Notes on Relativity and Gravitation

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Abstract

Introductory notes on Relativity (Special and General) for the Course "Gravitation", delivered at the Master de Física Teórica, IFT UAM/CSIC.

Contents

Chapter 1

Introduction and Preliminaries

We assume prior knowledge of mechanics and electrodynamics at the level of the first few chapters of Landau–Lifschitz vols I and II. In particular, we require certain familiarity with Lagrangian mechanics, Maxwell's equations and Special Relativity (SR). There are many books which can be used as a complement to these notes. A biased list is the following.

- **Elementary level:** Schutz (Cambridge 1980, 2nd ed 2009). Hartle (Benjamin 2003). Coleman (Cambridge 2022). Blundell-Lancaster (Oxford 2025).
- **Comprehensive:** Weinberg (Wiley 1972). Misner-Thorne-Wheeler (Freeman 1973). Landau-Lifshitz (Reverte 1981). Carroll (Benjamin 2003). Zee (Princeton 2013).
- **Advanced level:** Hawking-Ellis (Cambridge 1973). Wald (Chicago 1984). Poisson (Cambridge 2004).
- **Web:** Carroll, gr-qc/9712019. Tong, https://www.damtp.cam.ac.uk/user/tong/gr.html

Conventions

The acronym GR is used for General Relativity, SR is used for Special Relativity. Frames adapted to freely falling observers are sometimes termed *frefos*, while generic frames corresponding to fiducial observers are called *fidos*.

Four-dimensional indices transforming in the Lorentz group are labeled with latin letters from the beginning of the alphabet: $a, b, c, \ldots$. Four-dimensional indices transforming in the group of diffeomorphisms are labeled with greek letters: $\alpha, \beta, \ldots, \mu, \nu, \ldots$. Einstein's summation convention is used in these cases. Three-dimensional *spatial* indices, or more generally indices which do not include time, are labeled with latin letters from the middle of the alphabet: $i, j, k, \ldots$, and Einstein's convention is *not* used. Brackets in indices denote antisymmetrization,

$$A_{[\mu\nu]} = \frac{1}{2}(A_{\mu\nu} - A_{\nu\mu}) \ , \qquad A_{[\mu\nu\rho]} = \frac{1}{3!}(A_{\mu\nu\rho} + A_{\nu\rho\mu} + A_{\rho\mu\nu} - A_{\nu\mu\rho} - A_{\mu\rho\nu} - A_{\rho\nu\mu}) \ ,$$

and so on. Analogously, curved brackets denote symmetrization, $S_{(\mu\nu)} = \frac{1}{2}\left(S_{\mu\nu} + S_{\nu\mu}\right)$ and so on.

The metric signature convention is $(-+++)$, so that the proper time element satisfies

$$d\tau^2 = -ds^2 = -g_{\mu\nu}\, dx^\mu\, dx^\nu \ .$$

When ds^2 is positive, ds is interpreted as a proper distance element. When $d\tau^2$ is positive, $d\tau$ is interpreted as a proper time interval. We will often use the notation of a dot over a variable that is being differentiated with respect to τ.

The Levi–Civita symbol ϵ_{abcd} is defined as a completely antisymmetric object in four covariant indices, defining a tensor with respect to proper Lorentz transformations with the convention $\epsilon_{0123} = 1$. This implies that the contravariant version ϵ^{abcd} is completely antisymmetric with $\epsilon^{0123} = -1$. The three-dimensional version appearing in vector calculus is $\epsilon_{ijk} = \epsilon_{0ijk}$ and $\epsilon^{ijk} = -\epsilon^{0ijk}$.

The following set of conventions is adopted when writing the norm of a general four-vector $(U^\mu) = (U^0, \vec{U})$:

$$U^2 = g_{\mu\nu}\, U^\mu U^\nu = g^{\mu\nu}\, U_\mu U_\nu = U_\mu U^\mu \ .$$

For a Lorentz four-vector we have similar expressions in terms of Lorentz indices and the Lorentz metric η_{ab}. In addition, we also have $U^2 = U_a U^a = -(U^0)^2 + \vec{U}^{\,2} = -(U^0)^2 + \sum_{i=1}^3 (U^i)^2$. These conventions are also used for differential operators, such as the Laplacian $\nabla^2 \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$. Throughout the notes, the covariant derivatives ∇_μ are defined with respect to the Christoffel (also known as Levi–Civita) connection, the unique connection which is symmetric and metric compatible.

Riemann's tensor is defined with the appropriate sign so that the gravitational Lagrangian takes the form

$$\mathcal{L}_g = \frac{1}{16\pi G}\sqrt{-g}\, R \ ,$$

where $R = g^{\mu\nu}R_{\mu\nu}$ is the Ricci scalar, $g \equiv \det(g_{\mu\nu})$ and G is Newton's constant.

The integration measure which is invariant under diffeomorphisms is $d^4x\sqrt{-g}$. This generalizes for submanifolds in terms of the induced metric on the submanifold. For any scalar function $f(x)$, we often use the simplified notation

$$\int_X d^4x\, \sqrt{-g}\, f(x) = \int_X f \ .$$

Prelude

The Theory of Relativity, developed mainly by Einstein between 1905 and 1915, is the culmination of more than two millennia of reflections on the nature of space and time. The Einsteinian synthesis mixes space and time into a new entity, called *spacetime*, which is endowed with a dynamical geometry. This geometry is a generalization of both Euclidean and Riemannian geometry, preserving smoothness but allowing curvature. The dynamical content of the geometry of spacetime is identified with the physical phenomenon of gravitation. The whole picture is summarized in the famous *slogan* by J. A. Wheeler: *matter tells space how to curve, space tells matter how to move.*

The ancient greeks took the first giant step when they discovered the 'structure of space', as characterized by the laws of Euclidean geometry. They were so fascinated by this finding that incurred in the Platonic illusion that Euclidean geometry is a logical necessity. This belief started to shatter when XIX century mathematicians discovered logically consistent, yet non-Euclidean geometries, a revolutionary idea which culminated with the Riemannian description of geometries which are smooth, i.e. approximately Euclidean on short distances, but curved on generic distances. By the end of the XIX century the geometry of space was again considered an empirical question, but no suggestion of a dynamical nature was made at this point.

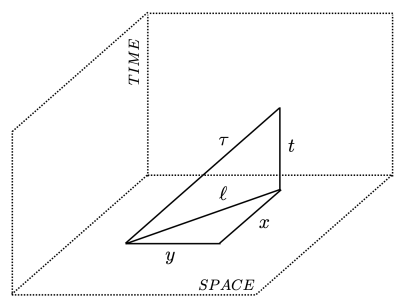

Figure 1.1: *The fundamental structure of Einstein's spacetime is revealed by considering pythagorean triangles extended in the time direction, representing the time lapses registered by moving clocks (ticking τ units) and static clocks (ticking t units). The purely spatial theorem $\ell^2 = x^2 + y^2$ is generalized to $-\tau^2 = -t^2 + \ell^2$, with time measured in, say seconds and distances measured in light-seconds.*

Greek geometry was extended by Einstein with the invention of the Theory of Special Relativity and its subsequent geometrical interpretation by Minkowski. Roughly, the behavior of light implies a generalization of the symmetries of classical mechanics. This generalization can be stated as a geometrical principle. Given a right triangle in space with sides x, y and ℓ, the Pythagorean theorem states $\ell^2 = x^2 + y^2$. Now, if we 'lift' this triangle into the time direction, by specifying two clocks, one at rest ticking to time t and another one in motion ticking to 'proper time' τ, one finds, in units where the speed of light is unity,

$$-\tau^2 = \ell^2 - t^2 = x^2 + y^2 - t^2 \ , \tag{1.1}$$

that is to say, the Pythagorean theorem is lifted into *spacetime* by simply sticking a minus sign in the time directions (see Figure (1.1)). This simple geometrical structure is one of the two or three fundamental principles of Physics. Finally, in 1915 Einstein brought together the Riemannian and the Minkowskian generalizations of Euclidean geometry with a new synthesis: spacetime is the curved (Riemannian) generalization of (1.1), with the curvature being determined by the gravitational field.

In these notes we will distill this deep result following a physical path, by showing that this is the only answer to the problem of constructing a relativistic version of the Newtonian law of gravitation, famously stating

$$|F_G| = G\frac{m_1 m_2}{r^2} \tag{1.2}$$

for the gravitational attraction between two masses separated a distance r. The potential energy at height r in the field of a mass M is given by $-GM/r$ per unit of test mass, up to an additive normalization constant. In orbital motion we have a kinetic energy of the same order of magnitude. The characteristic velocity of an orbit at height r is then $v(r) \sim \sqrt{GM/r}$, so that the relativistic character of this motion is measured by the ratio

$$\frac{v^2}{c^2} \sim \frac{GM}{c^2 r} \sim \frac{R_s}{r} \ ,$$

where c is the speed of light and we have defined a characteristic gravitational (so-called Schwarzschild) radius for the gravitational field of a mass M:

$$R_s(M) \equiv \frac{2GM}{c^2} \ .$$

The gravitational interaction becomes relativistic, i.e. non-Newtonian, for $\phi(r) \equiv R_s/r \sim 1$. The dimensionless potential ϕ also measures the order of magnitude of relativistic effects arising as corrections to the Newtonian theory.

For localized systems, the value of ϕ on Earth is of order $\phi_\oplus \sim 10^{-9}$. In the vicinity of the sun, $\phi_\odot \sim 10^{-6}$. On the surface of a white dwarf star we have $\phi_{\rm wds} \sim 10^{-4}$, the same order of magnitude of relativistic effects as the hydrogen atom. Finally, relativistic stars such as neutron stars reach $\phi_{\rm ns} \sim 0.1$ and black holes always have $\phi_{\rm bh} \sim 1$.

For non-localized systems, such a uniform-density distribution, we have $M(r) \sim \rho \cdot r^3$ for the mass enclosed by a sphere of size r. Then, we have $\phi \sim (H\,r)^2$, with H defining the Hubble parameter [1]

$$H^2(\rho) \equiv \frac{8\pi G\rho}{3c^2} \ .$$

Therefore, a cosmological model becomes relativistic at distances of order H^{-1}. This happens in our Universe for $H^{-1} \sim$ Gigaparsecs (Gpc).

In these notes we develop the current theory of the gravitational interaction in relativistic regimes where $\phi = O(1)$, essentially developed by Einstein in the decade prior to 1916. As indicated above, this theory can be tested to order 10^{-6} in solar system experiments, and it is essential to understand the dynamics of relativistic stars and the global properties of the universe on distance scales beyond Gigaparsecs. On the other hand, the effects of gravity are largely irrelevant at the scale of elementary particles. The reason is the extreme weakness of the gravitational interaction between subatomic particles.

[1] The precise numerical factors in the definitions of R_s and H are irrelevant for our present discussion of order-of-magnitude estimates, but they conform to the more precise definitions to come.

The ratio of electric to gravitational force between two protons with mass m_p is

$$\frac{|F_{\rm em}|}{|F_{\rm G}|} = \frac{e^2}{4\pi G m_p^2} \sim 10^{40} \ ,$$

which makes gravitation utterly negligible at the subatomic level.[2] We can write this ratio in terms of dimensionless quantities characterizing the intrinsic strength of the interaction, the fine structure constant $\alpha = e^2/4\pi\hbar c$, and the gravitational analog $\alpha_G = G m_p^2/\hbar c$. Since $\alpha \approx 1/137$ is a pure number, it turns out that $\alpha_G \approx 10^{-38}$ is also a pure number. Therefore we find that the quantity

$$M_{\rm Pl} \equiv \sqrt{\frac{\hbar c}{G}} \approx 10^{19}\, m_p$$

has dimensions of mass and it is called the Planck mass. The associated Planck energy is $M_{\rm Pl} c^2 \approx 10^{19}$ GeV and the Planck length $L_{\rm Pl} = \sqrt{G\hbar/c^3} \approx 10^{-19} \lambda_{\rm proton} \sim 10^{-33}$ cm, where $\lambda_{\rm proton} \sim \hbar/m_p c$ is the size of the proton.

The Planck scale has a quantum significance, marking the regime where quantum effects make gravity as strong as the other forces. To see this, notice that a quantum state localized on distance scales ℓ has typical momentum $\hbar/\ell$, according to Heisenberg's principle. When such a momentum is larger than mc for all masses in the problem, the state is highly relativistic, having energy $E \sim \hbar c/\ell$ and effective gravitating mass $\hbar/\ell c$. The dimensionless intensity of the gravitational field for such a mass is

$$\alpha_G(\ell) \sim \frac{G}{\hbar c} \cdot \frac{\hbar^2}{c^2 \ell^2} \sim \frac{G}{\hbar c^3 \ell^2} \sim \left(\frac{L_{\rm Pl}}{\ell}\right)^2 \ ,$$

which becomes of order unity when $\ell \sim L_{\rm Pl}$. In fact, it equilibrates the electromagnetic force at scales of the order of tens of Planck lengths.

We conclude that significant quantum gravitational effects would require Planck-mass elementary quantum objects or Planck lengths in the quantum resolution of scales implied by Heisenberg's principle. This means in practice that quantum gravitational effects are irrelevant from the point of view of feasible experiments.

It is possible to imagine situations where this conclusion is significantly modified. For example, it could be that the world is $4+n$ dimensional below some length scale ℓ_c. In this case Newton's force could actually scale like $1/r^{2+n}$ for $r \ll \ell_c$, which means a stronger growth at short distances. In practice this would imply a lower effective Planck mass and thus room for experimental hope. Currently there is no evidence for such scenarios, but they serve as indications that other possibilities exist.

More generally, the greatest revolutions in theoretical physics are associated to conceptual synthesis between seemingly incompatible theories, in a tight and essentially unique solution. Famous examples of this trend are the theory of quantum fields, the essentially unique unification of Special Relativity and quantum mechanics. Another example is Einstein's theory of gravitation, again the essentially unique theory of relativistic classical gravity. Many physicists hope that the final synthesis involving the three basic constants, $\hbar, c$ and G, will be born out in a conceptually 'tight' fashion, so that we may hope to 'corner' the answer even in the absence of crucial experimental input.

[2] In fact, on distance scales about a tenth of a millimeter and below, gravitation is already surpassed by Van der Waals forces.

While we wait for the fully fledged development of such a theory, we shall set $\hbar = 0$ for most of these lecture notes.

1.1 Particles, Fields, Lagrangians and Symmetries

Dynamical systems can be described classically by equations of motion or by specifying action functionals. Let the pair $(\mathcal{Q}(t), \mathcal{Q}_t(t))$ denote the *state* of a mechanical system (the data necessary to determine its future, usually initial positions and first time derivatives $\mathcal{Q}_t \equiv d\mathcal{Q}/dt$). Here we regard $\mathcal{Q}$ as an abstract notation for a set of position coordinates, particle label or local excitation of a field.

Given initial and final values of the configuration variables, $\mathcal{Q}(t_i) = \mathcal{Q}_i, \mathcal{Q}(t_f) = \mathcal{Q}_f$, we construct the action functional

$$S[\gamma] = \int_\gamma dt\, L(\mathcal{Q}, \mathcal{Q}_t) \ , \tag{1.3}$$

where γ denotes a given *history*, i.e. the function $\gamma : t \to \mathcal{Q}(t)$, and L is called the Lagrangian. Then, the equations of motion follow from the extrema of the action functional, $\delta S = 0$, for all variations $\mathcal{Q} \to \mathcal{Q} + \delta\mathcal{Q}$ with fixed initial and final values. This can be expressed as

$$0 = \delta S = \int_\gamma dt \left(\frac{\partial L}{\partial \mathcal{Q}} \delta\mathcal{Q} + \frac{\partial L}{\partial \mathcal{Q}_t} \delta\mathcal{Q}_t \right) \ .$$

Using that $\delta\mathcal{Q}_t = d(\delta\mathcal{Q})/dt$ and integrating by parts we find

$$0 = \int_\gamma dt\, \delta\mathcal{Q} \left(\frac{\partial L}{\partial \mathcal{Q}} - \frac{d}{dt}\frac{\partial L}{\partial \mathcal{Q}_t} \right) + 0 \ , \tag{1.4}$$

where the last vanishing term corresponds to the contribution of total derivatives in time, vanishing because of the boundary conditions $\delta\mathcal{Q}(t_i) = \delta\mathcal{Q}(t_f) = 0$. The extremal trajectory must satisfy (1.4) for all values of $\delta\mathcal{Q}$, implying the vanishing of the term in parenthesis. Therefore we arrive at the the so-called Euler–Lagrange equations of motion,

$$\frac{d}{dt}\frac{\partial L}{\partial \mathcal{Q}_t} - \frac{\partial L}{\partial \mathcal{Q}} = 0 \ . \tag{1.5}$$

Lagrangians are defined up to a total derivative in time. Given L, then $L + df/dt$ leads to the same equations of motion.

The elementary example is that of an isolated system of Newtonian particles with generalized coordinates $\mathcal{Q}(t) \to \vec{X}_p(t)$. The Lagrangian reads

$$L = \sum_p \tfrac{1}{2}\, m_p \left(\frac{d\vec{X}_p}{dt} \right)^2 - U(\vec{X}_1, \dots, \vec{X}_p, \dots) \ , \tag{1.6}$$

with U defining a potential energy function. This Lagrangian produces the well-known Newtonian system of equations

$$m_p \frac{d^2 \vec{X}_p}{dt^2} = -\frac{\partial U}{\partial \vec{X}_p} \ .$$

The Lagrangian formalism is very useful in the description of local degrees of freedom, their interactions and symmetries. It also provides one of the simplest jumping points from classical to quantum physics, since the fundamental quantum object is the probability amplitude for a transition between initial and final configurations, which has a simple formal relation to the classical action,

$$\mathcal{A}_{\,\mathcal{Q}_i \to \mathcal{Q}_f} = \sum_{\text{histories } \gamma_{i\to f}} e^{iS[\gamma]/\hbar} \ , \tag{1.7}$$

according to Feynman's formula. In these notes, we will not deal with quantum theory, but we will make full use of the Lagrangian formalism as the most powerful tool in elucidating local dynamics, whose most fundamental idea is the concept of *fields*.

Fields

We may take a formal continuum limit for systems of the form (1.6) with degrees of freedom pinned to points of space, leading to the notion of field theories. Here, the role of the particle index p is played by an 'approximately' continuous label corresponding to the point of space $\mathcal{Q}(t) \to q_{\vec{x}}(t)$. Intuitively, we imagine a field as an assembly of 'particles' attached to points of space. The function $q_{\vec{x}}(t)$ denotes 'oscillations' of this 'attached particle', either in physical space or in some internal space.

Given an equilibrium configuration $q_{\vec{x}}^{(0)}$ of the potential energy function U, we may expand it in powers of the local displacement variable $\phi(\vec{x}, t) = \sqrt{m_{\vec{x}}}\left(q_{\vec{x}}(t) - q_{\vec{x}}^{(0)}\right)$, which will be referred to as the *field*. In the continuum limit, the assumption of *locality* means that the functional $U[\phi(\vec{x}, t)]$ is the integral of a *density*, admitting an expansion around $\phi = 0$ in powers of derivatives of the field $\phi(\vec{x}, t)$:

$$U[\,\phi\,] = \int d^3x \,\left(V(\phi) + \tfrac{1}{2}\, c_s^2 (\vec{\partial}\phi)^2 + \text{higher derivatives}\right),$$

where we have further assumed translational and rotational invariance. Local stability of the equilibrium point requires that the so-called 'squared speed of sound', c_s^2, be positive and the non-derivative term, $V(\phi)$, be well approximated by a positive parabola near $\phi = 0$.

Notice that long distance physics is controlled by the terms with the smallest number of spatial derivatives. Consider for example two contributions with two and four derivatives respectively, $(\vec{\partial}\phi)^2 + \lambda^2(\vec{\partial}^2\phi)^2$. By dimensional analysis, λ must necessarily be a length scale, so that a field configuration with scale of variation of order ℓ contributes an amount of order $\ell^{-2}(1 + C(\lambda/\ell)^2)$ for some constant C. Hence, at long distances $\ell \gg \lambda$ the higher-derivative terms give a subleading contribution and in any case their effects can be gradually introduced by perturbation theory in the corresponding dimensionless effective couplings, such as $\lambda_{\text{eff}}(\ell) = \lambda/\ell$. This type of 'long-distance' expansion is called *Effective Field Theory* and it is the mathematical embodiment of the Principle of Locality.

Collecting all terms together we are led to a field theory Lagrangian

$$L = \int d^3x \, \mathcal{L}[\,\phi\,]\,, \tag{1.8}$$

where the Lagrangian *density* has the following structure

$$\mathcal{L}[\,\phi\,] = \tfrac{1}{2}\,(\partial_t\phi)^2 - \tfrac{1}{2}\, c_s^2(\vec{\partial}\phi)^2 - V(\phi) + \text{higher derivatives} + \text{boundary terms}\,. \tag{1.9}$$

In most applications, boundary terms are neglected assuming appropriate boundary conditions for the fields, so that we can integrate by parts at will. The physics of small fluctuations around equilibrium can be studied by polynomial approximations of the potential function $V(\phi)$. A linear term of the form $J\phi$, called a 'source term', is often considered for a general function $J(\vec{x}, t)$ as a way of 'driving' the system, to study perturbative response of the equilibrium state. The quadratic term is often called the 'mass' of the field. The Euler–Lagrange equations stemming from the long-distance approximation (1.9) are

$$\left(-\partial_t^2 + c_s^2\,\vec{\partial}^{\,2}\right)\phi = V'(\phi) = J\,c_s^2 + m^2 c_s^4\,\phi + \text{nonlinear terms}\,, \tag{1.10}$$

where the factors of c_s in the terms proportional to J and m^2 are conventional definitions. The elementary solution of the homogeneous equation for zero mass,

$$(-\partial_t^2 + c_s^2\,\vec{\partial}^{\,2})\phi = 0 \tag{1.11}$$

are waves $\phi_w(t, \vec{x}\,)$ propagating at speed c_s. The elementary solution for a static (time-independent) field created by a source J is the solution of Poisson's equation,

$$\phi(\vec{x}\,)_{\text{static}} = \frac{1}{\vec{\partial}^{\,2}} J = -\int d^3y\,J(\vec{y}\,)\int \frac{d^3k}{(2\pi)^3}\frac{e^{i\vec{k}(\vec{x}-\vec{y}\,)}}{\vec{k}^{\,2}} = -\int \frac{d^3y}{4\pi}\frac{J(\vec{y}\,)}{|\vec{x}-\vec{y}\,|}\,. \tag{1.12}$$

Time-dependent disturbances created by a time-dependent source $J(\vec{x}, t)$ are obtained from the static ones by simply recalling that these disturbances travel at speed c_s. Hence the solution is formally the same as (1.12) with the source replaced by its value at the 'retarded time' $t - |\vec{x}-\vec{y}\,|/c_s$,

$$\phi(\vec{x}, t)_{\text{retarded}} = \phi(\vec{x}, t)_{\text{wave}} - \int \frac{d^3y}{4\pi}\frac{J\left(t - \frac{|\vec{x}-\vec{y}\,|}{c_s}, \vec{y}\,\right)}{|\vec{x}-\vec{y}\,|}\,, \tag{1.13}$$

where ϕ_{wave} is a general solution of the homogeneous equation (1.11), which might be interpreted as a sort of 'primordial' field, coming from the remote past, which was there even before any currents existed that contributed to the field excitations. We can use the same argument to write a formal solution of the general non-linear equation with arbitrary potential:

$$\phi(\vec{x}, t)_{\text{retarded}} = \phi_{\text{wave}} + \frac{1}{\partial^2}\Big|_{\text{ret}} V'(\phi) = \phi_{\text{wave}} - \int \frac{d^3y}{4\pi}\frac{V'\left[\phi\left(t - \frac{|\vec{x}-\vec{y}\,|}{c_s}, \vec{y}\,\right)\right]}{|\vec{x}-\vec{y}\,|}\,. \tag{1.14}$$

Rather than an explicit solution, this is an integral equation for any potential with quadratic terms or higher non-linearities. This equation may be solved iteratively in the powers of the function $V'[\phi]$. For example, for a mass term we have $V'(\phi) = m^2 c_s^4 \phi$. The term of order m^{2n} in the iterative solution represents a wave sourced by J and averaged over $2n$ 'kicks' at which the wave is regenerated. This construction is analogous to Huygens' method of wave dispersion, so that one says that the mass term causes 'wave dispersion', as a result of which the effective propagation velocity over distances larger than c_s/m is *smaller* than c_s. Alternatively, we can look for solutions of definite frequency ω and momentum $\vec{k}$,

$$\phi(t, \vec{x})_{\omega, \vec{k}} \sim e^{-i\omega t + i\vec{k}\vec{x}}$$

and the equation of motion implies the so-called 'dispersion relation'

$$\omega^2 = m^2 c_s^4 + \vec{k}^{\,2} c_s^2\,.$$

The basic phenomenon of field dynamics is wave propagation. Locality translates into the finite propagation velocity c_s for any physical perturbations modeled by the field $\phi(t, \vec{x})$.

Symmetries

A classical system is said to enjoy a symmetry action when some group of transformations $\mathcal{G}$ acts on the space of solutions of the equations of motion, i.e. given one solution $\mathcal{Q}$, the transformed $g(\mathcal{Q})$ under $\mathcal{G}$ is also a solution of the equations of motion. One way to ensure this is to demand that the action functional be invariant under the action of $\mathcal{G}$:

$$S[\gamma] = S[g(\gamma)] \;, \tag{1.15}$$

for any history γ and any group element $g \in \mathcal{G}$. This is slightly more than strictly necessary in the classical realm, because one could do with an action of $\mathcal{G}$ just on the space of extrema of S, rather than the whole domain of definition of S. However, in quantum mechanics one really explores all possible trajectories and symmetries have to respect them all.

Invariance of the action is not always born out by invariance of the Lagrangian. Because of the ambiguity in Lagrangians by a total derivative, it is enough to demand that a Lagrangian is invariant up to a total time derivative,

$$L[g(\mathcal{Q})] = L[\mathcal{Q}] + \frac{df_g}{dt} \;. \tag{1.16}$$

For field theories, and under the technical assumption that boundary terms do not contribute to the dynamics, the Lagrangian density may be invariant up to a total derivative in time and space:

$$\mathcal{L}[\,\phi\,] \to \mathcal{L}[\,g(\phi)\,] = \mathcal{L}[\,\phi\,] + \partial_t f^t + \vec{\partial}\vec{f} \;, \tag{1.17}$$

for some functions f^t and $\vec{f}$.

The main use of Lagrangian methods is to implement economically the requirements of symmetry. At the same time, this formalism provides a very general link between symmetries and conservation laws, via a famous theorem by Noether.

Let the symmetry $\mathcal{G}$ act on the space of trajectories $\mathcal{Q}(t)$ as a continuous group and work near the identity,

$$g(\mathcal{Q}) = \mathcal{Q} + \delta_g \mathcal{Q} = \mathcal{Q} + \epsilon\, s[\mathcal{Q}] + O(\epsilon^2) \;, \tag{1.18}$$

where $s[\mathcal{Q}]$ might be a rather general functional of $\mathcal{Q}$. Since Lagrangians are defined up to a total time derivative, an invariant action is compatible with a variation of the Lagrangian

$$\delta_s L = \epsilon \frac{df_s}{dt} \;, \tag{1.19}$$

for some f_s. Rewriting this in differential form,

$$\delta_s L = \frac{\partial L}{\partial \mathcal{Q}} \delta_s \mathcal{Q} + \frac{\partial L}{\partial \mathcal{Q}_t} \delta_s \mathcal{Q}_t = \left(\frac{\partial L}{\partial \mathcal{Q}} - \frac{d}{dt} \frac{\partial L}{\partial \mathcal{Q}_t} \right) \delta_s \mathcal{Q} + \frac{d}{dt} \left(\frac{\partial L}{\partial \mathcal{Q}_t} \delta_s \mathcal{Q} \right) \;. \tag{1.20}$$

Using the equations of motion and equating (1.19) and (1.20), we establish the conservation of the Noether charge Q_s:

$$\frac{dQ_s}{dt} = 0 \;, \qquad Q_s = s[\mathcal{Q}] \frac{\partial L}{\partial \mathcal{Q}_t} - f_s \;. \tag{1.21}$$

Some examples of importance include the usual definitions of energy and momentum for a free Newtonian particle with Lagrangian $L = \frac{1}{2}\, m\vec{v}^{\,2}$, associated to translation symmetry in time and space, respectively

$$\vec{p} = \frac{\partial L}{\partial \vec{v}} = m\vec{v} \qquad E = \vec{v} \cdot \frac{\partial L}{\partial \vec{v}} - L = \tfrac{1}{2}\, m\vec{v}^{\,2} = \frac{\vec{p}^{\,2}}{2m} \;. \tag{1.22}$$

For field theories with Lagrangians of functional form $\mathcal{L}(\partial_t\phi, \vec{\partial}\phi, \phi)$, the statement of a continuous symmetry is that $\delta_s \mathcal{L} = \epsilon\, \partial_t f_s^t + \epsilon\, \vec{\partial} \vec{f}_s$ under a field transformation $\delta_s \phi = \epsilon\, s[\phi]$, where $s[\phi]$ is a functional of $\phi(\vec{x}, t)$ with similar structure as the Lagrangian. Repeating the previous derivation (1.20), neglecting total spatial derivatives, one finds a local 'continuity equation'

$$\partial_t J_s^t + \vec{\partial} \vec{J}_s = 0 \tag{1.23}$$

for the Noether current defined by

$$J_s^t = s[\phi] \cdot \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} - f_s^t \,, \qquad \vec{J}_s = s[\phi] \cdot \frac{\partial \mathcal{L}}{\partial(\vec{\partial}\phi)} - \vec{f}_s \,. \tag{1.24}$$

Whenever we encounter (1.23) we can interpret it as a statement of local conservation of charge, by assigning the role of *charge density* to J^t and the role of *flux-density* to $\vec{J}$. Namely, we define the Noether charge inside a three-dimensional region V^3 as the integral of the density,

$$Q_s(\mathrm{V}^3) = \int_{\mathrm{V}^3} J_s^t \,, \tag{1.25}$$

and the flux across the boundary $\partial\mathrm{V}^3$

$$\mathrm{Flux}_s(\partial \mathrm{V}^3) = \oint_{\partial \mathrm{V}^3} (J_s)_n \,, \tag{1.26}$$

where $(J_s)_n$ denotes the component of $\vec{J}_s$ along the outward pointing normal to $\partial\mathrm{V}^3$. Then, upon integration over V^3, (1.23) is equivalent to

$$\frac{d}{dt} Q_s(\mathrm{V}^3) = -\mathrm{Flux}_s(\partial \mathrm{V}^3) \,, \tag{1.27}$$

i.e. the variation of charge equals the inward flux through the boundary $\partial\mathrm{V}^3$, without any creation or destruction of charge inside V^3.

1.2 Crisis 'fin de siècle'

Circa 1900 one could summarize the Lagrangian of the 'fundamental' physics in terms of two components

$$L_{1900} \sim L_{\text{Newton}} + L_{\text{Maxwell-Lorentz}} \; , \tag{1.28}$$

where L_{Newton} stands for the model of pointlike masses interacting by instantaneous gravitational forces. The Maxwell–Lorentz sector concerns the electromagnetic phenomena and their coupling to charged massive particles. A mechanical interpretation was missing in this sector, although drawing inspiration from the theory of optics, it was natural to postulate a material medium, called *ether*, that would support the electromagnetic waves. This prompted an active search for mechanical models of the 'ethereal' substance. The continued failure of such models lead to a conceptual crisis at the turn of the century.

1.2.1 Newton

A model of matter in terms of pointlike objects characterized by positions $\vec{x}_p$ and velocities $\vec{v}_p = d\vec{x}_p/dt$, interacting by a Newtonian gravitational force is described by the Lagrangian

$$L_{\text{Newton}} = \sum_p \tfrac{1}{2}\, m_p \, \vec{v}_p^{\,2} + G \sum_{p<q} \frac{\mu_p \mu_q}{|\vec{x}_p - \vec{x}_q|} \; . \tag{1.29}$$

The most salient features of this Lagrangian are the instantaneous nature of gravitational forces (the potential energy only depends on positions of particles) and the equality of inertial and gravitational masses, $\mu_p = m_p$, the so-called "Principle of Equivalence". It also satisfies the Galilean principle of relativity, being invariant under a ten-parameter group of transformations

$$x_i \to \sum_j R_i^{\;j} \, (x_j + a_j + V_j t) \;, \qquad t \to t + a_t \; , \tag{1.30}$$

including boosts by a velocity $\vec{V}$ as well as $\mathbf{R}^3$-rotations $R_i^j \in SO(3)$ and translations in space and time.[3] This formulation of the symmetry principles of Newtonian mechanics assumes the existence of 'inertial frames' in which the dynamics actually takes the form expressed in (1.29). The pragmatic rule is that inertial frames are those in which test particles that are isolated (free from any forces) move in straight lines at constant velocity.

The equality of gravitational and inertial masses is famously exemplified by the (perhaps false) story of Galileo throwing objects from the top of Pisa's tower. In fact, if we denote m the inertial mass and μ the 'gravitational charge', it is enough to ensure that the ratio m/μ is a universal constant, equal for all possible bodies, since then this ratio can be set to unity by a rescaling of Newton's constant.

In the nineteen century, Eötvös was able to check this universality with great accuracy in a series of experiments comparing the balance of a gravitational force (responding to μ) and a purely inertial force (responding to m). A good example is a body at rest on the surface of the Earth, subject to gravitational and centrifugal forces. Tiny differences of the ratio m/μ between two objects would result in a slight misalignment of the net force acting on them. For the latitude of Budapest, home of Eötvös, one finds

$$\Delta_{m/\mu} \approx 600 \, |\sin\alpha\,| \; , \tag{1.31}$$

[3] This is an example of the case that the Lagrangian is invariant up to a total derivative in time.

where α is the misalignment angle. Eötvös was able to bound $\Delta_{m/\mu}$ to less than one part in 10^7. The bound was later improved by Dicke in the 60's to one part in 10^{11} and today we know it to be true to one part in 10^{15}. The conclusion is, therefore, that such a precision test should not be based upon a simple coincidence, but some deep principle. The interpretation of the Principle of Equivalence as the fundamental key to the theory of gravitation is the main contribution of Einstein.

Problem: Eötvös Experiment

Verify expression (1.31) by computing the misalignment of the total gravitational+centrifugal force for two objects of gravitational masses μ_1, μ_2 and inertial masses m_1, m_2. Give an expression for $|\sin\alpha|$ in terms of the surface gravity $g_\oplus = GM_\oplus/R_\oplus^2$, the Earth radius $R_\oplus$ and the Earth rotation angular velocity $\Omega_\oplus$.

Anticipating its use in the following, we note that Newton's gravitational law can be formally written as a field theory coupled to the mass density of matter. The equations of motion following from (1.29) are

$$m_p \frac{d\vec{v}_p}{dt} = \vec{F}_{\mathrm{N}} = -Gm_p \sum_{q\neq p} \frac{m_q(\vec{x}_p - \vec{x}_q)}{|\vec{x}_p - \vec{x}_q|^3} \ , \tag{1.32}$$

where we have already implemented the equality of gravitational and inertial masses, so that m_p drops from this equation. The force term on the right hand side can be written in terms of a gravitational potential, $\vec{F}_{\mathrm{N}} = -m_p\vec{\partial}\phi_{\mathrm{N}}$, where

$$\phi_{\mathrm{N}}(t,\vec{x}) = -\sum_q \frac{Gm_q}{|\vec{x} - \vec{x}_q(t)|} \ . \tag{1.33}$$

This form of the potential, defined up to a additive constant, is the solution of the so-called Poisson equation sourced by the mass density

$$\vec{\partial}^{\,2}\phi_{\mathrm{N}}(\vec{x},t) = 4\pi G \sum_q m_q\, \delta^{(3)}(\vec{x} - \vec{x}_q(t)) \equiv 4\pi G \rho_m(\vec{x},t) \ . \tag{1.34}$$

We see that Newtonian theory can be recast in the form of a field theory with a gravitational potential ϕ_{N} interacting with mass density through the equations

$$\frac{d\vec{v}_p}{dt} = -\vec{\partial}\,\phi_{\mathrm{N}} \ , \qquad \vec{\partial}^{\,2}\,\phi_{\mathrm{N}} = 4\pi G\,\rho_m \ , \tag{1.35}$$

which are equivalent to the Lagrangian

$$L_{\mathrm{Newton}} = \sum_p \tfrac{1}{2}\, m_p\vec{v}_p^{\,2} - \frac{1}{8\pi G}\int d^3x(\vec{\partial}\phi_{\mathrm{N}})^2 - \int d^3x\, \rho_m\phi_{\mathrm{N}} \ . \tag{1.36}$$

Solving for the potential ϕ_{N} using Poisson's equation yields precisely (1.33), and substituting back into (1.36) produces the 'action-at-a-distance' form of the Newtonian system (1.29). Notice that the absence of time derivatives from Poisson's equation is directly related to the instantaneous nature of the interactions in (1.29). In principle, a finite speed of propagation for the gravitational interaction can be introduced into this formalism by a simple replacement of Poisson's equation by a more conventional wave equation such as

$$\left(-\frac{1}{c_{\rm G}^2}\partial_t^2+\vec{\partial}^{\,2}\right)\phi_{\rm N}=4\pi G\,\rho_m\ ,$$

the standard Newtonian theory thus arising in the limit $c_G \to \infty$. At any rate, the extreme weakness of the gravitational interaction makes it very difficult to detect dynamical effects of the gravitational field, such as velocity-dependent forces. In comparing with the more familiar optical and magnetic effects, notice that the nominal intensity of gravity is only significant on human scales for planet-sized objects. If c_G is assumed to be in the ballpark of $c \approx 3\cdot 10^{10}$ cm/s, then any v/c_G effects visible to XIX-century physicists would require 'light-fast' planetary-sized objects hurling around. This is the kind of situation which finally led to the detection of gravitational waves as late as 2015.

1.2.2 Maxwell–Lorentz

Electrostatic phenomena can be described by a system of equations entirely analogous to (1.35) and (1.36), with the replacement of the Newtonian potential by the Coulomb potential produced by an electric charge e,

$$\phi_{\rm C}(\vec{x},t)=\frac{e}{4\pi|\vec{x}\,|}\ .$$

Magnetic phenomena may be incorporated by the introduction of velocity-dependent forces. However, it was found experimentally by Faraday and others that electrodynamics was very efficiently represented by the notion of 'electromagnetic fields' interacting with charged matter particles.

Maxwell found the set of local equations that explained all electromagnetic phenomena in terms of electric $\vec{E}$ and magnetic $\vec{B}$ fields in the presence of charges and currents,

$$\begin{aligned}\vec{\partial}\vec{E} &= \rho_e, \qquad c\,\vec{\partial}\times\vec{B}-\partial_t\vec{E}=\vec{J}_e\ ,\\ \vec{\partial}\vec{B} &= 0\ , \qquad c\,\vec{\partial}\times\vec{E}+\partial_t\vec{B}=0\ .\end{aligned} \tag{1.37}$$

The charge density and current are defined by

$$\rho_e(\vec{x},t)=\sum_p e_p\,\delta^{(3)}(\vec{x}-\vec{x}_p(t))\ ,\qquad \vec{J}_e(\vec{x},t)=\sum_p\,e_p\,\vec{v}_p(t)\ \delta^{(3)}(\vec{x}-\vec{x}_p(t)) \tag{1.38}$$

and satisfy the conservation law

$$\partial_t\rho_e+\vec{\partial}\vec{J}_e=0\ . \tag{1.39}$$

This conservation law for electric charge acquires a special character in this theory. Since electric charge sources the electric field, one can *detect* the charge inside a large region V^3 by measuring the electric field on the boundary of the region, $\partial\mathrm{V}^3$. This follows from Gauss's law, which is the first of the Maxwell equations in (1.37). Integrating this equation over V^3 one finds

$$Q_e(\mathrm{V}^3)=\oint_{\partial\mathrm{V}^3}E_n\ , \tag{1.40}$$

where E_n is the normal component of the electric field at $\partial\mathrm{V}^3$. If $\mathrm{V}^3=\mathbf{R}^3$, this integral computes the total electric charge and the electric flux can be regarded as piercing the so-called 'celestial sphere at infinity' $\mathbf{S}^2_\infty$.

Maxwell's equations depend on a numerical parameter, c, which can be given a physical interpretation by solving the second pair of equations in terms of potentials,

$$\vec{B} = \vec{\partial} \times \vec{A}\,, \qquad \vec{E} = -\frac{1}{c}\,\partial_t \vec{A} - \vec{\partial}\phi\,, \tag{1.41}$$

defined up to the *gauge* ambiguity,

$$\phi \to \phi - \frac{1}{c}\partial_t f\,, \qquad \vec{A} \to \vec{A} + \vec{\partial} f\,, \tag{1.42}$$

with $f(t, \vec{x})$ an arbitrary smooth function. Fixing this ambiguity (partially) by imposing the Lorenz condition [4]

$$\frac{1}{c}\,\partial_t\,\phi + \vec{\partial}\,\vec{A} = 0\,, \tag{1.43}$$

the first pair of Maxwell equations can be rewritten as wave equations

$$\begin{aligned} \left(\vec{\partial}^{\,2} - \frac{1}{c^2}\partial_t^{\,2}\right)\phi &= -\rho_e\,, \\ \left(\vec{\partial}^{\,2} - \frac{1}{c^2}\partial_t^{\,2}\right)\vec{A} &= -\frac{\vec{J}_e}{c}\,, \end{aligned} \tag{1.44}$$

which reveals the interpretation of c as the speed of electromagnetic waves. These waves also provide an electromagnetic theory of optics, since oscillating solutions of the electric and magnetic fields exist even outside charged matter and indeed the numerical value of c happens to coincide precisely with the velocity of light *in vacuo*.

The interaction between electromagnetic fields and charged particles is determined by the Lorentz force law

$$m_p \frac{d\vec{v}_p}{dt} = e_p\left(\vec{E} + \frac{\vec{v}_p}{c} \times \vec{B}\right)\,. \tag{1.45}$$

which in turn derives from the Lagrangian

$$\sum_p \tfrac{1}{2}\, m_p\, \vec{v}_p^{\,2} - \sum_p e_p \left(\phi(t, \vec{x}_p) - \frac{\vec{v}_p}{c}\ \vec{A}(t, \vec{x}_p)\right)\,, \tag{1.46}$$

The second term may be rewritten in terms of currents and charge densities to obtain the action principle behind Maxwell's equations (1.37),

$$S_{\text{Maxwell}-\text{Lorentz}} = \tfrac{1}{2}\int dt\, d^3x\ \left(\vec{E}^{\,2} - \vec{B}^{\,2}\right) - \frac{1}{c}\int dt\, d^3x\ \left(c\,\rho_e\,\phi - \vec{J}_e\cdot\vec{A}\right)\,, \tag{1.47}$$

with $\vec{E}$ and $\vec{B}$ implicitly written in terms of the potentials $\phi, \vec{A}$ via eq. (1.41). Notice that the potential formalism is very useful in writing the interactions with charges, and essential in the Lagrangian formalism.

The main problem of principle with this theory is the interpretation of the coupling constant c as a velocity of propagation of electromagnetic waves. Immediately the question arises: velocity with respect to what particular frame? Indeed, a Galilean transformation $\vec{x} \to \vec{x} + \vec{v}\,t$ cannot possibly be a symmetry of the Maxwell equations, because such a transformation would change

[4] Lorenz the *danish* instead of Lorentz the *dutch*.

the velocity of light to $\vec{c} - \vec{v}$. Hence, if we insist in adhering to a mechanical interpretation of electromagnetism (ether), it seems that Maxwell equations are only valid in the particular frame that sits at rest with respect to the ether.

In this situation, it becomes interesting to measure the velocity of the Earth with respect to the ether. The famous experiments of Michelson–Morley failed to reveal any such drift velocity. At the same time, stellar aberration and other optical data disfavored the possibility that the Earth could be dragging the ether along with its motion. So, there was a real problem that was tackled by Lorentz and others, with a number of *ad hoc* hypotheses.

Despite the fact that Lorentz had developed most of the right formulas of Special Relativity, it is notable that he was unable to interpret them correctly. Poincaré was also close to the solution, as he proposed that the Galilean group of transformations should be replaced by a different group and, in doing so, giving privilege to the electromagnetic part of the total Lagrangian. One would need to replace S_{Newton} with something different that would be compatible with the strange behavior of light. One could say that the Lorentz–Poincaré combination was bound to find the right answer sooner or later... but the first complete solution was given by Einstein in his famous 1905 work.

1.2.3 Einstein

Einstein's starting point was to take for granted the Galilean principle of relativity, i.e. that motion at constant velocity could not have absolute meaning, and at the same time assume as a feature of nature that the velocity of light is a constant, independent of the frame of reference.

Heuristically, one could argue that one simply needs the existence of a maximum velocity of propagation of signals. If one assumes the existence of this limiting velocity, the principle of relativity requires it to be the same in all frames in relative uniform motion, because otherwise one could assign an intrinsic velocity to a frame of reference by quoting the maximal velocity of signals in that frame. From this point of view, one could have Special Relativity even if light was not propagating right at the limiting velocity. For simplicity, let us assume in any case that c, the light's velocity *in vacuo*, is the maximal one. Then, the principle of relativity says that there must exist a group of transformations that leaves the physics invariant and that preserves the condition of uniform rectilinear motion. Using orthogonal cartesian coordinates, any such transformation,

$$(t, \vec{x}\,) \to (t', \vec{x}\,') = \widetilde{L}\,(t, \vec{x}\,) \tag{1.48}$$

must send straight trajectories $\vec{x}(t)$ into straight trajectories $\vec{x}'\,(t')$, and so it must be a *linear* map. Furthermore, the invariance of light's velocity means that

$$\vec{x}^{\,2} - c^2 t^2 = 0 = \vec{x}\,'^2 - c^2 t\,'^2 \,. \tag{1.49}$$

This equation, together with the linearity property, implies that $\vec{x}^{\,2} - c^2 t^2$ is invariant up to a constant factor,

$$\vec{x}^{2} - c^2 t^2 \to f(\vec{V}\,)\,(\vec{x}^{2} - c^2 t^2)\,, \tag{1.50}$$

where $f(\vec{V}\,)$ can only depend on the relative velocity of the frames $\vec{V}$. Spatial isotropy further imposes the condition that f must be a function of the modulus $|\vec{V}\,| = V$. Introducing now two boosts with velocities $\vec{V}_1$ and $\vec{V}_2$, and relative velocity $\vec{V}_{12}$, we find, upon performing two iterated transformations,

$$f(V_2) = f(V_1) f(V_{12})\,, \quad \text{equivalently} \quad f(V_{12}) = \frac{f(V_2)}{f(V_1)}\,, \tag{1.51}$$

but V_{12} depends on the relative angle between $\vec{V}_1$ and $\vec{V}_2$, unlike the right hand side of the previous equation. Hence, we conclude that f does not depend on $\vec{V}$ and in fact $f = 1$. We thus define the Lorentz group as those linear transformations that leave the quadratic form

$$c^2t^2 - \vec{x}^{\,2} = \text{invariant} \,. \tag{1.52}$$

This group is called $O(1,3)$. When supplemented by the spacetime translations it becomes the Poincaré group, a ten-parameter group which contracts to the Galilean group in the limit $V/c \to 0$. Factoring out the subgroup of $SO(3)$ rotations, we can align the boost velocity in a given direction, say x, and we are left with a two-dimensional problem of linear transformations leaving $c^2t^2 - x^2 = (ct-x)(ct+x)$ invariant. From this point on, we choose units so that $c = 1$. The required linear transformations (maintaining the origin fixed) are

$$x \pm t \to e^{\pm\psi}\,(x \pm t) \;,$$

or, in matrix notation

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh\psi & \sinh\psi \\ \sinh\psi & \cosh\psi \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} \,. \tag{1.53}$$

The parameter ψ is called *rapidity*. It can be related to the relative velocity by monitoring the transformation of the point $x = 0$, leading to $\tanh\psi = x'/t' = V$. Hence, we have the usual, so-called *boost* transformations [5]

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & \gamma V \\ \gamma V & \gamma \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} \,, \tag{1.54}$$

with $\gamma = (1 - V^2)^{-1/2}$. We see that, as expected, we obtain the Galilean transformations $t' \approx t$ and $\vec{x}' \approx \vec{x} + \vec{V}\,t$ whenever the relative frame velocities are small compared to the velocity of light, $|\vec{V}| \ll 1$.

Ships on Top of Ships

Using the rapidity parametrization, find a relativistic version of the formula for the addition of velocities when two Lorentz transformations are iterated. Imagine you construct a huge spaceship capable of maximal velocity $V_M < 1$. If one builds another spaceship with identical capabilities, to be lauched from the first one in the forward direction, what is the net speed that the second ship can reach? If this procedure is iterated, what is the maximum velocity attainable in the limit of an infinite series of ships launched on top of ships?

Minkowskian Geometry

The geometrical interpretation of Special Relativity was given by Minkowski in 1908. We can think of $\mathbf{R}^{1+3} \equiv \mathbf{R} \times \mathbf{R}^3$ as the spacetime, where the first factor is 'time' and the second factor is 'space'. We have the set of 'events' $(t, \vec{x}\,) \in \mathbf{R}^{1+3}$, equipped with a metric

$$\mathbf{g}(\Delta t, \Delta\vec{x}) = \Delta\vec{x}^{\,2} - \Delta t^{\,2} \tag{1.55}$$

[5] The boost with rapidity ψ is the analytic continuation of a standard Euclidean rotation by angle $\theta = i\psi$, leaving the Euclidean interval $I_E = (it)^2 + x^2$ invariant.

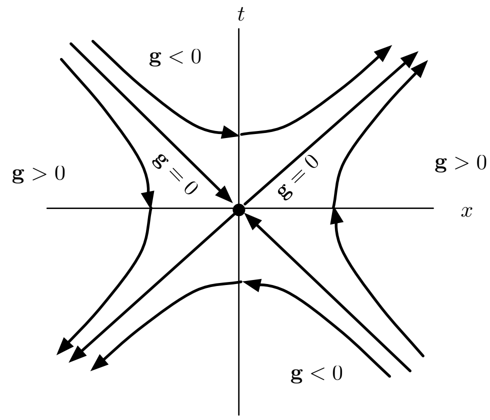

Figure 1.2: *The orbits of the Lorentz group in the (t, x) plane form a set of hyperbolae, defined by* $\mathbf{g} = -t^2 + x^2 = \text{constant}$. *Orbits in the spacelike region* $\mathbf{g} > 0$ *intersect the* $t = 0$ *axis, whereas orbits in the timelike region,* $\mathbf{g} < 0$ *intersect the* $x = 0$ *axis and preserve the sign of the times. The hyperbolae degenerate to straight lines in the light-like region,* $\mathbf{g} = 0$ *given by all the points connected by light rays from the origin. Rotating this picture into the rest of* $\mathbf{R}^3$ *gives the past and future light-cones for the* $\mathbf{g} = 0$ *'null' set.*

between pairs of events.[6] This generalizes the standard Euclidean metric of $\mathbf{R}^3$, which computes the rotationally invariant length-squared of the vector $\Delta\vec{x}$, by the addition of the term $-\Delta t^2$ which makes the Minkowski metric non-positive definite. Lorentz transformations introduce motions on $\mathbf{R}^{1+3}$ that preserve this metric, inducing a causal structure. To explain the physical meaning of the metric, we notice that any point in $\mathbf{R}^{1+3}$ must fall into one of three Lorentz-invariant sets, depending on the sign of $\mathbf{g}$.

If $\mathbf{g} < 0$, there exists a frame in which $\Delta\vec{x}^{\,2} = 0$ and the events 'occur' at the same point in space. In this case we say that the interval between such events is 'timelike' because in this rest frame, the interval can be written as $\mathbf{g} = -\Delta\tau^2$, where τ is the time as measured by a clock in that rest frame, called 'proper time'.

On the other hand, for $\mathbf{g} > 0$ there is a frame where $\Delta t^2 = 0$, and the events can be considered as 'simultaneous'. In this situation we speak of 'spacelike' separated events. In the special frame of simultaneity the metric equals the standard Euclidean metric $\mathbf{g} = \Delta\vec{x}^{\,2}$.

Finally, the events with $\mathbf{g} = 0$ are said to be 'lightlike' or 'null' separated and correspond to those that can be connected by light rays. Generally speaking, the physical content of the metric is that $\sqrt{\pm\mathbf{g}}$ can be interpreted as either a standard time lapse or a standard spatial distance.

It turns out that the sign of Δt is Lorentz invariant for timelike intervals, making it possible to establish causal relationships between such events. In particular, the velocity of any inertial motion between timelike events, $\vec{v} = \Delta\vec{x}/\Delta t$, is subluminal, $|\vec{v}\,| < 1$. On the other hand, the ordering of times for spacelike events is *not* Lorentz invariant, so that in that case we cannot agree to a Lorentz-invariant notion of causal influence. Effective velocities in that case would be superluminal. Hence, we conclude that only sub-luminal velocities can correspond to causal

[6]A common notation for infinitesimal intervals is $\mathbf{g}(dt, d\vec{x}\,) \equiv ds^2 = -dt^2 + d\vec{x}^{\,2}$.

processes. Given a point P in spacetime, the set of events that can be reached by a causal path (one with subluminal velocity at all times) lie on the future light-cone or in its interior. At the same time, all possible causal paths that can reach P from its past lie either on the past light cone or in its interior.

Minkowskian geometry is a generalization of Euclidean geometry which expresses the fact that the rotations of $\mathbf{R}^3$ generalize into 'rotations' of $\mathbf{R}^{1+3}$ in such a way that physics is independent of 'direction'. In the purely spatial directions we have the usual property of *isotropy* saying that no particular axis or orientation angle in $\mathbf{R}^3$ is special. In the spacetime directions mixing space and time by the boost velocity $\vec{V}$, the symmetry means that physics is independent of $\vec{V}$.

Problem: Twin Paradox

Given two timelike separated events, A and B, prove that any traveler making the journey from A to B will maximize her/his travel time by going at constant speed. Show that the subjective travel time can be made arbitrarily short if we tolerate arbitrary accelerations (Langevin's twin paradox).

Free Particle Dynamics

With the simple elements introduced so far we are ready to study the relativistic dynamics of free particles. Let us consider a causal path γ of a particle of mass m, specified as a function $\vec{x}(t)$ in such a way that the velocity $\vec{v} = d\vec{x}/dt$ is sub-luminal at all times, $|\vec{v}\,| < 1$. Following the general rules of Lagrangian dynamics, we define an action

$$S_{\mathrm{P}} = \int dt\, L_{\mathrm{P}}(\vec{x}(t), \vec{v}(t))$$

with the sole requirement that it gives the correct equations of motion for a free particle, i.e. $\vec{v} = \text{constant}$, and that it be Lorentz invariant. To find the right action, let us see the path as the limit of fine polygonal approximations. Then we can write the action as

$$S_{\mathrm{P}} \approx \sum_{\text{segments}} \Delta S_{\text{segment}}$$

On each straight segment, we can pick a frame with respect to which the particle is at rest. The clock on that comoving frame ticks to 'proper time' τ. Since, the state of the particle in this frame is that of 'rest', we can apply the standard policy from elementary mechanics and defining the Lagrangian as 'kinetic energy minus potential energy', we can write

$$\Delta S_{\text{segment}} = -C\,\Delta\tau_{\text{segment}}\ , \tag{1.56}$$

where C is a constant specifying the 'potential energy' of the particle at rest. Taking now the limit we find the action for a free particle to be

$$S_{\mathrm{P}} = -C \int_{\gamma} d\tau\ . \tag{1.57}$$

From the definition of proper time we have

$$d\tau^2 = dt^2 - d\vec{x}^{\,2} = dt^2 \left(1 - \left(\frac{d\vec{x}}{dt}\right)^2\right) \,,$$

where $(t, \vec{x})$ are coordinates in an external, fixed, but arbitrary Lorentz frame. We thus rewrite (1.56) in an arbitrary Lorentz frame in the form

$$S_{\rm P} = -C \int_\gamma dt\, \sqrt{1 - \vec{v}^{\,2}} \,, \tag{1.58}$$

and the value of the constant C can be obtained by expanding the Lagrangian at low velocities and matching to the Newtonian form: $-C\sqrt{1-\vec{v}^{\,2}} \approx -C + \frac{1}{2}\,C\,\vec{v}^{\,2} \approx -C + \frac{1}{2}\,m\,\vec{v}^{\,2}$, which determines $C = m$ and the final form

$$S_{\rm P} = -m \int_\gamma d\tau = -m \int_\gamma dt \sqrt{1 - \vec{v}^{\,2}} \,, \tag{1.59}$$

of the free particle action in Special Relativity. From here, we get the free equation of motion, $d\vec{v}/dt = 0$ and the relativistic versions of the conserved quantities in the motion, the momentum and the energy

$$\vec{p} = \frac{\partial L_{\rm P}}{\partial \vec{v}} = \frac{m\,\vec{v}}{\sqrt{1-\vec{v}^{\,2}}} \,, \qquad E = \vec{v}\cdot\vec{p} - L_{\rm P} = \frac{m}{\sqrt{1-\vec{v}^{\,2}}} = \sqrt{\vec{p}^{\,2} + m^2} \,. \tag{1.60}$$

In particular, a particle has divergent energy as it approaches the speed of light, giving another, more dynamical incarnation of the limiting character of the speed of light.

Notice that Lorentz invariance of the dynamics leads naturally to the notion of a potential energy at zero velocity, given by the rest mass, or the famous $E = mc^2$ if the conventional units were restored. Writing the last of (1.60) equations in the form $E^2 - \vec{p}^{\,2} = m^2$, we notice that the four-vector $(E, \vec{p})$ behaves under Lorentz transformations just as the coordinates $(t, \vec{x}\,)$.

Chapter 2

Relativistic Particles and Fields

Spacetime as described in the previous chapter, particularly in the geometrical picture of Minkowski, lies at the very foundation of contemporary physics. The only other pillar of comparable importance is Quantum Mechanics. Both foundational structures were laid down in the first quarter of the XX century.

In what regards the spacetime pillar, the basic concepts are the symmetry principles on one side and the dynamical Principle of Locality on the other side. The first one is of a more kinematical nature: one assumes homogeneity of space and time and the relativity principle embodied mathematically in the Lorentz group. Geometrically this is coached by saying that the metric of spacetime, as well as the action principle determining the dynamics, is invariant under the Poincaré group, a semi-direct product of the Lorentz group and the translations in space and time. Anticipating what is to come, the fact that gravitation will generalize the geometry of spacetime *does not* eliminate the importance of this principle, since it will be still true at sufficiently short distances, being part of the basic assumption of *smoothness* of spacetime.

The Principle of Locality is incorporated by the requirement that no action-at-a-distance exists, at least on scales accessible to experiment. In a relativistic theory this is equivalent to having a maximal speed of propagation of signals. Locality takes slightly different forms depending on the model of matter and forces that we adopt. For particles, we specify that their interactions occur at points in spacetime, either with other particles or with fields. Fields are the very embodiment of the notion of *local* degrees of freedom, since they are a mathematical abstraction from 'particles attached to points in space'. In their case, the Principle of Locality means that interactions of fields are specified at infinitesimally close points of spacetime or, in other words, the Lagrangian density is specified as a function at each point in spacetime, containing at most a finite number of derivatives of the fields. The expansion in derivatives is actually a systematic expansion, called *Effective Field Theory*, in which short-distance physics gives power-like corrections to long-distance physics. This notion of Effective Field Theory extends perfectly well into the quantum domain (Quantum Field Theory) and remains one of the backbones of our current description of nature in the XXI century.

In this chapter we will construct the actions specifying the dynamics in accordance to these principles, for both particles and fields. The particle-field dichotomy is crucial in classical physics. It was certainly a requirement of experiment at the turn of the XIX century. On the other hand, the existence of *both* particles and fields became a source of notorious mathematical troubles, caused by the fact that pointlike sources necessarily introduce singularities in field configurations. The classic example was the divergent electromagnetic self-energy of a particle due to the singularity of the Coulomb field. Despite attempts at unification in the framework of non-linear classical field theories, this basic dichotomy was not transcended until the arrival of Quantum Field Theory, and in some ways this conundrum still plagues the foundations of physics today.[1] These developments are, however, beyond the scope of these notes. Here we will introduce the Lorentz-covariant formalism to deal with particles interacting through the simplest field theories, guided by pedagogical simplicity and relevance to the actual description of nature.

2.1 Lorentz Covariance

The procedure outlined at the end of the previous chapter for free particles can be repeated for more complicated systems. In postulating dynamical principles based on Lorentz symmetry, we must always start by guessing a Lorentz-invariant action. Further constraints are brought by the principles of locality and the matching to known limits, such as the Newtonian approximation.

2.1.1 The Lorentz Group

In order to proceed with this program, it is useful to introduce some powerful notation to help in the search of Lorentz-invariant combinations of physical quantities.

The Lorentz group $O(1,3)$ was defined as the set of linear maps of $\mathbf{R}^{1+3}$ onto itself with the condition of keeping the Minkowski interval invariant. Working in differential notation

$$ds^2 = -d\tau^2 = -dt^2 + d\vec{x}^{\,2} \tag{2.1}$$

is left invariant. In the notation $(t, \vec{x}\,) = (x^0, \vec{x}\,) = (x^a)$, with $a = 0, 1, 2, 3$, we have

$$x^a \to x'^{\,a} = \sum_b L^a{}_b\, x^b \equiv L^a{}_b\, x^b \ , \tag{2.2}$$

where we have used Einstein's convention of implicit summation of repeated indices in top-down pairs. [2] Then, the Minkowski metric η_{ab},

$$ds^2 = \eta_{ab}\, dx^a\, dx^b \ , \qquad (\eta_{ab}) = \text{diag}\,(-1,1,1,1) \tag{2.3}$$

is left invariant by the Lorentz matrices,

$$\eta_{ab}\, L^a{}_c\, L^b{}_d = \eta_{cd} \tag{2.4}$$

or, in matrix notation $\eta = L^t\, \eta\, L$, where L^t stands for the transpose of the L matrix. Inverting this relation and noting that $\eta = \eta^{-1}$ we have $\eta = L^{-1}\eta(L^{-1})^t$ or, in index notation,

$$\eta_{ab}\, (L^{-1})^a{}_c\, (L^{-1})^b{}_d = \eta_{cd} \ . \tag{2.5}$$

It is common practice to write

$$(L^{-1})^a{}_b = L_b{}^a \ ,$$

[1] Under the modern name of *Hierarchy Problem*.

[2] In the following, we will use Einstein's summation convention for any set of four-dimensional indices, keeping the explicit sum notation for vector-like three-dimensional indices.

so that a switch in the ordering of upper and lower indices is tantamount to inversion. This is the Lorentzian version of the matrix equation $L^t = L^{-1}$, which is familiar for standard orthogonal matrices.

The transformations $x \to x' = L\,x$ can be viewed as a change of coordinates, in what is called the *passive* view. There is also an *active* view, in which we interpret $x \to x'$ as a motion in the spacetime, a map between different points of Minkowski spacetime, expressed in the *same* coordinate system. In this active view, the invariance of the metric (2.5) expresses the fact that any geometrical statement based on lengths or angles in actually invariant under this motion. This is usually called an *isometry* in the mathematical literature.

The full group of isometries of Minkowski spacetime is known as the Poincaré group, a semi-direct product of the Lorentz group and the spacetime translations. More explicitly,

$$x^b \longrightarrow x'^b = a^b + L^b{}_c\, x^c\;, \tag{2.6}$$

a ten-parameter group of transformations which becomes the Galilean group at velocities small compared to the speed of light.

Setting $a = b = 0$ in (2.4) we have

$$-1 = \eta_{00} = L^a{}_0\, L^b{}_0\, \eta_{ab} = -(L^0{}_0)^2 + \sum_i (L^i{}_0)^2\;, \tag{2.7}$$

so that $(L^0{}_0)^2 \geq 1$ and one cannot change continuously the sign of $L^0{}_0$. Similarly, from $\det\eta = \det\eta\;(\det L)^2$ we find that $\det L = \pm 1$ and we cannot smoothly change the sign of the determinant either. This divides the set of L matrices into four connected components according to the signs of $L^0{}_0$ and $\det(L)$. The subgroup of Lorentz transformations with both $L^0{}_0$ and $\det(L)$ positive are continuously connected to the identity and denoted $SO_+(1,3)$, and we will sometimes refer to them as *small* Lorentz transformations. We can further characterize the small transformations by expanding near the identity. Let $L^a{}_b = \delta^a{}_b + \theta^a{}_b + \dots$ with $|\theta^a{}_b| \ll 1$. Then, the isometry condition (2.5) is equivalent to requiring the matrix $\eta_{ab}\,\theta^b{}_c$ to be antisymmetric in the indices (a,c).

We can reach the other components of $O(1,3)$ by the action of $\mathcal{P}$ and $\mathcal{T}$ on the small transformations, where $\mathcal{P}$ stands for parity, $(t,\vec{x}) \to (t,-\vec{x})$, and $\mathcal{T}$ is time reversal, $(t,\vec{x}) \to (-t,\vec{x})$. In what follows, we will tacitly refer to *the* Lorentz transformations as the *small* ones, and state it explicitly whenever we are considering parity and time-reversal transformations. A good reason to restrict the Lorentz group to the small component is that, actually, neither $\mathcal{P}$ nor $\mathcal{T}$ are fundamental symmetries of nature, being violated in the quantum theory of elementary particles.

Upon analytic continuation $(t,x) \to (it,x)$ the Lorentzian metric becomes Euclidean $ds^2 = d(it)^2 + d\vec{x}^{\,2}$. The resulting connected group is $SO(4)$, which is isomorphic to $SU(2)_L \times SU(2)_R$. The operation of analytic continuation, being continuous, should not change the discrete structure of group representations. This means that all representations of the Lorentz group can be constructed as the pairs (j_L, j_R), with $j_{L,R}$ the standard spins of $SU(2)$ irreducible representations. This form is very useful to describe spinor degrees of freedom, although most of the applications in classical physics can be exhausted by the so-called *tensor* representations, which appear in the tensor products of vector representations.

We thus conclude that a relativistic model is given by a Poincaré-invariant Lagrangian defined over Minkowski spacetime, i.e. the pair $(\mathbf{R}^{3+1}, \eta)$, with dynamical variables in some representations of the small Lorentz group $SO_+(1,3)$.

2.1.2 Tensor Representations

The defining (vector) representation of the Lorentz group corresponds to any quantity that transforms as the coordinates:

$$U^a \to L^a{}_b U^b \,. \tag{2.8}$$

Any such quantity is called a 'contravariant' vector. Given two contravariant vectors, U, V, we can form a two-index object $T^{ab} = U^a V^b$ by 'tensor' product. It transforms as

$$T^{ab} \to L^a{}_c L^b{}_c T^{cd} \,.$$

Other combinations with the same transformation law are linear combinations of tensor products, $U^a V^b + X^a Y^b + \ldots$. Hence, any set of quantities transforming as above may be defined as a second-rank contravariant tensor, and the generalization to arbitrary rank is obvious.

A different transformation law is followed by the differential operators

$$\partial_a \equiv \frac{\partial}{\partial x^a} \,.$$

If we view $x'^{\,a}$ as functions of x^a, we have

$$\frac{\partial x'^{\,a}}{\partial x^b} = L^a{}_b \,, \quad \text{and the inverse} \quad \frac{\partial x^a}{\partial x'^{\,b}} = (L^{-1})^a{}_b \,. \tag{2.9}$$

We can now see why it was a good idea to introduce the notation

$$L_b{}^a \equiv (L^{-1})^a{}_b \,. \tag{2.10}$$

This notational trick dovetails nicely with the summation convention since, using the chain rule we may write

$$\partial'{}_a = \frac{\partial x^b}{\partial x'^{\,a}} \partial_b = L_a{}^b \partial_b \,. \tag{2.11}$$

Such transformation law, with the inverse Lorentz matrix L^{-1}, is conventionally called 'covariant'. General covariant tensors can be defined following the previous steps of their contravariant cousins, by tensor product and addition. For example, a rank-two covariant tensor transforms as

$$A_{ab} \to L_a{}^c L_b{}^d A_{cd} \tag{2.12}$$

and so on. Mixed tensors with p contravariant and q covariant indices can be constructed in straightforward fashion. They can be given an intrinsic definition by regarding the collections of indexed numbers as components in an abstract basis formed by the contravariant differentials dx^a and the covariant derivative operators ∂_b, i.e. we require the formal expression

$$\mathbf{T} = \sum_{a_1,\ldots,b_1,\ldots} T^{a_1\ldots a_p}{}_{b_1\ldots b_q}(x)\, dx^{b_1} \otimes \cdots \otimes dx^{b_q} \otimes \partial_{a_1} \otimes \cdots \otimes \partial_{a_p} \tag{2.13}$$

to be invariant under a change of basis

$$x^a \to x'^{\,a} = L^a{}_b x^b \,, \qquad dx^a \to L^a{}_b dx^b \,, \qquad \partial_a \to L_a{}^b \partial_b \,. \tag{2.14}$$

The transformation laws of the components follow suit:

$$T^{a_1\cdots}{}_{b_1\cdots}(x) \longrightarrow T'^{\,a_1\cdots}{}_{b_1\cdots}(x') = L^{a_1}{}_{c_1} \cdots L_{b_1}{}^{d_1} \cdots T^{c_1\cdots}{}_{d_1\cdots}(x) \,, \tag{2.15}$$

where we have written the functional dependence on the coordinates to expose how the functions transform, in the case the tensor quantities are defined locally point by point. For instance, a scalar function with no indices transforms as $\phi'(x') = \phi(x)$. Therefore, the value of the function is invariant but the function itself does transform.

This construction also shows that any pair of indices that are identified and summed up with Einstein's convention become irrelevant as far as the transformation rules is concerned. That is, the 'contracted' product $U^a V_a$ of a contravariant and a covariant vector is a Lorentz invariant, whereas the contraction $V_a T^{ab}$ transforms as a contravariant vector, according to the free index b. We can also construct invariant differential operators, such as

$$\partial^2 \equiv \partial_a \partial^a = \eta^{ab} \partial_a \partial_b = -\partial_t^2 + \vec{\partial}^{\,2} \ , \tag{2.16}$$

usually called Laplacian or d'Alembertian.

Invariant Tensors

The task of constructing Lorentz invariants is greatly aided by the definition of *invariant* tensors. These are tensors that retain their value under Lorentz transformations. The most obvious case is the Kronecker delta δ^a_b with upper-and-lower indices. It transforms as an invariant mixed tensor of rank two:

$$\delta^a_b = L^a{}_c \, L_b{}^d \, \delta^c_d \ . \tag{2.17}$$

Another important invariant tensor is the metric itself. According to (2.5) we have

$$\eta_{ab} L_c{}^a \, L_d{}^b = \eta_{cd} \ , \tag{2.18}$$

as corresponds to the transformation law of a rank-two covariant tensor. It is convenient to define the inverse metric (equal to itself) with upper indices

$$(\eta^{ab}) = \mathrm{diag}\,(-1, 1, 1, 1) \ . \tag{2.19}$$

Hence, $\eta^{ab}\eta_{bc} = \delta^a_c$, and given the tensor character of η_{ab} and δ^a_b, it follows that η^{ab} is an invariant contravariant tensor of rank two.

The invariant tensors constructed from the metric are especially interesting because they can be used to make covariant tensors out of contravariant tensors and viceversa. By the rules already stated it is very easy to check that, given a contravariant tensor, the contraction

$$\eta_{ab} \, T^{bc \cdots} = \overline{T}_a{}^{c \cdots} \tag{2.20}$$

transforms as a mixed tensor with the indices as indicated. We say that the metric can be used to 'lower' an index. It is also common to simplify the notation and use the same letter to denote the new tensor, relying on the position of the indices to distinguish it from the previous fully contravariant one, i.e. we usually write $\overline{T}_a{}^{c \cdots} = T_a{}^{c \cdots}$. Analogously, we can use the upper-index version to 'raise' covariant indices into contravariant ones. When playing the game of raising and lowering indices, the reader should be aware of the relative positions of the indices, for instance $T_a{}^b = \eta_{ac} \, T^{cb} \neq T^a{}_b = \eta_{bc} T^{ab}$ unless T^{ab} happens to be symmetric.

A final interesting object is the Levi–Civita tensor. Let us define the *permutation symbol* as the completely antisymmetric object $[abcd]$ with $[0123] = 1$. In other words, it is given by the even/odd parity of the permutation $\pi : (0, 1, 2, 3) \to (a = \pi(0),\, b = \pi(1),\, c = \pi(2),\, d = \pi(3))$:

$$[0123] = 1 \ , \qquad [\pi(0)\pi(1)\pi(2)\pi(3)] = (-1)^{\mathrm{parity}(\pi)} \ . \tag{2.21}$$

The Levi–Civita tensor is defined as a covariant object with the same numerical value as the permutation symbol [3]

$$\epsilon_{abcd} = [abcd] \ . \tag{2.22}$$

The identity

$$\epsilon_{abcd}\, L^a{}_e\, L^b{}_f\, L^c{}_g\, L^d{}_h = \det(L)\, \epsilon_{efgh} \tag{2.23}$$

implies that the Levi–Civita symbol is a true invariant tensor for small Lorentz transformations (more generally, all $SO(1,3)$ Lorentz transformations with $\det(L) = 1$), and flips its sign under parity or time reversal. The familiar Levi–Civita 3-tensor of vector algebra is

$$\epsilon_{ijk} = \epsilon_{0ijk} \ .$$

We can also define associated objects by raising/lowering indices with the metric η_{ab}. In particular, the completely contravariant version ϵ^{abcd} is also completely antisymmetric with

$$\epsilon^{0123} = \eta^{0a}\, \eta^{1b}\, \eta^{2c}\, \eta^{3d}\, \epsilon_{abcd} = \det{}^{-1}(\eta)\, \epsilon_{0123} = -1 \ ,$$

so that it differs from the permutation symbol by a crucial minus sign coming from the Lorentzian signature of spacetime:

$$\epsilon^{abcd} = -[abcd] \ . \tag{2.24}$$

A useful formula for contractions of arbitrary d-dimensional Levi–Civita tensors is

$$\epsilon^{c_1\cdots c_n a_{n+1}\cdots a_d}\, \epsilon_{c_1\cdots c_n b_{n+1}\cdots b_d} = (-1)^{\rm sig}\, n!\, \det\left(\delta^{a_j}{}_{b_k}\right) = (-1)^{\rm sig}\, n!(d-n)!\, \delta^{a_{n+1}}_{[b_{n+1}}\cdots \delta^{a_d}_{b_d]} \ , \tag{2.25}$$

where $(-1)^{\rm sig}$ is the signature, $+1$ for Euclidean metrics and -1 for Lorentzian ones, and the square brackets stand for the complete anti-symmetrization of indices. These give the Lorentzian generalization of the useful identities of vector calculus

$$\sum_k \epsilon_{ijk}\, \epsilon_{mnk} = \delta_{im}\, \delta_{jn} - \delta_{in}\, \delta_{jm} \ , \qquad \sum_{jk} \epsilon_{ijk}\, \epsilon_{mjk} = 2\delta_{im} \ , \qquad \sum_{ijk} \epsilon_{ijk}\, \epsilon_{ijk} = 6 \ . \tag{2.26}$$

Having one more invariant tensor at our disposal we can use it to construct new kinds of Lorentz invariants. Here it is convenient to work in arbitrary dimensions to see the emergence of combinatorial patterns.

Given a rank-p antisymmetric tensor $A_{[p]}$ with components $A_{a_1\cdots a_p} = A_{[a_1\cdots a_p]}$ (also known as a p-form) we can construct a so-called Poincaré dual tensor $\widetilde{A}^{a_{p+1}\cdots a_d}$ which is completely antisymmetric in $d-p$ indices and has components

$$\widetilde{A}^{a_{p+1}\cdots a_d} = \frac{1}{p!}\, \epsilon^{a_1\cdots a_p a_{p+1}\cdots a_d}\, A_{a_1\cdots a_d} \ . \tag{2.27}$$

It contains the same information as the original tensor (this is similar in spirit as the raising and lowering of indices). Poincaré duality is involutive up to a sign:

$$\widetilde{\widetilde{A}}_{[p]} = (-1)^{\rm sig}(-1)^{p(d-p)} A_{[p]} \ . \tag{2.28}$$

[3] The reader should be aware of the fact that sign conventions for Levi–Civita tensors differ significantly across the literature.

If $p = d/2$ we can now write an invariant contraction between A and $\widetilde{A}$. More generally, given a p-form A and a different $(d-p)$-form A' we can construct the Lorentz invariant

$$\frac{1}{p!(d-p)!}\,\epsilon^{a_1\cdots a_p a_{p+1}\cdots a_d}\,A_{a_1\cdots a_p}\,A'_{a_{p+1}\cdots a_d}\ . \tag{2.29}$$

When including Levi–Civita tensors in Lagrangians, we must remember that every time it appears we introduce a violation of parity and time reversal.

2.1.3 Integral Invariants

So far we have discussed invariants that can be defined locally, point by point in spacetime. This is the main requirement to construct a local action. There are, however, other invariants with a non-local character and with important physical interpretation. Two immediate examples coming to mind are the circulation of the vector potential $\oint \vec{A}\,d\vec{x}$ or the electric or magnetic flux $\oint \vec{E}\,d\vec{S}$ in electrodynamics. These are rotation invariants which are defined for extended lines and surfaces respectively. One important property they satisfy is that they depend on the particular submanifold of interest, but *do not* depend on the concrete parametrization we may use, i.e. they are intrinsically associated to the geometrical set. The prototypical example is the vector circulation, which admits an immediate generalization to a Lorentz-invariant line integral:

$$I[\gamma] = \int_\gamma A \equiv \int_\gamma A_a\,dx^a \equiv \int_\gamma A_a(x(\sigma))\,\frac{dx^a}{d\sigma}\,d\sigma\ , \tag{2.30}$$

which we have written in increasing degree of detail, to show that the integral is not only Lorentz-invariant but also invariant with respect to choices of path parameter σ. On the other hand, this is slightly non-trivial because, trying an obvious generalization to a two-tensor integrated over a two-surface $x^a(\sigma^1,\sigma^2)$

$$\int_{\Sigma^2} \tfrac{1}{2}\,T_{ab}\,dx^a\,dx^b = \int_{\Sigma^2} \tfrac{1}{2}\,T_{ab}(x(\sigma^1,\sigma^2))\sum_{m,n=1}^{2}\frac{\partial x^a}{\partial\sigma^n}\frac{\partial x^b}{d\sigma^m}\,d\sigma^n\,d\sigma^m\ , \tag{2.31}$$

the terms containing $(d\sigma^1)^2$ and $(d\sigma^2)^2$ do not correspond to a correct two-dimensional integration measure. Getting rid of these terms by restricting T_{ab} to be antisymmetric produces a standard reparametrization-invariant two-dimensional integral, where $d\sigma^n\,d\sigma^m$ can be replaced with $[n,m]\,d\sigma^1\,d\sigma^2$, in terms of the permutation symbol in two indices. This pattern generalizes and any covariant antisymmetric tensor, a p-form, can be integrated intrinsically over a p-surface Σ^p immersed in spacetime by functions $x^a(\sigma^1,\ldots,\sigma^p)$. The integral invariant is

$$I[\Sigma^p] = \int_{\Sigma^p} A_{[p]} = \int_{\Sigma^p}\frac{1}{p!}\,A_{a_1\cdots a_p}\,d\sigma^{a_1\cdots a_p}\ , \tag{2.32}$$

where

$$d\sigma^{a_1\cdots a_p} = [n_1\cdots n_p]\,\frac{\partial x^{a_1}}{\partial\sigma^{n_1}}\cdots\frac{\partial x^{a_p}}{\partial\sigma^{n_p}}\,d^p\sigma\ , \tag{2.33}$$

and we have used the permutation symbol in p indices $[n_1\cdots n_p]$ and $d^p\sigma = d\sigma^1\,d\sigma^2\cdots d\sigma^p$. In this form we recognize (2.33) as the volume element on the p-hyperplane tangent to Σ^p, since the vectors $E^a_r = (\partial x^a/\partial\sigma^r)\Delta\sigma^r$ are tangent to Σ^p and (2.33) computes their determinant, i.e.

the volume of a small polytope having these vectors on its edges. Using Poincaré duality (2.27) we can also write (2.32) as

$$I[\Sigma^p] = (-1)^{\rm sig} \int_{\Sigma^p} \frac{1}{(d-p)!} \widetilde{A}_{a_{p+1}\cdots a_d}\, \widetilde{d\sigma}^{a_{p+1}\cdots a_d} \ . \tag{2.34}$$

In this form, the tensor being integrated has indices transverse to the manifold Σ^p, so that we are representing a generalized flux. For $p = d-1$ this formula takes the familiar form of a vector flux over a codimension-one hypersurface,

$$I[\Sigma^{d-1}] = (-1)^{\rm sig} \int_{\Sigma^{d-1}} \widetilde{A}_a \, dS^a \tag{2.35}$$

with $\widetilde{d\sigma}^{\,a} = dS^a$ the oriented hypersurface element normal to $\Sigma^{(d-1)}$.

A fundamental object in p-form theory is the *exterior derivative*, which takes a p-form to a $(p+1)$-form by an antisymmetric derivation

$$\Big(dA_{[p]}\Big)_{a_0\, a_1\, \cdots\, a_p} \equiv (p+1)\, \partial_{[a_0}\, A_{a_1\cdots a_p]} \ . \tag{2.36}$$

It can only act once, since $d^2 = 0$ identically. One of its *raisons d'etre* is the generalized Stoke's theorem: if $\Sigma^p = \partial {\rm V}^{p+1}$ is a boundary, then

$$\int_{{\rm V}^{p+1}} dA_{[p]} = \int_{\Sigma^p} A_{[p]} \ . \tag{2.37}$$

A mundane example of all these definitions and theorems is the magnetic flux formula through a disc in ordinary space, ${\rm D} \in \mathbf{R}^3$:

$$\int_{\rm D} d\vec{S}\cdot \vec{B} = \int_{\rm D} \tfrac{1}{2}\, F_{ij}\, dx^i\, dx^j = \oint_{\partial {\rm D}} d\vec{x}\cdot \vec{A} \ , \tag{2.38}$$

where one uses (2.35) in the first equality and Stokes' theorem (2.37) in the second equality. We also assume the standard notation $F_{ij} = \partial_i A_j - \partial_j A_i = \sum_k \epsilon_{ijk} B_k$.

The Poincaré dual of the exterior derivative is the *divergence*, which we can denote by $d^\dagger$ and maps p-forms to $(p-1)$-forms by contracting one index with a derivative:

$$\Big(d^\dagger\, A_{[p]}\Big)_{a_2\cdots a_p} \equiv -\partial^{a_1}\, A_{a_1 a_2 \cdots a_p} \ . \tag{2.39}$$

The extra sign in this expression and the dagger notation is chosen so that $d^\dagger$ indeed acts as the adjoint of d, in the sense $(A, dA') = (d^\dagger A, A')$, with respect to the inner product

$$\Big(A_{[p]}, A'_{[p]}\Big) \equiv \frac{1}{p!} \int d^d x\ A^{a_1\cdots a_p}\, A'_{a_1\cdots a_p} \ . \tag{2.40}$$

With this definition, one finds that

$$\widetilde{dA}_{[p]} = (-1)^p\, d^\dagger\, \widetilde{A}_{[d-p]} \ ,$$

which allows us to derive analogous versions of Stoke's theorem for divergences of antisymmetric tensors, otherwise known as Gauss' integration theorems. The simplest and most important for the purposes of these notes is the volume integral of divergence of a current, which is usually written in the form

$$\int_{\mathrm{V}^{1+3}} \partial_a J^a = \int_{\partial \mathrm{V}^{1+3}} J_a n^a \ , \tag{2.41}$$

where V^{1+3} is the interior of a tubelike surface of finite extent in time, namely its boundary $\partial\mathrm{V}^{1+3}$ has cylinder shape with future and past caps, $\partial\mathrm{V}^{1+3} = \mathrm{V}^3_+ \cup \mathrm{V}^3_- \cup \mathrm{B}^{1+2}$. The caps $\mathrm{V}^3_\pm$ are spacelike (normal vector is timelike) and the tube B^{1+2} has everywhere spacelike normal vector. The formula (2.41) is slightly subtle in that the normal vector n^a is outward pointing when it is spacelike ($n_a n^a = 1$) and *inward* pointing when it is timelike ($n_a n^a = -1$).

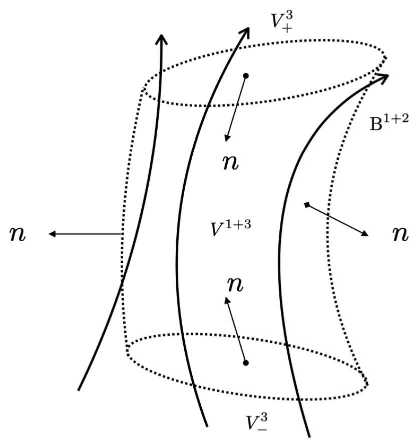

Figure 2.1: *Any increment of a locally conserved charge registered between* $\mathrm{V}^3_\pm$ *must be due to net inward flow through* B^{1+2}*. Notice the conventions for normal unit vectors.*

Then, Gauss' theorem applied to $\partial_a J^a = 0$ gives

$$Q(\mathrm{V}^3_+) - Q(\mathrm{V}^3_-) = -\mathrm{Flux}(\mathrm{B}^{1+2}) \ , \tag{2.42}$$

where $Q(\mathrm{V}^3) = -\int_{\mathrm{V}^3} J_a \tau^a$, with τ^a the future-directed normal to V^3, and $\mathrm{Flux}(\mathrm{B}^{1+2})$ is the outward flux of J^a through the cylindrical-shaped boundary.[4] When B^{1+2} is formally taken to infinity, so that $\mathrm{V}^3 = \Sigma^3$ is a spacelike, codimension-one submanifold without boundary, we learn that the charge is independent of the choice of Σ^3 (assuming as usual trivial field configurations at infinity). On the other hand, when the $\mathrm{V}^3_\pm$ collapse into one another we obtain the form (1.27).

2.1.4 Spinors

In known examples in nature, spinor representations are associated to fermionic fields. These are fundamentally more 'quantum' than their bosonic counterparts. By the spin-statistics theorem of Quantum Field Theory, we know that the bosonic fields come in representations of integer

[4]The easiest way of checking the (admittedly tricky) sign conventions is to write down (2.41) directly for a straight hypercube.

Lorentz spin, i.e of tensor type. For this reason we will not dwell on spinor representations in these notes, except for a quick review of standard notation.

The simplest spinor representation is carried by the so-called Dirac spinor, ψ, which transforms in the representation $(1/2, 0) \oplus (0, 1/2)$, of complex dimension four. To construct the Lorentz action in this representation, one usually starts from the four-dimensional Dirac matrices γ^a, satisfying the Clifford algebra,

$$\{\gamma^a, \gamma^b\} = -2\eta^{ab} \ ,$$

so that

$$\sigma^{ab} = \frac{i}{4}[\gamma^a, \gamma^b]$$

generate the $SO(1,3)$ Lie algebra. Then, the spinor representation is given by $\psi \to \mathcal{D}(L)\,\psi$, where $\mathcal{D}(L)$ is the four-dimensional matrix

$$\mathcal{D}(L) = \exp\left(\frac{i}{2}\sum_{a,b}\theta_{ab}\,\sigma^{ab}\right) \ ,$$

and the antisymmetric matrix of parameters θ_{ab} is defined in terms of the Lorentz transformation matrix L, by the relation $L = \exp(\theta')$, where $\theta'\ {}^a{}_b = \eta^{ac}\theta_{cb}$. The irreducible components are reached acting with the projectors $P_\pm = \frac{1}{2}(1 \pm \gamma_5)$, with $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. The resulting two-component spinors $\psi_\pm = P_\pm\psi$ are called left- and right-handed Weyl spinors. The two projectors are mapped into one another by parity. Hence, insisting on parity as a good symmetry forces the use of the four-component Dirac spinors.

2.2 Relativistic Particles

In general, a particle in classical physics is any system whose internal structure can be ignored on the scales of interest. For instance, planets are particles if we are interested in the long-range properties of their orbits. Therefore, the minimal description of a particle is given by its path in spacetime, a curve γ embedded in spacetime by functions $x^a(\sigma)$, parametrized by a real variable σ. It is often useful to use a proper-time parametrization $x^a(\tau)$, but this is only possible for massive particles. The only global constraint on the path γ is that it be *causal*, in the sense that the velocity is sub-luminal at all times for massive particles, and exactly at the speed of light when they are massless. In the following sections we describe the basic Lagrangian formalism for particles and their interactions, including their coupling to fields.

2.2.1 The Free

We can now recast the particle dynamics in a more covariant language. Let $x^a(\tau)$ be the trajectory of a particle in proper-time parametrization. The four-velocity defined as

$$u^a = \frac{dx^a}{d\tau}$$

is a contravariant four-vector under the Lorentz group, and the free equation of motion may be written as

$$\frac{d^2x^a}{d\tau^2} = \frac{du^a}{d\tau} = 0 \,,$$

solved by straight lines $x^a(\tau) = x_0^a + u^a\,\tau$, determined by initial conditions x_0^a and u^a.

Conserved energy and momentum can also be assembled into a contravariant four-vector, called the four-momentum $p^a = mu^a$, where $(p^0, \vec{p}) = (E, \vec{p})$. The so-called 'dispersion' or 'mass-shell' relation $E^2 - \vec{p}^{\,2} = m^2$ may be written in condensed form as $p^2 + m^2 = 0$. [5] Equivalently, the four-velocity is constrained to satisfy $u^2 = u_a u^a = -1$.

The free particle action over a trajectory γ can be written in the following variety of equivalent forms:

$$S_\mathrm{P} = -m\int_\gamma d\tau = -m\int_\gamma dt\frac{d\tau}{dt} = -m\int_\gamma dt\sqrt{1-\vec{v}^{\,2}} = -m\int_\gamma dt\sqrt{-\eta_{ab}\frac{dx^a}{dt}\frac{dx^b}{dt}}\,. \tag{2.43}$$

In particular, the second form shows that the coordinate time t is just an integration parameter which might be redefined at will. In many situations, the freedom to reparametrize the time variable does offer some rewards. So we consider the action in arbitrary parametrization

$$S_\mathrm{P} = -m\int d\tau = -m\int d\sigma\,\frac{d\tau}{d\sigma} = -m\int d\sigma\sqrt{-\eta_{ab}\frac{dx^a}{d\sigma}\frac{dx^b}{d\sigma}}\,. \tag{2.44}$$

There is a useful trick that simplifies the action by linearizing it and, at the same time, permitting a smooth massless limit, a convenient fact for the discussion of light propagation.

Let us consider the action

$$S_\mathrm{P}[x^a, e] = \tfrac{1}{2}\int d\sigma\left(\frac{1}{e(\sigma)}\eta_{ab}\frac{dx^a}{d\sigma}\frac{dx^b}{d\sigma} - m^2\,e(\sigma)\right)\,, \tag{2.45}$$

[5] Experimental measurements of dispersion relations, $p^2 + m^2 = 0$, provide the most stringent tests of Lorentz symmetry, down to the 10^{-20} level.

where the function $e(\sigma)$ is considered as an independent variable. Its equation of motion $\delta S_P/\delta e = 0$ leads to

$$\frac{dx^a}{d\sigma}\frac{dx_a}{d\sigma} = -m^2\, e(\sigma)^2\ . \tag{2.46}$$

Form this equation we can solve for $e(\sigma)$ and substitute back into (2.45), obtaining the original action (2.44). In this way, we show that both actions are physically equivalent.

The action (2.45) is invariant under reparametrizations of the path (just as (2.44) was) provided we let $e(\sigma)$ transform in such a way that $e(\sigma)d\sigma$ is left invariant. Worldline parameters for which $e(\sigma) =$ constant are called *affine*. Choosing $e(\sigma) = 1$ and taking the limit $m \to 0$ produces the equations of motion of a massless particle,

$$\frac{d^2x^a}{d\sigma^2} = 0\ , \qquad \frac{dx^a}{d\sigma}\frac{dx_a}{d\sigma} = 0\ .$$

The first equation says that the trajectory is 'straight' in the σ parametrization, and the second equation says that it lies on the light cone.

For massive particles, it is convenient to chose the proper time as the path parameter, $\sigma = \tau$, so that (2.46) sets $e(\tau) = 1/m$. This means that we can simply use the linearized version of the action

$$S_{\rm P} = \tfrac{1}{2}\, m \int d\tau\, \left(\eta_{ab}\frac{dx^a}{d\tau}\frac{dx^b}{d\tau} - 1\right)\ , \tag{2.47}$$

which is actually the most *naive* generalization of the Newtonian action. The resulting equations of motion, $d^2x^a/d\tau^2 = 0$, must be supplemented by the field equation of $e(\tau)$, which was lost when setting it to a constant in the action. This equation of motion is simply the mass-shell condition $u^2 = -1$, or equivalently $p^2 + m^2 = 0$. The energy of the particle in a 'laboratory' frame is $E = p^0 = -\eta_{00}\, p^0\, u^0$, so that the Lorentz-invariant expression for the energy measured by an observer at four-velocity u^a is

$$E_u(p) = -\eta_{ab}\, p^a u^b = -p_a u^a\ . \tag{2.48}$$

2.2.2 The Interacting

Introducing interactions in a relativistic theory of particles is a rather non-trivial issue, given the fact that a limit to the speed of information transfer is built into the very fabric of Special Relativity. This means that standard generalizations of many-particle interaction Lagrangians from Newtonian mechanics are bound to be problematic, since specifying potentials involving particles located at different points in spacetime is a sort of 'action at a distance'.

There are two basic solutions to this problem. The first solution is to take a phenomenological attitude and consider *contact* interactions in spacetime. In this view, particles are free except for sharply defined collisions at spacetime points, whose physical effect is to change the energy-momentum of each particle $p^a_{\rm in} \to p^a_{\rm out}$ at the collision point, with the constraint of energy-momentum conservation

$$\sum_{\text{colliding particles}} p^a_{\rm in} = \sum_{\text{colliding particles}} p^a_{\rm out}\ . \tag{2.49}$$

A clear shortcoming of this prescription is the arbitrariness of the detailed interaction law, since (2.49) does not determine the precise values of outgoing momenta, given the incoming

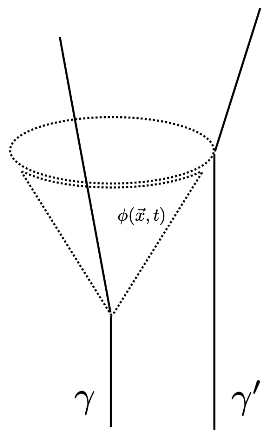

Figure 2.2: *Interaction of particles through fields. Particle on path γ has a kink (change in momentum) which locally produces disturbances on the field $\phi(\vec{x},t)$ propagating inside the future-directed spacetime cone of bounded velocities, and intersecting path γ' of the second particle at a later time, thereby 'transmitting' part of the momentum kick that was carried by the field in the interim. All interactions in the picture are local.*

ones.[6] A rather more satisfactory solution to the interaction problem is to follow the blueprint of the Maxwell–Lorentz theory and prescribe the particle interactions to be *mediated* by fields. Fields are assumed to satisfy *local* relativistic equations, ensuring that the solutions are waves traveling at most at the speed of light, thus automatically incorporating the required retardation effects between particles. Since the interaction of a fixed background field configuration with a given particle must be locally specified at the particle trajectory, we can make contact with the formal Lagrangian framework by describing such interaction with a world-line relativistic potential of the form

$$S_{\mathrm{P}} = -m\int d\tau - \int d\tau\, \mathcal{V}\left(x(\tau), \dot{x}(\tau)\right) \ , \tag{2.50}$$

where $\dot{x}^a = dx^a/d\tau$ and we have allowed for the potential to depend on the particle's four velocity as well as the particle coordinate. In order to find the equations of motion implied by this action we must be careful to use a general parametrization of the path, i.e. we write (2.50) in the form

$$S_{\mathrm{P}} = -m\int d\sigma \frac{d\tau}{d\sigma}\left[1 + \mathcal{V}\left(x(\sigma), \frac{d\sigma}{d\tau}\frac{dx}{d\sigma}\right)\right] \ , \tag{2.51}$$

with $d\tau/d\sigma = \sqrt{-\eta_{ab}\frac{dx^a}{d\sigma}\frac{dx^b}{d\sigma}}$.

Varying this action under fluctuations of the trajectory function $x^a(\sigma) \to x^a(\sigma) + \delta x^a(\sigma)$ and requiring it to be stationary we obtain the equations of motion in the form of a relativistic

[6] It is remarkable that this picture reemerges in Quantum Field Theory, where the indefiniteness of the interaction is just an embodiment of the quantum mechanical uncertainty, and the contact interaction described here is literally the so-called *vertex* in a Feynman diagram.

Newton's law

$$m\frac{d^2x^a}{d\tau^2} = F^a \ . \tag{2.52}$$

where the generalized force four-vector F^a depends on the potential $\mathcal{V}$ in a very complicated and non-linear way. Even for the case of a potential without dependence on four-velocities, $\mathcal{V}_0 = q_0\phi(x)$, we find the non-trivial velocity-dependent force

$$F_a = -q_0\, \partial_a\Phi - q_0\, \dot{x}_a\, \dot{x}^b\, \partial_b\Phi \ , \tag{2.53}$$

with

$$\Phi = \frac{m}{q_0}\, \log\left(1 + \frac{q_0}{m}\phi\right) \ . \tag{2.54}$$

The result is *very different* from the naive guess for the relativistic generalization of the Newtonian force, which would be $-q_0\partial_a\phi$.

We can generalize such purely scalar coupling by expanding the general potential $\mathcal{V}(x, \dot{x})$ in powers of four-velocities, obtaining a set of couplings of the particle's worldline to generalized fields transforming in symmetric tensor representations of the Lorentz group:

$$S_{\rm P} = -m\int d\tau - q_0\int d\tau\, \phi(x(\tau)) - q_1\int d\tau\, \dot{x}^a\phi_a(x(\tau)) - \tfrac{1}{2}\, q_2\int d\tau\, \dot{x}^a\dot{x}^b\, \phi_{ab}(x(\tau)) + \ldots \tag{2.55}$$

where we have separated conventional numerical coefficients, q_i, called 'charges' according to $q_n\phi_{a_1\ldots a_n} = \dot{\partial}_{a_1}\ldots\dot{\partial}_{a_n}\mathcal{V}$, and we use the notation $\dot{\partial}_a \equiv \frac{\partial}{\partial\dot{x}^a}$. . These terms can be interpreted as the interaction of the particle with an *external* scalar field ϕ, a vector field ϕ_a, a symmetric tensor field ϕ_{ab}, etc. In general, only symmetric tensors can be coupled in this way, unless the particle is decorated with some other degrees of freedom (such as for example a spin four-vector S^a).

The generic relativistic force obtained in this way is extremely non-linear in the external potentials, except for the case of a world-line potential linear in the four-velocities, i.e. $\mathcal{V}(x, \dot{x}) = q_1\, \dot{x}^a\, \phi_a$. In this case, the interaction potential has an intrinsic meaning as a purely geometrical line integral over the trajectory:

$$\int_\gamma d\tau\ \dot{x}^a\phi_a = \int_\gamma dx^a\phi_a \ ,$$

and the relativistic force is linear in velocities,

$$F_a = -q_1\, \Phi_{ab}\, \dot{x}^b \ , \tag{2.56}$$

and *also linear* in potentials $\Phi_{ab} \equiv \partial_a\phi_b - \partial_b\phi_a$. In fact, it is only for the coupling to vector fields that we find such simple force laws. The general case is similar to the situation with scalar fields (2.53) in the sense that we are led to extremely non-linear force laws. We shall see later that the vector-field force corresponds to the special case of electromagnetic theory.

The coupling of a system of particles to external fields can be summarized in field-theory Lagrangians by defining appropriate currents. For scalar couplings we have a scalar 'density'

$$J_{\rm S}(x) = \sum_p \int d\tau_p\, q_p\, \delta^{(4)}(x - x(\tau_p)) \ , \tag{2.57}$$

which is defined so that [7]

$$\int d^4x\, J_{\rm S}(x)\,\phi(x) = \sum_p q_p \int d\tau_p\, \phi(x(\tau_p))\ .$$

Vector fields are always coupled via standard vector currents

$$J^a_{\rm V}(x) = \sum_p \int d\tau_p\, q_p\, \dot{x}^a\, \delta^{(4)}(x - x(\tau_p)). \tag{2.58}$$

Vector currents are always associated to conservation laws. To see this, we pass to the non-covariant representation by performing the substitution $d\tau_p = dt\, d\tau_p/dt$ in the proper time integrals and using the temporal component of the delta function $\delta(t - t(\tau_p))$ to compute the integral over t. The result is the usual expression for the currents

$$J^0(t,\vec{x}\,) = \sum_p q_p\, \delta^{(3)}(\vec{x} - \vec{x}_p(t))\ , \qquad \vec{J}(t,\vec{x}\,) = \sum_p q_p \vec{v}_p\, \delta^{(3)}(\vec{x} - \vec{x}_p(t))\ . \tag{2.59}$$

The covariant conservation law $\partial_a J^a = 0$ translates then into the usual $\partial_t J^0 + \vec{\partial}\vec{J} = 0$.

We may also consider higher tensorial generalizations. The most interesting one is the two-index symmetric object

$$J^{ab}_{\rm T}(x) = \sum_p \int d\tau_p\, q_p\, \dot{x}^a\, \dot{x}^b\, \delta^{(4)}(x - x(\tau_p))\ , \tag{2.60}$$

which actually contains the scalar density above in the trace component, since $\dot{x}^a \dot{x}_a = -1$ implies $\eta_{ab} J^{ab}_{\rm T} = -J_{\rm S}$. The symmetric tensor (2.60) acquires an important physical interpretation when $q_p = m_p$, the rest mass of the particles.

2.2.3 The Local Energy-Momentum Tensor

The second order tensor

$$T^{ab}(x) = \sum_p \int d\tau_p\ m_p\, \dot{x}^a_p\, \dot{x}^b_p\ \delta^{(4)}(x - x(\tau_p)) \tag{2.61}$$

is called the energy-momentum tensor. According to the previous discussion, it measures the response of the system of particles to the introduction of an external field transforming as a symmetric tensor of rank two, and coupling proportionally to the mass of the particles.

The name 'energy-momentum' tensor is justified as follows. Using repeatedly the relation

$$\int d\tau F(x(\tau), \dot{x}(\tau))\ \frac{dt}{d\tau}\ \delta^{(4)}(x - x(\tau)) = F(x(t), \dot{x}(t))\ \delta^{(3)}(\vec{x} - \vec{x}_p) \tag{2.62}$$

we find that

$$T^{00} = \sum_p E_p\, \delta^{(3)}(\vec{x} - \vec{x}_p) \tag{2.63}$$

[7] The four-dimensional delta function $\delta^{(4)}(x - x(\tau)) = \delta(t - t(\tau))\delta^{(3)}(\vec{x} - \vec{x}(\tau))$ is Lorentz-invariant up to a sign of the transformation's determinant.

is the energy density of the particle system. In turn,

$$T^{i0} = \sum_p E_p \, v_p^i \, \delta^{(3)}(\vec{x} - \vec{x}_p) = \sum_p p_p^i \, \delta^{(3)}(\vec{x} - \vec{x}_p) \ , \tag{2.64}$$

which we can interpret either as the energy current, or as the momentum density. On the other hand

$$T^{ij} = \sum_p p_p^i \, v_p^j \, \delta^{(3)}(\vec{x} - \vec{x}_p) = \sum_p p_p^j \, v_p^i \, \delta^{(3)}(\vec{x} - \vec{x}_p) \tag{2.65}$$

is either the flux of i-th momentum in the j-the direction, or viceversa.

Since T^{ab} characterizes the density and flow of energy, it should be associated to a local conservation law, i.e. to a current that satisfies a 'continuity' equation $\partial_a J^a = 0$, just like the electromagnetic current. The natural current to be associated with the energy-momentum tensor is the four-momentum flux as measured by a local observer with four-velocity u^a,

$$\mathcal{P}_u^a = -u_b \, T^{ab} \ . \tag{2.66}$$

To justify this expression, notice that it reduces to the correct T^{0a} for an observer at rest and, being a tensor equation, it is valid in all Lorentz frames. We define the local current at all points by giving a family of fiducial observers with four-velocity field u^a. If these observers are stationary with respect to one another, all u^a are parallel, and $\partial_a u^b = 0$. Under these conditions, we obtain conservation of the local energy-momentum flux, $\partial_a \mathcal{P}_u^a = 0$, if the energy-momentum tensor satisfies $\partial_a T^{ab} = 0$.

A direct computation for the particle system yields

$$\partial_a T^{ab} = \sum_p m_p \int d\tau_p \, \dot{x}_p^a \, \dot{x}_p^b \, \partial_a \delta^{(4)}(x - x(\tau_p)) \ . \tag{2.67}$$

Acting on the delta function

$$\dot{x}^a \partial_a \, \delta^{(4)}(x - x(\tau)) = -\dot{x}^a \frac{\partial}{\partial x^a(\tau)} \, \delta^{4)}(x - x(\tau)) = -\frac{d}{d\tau} \, \delta^{(4)}(x - x(\tau)) \ ,$$

so that we can write

$$\partial_a T^{ab} = -\sum_p m_p \int d\tau_p \, \frac{d}{d\tau_p} \left(\dot{x}_p^b \, \delta^{(4)}(x - x(\tau_p)) \right) + \sum_p m_p \int d\tau_p \, \ddot{x}_p^b \, \delta^{(4)}(x - x(\tau_p)) \ . \tag{2.68}$$

The first term is supported on the endpoints and thus vanishes, whereas the last term can be written

$$\partial_a T^{ab} = \sum_p \int d\tau_p \, \frac{dp_p^b}{d\tau_p} \, \delta^{(4)}(x - x(\tau_p)) = \sum_p \frac{dp_p^b}{dt} \, \delta^{(3)}(\vec{x} - \vec{x}_p) \tag{2.69}$$

as the density of momentum *non-conservation.* Hence, the *local* energy-momentum conservation requires that particles are free $\ddot{x}^a = 0$, or perhaps that their interactions are localized in spacetime (in this case, $\sum_{\rm in} p = \sum_{\rm out} p$ at each interaction point). We conclude that, up to regions of measure zero in spacetime, the local conservation of the energy momentum tensor is equivalent to the equation of motion in a system of free particles. This property will be recurrent in all well-defined matter systems. The local conservation of the energy-momentum tensor is of paramount importance in relativistic theories, since it embodies the basic requirements of locality and Lorentz symmetry.

Angular Momentum and Spin

The local conservation of the energy-momentum tensor allows us to define the conserved four-momentum of a system of particles enclosed in a three-dimensional region V^3 (which could be all of $\mathbf{R}^3$).

$$P^a = \int_{\mathrm{V}^3} T^{0a} \; , \tag{2.70}$$

which behaves as the conserved four-momentum of a particle. In particular, we can define the total mass $M = \sqrt{-P^2}$ and the global four-velocity $U^a = P^a/M$. Analogously, we can define the total orbital angular momentum with respect to the origin as

$$L^{ab} = X^a P^b - X^b P^a \; , \tag{2.71}$$

where $X^a = (t, \vec{X})$ is the "center of mass" trajectory, where

$$\vec{X} = \frac{1}{M} \int_{\mathrm{V}^3} \vec{x}\, T^{00} \; .$$

The intrinsic angular momentum, or *spin*, is defined as the angular momentum with respect to the center of mass

$$S^{ij} = \int_{\mathrm{V}^3} \Big((x^i - X^i)\, T^{j0} - (x^j - X^j)\, T^{i0} \Big) \; , \tag{2.72}$$

which differs from the total angular momentum with respect to the origin

$$J^{ij} = \int_{\mathrm{V}^3} \left(x^i T^{j0} - x^j T^{i0} \right) \; ,$$

by

$$J^{ij} = L^{ij} + S^{ij} \; .$$

In vector notation, $S^i = \frac{1}{2} \sum_{jk} \epsilon^{ijk} S^{jk}$, i.e. $S^{jk} = \sum_i S^i \epsilon^{ijk}$. We can generalize this to a four-vector by stating that $S^0 = 0$ in the rest frame of the centre of mass. Hence, a Lorentz-invariant equation that reduces to these in the rest frame is

$$S_a = -\tfrac{1}{2}\, \epsilon_{abcd} S^{bc} U^d \; . \tag{2.73}$$

The vanishing of the temporal component in the rest frame translates into $U_a S^a = 0$.

The local tensor object

$$M^{abc} \equiv x^a T^{bc} - x^b T^{ac} \tag{2.74}$$

is conserved provided the energy-momentum tensor is *symmetric*, $\partial_c M^{abc} = T^{ba} - T^{ab} = 0$, and defines an angular momentum density $\mathcal{J}_u^{ab} = -u_c M^{abc}$ associated to a spacelike surface orthogonal to the field of four-velocities u^a. On a $t =$ constant surface it defines the conserved angular-momentum tensor

$$J^{ab} = -J^{ba} = \int_{\mathrm{V}^3} M^{ab0} = \int_{\mathrm{V}^3} (x^a T^{b0} - x^b T^{a0}) \; , \tag{2.75}$$

whose spatial components give J^{ij} above. We can project the spin four-vector out of J^{ab} by the formula (Pauli–Lubanski)

$$S_a = -\tfrac{1}{2}\, \epsilon_{abcd} J^{bc} U^d \; , \tag{2.76}$$

since the orbital part L^{ab} drops out from this expression.

Conservation of P^a, J^a and S^a implies that

$$\frac{dU^a}{d\tau_{\rm CM}} = \frac{dS^a}{d\tau_{\rm CM}} = 0 \ , \tag{2.77}$$

where $\tau_{\rm CM}$ is the proper time associated to the trajectory of the center of mass. If we abstract the internal structure of the system, we can regard the system as an effective particle with four velocity U^a and spin S^a. Hence, adding a spin degree of freedom is a minimal extension of the notion of point-like particle.

Fluids

It is interesting for applications to take the fluid limit. This is a particle system with very short-ranged collisions, so that a comoving observer sees the system as characterized by an energy density, a pressure, and perhaps gradients of the net velocity field, after one averages over the short-distance and short-time collisions. The fluid is called *perfect* if the comoving observer sees the fluid as isotropic, characterized solely by the density and the pressure:

$$\rho = \left\langle \sum_p E_p \, \delta^{(3)}(\vec{x} - \vec{x}_p) \right\rangle_{\rm fluid} \ , \qquad p = \frac{1}{3} \sum_i \left\langle \sum_p p_p^i \, v_p^i \, \delta^{(3)}(\vec{x} - \vec{x}_p) \right\rangle_{\rm fluid} \ . \tag{2.78}$$

Hence, in the fluid limit, we have $T^{00} = \rho$ and $T^{ij} = p\,\delta^{ij}$ in the comoving frame. [8] This determines the compact expression

$$T^{ab}(x)|_{\rm fluid} = p(x)\,\eta^{ab} + (p(x) + \rho(x))\; U^a(x)\, U^b(x) \tag{2.79}$$

in a general inertial frame. Here, U^a is the (spacetime dependent) field of four-velocities of the fluid. The conservation equation $\partial_a T^{ab} = 0$ implies, in the non-relativistic limit, $p \ll \rho$, $|\vec{v}\,| dp/dt \ll |\vec{\partial}\, p|$, the two Euler equations for the dynamics of a perfect fluid,

$$\partial_t \rho + \vec{\partial}(\rho\, \vec{v}\,) = 0 \ , \qquad \rho \left(\partial_t \vec{v} + (\vec{v} \cdot \vec{\partial}\,) \vec{v} \right) = -\vec{\partial}\, p \ . \tag{2.80}$$

[8] The fluid averages are defined as $\langle A \rangle_{\rm fluid} = \Delta V^{-1} \int_{\Delta V} A$ for a small volume element ΔV, still large enough to contain a macroscopic number of particles. Using kinetic theory, one can argue that $p_p^i v_p^i$ is the momentum transfer per unit area and time (i.e. the contribution to pressure) exerted by a particle over a surface orthogonal to the ith direction.

2.3 Relativistic Fields

A local Lagrangian of fields $\Psi(x)$ in arbitrary representations of the Lorentz group must be constructed as a Lorentz scalar (all indices must be contracted) and, by the Principle of Locality, may be organized by its dependence on derivatives $\mathcal{L}(\Psi, \partial_a\Psi, \partial_a\partial_b\Psi, \ldots)$. The terms with fewer derivatives will dominate the long-distance physics, and the terms with progressively higher derivatives will be gradually introduced in a perturbative expansion to recover effects at shorter distances, in the 'Effective Field Theory' philosophy outlined in the introduction to these notes.

Once the Lagrangian is specified, the corresponding equations of motion are obtained by requiring the action

$$S = \int \mathcal{L}[\Psi] \tag{2.81}$$

to be stationary under the variations $\Psi \to \Psi + \delta\Psi$. After repeated application of the identity $\delta(\partial_a\Psi) = \partial_a\delta\Psi$, integration by parts and drop of boundary terms coming from integrating total derivatives, one finds that stationarity of the action is equivalent to

$$0 = \frac{\partial\mathcal{L}}{\partial\Psi} - \partial_a\frac{\partial\mathcal{L}}{\partial(\partial_a\Psi)} + \partial_a\partial_b\frac{\partial\mathcal{L}}{\partial(\partial_a\partial_b\Psi)} - \ldots\ , \tag{2.82}$$

where the signs alternate according to the number of derivatives acting on Ψ. In most applications, one can restrict to Lagrangians depending polynomially upon one single derivative of the field, $\partial_a\Psi$ (perhaps after repeated application of partial integrations). In this case we obtain the more standard form of the Euler–Lagrange equations, which reduces to the first two terms in (2.82). As indicated above, terms in the Lagrangian with more than two derivatives must be safely considered as perturbative corrections in the sense of the local expansion, and their effect is only to be trusted when they are not dominant.

In what follows, we list the most common types of field theories, as established by their pedagogical features or their importance in the actual description of nature.

2.3.1 Scalar Fields

The prototype relativistic field theory with minimal derivative content is the scalar model (sometimes known as the Klein–Gordon field theory)

$$\mathcal{L}_{\mathrm{KG}} = -\tfrac{1}{2}\,\eta^{ab}\partial_a\phi\,\partial_b\phi - V(\phi)\ . \tag{2.83}$$

The first term is the only Lorentz-invariant combination of two derivatives of the field, whose detailed structure

$$-\tfrac{1}{2}\,(\partial\phi)^2 = \tfrac{1}{2}\,(\partial_t\phi)^2 - \tfrac{1}{2}\,(\vec{\partial}\phi)^2\ , \tag{2.84}$$

shows that the time derivative is the strict 'kinetic' term and the spatial derivative is really part of the 'potential energy'. Up to a total derivative, we also have

$$\mathcal{L}_{\mathrm{KG}} = \tfrac{1}{2}\,\phi\,\partial^2\,\phi - V(\phi) + \partial(\ldots)\ , \tag{2.85}$$

which leads to the field equation:

$$\partial^2\,\phi - V'(\phi) = 0\ . \tag{2.86}$$

One usually separates the linear and quadratic parts of the potential $V(\phi) = J\phi + \frac{1}{2}m^2\phi^2 + \ldots$ to interpret them as the coupling to a scalar 'source' and a 'mass' term.

To investigate basic properties of the solutions of the field equation, we start with the massless free field, $V(\phi) = 0$. The solution of the field equation $\partial^2\phi = 0$ is given by a linear superposition of plane waves $\phi_k(x) = \widetilde{\phi}_k \exp(ikx) + \text{c.c.}$ of wave vector $\vec{k}$ and frequency $\omega = k^0 = |\vec{k}\,|$. These waves are defined on all of $\mathbf{R}^{1+3}$, traveling at the speed of light from the infinite past to the infinite future.

The wave vector k^a is orthogonal to the constant-phase surfaces $k^a x_a = \text{constant}$. Consider the integral curves $x^a(\sigma)$ that solve the linear differential equation $dx^a/d\sigma = k^a$ and have, by definition, tangent vector k^a. These curves are orthogonal to the constant-phase surfaces and can be interpreted as trajectories of zero-mass particles, by the condition $k^2 = (dx^a/d\sigma)(dx_a/d\sigma) = 0$. The frequency measured in a rest frame is $\omega = k^0$, so that the frequency measured by an observer with four-velocity u^a is

$$\omega = -u_a k^a \; . \tag{2.87}$$

We notice the analogy with the formula for the energy of a particle, $E = -p_a u^a$. In the quantum theory, $E = \hbar\omega$ and $\vec{p} = \hbar\vec{k}$ for a photon, so that the particle 'model' for light propagation is no longer an analogy but a strict fact. [9] For the purposes of Special and General Relativity, we can formally treat the propagation of localized, massless wave packets in the 'ballistic' picture, as the zero-mass limit of particle propagation with the operator replacement

$$m\frac{d}{d\tau} \to \frac{d}{d\sigma} \; ,$$

where σ is an affine parameter along the 'light' ray.

The next level of complication is the case of an external 'source' $V(\phi) = J\phi$, with field equation $\partial^2\phi = J$, solved formally as

$$\phi = \phi_{\text{wave}} + \frac{1}{\partial^2} J \; , \tag{2.88}$$

where the first term is a solution of the massless wave equation *in vacuo*, $\partial^2\phi_{\text{wave}} = 0$, and the second term is a formal representation of a particular solution with structure similar to the Newtonian potential, except for the replacement of the operator $\vec{\partial}^2$ by the operator $\partial^2 = -\partial_t^2 + \vec{\partial}^2$, which takes the Poisson equation into the Klein–Gordon equation. Given a purely static solution of the Poisson equation $\vec{\partial}^2\phi = J$,

$$\phi_{\text{static}}(\vec{x}\,) = \frac{1}{\vec{\partial}^2} J = -\int \frac{d^3y}{4\pi} \frac{J(\vec{y}\,)}{|\vec{x} - \vec{y}\,|} \; ,$$

with no time dependence, we can obtain a solution of the relativistic equation by introducing time-dependence through the trick of the retarded potential, i.e. evaluating the source at the retarded time $t - |\vec{x} - \vec{y}\,|$, to represent the fact that disturbances on the field ϕ travel at the speed of light, as vacuum waves.[10]

$$\phi(t, \vec{x}\,)_{\text{retarded}} = \frac{1}{\partial^2}\Big|_{\text{ret}} J = -\int \frac{d^3y}{4\pi} \frac{J(t - |\vec{x} - \vec{y}\,|, \vec{y}\,)}{|\vec{x} - \vec{y}\,|} \; . \tag{2.89}$$

[9] Although we coach this discussion for the case of scalar fields and scalar particles, the propagation properties of massless scalar waves generalize readily to higher spin cases, including electromagnetic waves and the 'photon' particles.

[10] It should be noted that one can use *advanced* instead of retarded solutions (or a linear combination thereof). The difference between the two is an adjustment of the ϕ_{wave} term in the general solution (2.88). One usually works with the retarded potentials because they are more intuitive in representing causality.

We can now use this same intuition to interpret the general solution. For an arbitrary potential we may rewrite (2.86) as

$$\phi(t,\vec{x}\,)_{\rm retarded} = \frac{1}{\partial^2} V'(\phi)\Big|_{\rm retarded} = -\int \frac{d^3y}{4\pi} \frac{V'[\phi(t-|\vec{x}-\vec{y}\,|,\vec{y}\,)]}{|\vec{x}-\vec{y}\,|} \; . \tag{2.90}$$

Rather than a general solution, this is an integral equation which we may solve iteratively by redefining the source order by order as a function of the approximate solution at lower orders. In particular, each insertion of a term in $V'(\phi)$ represents a 'kick' of the free waves that transport the interaction at the speed of light. In this way, we can understand how successive kicks due to a mass term $V'(\phi) \sim m^2\phi$ have the collective effect of reducing the speed of propagation from that of light to a superposition of various speeds below light, as if the waves were a superposition of particles with dispersion $E = \sqrt{\vec{p}^{\,2}+m^2}$, a picture that is ultimately vindicated in the quantum version of these field theories.

A nonlinear generalization of (2.83) involving several scalar fields ϕ^I but still two-derivatives is the so-called nonlinear sigma-model, with many applications in particle physics and condensed matter physics

$$\mathcal{L}_{\sigma-{\rm model}} = -\tfrac{1}{2} \sum_{IJ} G_{IJ}(\phi)\, \partial_a \phi^I\, \partial^a \phi^J - V(\phi^I) \; . \tag{2.91}$$

2.3.2 Spinor Fields

Fields in spinor representations of the Lorentz group are also very important in nature (all fermions) but the appropriate Lagrangians require the Dirac formalism and are, strictly speaking, part of the quantum version of the theory. Here, we shall simply quote for completeness the case of the basic free Dirac field. A correct relativistic propagation is ensured if we write

$$(\partial^2 - m^2)\psi = 0$$

as the field equation. However, using the properties of Dirac matrices, one can see that $\partial^2 - m^2 = (i\gamma^a\partial_a + m)(i\gamma^a\partial_a - m)$. Hence, the relativistic equation on spinors with the most general set of solutions is the so-called Dirac equation

$$(i\gamma^a\partial_a - m)\psi = 0,$$

which follows from the Lagrangian

$$\mathcal{L}_{\rm Dirac} = \bar{\psi}(i\gamma^a\partial_a - m)\psi \; ,$$

where $\bar{\psi} = \psi^\dagger\gamma^0$ is the Lorentz-conjugated representation. The most salient feature of this Lagrangian is its first order character in derivatives, as opposed to the bosonic counterparts.

2.3.3 Vector Fields

Higher-spin tensor fields always lead to subtleties. Consider, for example, a free vector field ϕ^a of mass m, with equation of motion

$$(\partial^2 - m^2)\,\phi^a = 0 \; , \tag{2.92}$$

consisting of four copies of the Klein–Gordon equation. A candidate Lorentz-invariant Lagrangian with the correct equations of motion would be

$$\mathcal{L}[\,\phi^a\,]_{\rm KG} = \tfrac{1}{2}\,\eta_{ab}\,\phi^a\,\partial^2\,\phi^b - \tfrac{1}{2}\,m^2\,\eta_{ab}\,\phi^a\phi^b \tag{2.93}$$

as a direct generalization of the scalar Klein–Gordon Lagrangian. However, the fact that η_{ab} is *not* positive definite leads to problems, since the whole Lagrangian of the ϕ^0 field has the wrong sign. This field may then store arbitrarily large amounts of negative energy, and any interaction term would render the dynamics unstable. So, we face an interesting dilemma: the field equation following from this Lagrangian seems to be correct and yet the energy density appears to be ill-defined.

We cannot simply drop the offending component ϕ^0, because that would spoil the Lorentz symmetry. However, for a plane-wave solution in momentum space, $\phi^a = \widetilde{\phi}^{\,a} e^{ipx}$,

$$(p^2 + m^2)\widetilde{\phi}^a = 0\ ,$$

we may impose the Lorentz-invariant condition of transversality between the polarization vector $\widetilde{\phi}^a$ and the propagation vector p^a, i.e.

$$p_a\,\widetilde{\phi}^{\,a} = 0\ . \tag{2.94}$$

This condition brings $\widetilde{\phi}^0$ to vanish in the rest frame $(p^a) = (m, 0, 0, 0)$. Therefore, we conclude that such 'transverse waves' satisfying (2.94) will not have energy stability problems.

This suggests that we add the transversality condition $\partial_a\phi^a = 0$ as an extra equation, together with the Klein–Gordon equation (2.92). In fact, in this case we may redefine (2.92) by any multiple of $\partial_b\,\partial_a\phi^a$, and the particular choice (known as the Proca Equation)

$$\Big(\eta_{ab}(\partial^2 - m^2) - \partial_a\partial_b\Big)\,\phi^b = 0 \tag{2.95}$$

does enforce the transversality condition automatically, provided $m \neq 0$. To see this, we take its divergence to find $m^2\partial_a\phi^a = 0$. The Lagrangian associated to the equation of motion (2.95) is

$$\mathcal{L}_{\rm Proca} = -\tfrac{1}{4}\Phi_{ab}\,\Phi^{ab} - \tfrac{1}{2}\,m^2\,\phi_a\,\phi^a\ , \tag{2.96}$$

where we have defined $\Phi_{ab} = \partial_a\phi_a - \partial_b\phi_a$. We conclude that (2.96) is the consistent Lagrangian of a free vector field, describing the propagation of *three* polarization degrees of freedom.

More subtleties lurk in the zero-mass limit $m \to 0$. In this case, the 'rest frame' of a plane wave does not exist, as it propagates at the speed of light. It is possible to reach the standard null frame $(p^a) = (\omega, 0, 0, \omega)$, but then the transversality condition only enforces $\widetilde{\phi}^0 = \widetilde{\phi}^3$, which is not enough to cancel out the offending time-like component ϕ^0. On the other hand, if (2.96) has no energy balance problems, the same should be true of the $m^2 \to 0$ limit.

There is an interesting escape out of this paradox. Suppose we *declare* by hand that $\widetilde{\phi}^{\,0} = 0$ on the standard null frame. This means that, because of the transversality constraint, we are actually demanding the wave polarization to be transverse in the spatial sense, $\sum_i \widetilde{\phi}_i p^i = \widetilde{\phi}_0 = 0$, with only two polarization degrees of freedom remaining. The problem with this prescription is of course the violence to Lorentz invariance, since the Lorentz transformation of such a polarization vector will in general produce longitudinal and time-like components. However, whatever the Lorentz transformation does, it must preserve the transversality constraint $p^a\widetilde{\phi}_a = 0$, which precisely for $p^2 = 0$ has a *redundancy*: if $\widetilde{\phi}^a$ is transverse, so is $\widetilde{\phi}^a + \tilde{f}\, p^a$, with $\tilde{f}$ an arbitrary constant. This means that $\widetilde{\phi}_a$ will shift by a term proportional to p_a under Lorentz transformations.

The way to remove the offending degree of freedom in a Lorentz-invariant fashion is simply to declare *all* configurations related by such shifts as *physically equivalent*, in the sense that all physical observables should be exactly invariant under $\widetilde{\phi}^a \to \widetilde{\phi}^a + \tilde{f}\, p^a$, or its position-space counterpart

$$\phi_a \longrightarrow \phi_a + \partial_a f \; . \tag{2.97}$$

This is called *gauge invariance*.

We may then use the degree of freedom in $\tilde{f}$ to remove $\widetilde{\phi}_0$ in the standard null frame. In doing so, we remove $\widetilde{\phi}_3$ too, as they are linked by the transversality constraint. Therefore, we end up with a consistent Lorentz-invariant theory of massless vector fields with only two physical degrees of freedom. We can find the Lagrangian of this theory by working with the explicitly transverse fields obtained by the linear projection

$$\widetilde{\phi}^a_{\rm T} = \widetilde{P}^a{}_b\, \widetilde{\phi}^b \equiv \left(\delta^a{}_b - \frac{p^a\, p_b}{p^2}\right) \widetilde{\phi}^b \; , \tag{2.98}$$

which translates into a *non-local* projector in position space,

$$\phi^a_{\rm T} = P^a{}_b\, \phi^b \equiv \left(\delta^a_b - \frac{\partial^a\, \partial_b}{\partial^2}\right) \phi^b. \tag{2.99}$$

Notice that such transverse fields are automatically gauge-invariant since

$$P^a{}_b\, \phi^b = P^a{}_b (\phi^b + \partial^b f) \; . \tag{2.100}$$

Evaluating then the massless Klein–Gordon Lagrangian on transverse gauge-invariant fields we find the Lagrangian

$$\tfrac{1}{2}\, \eta_{ab}\, \phi^a_{\rm T}\, \partial^2\, \phi^b_{\rm T} = -\tfrac{1}{4}\, \Phi_{ab}\, \Phi^{ab} \; , \tag{2.101}$$

corresponding to the massless limit of (2.96). It is important to keep in mind, however, that the massless vector model is assumed to be defined with the built-in redundancy under $\phi_a \to \phi_a + \partial_a f$, for an arbitrary function $f(x)$, and in particular it has one less degree of freedom than the naive massless limit of the massive model. So, even if (2.101) is the smooth $m^2 \to 0$ limit of (2.96) at the level of Lagrangians, there is a discontinuous jump in degrees of freedom as we go to the massless limit.

The gauge redundancy is very useful in constructing dynamical laws for massless vector fields. For example, a linear coupling of the form

$$\mathcal{L}_J = \phi_a\, J^a$$

is bound to break gauge invariance unless we demand the conservation of the 'current': $\partial_a J^a = 0$. Should the current *not* be conserved, so that gauge invariance is broken, then the theory with this coupling would simply describe three degrees of freedom instead of two.

The gauge redundancy is often called a 'local symmetry', a rather misleading name, since its real role is to ensure that the theory describes the minimal number of degrees of freedom (two for a vector field) even after Lorentz-invariant couplings are specified.

Problem: Gauge Is Not a Symmetry

Consider the massive vector field model with Lagrangian

$$\mathcal{L}_{\text{Proca}} = -\tfrac{1}{4}\,\Phi_{ab}\,\Phi^{ab} - \tfrac{1}{2}\,m^2\,\phi_a\,\phi^a\ . \tag{2.102}$$

Show that it is *not* gauge invariant. Define the so-called Stueckelberg model, which couples a massless vector field and a massless scalar field φ with Lagrangian

$$\mathcal{L}[\,\phi_a,\varphi\,] = -\tfrac{1}{4}\Phi_{ab}\,\Phi^{ab} - \tfrac{1}{2}\,\eta^{ab}(m\phi_a + \partial_a\varphi)(m\phi_b + \partial_b\varphi)\ . \tag{2.103}$$

Show that the Stueckelberg model is gauge-invariant under the transformations

$$\phi_a(x) \to \phi_a(x) + \partial_a f(x)\ , \qquad \varphi(x) \to \varphi(x) - f(x)/m\ .$$

Show that the Stueckelberg model is equivalent to (2.102) by an appropriate fixing of the gauge redundancy.

Hence, we find that a model with three degrees of freedom and no gauge invariance is equivalent to a gauge-invariant model with two degrees of freedom, plus one explicit degree of freedom, so that gauge ‘symmetry’ is not real... just a matter of bookkeeping conventions in describing the true degrees of freedom.

‘Integrate out’ the scalar field φ in (2.103), by solving its equation of motion and substituting back into the action. Show that the resulting equivalent Lagrangian is

$$\mathcal{L}_{\text{non local}} = -\tfrac{1}{4}\,\Phi_{ab}\,\Phi^{ab} + \tfrac{1}{4}\,m^2\,\Phi_{ab}\frac{1}{\partial^2}\,\Phi^{ab}\ . \tag{2.104}$$

That is, a massless vector gauge theory, corrected by a non-local term proportional to the mass squared. Notice that *it is* gauge invariant. The non-local term is the result of the existence of more degrees of freedom than just the two massless polarizations. This non-local model is actually the result of imposing the gauge invariance ‘by hand’ in the naive Klein–Gordon Lagrangian, i.e. show that

$$\mathcal{L}_{\text{non local}} = \mathcal{L}[\,\phi^a\,]_{\text{TKG}} = \tfrac{1}{2}\,\eta_{ab}\,\phi^a_{\text{T}}\,\partial^2\,\phi^b_{\text{T}} - \tfrac{1}{2}\,m^2\,\eta_{ab}\,\phi^a_{\text{T}}\phi^b_{\text{T}}\ .$$

Hence, insisting on gauge invariance for a *massive* vector theory requires either a non-local model or a local model with an extra degree of freedom. This is natural, since we know that a massive vector field must have three polarization degrees of freedom, forcing a gauge-invariant description with two polarizations requires the introduction of the third polarization as an explicit field degree of freedom, the so-called Stueckelberg field, a precursor of the Higgs field.

Maxwell in the Lorentz-Invariant Way

We now reproduce the Lagrangian structure of massless vector fields from the direct physical construction of Maxwell’s theory. One can exhibit the Lorentz covariance of Maxwell’s equations by defining the electromagnetic tensor F_{ab}, via

$$F^{0i} = E^i\ , \qquad F_{ij} = \sum_k \epsilon_{ijk}B_k$$

in terms of the electric and magnetic field vectors $\vec{E},\vec{B}$. Defining also the current four-vector $(J^a) = (\rho_e, \vec{J}_e)$, the equations can be cast into the system

$$\partial_a F^{ab} = -J^b\ , \qquad \epsilon^{abcd}\partial_b F_{cd} = 0\ , \tag{2.105}$$

which are manifestly Lorentz-covariant. Thus, we confirm Poincaré's dictum that the Maxwell system remains untouched in the new relativistic formalism, whereas the Newtonian particle dynamics had to yield. Curiously, it was Maxwell the first to write down a relativistic equation in the mid XIX century!

In order to derive these equations from a Lagrangian, we introduce the potential A^a by solving the second Maxwell equation (sometimes referred to as the Bianchi identity) via the Poincaré lemma: locally, an antisymmetric tensor with vanishing antisymmetrized divergence can always be written as [11]

$$F_{ab} = \partial_a A_b - \partial_b A_a \tag{2.106}$$

in terms of some vector field A^a, which always comes with a built-in redundancy, i.e. a gauge ambiguity by the redefinition $A^a \to A^a + \partial^a f$. The rest of Maxwell's equations then follow upon varying the Maxwell Lagrangian

$$\mathcal{L}_{\rm Maxwell} = -\tfrac{1}{4} F_{ab} F^{ab} + J_a A^a \tag{2.107}$$

with respect to A^a. As explained before, the coupling to the electric current does not spoil gauge invariance (up to total derivatives) provided the current is indeed conserved, $\partial_a J^a = 0$.

A convenient Lorentz-invariant convention to fix partially the gauge redundancy is the Lorenz condition $\partial_a A^a = 0$, whose main virtue is the simplification of the Maxwell equations, simply collapsing them to four Klein–Gordon equations

$$\partial^2 A^a = -J^a \ , \tag{2.108}$$

whose standard solution was discussed before

$$A^a(t, \vec{x})_{\rm retarded} = -\frac{1}{\partial^2} J^a \Big|_{\rm retarded} = \int \frac{d^3 y}{4\pi} \frac{J^a(t - |\vec{x} - \vec{y}\,|, \vec{y}\,)}{|\vec{x} - \vec{y}\,|} \ , \tag{2.109}$$

for fields that vanish at infinity and with causal (retarded) boundary conditions. This relation accounts for most of classical electrodynamics.

Free electromagnetic waves can be added to A^a as solutions of the homogeneous equation with $J^a = 0$. By the linearity property of Maxwell's equations, any such solution is a superposition of plane waves of the form

$$A^a(x) = \tilde{A}^a(k)\, e^{ikx} + \text{c.c.} \ ,$$

where $\tilde{A}^a(k)$ is now a constant vector and $k\,x = \eta_{ab} k^a x^b = -\omega\, t + \vec{k} \cdot \vec{x}$. The four-vector k^a includes the wave vector and the frequency. Maxwell equations and the gauge condition impose $k^a \tilde{A}_a(k) = 0$ and $k^2 = 0$. By further use of the gauge redundancy one can set $\tilde{A}^0(k) = 0 = \sum_i k^i \tilde{A}^i(k)$, leaving fully transverse waves with two polarization degrees of freedom.

The coupling to charges, given by the linear term in the field-theory action $\int d^4x\, J_a A^a$, with the *conserved* electromagnetic current

$$J^a(x) = \sum_p e_p \frac{dx^a}{d\tau_p}\, \delta^{(4)}(x - x_p(\tau_p)) \ ,$$

[11] An explicit proof is obtained by the following *ansatz*: $A^a(x) = -\int_0^1 d\lambda\, \lambda\, F^{ab}(\lambda x)\, x^b$. Notice that the λ integral over points of the form λx requires that the region where we define the fields be 'star shaped', in the sense that it can be contracted to a point through straight lines, each of which being fully contained in the original region. This is what is meant by saying that the Poincaré lemma applies *locally*.

produces a correction to the free particle Lagrangian of the form

$$S_{\mathrm{P}} = -\sum_p m_p \int d\tau_p + \sum_p e_p \int d\tau_p A_a \frac{dx^a}{d\tau_p} \ , \tag{2.110}$$

which in turn leads to the so-called Lorentz force in its fully relativistic incarnation

$$m_p \frac{d^2 x_p^a}{d\tau^2} = e_p \, F^a{}_b \, \frac{dx_p^b}{d\tau} \ . \tag{2.111}$$

The simplicity of (2.105) and (2.111) is a bit deceptive. Together they form a set of non-linear equations, because the fields determining the motion of the particles do depend on the history of the particles themselves. There are thorny issues of principle, related to the existence of pointlike objects with formally infinite energy density (the particles) and the fact that the energy carried by their own electric field also has mechanical properties. This complicates the calculation of how a particle reacts when it radiates.

Problem: Parity Violating Photons?

Consider adding a Lorentz-invariant term to the Lagrangian of the Maxwell theory of the form

$$\mathcal{L}_\vartheta = \vartheta \, \widetilde{F}_{ab} F^{ab}$$

where

$$\widetilde{F}_{ab} = \tfrac{1}{2} \, \epsilon_{abcd} F^{cd}$$

is the Poincaré dual. Check that the dual tensor has the roles of electric and magnetic fields reversed. Write the Lagrangian $\mathcal{L}_\vartheta$ in terms of electric and magnetic fields and study its behavior under parity and time reversal. Why is this Lagrangian not usually considered as a correction to Maxwell theory?

2.3.4 Higher Rank Tensor Fields

The construction of field theories in higher tensor representations reproduces some of the problems already present in the vector case. Lorentz invariance tends to introduce unstable modes, which have to be dealt with via constraints and gauge redundancies. There are general no-go arguments that limit the possibilities of constructing such field theories if we require long-range interactions and, in any case, they do not appear in our current fundamental description of nature, except for two cases, both of them related to rank-two tensor fields.

We consider here the massless case, which is the most interesting from the physical point of view. A naive action for a massless two-index tensor field ϕ_{ab} of the form

$$\mathcal{L}[\,\phi_{ab}\,]_{\mathrm{KG}} = \tfrac{1}{2} \, \phi_{ab} \, \partial^2 \, \phi^{ab}$$

has the proverbial stability issues, which may be solved by introducing gauge redundancies. In particular, for the antisymmetric case, $b_{cd} = -b_{dc}$, we introduce a redundancy by an anti-symmetric derivative $b_{cd} \to b_{cd} + \partial_{[c} f_{d]}$, whereas the symmetric case, $h_{ab} = h_{(ab)}$, requires a corresponding 'symmetric' redundancy $h_{ab} \to h_{ab} + \partial_{(a} f_{b)}$.

The resulting gauge-invariant action for the antisymmetric case is a simple generalization of the Maxwell construction:

$$\mathcal{L}[\, b_{cd}\,] = -\tfrac{1}{12} H_{abc} H^{abc} \;, \qquad H_{abc} \equiv 3\, \partial_{[a} b_{bc]} \;. \tag{2.112}$$

Here, H_{abc} is a generalization of the Maxwell tensor F_{ab} and b_{ab} is a generalization of the potential four-vector A_a. The equation of motion $\partial_a H^{abc} = 0$ implies that the pseudovector Poincaré-dual field $\widetilde{H}_a = -\frac{1}{6}\epsilon_{abcd} H^{bcd}$ has vanishing antisymmetric derivative and, using Poincaré's lemma, it admits the representation $\widetilde{H}_b = \partial_b\, \widetilde{a}$ as the divergence of a pseudoscalar field $\widetilde{a}$, usually called the *axion*.[12] Then, the Bianchi identity $\partial_{[a} H_{bcd]} = 0$ is equivalent to the standard wave equation for the axion: $\partial^b \partial_b \widetilde{a} = 0$. This chain of arguments shows that the theory (2.112) only has *one* physical polarization degree of freedom, that is to say, it is equivalent to the theory of a massless pseudoscalar field.

If we try the same strategy with the 3-form field C_{abc} with field strength $D_{abcd} = 4\partial_{[a} C_{bcd]}$, the equation of motion is equivalent to requiring $\partial_a \widetilde{D} = 0$, where $\widetilde{D}$ is the Poincaré dual of D_{abcd}. Since the solutions are constant fields, there is no interesting propagating degree of freedom associated to this theory. This pattern repeats for higher p-form fields in d dimensions,

The symmetric case is more complicated, and more interesting for the theory of gravity. There are four independent Lorentz-invariant structures at the quadratic, two-derivative level:

$$\partial_a h_{bc}\, \partial^a h^{bc} \;, \qquad \partial_a h^{ab}\, \partial^c h_{cb} \;, \qquad \partial_a h^b_b\, \partial^a h^c_c \;, \qquad \partial_a h^c_c\, \partial_b h^{ab} \;,$$

up to total derivatives. Imposing now gauge invariance under $h_{ab} \to h_{ab} + \partial_{(a} f_{b)}$ one finds a unique answer up to overall normalization, the so-called Fierz–Pauli Lagrangian

$$\mathcal{L}[\, h_{ab}\,]_{\rm FP} = -\tfrac{1}{4} \partial_a h_{bc}\, \partial^a h^{bc} + \tfrac{1}{4} \partial_a h^c_c\, \partial^a h^c_c - \tfrac{1}{2}\, \partial_a h^c_c\, \partial_b h^{ab} + \tfrac{1}{2}\, \partial_a h^{ab}\, \partial^c h_{bc} \;, \tag{2.113}$$

which will turn out to be the linearized approximation to Einstein's theory! Therefore, it describes linearized gravitational waves. For that reason, we will defer further discussion to the section on gravitational radiation.

[12] A pseudoscalar field is a scalar under small Lorentz transformations which changes sign under a parity transformation. The axion is a hypothetical particle whose existence is not established, but pseudoscalar fields are routinely used to describe, for instance, the π-mesons.

2.4 Symmetry and the Field

One of the great advantages of the Lagrangian formalism is the efficient handling of symmetries. This fact is exacerbated in the case of field theories. The powerful theorem of Noether which links symmetry and conservation becomes an essential tool in analyzing the general complicated dynamics of field theories. In this section we start with a covariant proof of a general version of Noether's theorem, and then we focus on the example of major interest: the definition of the energy-momentum tensor for general field theories.

2.4.1 Noether

Let $\mathcal{L}[\Psi]$ be a general functional of fields $\Psi(x)$ in arbitrary tensor representations of the Lorentz group, with polynomial dependence on ∂_a, but otherwise generic. We consider a continuous symmetry whose action on the fields, $\Psi \to \Psi'$, when expanded close to the identity is

$$\Psi'(x) - \Psi(x) = \delta_s \Psi(x) + O(\epsilon^2) = \epsilon\, s[\Psi] + O(\epsilon^2) \ , \tag{2.114}$$

where the functional $s[\Psi]$ is also assumed to have polynomial dependence on derivatives. Neglecting boundary terms coming from total derivatives, invariance of the action implies that the Lagrangian density transforms as a total derivative $\delta_s \mathcal{L} = \epsilon\, \partial_a f_s^a[\Psi]$.

We will now implement a formal trick. Consider a *more general* transformation where the symmetry parameter is given a functional dependence on spacetime coordinates, i.e. $\delta_s \Psi = \epsilon(x) s[\Psi]$. The variation of the Lagrangian density can be expanded in derivatives of $\epsilon(x)$ as

$$\delta_s \mathcal{L} = \epsilon(x)\, K_s[\Psi] + \partial_a \epsilon(x)\, K_s^a[\Psi] + \partial_a \partial_b \epsilon(x)\, K_s^{ab}[\Psi] + \cdots \ , \tag{2.115}$$

for some functionals K_s, K_s^a, etc. In particular, by setting $\epsilon =$ constant we learn that

$$K_s[\Psi] = \partial_a f_s^a[\Psi] \ . \tag{2.116}$$

We emphasize that the spacetime dependence of $\epsilon(x)$ is just a mathematical trick, not a generalization from a global to a local symmetry. Indeed, (2.115) shows that the Lagrangian is *not* invariant under the local transformation. The action is only invariant under the global transformation with constant ϵ.

Neglecting total derivatives (by choosing for instance $\epsilon(x)$ with compact support), we may now integrate by parts (2.115) and find

$$\delta_s \int \mathcal{L}[\Psi] = -\int \epsilon(x)\, \partial_a J_s^a \ , \tag{2.117}$$

where

$$J_s^a = -f_s^a[\Psi] + K_s^a[\Psi] - \partial_b K_s^{ab}[\Psi] + \cdots \tag{2.118}$$

Since the variation $\delta_s \Psi = \epsilon(x) s[\Psi]$ is a particular case of a general variation of the fields, on a configuration that extremizes the action we conclude that this current is conserved, $\partial_a J_s^a = 0$. Hence, in (2.118) we have a constructive definition of the Noether current, valid in an arbitrary derivative expansion. To construct the functionals $K_s^a, K_s^{ab}, \ldots$ appearing in (2.115) one can make repeated use of the identity

$$\delta_s(\partial_a \Psi) = \partial_a(\delta_s \Psi) = \partial_a(\epsilon(x) s[\Psi]) = \epsilon(x)\partial_a s[\Psi] + s[\Psi]\partial_a \epsilon(x)$$

in the formal expansion

$$\delta_s \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \Psi}\, \delta_s \Psi + \frac{\partial \mathcal{L}}{\partial(\partial_a \Psi)}\, \delta_s(\partial_a \Psi) + \frac{\partial \mathcal{L}}{\partial(\partial_a \partial_b \Psi)}\, \delta_s(\partial_a \partial_b \Psi) + \dots$$

In the simple case where the Lagrangian depends polynomially on a single derivative, the only non-vanishing functional is

$$K^a_s[\Psi] = \frac{\partial \mathcal{L}}{\partial(\partial_a \Psi)}\, s[\Psi]\ . \tag{2.119}$$

The resulting Noether current is then the familiar expression

$$J^a_s = \frac{\partial \mathcal{L}}{\partial(\partial_a \Psi)}\, s[\Psi] - f^a_s[\Psi]\ . \tag{2.120}$$

For an illustrative example of how Noether's theorem works in field theory, consider a set of N scalars with Lagrangian

$$\mathcal{L}_{O(N)} = -\tfrac{1}{2} \int \sum_{I=1}^{N} \partial_a\, \phi_I\, \partial^a\, \phi_I\ , \tag{2.121}$$

which is invariant under the $O(N)$ transformations $\phi_I(x) \to \sum_J R_{IJ}\phi_J(x)$ for any constant orthogonal matrix R_{IJ}. To obtain the Noether current we work near the identity $R_{IJ} = \delta_{IJ} + \Omega_{IJ} + O(\Omega^2)$ with Ω_{IJ} antisymmetric. Picking a basis of $N(N-1)/2$ matrices T^s_{IJ} for the Lie algebra of $O(N)$ we can write $\Omega_{IJ} = \sum_s \epsilon_s T^s_{IJ}$ and the corresponding $N(N-1)/2$ Noether currents taking the form

$$J^s_a = \sum_{IJ} \phi_I\, T^s_{IJ}\, \partial_a\, \phi_J \tag{2.122}$$

are conserved $\partial^a J^s_a = 0$.

The previous derivation implicitly defines the field variations as *functional* variations, in the sense that a field configuration is compared to the transformed field configuration *at the same point*: $\delta\Psi(x) + O(\epsilon^2) = \Psi'(x) - \Psi(x)$. This is the standard situation for symmetries which do not affect the spacetime coordinates (called *internal symmetries*). On the other hand, spacetime symmetries are often presented in tensorial form, which does involve comparing functions at different values of the variables:

$$\Psi'{}^{\cdots a \cdots}{}_{\cdots b \cdots}(x') = \cdots \frac{\partial x'^{\,a}}{\partial x^c}\Big|_x \cdots \frac{\partial x^d}{\partial x'^{\,b}}\Big|_{x'} \cdots\ \Psi^{\cdots c \cdots}{}_{\cdots d \cdots}(x)\ , \tag{2.123}$$

where we have been very explicit about the functional dependence of all pieces on the coordinates. This formula gives the transformation law for any tensor field under a Poincaré transformation $x^b \to x'^{\,b} = x^b + a^b + L^b{}_c\, x^c$. In the *active* view of the transformations, it gives the relation between the tensor *functions* at different points in Minkowski spacetime, labeled x and x'.

Therefore, in order to use (2.123) to derive $\delta\Psi(x)$, we need to expand it near the identity and evaluate the variation at the *same* point. In general, any continuous transformation of the coordinates can be written in infinitesimal form as

$$x'^{\,a} = x^a - \epsilon\, \xi^a(x) + O(\epsilon^2)\ , \tag{2.124}$$

for some vector field $\xi^a(x)$. Inserting this expression into (2.123) and keeping only terms linear in ϵ we find

$$\delta_\xi \Psi = \epsilon\, \pounds_\xi\, \Psi\ , \tag{2.125}$$

where $\pounds_\xi \Psi$ is the so-called *Lie derivative* along the vector field $\xi^a(x)$:

$$\pounds_\xi \, \Psi \,{}^{\cdots a \cdots}_{\cdots b \cdots} = \xi^c \partial_c \Psi \,{}^{\cdots a \cdots}_{\cdots b \cdots} - \cdots - \partial_c \xi^a \, \Psi \,{}^{\cdots c \cdots}_{\cdots b \cdots} - \cdots + \partial_b \xi^d \, \Psi \,{}^{\cdots a \cdots}_{\cdots d \cdots} + \cdots \, . \tag{2.126}$$

The Lie derivative is essentially a directional derivative along the vector field, corrected by terms that depend on the spacetime variation of the vector field itself, with signs that depend on the covariant or contravariant nature of the indices.

With this definition of the field variations, we can write the Noether current for any spacetime symmetry $\delta_\xi \, x^a = -\epsilon \, \xi^a(x)$ reinterpreting (2.120) in the form

$$J^a_\xi = \frac{\partial \mathcal{L}}{\partial(\partial_a \Psi)} \, \pounds_\xi[\Psi] - f^a_\xi \, . \tag{2.127}$$

Finally, when a spacetime symmetry has also a simultaneous 'internal' action beyond the tensorial form (2.123) we can write the field variation to linear order in ϵ as $\delta_\xi \Psi = \epsilon s_\xi[\Psi] + \epsilon \pounds_\xi[\Psi]$ and we can write for the corresponding Noether current:

$$J^a_\xi = \frac{\partial \mathcal{L}}{\partial(\partial_a \Psi)} \, (s_\xi[\Psi] + \pounds_\xi[\Psi]) - f^a_\xi \, , \tag{2.128}$$

a formula that one encounters when discussing scale and conformal transformations.

Armed with these expressions for the conserved currents, we could write down all the canonical forms of currents associated to the ten independent symmetries of the Poincaré group. The most important of all is the local energy-momentum tensor, T_{ab}, which is none other than the set of Noether currents for translational symmetry. Once T_{ab} is known, it can be used to build all other currents in a constructive fashion.

Noether and the Gauge

The mathematical trick of using a spacetime-dependent parameter $\epsilon(x)$ brings up the following question. What would it take to make the theory invariant under the *local* transformation? Considering the simple case of Lagrangians with explicit dependence on first derivatives of the field, we saw that the variation of the action can be written as

$$\delta_s \int \mathcal{L} = \int J^a_s \, \partial_a \epsilon(x) \, , \tag{2.129}$$

that is to say, the Noether current characterizes directly the lack of invariance when the symmetry parameter ϵ is given a functional dependence on spacetime position. A way to make the theory symmetric is to add a new term to the Lagrangian, depending on an extra vector field A^a_s, through the coupling

$$\mathcal{L} \to \mathcal{L} - \int \eta_{ab} \, J^a_s \, A^b_s \, , \tag{2.130}$$

where the vector field A_s is required to transform as

$$\delta_s A^a_s = \partial^a \epsilon(x) + \ldots \, , \tag{2.131}$$

so that the complete Lagrangian is now invariant under the local transformation (the dots stand for extra terms that may be needed but would not depend on derivatives of $\epsilon(x)$). This procedure is called the Noether method of gauging, and it manufactures a gauged version out of any global symmetry. We may leave A_s as a mere background field, or we can postulate new terms in the Lagrangian that provide a kinetic term for A_s, such as a Maxwell term in terms of the field strength $\partial^a A_s^b - \partial^b A_s^a$, giving us a fully fledged gauge theory. Notice that the resulting Lagrangian for the gauge field must respect the local symmetry implying, among other things, that it must be massless.

2.4.2 Energy-Momentum Tensor in Field Theories

Having defined the energy-momentum tensor of a particle system with contact interactions, we can also monitor the local exchanges of energy and momentum in the more physical case where the interactions are mediated by fields. Consider for example a system of charged particles whose electromagnetic interactions are governed by Lorentz forces

$$m_p\, \ddot{x}_p^a = e_p\, F^a{}_b\, \dot{x}_p^b \ . \tag{2.132}$$

Then, the energy-momentum tensor of the particle system is not conserved alone, as some energy flows into the field degrees of freedom. To see this, we insert (2.132) into (2.68) to find

$$\partial_b T^{ab}_{\text{particles}} = F^a{}_b\, J^b \ . \tag{2.133}$$

Thus, local energy conservation requires that we add the intrinsic energy-momentum tensor of the electromagnetic field, canceling the source term in (2.133). We know from elementary discussions of electrodynamics that the energy density of the electromagnetic field is given by $\frac{1}{2}\,(\vec{E}^{\,2} + \vec{B}^{\,2})$, i.e. it is quadratic in components of the electromagnetic tensor F_{ab}. This follows immediately from a glance at the basic Lagrangian (1.47); since the electric field is proportional to the time derivative of the vector potential, the structure of the Lagrangian suggests the structure of the energy density. Hence we may try a quadratic *ansatz* for the energy-momentum tensor of the electromagnetic field:

$$c_1\, F^{ac} F_c{}^b + c_2\, \eta^{ab}\, F_{cd} F^{cd}$$

depending on the two independent quadratic combinations of F_{ab} which are allowed by Lorentz symmetry and the antisymmetry of the electromagnetic tensor. Imposing now the condition

$$\partial_b T^{ab}_{\text{Maxwell}} = -F^{ab}\, J_b$$

given the equations of motion $\partial_a F^{ab} = -J^b$ we fix the constants $c_1 = -1$ and $c_2 = -1/4$ to obtain the Maxwell energy-momentum tensor

$$T^{ab}_{\text{Maxwell}} = -F^{ac} F_c{}^b - \tfrac{1}{4} \eta^{ab}\, F_{cd} F^{cd} \ . \tag{2.134}$$

which recovers the standard expressions for the energy density, $T^{00} = \frac{1}{2}\,(\vec{E}^{\,2} + \vec{B}^{\,2})$, and energy flux, $T^{0i} = (\vec{E} \times \vec{B})^i$, the so-called Poynting vector.

This *ansatz* method is efficient in finding the expression for T_{ab}. It is often the case that listing the Lorentz structures with two derivatives and requiring conservation $\partial_a T^{ab} = 0$ fixes the energy-momentum tensor for most field theories of interest. For instance, for the Klein–Gordon scalar model (2.83) we find in this way

$$T^{ab}_{\text{KG}} = \partial^a \phi \partial^b \phi - \eta^{ab} (\tfrac{1}{2}\,(\partial\phi)^2 + V(\phi)) \ . \tag{2.135}$$

As a check, we see that T_{00} indeed gives the energy density of the scalar field

$$T_{00} = \rho_\phi = \tfrac{1}{2}\,(\partial_t \phi)^2 + \tfrac{1}{2}\,(\vec{\partial}\phi)^2 + V(\phi)\ .$$

The energy-momentum tensor of the Maxwell field has the important property of being *traceless*, $T^a{}_a = 0$. For the massive vector theory this trace is instead proportional to the 'photon mass':

$$T^{ab}_{\rm Proca} = T^{ab}_{\rm Maxwell} - \tfrac{1}{2}\, m^2\, \eta^{ab}\, A_c\, A^c\ , \qquad \eta_{cd}\, T^{cd}_{\rm Proca} = -2m^2\, A_a\, A^a\ .$$

The trace of the energy-momentum tensor is in general sensitive to the mass terms in the problem. For example, for a system of particles, we already noticed that the trace of the energy-momentum tensor is controlled by the scalar density of 'rest mass':

$$T^a{}_a\Big|_{\rm particles} = -J_m = -\sum_p \int d\tau_p\, m_p\, \delta^{(4)}(x - x(\tau_p))\ .$$

An important exception to this rule is the case of massless scalars, for which T^a_a *does not* vanish except in two dimensions.

The Noether Way

On general grounds, energy and momentum are ultimately defined as Noether charges for the symmetries of time and space translations. So we can determine the energy-momentum tensor as a set of Noether currents for Minkowski spacetime translations following the formula (2.127).

Let the translation along the x^b direction be expressed as $\delta_b x^a = -\epsilon\, \xi^a_{(b)}$, for the vector field $\xi^a_{(b)} = \delta^a_b$. Since any Lagrangian density is a Lorentz scalar, we have

$$\delta_b \mathcal{L} = \epsilon\, \pounds_{\xi_{(b)}} \mathcal{L} = \epsilon\, \xi^a_{(b)} \partial_a \mathcal{L} = \epsilon\, \delta^a_b \mathcal{L}\ ,$$

so that the Lagrangian varies by a total derivative with $f^a_{\xi_{(b)}} = \delta^a_b \mathcal{L}$. The variation of any Lorentz-tensor field is given by

$$\delta_b \Psi = \epsilon\, \pounds_{\xi_{(b)}} \Psi = \epsilon\, \xi^c_{(b)} \partial_c \Psi = \epsilon \partial_b \Psi\ ,$$

where we have suppressed the Lorentz indices of Ψ and used the fact that $\xi^a_{(b)}$ has vanishing derivatives. Applying then the general formula (2.127) we obtain a set of four Noether currents

$$\mathcal{J}^a{}_{(b)} = \frac{\partial \mathcal{L}}{\partial(\partial_a \Psi)}\, \partial_b \Psi - \delta^a_b \mathcal{L}\ . \tag{2.136}$$

Evaluating for the scalar case and comparing with (2.135) we find that

$$T_{ab} = -\mathcal{J}_{a\,(b)}\ . \tag{2.137}$$

Notice the different status of the indices in this expression. The index a is an ordinary vector index of each current, but the index b is a label of the particular current we are considering. In fact, the so-defined tensor T_{ab} is not symmetric in general. To see the tension more explicitly, we can now apply the formalism to the Maxwell theory, with Lagrangian

$$\mathcal{L}_{\rm Maxwell} = -\tfrac{1}{4}\, F_{ab} F^{ab}\ .$$

The naive energy-momentum tensor for $\delta_a A_b = \epsilon\, \partial_a A_b$ turns out to be given by

$$F_a{}^c\, \partial_b A_c + \eta_{ab}\, \mathcal{L}_{\text{Maxwell}} \ , \tag{2.138}$$

which is neither symmetric nor gauge-invariant. The lack of symmetry is inconvenient, but the lack of gauge-invariance is just not acceptable, since the energy momentum tensor gives physical energy and momentum densities, and these cannot depend on our choice of vector potential. We can fix both problems by defining a gauge-invariant field variation:

$$\bar{\delta}_b A_c \equiv \delta_b A_c + \delta'_b A_c$$

with $\delta'_b A_c \equiv \partial_c(-\epsilon\, A_b)$ an appropriate field-dependent infinitesimal gauge transformation. The new gauge-invariant translation is $\bar{\delta}_b A_c = \epsilon\, F_{bc}$ which, if used in the Noether procedure, yields the correct gauge-invariant energy-momentum tensor,

$$T^{ab}_{\text{Maxwell}} = -F^a_c\, F^{cb} + \eta^{ab}\, \mathcal{L}_{\text{Maxwell}} \ . \tag{2.139}$$

An interesting comment to make is that the difference between the naive (2.138) and the true (2.139) Maxwell energy-momentum tensors is the term $F_{ca}\, \partial^c A_b$ which, up to the equation of motion $\partial^c F_{ca} = 0$, is equivalent to the total derivative $\partial^c(F_{ca} A_b)$. This is an example of a subterfuge called 'improvement'. Just as Lagrangians are not completely determined by the equations of motion, allowing redefinitions by total derivatives, local currents are not completely determined by the sole requirements of symmetry and conservation. In principle, any additive redefinition by a local functional (depending on derivatives of fields at most polynomially) which is a total derivative and preserves conservation (using the equations of motion), introduces a valid current. Thus for any Noether current J^a_s, the 'Noether ambiguity' $J^a_s \to J^a_s + \partial_c N^{[ca]}$ allows us to redefine the current, provided the tensor $N^{[ca]}$ is antisymmetric to ensure conservation. Being a total derivative, the added term does not affect the global charges obtained by integrating the current over any constant-time hypersurface (assuming as usual that we do not pick boundary terms at infinity).

The Noether ambiguity for the energy-momentum tensor is any additive redefinition of the form

$$T_{ab} \longrightarrow T_{ab} + \partial^c N_{[ca]b} \ , \tag{2.140}$$

where the tensor N_{cab} is assumed antisymmetric in the first two indices. For the Maxwell case we had $N_{cab} = F_{ca} A_b$. More generally, it was shown by Belifante that one can always use the Noether ambiguity (2.140) to *improve* the canonical energy momentum tensor (2.137) into a symmetric one.

Why the insistence on a symmetric energy-momentum tensor? One reason is pure convenience. As indicated in (2.74) in the section on the energy-momentum of particle systems, given a conserved and *symmetric* T_{ab}, the set of currents $M^{abc} \equiv x^a T^{bc} - x^b T^{ac}$ are automatically conserved $\partial_c M^{abc} = 0$ and their associated charges,

$$J^{ab} = \int_{\mathbf{R}^3} M^{ab0} \ , \tag{2.141}$$

define the relativistic generalization of angular momenta. They give literally the angular momenta for spatial indices J^{ij}, and give the charges associated to the Lorentz boosts in the J^{0i} components. Thus, the set T^{ab}, M^{abc} gives the complete list of currents of the ten-parameter

Poincaré isometries, and all of this is still true for any field theory whose energy-momentum tensor is symmetric. Notwithstanding all this, the most important reason for insisting on symmetric T_{ab} is the starring role of the energy-momentum tensor in Einstein's theory of gravitation. It plays for gravity the same role that the electric current plays for electromagnetism, but the version of the energy-momentum tensor that enters the Einstein equations is *always* a symmetric one. Therefore, it is reassuring that one can always choose a symmetric T_{ab} for any reasonable system of particles and fields.

The art of energy-momentum tensor improvement does not stop at Noether ambiguities though. As we noticed above, the relation between the absence of masses and the vanishing trace $T^a_a = 0$ is broken in the case of massless Klein–Gordon field, which satisfies

$$T_{\mathrm{KG}}{}^{a}{}_{a} = -(\partial\phi)^2 \ .$$

There is, however, an improvement of the energy-momentum tensor that makes it traceless (as always, after using the equations of motion). It is given by

$$T_{ab} \longrightarrow T_{ab} - \frac{1}{6}\left(\partial_a\,\partial_b - \eta_{ab}\,\partial^2\right)\,\phi^2 \ . \tag{2.142}$$

This improvement term is different from the Noether ambiguity types but, like those, it also preserves the local conservation and it is a total derivative, so that it does not contribute to global conserved charges.

Problem: Energy-Momentum Conservation and Equations of Motion

Show that the local conservation of the energy-momentum tensor, $\partial_a T^{ab} = 0$, implies the equations of motion for both scalar and electromagnetic fields, provided certain genericity conditions are met. Namely the equations of motion are satisfied except at points where $\partial_a\phi = 0$ in the scalar field case. For Maxwell fields one must require that the matrix F^a_b be invertible.

Problem: Scale Invariance

Find the transformation rules that make the action of a free massless scalar field invariant under the scale transformations (dilatations) $x^a \to e^{-\epsilon}\, x^a$, with real ϵ. Show that a mass term breaks the symmetry but there exists one particular polynomial potential which still preserves scale invariance. Apply Noether's theorem to this theory to define the conserved dilatation current D^a. How is it related to the energy-momentum tensor? Find an improvement that brings the dilatation current to be of the form $D^a = x_b T^{ba}$, which is indeed conserved when $T^a_a = 0$.

Chapter 3

The Principle of Equivalence

It is certainly tempting to apply the previous machinery of relativistic field theory to the problem of gravitation. That is to say, to postulate some tensorial potential that generalizes the Newtonian gravitational potential.

Given the formal similarity between the Newton potential and the Coulomb potential, it looks natural to follow the blueprint of Maxwell's theory by postulating a vector-like 'gravitational potential' g_a with field strength $G_{ab} = \partial_a g_b - \partial_b g_a$ and a time-like component proportional to the Newtonian potential $g^0 \sim \phi_{\rm N}$. A Lagrangian for such a theory would read

$$\mathcal{L}_{\rm G-vector} = -\frac{1}{16\pi G}\, G_{ab}\, G^{ab} - g_a\, J_m^a \ ,$$

where J_m^a is the current of mass. In particular, its time component is exactly the rest-mass density,

$$J_m^0 = \sum_p m_p\, \delta^{(3)}(\vec{x} - \vec{x}_p(t))$$

for a system of free particles. So this theory has the peculiarity that the potential of a static source depends on rest masses rather than energies. This would imply that a bucket of radiation with internal energy E would not gravitate proportionally to a mass E/c^2, a situation that would run counter to the fundamental equivalence between mass and energy that is observed in non-gravitational relativistic physics, including quantum mechanical processes. In any case, a more urgent problem of the vector theory is the fact that equal-sign masses will repel one another! This can be momentarily fixed by changing the sign of the Newton constant, $G \to -G$, but then one finds the gravitational energy density to be *negative definite* [1]

$$T^{00}_{\rm grav} = -\frac{1}{8\pi |G|}\left(\vec{E}_G^2 + \vec{B}_G^2\right) < 0 \ ,$$

a severe problem for stability.

A much better attempt at a relativistic generalization of Newton's theory would promote $\phi_{\rm N}$ to a fully fledged Lorentz scalar field $\mathrm{g}(x)$ with Lagrangian

$$\mathcal{L}_{\rm G-scalar} = -\frac{1}{8\pi G}(\partial \mathrm{g})^2 - \mathrm{g}\, J_m \ ,$$

[1] This particular problem was known to Maxwell himself, who tried to construct such a theory of gravitation.

where now J_m is the Lorentz-invariant mass 'density'. For a system of free particles we have

$$J_m = \sum_p \int d\tau_p \, m_p \, \delta^{(4)}(x - x_p(\tau_p)) \; .$$

In fact, as shown in the discussion after (2.60), such density is essentially the trace of the energy-momentum tensor: $J_m = -T^a_a$. This means that such scalar gravity field, while leading to attractive forces, fails to couple to electromagnetic fields and for a fluid of density ρ and pressure p, it couples to the net combination $\rho - 3p$, which vanishes for ultrarelativistic fluids (radiation). Therefore, the scalar theory also fails to incorporate the gravitation of 'internal energies' in a satisfactory way. It is also in disagreement with experiment, since it predicts no bending of light in a gravitational field and a wrong precession effect for planetary orbits. Nevertheless, the scalar gravity theory is logically consistent, and was indeed seriously considered by many authors, including the likes of Abraham, Nördstrom and Einstein. Today, it survives in the so-called Jordan–Brans–Dicke form as a possible extension of General Relativity, albeit significantly constrained by precision experimental tests.

If we insist that gravity should couple to energy in the most natural way, we should use the energy-momentum tensor in the role of 'matter current', since it is T_{ab} the object that encapsulates the local conservation of energy in SR. In this way, one is naturally postulating a gravitational potential of tensor nature, i.e. a minimal coupling of the form

$$S_{\text{int}} \sim \int d^4x \; h_{ab} \, T^{ab} \; , \tag{3.1}$$

between T_{ab} and a symmetry rank-two gravitational potential h_{ab}. The local conservation of the energy-momentum tensor, $\partial_a T^{ab} = 0$, implies that this Lagrangian interaction is 'gauge invariant' under the transformations

$$h_{ab} \to h_{ab} + \partial_a f_b + \partial_b f_a \; , \tag{3.2}$$

with $f_a(x)$ an arbitrary four-vector function. Hence, we are led to the development of a symmetric two-index generalization of Maxwell's theory, the Fierz–Pauli theory (2.113), coupled to the matter energy-momentum tensor through (3.1).

However, such a program quickly becomes treacherous. The reason is that any field coupling to energy must necessarily couple to itself, resulting in a nonlinear theory. The gist of the argument is as follows. The Fierz–Pauli Lagrangian contributes a term to the total energy-momentum tensor quadratic in the h-field, i.e. $T_{(h)} \sim (\partial h)(\partial h)$, suppressing the detailed index structure. Since h_{ab} must couple to all forms of energy, including the energy contained in the h-field itself, we must write (3.1) in the 'improved' form

$$S_{\text{int}} \sim \int h_{ab} \left(T^{ab}_{\text{matter}} + T^{ab}_{(h)} \right) \; ,$$

which induces a cubic interaction term in the h_{ab} field with the structure $h \, \partial h \, \partial h$. In turn, taking this term into account in the computation of the h-field energy momentum tensor, we will obtain a similar cubic term in $T^{ab}_{(h)}$, which induces a quartic term when fed into the interaction term, and so on... Hence, we are led to an iterative construction of a full non-linear theory, with arbitrarily large powers of h_{ab}, albeit with only two derivatives. Feynman and others were able to show that this iterative construction indeed converges to Einstein's theory of General Relativity. We shall return to this issue later on, but for the time being we will be following another route.

3.1 From Galileo to Einstein

Einstein chose to develop the theory of gravity on top of the Principle of Equivalence between gravitation and inertia. A principle (going back to the tales of Galileo at the Pisa tower) stating, in its weakest version, that the gravitational 'charge' of particles is universally proportional to its inertial mass.

When a Lagrangian shows some constraint in its couplings to a great accuracy (recall Eötvös measurements), it is natural to try representing this constraint as the result of some deeper fact. In the case at hand, this is possible in a rather interesting way. The Newtonian equation of motion for a particle of mass m on some gravitational potential is

$$m\,\frac{d\vec{v}}{dt} = -m\,\vec{\partial}\,\phi_{\rm N}\ , \tag{3.3}$$

and the Principle of Equivalence simply means that m drops from this equation, so that trajectories are intrinsically determined by initial conditions, independently of the particular mass considered. This suggest that these trajectories have an *intrinsic* geometrical meaning. Consider, for example, a constant field with uniform gravity acceleration $\vec{g}$. The equation of motion of a falling particle is

$$\frac{d^2\vec{x}}{dt^2} = \vec{g}\ . \tag{3.4}$$

Now, in a reference frame in free fall,

$$\vec{x}_{\rm ff} = \vec{x} - \tfrac{1}{2}\,\vec{g}\,t^2 \tag{3.5}$$

we have $d^2\vec{x}_{\rm ff}/dt^2 = 0$, recovering the familiar feature that the gravitational field disappears in free fall. However, this was ultimately possible due to the particle mass dropping from the equations. Hence, we can represent the Principle of Equivalence as a statement about the change of reference frames, by saying that all systems in free fall are inertial, in the sense that they feel no gravitational forces.

Conversely, a frame of reference under constant acceleration of magnitude $\vec{g}$ is indistinguishable from a rest frame immersed in a constant gravitational field of the same intensity. In general, this simulation of gravitational fields by accelerations is only possible locally. We shall refer to such gravitational fields that can disappear globally by entering free fall as 'fictitious' gravitational fields. In 'true' ones, one can still eliminate the gravitational field, but only locally. If the gravitational acceleration at the spacetime point P is given by $\vec{g}_P = -\vec{\partial}\phi_{\rm N}(P)$, the frame transformation $\vec{x}_{\rm ff} = \vec{x} - \frac{1}{2}\,\vec{g}_P\,t^2$ gives

$$\frac{d^2\vec{x}_{\rm ff}}{dt^2} = \vec{g}(\vec{x},t) - \vec{g}_P\ , \tag{3.6}$$

so that the acceleration vanishes at the point P. Away from the free-fall point P, there is a residual 'tidal' acceleration, proportional to the first derivatives of the gravitational force, or the second derivative of the potential. We can define the 'tidal tensor' for static fields as

$$\mathcal{R}_{ij} \equiv -\partial_i\partial_j\phi_{\rm N}\ , \tag{3.7}$$

which provides a local diagnostic of whether a gravitational field is 'true' or 'fictitious'. In particular, when the tidal tensor of a *static* field vanishes everywhere, the potential is linear in the spatial coordinates, $\phi_{\rm N}(\vec{x}) = a + \vec{b}\cdot\vec{x}$. In this case the gravitational force is $\vec{g}(\vec{x}) = -\vec{b}$, a uniform field, which may be removed by the free-fall frame $\vec{x}_{\rm ff} = \vec{x} + \frac{1}{2}\,\vec{b}\,t^2$. The inverse criterion is slightly subtle, since a rotating frame with constant angular velocity Ω has a centrifugal 'inertial' force $m\Omega^2\vec{x}$, formally following from a 'centrifugal' potential $\phi_{\rm cen} = -\frac{1}{2}\,m\Omega^2|\vec{x}|^2$, which *does* have a non-vanishing tidal tensor. In this case, it is the traceless part of the tidal tensor that vanishes in this particular fictitious field.[2]

In the relativistic theory, the Principle of Equivalence admits the same formulation: *gravitational fields can be locally eliminated by entering free fall*, except that this time it is Special Relativity (SR), rather than Newtonian particle mechanics, what is assumed to be valid at the freely falling frames. The Principle of Equivalence is sometimes qualified as 'weak' of 'strong' depending on the scope of its application. The weak version only refers to the mass of a probe particle dropping from the equation of motion. Stronger versions differ by the generality of phenomena becoming 'blind' to gravitation in free fall, from purely mechanical to electromagnetic, gravitational self-energy effects or even quantum effects. Experimental tests of the Principle of Equivalence show no deviation from the strongest versions, so we will not make distinctions unless otherwise stated and speak of *the* Principle of Equivalence.

Since the Principle of Equivalence must be applied locally in generic gravitational fields, there is a degree of vagueness as to the size of the region where free-fall conditions apply. We shall give a more mathematical definition of the principle in coming sections, but for the time being we will continue exploring fictitious gravitational fields in order to gain some intuition in the relativistic regime.

3.1.1 Relativistic Inertial Fields

There are at least two novel relativistic effects with far-reaching consequences in relation to the Principle of Equivalence. The first is the existence of a novel kind of time dilation effects, associated to gravitational fields, and second is the occurrence of non-euclidean geometry. Both effects can be illustrated in a simple heuristic way by studying appropriate fictitious gravitational fields.

The Relativistic Elevator and Rindler Spacetime

We saw that, according to the Principle of Equivalence, uniform and constant gravitational fields can be manufactured by constructing an elevator undergoing constant acceleration. In the relativistic theory, such an elevator would reach eventually the speed of light, so it is not obvious what is the relativistic concept associated to 'constant acceleration'. Given a timelike trajectory $x(\tau)$, with four-velocity

$$u^a = \frac{dx^a}{d\tau}\;, \qquad u^2 = -1\;, \tag{3.8}$$

we define the acceleration four-vector by

$$a^b = \frac{du^b}{d\tau}\;, \qquad a^2 = a_b\, a^b\;. \tag{3.9}$$

[2] We hope the reader is not too confused about the usual terminology in Newtonian theory, where one speaks of 'inertial forces' felt in 'non-inertial frames'.

Taking the derivative of $u^2 = -1$ we find that acceleration and velocity are orthogonal $a^b\, u_b = 0$. In an inertial frame x'^a momentarily at rest with respect to the accelerated trajectory we have $u = (1, \vec{0})$ and the orthogonality implies $a = (0, \vec{a}')$, $\vec{a}' \cdot \vec{a}' = a^2$. Hence, we shall define constant acceleration as $a^2 = g^2$ with g a pure number, constant in time.

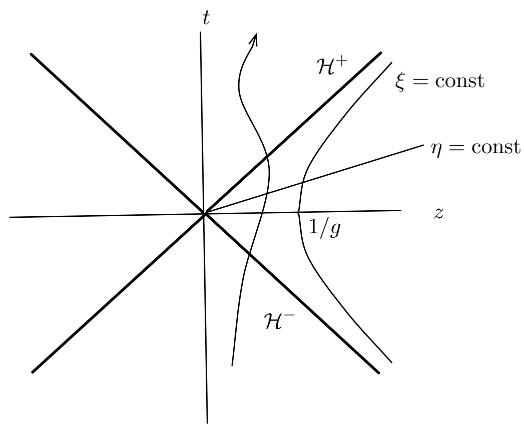

Figure 3.1: *The so-called Rindler wedge is the region* $0 < |t| < z$. *Uniformly accelerated observers follow the* $\xi =$ constant *hyperbolic trajectories, which degenerate into the past and future particle horizons* $\mathcal{H}^\pm$. *We also show the timelike trajectory of another observer. The part of her history to the future of* $\mathcal{H}^+$ *is completely inaccesible to the accelerated observer, whereas her crossing of the horizon takes an infinite subjective time for him.*

Now we go back to the 'laboratory' frame and take an accelerated rectilinear motion in the z-direction, $\vec{a} = d^2\vec{x}/d\tau^2 = (0, 0, a^z)$. We take $a^z > 0$ by convention. From the three algebraic equations

$$u^2 = -1 \;, \qquad u \cdot a = 0 \;, \qquad a^2 = g^2$$

we derive

$$a^0 = \frac{du^0}{d\tau} = g\, u^z \;, \qquad a^z = \frac{du^z}{d\tau} = g\, u^0 \;. \tag{3.10}$$

The solution of these ordinary differential equations is

$$(u^0 \pm u^z) = (u^0 \pm u^z)(0)\; e^{\pm g\tau} \;.$$

Choosing now $u^0(0) = 1$ and $u^z(0) = 0$ (zero velocity at $\tau = 0$) we obtain the parametric form of the trajectory,

$$t - t(0) = \frac{1}{g} \sinh(g\tau) \;, \qquad z - z(0) = \frac{1}{g}\left(\cosh(g\tau) - 1\right) \;, \tag{3.11}$$

and choosing the origin or coordinates so that $t(0) = 0, z(0) = 1/g$ we see that 'constantly accelerated' trajectories lie on the hyperbolae

$$z^2 - t^2 = \frac{1}{g^2} \;. \tag{3.12}$$

Now consider a family of accelerated observers running on hyperbolae $\xi =$ constant, according to the change of coordinates

$$z \pm t = \frac{1}{g}\, e^{g\xi}\, e^{\pm g\eta} \;. \tag{3.13}$$

Each such observer feels a constant acceleration $g_\xi = g e^{-g\xi}$. If we arrange all of them to pass through the $t = \eta = 0$ line simultaneously, then they will always share the same value of the η coordinate at all times. Any such line is related to the $t = 0$ line by a Lorentz transformation, so that the relative rest condition of the accelerated observers is maintained in time. Hence, the constructed family of accelerated observers furnishes a 'relativistic elevator'.

The space (η, ξ) with arbitrary real values of the coordinates is called *Rindler space*. It covers the wedge $0 < |t| < z$ of Minkowski space, and represents the part of it accesible to accelerated observers moving with positive uniform acceleration in the z-direction. Locally, Rindler space has identical geometry to Minkowski space, although the metric in Rindler coordinates reads

$$ds^2 = -dt^2 + dz^2 + dx^2 + dy^2 = e^{2g\xi}(-d\eta^2 + d\xi^2) + dx^2 + dy^2 \;. \tag{3.14}$$

The future boundary of the Rindler wedge $\mathcal{H}^+$ is called the future *horizon*. Beyond this horizon, we find all the events that are causally disconnected from any accelerated observer in Rindler space. An observer at rest at $z > 1/g$ will eventually be passed by our accelerated elevator and by the time her worldline passes through the curve $z = t$, any subsequent signal that might emit will never reach the elevator. At the same time, the observer on the elevator will never see her cross the horizon, her motions being constantly elongated in time by an infinite time dilation factor.

There is an analogous notion of 'past horizon', $\mathcal{H}^-$, which is defined by the time-reversal of the future horizon. Such notions of causal horizons feature prominently in the theory of black holes and also on the global properties of cosmological spacetimes.

The proper time of each observer at fixed ξ can be computed from the relation (3.11)

$$t(\tau_\xi) = \frac{1}{g} e^{g\xi} \sinh\left(g\eta(\tau_\xi)\right) = \frac{1}{g_\xi} \sinh\left(g_\xi\, \tau_\xi\right)$$

which results in $g\eta(\tau_\xi) = g_\xi\, \tau_\xi$, or

$$\tau_\xi = \eta\, e^{g\xi} \;. \tag{3.15}$$

Using this result, we find the ratio of proper times for observers at different heights in the elevator to be

$$\frac{\Delta\tau(\xi)}{\Delta\tau(\xi')} = e^{g(\xi - \xi')} = \frac{g_{\xi'}}{g_\xi} \;. \tag{3.16}$$

According to the Principle of Equivalence, this means that clocks run slower the more intense the gravitational field. This effect is one of the classic predictions of GR, called 'gravitational red shift'. We can derive a local version of this important fact by considering two clocks 'at rest' in the elevator frame (that is to say, at equal values of η), and small proper distance apart in 'elevator height'. Putting them at the same location in the transverse (x, y) plane, their height positions are ξ and $\xi' = \xi - \Delta\xi$. To leading order in $\Delta\xi$ their proper distance is then $h = \Delta s \approx e^{g\xi}\Delta\xi$. Applying (3.16) to leading order in the height, h, we find for the relative period shift of clocks

$$\frac{\Delta T}{T} = \frac{\Delta\tau_\xi - \Delta\tau_{\xi-\Delta\xi}}{\Delta\tau_\xi} \approx g_\xi\, h \;. \tag{3.17}$$

A local observer sitting in the elevator interprets g_ξ as the intensity (acceleration) of the gravitational field at point ξ.

Rotation and Non-Euclidean Geometry

Consider two frames of reference in SR. K is inertial and K' rotates at angular velocity Ω. Imagine setting up an operational measure of π, i.e. we draw a circumference of radius R and measure its length. The ratio being 2π in the inertial system K. An observer on K' would use measuring rods that, according to K, suffer Lorentz contraction when laid along the circumference, but remain intact when measuring the radius. Hence, the observer on K' will measure a larger ratio of circumference to diameter, by a factor of the Lorentz contraction $\sqrt{1-v^2} = \sqrt{1-\Omega^2 R^2}$. This leads to a 'measurement' of π as

$$\pi_{K'} = \frac{\pi}{\sqrt{1-\Omega^2 R^2}} > \pi \ . \tag{3.18}$$

This is characteristic of spaces of *negative* curvature. According to the Principle of Equivalence, there is no local difference between accelerated frames and true gravitational fields, the two differing only at the level of global properties. Hence we should expect non-euclidean geometry to arise in true gravitational fields as a general rule. This simple *gedanken experiment* was pivotal in Einstein's own heuristic path to GR.

Incidentally, we can also use this example of the rotating frame to give a derivation of the gravitational time delay. Distributing clocks at different radial positions, their rhythm will be proportional to $\sqrt{1-v^2} = \sqrt{1-\Omega^2 R^2}$ as witnessed by the observers on K. Observers on K' will notice this delay and, seeing the clocks at rest, will interpret it as a consequence of the local centrifugal 'gravitational' field. The larger the centrifugal field, the slower the clock rate.

Problem: Rotating Clock Delay

Given two clocks in the rotating frame at radii r and $r-h$, with small h, show that the relative period shift to leading order in h is given by

$$|\Delta T/T| = g(r)\, h$$

where $g(r)$ is the local centrifugal acceleration measured at radius r. Check that $g(r)$ is given by the Newtonian formula $g(r) = \Omega^2 r$ in the non-relativistic approximation.

3.1.2 A Tale of Two Frames: a First Look at General Covariance

We have seen that, according to the Principle of Equivalence, we can regard any field of inertial forces, induced by a change of frame, as a 'fictitious' gravitational field. It is fictitious in the sense that we can make it disappear by simply changing the frame back to the inertial one, which looks to be in free fall from the point of view of the non-inertial frame. In other words, we are just describing Lorentz-invariant physics of Minkowski spacetime, in some general coordinate frame, a spacetime generalization of the well-known fact that we may describe Newtonian physics in special 'curvilinear' coordinates such as spherical, accelerated or rotating coordinates.

It will be useful to develop some formalism about such fictitious fields, in a manner that is as coordinate-independent as possible. The idea is to generalize the two cases of accelerated/rotating observers studied in the last two sections. We will refer to the inertial frame with Minkowskian coordinates x^a as the *frefo* frame, indicating that it is in 'free fall', in the sense of the Principle of Equivalence. We will also consider a frame y^α of curvilinear and/or accelerated coordinates, called the *fido* frame, for 'fiducial observer', referring to the general situation exemplified by the families of observers in uniform acceleration or rotation in the previous subsections. It could be that the *fido* frame has a physical interpretation in terms of standard observers, like the cases above, but in general we will interpret the y^α frame in purely mathematical terms, just as we do for polar spherical coordinates for instance. We will only ask for smoothness of the transformation $x^a \to y^\alpha$.

Differentiating the functions $x^a(y)$ we obtain the expression of the interval in arbitrary coordinates:

$$ds^2 = \eta_{ab}\, dx^a\, dx^b = g_{\alpha\beta}\, dy^\alpha\, dy^\beta \ , \tag{3.19}$$

where the 4×4 symmetric matrix of functions

$$g_{\alpha\beta}(y) \equiv \eta_{ab}\, \frac{\partial x^a}{\partial y^\alpha}\, \frac{\partial x^b}{\partial y^\beta} \tag{3.20}$$

is the metric transformed to the new coordinates. If we want to relate the metric in different choices of generalized coordinates we just apply the chain rule of differentiation to obtain

$${}^{(z)}g_{\alpha\beta}(z) = {}^{(y)}g_{\gamma\delta}(y)\, \frac{\partial y^\gamma}{\partial z^\alpha}\, \frac{\partial y^\delta}{\partial z^\beta} \ . \tag{3.21}$$

The transformation matrices $\partial y^\alpha / \partial z^\beta$ are general invertible matrices in $GL(4, \mathbf{R})$, with smooth dependence on the points of spacetime, for any smooth invertible map $z \to y(z)$, a so-called Diffeomorphism (Diff).[3] We see that the metric functions $g_{\alpha\beta}$ transform like a covariant tensor of rank two under this Diff group. The transformation rules are formally the same that we had for Lorentz tensors, with the $O(1,3)$ matrices replaced with the $GL(4)$ matrices, point by point. Notice however that, while the η_{ab} were Lorentz-invariant, the $g_{\alpha\beta}$ are only Diff-covariant, but *both* are valid representations of the Minkowski metric, because they compute the same interval (3.19).

Diff Tensor Calculus

Any Lorentz-covariant equation in SR, written in terms of Lorentz tensors, can written as a Diff-covariant equation by passing to Diff-covariant tensors. To do this, focus on the transformation matrices between a *frefo* and a *fido*:

$$E^a_\alpha(y) \equiv \frac{\partial x^a}{\partial y^\alpha}(y) \ . \tag{3.22}$$

The matrix of functions $E^a_\alpha(y)$ transform as contravariant Lorentz-vectors in the *frefo* index and as covariant Diff-vectors in the *fido* index. One can consider quantities related to these by an operation of raising and lowering of indices, with the convention that *frefo* indices are raised and

[3] We will always consider in these notes the 'small' Diff group, namely smooth maps continuously deformable to the identity. In particular, all these maps are orientation-preserving.

lowered with the Lorentz metric and the *fido* indices are lowered with the $g_{\alpha\beta}$ metric and raised with its inverse $g^{\alpha\beta} \equiv (g^{-1})^{\alpha\beta}$:

$$E_{a\alpha} = \eta_{ab}\, E^b_\alpha\ , \qquad E^{a\alpha} = g^{\alpha\beta}\, E^a_\beta\ , \qquad E^\alpha_a = \eta_{ab}\, g^{\alpha\beta}\, E^b_\beta\ , \tag{3.23}$$

transform as indicated by the position of the indices. Our definition of covariant vs contravariant transformations for Diff are the same we adopted for Lorentz, namely we say that $\partial_\alpha = \partial/\partial y^\alpha$ transforms *covariantly* in the index α, whereas dy^α transforms *contravariantly* in the index α, when these transformations are effected by applying the chain rule of differentiation:

$$\frac{\partial}{\partial y_\alpha} \to \frac{\partial z^\beta}{\partial y^\alpha}\,\frac{\partial}{\partial z^\beta}\ , \qquad dy^\alpha \to \frac{\partial y^\alpha}{\partial z^\beta}\, dz^\beta\ . \tag{3.24}$$

Then, given any Lorentz tensor of components $T^{\cdots a \cdots}_{\cdots b \cdots}(x)$, the related set of quantities

$$T^{\cdots\alpha\cdots}_{\cdots\beta\cdots}(y) \equiv T^{\cdots a\cdots}_{\cdots b\cdots}(x) \ \ \cdots E^\alpha_a(x) \cdots E^b_\beta(y)\ \cdots\ , \tag{3.25}$$

contains the same information as the $T^{\cdots a\cdots}_{\cdots b\cdots}(x)$, once the diffeomorphism $x \to y(x)$ is known. This expression defines the Diff-tensor version of the formal object

$$\mathbf{T} = T^{\cdots\alpha\cdots}_{\cdots\beta\cdots}(y)\ \cdots \otimes \frac{\partial}{\partial y^\alpha} \otimes \cdots \otimes dy^\beta \otimes \cdots \tag{3.26}$$

which is Diff-invariant.

The definition of the metric itself (3.20) is a particular case of the general rule (3.25) to construct Diff tensors out of Lorentz tensors. As we mentioned above, the metric loses its 'invariant-tensor' character when generalized to the Diff group. On the other hand, the Kronecker delta $\delta^a{}_b$ retains its invariance property, since its Diff version $\delta^\alpha{}_\beta = E^a_\alpha E^\beta_b \delta^b_a$ remains a Kronecker delta. The situation with the other classic invariant tensor, the Levi–Civita tensor ϵ_{abcd} is slightly different. Recall our conventions: $\epsilon_{abcd} = [abcd]$, the permutation symbol with $[0123] = 1$. Then, we notice that its Diff-covariantization

$$\epsilon_{abcd}\, E^a_\alpha\, E^b_\beta\, E^c_\gamma\, E^d_\delta \tag{3.27}$$

is completely antisymmetric in the indices $\alpha, \beta, \gamma, \delta$. We may then write

$$\epsilon_{abcd}\, E^a_\alpha\, E^b_\beta\, E^c_\gamma\, E^d_\delta = [\alpha\beta\gamma\delta]\, \epsilon_{abcd}\, E^a_0\, E^b_1\, E^c_2\, E^d_3 = [\alpha\beta\gamma\delta]\,\det(E)\ ,$$

where $\det(E)$ is a shorthand for the determinant of the matrix of functions $E^a{}_\beta$. Since the transformation $x^a \to y^\alpha$ is invertible, $\det(E)$ cannot vanish. In fact, its sign is positive if the *frefo* $\to$ *fido* map preserves orientation, and negative otherwise. Assuming that orientation is indeed preserved, we have $\det(E) > 0$. We will also adopt the following shorthand notation for the determinant of the metric matrix of functions:

$$g \equiv \det(g_{\alpha\beta})\ . \tag{3.28}$$

Then (3.20) implies that $g = \det(E)^2 \det(\eta) = -|\det(E)|^2$, so that we can write $\det(E) = (-g)^{1/2}$ and

$$\epsilon_{\alpha\beta\gamma\delta} \equiv \epsilon_{abcd}\, E^a_\alpha\, E^b_\beta\, E^c_\gamma\, E^d_\delta = \sqrt{-g}\,[\alpha\beta\gamma\delta] \tag{3.29}$$

defines a rank-four covariant Diff-tensor. Analogously, we can define the contravariant Levi–Civita tensor $\epsilon^{\alpha\beta\gamma\delta}$ by the usual raising of indices with the inverse metric $g^{\alpha\beta}$, and obtain an associated rank-four contravariant Diff-tensor

$$\epsilon^{\alpha\beta\gamma\delta} = -\frac{1}{\sqrt{-g}}\,[\alpha\beta\gamma\delta]\;. \tag{3.30}$$

In general, quantities that are tensors up to a power of the Jacobian of the transformation are called tensor densities. The object $\mathsf{T}^{(w)\,\cdots\alpha\cdots}_{\,\cdots\beta\cdots}$ is called *tensor density* of weight w when it transforms as

$$\mathsf{T}'^{\,(w)\,\cdots\mu\cdots}_{\,\cdots\nu\cdots}(x') = \left|\frac{\partial x}{\partial x'}\right|^{w}\;\cdots\frac{\partial x'^{\,\mu}}{\partial x^{\alpha}}\cdots\frac{\partial x^{\beta}}{\partial x'^{\,\nu}}\cdots\mathsf{T}^{(w)\,\cdots\alpha\cdots}_{\,\cdots\beta\cdots}(x)\;. \tag{3.31}$$

In addition to the permutation symbol, one notable example is the delta function $\delta^{(4)}(x-x_0)$, which transforms as a scalar density of weight one. In particular $(-g)^{-1/2}\delta^{(4)}(x-x_0)$ is a true scalar.

The Covariant Derivative

So far the construction of Diff tensors out of Lorentz tensors was quite straightforward. The next slightly non-trivial step involves defining a Diff-tensor version of the derivative operator. When acting on Diff-scalar functions, the ordinary *fido* derivative $\partial_\alpha\phi$ is related to the *frefo* derivative $\partial_a\phi$ by the standard formula (3.25), namely

$$\partial_\alpha\phi = E^a{}_\alpha\,\partial_a\phi\;.$$

It is interesting that this *does not* generalize to derivative operators acting on higher-rank Diff-tensors. In particular, the object $\partial_\alpha V_\beta$ is not a Diff-tensor, even if we define V_β as a covariant Diff-vector. The trouble arises when the derivative acts on the transformation matrices, which now are general functions of position. On the other hand, the general rule (3.25) does produce a Diff-tensor out of the Lorentz tensor $\partial_a V_b$, namely the object

$$\nabla_\alpha\,V_\beta \equiv E^a{}_\alpha\,E^b{}_\beta\,\partial_a\,V_b \tag{3.32}$$

is a rank-two covariant Diff-tensor by construction. By direct computation, we can easily find that

$$\partial_\alpha\,V_\beta = \partial_\alpha\,(E^a{}_\beta\,V_a) = (\partial_\alpha\,E^a{}_\beta)\,V_a + E^b{}_\beta\,\partial_\alpha\,V_b = E_a{}^\delta\,\partial_\alpha\,E^a{}_\beta\,V_\delta + E^a{}_\alpha\,E^b{}_\beta\,\partial_a\,V_b$$

Therefore, we can write the so-called *covariant derivative* acting on covariant Diff-vectors as

$$\nabla_\alpha\,V_\beta = \partial_\alpha\,V_\beta - \Gamma^\delta_{\alpha\beta}\,V_\delta\;, \tag{3.33}$$

where the coefficients

$$\Gamma^\delta_{\alpha\beta} \equiv (\partial_\alpha\,E^a{}_\beta)\,E_a{}^\delta = \frac{\partial^2 x^a}{\partial y^\alpha\partial y^\beta}\,E_a{}^\delta \tag{3.34}$$

depending on second derivatives of the *frefo*-*fido* mapping, are usually referred to as Christoffel symbols, or affine connection coefficients. They provide the correction term that is needed to write a covariant version of the derivative operator.

Straight from the definition (3.34) one can prove that the Christoffel symbols can be entirely expressed as linear combinations of the first derivatives of the metric:

$$\Gamma^{\delta}_{\alpha\beta} = \frac{1}{2}\, g^{\delta\gamma}\left(\partial_\alpha\, g_{\gamma\beta} + \partial_\beta\, g_{\alpha\gamma} - \partial_\gamma\, g_{\alpha\beta}\right)\ . \tag{3.35}$$

It is very important to remember that, despite the notation, the Christoffel symbols *do not* define a Diff tensor. This is just natural, since their purpose is to cancel the non-tensor components of the ordinary derivative when acting on Diff-tensors. Indeed, a direct calculation from either (3.34) or (3.35) show that, under a diffeomorphism $y \to z$, the connection coefficients transform as

$$^{(y)}\Gamma^{\delta}_{\alpha\beta} = {}^{(z)}\Gamma^{\rho}_{\gamma\sigma}\ \frac{\partial z^\gamma}{\partial y^\alpha}\frac{\partial y^\delta}{\partial z^\rho}\frac{\partial z^\sigma}{\partial y^\beta} + \frac{\partial z^\rho}{\partial y^\alpha}\frac{\partial z^\sigma}{\partial y^\beta}\frac{\partial^2 y^\delta}{\partial z^\rho\,\partial z^\sigma}\ . \tag{3.36}$$

There is a tensorial part and an inhomogenous part depending on second derivatives of the diffeomorphism. This non-tensorial character is responsible for the fact that the Christoffel symbols can vanish in a particular frame, such as the *frefo*, and not vanish in a generic *fido*.

Entirely similar manipulations allow us to define covariant derivatives acting on contravariant Diff-vectors, the result differing from (3.32) only by a sign in the correction term:

$$\nabla_\alpha\, V^\beta \equiv E^a{}_\alpha\, E_b{}^\beta\, \partial_a\, V^b = \partial_\alpha\, V^\beta + \Gamma^\beta_{\alpha\delta}\, V^\delta\ . \tag{3.37}$$

Upon further generalization along the same lines, we obtain the general rule for the covariant derivative of an arbitrary Diff tensor:

$$\nabla_\delta\, T^{\cdots\,\alpha\,\cdots}_{\cdots\,\beta\,\cdots} = \partial_\delta\, T^{\cdots\,\alpha\,\cdots}_{\cdots\,\beta\,\cdots} + \cdots + \Gamma^\alpha_{\delta\gamma}\, T^{\cdots\,\gamma\,\cdots}_{\cdots\,\beta\,\cdots} + \cdots - \Gamma^\gamma_{\delta\beta}\, T^{\cdots\,\alpha\,\cdots}_{\cdots\,\gamma\,\cdots} - \cdots\ . \tag{3.38}$$

It is not difficult to check that these definitions give a covariant derivative operator ∇_α satisfying Leibnitz's rule, namely $\nabla(T \otimes S) = (\nabla T) \otimes S + T \otimes (\nabla S)$ for any tensors T and S.

Using these expressions, one can derive a number of useful identities. The first and most important is the so-called *metric compatibility* of the covariant derivative, which simply means that the covariant derivative of the metric vanishes identically.

$$\nabla_\delta\, g_{\alpha\beta} = 0\ , \tag{3.39}$$

a property implying that the covariant derivative commutes with the operation of raising and lowering Diff-indices.

A second important property is that the Christoffel connection drops from the antisymmetric derivatives. Given an antisymmetric tensor $A_{\alpha\beta\cdots} = A_{[\alpha\beta\cdots]}$, the antisymmetric derivative satisfies

$$\nabla_{[\alpha} A_{\beta\cdots]} = \partial_{[\alpha} A_{\beta\cdots]}\ . \tag{3.40}$$

This follows technically from the symmetry of the connection in lower indices, $\Gamma_\alpha{}^\gamma{}_\beta = \Gamma_\beta{}^\gamma{}_\alpha$. A related identity applies to the contracted derivative, again acting on antisymmetric tensors:

$$\nabla_\alpha\, A^{[\alpha\cdots]} = \frac{1}{\sqrt{-g}}\partial_\alpha\left(\sqrt{-g}\, A^{[\alpha\cdots]}\right)\ . \tag{3.41}$$

These two relations allow us to generalize the exterior calculus on p-forms (the machinery associated to the exterior derivative operator d and its adjoint $d^\dagger$) in a Diff-covariant manner.

Together with the Diff versions of the Levi–Civita tensors, we can also define the Diff-covariant version of Poincaré duality and all the theory of integral invariants.

A particular case of (3.41) is the very useful expression for the Laplacian operator acting on a scalar function:

$$\nabla^2\phi = \nabla_\alpha \nabla^\alpha \phi = \frac{1}{\sqrt{-g}} \partial_\alpha \left(\sqrt{-g} g^{\alpha\beta} \partial_\beta \right) \phi \; . \tag{3.42}$$

Another important particular case is that of the covariant divergence of a vector field J^μ, which appears in the Diff-covariant version of the continuity equation $\nabla_\mu J^\mu = 0$. In this case we can write

$$\nabla_\mu J^\mu = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} J^\mu) \; . \tag{3.43}$$

This form can be used to prove the Diff-covariant version of Gauss' integration theorem for a total derivative over a four-dimensional region with boundary:

$$\int_{\mathrm{V}^{1+3}} \nabla_\mu J^\mu = \int_{\partial \mathrm{V}^{1+3}} n_\mu J^\mu \; , \tag{3.44}$$

where n^μ with $n_\mu n^\mu = \pm 1$ is a normal unit vector to $\partial \mathrm{V}^{1+3}$ with the convention that it is outward pointing when n^μ is spacelike and inward pointing when n^μ is timelike.

Another handy device is the ability to compute a derivative of a tensor along a curve γ, parametrized by functions $x^a(\sigma)$ or $y^\alpha(\sigma)$. In the *frefo* we have

$$\frac{d}{d\sigma} T^{\;\cdots\, a\, \cdots}_{\;\cdots\, b\, \cdots} = \frac{dx^c}{d\sigma} \partial_c T^{\;\cdots\, a\, \cdots}_{\;\cdots\, b\, \cdots} \; .$$

Now the expression on the right hand side can be Diff-covariantized into

$$\frac{\nabla}{d\sigma} T^{\;\cdots\, \alpha\, \cdots}_{\;\cdots\, \beta\, \cdots} = \frac{dy^\delta}{d\sigma} \nabla_\delta T^{\;\ldots\alpha\ldots}_{\;\ldots\beta\ldots} \tag{3.45}$$

where we have introduced a new notation, $\nabla/d\sigma$, for the notion of 'covariant derivative along the parametrized curve'. Since the tangent vector γ_σ to the parametrized curve γ, with components $dy^\delta/d\sigma$, is a Diff-vector, we can also write this as a 'directional covariant derivative' in the direction of the tangent vector,

$$\frac{\nabla}{d\sigma} T^{\ldots\alpha\ldots}_{\ldots\beta\ldots} = \nabla_{\gamma_\sigma} T^{\;\cdots\alpha\cdots}_{\;\ldots\beta\cdots} \tag{3.46}$$

where

$$\nabla_V = V^\alpha \nabla_\alpha \tag{3.47}$$

is the directional covariant derivative operator along the vector V.

An example where we can put these definitions to work is the equation of motion for a free massive particle in SR. In a *frefo* frame it reads

$$\frac{d^2 x^a}{d\tau^2} = \frac{du^a}{d\tau} = 0 \; , \tag{3.48}$$

where τ is the proper time along the parametrized spacetime path $x^a(\tau)$ of the particle and $u^a = dx^a/d\tau$ is the four-velocity. In a *fido* frame the path is described by functions $y^\alpha(\tau)$ and the equation of motion can be written in a variety of forms:

$$\frac{\nabla^2 y^\alpha}{d\tau^2} = \frac{\nabla u^\alpha}{d\tau} = u^\beta \nabla_\beta u^\alpha = \nabla_u u^\alpha = 0 \; , \tag{3.49}$$

or, in full component glory:

$$\frac{d^2y^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\delta}\,\frac{dy^\beta}{d\tau}\,\frac{dy^\delta}{d\tau} = 0\ . \tag{3.50}$$

Another example is Maxwell's equation, $\partial_a\, F^{ab} = -J^b$, which can be covariantized into

$$\nabla_\alpha F^{\alpha\beta} = -J^\beta\ . \tag{3.51}$$

A further useful presentation of the same equation is

$$\frac{1}{\sqrt{-g}}\partial_\alpha(\sqrt{-g}\,F^{\alpha\beta}) = -J^\beta\ . \tag{3.52}$$

Physics, what Physics?

We have introduced in the previous sections a large part of the mathematical machinery that is used in the theory of General Relativity, yet one can say that the exercise is purely formal. Indeed, all we were doing is describing good old SR in a Diff-covariant way. We never left Minkowski spacetime, and all the Diff-machinery could evaporate immediately if we decided to use a *frefo* frame throughout $\mathbf{R}^{1+3}$.

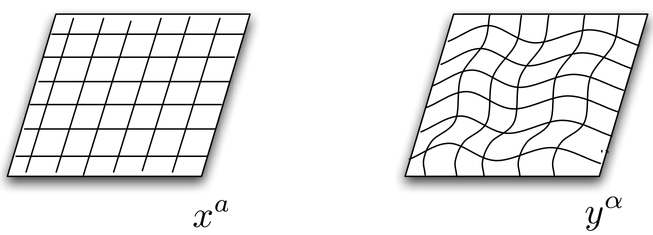

Figure 3.2: *An arbitrary 'fictitious' field is just Minkowski spacetime (left) disguised in a general system of curvilinear and/or accelerated coordinates (right).*

The ability to express physical equations in terms of a general coordinate system, the Diff-covariant description outlined above, is called *general covariance* and it is by itself devoid of any physical content. Every equation can be made Diff-covariant without changing its physical content. If the equation is local, it can be written in terms of local quantities over spacetime and its derivatives. Then, the dependence on the arbitrary frame is for the most part encapsulated in the metric matrix $g_{\alpha\beta}$ and the Christoffel symbols $\Gamma^\delta_{\alpha\beta}$.

Let us take up the example of the particle equation of motion (3.50). In the *frefo* we have the standard $d^2x^a/d\tau^2 = 0$ which is solved by straight lines $x^a(\tau) = x^a_0 + u^a\tau$ with sub-luminal velocity. In the *fido*, the histories $y^\alpha(\tau)$ will not be straight lines in general of course, but they will be the images of the straight lines, under the *frefo* → *fido* map. Their generalized acceleration will be equal to a non-trivial force proportional to four-velocities which is nothing but relativistic generalization of the good old Newtonian 'inertial' forces:

$$\frac{d^2y^\alpha}{d\tau^2} = \frac{1}{m}F^\alpha_{\text{inertial}} = -\Gamma^\alpha_{\delta\,\gamma}\,\frac{dy^\gamma}{d\tau}\frac{dy^\delta}{d\tau}\ . \tag{3.53}$$

With regard to gravitation, the physical content of this discussion is just the *interpretation* of inertial forces as 'gravitational' in the sense that, locally there is no way to tell the difference. The reader may feel that the generally covariant formalism introduced in this section is a bit heavy handed and not quite worth the effort, but we will reap the benefits in the next section.

Problem: Relativistic Rotation

We discuss here relativistic inertial forces under rotation. Let (t, z, r, θ) be standard cylindrical coordinates in Minkowski spacetime, with metric

$$ds^2 = -dt^2 + dz^2 + dr^2 + r^2 d\theta^2 \ ,$$

and pass to a rotating frame (t, z, r, ϕ) with $\phi = \theta - \Omega\, t$. Find the metric in the rotating frame, compute the Christoffel symbols and write down the equation of motion for massive particles. Interpreting (3.53) as the relativistic Newton law with inertial forces, take the non-relativistic limit and find the expected centrifugal and Coriolis inertial forces.

3.2 Riemannian Geometry and the Principle of Equivalence

Having exploited the Principle of Equivalence for the particular case of fictitious gravitational fields, those that can be regarded as having purely inertial forces, we are now ready to consider the more general case in which the condition of 'free fall' can only be implemented locally around a point in spacetime.

Let us consider a spacetime X^{1+3} with a general gravitational field. We characterize the Principle of Equivalence in a pragmatic way as follows: Around any point $P \in \mathrm{X}^{1+3}$, we can erect a *frefo* frame through P, denoted X^a_P, with the property that Special Relativity is locally valid in a small neighborhood of P, when described in this particular frame. The whole spacetime X^{1+3} can be covered by such *frefos*. We can also refer all phenomena to an external frame, x^μ, the *fido*, to signify the fact that it may be associated to a fixed set of fiducial observers. Notice that no restrictions are placed on the properties of the *fido*, and no canonical choice of *frefos* throughout spacetime exists in general, if only because any *frefo* can be rotated by a Lorentz transformation and remain a valid *frefo*, and this can be done independently at each point in spacetime.

3.2.1 Particle Probes

There is some uneasy vagueness to the previous statements; just how small must the *frefo* region around P be in order claim that SR applies? In order to make more precise what is meant by X^a_P being 'in free fall at P', we send a particle probe of mass m. Its spacetime path is parametrized in the *fido* by functions $x^\alpha(\sigma)$. We can partition the path as $\gamma = \cup_P \gamma_P$, with γ_P small 'pieces' around points P in the path, each of them parametrized in the *frefos* by the functions $X^a_P(\sigma)$. According to the Principle of Equivalence, the equation of motion at point P, when referred to the *frefo* at P, takes the special-relativistic form:

$$\frac{d^2 X^a_P}{d\tau^2}\Big|_P = 0\ . \tag{3.54}$$

But we can now use the Diff-covariant formalism that we developed for fictitious fields in the previous section, to map this equation into the *fido* frame. Defining the matrices of partial derivatives

$$E_P{}^a{}_\alpha(x) \equiv \frac{\partial X^a_P}{\partial x^\alpha}(x)\ , \tag{3.55}$$

we can carry over the formalism of Christoffel connections to this case, replacing the previous matrices of functions E^a_α by the new ones (3.55), referred to the local *frefo* at P.[4]The result is

$$\left(\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\,\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\right)\Big|_P = 0\ . \tag{3.56}$$

In this expression, the Christoffel symbols have the same definition as they had for fictitious fields:

$$\Gamma^\delta_{\alpha\beta}(P) = E_P{}_a{}^\delta(P)\ (\partial_\alpha\, E_P{}^a{}_\beta)|_P = E_P{}_a{}^\delta(P)\ \frac{\partial^2 X^a_P}{\partial x^\alpha \partial x^\beta}\Big|_P\ , \tag{3.57}$$

[4]We emphasize that the functions (3.55) do extend throughout a neighborhood P, but they are *different* functions for different anchoring points P.

and also the same expression (3.34) in terms of first derivatives of the metric. The metric itself in the *fido* frame is given by

$$g_{\alpha\beta}(P) = E_P{}^a{}_\alpha(P)\, E_P{}^b{}_\beta(P)\, \eta_{ab}\ , \tag{3.58}$$

and it computes the same proper time interval as the Minkowski metric, provided $d\tau$ is small enough to be well-approximated by a given *frefo*. The crucial difference with the case of fictitious fields is the absence of a global, unique *frefo*. Rather, in this case the metric can be computed using data from each *frefo* at P, but the *frefo* changes from point to point. In general, there will *not* exist a global *frefo* on which the metric takes the form η_{ab} throughout a finite region.

We conclude that the Principle of Equivalence, together with Special Relativity, just forces upon us the description of gravitation in terms of a Riemannian metric, namely a set of functions $g_{\mu\nu}$ which compute the proper time as

$$d\tau^2 = -g_{\mu\nu}\, dx^\mu dx^\nu\ , \tag{3.59}$$

and that transform as a rank-two covariant Diff-tensor under a smooth change of coordinates:

$$g_{\mu\nu}(x) \longrightarrow g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\,\frac{\partial x^\beta}{\partial x'^\nu}\, g_{\alpha\beta}(x)\ . \tag{3.60}$$

All the formulas involving Diff-covariant quantities which we derived for fictitious fields, including the rules for covariant differentiation (3.38) and the metric compatibility property (3.39), *do* apply in a general gravitational field, provided we use as fundamental building block the general Riemannian metric (3.59).

We are now in position to reduce the vagueness of the previous discussion. The Equivalence Principle states that, through any point P in spacetime, there exist a *frefo* X^a_P, with the property that, in this frame, the metric at P reduces to the Minkowski form and the Christoffel symbols vanish at P. Equivalently, all first derivatives of the metric vanish at P:

$$g_{ab}(P) = \eta_{ab}\ , \qquad \partial_c\, g_{ab}(P) = 0 = \Gamma^a{}_{bc}(P)\ . \tag{3.61}$$

Therefore, the 'size' of the *frefo* at P just depends on how far from P we may want to go, while ignoring the second order term in the Taylor expansion of the metric around P (controlled by the second derivatives of the metric). Fictitious fields are those for which a *frefo* exists such that this equations hold exactly a finite region of spacetime, so that the metric can be exactly put in Minkowskian form throughout that region.

Covariant Particle Action

We can obtain the same results by focusing on the action of the probe particle, rather than its equations of motion. We shall reproduce this reasoning here because it gives an interesting perspective on the formalism and it serves as a blueprint for the 'minimal coupling' prescriptions that will be introduced later. Let us split the path γ in small segments, each of them passing through *frefos* X_P. On each segment, the Principle of Equivalence dictates the form of the action to be the same as in SR. Piecing together all such segments and taking the limit of a fine partition we obtain

$$S[\gamma] = \lim_{\{P\}\to\gamma} \sum_P \Delta S[\gamma_P]\ , \tag{3.62}$$

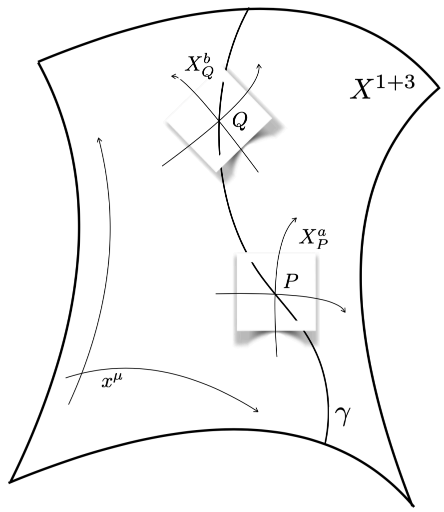

Figure 3.3: *A particle path γ through X^{1+3} with two frefos at points P and Q and an external fido frame x^μ. Notice that the frefos may extend beyond the region where they are approximately in free fall. Beyond this region, they will lose the frefo character except for fictitious fields.*

where

$$\Delta S[\gamma_P] \approx -m\Delta\tau_P = -m\Delta\sigma_P\sqrt{-\eta_{ab}\frac{dX^a_P}{d\sigma}\frac{dX^b_P}{d\sigma}}\ , \tag{3.63}$$

which gives in the limit

$$S[\gamma] = -m\int_\gamma d\tau = -m\int_\gamma d\sigma\sqrt{-\eta_{ab}\frac{dX^a_{P(\sigma)}}{d\sigma}\frac{dX^b_{P(\sigma)}}{d\sigma}}\ . \tag{3.64}$$

Hence the Principle of Equivalence establishes the particle action as proportional to the elapsed proper time, just as in ordinary SR, and provides a way to compute it, given a family of *frefos* along γ. This parametrization serves the purpose of *defining* the action through the Principle of Equivalence, but it is not particularly 'user-friendly', since it contains a large amount of redundant information, related to the arbitrary choice of *frefos* at each point in γ. All this is simplified by passing to the *fido* description using

$$\frac{dX^a_P}{d\sigma} = \frac{\partial X^a_P}{\partial x^\alpha}\frac{dx^\alpha}{d\sigma} = E_P{}^a{}_\alpha\frac{dx^\alpha}{d\sigma}\ , \tag{3.65}$$

which allows us to write

$$\eta_{ab}\frac{dX^a_{P(\sigma)}}{d\sigma}\frac{dX^b_{P(\sigma)}}{d\sigma} = g_{\alpha\beta}(x(\sigma))\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma}\ ,$$

and finally obtain the particle action in *fido* parametrization:

$$S[\gamma] = -m\int_\gamma d\tau = -m\int_\gamma d\sigma\sqrt{-g_{\alpha\beta}(x(\sigma))\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma}}\ . \tag{3.66}$$

All the redundancy implicit in the choice of *frefo* at each point has been removed and the physical description of the particle dynamics is completely specified once the Riemannian metric $g_{\mu\nu}$ is known.

We can actually go further and derive the equation of motion stemming from (3.66) by demanding the action to be an extremum with respect to variations of the trajectory $x^\alpha(\sigma) \to x^\alpha(\sigma) + \delta x^\alpha(\sigma)$. We find

$$0 = \delta S_{\rm P} = \int \frac{d\sigma}{2\sqrt{-g_{\alpha\beta}\,\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma}}}\,\delta\left(g_{\alpha\beta}\,\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma}\right) \,.$$

We may now take advantage of the reparametrization invariance to restore the proper time parameter $\sigma \to \tau$ to remove the square root in the denominator and write

$$0 = \int d\tau \left(\tfrac{1}{2}\,\partial_\mu g_{\alpha\beta}\,\dot{x}^\alpha \dot{x}^\beta\,\delta x^\mu + g_{\alpha\beta}\,\dot{x}^\alpha \frac{d}{d\tau}\delta x^\beta\right) \,.$$

Reshuffling the indices, integrating by parts and multiplying the resulting equation of motion by the inverse of the metric matrix we obtain the, by now familiar, equation of motion

$$0 = \frac{d^2 x^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \,, \tag{3.67}$$

where the Christoffel symbols appear directly in their expression in terms of the first derivatives of the metric:

$$\Gamma^\alpha_{\mu\nu} = \tfrac{1}{2}\,g^{\alpha\beta}\left(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}\right) \,. \tag{3.68}$$

3.2.2 Gravitational Forces

Writing the free fall equation of motion (3.67) in morally Newtonian form

$$m\frac{d^2 x^\alpha}{d\tau^2} = F^\alpha = -m\,\Gamma^\alpha_{\mu\nu}\,\frac{dx^\mu}{d\tau}\,\frac{dx^\nu}{d\tau} \,, \tag{3.69}$$

we see that relativistic gravitational forces are velocity-dependent. We already noticed this fact when considering fictitious gravitational fields, otherwise known as 'inertial forces', as in (3.53). It is certainly not surprising that inertial forces should depend on velocities, because they certainly do in the Newtonian case (think of the formula for Coriolis forces). The remarkable fact is that the full equation of motion in an arbitrary gravitational field, does have exactly the same structure as the relativistic generalization of inertial forces. In hindsight, this is just natural, because the Principle of Equivalence is predicated on the identification of gravitational and inertial forces at a local level. Since the equation of motion is certainly local, this is the expected result.

Structurally, the forces are also proportional to the connection coefficients Γ, so that the metric functions acquire the interpretation of 'gravitational potentials', since their first derivatives determine the 'forces'. We can make these intuitions more precise by taking the weak-field limit and further the Newtonian limit.

Weak Fields

In keeping with the idea that gravitational physics amounts to replacing the Minkowski metric η_{ab} by a general Riemannian metric $g_{\alpha\beta}$ we may study the case where gravitational effects are small, so that we write $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$, with $|h_{\alpha\beta}| \ll 1$, a condition that requires restricting the reference frames, since we may induce large components of the metric by simply using a frame with large accelerations. Hence, we assume that an 'almost' *frefo* frame extends over the region of interest and we only allow further coordinate transformations that maintain the 'small field' property. Writing those transformations as $x^\alpha \to x'^\alpha = x^\alpha - \varepsilon^\alpha(x)$, the metric transformations (3.60) imply

$$\eta_{\alpha\beta} + h_{\alpha\beta}(x) \to \eta_{\alpha\beta} + h'_{\alpha\beta}(x') = \frac{\partial x^\gamma}{\partial x'^\alpha}\frac{\partial x^\delta}{\partial x'^\beta}\left(\eta_{\gamma\delta} + h_{\gamma\delta}(x)\right) .$$

Upon explicit computation to leading order in ε^α one finds that the *functional* shift in $h_{\alpha\beta}$ is given by

$$h'_{\alpha\beta}(x) - h_{\alpha\beta}(x) = \partial_\alpha \varepsilon_\beta(x) + \partial_\beta \varepsilon_\alpha(x) + O(\varepsilon^2) . \tag{3.70}$$

Therefore, the condition on the allowed 'weak-diffeomorphisms' is $|\partial_\alpha \varepsilon_\beta| \ll 1$. Incidentally, the weak-diffeormorphisms act on the weak gravitational field $h_{\alpha\beta}$ as a gauge symmetry for a Lorentz-covariant symmetric tensor field. Expanding now the particle action to leading order in $h_{\alpha\beta}$ one finds

$$-m\int_\gamma d\sigma\sqrt{-(\eta_{\alpha\beta}+h_{\alpha\beta})\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma}} = S[\gamma]_{\text{Minkowski}} + \tfrac{1}{2}\, m\int d\tau\, h_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} + O(h^2) , \tag{3.71}$$

where τ in the last term is defined as the proper time with respect to the Minkowski metric. The term linear in $h_{\alpha\beta}$ can be rewritten as

$$\tfrac{1}{2}\int d^4x\, h_{\alpha\beta}\, T^{\alpha\beta} ,$$

where $T^{\alpha\beta} = m\int d\tau \dot{x}^\alpha\, \dot{x}^\beta\, \delta^{(4)}(x - x(\tau))$ is the standard definition of the particle's energy momentum tensor in SR (cf. (3.1)). Hence, we conclude that the leading perturbative coupling of gravity, as dictated by the Principle of Equivalence, conforms to the general arguments based on SR. However, the Principle of Equivalence is much more powerful, giving a precise resummation of powers of $h_{\alpha\beta}$ in the matter-gravity coupling.

Newtonian Limit

We now sharpen the interpretation of Γ as gravitational forces by taking the non-relativistic limit of the equation of motion (3.67). It is useful to transform it from the proper-time parametrization to coordinate time parametrization. Using the chain rule one finds

$$\frac{d^2x^\alpha}{dt^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} = f(t)\frac{dx^\alpha}{dt} , \tag{3.72}$$

with

$$f(t) = -\frac{d^2t/d\tau^2}{(dt/d\tau)^2} .$$

We may consider now the spatial components of (3.72) in the weak-field regime, for *stationary* fields, $\partial_t h_{\alpha\beta} \ll \partial_i h_{\alpha\beta}$, and in the non-relativistic limit, $v^i = dx^i/dt \ll 1$. Keeping terms up to first order in velocities we obtain

$$\frac{dv_i}{dt} \approx \tfrac{1}{2}\,\partial_i\,h_{00} - \sum_j v_j(\partial_j h_{0i} - \partial_i h_{0j})\;. \tag{3.73}$$

The first term is the standard gradient gravitational force for a field with potential

$$\phi_{\rm N} \approx -\tfrac{1}{2}\,h_{00}\;, \tag{3.74}$$

whereas the second term has the form of a Coriolis force. To see this, notice that we may define the effective 'angular velocity' vector

$$\vec{\Omega} = \tfrac{1}{2}\,\vec{\partial} \times \vec{h}\;, \tag{3.75}$$

where $\vec{h} = (h_{0i})$. The effective Newton equation takes then the form

$$\frac{d\vec{v}}{dt} \approx -\vec{\partial}\,\phi_{\rm N} + 2\,\vec{v} \times \vec{\Omega}\;, \tag{3.76}$$

which contains the potential and Coriolis terms.[5] Hence, we confirm that the Principle of Equivalence *unifies* what in Newtonian theory is regarded as separate 'inertial' and 'gravitational' forces.

In practice this analysis shows that, once we pick a non-rotating frame, the leading Newtonian approximation to the motion of test particles is characterized by an effective metric

$$ds^2_{\rm Newtonian} \approx -(1 + 2\phi_{\rm N})\,dt^2 + d\vec{x}^{\,2}\;. \tag{3.77}$$

[5]If $\vec{\Omega}$ really represents the angular velocity of a rotating frame, there should be a centrifugal force proportional to Ω^2 in (3.76). This term is indeed found by going to second order in the weak field expansion (see the Problem on Relativistic Rotation).

3.3 Minimal Coupling

We are ready to use the formal machinery introduced so far to couple a generic Lorentz-invariant Lagrangian $\mathcal{L}(\Psi^{\cdots a \cdots}_{\cdots b \cdots}, \partial_a, \cdots)$ with arbitrary 'matter' fields in tensor representations of the Lorentz group, to an external gravitational field characterized by a metric field $g_{\alpha\beta}$. The strategy is identical to the one that worked before for free particles. We start by covering the spacetime X^{1+3} with *frefos* and specify the action in terms of the limit of a fine partition:

$$S = \lim_{\{P\} \to \mathrm{X}^{1+3}} \sum_P \Delta_P S \ , \tag{3.78}$$

with

$$\Delta_P S \approx (\Delta^4 X_P) \, \mathcal{L}(\Psi^{\cdots a \cdots}_{\cdots b \cdots}, \ \partial/\partial X^a_P, \cdots) \, |_P \tag{3.79}$$

the element of action in a *frefo*. We now convert to the reference *fido* with the translation of Lorentz tensorial structures to Diff tensorial structures. As for the volume measure we have

$$(\Delta^4 X_P) \approx (\Delta^4 x) \left| \det \left(\frac{\partial X_P}{\partial x} \right) \right| = \Delta^4 x \ |\det(E_P)| = \Delta^4 x \sqrt{-\det(g_{\alpha\beta})} \ ,$$

or, in the more familiar differential form $d^4 X = d^4 x \sqrt{-g}$. Hence, the coupling to gravity is reduced to a mere exercise in notation:

$$S[\Psi, g_{\alpha\beta}] = \int_{\mathrm{X}^{1+3}} d^4 x \sqrt{-g} \ \mathcal{L}(\Psi^{\cdots \alpha \cdots}_{\cdots \beta \cdots}, \nabla_\alpha, \cdots) \ . \tag{3.80}$$

Namely, we replace Lorentz tensors $T^{a \cdots}$ by their Diff-tensor forms $T^{\alpha \cdots}$ using the map in (3.25), we replace the ordinary derivatives ∂_a by the covariant derivatives ∇_α, and we integrate the Lagrangian over spacetime with a Diff-invariant measure $d^4 x \sqrt{-g}$. If we want to include parity-violating interactions, we use the Diff-tensor Levi–Civita, $\epsilon_{\alpha\beta\gamma\delta}$, in place of the Minkowski one, ϵ_{abdc}. This algorithmic procedure is known as the 'minimal coupling prescription'.

Examples of the procedure include the Lagrangian of a scalar field,

$$S_{\mathrm{KG}} = - \int d^4 x \sqrt{-g} \left(\tfrac{1}{2} \, g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + V(\phi) \right) \ , \tag{3.81}$$

with field equation

$$\nabla^2 \phi = V'(\phi) \ ,$$

or the Maxwell theory in a gravitational field,

$$S_{\mathrm{Maxwell}} = - \int d^4 x \sqrt{-g} \left(\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - J_\mu A^\mu \right) \ , \tag{3.82}$$

where $A_\mu = e^a_\mu A_a$, $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$, $F^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta}$. The field equations

$$\nabla_\mu F^{\mu\nu} = -J^\nu$$

may also be written as

$$\partial_\mu (\sqrt{-g} F^{\mu\nu}) = -\sqrt{-g} \ J^\nu \ .$$

We have already seen the minimal coupling algorithm in action in deriving the point particle dynamics, which we quote here again in its more general version (which allows the study of the massless limit) as a covariant form of (2.45):

$$S_{\rm P}(x^a, e) = \frac{1}{2}\int d\sigma \left(\frac{1}{e(\sigma)} g_{\alpha\beta}\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma} - m^2\, e(\sigma)\right) \tag{3.83}$$

with the Minkowski equation of motion $du^a/d\tau = 0$ transformed into $u^\beta \nabla_\beta\, u^\alpha = 0$. In an analogous fashion, the free equation for the spin degree of freedom of a particle, $dS^a/d\tau = 0$, generalizes to

$$\frac{\nabla S^\alpha}{d\tau} = u^\beta \nabla_\beta S^\alpha = 0 \tag{3.84}$$

in a general gravitational field. In components, we have

$$\frac{dS^\alpha}{d\tau} + \Gamma^\alpha_{\beta\gamma} u^\beta S^\gamma = 0\ , \tag{3.85}$$

an equation describing the precession of gyroscopes in a gravitational field.

Energy-Momentum Tensor

We can now repeat the weak-gravity analysis which was already done for the case of point particles. Setting $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ and keeping the linear term in $h_{\alpha\beta}$ one finds

$$S[\Psi, \eta + h] = S[\Psi, \eta] + \frac{1}{2}\int d^4x\, h_{\alpha\beta} \left(2\frac{\partial \mathcal{L}}{\partial h_{\alpha\beta}} + \eta^{\alpha\beta}\mathcal{L}\right)\Bigg|_{h=0} + O(h^2)\ , \tag{3.86}$$

where we have used $\sqrt{-\det(\eta + h)} = 1 + \frac{1}{2}\,\eta^{\alpha\beta}h_{\alpha\beta} + O(h^2)$, easily proved in the eigenvalue basis. The quantity in parenthesis in (3.86) measures the *linear response* of the matter system to the perturbation by a gravitational field. By explicit computation we see that it coincides with the energy-momentum tensor of the matter system in all the usual cases involving point particles, scalars and electromagnetic fields. Rather than quoting these results for the case of weak fields, we can actually generalize the result for the case of linear response in an arbitrary gravitational field, i.e. the behavior under a perturbation $g_{\alpha\beta} \to g_{\alpha\beta} + \delta g_{\alpha\beta}$. In this case we have

$$S[\Psi, g + \delta g] = S[\Psi, g] + \frac{1}{2}\int d^4x\, \sqrt{-g}\, \delta g_{\alpha\beta}\, T^{\alpha\beta} + O(\delta g^2)\ , \tag{3.87}$$

where the symmetric tensor $T^{\alpha\beta}$ is given by

$$T^{\alpha\beta} = 2\frac{\partial \mathcal{L}}{\partial g_{\alpha\beta}} + g^{\alpha\beta}\mathcal{L}\ . \tag{3.88}$$

In obtaining this equation we used $\sqrt{-(g + \delta g)} = \sqrt{-g} + \delta\sqrt{-g} = \sqrt{-g} + \frac{1}{2}\sqrt{-g} g^{\alpha\beta}\delta g_{\alpha\beta} + O(\delta g^2)$, which is again easily proved in the eigenvalue basis. Another useful identity which is proved in a similar way is $\delta g^{\alpha\beta} = -g^{\alpha\mu}g^{\beta\nu}\delta g_{\mu\nu}$. Using all these results, it is easy to show that, for a system of particles, scalars and electromagnetic fields, we get a total linear-response function

$$T^{\alpha\beta} = T^{\alpha\beta}_{\rm particles} + T^{\alpha\beta}_{\rm scalar} + T^{\alpha\beta}_{\rm Maxwell}\ ,$$

where

$$
\begin{aligned}
T^{\alpha\beta}_{\text{particles}} &= \sum_p m_p \frac{dx^\alpha}{d\tau_p}\frac{dx^\beta}{d\tau_p}\frac{\delta^{(4)}(x-x_p)}{\sqrt{-g}} \\
T^{\alpha\beta}_{\text{scalar}} &= \partial^\alpha\phi\,\partial^\beta\phi - g^{\alpha\beta}\left(\tfrac{1}{2}\,(\partial\phi)^2 + V(\phi)\right) \\
T^{\alpha\beta}_{\text{Maxwell}} &= -F^{\alpha\gamma}F_\gamma^{\ \beta} - \tfrac{1}{4}g^{\alpha\beta}F^{\mu\nu}F_{\mu\nu}\ .
\end{aligned}
\tag{3.89}
$$

The answer is just the Diff-tensor forms of the Noetherian energy-momentum tensors for each of these systems. Hence, we conclude that the energy-momentum tensor measures the response of the matter system to a general gravitational perturbation in an arbitrary gravitational field. In fact, the simplicity of the minimal coupling prescription implies that it is often better to compute the energy momentum tensor through the formulas (3.87) and (3.88) than using the canonical methods based on Noether's theorem.

Limitations of Minimal Coupling and General Covariance

The minimal coupling algorithm described in the previous sections follows from the Principle of Equivalence and allows us to write down the effects of gravitation for an arbitrary system. The question thus arises, why is it called *minimal* ?

The reason is that there could be higher order terms in the Lagrangian, depending on second derivatives of the metric tensor or higher, that vanish in Minkowski space (hence are invisible in SR) but do not vanish locally in a *frefo* (recall that the Principle of Equivalence only requires the first derivatives of the metric to vanish in free fall). In other words, there could be different scalar Lagrangians with the same Minkowskian limit. This phenomenon occurs for example when considering powers of covariant derivatives of tensors.

Let V_ρ by a covariant tensor field and consider the double covariant derivative $\nabla_\mu\nabla_\nu V_\rho$. This is rank-three covariant tensor by construction. Its flat-space limit, obtained setting $g=\eta$, is the Lorentz tensor $\partial_a\partial_b V_c$. This Lorentz tensor is obviously symmetric in the indices a, b. However, the original general tensor is not always symmetric in the indices μ, ν. Noting that $\nabla_\mu \sim \partial_\mu + \Gamma_\mu$, we see that this tensor contains terms of the form $\partial_\mu\partial_\nu V$ that are indeed symmetric in μ, ν, and terms of the form $\Gamma_\mu\partial_\nu V$ and $\Gamma_\mu\Gamma_\nu V$, that vanish in a local *frefo*, but there are also terms of the form $(\partial_\mu\Gamma_\nu)V$ that *do not* vanish in a local *frefo*, and are not necessarily symmetric in μ, ν. These terms do vanish in a global *frefo* for a fictitious gravitational field, but not in a general one.

Hence, any term in the Lagrangian involving more than one covariant derivative of tensors, with some antisymmetry property, will not be generated by the minimal coupling algorithm. Such terms are perfectly possible and do not violate the strictly formulated Principle of Equivalence, because the latter specifies that free fall removes gravitational effects up to second derivatives of the metric (except in fictitious fields, where we remove the gravitational effects to all orders). All these terms involve higher powers of derivatives in the Lagrangian, and thus are suppressed by inverse powers of the spatial range of variation of the gravitational field.

In order to cope with these subtleties, one usually extends the method of minimal coupling to the so-called 'Principle of General Covariance', which states that one should write down all possible terms in the Lagrangian that are Diff scalars and that reduce to Lorentz scalars in the flat space limit $g \to \eta$. This may include terms that effectively reduce to vanishing Lorentz tensors, such as the antisymmetric second covariant derivative of tensor fields. In keeping with the strategy of the 'locality principle', all such terms should be organized in a long-distance expansion and their effects only show up at sufficiently small scales.

Spin Connection

There is an alternative view of geometry, due essentially to E. Cartan, in which one shifts focus from the metric to its formal 'square root', namely the vector fields that specify the local *frefos* at each point. Recall the basic definition of the Riemannian metric out of the Principle of Equivalence:

$$g_{\mu\nu}(P) = E_P{}^a{}_\mu(P)\, E_P{}^b{}_\nu(P)\, \eta_{ab}\,, \qquad E_P{}^a{}_\alpha(P) \equiv \frac{\partial X_P^a}{\partial x^\alpha}\Big|_P\,. \tag{3.90}$$

The four covariant Diff-vector fields

$$e^a_\alpha(P) = E_P{}^a{}_\alpha(P) \equiv \frac{\partial X_P^a}{\partial x^\alpha}\Big|_P\,, \tag{3.91}$$

labeled by $a = 0, 1, 2, 3$ are called the *tetrads* or 'moving frames' (*repère mobile*), and they can be regarded as a 'Lorentzian' square-root of the metric. They contain the same information as the metric, but with larger redundancy, since any choice of tetrads can be rotated by a different Lorentz transformation at each point and still produce the same metric.

Notice the subtle difference between matrix of functions $E_P{}^a{}_\alpha(x)$ and $e^a_\alpha(P)$. The first is a different function for each P, but each such function extends beyond P. The second is just the value of these functions at P. In practice, anytime we evaluate $E_P{}^a{}_\alpha(x)$ at P, we can use instead $e^a_\alpha(P)$, like in the definition of Diff tensors out of Lorentz tensors (3.25). On the other hand, derivatives of $E_P{}^a{}_\alpha(x)$ *do not* coincide with derivatives of $e^a_\alpha(x)$, since the first type of derivative gives the shape around P of a function which is different for different choices of P, whereas the second type of derivative just monitors how the P-values change as we change P. The main situation where the subtlety arises is in the definition of the Christoffel symbols (3.57), where the expression

$$\partial_\mu E_P{}^a{}_\alpha|_P = \frac{\partial^2 X_P^a}{\partial x^\mu \partial x^\alpha}\Big|_P$$

appears, in which one evaluates the functions at the point P only *after* the second derivative of $X_P^a(x)$ has been taken. In particular, this expression is different from $\partial_\mu e^a_\alpha$. It turns out that the parametrization of this subtle difference is a useful object to carry around, called the *spin connection*:

$$\omega_\mu{}^a{}_b \equiv e^\nu_b\, (\partial_\mu e^a_\nu - \partial_\mu E_P{}^a{}_\nu)\,. \tag{3.92}$$

We may remove all reference to local *frefos* by expressing the E_P part in terms of the Christoffel symbols to get

$$\omega_\mu{}^a{}_b = e^\nu_b \left(\partial_\mu e^a_\nu - \Gamma^\alpha_{\mu\nu} e^a_\alpha\right) \equiv e^\nu_b\, \nabla_\mu\, e^a_\nu\,. \tag{3.93}$$

the last form, in terms of the covariant derivative of the tetrads, shows that, unlike the Christoffel symbol, the spin connection does transform as a covariant vector in the Diff index μ. It also shows that the Lorentz covariant form $\omega_{\mu\, ab}$ is antisymmetric in the ab Lorentz indices, as a result of the metric compatibility of the Christoffel connection.

On the other hand, the spin connection does not have a homogeneous transformation rule with respect to the local Lorentz rotations, due to the derivatives of the tetrads:

$$\partial_\mu e^a_\alpha \to \partial_\mu (L^a{}_b\, e^b_\alpha) = L^a{}_b\, \partial_\mu e^b_\alpha + e^b{}_\alpha\, \partial_\mu L^a{}_b\,.$$

In particular, under $e^a_\mu \to L^a{}_b\, e^b_\mu$, the spin connection transforms as

$$\omega_\mu \to L\,(\omega_\mu + \partial_\mu)\, L^{-1}\ , \tag{3.94}$$

in matrix notation for the Lorentz indices. Therefore, the operator $D_\mu = \partial_\mu + \omega_\mu$ is a covariant derivative with respect to the local (gauge) Lorentz symmetry! Namely, given a vector field $V^a = e^a_\mu V^\mu$, the derivative

$$D_\mu V^a = \partial_\mu V^a + \omega_\mu{}^a{}_b\, V^b \tag{3.95}$$

is Lorentz-covariant, $D_\mu V^a \to L^a_b D_\mu V^b$.

These definitions imply that the Christoffel symbols can also be calculated in terms of the Lorentz-covariant derivatives of the tetrads:

$$\Gamma^\alpha_{\mu\nu} = e^\alpha_a\, D_\mu\, e^a_\nu \equiv e^\alpha_a\, \left(\partial_\mu\, e^a_\nu + \omega_\mu{}^b{}_a\, e^b_\nu\right) \tag{3.96}$$

in a general gravitational field. In this way, the two representations of a tensor in terms of 'curved' or 'flat' indices have ∇_μ and D_μ as the respective covariant derivative operators:

$$D_\mu V^a = e^a_\alpha \nabla_\mu V^\alpha\ . \tag{3.97}$$

This relation expresses the fact that the spin connection does not introduce new degrees of freedom and could be used to 'define' ω_μ.

So, the subtlety in handling the derivatives of the tetrads tuned out to hold the solution of the problem of implementing the local Lorentz rotations as a conventional gauge redundancy, the spin connection being the corresponding gauge field. All these is very nice, but a bit formal. What is the use of the Cartan formalism in physics? While it offers some simplifications in the computation of curvature tensors, by far the main application is to the extension of the minimal coupling procedure to spinor fields.

The simplicity of the minimal coupling prescription is based on the deceptively 'trivial' replacement of latin (Lorentz) indices by greek (Diff) indices. This is ultimately possible because the point-by-point transformation matrices of Diff tensors belong to the $GL(4,\mathbf{R})$ group and, the Lorentz group being a subgroup, we can always get a representation of the Lorentz group by restricting a representation of the general linear group. Alas, the converse is not true, as there are representations of $O(1,3)$ that *cannot* be extended into representations of $GL(4,\mathbf{R})$. These are precisely the spinorial representations, used to describe fermions in nature. For this reason, to couple spinors to gravity one must use the spin connection, which emphasizes the Lorentz indices, instead of the Christoffel one.

In particular, the Dirac operator in the presence of gravitation is defined as

$$\not{D} \equiv \gamma^\mu D_\mu = \gamma^\mu \left(\partial_\mu + \tfrac{1}{2}\,\omega^{ab}_\mu \sigma_{ab}\right)\ , \tag{3.98}$$

where $\gamma^\mu = e^\mu_a \gamma^a$ is a basis of curved-space Dirac matrices satisfying the generalized Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}$.

3.4 An Interlude: Vernacular Riemannian Geometry

We have argued on physical terms that the Principle of Equivalence, together with Special Relativity, sets the spacetime arena to be given by Riemannian geometry. In this section we offer some contact with the traditional geometrical language in which Riemannian geometry is discussed.

In the mathematical literature, spaces with a positive-definite metric determining local distances

$$d\ell^2 = \sum_{ij=1}^{d} g_{ij}\, dy^i\, dy^j \tag{3.99}$$

are called d-dimensional Riemannian manifolds.[6] The meaning of this expression is that any curve γ, parametrized by $y^i(\sigma)$, has length

$$\ell(\gamma) = \int_\gamma d\sigma \sqrt{\sum_{ij} g_{ij}(y(\sigma)) \frac{dy^i}{d\sigma}\frac{dy^j}{d\sigma}}\ . \tag{3.100}$$

An intuitive interpretation of a Riemannian metric can be given by visualizing the manifold X^d as a curved submanifold embedded[7] in a higher-dimensional flat space $\mathbf{R}^D$, with a sufficiently large $D > d$. Let X^I denote cartesian coordinates on $\mathbf{R}^D$ and let the submanifold X^d of interest be parametrized by functions $X^J(y^j)$. At a given point on X^d we can consider the tangent hyperplane, generated by the vectors of components

$$E_i^I = \frac{\partial X^I}{\partial y^i}\ ,$$

with scalar product

$$E_i \cdot E_j = \sum_{I=1}^{D} E_i^I E_j^I\ .$$

The length of a curve $X^I(\sigma)$ in $\mathbf{R}^D$ is given by

$$\ell(\gamma) = \int_\gamma d\sigma \sqrt{\sum_I \frac{dX^I}{d\sigma}\frac{dX^I}{d\sigma}}\ . \tag{3.101}$$

However, when the curve lies inside X^d we can also describe it by functions $y^i(\sigma)$, then

$$\frac{dX^I}{d\sigma} = \sum_j \frac{dy^j}{d\sigma}\frac{\partial X^I}{\partial y^j} = \sum_j E_j^I \frac{dy^j}{d\sigma}$$

and the formula (3.100) follows with the metric "induced" from the embedding:

$$g_{ij}^{(\text{induced})} = \sum_{I=1}^{D} E_i^I E_j^I\ . \tag{3.102}$$

[6]Technically, the notion of manifold requires that the space be locally like a subset of $\mathbf{R}^d$, in the topological and differential sense. In practice, this means that it can be parametrized by well-behaved coordinate systems.

[7]We use the word 'embedded' as opposed to 'immersed' to emphasize that X^d is required to have no self-intersections when viewed from $\mathbf{R}^D$, so that it will look locally like $\mathbf{R}^d$ and thus be a manifold.

The starting point of Riemannian geometry is the abstraction of the embedding space. Once the intrinsic metric g_{ij} is defined, Riemann concentrates on those properties that follow from the metric alone. [8] The vectors E_i at each point P generate the so-called tangent space of the manifold at P denoted $T_P \mathrm{X}^d$. Smoothness of X^d means that $T_P \mathrm{X}^d$ is a good linear approximation to X^d around any point P.[9]

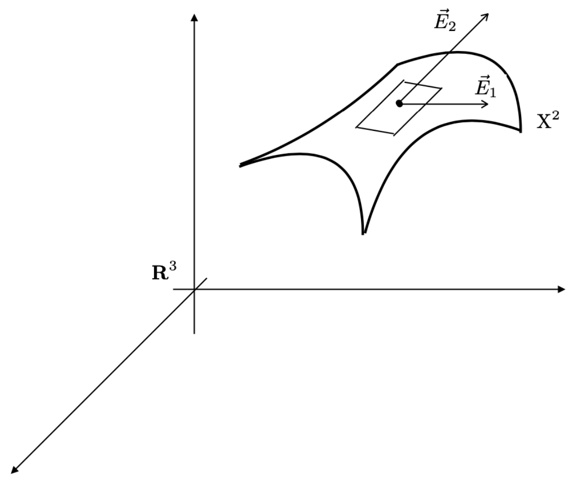

Figure 3.4: *A two-dimensional surface* X^2 *embedded in* $\mathbf{R}^3$. *The vectors* $\vec{E}_1$ *and* $\vec{E}_2$ *give a basis of a local tangent space. The induced metric is given by* $g_{ij} = \vec{E}_i \cdot \vec{E}_j$.

The appropriate generalization for GR is that of a Riemannian manifold with Lorentzian signature, meaning that the eigenvalues of the metric in a special (free fall) frame, have signs $(- + ++)$ (these are sometimes called Pseudo-Riemannian manifolds). Instead of an Euclidean tangent space at each point, the Lorentzian manifolds have tangent spaces isomorphic to Minkowski spacetime at each point. Again, we can say that Minkowski spacetime (together with SR) is a good linear approximation to the spacetime manifold, in the neighborhood of any point P. Hence, we see explicitly how the Principle of Equivalence embodies the geometrical interpretation of GR. From this standpoint, we could derive GR by simply providing a physical interpretation of each element of Riemannian geometry, such as the notions of connection, torsion, curvature, isometries, etc.

In the mathematical literature, the free-fall equation of motion goes under the name of 'geodesic equation'. This refers to the related problem of finding 'smallest length' paths between two points in a curved manifold. If our manifold is equipped with a metric $d\ell^2 = \sum_{ij} g_{ij}\, dy^i dy^j$, the length along a given path can be obtained by (3.100). Extrema of this functional define the

[8] Incidentally, a famous theorem by the even more famous John Nash shows that any given Riemannian metric g_{ij} can be induced by embedding the manifold X^d in $\mathbf{R}^D$, for a sufficiently large dimension D.

[9] The attentive reader will notice the similarity between the vectors E_i and the analogous objects in the *frefo/fido* correspondence. Indeed, they play essentially the same role.

locally shortest paths, i.e. the geodesics. But we see that the mathematical problem is identical to ours, except for the use of positive definite metrics, rather than Lorentzian ones. Accordingly, the local geodesic equation has the form

$$\sum_i T^i \nabla_i T^j = 0 \ , \tag{3.103}$$

where $\nabla \sim \partial + \Gamma$ in a completely analogous fashion and $T^i = dy^i/d\ell$ is the tangent vector to the geodesic curve, parametrized by the proper length ℓ. In particular, the explicit formula for the Christoffel symbols in terms of the first derivatives of the metric is exactly the same. So, we can adopt the geometry language and say that the motion in a gravitational field is equivalent to the motion along geodesics of a curved spacetime manifold. In our case, the geodesics are not 'minimal length' paths, but rather 'maximum proper-time' ones.

There are surprisingly few differences at the local level between the Riemannian formalism of Euclidean and Lorentzian signature manifolds. Since the signature of the metric can be changed by a formal analytic continuation in the eigenvalues of the metric, and this is a continuous procedure, the tensor differential calculus remains largely intact, up to some signs. At the global level, there are very important differences, associated to the existence of a causal structure in the Lorentzian signature case. This is related to the existence of restrictions on physical paths of particles and light rays, namely the locally measured speed of light must be maximal. These global differences becomes largest when dealing with cosmology and black holes.

The construction of Riemannian geometry from embeddings can be further generalized to define an induced metric for any p-dimensional submanifold Σ^p of X^d, with $p < d$, which of course will also be a submanifold of $\mathbf{R}^D$. Iterating the procedure gives the induced metric on Σ

$$g^{(\Sigma)}_{uv} = \sum_I \frac{\partial X^I}{\partial \sigma^u} \frac{\partial X^I}{\partial \sigma^v} = \sum_{ij} g_{ij} \frac{\partial y^i}{\partial \sigma^u} \frac{\partial y^j}{\partial \sigma^v} \ , \tag{3.104}$$

where σ^u label coordinates on Σ^p which can be specified by embedding functions $y^j(\sigma)$ into X^d or $X^I(x(\sigma))$ into $\mathbf{R}^D$.[10]

It is very important that no confusion arises between the four-dimensional spacetime geometry, characterized by the Lorentzian metric $g_{\mu\nu}$, and the metric on spatial submanifolds. In our previous example of relatively rotating frames, a non-Euclidean *spatial* geometry was operationally demonstrated in the rotating frame. However, the four-dimensional *spacetime* geometry remained just Minkowski space expressed in a rotating frame.

The trick of 'inducing' the metric from an ambient space can be repeated for any tensor. Given a covariant Diff tensor in X^d, with components $T_{ij\dots}$, we can define a Diff tensor on any submanifold Σ^p by

$$\left(T^{(\Sigma)}\right)_{uv\cdots} = T_{uv\cdots} \equiv \sum_{ij\cdots} T_{ij\cdots} \frac{\partial y^i}{\partial \sigma^u} \frac{\partial y^j}{\partial \sigma^v} \cdots \ , \tag{3.105}$$

an operation which is also known as 'pulling-back' the tensor from X^d onto Σ^p. Once the tensor is pulled back on Σ^p, one can operate as usual for any Σ-intrinsic calculations, including the raising and lowering of indices with g_{uv}, taking Diff(Σ)-covariant derivatives, or defining invariant integrals and exterior calculus. The volume form for integration of any

[10]We will often abuse notation and identify tensors on Σ^p by the indices alone. In particular, we write $g^{(\Sigma)}_{uv} = g_{uv}$ when no confusion arises.

Diff-scalar over Σ^p is $d^p\sigma\sqrt{|\det(g_{uv})|}$ and the Levi–Civita tensor for integration of forms is $\epsilon_{u_1\cdots u_p} = \sqrt{|\det(g_{uv})|}\,[u_1\cdots u_p]$.

Parallel Transport, Holonomy and Curvature

In general, if a vector V^i is defined at each point of a curve $x^j(\sigma)$ with tangent vector $T^i = dx^i/d\sigma$, we can define the covariant derivative along the curve as

$$\frac{\nabla V^j}{d\sigma} \equiv \nabla_T V^i \equiv \sum_j T^j \nabla_j V^i \; . \tag{3.106}$$

If this covariant derivative vanishes, one says that the vector V is being defined by *parallel transport* along the curve $x^i(\sigma)$. The justification for this name comes from the fact that any two vectors V and W, carried by parallel transport along the curve, $\nabla_T V = \nabla_T W = 0$, satisfy

$$\frac{d}{d\sigma}\left(\sum_{ij} g_{ij} V^i W^j\right) = \sum_{ijk} T^k \nabla_k (g_{ij} V^i W^j) = 0 \; ,$$

as a consequence of the metric compatibility property, $\nabla_k g_{ij} = 0$. Hence, the local metric definition of angles and norms is preserved under parallel transport. [11] We can write more explicitly the parallel-transport equation as

$$\frac{dV^i}{d\sigma} + \sum_j (\Gamma_T)^i_j V^j = 0 \; ,$$

where we define the matrix $(\Gamma_T)^i_j \equiv \sum_k T^k \Gamma^i_{kj}$. This equation can be solved to give the vector at some point Q, parallel-transported from some other point P, along the curve γ_{PQ}, in terms of a formal path-ordered product

$$V(Q) = \mathcal{P}\,\exp\left(-\int_{\gamma_{PQ}} \Gamma_T\right)\, V(P) \; . \tag{3.107}$$

For an infinitesimal displacement we have $V(Q) = (1 - \Delta\sigma\Gamma_T)V(P)$, so that the path-ordered product can be defined as the formal limiting product over an infinitesimal partition of the path

$$\mathcal{P}\,\exp\left(-\int_{\gamma_{PQ}} \Gamma_T\right) = \lim_{\Delta\sigma_s \to 0} \prod_s e^{-\Delta\sigma_s \Gamma_T(s)} \; .$$

The concept of parallel transport can be used to define a notion of curvature. Consider a closed curve γ_P, based at P, given by the function $x^i(\sigma)$ that starts and returns to a point P, i.e. $x^i(0) = x^i(2\pi)$, with σ an angular variable. The corresponding closed-path parallel transporter

$$U[\gamma_P] = \mathcal{P}\exp\left(-\oint_{\gamma_P} \Gamma_T\right)$$

depends both on the curve and the point P, and is called *holonomy*. Since parallel transport preserves norms and angles, $U(\gamma_P)$ is some matrix in the $SO(d)$ group, where d is the dimension

[11] Notice that a geodesic is defined as a proper-length parametrized curve, $x^i(\ell)$, whose tangent vector, $T^i = dx^i/d\ell$, is 'carried forward' by parallel transport along itself, $\nabla_T T = 0$.

of the Riemannian manifold. [12] In Minkowski space, or a 'flat' space in general, the holonomy of any curve is the identity matrix. In this case, the parallel transport of vectors between two arbitrary points is independent of the path joining them. A local measure of the deviation from flatness, i.e. a mathematical definition of 'curvature', is obtained by considering the limit of very small closed curves.

Let us make one little 'square' composed of four curves intersecting in pairs, using the parameters of these curves as coordinates (x, y) of a two-dimensional submanifold with size of order ϵ. We consider tangent vector fields X and Y whose only nonvanishing components are $X^x = Y^y = 1$ respectively. Denoting $X^i = \partial z^i/\partial x$ and $Y^i = \partial z^i/\partial y$ the components in an arbitrary coordinate system, we have the following identity

$$[X, Y]^i \equiv \sum_j (X^j \partial_j Y^i - Y^j \partial_j X^i) = 0 \;, \tag{3.108}$$

where the so-defined vector field $[X, Y]$ is known as the *Lie bracket* of X and Y. Hence, when two vector fields are used to coordinate a two-dimensional submanifold, their Lie bracket vanishes.[13]

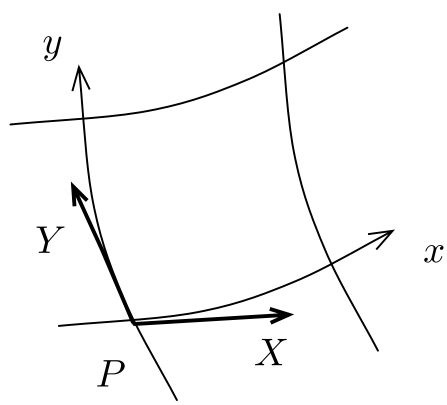

Figure 3.5: *A quadrilateral round-trip at P with direction vectors X, Y and holonomy $U(\gamma_P) = U_{-\epsilon Y} U'_{-\epsilon X} U'_{\epsilon Y} U_{\epsilon X}$.*

The holonomy around this quadrilateral round trip is given by

$$U(\gamma_P) = U_{-\epsilon Y} U'_{-\epsilon X} U'_{\epsilon Y} U_{\epsilon X}$$

where we have, with quadratic precision in ϵ,

$$U_{\epsilon X} = e^{-\epsilon\,\Gamma_X} = 1 - \epsilon\,\Gamma_X + \tfrac{1}{2}\,\epsilon^2\,(\Gamma_X)^2 + O(\epsilon^3)$$

with a similar definition for $U_{\epsilon Y}$. The primed transporters are computed from the displaced versions of the connection

$$\Gamma'_Y = \Gamma_Y + \epsilon\,\partial_X \Gamma_Y + O(\epsilon^2) \;, \qquad \Gamma'_X = \Gamma_X + \epsilon\,\partial_Y \Gamma_X + O(\epsilon^2) \;,$$

[12] In four-dimensional Lorentzian signature the holonomy is a matrix in the Lorentz group $O(1, 3)$.
[13] The converse statement is also true and it is a particular case of a theorem by Frobenius.

with the notation $\partial_X = \sum_i X^i \partial_i$ and analogously for Y. Upon explicit evaluation of the holonomy we find the first non-trivial term occurring at order ϵ^2 and given by

$$U(\gamma_P) = 1 - \epsilon^2 \left(\partial_X \Gamma_Y - \partial_Y \Gamma_X + \Gamma_X \Gamma_Y - \Gamma_Y \Gamma_X \right) + O(\epsilon^3) \; .$$

Evaluating now the derivatives in index notation and using $[X, Y] = 0$ one finds

$$U(\gamma_P)^i_j = \delta^i_j - \epsilon^2 \sum_{k,l} X^k \, Y^l \, R^i_{\ jkl} + O(\epsilon^3) \; , \tag{3.109}$$

where the coefficients

$$R^i_{\ jkl} = \partial_k \Gamma^i_{lj} + \sum_n \Gamma^i_{kn} \Gamma^n_{lj} - (k \leftrightarrow l) \tag{3.110}$$

define the Riemann tensor. Despite its explicit definition in terms of non-tensor objects, it is a Diff tensor with 20 independent components in four dimensions. [14] Equivalently, it can be represented in terms of the spin connection via the mixed Diff-Lorentz tensor $R_{kl}{}^{ab} = \sum_{ij} e^a_i \, e^{jb} \, R^i_{\ jkl}$,

$$R_{kl}{}^{ab} = \partial_k \omega_l^{ab} - \partial_l \omega_k^{ab} + [\omega_k, \omega_l]^{ab} \; , \tag{3.111}$$

thus taking the form of a standard non-abelian field strength of Yang–Mills type, associated to the local $SO(1, 3)$ gauge group.

The Riemann tensor has the purely geometrical interpretation of characterizing completely the local curvature: a Riemannian space is flat, i.e. metrically $\mathbf{R}^d$, if and only if the Riemann tensor vanishes identically. On the other hand, the Riemann tensor describes the deviation from flatness, isolating the combinations of second derivatives of the metric that cannot be eliminated by a clever choice of coordinates.

Hypersurfaces

For submanifolds Σ there is an important distinction between *intrinsic* and *extrinsic* curvature. The intrinsic curvature is the Riemannian curvature of the induced metric $g^{(\Sigma)}_{uv}$, whereas the extrinsic curvature defines how the submanifold is 'bent' inside the ambient manifold. When Σ^p is a codimension-one submanifold, $p = d - 1$, also called a *hypersurface*, the 'bending' is determined by the normal unit vector to Σ, which we denote by n^i. The two-index object $\Pi_{ij} = g_{ij} - n_i n_j$ is a projector that gives the induced metric when restricted to the tangent space of Σ, i.e. in intrinsic coordinates for Σ one has $g^{(\Sigma)}_{uv} = \Pi^{(\Sigma)}_{uv}$.

The bending of Σ inside X^d is then related to the failure of n^i to follow parallel transport when moved around Σ. Hence we define the *extrinsic curvature* as the hypersurface-tensor 'induced' on Σ by $\nabla_i n_j$:

$$K_{uv} = \sum_{ij} \nabla_i n_j \, \frac{\partial y^i}{\partial \sigma^u} \frac{\partial y^j}{\partial \sigma^v} ; . \tag{3.112}$$

To compute the covariant derivative of the normal vector, one can define n^i slightly away from Σ by extending it as the unit tangent vector of geodesics orthogonal to Σ. Then, one can see by explicit computation that $K_{uv} = K_{vu}$ is symmetric, using the symmetry of the Christoffel symbols.

[14] An equivalent, if more elegant expression with manifest tensorial form is $\left([\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \right) V^i = \sum_{klj} X^k Y^l R^i_{\ jkl} \, V^j$, valid even for the case that $[X, Y] \neq 0$.

A useful device when dealing with hypersurfaces is the 'Gaussian normal coordinate system'. Given the hypersurface Σ, it is always possible to choose coordinates locally (near a point in Σ) so that the full metric takes the form

$$ds^2 = dn^2 + \sum_{u,v} g_{uv}(n,\sigma)\, d\sigma^u\, d\sigma^v \ , \tag{3.113}$$

where n denotes a proper-distance coordinate normal to Σ, so that we can write the normal unit vector as $\sum_i n^i \partial_i = \partial/\partial n$. The orientation of n^i is conventionally chosen such that the system of coordinates (n, σ^u) is positively oriented in X^d. The lines of constant σ^u are geodesics orthogonal to Σ. In these coordinates $g_{nn} = 1$ and $g_{nu} = 0$, giving d conditions on the metric functions that can be fixed by adjusting the d functions defining a diffeomorphism.

Gaussian normal coordinates make the symmetry of K_{uv} completely manifest:

$$K_{uv} = \frac{1}{2}\,\partial_n\, g_{uv} = \frac{1}{2}\sum_{ij} \pounds_n\, g_{ij}\, \frac{\partial y^i}{\partial \sigma^u}\frac{\partial y^j}{\partial \sigma^v} \ , \tag{3.114}$$

where the second expression is the Lie derivative as defined in previous chapters.

Hypersurfaces can be boundaries of proper dimension-d subregions of the manifold. Then, using again the Gaussian normal coordinates, it is easy to prove Gauss' integration theorem on a general manifold, for the integral of a total derivative:

$$\int_{\mathrm{V}^d} \sum_i \nabla_i J^i = \int_{\partial \mathrm{V}^d} J_n \ , \tag{3.115}$$

where $J_n = \sum_i J_i\, n^i$ is the normal component of the vector field J^i, where now the normal vector is pointing 'outward' from the inside of V^d.

For Lorentzian signature, hypersurfaces can be spacelike, timelike or null according to whether the normal vector n^μ is respectively timelike, spacelike or null. This leads to some sign differences and specific constructions (not discussed here) for the null case. For timelike normal, $\tau^\mu \tau_\mu = -1$, the projector that defines the intrinsic metric is $\Pi_{\mu\nu} = g_{\mu\nu} + \tau_\mu \tau_\nu$ and the Gaussian normal coordinates are sometimes termed 'synchronous gauge':

$$ds^2 = -d\tau^2 + \sum_{ij} g_{ij}(\tau, \sigma)\, d\sigma^i\, d\sigma^j \ , \tag{3.116}$$

where τ now stands for the proper time along the timelike geodesics at fixed σ^i and orthogonal to Σ. One subtlety regarding sign conventions is the Lorentzian generalization of the divergence theorem, which retains the form (3.115) applied to regions such that $\partial \mathrm{V}^d$ is the union of piecewise timelike or spacelike hypersurfaces (no null pieces), with the rule that the normal vector is outward-pointing when spacelike and inward-pointing when timelike.

Non-Metrical Connections and Derivatives

The concept of connection (covariant derivative) is actually more primitive than the notion of metric. One can define ∇_μ axiomatically and from there one can reach the concepts of parallel transport, holonomy and curvature, since all these are concepts of the general theory of fiber bundles. This conceptual track emphasizes the similarity between GR and other gauge theories. The most general covariant derivatives are still of the form $\nabla \sim \partial + \Gamma$, but now the connection coefficients $\Gamma^{\alpha}_{\mu\nu}$ need not be symmetric in the lower indices, with the antisymmetric part $T^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu} - \Gamma^{\alpha}_{\nu\mu}$ defining the *torsion*. One can check explicitly that this is always a tensor.

Even in the presence of a metric, it is possible to have non-standard connections, in the sense that they might not be metric compatible, i.e. $\nabla_\alpha g_{\mu\nu} \neq 0$, and/or they might have torsion. However, demanding the connection to be symmetric (torsion free) and metric compatible, fixes it completely to be the Levi–Civita connection, defined by the Christoffel symbols. In practice, one can always write an arbitrary connection as the sum of the Levi–Civita connection and a remainder. This remainder may depend on torsion terms and/or contain the degrees of freedom that violate the metric compatibility, and it can be regarded as a non-minimal coupling in covariant derivatives. The important fact about this remainder is that it is always a tensor. In this way, we may still use the standard Riemannian language for physics applications, with the non-standard connection degrees of freedom interpreted as a type of exotic tensor 'matter' that does not follow the minimal coupling prescription.

This freedom in the use of non-standard connections shows up in modern generalizations of GR, in the context of supergravity theories. Historically, this was also the arena for various forgotten alternative theories, tried by the 'founding fathers'. In particular, Einstein contemplated connections with torsion in some of his attempts at constructing a 'unified theory' of gravity and electromagnetism. Even earlier, Weyl considered connections satisfying $\nabla_i g_{jk} = A_i\, g_{jk}$ for some vector field A_i. Such connections only preserve angles under parallel transport (rather than lengths).

3.5 Curvature

The Principle of Equivalence says that the gravitational field trivializes locally when entering free fall. Therefore, the interesting gravitational dynamics must be encoded in the first non-trivial correction, the tidal forces. In this chapter we find that the relativistic treatment of tidal forces has a neat geometrical character in the formalism of Riemannian geometry: it is none other than curvature.

In our previous introduction to the notion of Riemannian curvature we emphasized the modern track based on parallel transport of vectors through round trips. However, the first encounter with the notion of curvature was historically associated to the famous Euclid's 'fifth postulate', stating that initially parallel straight lines would stay so forever. The failure of this fifth postulate is an equivalent characterization of curvature. Interestingly, it is within this avatar that curvature arises most naturally in the physics of gravitation. Namely, geodesics that fail to stay parallel *accelerate* relative to one another. This is the tidal acceleration.

3.5.1 Relativistic Tidal Acceleration

The basic property of a fictitious gravitational field is the existence of a global free-fall frame that 'transforms away' gravity globally. In the free-fall frame, initially parallel trajectories stay so in time, with vanishing relative acceleration. On the other hand, in a real gravitational field, the Principle of Equivalence only removes the gravitational field at a point, leaving a residual relative acceleration between nearby particle trajectories. The corresponding residual forces are called 'tidal forces'.

It should be clear that the tidal accelerations we are discussing are a property of the background gravitational field, and not a result of the gravitational attraction between the probe particles. What we have in mind is the radial infall trajectories around Earth for instance: two test particles falling with a relative angle with respect to the center of planet will accelerate towards each other, even when we neglect their mutual gravitational attraction.

We can discuss tidal forces in the Newtonian theory by considering a family of free-falling particles along a curve $\vec{x}(s)$ at time $t = 0$. We also assume that the initial velocities of these particles varies continuously along the family, i.e. the function

$$\vec{v}(s,0) = \frac{d\vec{x}}{dt}(s,0)$$

is a continuous vector function of s. Now, we let them develop in time, so that the set of trajectories span a two-dimensional surface coordinated by (s,t). The relative separation at time t of two particles with parametric distance Δs is

$$\Delta\vec{x} = \Delta s\,\frac{d\vec{x}}{ds}(s,t)\;,$$

to first order in Δs. Denoting the transverse separation vector $\vec{\ell} = d\vec{x}/ds$, the rate of separation of the falling trajectories is $\Delta s\, d\vec{\ell}/dt$ and the relative acceleration $\Delta s\, d^2\vec{\ell}/dt^2$. Using now the equation of motion

$$\frac{d^2\vec{x}}{dt^2}(s,t) = -\vec{\partial}\,\phi_{\rm N}(\vec{x}(s,t))$$

we obtain the final expression for the tidal accelerations,

$$a^i_{\rm tidal} = \frac{d^2\ell^i}{dt^2} = -(\vec{\ell}\cdot\vec{\partial})\,\partial^i\phi_{\rm N} = \sum_j \mathcal{R}^i_{\ j}\,\ell^j \ , \tag{3.117}$$

which indeed depend on the second derivatives of the gravitational potential and reproduces our previously introduced tidal tensor $\mathcal{R}^i_{\ j} = -\partial^i\partial_j\phi_{\rm N}$.

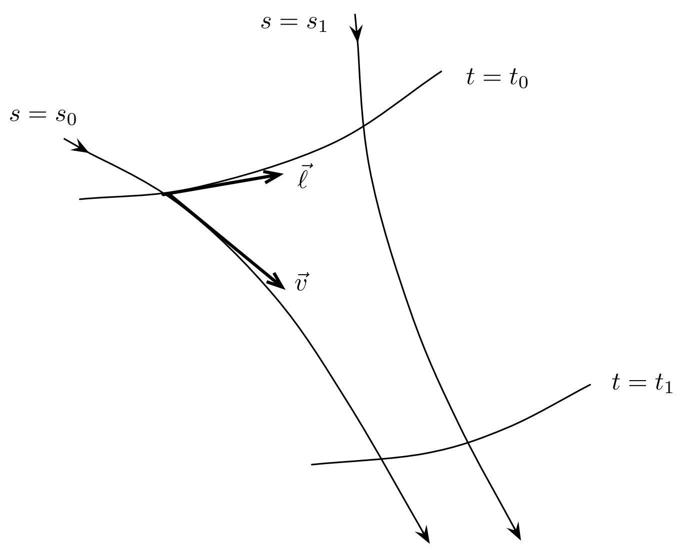

Figure 3.6: *Family of free-fall trajectories parametrized by global time and fiducial parameter s. The relative separation of the falling points is given by $\Delta s\, d\vec{x}/ds$.*

In the relativistic case, we consider an entirely analogous situation, i.e. a family of free-falling geodesics $x^\mu(\tau, s)$, parametrized by the geodesic label, s, and proper time τ. The separation and velocity four-vectors are defined by

$$\ell^\mu = \frac{dx^\mu}{ds} \ , \qquad u^\nu = \frac{dx^\nu}{d\tau} \ . \tag{3.118}$$

We can now define covariant derivatives along parametrized curves as

$$\frac{\nabla}{ds} \equiv \nabla_\ell \equiv \ell^\mu\nabla_\mu \ , \qquad \frac{\nabla}{d\tau} \equiv \nabla_u \equiv u^\mu\nabla_\mu \ . \tag{3.119}$$

With this definition, the free-fall equation of motion may be rewritten as

$$\frac{du^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta}u^\alpha u^\beta = \nabla_u\, u^\mu = u^\alpha\nabla_\alpha u^\mu = 0 \ . \tag{3.120}$$

Using these formulas, one can explicitly write

$$\nabla_u\,\ell^\alpha = u^\mu\nabla_\mu\ell^\alpha = \frac{d\ell^\alpha}{d\tau} + \Gamma^\alpha_{\mu\nu}u^\mu\ell^\nu = \frac{d^2x^\alpha}{d\tau ds} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{ds} \ .$$

Since this expression is symmetric in s, τ, it is actually equal to

$$\frac{d^2 x^\alpha}{ds d\tau} + \Gamma^\alpha_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{d\tau} = \ell^\mu \nabla_\mu u^\alpha \ .$$

Hence, we have proven

$$\nabla_u \, \ell^\alpha = u^\mu \nabla_\mu \ell^\alpha = \ell^\mu \nabla_\mu u^\alpha = \nabla_\ell \, u^\alpha \tag{3.121}$$

With these preliminaries, we are ready to define the relative acceleration of neighboring geodesics as

$$a^\mu_{\rm tidal} = (\nabla_u)^2 \, \ell^\mu = u^\alpha \nabla_\alpha (u^\beta \nabla_\beta \, \ell^\mu) = u^\alpha \nabla_\alpha (\ell^\beta \nabla_\beta \, u^\mu) \ , \tag{3.122}$$

where we have used (3.121) in the last equality. Manipulating this expression we find

$$a^\alpha_{\rm tidal} = u^\mu (\nabla_\mu \ell^\nu) \nabla_\nu u^\alpha + u^\mu \ell^\nu \nabla_\nu \nabla_\mu u^\alpha + u^\mu \ell^\nu \, [\nabla_\mu, \nabla_\nu \,] \, u^\alpha \ . \tag{3.123}$$

Using again (3.121) in the first term and the geodesic condition $u^\alpha \nabla_\alpha u^\beta = 0$ we finally obtain

$$a^\alpha_{\rm tidal} = u^\mu \ell^\nu \, [\, \nabla_\mu, \nabla_\nu \,] \, u^\alpha \ . \tag{3.124}$$

Thus, we have proven that the physical distinction between real and fictitious gravitational fields can be locally parametrized by the commutator of covariant derivatives acting on vector fields. Such an object defines the relativistic version of the tidal tensor, the so-called Riemann tensor.

3.5.2 The Riemann Tensor

The commutator of covariant derivatives acting on an arbitrary vector field, $[\nabla_\mu, \nabla_\nu \,] \, V^\alpha$, defines a tensor object that vanishes in a fictitious gravitational field (i.e. a reparametrized Minkowski space) because its flat-space version satisfies $[\partial_a, \partial_b] V^c = 0$ identically. Upon direct evaluation, we find

$$[\nabla_\mu, \nabla_\nu \,] \, V^\alpha = R^\alpha{}_{\rho\mu\nu} \, V^\rho \ , \tag{3.125}$$

where

$$R^\alpha{}_{\rho\mu\nu} = \partial_\mu \, \Gamma^\alpha_{\nu\rho} + \Gamma^\alpha_{\mu\sigma} \, \Gamma^\sigma_{\nu\rho} - (\mu \leftrightarrow \nu) \tag{3.126}$$

is the Riemann tensor. Although this expression in terms of the connection coefficients does not look obviously tensorial, the formula (3.125) ensures that the collection of coefficients $R^\alpha{}_{\rho\mu\nu}$ indeed transform as a tensor. As expected, it contains second derivatives of the metric that need not vanish anywhere in a real gravitational field, even in a *frefo* frame. The explicit calculation above can be repeated for the case of a covariant vector field,

$$[\nabla_\mu, \nabla_\nu \,] \, V_\alpha = -R^\rho{}_{\alpha\mu\nu} \, V_\rho \ , \tag{3.127}$$

and for a general tensor

$$[\nabla_\mu, \nabla_\nu \,] \, T^{\cdots\alpha\cdots}_{\cdots\beta\cdots} = R^\alpha{}_{\rho\mu\nu} \, T^{\cdots\rho\cdots}_{\cdots\beta\cdots} + \cdots - R^\rho{}_{\beta\mu\nu} \, T^{\cdots\alpha\cdots}_{\cdots\rho\cdots} - \dots \ . \tag{3.128}$$

The Riemann tensor controls tidal gravitational effects in the relativistic theory. Substituting back in (3.124) we find

$$a^\alpha_{\rm tidal} = u^\mu \ell^\nu \, R^\alpha{}_{\rho\mu\nu} \, u^\rho \ , \tag{3.129}$$

whose non-relativistic approximation at low velocities and static weak fields $g_{00} \approx -1 - 2\phi_{\rm N}$ recovers the Newtonian tidal tensor for static fields

$$R^i{}_{\mu\nu j} \, u^\mu u^\nu \approx R^i{}_{00j} = -R^i{}_{0j0} \approx -\partial_j \partial^i \phi_{\rm N} \ . \tag{3.130}$$

Identities

The Riemann tensor has in principle 256 components, which are reduced to 20 independent ones due to a large set of symmetries. First, from the definition we immediately have antisymmetry in the second pair of indices:

$$R^{\alpha}{}_{\rho\mu\nu} = -R^{\alpha}{}_{\rho\nu\mu} \; . \tag{3.131}$$

We also have the so-called cyclic identity of the last three indices:

$$R^{\alpha}{}_{\rho\mu\nu} + R^{\alpha}{}_{\nu\rho\mu} + R^{\alpha}{}_{\mu\nu\rho} = 0 = R^{\alpha}{}_{[\rho\mu\nu]} \; . \tag{3.132}$$

In addition, working with the completely covariant form $R_{\sigma\rho\mu\nu} = g_{\sigma\alpha}R^{\alpha}{}_{\rho\mu\nu}$, there is antisymmetry in the first pair of indices and symmetry in the swap of the first and second pairs of indices:

$$R_{\alpha\rho\mu\nu} = -R_{\rho\alpha\mu\nu} \; , \qquad R_{\alpha\rho\mu\nu} = R_{\mu\nu\alpha\rho} \; . \tag{3.133}$$

The simplest way of proving all these identities is to pass to a *frefo* frame at a given point P. In this frame the connection coefficients vanish at P and the Riemann tensor can be written:

$$R_{abcd}\Big|_{P-\text{frefo}} = \frac{1}{2}\left(\partial_b\partial_c\, g_{ad} + \partial_a\partial_d\, g_{bc} - \partial_b\partial_d\, g_{ac} - \partial_a\partial_c\, g_{bd}\right) \; , \tag{3.134}$$

so that the algebraic identities can be verified directly. Once established in the *frefo* frame, being tensorial relations, they are valid in an arbitrary frame.

In addition to these purely algebraic properties, the Riemann tensor also satisfies a set of differential identities called *Bianchi identities*:

$$\nabla_{[\sigma}R_{\alpha\rho]\mu\nu} = 0 \; . \tag{3.135}$$

These too can be proven using the trick of the *frefo* frame, since the covariant derivative of the Riemann tensor simplifies as

$$\nabla_e R^{d}{}_{cab}\Big|_{P-\text{frefo}} = \partial_e\partial_a\,\Gamma^{d}_{bc} - \partial_e\partial_b\,\Gamma^{d}_{ac} \; . \tag{3.136}$$

In more mathematical terms, the Bianchi identities are nothing but the Jacobi identity for the covariant derivative operator:

$$0 = [\nabla_\mu, [\nabla_\nu, \nabla_\rho]] + \text{cyclic permutations.}$$

It is worth mentioning at this point that the notion of curvature, and the associated Riemann tensor, can be introduced provided one is handed a connection (a covariant derivative operator) independently of the existence of a metric. However, some of the properties listed above, such as the quoted forms of the cyclic (3.132) and Bianchi (3.135) identities and the simplifying *frefo* expression (3.136), require that the torsion vanishes (symmetric connections). The rest of symmetry properties which depend upon lowering indices and using (3.134) further require that the connection be metric compatible, $\nabla_\alpha g_{\mu\nu} = 0$. Since we only use the Levi–Civita connection in these notes, we make no such distinctions, but the reader should be aware of these subtleties when consulting mathematical references.

Riemann Is all You Need

At this point we have seen that the Riemann tensor gives a mathematical characterization of curvature and at the same time controls tidal gravitational effects. It is a Diff tensor depending on second derivatives of the metric. Since the metric defines the relativistic analog of a gravitational potential, the Riemann tensor is a natural candidate to enter the Lagrangian of the gravitational field. One natural question at this point is whether there might exist other independent tensors constructed from second derivatives of the metric which deserved consideration. We will give a negative answer, using an argument that is motivated by a somewhat different question.

The condition (3.61) gives a mathematically precise statement of the Principle of Equivalence. It is interesting to ask whether the required *frefo* can always be introduced at an arbitrary point, for a *generic* $g_{\alpha\beta}$, or some restriction must be put on the metric of spacetime. In other words, is it always possible to enter free fall in any gravitational field? To study this, consider constructing the map between the *frefo* frame X_P^a and the *fido* frame x^α in a perturbative expansion. Without loss of generality we may define the additive normalization of the coordinates so that $x^\alpha(P) = X_P^a(P) = 0$. Then we can expand the function $x^\alpha(X_P^a)$ as

$$x^\alpha(X_P^a) = A_a^\alpha\, X_P^a + \frac{1}{2}\, B_{ab}^\alpha\, X_P^a\, X_P^b + \frac{1}{6} C_{abc}^\alpha\, X_P^a\, X_P^b\, X_P^c + O(X_P^4)\;, \tag{3.137}$$

where the A, B, C are collections of constant coefficients arising from the Taylor expansion. Using now (3.60) we find for the metric in the *frefo* frame

$$g_{ab}(P) = A_a^\alpha\, A_b^\beta\, g_{\alpha\beta}(P)\;,$$

so that $g_{ab}(P) = \eta_{ab}$ by simply choosing the A_a^α to diagonalize and rescale appropriately the original metric matrix at point P. There is in fact an ambiguity by multiplication of A_a^α by a Lorentz transformation, since η_{ab} is invariant under those. Computing now the Christoffel symbols in the X_P-frame, using for example (3.36), we find

$$\Gamma_{bc}^a(P) = (A^{-1})_\alpha^a\, A_b^\beta\, A_c^\gamma\, \Gamma_{\beta\gamma}^\alpha(P) + (A^{-1})_\alpha^a\, B_{bc}^\alpha\;,$$

so that $\Gamma_{bc}^a(P) = 0$ can be readily arranged by setting

$$B_{bc}^\alpha = -A_b^\beta\, A_c^\gamma\, \Gamma_{\beta\gamma}^\alpha(P)\;.$$

We conclude that the Principle of Equivalence, as given by (3.61), can always be reached in an appropriate *frefo*. On the other hand, the second derivatives of the metric at P, or equivalently the first derivatives of the connection, will be affected by the 80 independent coefficients C_{abc}^α. All in all there are 100 independent components in the second derivatives of the metric $\partial_\alpha\partial_\beta g_{\mu\nu}(P)$, so that we cannot remove all of them by a clever choice of *frefo*. The offset of 20 components gives the degrees of freedom of the *curvature* in a four-dimensional spacetime. The argument just given here is known in the mathematical literature as the 'local flatness theorem', since it specifies that any Riemannian space is 'flat' to linear order in a local expansion, with 'intrinsic' curvature effects arising only in second order in this local expansion.

These 20 non-removable degrees of freedom in second derivatives of the metric equal in number the 20 independent components of the Riemann tensor, and this is no coincidence. Expanding the metric to second order in the expansion (3.137) we have

$$g_{ab}(X_P)\Big|_{P-\text{frefo}} = \eta_{ab} - \frac{1}{3} E_{acbd}\, X_P^c X_P^d + O(X_P^3)\;, \tag{3.138}$$

where the set of coefficients E_{acbd}, proportional to the second derivatives of the metric, must be symmetric in the (ab) and (cd) pairs of indices. Computing these second derivatives we find

$$\partial_c\partial_d\, g_{ab}\Big|_P = -\frac{1}{3}\left(E_{acbd} + E_{adbc}\right)$$

which we can substitute back into (3.134) to prove explicitly that $E_{abcd} = R_{abcd}$. Therefore, we see that the Riemann tensor characterizes completely the quadratic deviation from the Minkowski spacetime. More invariantly, we say that $T_P\mathrm{X}^{1+3}$, the tangent space of X^{1+3} at P is not only a copy of $\mathbf{R}^{1+3}$, Minkowski spacetime, but it also provides a linear approximation to the geometry of X^{1+3}, with the deviation from flatness arising at the level of second derivatives of the metric, and invariantly characterized by the Riemann tensor.

From the physical point of view, the main conclusion is that Riemann is the essentially unique tensor that can be constructed with up to two derivatives in the metric tensor.

Ricci, Einstein and Weyl Tensors

The symmetry properties imply that the Riemann tensor produces only one independent second rank tensor under contraction, the so called Ricci tensor

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}\ , \tag{3.139}$$

which is symmetric $R_{\mu\nu} = R_{\nu\mu}$. There is also a unique scalar that can be formed from contraction of Ricci, the so-called Ricci scalar

$$R = g^{\mu\nu} R_{\mu\nu}\ . \tag{3.140}$$

Contracting the Bianchi identities we obtain their projection over the Ricci tensors,

$$\nabla^{\mu} R_{\mu\nu} = \frac{1}{2}\partial_\nu R\ , \tag{3.141}$$

so that it becomes natural to define the so-called Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}\, g_{\mu\nu} R \tag{3.142}$$

for which the Bianchi identities amount to 'covariant conservation':

$$\nabla^{\mu} G_{\mu\nu} = 0\ . \tag{3.143}$$

Another interesting object is the Weyl tensor, defined by

$$C_{\mu\nu}{}^{\rho\sigma} \equiv R_{\mu\nu}{}^{\rho\sigma} - 2R_{[\mu}{}^{[\rho}\delta_{\nu]}{}^{\sigma]} + \frac{1}{3}\, R\, \delta_{[\mu}{}^{\rho}\delta_{\nu]}{}^{\sigma}\ . \tag{3.144}$$

It can be interpreted as the part of the Riemann tensor that is not controlled by the Ricci component, as all its contractions vanish identically. Its main application is as a criterion for conformal equivalence of metrics, namely two metrics related by a local rescaling $g_{\mu\nu}(x) \to \Omega^2(x) g_{\mu\nu}(x)$ have the same Weyl tensor.

Chapter 4

Einstein's Equations

With the formal machinery introduced so far, we are able to solve for the dynamics of arbitrary matter systems in a prescribed gravitational field, as specified by some metric tensor $g_{\mu\nu}$. In this so-called *probe approximation*, the spacetime metric is considered as a fixed external background. The remaining part of the problem is the specification of the dynamics of the gravitational field itself or, in other words, the Lagrangian of the metric tensor, which ultimately determines which metrics actually occur depending on the energy content of spacetime.

4.1 The Matter-Gravitation Dynamics

Since $g_{\mu\nu}$ plays the role of a gravitational potential, we seek the relativistic analog of the Newtonian action

$$S_{\text{Newton}} = \frac{1}{8\pi G}\int dt\, d^3x\; \phi_{\text{N}}\, \vec{\partial}^2\, \phi_{\text{N}}$$

in a scalar Lagrangian linear in second derivatives of the metric, i.e.

$$S = S_G + S_m = \int d^4x\sqrt{-g}\;\left(\mathcal{L}_G(g_{\mu\nu}) + \mathcal{L}_m(\Psi, \nabla_\mu)\right)\;, \tag{4.1}$$

with $\mathcal{L}_m$ the Lagrangian of generic matter degrees of freedom Ψ, minimally coupled to $g_{\mu\nu}$ as in the previous sections, and $\mathcal{L}_G \sim \partial\partial g$ is the purely gravitational Lagrangian, a Diff scalar. The obvious choice $\nabla^2 g$ vanishes because of the metric compatibility of the affine connection. However, we have argued that there is essentially a unique tensor containing up to two derivatives of the metric, the Riemann tensor. Therefore, we are forced to choose as the Lagrangian for the gravitational field the unique scalar constructed from the Riemann tensor, the Ricci scalar. This leads to the Hilbert action [1]

$$S_{\text{H}} = \frac{1}{2\kappa^2}\int d^4x\,\sqrt{-g}\,R\,, \tag{4.2}$$

where κ^2 is proportional to G, the precise constant of proportionality to be determined below.

In fact, there is a superficially more important term that can be added to the purely gravitational Lagrangian: an additive renormalization of R by a constant. It is more important because

[1] In a dramatic episode in November 1915, Hilbert obtained the correct field equations from this action at the same time, if not before, Einstein himself.

it does not contain derivatives and thus dominates the long distance physics. This is the famous *cosmological constant*, Einstein's 'biggest blunder', by his own admission and, in hindsight, one of his major discoveries!

The complete Lagrangian of gravity and matter then reads,

$$S[g,\Psi] = S_{\rm H} + S_m = \int d^4x\,\sqrt{-g}\,\left(\frac{1}{2\kappa^2}(R-2\Lambda) + \mathcal{L}_m(\Psi,\nabla)\right) + \cdots\ , \qquad (4.3)$$

with the dots standing for corrections in higher derivatives, in the spirit of Effective Field Theory. It can be shown that, in the purely gravitational sector, the terms that can appear are any Diff-scalars made of polynomial combinations of covariant derivatives and powers of the Riemann tensor so that, to some extent, the Riemann tensor is the fundamental building block of all terms in the higher-derivative Lagrangian.

4.1.1 Einstein's Equations

We are now ready to derive Einstein's equations for the gravitational field, by simply stabilizing the variation of the Lagrangian with respect to variations of the metric $g_{\mu\nu}$. We first consider the variation of the gravitational Lagrangian. It contains three terms,

$$\delta(\sqrt{-g}\,(R-2\Lambda)) = \delta(\sqrt{-g})(R-2\Lambda) + \sqrt{-g}\,\delta g^{\mu\nu}R_{\mu\nu} + \sqrt{-g}\,g^{\mu\nu}\delta R_{\mu\nu}\ .$$

Using the relations $\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu}$ and $\delta g^{\mu\nu} = -g^{\mu\alpha}g^{\nu\beta}\delta g_{\alpha\beta}$ we can evaluate the first two terms. As for the last one we have

$$\delta R_{\mu\nu} = \partial_\alpha\delta\Gamma^\alpha_{\mu\nu} - \partial_\mu\delta\Gamma^\alpha_{\alpha\nu} + O(\Gamma\delta\Gamma)\ .$$

The terms proportional to $\Gamma\delta\Gamma$ vanish on a free fall frame at a given point. Now, although the connection Γ is not a tensor, the difference between two connections is a tensor, since the offending inhomogeneous terms vanish. So we can find the complete expression by covariantizing the previous one,

$$\delta R_{\mu\nu} = \nabla_\alpha\delta\Gamma^\alpha_{\mu\nu} - \nabla_\mu\delta\Gamma^\alpha_{\alpha\nu}\ . \qquad (4.4)$$

The same type of reasoning shows that

$$\delta\Gamma^\alpha_{\mu\nu} = \tfrac{1}{2}\,g^{\alpha\sigma}\left(\nabla_\mu\delta g_{\sigma\nu} + \nabla_\nu\delta g_{\sigma\mu} - \nabla_\sigma\delta g_{\mu\nu}\right)\ . \qquad (4.5)$$

So, the final result is that $g^{\mu\nu}\delta R_{\mu\nu} = \nabla_\mu f^\mu$, with f^μ a four-vector field linear in covariant derivatives of $\delta g_{\mu\nu}$. Since this variation contributes a total derivative, it is tempting to neglect it on the grounds that the metric is fixed on the boundary of spacetime, in accord with common practice in the Lagrangian formalism. There is a subtlety though, in that f^μ depends on $\delta g_{\mu\nu}$ through its covariant derivative, and the condition that these vanish is not quite the same as the vanishing of $\delta g_{\alpha\beta}$. In fact, as remarked by York, Gibbons and Hawking (YGH), one can add a boundary term to Hilbert's action in such a way that those terms are cancelled out, and one is left with a standard variational principle. [2] The resulting equations are identical to those obtained by a naive procedure in which one simply neglects the f^μ term, so that we will not worry about this issue at this point.

[2] The York–Gibbons–Hawking term is given by $S_{\rm YGH} = \frac{1}{\kappa^2}\oint K$, where the integral is defined over the boundary, and K is the trace of the extrinsic curvature associated to the induced metric.

Collecting all terms, the variation of the gravitational action is

$$\delta S_{\rm H} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left(-R^{\mu\nu} + \tfrac{1}{2}\, g^{\mu\nu}(R - 2\Lambda) \right) \delta g_{\mu\nu} \ . \tag{4.6}$$

Next we turn to the matter sector. Here, the first order variation defines the *energy-momentum tensor*, according to (3.87) and (3.88), through

$$\delta S_m = \int d^4x \sqrt{-g}\, \tfrac{1}{2}\, T^{\mu\nu} \delta g_{\mu\nu} \ . \tag{4.7}$$

In terms of the Lagrangian,

$$T^{\mu\nu} = 2 \frac{\partial \mathcal{L}_m}{\partial g_{\mu\nu}} + g^{\mu\nu}\, \mathcal{L}_m \ . \tag{4.8}$$

Finally, we can put all the terms together and write down Einstein's equations as

$$R_{\mu\nu} - \tfrac{1}{2}\, g_{\mu\nu}(R - 2\Lambda) = \kappa^2\, T_{\mu\nu} \ , \tag{4.9}$$

a set of ten non-linear second-order partial differential equations. The non-linearity, which persists even for vacuum solutions, makes it highly non-trivial to find exact solutions. Flat Minkowski space $g_{\alpha\beta} = \eta_{\alpha\beta}$ is of course a solution with $T_{\mu\nu} = \Lambda = 0$. The vacuum equations are obtained by setting to zero the matter sector and tracing the equation to obtain $R - 2(R - 2\Lambda) = -R + 4\Lambda = 0$. Hence,

$$R_{\mu\nu} = \Lambda g_{\mu\nu} \tag{4.10}$$

gives the vacuum equations. Manifolds solving the vacuum Einstein equations are called *Einstein manifolds*. For $\Lambda = 0$ we have the so-called *Ricci-flat manifolds*.

The cosmological constant term is equivalent to a vacuum energy on the matter energy-momentum tensor. For example, if the matter sector contains a scalar field with energy-momentum tensor

$$T^{(\phi)}_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \left(\tfrac{1}{2}\, (\partial\phi)^2 + V(\phi) \right) \ ,$$

we see that the shift $V(\phi) \to V(\phi) + \Lambda/\kappa^2$ in the origin of potential energies introduces the cosmological constant if it was zero on the left hand side of Einstein's equations. Alternatively, for a perfect fluid *ansatz*,

$$T^{\rm (fluid)}_{\mu\nu} = (p + \rho)\, U_\mu U_\nu + p\, g_{\mu\nu} \ ,$$

the cosmological constant is equivalent to a matter component with the equation of state $p = -\rho = -\Lambda/\kappa^2$, so that we may fold Λ into the non-gravitational Lagrangian as a matter of convention and write the equations as [3]

$$G_{\mu\nu} = \kappa^2\, T_{\mu\nu} \ . \tag{4.11}$$

It remains to determine the absolute normalization of the action, i.e. the relation between κ and ordinary Newton's constant G. This can be fixed by taking the Newtonian limit. We begin by rewriting (4.11) in the equivalent form

$$R_{\mu\nu} = \kappa^2\, (T_{\mu\nu} - \tfrac{1}{2}\, g_{\mu\nu} T) \ , \tag{4.12}$$

[3] This innocent-looking statement causes the so-called 'cosmological constant problem'; the astonishing fact that the measured value of Λ, given by $G\,\Lambda \sim 10^{-120}$, is so incredibly small, despite the multitude of independent contributions to Λ from the matter sector, each of them potentially much larger than the experimental bound.

where $T = g^{\alpha\beta}T_{\alpha\beta}$ is the trace of the energy-momentum tensor. In the non-relativistic limit for static and small fields, we can replace the metric with the Newtonian form in computing the Ricci tensor,

$$ds^2 \approx -(1 + 2\phi_{\rm N})\, dt^2 + d\vec{x}^{\,2} \; . \tag{4.13}$$

for which the time-time component of Einstein's equations gives

$$\vec{\partial}^{\,2}\phi_{\rm N} \approx R_{00} = \kappa^2(T_{00} + \tfrac{1}{2}\,T) \; , \tag{4.14}$$

where we have used $1 + 2\phi_{\rm N} \approx 1$ on the right hand side. For a matter source composed of a non-relativistic system of slow particles, we have $T_{00} \approx \rho_m$ and $T = -T_{00} + \sum_i T_{ii} \approx -T_{00}$, so that the right-hand side of (4.14) is $\frac{\kappa^2}{2}\,\rho_m$, which agrees with the Poisson equation for

$$\kappa^2 = 8\pi G \; . \tag{4.15}$$

Einstein's equations can be viewed as a set of partial differential equations determining the metric $g_{\mu\nu}$ on a spatial surface at time t, given initial data on a spatial surface at time $t = t_0$. The initial data are 'positions' $g_{\mu\nu}(t_0)$, and 'velocities' $\partial_t g_{\mu\nu}(t_0)$. Apparently, the system of ten equations just determines the ten components of the metric, but this is actually deceptive, since there are four degrees of freedom not determined by the equations, related to Diff covariance. Indeed, only the six spatial equations

$$G_{ij} = 8\pi G\, T_{ij}$$

contain second derivatives in time and are true 'dynamical equations'. The remaining four,

$$G^{0\mu} = 8\pi G\, T^{0\mu} \tag{4.16}$$

only contain up to first derivatives in time of $g_{\mu\nu}$. This follows from the Bianchi identities, $\nabla_\mu G^{\mu\nu} = 0$, which can be written as

$$\nabla_0 G^{0\mu} = -\sum_i \nabla_i G^{\mu i} \; .$$

The right hand side contains derivatives up to at most second order in time, thus $G^{0\mu}$ can only contain, at most, *first* time derivatives of the metric. The conclusion is that the four equations (4.16) must be imposed as constraints on initial data. This remark is very useful in practice, when faced with the task of solving Einstein's equations. In many situations with high symmetry, it is useful to evaluate the $G_{00} = 8\pi G\, T_{00}$ equation first, since it often expresses "constants of the motion" in the combined gravity/matter system. More generally, the method of solving Einstein's equations in 'Hamiltonian' form, emphasizing the initial-value formulation, is the starting point for numerical simulations of Einstein equations, a subject that has grown enormously in precision and scope in the last decade.