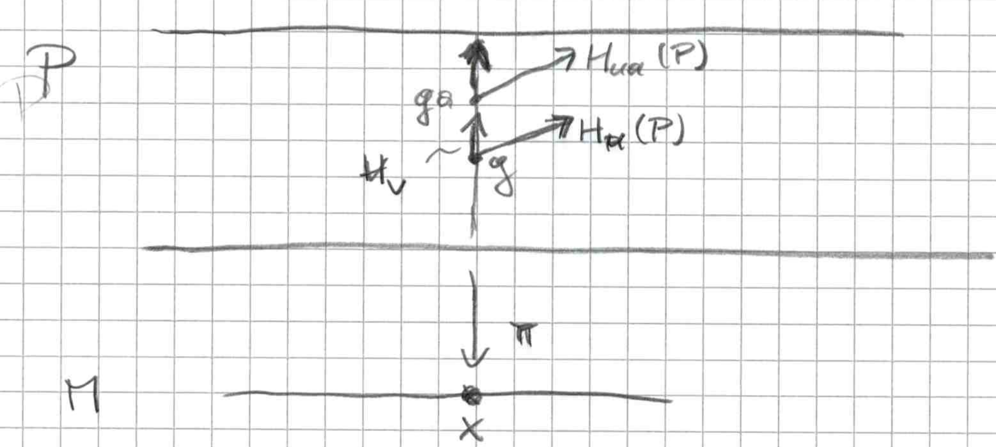
<u>Def</u>: A connection is a unique prescription of writing $T_u(P) = V_u(P) \oplus H_u(P)$

i.e. dividing the tangent space a u to P into vertical and horizontal parts

$$H_{ua}(P) = R_{a*} H_u(P) \qquad \forall\, a \in G,\ u \in P$$

where R_{a*} is the push ford induced by the right-action of G.

Every vector $X = \underbrace{X_V}_{\text{"vertical"}} + \underbrace{X_H}_{\text{"horizontal"}}$



<u>Def</u>: Connection 1-form ω is a Lie-algebra valued 1-form $\omega: T^*P \to \mathfrak{X}(G)$

- $\omega(A^\#) = A$
- $R_a^*(\omega) = Ad_{a^{-1}}\omega = a^{-1}\omega a$

$$R_a^* \omega_{ua}(X) = \omega_{ua}(R_{a*}X) = a^{-1}\omega(X) a$$

The horizontal subspace is the kernel of ω

$$H_uP = \{X \in T_uP \,/\, \omega(X) = 0\}$$

Fulfills: $R_{a*} H_u(P) = H_{ua}(P)$

since $\omega(R_{a*}X) = R_a^* \omega(X) = a^{-1}\omega(X) a = 0$

(<u>Note</u>: $a^{-1} A a$ with $A \in \mathfrak{X}(G)$ $a \in G$ is a right-action isomorpism on $\mathfrak{X}(G)$)

Assume a local trivialisation (ϕ_α, U_α) : $M \times G$ and a Lie-algebra valued 1-form A on M

$A: T(M) \to \mathfrak{X}(G)$

then $\omega = g_\alpha^{-1} dg_\alpha + g_\alpha^{-1} A_\alpha g_\alpha$

defines a connection 1-form on P.

- $\omega(A^\#) = g_\alpha^{-1} d(g_\alpha(ue^{tA})) = g_\alpha^{-1} g_\alpha(u) \frac{d}{dt} e^{tA} = A$ ✓
- $R_a^*(\omega(x)) = \omega(R_{a*}X) =$

$$g_{\alpha\, ua}^{-1}\, dg_{\alpha ua}(R_{a*}X) + g_{\alpha ua}^{-1} A(\pi_* R_{a*}X)\, g_{\alpha ua} =$$

$$= a^{-1} g_\alpha dg_\alpha(X)\, a + a^{-1} g_\alpha^{-1} A(X)\, g_\alpha a$$

and we used

$$g_{\alpha ua}^{-1}\, d g_{\alpha ua} (R_{a*} X) = a^{-1} g_{\alpha}^{-1} \frac{d}{dt} g_{\alpha\, \gamma(t)\cdot a}$$

$\gamma(t)$ curve corresponding to vector field X

$\gamma(t)\cdot a$ is the curve corresponding to $R_{a*} X$

$g_{\alpha\, \gamma(t)\cdot a} = g_\alpha(t)\cdot a$ in local trivialization

$$= a^{-1} g_\alpha^{-1} \frac{d}{dt} g_\alpha(t)\, a^{-1} = a^{-1} g_\alpha^{-1} d g_\alpha (X)\, a^{-1}$$

$\Rightarrow$ Indeed $\omega_\alpha = g_\alpha^{-1} d g_\alpha + g_\alpha^{-1} A_\alpha g_\alpha$

is a connection.

Compatibility $\omega_\alpha = \omega_\beta$ for two local trivializations with $U_\alpha \cap U_\beta \neq \{\}$

$g_\alpha \to t_{\alpha\beta}\, g_\beta$

$$\omega_\alpha = g_\beta^{-1} t_{\alpha\beta}^{-1} d t_{\alpha\beta}\, g_\beta + g_\beta^{-1} d g_\beta +$$

$$+ g_\beta^{-1} t_{\alpha\beta}^{-1} A_\alpha t_{\alpha\beta}\, g_\beta =$$

$$= g_\beta^{-1} d g_\beta + g_\beta^{-1} A_\beta\, g_\beta$$

$$\Rightarrow \boxed{A_\beta = t_{\alpha\beta}^{-1} d t_{\alpha\beta} + t_{\alpha\beta}^{-1} A_\alpha\, t_{\alpha\beta}}$$

<u>Note:</u> this looks like a gauge transformation! but it is actually the transformation of the local connection under a change of local trivialization

A strict gauge-transformation is a change of fiber coordinates by a left-action

$$(x, g) \in U_\alpha \times G$$

$$g_2 \to h(x)\, g(x)$$

$$\boxed{A_2 = h^{-1} dh + h^{-1} A_1 h}$$

Horizontal lift of a curve

Let $P \xrightarrow{\pi} M$ be a principle bundle

let $\gamma(t)$ be a curve in M.

The horizontal lift $\tilde{\gamma}(t)$ is the unique curve in P such that $\pi(\tilde{\gamma}(t)) = \gamma(t)$ and the vector $\frac{d}{dt}\tilde{\gamma}(t)$ is always horizontal.

$$\omega(X) = 0$$

in a local trivialization

$$g_\alpha^{-1} \frac{dg_\alpha}{dt} + g_\alpha^{-1} A_\alpha(x)\, g_\alpha = 0$$

$$A(x) = (A_\alpha)_i \frac{dx^i}{dt}$$

$$\boxed{\frac{dg_\alpha}{dt} = - A_\alpha(x(t))\, g_\alpha(t)}$$

$$\frac{dg_\alpha}{dt} = \frac{\partial g_\alpha}{\partial x^i} \dot{x}^i \qquad A_\alpha(x(t)) = (A_\alpha)_i \dot{x}^i$$

$$\Rightarrow \quad \dot{x}^i \underbrace{\left(\partial/\partial x^i + (A_\alpha)_i\right)}\, g_\alpha(x) = 0$$

"covariant derivative"

Now we know the connection on the principle bundle. For the associated bundle with fibre F :

$$(u, f) \cong (u g, \rho(g^{-1}) f) \qquad \begin{matrix} u \in P \\ f \in F \end{matrix}$$

$\rho(g)$ is a representation of G

Fibre bundle (x_α, f_α) is parallel transported along the horizontal lift :

$\tilde{\gamma}(t)$ is the horizontal lift

$(x(t), \rho(\tilde{\gamma}(t)) f)$ "Parallel transport"

section of the fibre is constant along the horizontal lift

locally $f(x)$ $\boxed{\left[\partial_i + \rho(A_i)\right] f(x) = 0}$

here $\rho(A_i)$ is the induced representation on the Lie-algebra.

Curvature & parallel transport around a small closed curve

$$[D_\mu, D_\nu] = F_{\mu\nu}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

or in form-language $F = \frac{1}{2} F_{\mu\nu}\, dx^\mu \wedge dx^\nu = dA + A \wedge A$

$$(d + A) \wedge (d \wedge A) = dA + A \wedge A$$

Note that $\boxed{F_\beta = t^{-1}_{\alpha\beta} F_\alpha t_{\alpha\beta}}$

Important examples of connections in Physics

$T(M)$: $v^i \frac{\partial}{\partial x^i}$, $F = \{v^i\}$

covariant derivative $(d + A) \underset{\sim}{v}$

A: $A^k_{i\,e}\, dx^i \in GL(N,R)$ valued 1-form

it is customary to write $\underbrace{\Gamma^k_{i\,e}}\, dx^i$

affine connection

$v^i w_i$ = invariant = scalar

$$d(v^i w_i) = (d + \Gamma) v^i w_i + v^i (d + \tilde{A}) w_i = d(v^i w_i)$$

$$\Rightarrow \Gamma^i_{k\,j}\, dx^k v^j w_i + v^j A_k{}^i{}_j\, dx^k w_i = 0$$

$$A_k{}^i{}_j = - \Gamma^i_{k\,j}$$

contra-veriant vectors $\quad \partial_i v^j + \Gamma^j_{i\,k} v^k$

co-veriant vectors $\quad \partial_i w_j - \Gamma^k_{i\,j} w_k$

Co-Frame $\quad e^a \sim \underbrace{de^a}_{\text{2-form}} + \underbrace{A^a{}_b \wedge e^b}_{\text{2-form}}$

$A^a{}_b$ = spin-connection

costumorily denoted by $\omega^a{}_b = \omega_\mu{}^a{}_b \, dx^\mu$

We usually impose

$$D_\mu e^a_\nu = \partial_\mu e^a_\nu - \Gamma^\lambda_{\mu\nu} e^a_\lambda + \omega_\mu{}^a{}_b \, e^b_\nu = 0$$

such that $\quad g_{\mu\nu} = \delta_{ab} e^a_\mu e^b_\nu \quad$ is

covariantly constant $\quad D_\mu g_{\rho\lambda} = 0$

Torsion $\quad de^a + \omega^a{}_b \wedge e^b = T^a = \frac{1}{2} T^a_{\mu\nu} dx^\mu \wedge dx^\nu$

also from $\left.\begin{array}{l} D_\mu e^a_\nu = 0 \\ D_\nu e^a_\mu = 0 \end{array}\right] - \Rightarrow e^a_\lambda (\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}) = T^a_{\mu\nu}$

→ torsion is the antisymmetric part of $\Gamma^\lambda_{\mu\nu}$

If we impose that the vielbein is parallel transported then $T^a = 0$

$\Gamma^\lambda_{\mu\nu}$ is symmetric.

We get a one-to-one relation between Γ and ω

$$\boxed{\omega_\mu{}^a{}_b = E^\nu_b \partial_\mu e^a_\nu - E^\nu_b \Gamma^\lambda_{\mu\nu} e^a_\lambda}$$

solve $D_\mu e^a_\nu = 0$

$$\Rightarrow \boxed{\omega_\mu{}^a{}_b = \frac{1}{2} E^{\nu a} (\partial_\mu e^b_\nu - \partial_\nu e^b_\mu) - \frac{1}{2} E^{\nu b} (\partial_\mu e^a_\nu - \partial_\nu e^a_\mu) - \frac{1}{2} e_{\mu c} E^{\lambda a} E^{\rho b} (\partial_\lambda e^c_\rho - \partial_\rho e^c_\lambda)}$$

solve $D_\mu g_{\rho\lambda} = 0$

$$\boxed{\Gamma^\mu_{\nu\lambda} = \frac{1}{2} g^{\mu\rho} (\partial_\nu g_{\rho\lambda} + \partial_\lambda g_{\rho\nu} - \partial_\rho g_{\nu\lambda})}$$

How to write the Dirac-action on a curved manifold:

Ψ ... spinor under $O(N)$

$$\boxed{\gamma^a E^\mu_a \left(\partial_\mu - \frac{i}{2} \omega_\mu^{ab} \Sigma_{ab}\right) \Psi = 0}$$

$\Sigma_{ab} = \frac{i}{4} [\gamma_a, \gamma_b]$ generators of $so(n)$

$\{\gamma_a, \gamma_b\} = 2\delta_{ab}$

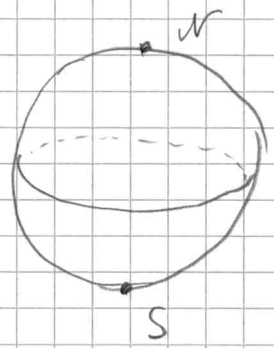
E^μ_a ... orthonormal basis of $T(M)$

$[(e^a, E_b) = \delta_a^b]$ $e^a(E_b) = \delta_a^b$

Some important bundles

REMs Physics convention here! $A \to iA$

a) The monopole bundle



S^2 $U(1)$ - connection A

the sphere has to be coverded by at least 2 open charts: northern and southern hemisphere H_N, H_S

Let us calculat the magnetic flux

$$\int_{S^2} F = \int_{H_N} F + \int_{H_S} F = \int_{H_N} dA_N + \int_{H_S} dA_S$$

whatever the precices forms of H_N and H_S the overlap on a set that is topologically a cylinder $\to$ contractible to S^1

(think of the S^1 as the equator and H_N and H_S glued together at the equator)

By Stokes theorem

$$\oint_{H_N} dA_N = \oint_{\mathcal{E}} A_N$$

$$\oint_{H_S} dA_S = -\oint_{\mathcal{E}} A_S$$

$$\Rightarrow \int F = \int_{\mathcal{E}} A_N - A_S$$

but A_N and A_S have to be related by

$$A_S = A_N + i g_{SN}^{-1} d g_{SN}$$

with $g_{SN} \in U(1)$

$$g_{SN} = \exp\left(i\,\phi_{SN}(\theta)\right)$$

g_{SN} has to be single valued on S^1

$$\Rightarrow \quad \phi_{SN}(\theta + 2\pi) = 2\pi n \qquad n \in \mathbb{Z}$$

$$\int_{S^2} F = -i \oint_{\mathcal{E}} g_{SN}^{-1}\, dg_{SN} = \int_0^{2\pi} \frac{\partial \phi}{\partial \theta}\, d\theta = 2\pi n$$

$$\Rightarrow \quad \boxed{\frac{1}{2\pi} \int F = n} \qquad n \in \mathbb{Z}$$

$U(1)$ bundles on S^2 are classified b 1^{st} Chern-number

Instanton bundle

Now we consider the manifold S^4 and the integral

$$\int \mathrm{tr}\, F \wedge F \qquad \left[\text{actually } \kappa\,(F \wedge F)\right]$$

we can devide S^4 in a northern hemisphere and a southern one just like before.

we also observe

$$d\,\mathrm{tr}\left(A\, dA + \tfrac{2}{3} i\, A \wedge A \wedge A\right) =$$

$$= \mathrm{tr}\left(dA\, dA + \tfrac{2}{3} i\, dA \wedge A \wedge A - \tfrac{2}{3} i\, A\, dA\, A + \tfrac{2}{3} i\, A\, A\, dA\right)$$

$$= \mathrm{tr}\left(dA\, dA + 2i\, dA \wedge A \wedge A\right)$$

→ graded cyclic permutation under the trace

$$\mathrm{tr}\left(A \wedge A \wedge A \wedge A\right) = -\,\mathrm{tr}\left(A \wedge A \wedge A \wedge A\right) = 0$$

$$= d\,\mathrm{tr}\left(A \wedge dA + \tfrac{2}{3} i\, A \wedge A \wedge A\right) = \mathrm{tr}\left(F \wedge F\right)$$

so again we can write

$$\int \mathrm{tr}(F\wedge F) = \int_{\mathcal{E}} \Big(A_N\, dA_N + \tfrac{2}{3} i A_N \wedge A_N \wedge A_N -$$
$$- A_S\, dA_S + \tfrac{2}{3} i A_S \wedge A_S \wedge A_S \Big)$$

the equator is 3-sphere S^3

$$A_N = g_{NS}^{-1} A_S\, g_{SN} - i g_{NS}^{-1} dg_{NS}$$

$$CS(g^{-1}Ag + g^{-1}dg) = CS(A) - d(g^{-1}A\,dg) + \frac{1}{3}(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg)$$

since the equator is closed $\partial S^3 = 0$

we have $\displaystyle\int_{S^4} \mathrm{tr}(F\wedge F) = + \frac{1}{3} \int_{S^3} \mathrm{tr}(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg)$

suppose structure group is $SU(2)$: we have already that

$SU(2)$: $\begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}$ $\quad |\alpha|^2 + |\beta|^2 = 1$ $\leftarrow$ also an S^3

g_{NS} is a map from S^3 (the equator) to S^3, the group.

Normalization $\mathrm{tr}(T_a T_b) = \frac{1}{2}\delta_{ab}$

$$\frac{1}{3}\int \mathrm{tr}(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg) = 8\pi^2 n$$

n .. winding number of map g $S^3 \to S^3$

$$\frac{1}{8\pi^2}\int \mathrm{tr}(F\wedge F) = \text{Instanton number}$$

$g(\theta) = \begin{pmatrix} \theta_4 + i\theta_1 & \theta_2 + i\theta_3 \\ \theta_2 + i\theta_3 & \theta_4 \end{pmatrix}$