Introduction to Noncommutative Field Theory

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Abstract

These Lecture Notes give an intuitive introduction to noncommutative field theory with an emphasis on the physics ideas and methods. We pay special attention to those aspects of noncommutative field theory that represent genuine novelties from the physical point of view, such as the UV/IR mixing. We also include brief discussions of possible applications of these ideas to phenomenology as well as the connection to string theory.

Contents

1 Introduction

Noncommutative Field Theory (NCFT) is a field theory defined over a space-time endowed with a Noncommutative Geometry (NCG) (c.f. [1, 2, 3]).

Although the motivations for considering NCG are mostly mathematical, aspects of the formalism do show up in various physical situations and, in priciple, it is a relevant generalization of the standard framework of local quantum field theory. Indeed, the existence of a nonlocal, and yet tractable, generalization of quantum field theory is a highly non-trivial fact of great intrinsic interest. This is not only linked to interesting mathematics but it is also related to the non-locality present in string theory [4].

In this vein, the recent discovery of subtle quantum mechanical effects in NCFT, having to do with the interplay between locality and renormalization (c.f. [5]), has prompted a wide interest in NCFT as a toy model for the most widely studied nonlocal theory: string theory. Other potential applications of the formalism to the study of large-N limits of ordinary gauge theories (c.f. [2, 6, 7]), as well as the Quantum Hall Effect [8], only add to the interest of these ideas.

Here we give a very basic introduction to NCFT, emphasizing the physical methods and motivations, at the price of being considerably sloppy on the mathematical niceties of the subject. Other reviews with a much more comprehensive scope exist. See for example [9]. Reviews with a more mathematical outlook are for example [10, 11].

In preparing these notes, no attemp has been made of giving a careful set of references. Rather complete sets of references can be found in the reviews just quoted. In the text, we will only refer explicitly to some works that are particularly relevant to the discussion.

1.1 Noncommutative Geometry

Intuitively, NCG is the generalization of standard geometry ideas, such as manifolds, metrics and fiber bundles, to spaces where the "coodinates" are operators rather than c-numbers. In particular, they do not commute, but satisfy some operator algebra

$$[\hat{x}^i, \hat{x}^j] = C^{ij}(\hat{x}). \tag{1}$$

It is useful to think of the operators $\hat{x}^k$ as "generators" of an algebra $\mathcal{A}$, in the sense that the general element of $\mathcal{A}$ can be thought of as a function of the basic variables, $f(\hat{x})$, satisfying certain constraints. In this case, the functions $C(\hat{x})$ acquire the interpretation of "structure functions", generalizations of the notion of structure constants for ordinary Lie algebras. The basic idea of the development of NCG is then the recovery of geometrical notions about the "base space" (the space parametrized by the "coordinates" $\hat{x}_k$) in terms of the algebra $\mathcal{A}$ of functions on that space, where this algebra is required to be associative but in general non-commutative.

1.1.1 Examples

Rather than developing these ideas in full generality, here we collect some simple examples that are motivated by the applications of the formalism to physics.

- The trivial example is $C_k^{ij} = 0$, a commutative algebra. Then $\mathcal{A}$ is the algebra of (say smooth) functions $C(M)$ on the base manifold parametrized by the c-numbers x^k.

- The next example in order of triviality is when the noncommutative algebra is a direct product of a commutative algebra and a *finite-dimensional* noncommutative algebra, such as some Lie algebra $\mathcal{G}$:

$$\mathcal{A} = C(M) \otimes \mathcal{G}. \tag{2}$$

  This is the case of ordinary gauge theory, where fields are just matrix-valued functions.

- Another simple, albeit somewhat exotic example is the "fuzzy sphere". If we define $\mathbf{S}^2$ as the solution in $\mathbf{R}^3$ of

$$x_1^2 + x_2^2 + x_3^2 = R^2, \tag{3}$$

  the obvious definition of the fuzzy sphere whould be in terms of three non-commuting operators $\hat{x}_1, \hat{x}_2, \hat{x}_3$ that satisfy

$$\hat{x}_1^2 + \hat{x}_2^2 + \hat{x}_3^2 = R^2\,\mathbf{1}, \tag{4}$$

  with $\mathbf{1}$ the unit operator of the algebra. An obvious choice is

$$\hat{x}_a = \frac{R}{\sqrt{j(j+1)}}\, J_a, \tag{5}$$

where J_a are $SU(2)$ angular momenta in the spin-j representation. Hence, this is the particular choice

$$C_{ab}(\hat{x}) = i\,\frac{R}{\sqrt{j(j+1)}}\sum_c \epsilon_{abc}\,\hat{x}_c \tag{6}$$

for the structure "constants". Notice that the resulting operator space respects the $SO(3)$ isometry that characterizes $\mathbf{S}^2$. The space is "discrete" in some sense, because the spectrum of eigenvalues of any position operator $\hat{x}_a$ has dimension $2j+1$. So, it looks like some kind of "lattice approximation" to $\mathbf{S}^2$. Strictly speaking, we cannot build a quantum field theory with an infinite number of degrees of freedom on such space. In the limit $j \to \infty$ at fixed R, the number of degrees of freedom does diverge, but then we recover the commutative algebra of functions on $\mathbf{S}^2$.

- In the previous examples, the noncommutative character of $\mathcal{A}$ was "finite-dimensional", which leads to somewhat trivial examples. The next step in complexity is to regard $\hat{x}_j$ as operators represented in some infinite-dimensional Hilbert space, with continuous spectrum, *i.e.* we want to regard their eigenvalues are parametrizing standard flat space $\mathbf{R}^d$. Then the simplest choice of structure constants is a simple central extension:

$$[\hat{x}^j, \hat{x}^k] = i\,\theta^{jk}, \tag{7}$$

with θ^{jk} an antisymmetric matrix of constants with length-dimension two. This defines noncommutative flat space or $\mathbf{R}^d_\theta$, and an obvious restriction to periodic angular coordinates defines the noncommutative torus $\mathbf{T}^d_\theta = \mathbf{R}^d_\theta/\mathbf{Z}^d$.

1.2 Examples from Physics

NCG may arise in physical systems when some *effective* position operator becomes non-commutative as a result of interactions.

$$\left[\hat{X}^\mu_{\text{eff}}, \hat{X}^\nu_{\text{eff}}\right] \neq 0. \tag{8}$$

This involves typically non-relativistic systems in first-quantization and the non-commutatitivity of the position operator may or may not vanish in the classical limit $\hbar \to 0$. We will illustrate this with two examples: electrons in a strong magnetic field and D-branes.

1.2.1 Electrons in a Strong Magnetic Field

Let us consider planar electrons in a strong uniform magnetic field B_{ij}, with Hamiltonian

$$H = \frac{1}{2m_e} \left(\vec{p} - e\,\vec{A}\right)^2 , \tag{9}$$

where

$$A_i = -\frac{1}{2}\, B_{ij}\, x^j \tag{10}$$

in an appropriate gauge. Defining

$$z = \sqrt{\frac{e|B|}{2\hbar}}\,(x + iy) \tag{11}$$

and the operators

$$a = \partial_{\bar z} + \frac{z}{2}, \qquad a^\dagger = -\partial_z + \frac{\bar z}{2}, \tag{12}$$

one finds a harmonic oscillator system

$$[a, a] = [a^\dagger, a^\dagger] = 0, \qquad [a, a^\dagger] = 1, \tag{13}$$

and the Hamiltonian

$$H = \hbar\,\omega_c \left(a^\dagger\, a + \frac{1}{2}\right) , \tag{14}$$

with spectrum $E_\ell = \hbar\omega_c(\ell + \frac{1}{2}), \ell \in \mathbf{Z}$, where $\omega_c = e|B|/m_e$ denotes the cyclotron (Larmor) frequency. Each energy (Landau) level has an infinite degeneracy; the ground states satisfy:

$$a\,\psi(z, \bar z) = \left(\partial_{\bar z} + \frac{z}{2}\right) \psi(z, \bar z) = 0. \tag{15}$$

A basis of the lowest Landau level (LLL) can be chosen as

$$\psi_m(z, \bar z) = \frac{z^m}{\sqrt{m!}}\, e^{|z|^2/2}. \tag{16}$$

We can concentrate on the LLL wave functions if the magnetic field is large enough, so that mixing with the higher Landau levels is suppressed by the high cyclotron frequency gap. The interesting feature of the LLL wave functions is that they are almost analytic. We can consider analytic functions $v_m(z)$ by stripping off the exponential term:

$$v_m(z) \equiv e^{|z|^2/2}\, \psi_m(z, \bar z). \tag{17}$$

If we further define a specific inner product on the LLL:

$$(v_n|v_m) \equiv \int d\mu(z,\bar{z})\, \bar{v}_n(\bar{z})\, v_m(z) = \langle \psi_n|\psi_m\rangle \tag{18}$$

with the non-holomorphic exponential term in the measure:

$$d\mu(z,\bar{z}) = e^{-|z|^2} dz d\bar{z}, \tag{19}$$

then we have, integrating by parts:

$$(f|\partial_z|g) = (f|\bar{z}|g), \tag{20}$$

so that, on the LLL:

$$(\partial_z)_{LLL} = (\bar{z})_{LLL}. \tag{21}$$

Hence, $[\partial_z, z] = 1$ implies

$$[\bar{z}, z]_{LLL} = 1 \tag{22}$$

or, back to the original variables

$$[\hat{x}, \hat{y}]_{LLL} = i\,\theta_B, \qquad \theta_B = \frac{\hbar}{e|B|}. \tag{23}$$

Thus, the motion of electrons in the lowest Landau level is effectively described by a noncommutative plane. NCG is relevant to the physics of the Quantum Hall Effect.

It is worth deriving this result in a more heuristic fashion, using a Lagrangian argument. The Lagrangian of the system is

$$L = \frac{1}{2} m_e \dot{\vec{x}}^2 - \frac{e}{2} B_{ij}\, x^i\, \dot{x}^j. \tag{24}$$

In a situation where the kinetic energy term is negligible $|m_e \dot{x}^i| \ll |B_{ij} x^j|$, we may approximate the dynamics by the degenerate Lagrangian

$$L \approx -\frac{e}{2} B_{ij} x^i\, \dot{x}^j. \tag{25}$$

The canonical momenta are proportional to the coordinates themselves:

$$\pi_j = \frac{dL}{d\dot{x}^j} = -eB_{jk} x^k. \tag{26}$$

Upon canonical quantization

$$[\hat{\pi}_j, \hat{x}^l] = -i\hbar\, \delta_j^l = -e\, B_{jk} [\hat{x}^k, \hat{x}^l], \tag{27}$$

and finally:

$$[\hat{x}^k, \hat{x}^l] = i\hbar \left(\frac{1}{e|B|}\right)^{kl}. \tag{28}$$

1.2.2 D-branes

Dp-branes are specific states of string theory that resemble non-relativistic solitons extended in p spatial dimensions [12]. For the case of D-particles, their low-energy dynamics is primarily characterized by the position collective coordinates. For a system of *distant* N D-particles, we have a collection N vectors of positions $\vec{x}_i, i = 1, \dots, N$. When the D-particles' separation is in the stringy domain, $|\vec{x}_i - \vec{x}_j| < \ell_s$, with ℓ_s the string length, new light degrees of freedom appear, corresponding to open strings stretched between neighboring D-particles. Therefore, the number of collective coordinates is enlarged to N^2 and we may assemble them into a hermitian matrix $\mathbf{X}_{ij}$.

Thus, the notion of positon becomes "fuzzy" at short distances. An operational definition of the i-th particle position is

$$\langle \mathbf{X} \rangle_i \equiv \langle i|\mathbf{X}|i\rangle = \mathbf{X}_{ii}. \tag{29}$$

With this definition, any non-diagonal matrix of collective coordinates assigns a nonvanishing dispersion to the possition of the i-th particle:

$$(\Delta \mathbf{X})_i^2 = \langle \mathbf{X}^2 \rangle_i - \langle \mathbf{X} \rangle_i^2 = \sum_{j \neq i} |\mathbf{X}_{ij}|^2 \geq 0. \tag{30}$$

Once the positions are promoted to a matrix, the statistical permutation group of N particles, S_N, is naturally promoted to $U(N)$, whose Weyl subgroup is precisely S_N.

In fact, for a one-dimensional system we just have a single "position matrix" and we can always agree to define the positions in terms of the eigenvalues of this matrix. Starting with two spatial dimensions we have more than one position matrix and it is not possible to diagonalize all of them in the same basis, unless they commute. In D-brane theory, this condition is selected dynamically by the minima of the static interaction potential of a system of D-particles:

$$V(\mathbf{X}) = -\frac{1}{g_s \ell_s} \sum_{a,b=1}^{d} \left[X^a, X^b\right]^2. \tag{31}$$

Thus, in this case the noncommutativity survives the classical limit of the theory. In fact, taking into account the "statistical symmetry" $U(N)$ we are just constructing a $U(N)$ gauge theory with Higgs fields in the adjoint representation, and interpreting the expectation values of these scalar fields as generalized position coordinates of the soliton. Thus, from the point of view of the earlier list of simple NCG examples, the D-branes represent the noncommutative algebra $C(M) \times U(N)$. Notice however that here M is only the world-volume of the D-brane, whereas the space transverse to the D-brane is constructed out of the matrix degrees of freedom, via the Higgs fields X^a in the adjoint of $U(N)$.

In certain situations, the interaction potential depends on a background field through a "dielectric coupling" [13]:

$$\delta V(\mathbf{X}) = if\,\epsilon^{abc}\,\mathrm{tr}\; X_a X_b X_c. \tag{32}$$

In this example, it depends on a single constant parameter f and we take $d = 3$. The equations of motion become

$$\Big[[X^a, X^b], X_b\Big] + if\,\epsilon^{abc}\,[X_b, X_c]. \tag{33}$$

Although commuting (diagonal) matrices are still a solution, we see that the fuzzy sphere (6) is a solution with

$$X_a = f\,J_a, \tag{34}$$

and J_a in the spin-j representation of $SU(2)$.

2 Noncommutative Field Theory

In constructing NCFT we go one step further. As in the D-brane example, the underlying NCG is taken as a passive "arena", or background choice, for the dynamics, but we formally generalize the noncommutativity to infinite matrices, i.e. operator algebras. In these lectures we concentrate on the simple example of $\mathbf{R}^d_\theta$. The nontrivial structure

$$[\hat{x}^j, \hat{x}^k] = i\,\theta^{jk} \tag{35}$$

can be interpreted by regarding $\hat{x}^k$ as phase-space variables represented on a Hilbert space $\mathcal{H}_\theta$. This Hilbert space has nothing to do with the standard Quantum Hilbert space $\mathcal{H}_\hbar$ that arises upon quantization. In fact, $\mathcal{H}_\theta$ is part of the specification of the classical field theory, i.e. the classical field configurations are functions $\phi(\hat{x}^k)$ on the algebra of operators $\mathcal{A}_\theta$ that are represented on $\mathcal{H}_\theta$.

It is clear that such a structure imposes a physical nonlocality on length scales of $\mathcal{O}(\sqrt{\theta})$. There is a minimal area unit of $\mathcal{O}(\theta)$ in the sense of the Heisenberg uncertainty relation:

$$\Delta x^j \, \Delta x^k \geq \frac{1}{2} |\theta^{jk}|. \tag{36}$$

Thus, we may hope that $\sqrt{\theta}$ is an interesting physical cutoff in quantum field theory, presumably with interesting applications to the quantum gravity realm. Meanwhile, if space-time satisfies (35) at short distances, the most characteristic hint at low energies would be the short-distance breakdown of Lorentz invariance, a very well-tested symmetry.

2.1 Elementary Construction of Classical NCFT

For simplicity, we begin with a single noncommutative plane with coordinates x, y satisfying

$$[\hat{x}, \hat{y}] = i\,\theta. \tag{37}$$

We consider the standard representation on "wave functions" on $L^2(\mathbf{R})$. The operator $\hat{x}$ is diagonal and represented multiplicatively, whereas $\hat{y}$ is the corresponding 'conjugated momentum':

$$\hat{x}\,\psi(x) = x\,\psi(x), \qquad \hat{y}\,\psi(x) = -i\theta\,\partial_x\,\psi(x). \tag{38}$$

We have then the standard operator identities:

$$e^{ip\hat{y}}\,f(\hat{x}) = f(\hat{x} - p\theta)\,e^{ip\hat{y}}, \tag{39}$$

so that $\hat{y}$ generates translations of $\hat{x}$ eigenvalues. Straightforward application of the Baker–Campbell–Hausdorff formula yields the plane-wave composition rule:

$$e^{ip_\mu \hat{x}^\mu}\,e^{iq_\mu \hat{x}^\mu} = e^{-\frac{i}{2} p\times q}\,e^{i(p+q)_\mu \hat{x}^\mu}, \tag{40}$$

where we have returned to a general $\theta^{\mu\nu}$ matrix and defined

$$p \times q \equiv p_\mu\,\theta^{\mu\nu}\,q_\nu. \tag{41}$$

A convenient way of manipulating the operator algebra is to map it to some deformed function algebra. This in turn allows a much more intuitive development of the physical set up for NCFT.

The basic idea is to work with the "components" of the operators in a conventionally chosen basis. This is the infinite-dimensional generalization of the standard choice of a basis in a finite-dimensional $U(N)$ Lie algebra:

$$A = \sum_{a=1}^{N^2} A^a \, T^a. \tag{42}$$

In this case, we say that the hermitian matrix A has "vector components" A^a in the basis of generators $\{T^a\}$. For a general operator $\hat{O}$ acting on $\mathcal{H}_\theta$ the "vector of components" in a given basis is in general a function of a continuous label $f_{\hat{O}}(x^\mu)$. This establishes a map from the operator algebra to the space of ordinary functions:

$$\hat{O} = \int d^d x \, f_{\hat{O}}(x^\mu) \, \hat{T}_{x^\mu}, \tag{43}$$

where $\hat{T}_{x^\mu}$ is a basis of the operator algebra. Associated to this choice of basis, there is a representation of the operator product "in components". This is a product in the space of component functions, the "star product", defined by the identity:

$$f_{\hat{O}\hat{O}'}(x) = f_{\hat{O}}(x) \star f_{\hat{O}'}(x). \tag{44}$$

For illustrative purposes, it is interesting to work out the star product in the finite-dimensional $U(N)$ Lie algebra. In a conventional basis of generators T^a we have

$$T^a \, T^b = \sum_c C_c^{ab} \, T^c \tag{45}$$

for some constants C_c^{ab}. Given two hermitian matrices $A = \sum_a A_a T^a, B = \sum_a B_a T^a$, the product can be written as

$$AB = \sum_{a,b} A_a \, B_b \sum_c C_c^{ab} \, T^c = \sum_c (AB)_c \, T^c. \tag{46}$$

So that the definition of "star product" is simply

$$A_c \star B_c \equiv (AB)_c = \sum_{a,b} C_c^{ab} \, A_a \, B_b. \tag{47}$$

A convenient choice for $\mathbf{R}_\theta^d$ is the so-called Weyl map, defined by choosing the operator basis as

$$\hat{T}_{x^\mu} = \int \frac{d^d k}{(2\pi)^d} \, e^{ik\cdot(\hat{x}-x)}, \tag{48}$$

with inverse

$$f_{\hat{O}}(x^\mu) = \int \frac{d^d k}{(2\pi)^d} \,\mathrm{Tr}\, e^{ik_\mu (x-\hat{x})^\mu}\, \hat{O}(\hat{x}^\mu). \tag{49}$$

The specific property of the Weyl map that makes it useful is that the plane-wave operator

$$\exp(ip\cdot\hat{x}) \tag{50}$$

is associated to the plane-wave function

$$\exp(ip\cdot x). \tag{51}$$

In particular, the composition law (44) holds for the star product of the component functions:

$$e^{ip\cdot x} \star e^{iq\cdot x} = e^{-\frac{i}{2}\, p\times q}\, e^{i(p+q)\cdot x}. \tag{52}$$

A general expression for arbitrary functions can be obtained by simple superposition of plane waves:

$$f(x) \star g(x) = f(x) \exp\left(\frac{i}{2}\, \overleftarrow{\partial}_\alpha\, \theta^{\alpha\beta}\, \overrightarrow{\partial}_\beta\right) g(x). \tag{53}$$

This associative, but noncommutative product is known as the Moyal product of functions. In this language, NCG amounts to a smooth deformation of the classical algebra of functions, i.e. we just change the composition rules, but not the elements of the algebra.

Since $\mathcal{A}_\theta$ can be viewed as a deformation of the ordinary algebra of functions on $\mathbf{R}^d$, we can construct NCFT by deforming action functionals in a straightforward way.

Therefore, a prescription to construct NCFT's is to exercise a "correspondence principle" in terms of the noncommutativity deformation parameter θ^{jk}: one just replaces ordinary products by Moyal products all over the place, i.e. for a scalar field:

$$S[\phi] = \int d^d x \left(\frac{1}{2}\partial_\mu\phi \star \partial^\mu\phi - \frac{1}{2}m^2\, \phi\star\phi - \frac{\lambda}{4!}\, \phi\star\phi\star\phi\star\phi\right). \tag{54}$$

An important property of any such action is the cyclic property of the Moyal product inside integrals:

$$\int d^d x\, f(x)\star g(x)\star h(x) = \int d^d x\, g(x)\star h(x)\star f(x), \tag{55}$$

provided one can neglect boundary terms at infinity. In particular, under the same conditions one can remove *one* Moyal product inside integrals:

$$\int d^dx\, f(x) \star g(x) = \int d^dx\, f(x)\, g(x). \tag{56}$$

As with any correspondence principle, the noncommutativity of products implies some ambiguities in translating actions. For example, for fields with indices, the interaction term

$$\int d^dx\ \phi_i \star \phi^i \star \phi_j \star \phi^j \tag{57}$$

is *not equivalent* to

$$\int d^dx\ \phi_i \star \phi_j \star \phi^i \star \phi^j. \tag{58}$$

2.2 Noncommutative Gauge Theories

The previous construction of a classical scalar field theory admits straightforward generalizations to other theories with polynomial interactions involving fermions and scalars with Yukawa-type couplings. Special features arise in the case of gauge fields.

Starting from an ordinary gauge theory based on a Lie group G, the naive correspondence principle yields

$$S_{\rm NCYM} = -\frac{1}{4g^2} \int d^dx\ {\rm tr}\ F_{\mu\nu} \star F^{\mu\nu}, \tag{59}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i\, A_\mu \star A_\nu - i\, A_\nu \star A_\mu, \tag{60}$$

with infinitesimal gauge transformations acting as

$$\delta A_\mu = D_\mu\, \epsilon = \partial_\mu\, \epsilon + i\, A_\mu \star \epsilon - i\, \epsilon \star A_\mu. \tag{61}$$

Notice that taking the gauge fields valued on the standard Lie algebra of G is not in general consistent with the noncommutative deformation. Let us write $A(x) = \sum_a A_a(x) T^a$ for some basis of $\mathcal{G}$. Then the non-commutative character of the Moyal product implies that gauge transformations depend on the anticommutator $\{T^a, T^b\}$, together with the usual commutator terms $[T^a, T^b]$. In general, the anticommutator of two generators belongs to the Lie algebra only in the case of $U(N)$ in the fundamental representation. Thus, the discussions of NCYM theories are normally restricted to $U(N)$ groups. In principle, other options are possible (such as working with the univeral enveloping algebra that contains the products of all generators [14]) at the price of working with a theory whose degree of non-locality is considerably *larger* than that implied by the Moyal product.

Restricting the generators to the fundamental representation of $U(N)$ yields further constraints on the possible matter representations. These are restricted to the adjoint, Ψ, the fundamental, ψ, and the antifundamental, $\bar{\psi}$. If $g(x)$ is an $N \times N$ matrix-valued function satisfying

$$g(x) \star g(x)^\dagger = g(x)^\dagger \star g(x) = \mathbf{1}, \tag{62}$$

the finite gauge transformations are

$$A_\mu \to g\star(A_\mu - i\,\partial_\mu)\star g^\dagger, \qquad \Psi \to g\star\Psi\star g^\dagger, \qquad \psi \to g\star\psi, \qquad \bar{\psi} \to \bar{\psi}\star g^\dagger. \tag{63}$$

An important property that follows from these expressions is the non-existence of naive *local* gauge-invariant operators, i.e. $F^2 \to g \star F^2 \star g^\dagger$ under gauge transformations, but in order to cancel out $g(x)$ against $g(x)^\dagger$ we need to use the cyclic property of the trace. Since the "trace" for the Moyal product includes the ordinary integral, we conclude that standard local operators must be integrated over in order to remain gauge-invariant after the noncommutative deformation.

On can do slightly better and define quasi-local operators by using the so-called "open Wilson lines". Consider a Wilson line operator associated to the path γ_x with initial point x:

$$W(\gamma_x) = P_\star \exp\left(i\int_\gamma A\right), \tag{64}$$

with $P_\star$ denoting the instruction of path-ordering with respect to the Moyal product. Then, given any local operator $\mathcal{O}(x)$, formally constructed out of the field strength and covariant derivatives, the noncommutative Fourier transform

$$\tilde{\mathcal{O}}(k) = \int d^d x \,\mathrm{tr}\; \mathcal{O}(x) \star W(\gamma_x) \star e^{ikx} \tag{65}$$

is gauge-invariant provided the endpoint of the path γ_x lies at the point $x^\mu + k_\nu \theta^{\mu\nu}$ (c.f. for example [15]).

There are some interesting consequences of these algebraic restrictions when considering the rank-one NCYM theory, i.e. the one based on "$U(1)$

gauge fields". Notice that this theory contains non-linear interactions much like any other non-abelian gauge theory, such that the theory becomes free in the classical commutative limit $\theta \to 0$. Once the Yang–Mills coupling e is fixed, the restriction on the matter representations implies that the charge of the matter fields cannot be adjusted further. Thus, the $U(1)$ charge assignments of the Standard Model are not easily implemented in a noncommutative deformation (c.f. for example [16], and [17] for alternative constructions).

2.3 Perturbative Quantization

In carrying the quantization of the *classical* theory (54) we may proceed with a formal canonical quantization provided $\theta^{0i} = 0$. Otherwise, the infinite number of time derivatives in the action makes the canonical program rather ackward.

An alternative is to write down a formal path integral

$$Z[J] = \int d\mu[\phi]\; e^{iS[\phi]}\; e^{i\int d^dx\, J\star\phi} \tag{66}$$

with some specification of the integration measure. For the time being, we will restrict ourselves to the perturbative evaluation of $Z[J]$.

The crucial observation is that the free approximation is locally θ-independent:

$$S[\phi]_{\text{free}} = \frac{1}{2}\int d^dx\; \left(\partial_\mu\phi\star\partial^\mu\phi - m^2\phi\star\phi\right) = \frac{1}{2}\int d^dx\; \left(\partial_\mu\phi\,\partial^\mu\phi - m^2\phi^2\right). \tag{67}$$

Therefore, in evaluating perturbation-theory integrals, we can consider the standard Gaussian θ-independent measure. This prescription gives a set of Feynman rules. We have standard propagators

$$\frac{i}{p^2 - m^2 + i\,0}, \tag{68}$$

and non-standard interaction terms. Upon Fourier transformation:

$$\int d^dx\; \phi(x)\star\ldots\star\phi(x) = \int d^dp\,(2\pi)^d\,\delta(\textstyle\sum p)\;\tilde{\phi}(p_1)\ldots\tilde{\phi}(p_n)\;W(p_1,\ldots,p_n), \tag{69}$$

where

$$W(p_1,\ldots,p_n) = \exp\left(-\frac{i}{2}\sum_{i<j} p_i\times p_j\right) \tag{70}$$

is the so-called Moyal phase. Thus, we are led to a simple Feynman rule for the interactions. We just need to "decorate" the standard Feynman vertex with the non-local Moyal phase:

$$-i\,\lambda_n \longrightarrow -i\,\lambda_n\, W(p_1, \dots, p_n). \tag{71}$$

Notice that the Moyal phase spoils the Bose symmetry of the vertex, the noncommutative vertex being only cyclically symmetric. This modifies the symmetry factors associated to the Feynman rules.

Since the vertices written as in (71) are only cyclically symmetric, they satisfy the same topological properties as planar vertices in 't Hooft's double line notation for gauge-theory Feynman rules [18]. Thus, diagrams in non-commutative field theories admit a similar topological classification by the genus of the surface on which they can be drawn.

Using simple topological arguments, plus momentum conservation at each vertex, one can prove a general result regarding the θ-dependence of the Feynman diagram integrands: the class of *planar* diagrams has a θ-dependence saturated by the external legs, i.e. the overall Moyal phase of the diagram with a given set of external legs equals the phase of a single-vertex diagram with the same external legs (c.f. [19]).

For *nonplanar* diagrams, the θ-dependence remains in non-trivial phases in the integrand. Nonplanar loop integrations are then sensitive to the Moyal phases.

2.3.1 Two Examples

Having noticed that the bosonic Feynman vertices are not Bose-symmetric in general, it is still useful in practice to symmetrize them in order to manipulate them in a standard fashion, without paying special attention to the different topological classes of diagrams. We can illustrate this with two examples.

Consider first ϕ^3 theory. The vertex can be obtained directly by considering the Moyal product of two plane-waves

$$\phi(p_1) e^{ip_1 x} \star \phi(p_2)\, e^{ip_2 x} = \phi(p_1)\phi(p_2)\; e^{-\frac{i}{2} p_1 \times p_2}\, e^{i(p_1+p_2)x}. \tag{72}$$

Since the momentum variables p_1, p_2 are integrated over in writing the interaction action, they are dummy variables can be switched over. So we can symmetrize the Moyal product above and write

$$\phi(p_1)\phi(p_2)\,\cos\left(\frac{p_1 \times p_2}{2}\right)\; e^{i(p_1+p_2)x}. \tag{73}$$

Therefore, we can use the Feynman rule

$$\text{Vertex } = -i\,\lambda\,\cos(p_1 \times p_2/2), \tag{74}$$

where Bose symmetry is restored. Consider now the one-loop contribution to the two-point function. It contains a factor of $\cos^2(p_1 \times p_2/2)$ from the vertices. The two structures, planar and non-planar, arise upon writing:

$$\cos^2\left(\frac{p_1 \times p_2}{2}\right) = \frac{1}{2} + \frac{1}{2}\cos\left(p_1 \times p_2\right). \tag{75}$$

The first term, θ independent, yielding the planar part.

A second example of the same nature involves the Feynman rules of a $U(N)$ NCYM theory. Let us write for the plane-wave field:

$$A_\mu(x) = \sum_{a=1}^{N^2} A_\mu^a(p)\,T^a\,e^{ipx} \tag{76}$$

and reduce the commutator:

$$[A_\mu, A_\nu]_\star = \frac{1}{2}\sum_{a,b}\{T^a, T^b\}\,[A_\mu^a, A_\nu^b]_\star + \frac{1}{2}\sum_{a,b}[T^a, T^b]\,\{A_\mu^a, A_\nu^b\}_\star \tag{77}$$

Defining now the usual symmetric and antisymmetric tensor structures:

$$[T^a, T^b] = i\sum_c f^{abc}\,T^c, \qquad \{T^a, T^b\} = \sum_c d^{abc}\,T^c, \tag{78}$$

one obtains

$$\begin{aligned}[A_\mu, A_\nu]_\star \quad &= \quad \sum_c \Big(i\,d^{abc}\,T^c\sin\left(p_1 \times p_2/2\right) + i\,f^{abc}\,T^c\cos\left(p_1 \times p_2/2\right)\Big) \\ &\qquad\qquad \times A_\mu^a(p_1)\,A_\nu^b(p_2)\;e^{i(p_1+p_2)x}\end{aligned} \tag{79}$$

It follows that the Feynman rule for a $U(N)$ noncommutative gauge theory can be constructed from the Feynman rule of the ordinary $SU(N)$ theory by the substitution of the structure constants:

$$f^{abc} \longrightarrow f^{abc}\cos\left(\frac{p_a \times p_b}{2}\right) + d^{abc}\sin\left(\frac{p_a \times p_b}{2}\right). \tag{80}$$

where now the group indices a, b, c include also the diagonal $U(1)$ subgroup of $U(N)$. For example, the noncommutative rank-one theory, $U(1)$, has a three-point coupling of the photon given by

$$\begin{aligned}V_{\gamma\gamma\gamma} \quad &= \quad -2g\sin\left(\frac{p_1 \times p_2}{2}\right)\,\left[(p_1 - p_2)^{\mu_3}\,\eta^{\mu_1\mu_2} + (p_2 - p_3)^{\mu_1}\,\eta^{\mu_2\mu_3}\right. \\ &\qquad\qquad \left. + (p_3 - p_1)^{\mu_2}\,\eta^{\mu_1\mu_3}\right].\end{aligned} \tag{81}$$

2.3.2 Asymptotically Free Photons

As an example of the peculiar new features introduced by noncommutativity we make a heuristic discussion of a surprising fact: the rank-one noncommutative Yang–Mills theory (pure noncommutative photons) is asymptotically free (see for example [20]). According to the previous paragraph, the perturbative structure of this theory is rather similar to that of $SU(N)$ Yang–Mills theory in the limit $N \to 1$. The perhaps surprising fact is that a characteristic dynamical feature such as asymptotic freedom does survive in the limit.

Consider the ordinary $SU(N)$ Yang–Mills theory with Wilsonian cutoff Λ and bare coupling g_Λ (we now switch to Euclidean signature):

$$S = \frac{1}{4g_\Lambda^2} \int^\Lambda \mathrm{tr}\, |F|^2. \tag{82}$$

Integrating out quantum fluctuations in a momentum slice $|k| < |q| < \Lambda$, the operator $|F|^2$ is renormalized as

$$S_{\mathrm{eff}} = \frac{1}{4} \int^{|k|} \frac{1}{g^2(k)}\, \mathrm{tr}\, |F|^2 + \dots, \tag{83}$$

where the effective coupling is given, with logarithmic precission, by

$$\frac{1}{g^2(k)} \sim \frac{1}{g_\Lambda^2} + N \int_{|k|}^\Lambda \frac{d^4 q}{(p-q)^2 q^2} + \dots = \frac{1}{g_\Lambda^2} + \frac{\beta_0 N}{(4\pi)^2} \log\left(|k|^2/\Lambda^2\right) + \text{finite} \tag{84}$$

For $SU(N)$ gauge group, we have $\beta_0 = 22/3$, the usual one-loop beta function coefficient. Notice that the effective coupling corrected by the effect of quantum fluctuations grows towards the infrared, the behaviour that signals asymptotic freedom. Perturbation theory is then expected to break down at scales of order

$$\Lambda_{\mathrm{QCD}} \sim \Lambda\, \exp\left(-\frac{8\pi^2}{N\beta_0 g_\Lambda^2}\right). \tag{85}$$

For an ordinary $U(N)$ gauge theory, the same running takes place, except for the coupling of the global $U(1)$ subgroup, that remains decoupled. Separating this part through the identity

$$\mathrm{tr}\, F^2 = \frac{1}{N} \left(\mathrm{tr}\, F\right)^2 + \mathrm{tr}\, F_{SU(N)}^2 \tag{86}$$

we end up with a one-loop corrected effective action:

$$\begin{aligned} S_{\text{eff}}^{U(N)} &= \frac{1}{4}\int^{|k|}\left(\frac{1}{g_\Lambda^2}+\frac{\beta_0 N}{(4\pi)^2}\log(|k|^2/\Lambda^2)\right)\text{tr }|F|^2 \\ &\quad -\frac{1}{N}\frac{\beta_0 N}{(4\pi)^2}\log(|k|^2/\Lambda^2)\,|\text{tr }F|^2, \end{aligned} \tag{87}$$

where the second term subtracts the running of the $U(1)$ coupling. It can be thought of as the contribution of the one-loop non-planar diagram to the two point function of the field strength.

We now consider the noncommutative theory with $\theta \neq 0$. The integrand has a factor of

$$\sin^2\left(\frac{k\times q}{2}\right)=\frac{1}{2}-\frac{1}{2}\sin\,(k\times q) \tag{88}$$

from (88). The planar diagram contribution is identical to the first term in (87), since θ-dependence only affects external legs. On the other hand, the nonplanar contribution has a surviving factor of

$$\sin\,(k\times q)$$

from the Feynman rules. This factor oscillates very fast for large values of the loop momentum $|q|$. Thus, the loop momentum integral in the nonplanar graph is effectively cut-off at

$$\Lambda_{\text{eff}} \sim \frac{1}{|\tilde{k}|^2}, \tag{89}$$

where we have defined

$$\tilde{k}^\mu \equiv k_\nu\,\theta^{\nu\mu}. \tag{90}$$

In other words, for $|k|^2\theta \gg 1$ the effective coupling runs only at the planar level, with

$$S(|k|^2\theta \gg 1)_{\text{eff}} \approx \frac{1}{4}\int^{|k|}\left(\frac{1}{g_\Lambda^2}+\frac{\beta_0 N}{(4\pi)^2}\log(|k|^2/\Lambda^2)\right)\text{ tr }|F|^2. \tag{91}$$

This still makes sense for $N=1$, so we learn that the NC $U(1)$ theory is asymptotically free! The NC $U(N)$ theory has in fact the same beta function as the ordinary $SU(N)$ theory:

$$\beta(g^2)_{U(N)_\star}=\frac{dg_\Lambda^2}{d\,\log\,\Lambda}=\beta(g^2)_{SU(N)}=-\frac{11g^4N^2}{12\pi^2}. \tag{92}$$

In particular, this would suggest that the NC $U(1)$ theory becomes strongly coupled for

$$\Lambda_{\text{strong}} \sim \Lambda \exp\left(-\frac{12\pi^2}{11 g_\Lambda^2}\right). \tag{93}$$

On the other hand, perhaps we should expect some kind of threshold effect at the classical scale of noncommutativity $|k|^2\theta \sim 1$. In fact, this is the case. Recall that the effective ultraviolet cutoff of the nonplanar diagram was $\Lambda_{\text{eff}} = 1/|\tilde{k}|^2$. So, for $|k|^2\theta \leq 1$ the logarithmic divergence in the nonplanar diagram gives a term proportional to

$$\log\left(\frac{|k|^2}{\Lambda_{\text{eff}}^2}\right) = \log\left(|k|^2\, |\tilde{k}|^2\right), \tag{94}$$

and we obtain

$$\begin{aligned} S(|k|^2\theta \leq 1)_{\text{eff}} \quad &\approx \quad \frac{1}{4}\int^{|k|} \left(\frac{1}{g_\Lambda^2} + \frac{\beta_0 N}{(4\pi)^2} \log\left(|k|^2/\Lambda^2\right)\right) \operatorname{tr}|F|^2 \\ &\quad - \frac{\beta_0}{(4\pi)^2} \log\left(|k|^2 |\tilde{k}|^2\right) |\operatorname{tr}(\partial A)|^2 \end{aligned} \tag{95}$$

In the second term we have written ∂A instead of F because the effective action is evaluated at quadratic order only, and in fact the gauge-invariant completion of (95) cannot be written entirely in terms of the field strength F (c.f. [21]). For us, the important point about (95) is that the second term grows at low energies and produces screening rather than the antiscreening that is characteristic of asymptotic freedom. Thus, we can combine these results and extract the effective coupling of the diagonal $U(1)$ degrees of freedom with running

$$\left(\frac{1}{g_{U(1)}^2}\right)_{|k|^2\theta \leq 1} \approx \frac{1}{g_\Lambda^2} - \frac{\beta_0}{(4\pi)^2} \log\left(|\tilde{k}|^2\, \Lambda^2\right). \tag{96}$$

The result is that the effective $U(1)$ coupling grows towards the infrared, with the running induced by the planar contribution, as in an $SU(N)$ theory in the formal $N \to 1$ limit, up to energies of order $1/\sqrt{\theta}$. At this threshold, the screening effects start to dominate and the effective coupling grows back up. At energies of order $1/\Lambda\theta$ the effective coupling has again the ultraviolet value g_Λ and ceases to run. In principle, one can still have an infrared Landau pole in the pure $U(1)$ noncommutative theory provided $\Lambda_{\text{strong}}\sqrt{\theta} > 1$.

The phenomenon just discussed is the first example of a "mild" UV/IR effect, since we see that, after removal of the UV cutoff Λ, the $\theta \to 0$ limit of the theory is no longer the ordinary free $U(1)$ Maxwell model.

2.4 Physical Interpretation of the Moyal Product

Consider a particle described by a noncommutative field $\phi(x)$, interacting with a fixed external potential $V(x)$ by a term

$$\int d^d x \; (V(x) \star \phi(x) - \phi(x) \star V(x)). \tag{97}$$

For a plane wave configuration $\phi(x) \sim e^{ip \cdot x}$ we have

$$V(x) \star e^{ip\cdot x} - e^{ip\cdot x} \star V(x) = (V(x + p \cdot \theta/2) - V(x - p \cdot \theta/2)) \; e^{ip\cdot x}. \tag{98}$$

Thus, the noncommutative interaction is exactly reproduced by that of a rigid dipole oriented along the vector

$$L^\mu = \theta^{\mu\nu} \, p_\nu, \tag{99}$$

interacting ordinarily through the end-points, exactly like a rigid open string. This analogy is actually rather literal, as we will see in the next section.

Fields interacting in the "fundamental representation" as

$$\int d^d x \; V(x) \star \phi(x) \tag{100}$$

behave as half-dipoles of length $L^\mu/2$ (c.f. [22]).

Therefore, the non-locality of the noncommutative theories constructed out of Moyal products amounts to reinterpreting the elementary excitations as extended rigid objects [23]. This leads to an interesting extension of the heuristic Heisenberg principle. The effecive size of a noncommutative particle grows linearly with the momentum at very high velocity:

$$L_{\text{eff}} = \max \left(\frac{1}{|p|}, |\theta \cdot p| \right). \tag{101}$$

This type of relation is known to appear in string theory with the noncommutativity scale replaced by the Regge slope parameter α' (c.f. for example [24]). This is essentially the reason why NCFT is an interesting toy model of string dynamics; it combines some essential features of strings with a much simpler dynamics with finite particle degrees of freedom.

2.5 Connection to String Theory

The dipole picture implies that elementary quanta of NCFT are analogous to open strings. This analogy is actually the source of one of the most important recent developments in the subject.

Indeed, oriented open strings are naturally dipoles. The coupling of an electromagnetic $U(1)$ vector potential to an open string is given by a Wilson line coupling to the end-points of the string. Consider a string worldsheet with proper time τ and string coordinate σ, the endpoints given by $\sigma = 0$ and $\sigma = \pi$. The $U(1)$ coupling is then

$$S_{U(1)} = \int_{\sigma=0} A_\mu dx^\mu - \int_{\sigma=\pi} A_\mu dx^\mu = \int_{\partial\Sigma} A_\mu dx^\mu = \frac{1}{2}\int_\Sigma F_{\mu\nu}\, dx^\mu \wedge dx^\nu . \tag{102}$$

The complete sigma-model action for a string moving in a background metric $g_{\mu\nu}$ and background magnetic field B_{ij} is

$$S = \frac{1}{4\pi\alpha'}\int_\Sigma g_{\mu\nu}\, dx^\mu\, dx^\nu + \frac{1}{2}\int_\Sigma B_{\mu\nu}\, dx^\mu \wedge dx^\nu , \tag{103}$$

where $(2\pi\alpha')^{-1}$ is the tension of the string.

Let us now suppose that B_{ij} is constant and moreover $|g_{ij}| \ll |\alpha' B_{ij}|$, so that we can approximate the action by

$$S \approx \frac{1}{2}\int B_{ij}\, dx^i \wedge dx^j = \frac{1}{2} B_{ij} \int_{\partial\Sigma} x^i \partial_\tau\, x^j . \tag{104}$$

Thus, we see that the endpoints of the open string behave like electrons in the LLL in this limit! The same arguments as in the electron case yield then

$$[x^j, x^k]_{\partial\Sigma} = i\, \theta^{jk} \tag{105}$$

with

$$\theta^{jk} = \left(\frac{1}{B}\right)^{jk} . \tag{106}$$

In order to obtain a NCFT of rigid dipoles we would like to project out all the massive (oscillatory) degrees of freedom of the open string theory, i.e. we would like to take the zero-slope limit $\alpha' \to 0$. But we just have learnt that at the same time we must keep $\theta \sim 1/B$ constant and also $|g_{ij}| \ll |\alpha' B_{ij}|$. A scaling limit that satisfies these constraints and produces NCFT interaction Lagrangians out of the open-string perturbative interactions is the so-called Seiberg–Witten limit [25]:

$$g_{ij} \sim (\alpha')^2\, B_{ij} B^{ij} \longrightarrow 0 \tag{107}$$

at fixed $B_{ij} = (1/\theta)_{ij}$. Physically, what is being stated is very simple. In order to make the open string into a rigid dipole, we must take the nominal tension to infinity to decouple all oscillator modes (rigidity). Normally this produces the effective collapse of the open string to a pointlike object. However, if the magnetic field is kept large in the scaling limit, the Lorentz force tending to stretch the open string endpoints can compensate for this effect and one reaches a rigid open string of finite extent given by $L \sim \theta p$.

2.6 The UV/IR Mixing

The phenomenon of UV/IR mixing represents the most radical departure of NCFT from the standard behaviour of ordinary field theories. It occurs in perturbation theory, so that it can be studied with considerable detail, and represents the fact that the two deformation operations: the noncommutative deformation $\theta \neq 0$, and the quantum deformation $\hbar \neq 0$, do not commute [5].

The UV/IR mixing is a lack of Wilsonian decoupling between UV and IR scales, even in the presence of explicit masses. Technically, it comes about in a rather elementary fashion. Recall that nonplanar diagrams have improved convergence properties because of Moyal phases that depend on loop momenta. For example, two loop momenta q, q' tied by a Moyal phase

$$e^{-\frac{i}{2} q \times q'}$$

will introduce an effective cutoff in the diagram at the scale $\Lambda_{\text{eff}} \sim 1/\sqrt{\theta}$. On the other hand, a loop momentum q tied to an exteral momentum p will introduce

$$e^{-\frac{i}{2} q \times p},$$

which in turn gives an effective cutoff $\Lambda_{\text{eff}} \sim 1/|p \cdot \theta|$. Since the corresponding UV divergences are absent, they are not explicity subtracted in the renormalization procedure. However, since the effective cutoff is non-analytic in θ, these singularities in physical quantities show up in the $\theta \to 0$ limit. Alternatively, in Green's functions depending on external momenta, they show up in the limit $|\theta \cdot p| \to 0$. This may be interpreted as non-analytic behaviour in the $\theta \to 0$ limit at finite $|p|$, or as an infrared singularity at fixed θ.

Therefore, we see that in general the noncommutative *quantum* field theory is *not* a smooth deformation of the ordinary $\theta = 0$ theory, even if it was so in the classical approximation. We also learn that, at fixed non-zero

θ, the NCFT is IR singular as a result of divergences that originally had an UV interpretation, hence the name UV/IR mixing.

2.6.1 A Simple Example

In order to illustrate this important phenomenon, we consider the simplest setting in which it arises: the one-loop mass renormalization of the ϕ^4 model in four dimensions (in this section we work in Euclidean signature):

$$S = \int \left(\frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi \star \phi \star \phi \star \phi \right) . \qquad (108)$$

In the ordinary ($\theta = 0$) model the leading mass renormalization comes from the normal-ordering diagram contribution to the self-energy:

$$\Sigma = \frac{\lambda}{2} \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + m^2} \approx \frac{\lambda}{32\pi^2} \left(\Lambda^2 - m^2 \log(\Lambda^2/m^2) + \text{finite} \right) , \qquad (109)$$

in terms of the ultraviolet cutoff Λ. We find the standard quadratic renormalization together with a subleading logarithmic piece.

In the noncommutative theory, we have two contributions, planar and nonplanar. The planar diagram gives exactly the contribution (109), except for the different symmetry factor of the diagram, which is 1/3 instead of 1/2. On the other hand, the nonplanar diagram has a surviving Moyal phase that makes it finite:

$$\Sigma_{\rm NP} = \frac{\lambda}{6} \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\times p}}{k^2 + m^2} = \frac{\lambda}{24\pi^2} \frac{m^2}{\sqrt{m^2\tilde{p}^2}} K_1 \left[\sqrt{m^2\tilde{p}^2} \right] . \qquad (110)$$

In order to compare the planar and nonplanar parts, we introduce an ultraviolet cutoff via a Schwinger proper-time parametrization:

$$\left[\frac{1}{k^2 + m^2} \right]_\Lambda = \int_0^\infty ds \, e^{-s(k^2+m^2)} \, e^{-1/\Lambda^2 s} . \qquad (111)$$

We find

$$\Sigma_{\rm NP} = \frac{\lambda}{96\pi^2} \left(\Lambda_{\rm eff}^2 - m^2 \log\left(\Lambda_{\rm eff}^2/m^2\right) + \dots \right) , \qquad (112)$$

where the effective cutoff is given by

$$\Lambda_{\rm eff}^2 = \frac{1}{\tilde{p}^2 + 1/\Lambda^2} . \qquad (113)$$

Notice that $\Lambda_{\text{eff}} \approx \Lambda$ for $|p| \ll 1/\Lambda\theta$, whereas $\Lambda_{\text{eff}} \approx 1/|\tilde{p}|$ for $|p| \gg 1/\Lambda\theta$. So, if we renormalize the theory at fixed p and fixed θ, by subtracting the planar divergence in the $\Lambda \to \infty$ limit:

$$m^2 \to M^2 = m^2 + \frac{\lambda}{48\pi^2}\Big(\Lambda^2 - m^2 \log\left(\Lambda^2/m^2\right)\Big) + \text{constant} \tag{114}$$

we have a quadratic 1PI effective action:

$$\Gamma_{\text{1PI}} = \int d^4p\, \phi(-p)\, \Gamma^{(2)}(p)\, \phi(p) + \dots \tag{115}$$

with

$$\Gamma^{(2)}(p) = p^2 + M^2 + \frac{\lambda}{96\pi^2\tilde{p}^2} - \frac{\lambda M^2}{96\pi^2} \log\left(1/M^2\tilde{p}^2\right) + \dots \tag{116}$$

Thus, as promised, the effective action has a singularity at $p = 0$ that can be interpreted either as an IR singularity at fixed θ or as a non-analiticity as a function of θ at fixed p.

We may wonder to what extent the leading IR-singular term

$$\Sigma_{\text{NP}} \sim \frac{1}{|\tilde{p}|^2}$$

can be reliably calculated in perturbation theory. An indication is given by the following estimation. Higher-order perturbative corrections to the leading $1/\tilde{p}^2$ behaviour have the form

$$\frac{\lambda}{\tilde{p}^2}\left[\lambda \log\left(M^2\,\tilde{p}^2\right)\right]^n.$$

These corrections are significant only for momenta such that the term in brackets is of $\mathcal{O}(1)$. Thus, we see that perturbation theory will break down at nonperturbatively small momenta of order

$$|p|_{\text{breakdown}} \sim \frac{1}{M\theta}\, e^{-C/\sqrt{\lambda}}. \tag{117}$$

For the present model, we can give a simple physical interpretation of the UV/IR mixing provided the noncommutativity is purely spatial, i.e. $\theta^{0i} = 0$. Notice that the just computed 1PI effective action implies a modified dispersion relation for the ϕ-quanta of the form:

$$p^2 + M^2 + \frac{\lambda}{96\pi^2\tilde{p}^2} = 0. \tag{118}$$

After Wick rotation back to $(-+++)$ signature one finds:

$$\omega = \sqrt{|\vec{p}|^2 + M^2 + \frac{c}{\theta^2 |\vec{p}|_\theta^2}} \tag{119}$$

where $c = \lambda/96\pi^2$ and $\vec{p}_\theta$ is the projection of the spatial momentum onto the plane of noncommutativity.

This expression shows dramatically the UV/IR mixing effects, since the entire energy spectrum below noncommutative momenta of order $\lambda^{1/4}/\sqrt{\theta}$ has been removed!

2.6.2 The Case of Gauge Theories

The UV/IR mixing in the case of gauge theories shows some specific features of interest [26]. Consider the polarization tensor of the NC $U(1)$ theory:

$$S^{(2)} = \frac{1}{2} \int A_\mu(k)\, \Pi^{\mu\nu}(k)\, A_\nu(-k). \tag{120}$$

In the ordinary (or planar) case, gauge invariance together with Lorentz invariance forbids a quadratic divergence in the polarization $\Pi_{\mu\nu} \sim \eta_{\mu\nu}\,\Lambda^2$. It would violate transversality. In fact

$$\Pi_{\mu\nu}(k) = \left(k_\mu\, k_\nu - \eta_{\mu\nu}\, k^2\right)\, \Pi(k), \tag{121}$$

where

$$\Pi(k) \sim \log\left(\frac{|k|^2}{\Lambda^2}\right) + \text{finite}. \tag{122}$$

The nonplanar contribution has the standard effective cutoff $\Lambda_{\text{eff}} = \min(\Lambda, 1/|\tilde{k}|)$. Because of gauge invariance at $\theta = 0$, we would expect that UV/IR phenomena would only appear at logarithmic level $\Pi(k) \sim \log(|k|^2|\tilde{k}|^2)$, and indeed we found such terms in the previous section in our discussion of asymptotic freedom.

However, the explicit breaking of Lorentz symmetry allows now for other kinematical structures with IR singularity stronger than logarithmic and still transverse. In particular, quadratic divergences do appear with the structure

$$\Pi^{\text{NP}}_{\mu\nu} = -g^2\, C\, \frac{\tilde{k}^\mu \tilde{k}^\nu}{\tilde{k}^2}\, \Lambda_{\text{eff}}^2 = -g^2\, C\, \frac{\tilde{k}^\mu \tilde{k}^\nu}{\tilde{k}^4}. \tag{123}$$

Notice that transversality is ensured by $k^\mu \tilde{k}_\mu = k_\mu \theta^{\mu\nu} k_\nu = 0$. At one-loop, the constant C has been calculated to be

$$C = \frac{2N}{\pi^2}\,(2 + n_s - 2n_f), \tag{124}$$

where N is from the $U(N)$ gauge group, n_s is the number of complex scalars in the adjoint representation and n_f is the number of Majorana fermions also in the adjoint representation. Notice that $C = 0$ for supersymmetric or soflty broken supersymmetric spectra.

Thus, we learn that the strength of the UV/IR mixing responds to the naive power-counting rather than to the effective divergence structure of the $\theta = 0$ model. In particular, one finds unstable dispersion relations in NC $U(1)$

$$\omega(k) = \sqrt{\vec{k}^{\,2} - \frac{g^2 C}{\theta^2 |\vec{k}_\theta|^2}} \tag{125}$$

with low-momentum tachyonic excitations as soon as $C > 0$.

2.6.3 Heuristic Explanation of the UV/IR Mixing

The dipole picture of NCFT that was developed before provides a simple heuristic explanation of the UV/IR mixing. Since a virtual loop of momentum p carries dipoles of transverse length $|\theta \cdot p|$, we understand that the loop corrections to the Green's functions will have strong θ-dependence down to arbitrarily low energies, unless these effects are cancelled by some mechanism (such as enough amount of supersymmetry).

Notice that, if an explicit UV cutoff is present, Λ, it sets the maximum possible momentum of the virtual dipoles circulating in the loop. This in turn means that significant θ-dependence only appears down to momenta of order $1/\Lambda\theta$.

Thus, we have the following general hierarchycal structure. NCFT with ultraviolet cutoff $\Lambda\sqrt{\theta} \gg 1$ has significant classical effects (tree level) associated to noncommutativity up to length scales of $\mathcal{O}(\sqrt{\theta})$. However, one-loop effects "transport" the effects of noncommutativity to the larger length scale of $\mathcal{O}(\Lambda\theta)$. This larger length scale is true dynamical scale of noncommutativity. Of course, this picture would be invalidated if perturbation theory would break down at some intermediate scale. For example, if we insist on removing the ultraviolet cutoff $\Lambda \to \infty$ at fixed θ, necessarily $\Lambda\theta \to \infty$ and perturbation theory is bound to break down before we reach the deep infrared domain.

2.6.4 UV/IR Mixing and Unitarity

There is an interesting interplay between the UV/IR mixing and the violation of unitarity in the case that the noncommutativity affects time. Instead of developing the general theory we will simply explain the basic phenomena by looking at a simple example. Let us consider a noncommutativity matrix of the skew-diagonal form $(\theta^{\mu\nu}) = i\,\mathrm{diag}\,(\sigma_2\,\theta_e, \sigma_2\,\theta_m)$. That is, we have the noncommutativity relations:

$$[t, x] = i\theta_e, \qquad [y, z] = i\theta_m. \tag{126}$$

We return now to the ϕ^4 theory studied in the previous section and we consider the massless model for simplicity. The normal-ordering tadpole diagram has no interesting dynamical interpretation in the ordinary theory, simply inducing the quadratic renormalization of the mass parameter. However, this is no longer the case for the noncommutative theory, since the nonplanar tadpole diagram *does* have an interesting singularity structure when interpreted as a $1 \rightarrow 1$ scattering amplitude:

$$i\mathcal{M}(p \rightarrow p) = -i\frac{\lambda}{6}\int \frac{d^4q}{(2\pi)^4} e^{-i\tilde{p}\cdot q}\,\frac{i}{q^2 + i0} = -i\frac{\lambda}{24\pi^2}\frac{1}{-\tilde{p}^2 + i0}. \tag{127}$$

The striking fact about this explicit expression is that the imaginary part of the amplitude is a non-trivial distribution, i.e.

$$2\,\mathrm{Im}\,\mathcal{M}(p) = \frac{\lambda}{12\pi}\,\delta(-\tilde{p}^2). \tag{128}$$

Therefore, if unitarity is to be satisfied, this imaginary part should be understadable in terms of a product of on-shell amplitudes corresponding to all the non-trivial cuttings of the diagram. Since the tadpole has no on-shell cuttings, it seems that we find a violation of unitarity [27].

Despite this fact, one can still manipulate $\mathrm{Im}\,\mathcal{M}$ in a purely formal fashion so that it looks like a contribution from the optical theorem. Take $\theta_m = 0$ and $\theta_e \neq 0$, and introduce

$$1 = \int d^4k\,\delta(p - k)$$

to obtain

$$2\,\mathrm{Im}\,\mathcal{M}(p) = \frac{\lambda}{12\pi}\int d^4k\,\delta(k-p)\,\delta(-\tilde{p}^2) = \int \frac{d^3\vec{k}}{2(2\pi)^3|k_1|}\left(\frac{(2\pi)^2\lambda}{6\theta_e^2}\right)\,\delta(p-k). \tag{129}$$

This formula can be interpreted as the amplitude for the mixing of the ϕ quanta with particle states $|\chi\rangle$ with dispersion relation $|k_0| = |k_1|$. The $\phi - \chi$ coupling is given by

$$\lambda_{\phi\chi} = \sqrt{\frac{(2\pi)^2 \lambda}{6\theta_e^2}}. \tag{130}$$

Thus, it seems that we can save unitarity at the expense of enlarging the Hilbert space of asymptotic states, just like one can make the S-matrix of open-string theory unitary by introducing the closed-string states. In fact, while this is true at a formal level, it turns out that the added Hilbert space of 'closed-string' states $|\chi\rangle$ does not satisfy appropriate physical conditions. In particular these states come with a continuous spectrum, they are tachyonic and moreover have negative norm in general.

For example, just considering the more general case with $\theta_m \neq 0$ in our example above yields

$$2\,\mathrm{Im}\,\mathcal{M}(p) = \int \frac{d^3\vec{k}}{2(2\pi)^3 \omega_\chi} \left(\frac{(2\pi)^2\lambda}{6\theta_e^2}\right) \delta(p-k), \tag{131}$$

where the frequency of the χ particles is:

$$\omega_\chi = \sqrt{|\vec{p}_e|^2 - \frac{\theta_m^2}{\theta_e^2}\,|\vec{p}_m|^2}. \tag{132}$$

This dispersion relation shows clearly that the χ particles have an unbounded-below spectrum of tachyonic excitations [28]. Thus, timelike noncommutative theories are generically inconsistent in perturbation theory, at least to the extent that UV/IR mixing is present.

2.7 Remarks on θ-Phenomenology

The most obvious application of NCFT is to entertain the possibility that the noncommutativity of spacetime might be real and could be detected experimentally. In such a situation the most notorious feature of the physics is the breakdown of Lorentz invariance. Even if $\theta^{0i} = 0$, the spatial noncommutativity $\theta^{ij} = \epsilon^{ijk}\theta_k$ determines a privileged direction *in vacuo* $\vec{\theta} = (\theta_k)$. Thus, collider experiments put a bound of order

$$|\theta| < (100 \text{ GeV})^{-2} \tag{133}$$

to begin with. In fact, it is not easy to be more specific since the Standard Model doesn't fit naturally into a NCFT with Lorentz violation (recall the problem of $U(1)$ charge assignments). For this reason, most of the phenomenological discussions of NCG effects have been carried out in the noncommutative generalization of the QED sector.

The bound (133) can be improved by application of some elementary constraints from atomic physics. Because of the dipole picture given before, the leading interaction of electrons with the field of the atomic nucleus has a dipole moment induced by the substitution

$$x^\mu \longrightarrow x^\mu - \frac{1}{2} p_\alpha \, \theta^{\alpha\mu},$$

so that the Coulomb potential has terms:

$$V_C(|\vec{x} - \tfrac{1}{2}\,\vec{p}\cdot\vec{\theta}|) = -\frac{\alpha_{em} Z}{\sqrt{(\vec{x} - \frac{1}{2}\,\tilde{p})^2}} \approx -\frac{\alpha_{em} Z}{|\vec{x}|} + \frac{1}{2}\frac{\alpha_{em} Z}{|\vec{x}|^3}\,\vec{\theta}\cdot\vec{L} + \mathcal{O}(\theta^2 p^4), \quad (134)$$

where $\vec{L} = \vec{x} \wedge \vec{p}$. Thus, this term induces a "noncommutative hyperfine splitting" [29]. From limits on the Lamb shift we can put a bound of order

$$|\theta| < (10 \text{ TeV})^{-2}. \quad (135)$$

Constraints from collider experiments are not actually much better than this, if evaluated at tree level. Dependence on the noncommutativity parameter in the vertices comes with two powers of momenta (derivatives) and thus it corresponds generally to dimension five or six effective operators. For example, a leading correction to the $e^+\gamma e^-$ vertex is given by the operator

$$\theta^{\alpha\beta}\,\partial_\alpha\,\bar{\psi}\,\gamma^\mu A_\mu\,\partial_\beta\,\psi. \quad (136)$$

Corrections from such operators are or relative order $\mathcal{O}(\theta E^2)$ for processes at typical energies of $\mathcal{O}(E)$. Thus, collider physics at $E \sim 100$ GeV, known to within a few percent errors, give bounds of order

$$|\theta| < \frac{1}{100\,E^2} \sim (\text{TeV})^{-2}. \quad (137)$$

When quantum corrections are considered, the situation changes dramatically. The UV/IR mixing arising at one-loop order implies that noncommutative effects show up at energies much below $1/\sqrt{\theta}$. In fact, noncommutative QED has tachyonic photon excitations induced at one-loop order and therefore it is incompatible, not only with experiments, but with simple observations of everyday life. This means that, in exploring applications of NCG to phenomenology in the context of weakly coupled NCFT, we must assume the existence of an UV cutoff beyond which the effects of UV/IR mixing dissappear.

Since UV/IR mixing affects dispersion relations, this means that the breakdown of Lorentz symmetry is not restricted to (nonrenormalizable) operators of high dimension, but rather creeps in the operators of dimension two and three at the one-loop level. Correspondingly, the violations of Lorentz symmetry that affect dispersion relations are the subject of fantastic constraints from both low and high energy physics (see for example [30]).

Consider, for example, the dispersion relation of photons corrected at one loop in the pure NC $U(1)$ theory. The leading terms in the polarization tensor at low momentum are (we neglect the logarithmic corrections that only renormalize the coupling):

$$\Pi_{\mu\nu} = (p_\mu \, p_\nu - p^2 \, \eta_{\mu\nu}) + \tilde{p}_\mu \, \tilde{p}_\nu \, \Pi_{nc}, \tag{138}$$

where

$$\Pi_{nc} = -\frac{C \, g^2}{|\tilde{p}|^4}. \tag{139}$$

Considering transverse photons with polarization $A_\mu \sim \tilde{p}_\mu$ we obtain a mass-shell condition

$$p^2 - \frac{C \, g^2 \, \tilde{p}^2}{|\tilde{p}|^4} = 0. \tag{140}$$

Since $C > 0$ we find tachyonic excitations at low momentum. Therefore, we must assume some UV cutoff that eliminates the UV/IR mixing due to very long dipoles in the virtual loop. One such cutoff is provided for example by a softly broken supersymmetric spectrum broken at scale Λ_s. Then, we have an effective cutoff for the nonplanar diagram given by

$$\Lambda^2_{\text{eff}} = \frac{1}{(-\tilde{p}^2 + 1/\Lambda_s^2)^2} \tag{141}$$

and a corrected dispersion relation for photons polarized as $A_\mu \sim \tilde{p}_\mu$ given by

$$\omega^2 = |\vec{p}|^2 - \frac{C g^2 |\vec{p}|^2 \theta^2}{(\theta^2 |\vec{p}|^2 + 1/\Lambda_s^2)^2}, \tag{142}$$

where we assume that the photon propagates parallel to the noncommutative directions. Expanding this dispersion relation around low momenta we see that it produces a correction to the speed of light for these photons given by

$$c_s = 1 - C\, g^2\, \theta^2\, \Lambda_s^4. \tag{143}$$

This means, in particular, that we must have $\Lambda_s \sqrt{\theta} \gg 1$ in order not to conflict with observations. So we actually have an inverted hierarchy in which the noncommutativity scale is forced to be much higher than the supersymmetry breaking scale. Even in this situation, a variety of phenomenological constraints put bounds of order

$$|c_s - 1| < 10^{-15} \tag{144}$$

or even stronger, depending on how model-independent we wish to be (see for example [31]). This translates into bounds on the hierarchy between Λ_s and θ that easily render the classical bounds irrelevant.

In any case, the lesson to be learned from these considerations is that noncommutative phenomenology is probably a premature exercise. The absence of natural models and the strong bounds to be put on θ at the level of perturbative dynamics are rather neat arguments against the prospects of such phenomenonlogical exercises.

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