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Notes on Gravitation[1]

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Abstract

We present an elementary introduction to Einstein's theory of General Relativity (GR). These notes are an informal rendering of the handwritten guidelines for the classes, with some editing and additions. They are in constant modification, as typos or more serious errors are discovered.

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Contents

Chapter 1

Introduction and preliminaries

We assume prior knowledge of mechanics and electrodynamics at the level of Landau–Lifschitz vols I and II (the chapters on fundamentals). In particular, we require certain familiarity with Lagrangian mechanics, Maxwell's equations and Special Relativity (SR). There are many books which can be used as a complement to these notes. A biased list is the following.

- **Elementary level:** Schutz (Cambridge 1980, Cambridge 1985), Hartle (Benjamin 2003).
- **Comprehensive:** Weinberg (Wiley 1972), Misner-Thorne-Wheeler (Freeman 1973), Landau-Lifshitz (Reverte 1981). Carroll (Benjamin 2003).
- **Advanced level:** Wald (Chicago 1984), Hawking-Ellis (Cambridge 1973). Chandrasekhar (Oxford 1992).
- **Web:** 't Hooft, http://www.phys.uu.nl/ thooft/. Carroll, gr-qc/9712019. Townsend, gr-qc/9707012.

Conventions

The metric signature convention is $(-+++)$. Four-dimensional indices transforming in the Lorentz group are labeled with latin letters: $a, b, c, \ldots$. Four-dimensional indices transforming in the group of diffeomorphisms are labeled with greek letters: $\alpha, \beta, \ldots, \mu, \nu, \ldots$. Einstein's summation convention is used in these cases. Three-dimensional *spatial* indices are labeled with latin letters: $i, j, k, \ldots$, and Einstein's convention is *not* used.

Riemann's tensor is defined with the appropriate sign so that the gravitational Lagrangian takes the form $\mathcal{L}_g = \sqrt{-g}\, R$, where $R = g^{\mu\nu} R_{\mu\nu}$ is the Ricci scalar and $g \equiv \det(g_{\mu\nu})$.

The following set of conventions is adopted when writing the norm of a general four-vector $U^\mu = (U^0, \vec{U})$:

$$U^2 = g_{\mu\nu}\, U^\mu U^\nu = g^{\mu\nu}\, U_\mu U_\nu = U_\mu U^\mu \ .$$

For a Lorentz four-vector we have similar expressions in terms of Lorentz indices and the Lorentz metric η_{ab}. In addition, we also have $U^2 = U_a U^a = -(U^0)^2 + \vec{U}^{\,2} = -(U^0)^2 + \sum_{i=1}^3 (U^i)^2$. These conventions are also used for differential operators, such as the Laplacian $\nabla^2 \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu$.

A dot superscript denotes derivation with respect to proper time: $\dot{x}^\mu \equiv dx^\mu / d\tau$, whereas derivation with respect to coordinate time is written explicitly.

Prelude

The first comprehensive treatment of gravitational phenomena goes back to Newton in the XVII century, with his famous force law

$$|F_G| = G\frac{m_1 m_2}{r^2} \tag{1.1}$$

for the gravitational attraction between two masses separated a distance r. The potential energy at height r in the field of a mass M is given by $-GM/r$ per unit of test mass, up to an additive normalization constant. In orbital motion we have a kinetic energy of the same order of magnitude. The characteristic velocity of an orbit at height r is then $v(r) \sim \sqrt{GM/r}$, so that the relativistic character of this motion is measured by the ratio

$$\frac{v^2}{c^2} \sim \frac{GM}{c^2 r} \sim \frac{R_s}{r} \;,$$

where c is the speed of light and we have defined a characteristic gravitational (so-called Schwarzschild) radius for the gravitational field of a mass M:

$$R_s(M) \equiv \frac{2GM}{c^2} \;.$$

The gravitational interaction becomes relativistic, i.e. non-Newtonian, for $\phi(r) \equiv R_s/r \sim 1$. The dimensionless potential ϕ also measures the order of magnitude of relativistic effects arising as corrections to the Newtonian theory.

For localized systems, the value of ϕ on Earth is of order $\phi_\oplus \sim 10^{-9}$. In the vicinity of the sun, $\phi_\odot \sim 10^{-6}$. On the surface of a white dwarf star we have $\phi_{\rm wds} \sim 10^{-4}$, the same order of magnitude of relativistic effects as the hydrogen atom. Finally, relativistic stars such as neutron stars reach $\phi_{\rm ns} \sim 0.1$ and black holes always have $\phi_{\rm bh} \sim 1$.

For non-localized systems, such a uniform-density distribution, we have $M(r) \sim \rho \cdot r^3$ for the mass enclosed by a sphere of size r. Then, we have $\phi \sim (H\,r)^2$, with H defining the Hubble parameter [1]

$$H^2(\rho) \equiv \frac{8\pi G\rho}{3c^2} \;.$$

Therefore, a cosmological model becomes relativistic at distances of order H^{-1}. This happens in our Universe for $H^{-1} \sim$ Gpc.

In these notes we develop the current theory of the gravitational interaction in relativistic regimes where $\phi = \mathcal{O}(1)$, essentially developed by Einstein in the decade prior to 1916. As indicated above, this theory can be tested to order 10^{-6} in solar system experiments, and it is essential to understand the dynamics of relativistic stars and the global properties of the universe on distance scales beyond gigaparsecs. On the other hand, the effects of gravity are largely irrelevant at the scale of elementary particles. The reason is the extreme weakness of the gravitational interaction between subatomic particles.

Despite the intuitive idea that subatomic particles are essentially ‘point-like’ and thus should behave as tiny black holes, quantum effects prevent any physical localization of a particle below

[1] The precise numerical factors in the definitions of R_s and H are irrelevant for our present discussion of order-of-magnitude estimates, but they conform to the more precise definitions to come.

its Compton wavelength, which sets its minimum 'quantum size'. For a particle of mass m we have a Compton length scale

$$\lambda_{\rm C} \sim \frac{\hbar}{mc} ,$$

where $\hbar$ is the reduced Planck constant. The value of the gravitational potential at the Compton scale is

$$\phi_{\rm Compton} \sim \frac{R_s(m)}{\lambda_{\rm C}(m)} \sim \left(\frac{m}{M_{\rm Pl}}\right)^2 \sim \left(\frac{\ell_{\rm Pl}}{\lambda_{\rm C}}\right)^2 ,$$

where $M_{\rm Pl} = \sqrt{\hbar c/G} \sim 10^{19} m_{\rm proton}$ is the Planck mass and $\ell_{\rm Pl} = \sqrt{G/\hbar c^3} \sim 10^{-19}\lambda_{\rm proton}$ is the Planck length. We see that gravitational effects in the behavior of known quantum particles are utterly negligible, of $\mathcal{O}(10^{-40})$. Significant quantum gravitational effects would require Planck-mass elementary quantum objects or Planck lengths in the quantum resolution of scales implied by Heisenberg's principle. This means in practice that quantum gravitational effects are irrelevant from the point of view of feasible experiments.

It is possible to imagine situations where this conclusion is significantly modified. For example, it could be that the world is $4+n$ dimensional below some length scale ℓ_c. In this case Newton's force could actually scale like $1/r^{2+n}$ for $r \ll \ell_c$, which means a stronger growth at short distances. In practice this would imply a lower effective Planck mass and thus room for experimental hope. Currently there is no evidence for such scenarios, but they serve as indications that other possibilities exist.

More generally, the greatest revolutions in theoretical physics are associated to conceptual synthesis between seemingly incompatible theories, in a tight and essentially unique solution. Famous examples of this trend are the theory of quantum fields, the essentially unique unification of special relativity and quantum mechanics. Another example is Einstein's theory of gravitation, again the essentially unique theory of relativistic classical gravity. Many physicists hope that the final synthesis involving the three basic constants, $\hbar, c$ and G, will be born out in a conceptually 'tight' fashion, so that we may hope to 'corner' the answer even in the absence of crucial experimental input.

While we wait for the fully fledged development of such a theory, we shall set $\hbar = 0$ for most of these lecture notes.

1.1 Lagrangians

Dynamical systems can be described classically by equations of motion or by specifying action functionals. Let the pair $(\mathcal{Q}(t), d\mathcal{Q}(t)/dt)$ denote the *state* of a mechanical system (the data necessary to determine its future, usually initial positions and first time derivatives)[2].

Given initial and final values of the configuration variables, $\mathcal{Q}(t_i) = \mathcal{Q}_i, \mathcal{Q}(t_f) = \mathcal{Q}_f$, we construct the action functional

$$S[\gamma] = \int_\gamma dt\, L(\mathcal{Q}, d\mathcal{Q}/dt) \ , \tag{1.2}$$

where γ denotes a given trajectory, i.e. the function $\gamma : t \to \mathcal{Q}(t)$, and L is called the Lagrangian. Then, the equations of motion follow from the extrema of the action functional, $\delta S = 0$, for all variations $\mathcal{Q} \to \mathcal{Q} + \delta\mathcal{Q}$ with fixed initial and final values.

By direct calculation we have

$$0 = \delta S = \int_\gamma dt \left(\frac{\partial L}{\partial \mathcal{Q}} \delta\mathcal{Q} + \frac{\partial L}{\partial \mathcal{Q}_t} \delta\mathcal{Q}_t \right) \ ,$$

where we set $\mathcal{Q}_t \equiv d\mathcal{Q}/dt$. Using that $\delta\mathcal{Q}_t = (\delta\mathcal{Q})_t$ and integrating by parts we find

$$0 = \int_\gamma dt\, \delta\mathcal{Q} \left(\frac{\partial L}{\partial \mathcal{Q}} - \frac{d}{dt}\frac{\partial L}{\partial \mathcal{Q}_t} \right) + 0 \ , \tag{1.3}$$

where the last vanishing term corresponds to the contribution of total derivatives in time, vanishing because of the boundary conditions $\delta\mathcal{Q}(t_i) = \delta\mathcal{Q}(t_f) = 0$. The extremal trajectory must satisfy (1.3) for all values of $\delta\mathcal{Q}$, implying the vanishing of the term in parenthesis. Therefore we arrive at the the so-called Euler–Lagrange equations of motion,

$$\frac{d}{dt}\frac{\partial L}{\partial \mathcal{Q}_t} - \frac{\partial L}{\partial \mathcal{Q}} = 0 \ . \tag{1.4}$$

Lagrangians are defined up to a total derivative in time. Given L, then $L + df/dt$ leads to the same equations of motion.

The elementary example is that of an isolated system of Newtonian particles with generalized coordinates $\mathcal{Q}(t) \to q_p(t)$. The Lagrangian reads

$$L = \sum_p \tfrac{1}{2}\, m_p \left(\frac{dq_p}{dt} \right)^2 - U(q_1, \ldots, q_p, \ldots) \ , \tag{1.5}$$

which induces the well-known Newtonian system of equations

$$m_p \frac{d^2 q_p}{dt^2} = -\frac{\partial U}{\partial q_p} \ .$$

[2] Here $\mathcal{Q}$ is an abstract notation for a set of position coordinates, particle number or local excitation of a field.

Fields

We may take a formal continuum limit for systems of the form (1.5) with degrees of freedom pinned to points of space, leading to the notion of field theories. Here, the role of the particle index p is played by an 'approximately' continuous label corresponding to the point of space $\mathcal{Q}(t) \to q_{\vec{x}}(t)$, where we continue using a condensed notation, suppressing the extra degrees of freedom of $q_{\vec{x}}$ at each point.

Expanding the interaction potential energy U around a stable equilibrium configuration and defining the local displacement variable $\phi(\vec{x},t) = \sqrt{m_{\vec{x}}}\left(q_{\vec{x}}(t) - q_{\vec{x}}^{(0)}\right)$, the assumption of equilibrium means that $U[\phi(\vec{x},t)]$ is a functional with a local minimum at $\phi = 0$. The assumption of *locality* of the interactions means that we can organize the expansion in powers of derivatives of $\phi(\vec{x},t)$, so that the potential energy can be formally written as

$$U[\,\phi\,] = \int d^3x \,\left(V(\phi) + \tfrac{1}{2}\,c_s^2(\vec{\partial}\phi)^2 + \text{higher derivatives}\right),$$

where we have further assumed translational and rotational invariance. Local stability of the equilibrium point requires that the so-called 'squared speed of sound', c_s^2, and 'field mass squared', $m^2 = d^2V(\phi)/d\phi^2|_{\phi=0}$, be positive.

Notice that long distance physics is controlled by the terms with the smallest number of spatial derivatives. Consider for example two contributions with two and four derivatives respectively, $(\vec{\partial}\phi)^2 + \lambda^2(\vec{\partial}^2\phi)^2$. By dimensional analysis, λ must necessarily be a length scale, so that a field configuration with scale of variation of order L contributes an amount of order $L^{-2}(1 + C(\lambda/L)^2)$ for some constant C. Hence, at long distances $L \gg \lambda$ the higher-derivative terms give a subleading contribution and in any case their effects can be gradually introduced by perturbation theory in the corresponding dimensionless effective couplings, such as $\lambda_{\rm eff}(L) = \lambda/L$. This type of "short-distance" expansion is called *effective field theory* and permeates all modern approaches to fundamental physics.

Collecting all terms together we are led to a field theory Lagrangian

$$L = \int d^3x \,\mathcal{L}[\,\phi\,]\,, \tag{1.6}$$

where the Lagrangian *density* has the following structure

$$\mathcal{L}[\,\phi\,] = \tfrac{1}{2}\,(\partial_t\phi)^2 - \tfrac{1}{2}\,c_s^2(\vec{\partial}\phi)^2 - V(\phi) + \text{higher derivatives} + \text{boundary terms}\,. \tag{1.7}$$

In most applications, boundary terms are neglected assuming appropriate boundary conditions for the fields, so that we can integrate by parts at will. The function $V(\phi)$ stands for the non-derivative part of the potential energy, and takes a polynomial form in an expansion around the equilibrium configuration at $\phi = 0$. A linear term of the form $J\phi$, called a 'source term', is often considered as a way of 'driving' the system, to study perturbative response of the equilibrium state. The Euler–Lagrange equations stemming from the long-distance approximation (1.7) are

$$\left(-\partial_t^2 + c_s^2\,\vec{\partial}^{\,2}\right)\phi = V'(\phi) = J + m^2\,\phi + \text{nonlinear terms}\,. \tag{1.8}$$

The elementary solution of the homogeneous equation for zero mass,

$$(-\partial_t^2 + c_s^2\,\vec{\partial}^{\,2})\phi = 0 \tag{1.9}$$

are waves $\phi(t, \vec{x}\,)$ propagating at speed c_s. The elementary solution for a static (time-independent) field created by a source J is the solution of Poisson's equation,

$$\phi(\vec{x}\,)_{\text{static}} = \frac{1}{\vec{\partial}^2} J = -\int d^3y\, J(\vec{y}\,) \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}(\vec{x}-\vec{y}\,)}}{\vec{k}^{\,2}} = -\int \frac{d^3y}{4\pi} \frac{J(\vec{y}\,)}{|\vec{x}-\vec{y}\,|} \,. \tag{1.10}$$

Time-dependent disturbances created by a time-dependent source $J(\vec{x}, t)$ are obtained from the static ones by simply recalling that these disturbances travel at speed c_s. Hence the solution is formally the same as (1.10) with the source replaced by its value at the 'retarded time' $t - |\vec{x} - \vec{y}\,|/c_s$,

$$\phi(\vec{x}, t) = \phi(\vec{x}, t)_{\text{wave}} - \int \frac{d^3y}{4\pi} \frac{J\left(t - \frac{|\vec{x}-\vec{y}\,|}{c_s}, \vec{y}\,\right)}{|\vec{x}-\vec{y}\,|} \,, \tag{1.11}$$

where ϕ_{wave} is a general solution of the homogeneous equation (1.9). We can use the same argument to write a formal solution of the general non-linear equation with arbitrary potential:

$$\phi(\vec{x}, t) = \phi_{\text{wave}} + \frac{1}{\partial^2} V'(\phi) = \phi_{\text{wave}} - \int \frac{d^3y}{4\pi} \frac{V'\left[\phi\left(t - \frac{|\vec{x}-\vec{y}\,|}{c_s}, \vec{y}\,\right)\right]}{|\vec{x}-\vec{y}\,|} \,. \tag{1.12}$$

Rather than an explicit solution, this is an integral equation for any potential with quadratic terms or higher non-linearities. This equation may be solved iteratively in the powers of the function $V'[\phi]$. For example, for a mass term we have $V'(\phi) = m^2\phi$. The term of order m^{2n} in the iterative solution represents a wave sourced by J and averaged over $2n$ 'kicks' at which the wave is regenerated. This construction is analogous to Huygens' method of wave dispersion, so that one says that the mass term causes 'wave dispersion', as a result of which the effective propagation velocity over distances larger than c_s/m is *smaller* than c_s.

1.1.1 Symmetries

A classical system is said to enjoy a symmetry action when some group of transformations G acts on the space of solutions of the equations of motion, i.e. given one solution $\mathcal{Q}$, the transformed $g(\mathcal{Q})$ under G is also a solution of the equations of motion. One way to ensure this is to demand that the action functional be invariant under the action of G:

$$S[\gamma] = S[g(\gamma)] \,, \tag{1.13}$$

for any trajectory γ and any group element $g \in G$. This is slightly more than strictly necessary in the classical realm, because one could do with an action of G just on the space of extrema of S, rather than the whole domain of definition of S. However, in quantum mechanics one really explores all possible trajectories and symmetries have to respect them all.

Invariance of the action is not always born out by invariance of the Lagrangian. Because of the ambiguity in Lagrangians by a total derivative, it is enough to demand that a Lagrangian is invariant up to a total time derivative,

$$L(g(\mathcal{Q})) = L(\mathcal{Q}) + \frac{df_g}{dt} \,. \tag{1.14}$$

For field theories, and under the technical assumption that boundary terms integrate to zero, the Lagrangian density may be invariant up to a total derivative in time and space:

$$\mathcal{L}[\,\phi\,] \to \mathcal{L}[\,g(\phi)\,] = \mathcal{L}[\,\phi\,] + \partial_t f^t + \vec{\partial}\vec{f} \,, \tag{1.15}$$

for some functions f^t and $\vec{f}$.

The main use of Lagrangian methods is to implement economically the requirements of symmetry. At the same time, this formalism provides a very general link between symmetries and conservation laws, via a theorem by Noether.

Noether's theorem

Let G act on the space of trajectories $\mathcal{Q}(t)$ as a continuous group and work near the identity,

$$g(\mathcal{Q}) = \mathcal{Q} + \delta_\epsilon \mathcal{Q} = \mathcal{Q} + \epsilon \xi(\mathcal{Q}) + O(\epsilon^2) \; . \tag{1.16}$$

Since Lagrangians are defined up to a total time derivative, an invariant action is compatible with a variation of the Lagrangian

$$\delta_\epsilon L = \epsilon \frac{df_\xi}{dt} \; , \tag{1.17}$$

for some f_ξ. Rewriting this in differential form,

$$\delta_\epsilon L = \frac{\partial L}{\partial \mathcal{Q}} \delta_\epsilon \mathcal{Q} + \frac{\partial L}{\partial \mathcal{Q}_t} \delta_\epsilon \mathcal{Q}_t = \left(\frac{\partial L}{\partial \mathcal{Q}} - \frac{d}{dt} \frac{\partial L}{\partial \mathcal{Q}_t} \right) \delta_\epsilon \mathcal{Q} + \frac{d}{dt} \left(\frac{\partial L}{\partial \mathcal{Q}_t} \delta_\epsilon \mathcal{Q} \right) \; . \tag{1.18}$$

Using the equations of motion we find the conservation of the Noether charge Q_ξ,

$$\frac{dQ_\xi}{dt} = 0 \; , \qquad Q_\xi = \xi(\mathcal{Q}) \frac{\partial L}{\partial \mathcal{Q}_t} - f_\xi \; . \tag{1.19}$$

Some examples of importance include the usual definitions of energy and momentum for a free Newtonian particle with Lagrangian $L = \frac{1}{2} m \vec{v}^2$, associated to translation symmetry in time and space, respectively

$$\vec{p} = \frac{\partial L}{\partial \vec{v}} = m\vec{v} \qquad E = \vec{v} \cdot \frac{\partial L}{\partial \vec{v}} - L = \tfrac{1}{2} m \vec{v}^2 = \frac{\vec{p}^2}{2m} \; . \tag{1.20}$$

For field theories with Lagrangians of functional form $\mathcal{L}(\partial_t \phi, \vec{\partial} \phi, \phi)$, the statement of a continuous symmetry is that $\delta_\epsilon \mathcal{L} = \epsilon \, \partial_t f^t_\xi + \epsilon \, \vec{\partial} \vec{f}_\xi$ under a field transformation $\delta_\epsilon \phi = \epsilon \, \xi(\phi)$. Repeating the previous derivation (1.18), neglecting total spatial derivatives, one finds a local 'continuity equation'

$$\partial_t J^t_\xi + \vec{\partial} \vec{J}_\xi = 0 \tag{1.21}$$

for the Noether current defined by

$$J^t_\xi = \xi(\phi) \cdot \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} - f^t_\xi \; , \qquad \vec{J}_\xi = \xi(\phi) \cdot \frac{\partial \mathcal{L}}{\partial(\vec{\partial} \phi)} - \vec{f}_\xi \; . \tag{1.22}$$

To interpret (1.21) as a conservation equation we define the Noether charge inside a three-dimensional region V_3,

$$Q_\xi(V_3) = \int_{V_3} J^t_\xi \; , \tag{1.23}$$

and the flux across the boundary ∂V_3

$$\Phi_\xi(\partial V_3) = \oint_{\partial V_3} (J_\xi)_n \; , \tag{1.24}$$

where $(J_\xi)_n$ denotes the component of $\vec{J}_\xi$ along the outward pointing normal to ∂V_3. Then (1.21) is equivalent to

$$\frac{d}{dt} Q_\xi(V_3) = -\Phi_\xi(\partial V_3) \ , \tag{1.25}$$

i.e. the variation of charge equals the flux through the boundary ∂V_3, without any creation or destruction of charge inside V_3.

1.2 Crisis 'fin de siecle'

Circa 1900 one could summarize the Lagrangian of the world in terms of two components

$$S_{1900} \sim S_{\rm Newton} + S_{\rm Maxwell-Lorentz} \;, \tag{1.26}$$

where $S_{\rm Newton}$ stands for the model of pointlike masses interacting by instantaneous gravitational forces. The Maxwell–Lorentz sector concerns the electromagnetic phenomena and their coupling to charged massive particles. A mechanical interpretation was missing in this sector, although drawing inspiration from the theory of optics, it was natural to postulate a material medium, called *ether*, that would support the electromagnetic waves. This prompted an active search for mechanical models of the 'ethereal' substance. The continued failure of such models lead to a conceptual crisis at the turn of the century.

1.2.1 Newton

A model of matter in terms of pointlike objects characterized by positions $\vec{x}_p$ and velocities $\vec{v}_p = d\vec{x}_p/dt$, interacting by a Newtonian gravitational force is described by the Lagrangian

$$L_{\rm Newton} = \sum_p \tfrac{1}{2}\, m_p\, \vec{v}_p^{\,2} + G \sum_{p<q} \frac{\mu_p \mu_q}{|\vec{x}_p - \vec{x}_q|} \;. \tag{1.27}$$

The most salient features of this Lagrangian are the instantaneous nature of gravitational forces (the potential energy only depends on positions of particles) and the equality of inertial and gravitational masses, $\mu_p = m_p$, the so-called "equivalence principle". It also satisfies the Galilean principle of relativity, being invariant under a ten-parameter group of transformations

$$x_i \rightarrow \sum_j R_i^{\;j}\, (x_j + a_j + V_j t) \;, \qquad t \rightarrow t + c \;, \tag{1.28}$$

including boosts by a velocity $\vec{V}$ as well as $\mathbf{R}^3$-rotations and translations in space and time. [3]

The equality of gravitational and inertial masses is famously exemplified by the (surely false) story of Galileo throwing objects from the top of Pisa's tower. In fact, if we denote m the inertial mass and μ the 'gravitational charge', it is enough to ensure that the ratio m/μ is a universal constant, equal for all possible bodies, since then this ratio can be set to unity by a rescaling of Newton's constant.

In the nineteen century, Eötvös was able to check this universality with great accuracy in a series of experiments comparing the balance of a gravitational force (responding to μ) and a purely inertial force (responding to m). A good example is a body at rest on the surface of the Earth, subject to gravitational and centrifugal forces. Tiny differences of the ratio m/μ between two objects would result in a slight misalignment of the net force acting on them. For the latitude of Budapest, home of Eötvös, one finds

$$\Delta_{m/\mu} \approx 600\, |\sin\alpha\,| \;, \tag{1.29}$$

where α is the misalignment angle. Eötvös was able to bound $\Delta_{m/\mu}$ to less than one part in 10^7. The bound was later improved by Dicke in the 60's to one part in 10^{11} and the satellite STEP

[3]**Exercise:** Prove that the Newtonian Lagrangian (1.27) transforms under (1.28) into itself plus a total time derivative.

is expected to improve it to 10^{17}. The conclusion is, therefore, that such a precision test should not be based upon a simple coincidence, but some deep symmetry principle. The interpretation of the equivalence principle in terms of symmetry will be pivotal in the theory of gravitation.

Problem: Eötvös experiment

Verify expression (1.29) by computing the misalignment of the total gravitational+centrifugal force for two objects of gravitational masses μ_1, μ_2 and inertial masses m_1, m_2. Give an expression for $|\sin\alpha|$ in terms of the surface gravity $g_\oplus = GM_\oplus/R_\oplus^2$, the Earth radius $R_\oplus$ and the Earth rotation angular velocity $\Omega_\oplus$.

Anticipating its use in the following, we note that Newton's gravitational law can be formally written as a field theory coupled to the mass density of matter. The equations of motion following from (1.27) are

$$m_p \frac{d\vec{v}_p}{dt} = \vec{F}_{\rm N} = -G m_p \sum_{q \neq p} \frac{m_q(\vec{x}_p - \vec{x}_q)}{|\vec{x}_p - \vec{x}_q|^3} \ , \tag{1.30}$$

where we have already implemented the equality of gravitational and inertial masses, so that m_p drops from this equation. The force term on the right hand side can be written in terms of a gravitational potential, $\vec{F}_{\rm N} = -m_p \vec{\partial} \phi_{\rm N}$, where

$$\phi_{\rm N}(t, \vec{x}) = -\sum_q \frac{G m_q}{|\vec{x} - \vec{x}_q(t)|} \ . \tag{1.31}$$

This form of the potential, defined up to a additive constant, is the solution of the so-called Poisson equation sourced by the mass density

$$\vec{\partial}^{\,2} \phi_{\rm N}(\vec{x}, t) = 4\pi G \sum_q m_q \, \delta^{(3)}(\vec{x} - \vec{x}_q(t)) \equiv 4\pi G \rho_m(\vec{x}, t) \ . \tag{1.32}$$

We see that Newtonian theory can be recast in the form of a field theory with a gravitational potential $\phi_{\rm N}$ interacting with mass density through the equations

$$\frac{d\vec{v}_p}{dt} = -\vec{\partial}\, \phi_{\rm N} \ , \qquad \vec{\partial}^{\,2}\, \phi_{\rm N} = 4\pi G \, \rho_m \ , \tag{1.33}$$

which are equivalent to the Lagrangian

$$L_{\rm Newton} = \sum_p \tfrac{1}{2}\, m_p \vec{v}_p^{\,2} - \frac{1}{8\pi G} \int d^3x (\vec{\partial} \phi_{\rm N})^2 - \int d^3x \, \rho_m \phi_{\rm N} \ . \tag{1.34}$$

Solving for the potential $\phi_{\rm N}$ using Poisson's equation yields precisely (1.31), and substituting back into (1.34) produces the 'action-at-a-distance' form of the Newtonian system (1.27). Notice that the absence of time derivatives from Poisson's equation is directly related to the instantaneous nature of the interactions in (1.27). In principle, a finite speed of propagation for the gravitational interaction can be introduced into this formalism by a simple replacement of Poisson's equation by a more conventional wave equation such as

$$\left(-\frac{1}{c_{\rm G}^2} \partial_t^2 + \vec{\partial}^{\,2} \right) \phi_{\rm N} = 4\pi G \, \rho_m \ ,$$

the standard Newtonian theory thus arising in the limit $c_G \to \infty$.

1.2.2 Maxwell–Lorentz

Electrostatic phenomena can be described by a system of equations entirely analogous to (1.33) and (1.34), with the replacement of the Newtonian potential by the Coulomb potential produced by an electric charge Q_e,

$$\phi_{\mathrm{C}}(\vec{x},t) = \frac{Q_e}{4\pi|\vec{x}\,|}\ .$$

Magnetic phenomena may be incorporated by the introduction of velocity-dependent forces. However, it was found experimentally by Faraday and others that electrodynamics was very efficiently represented by the notion of 'electromagnetic fields' interacting with charged matter particles.

Maxwell found the set of local equations that explained all electromagnetic phenomena in terms of electric $\vec{E}$ and magnetic $\vec{B}$ fields in the presence of charges and currents,

$$\begin{aligned} \vec{\partial}\vec{E} &= \rho_e, \qquad c\,\vec{\partial}\times\vec{B} - \partial_t\vec{E} = \vec{J}_e\ , \\ \vec{\partial}\vec{B} &= 0\ , \qquad c\,\vec{\partial}\times\vec{E} + \partial_t\vec{B} = 0\ . \end{aligned} \tag{1.35}$$

The charge density and current are defined by

$$\rho_e(\vec{x},t) = \sum_p e_p\,\delta^{(3)}(\vec{x}-\vec{x}_p(t))\ , \qquad \vec{J}_e(\vec{x},t) = \sum_p e_p\,\vec{v}_p(t)\,\delta^{(3)}(\vec{x}-\vec{x}_p(t)) \tag{1.36}$$

and satisfy the conservation law

$$\partial_t\rho_e + \vec{\partial}\vec{J}_e = 0\ . \tag{1.37}$$

The coupling constant, c, can be given a physical interpretation by solving the second pair of equations in terms of potentials,

$$\vec{B} = \vec{\partial}\times\vec{A}\ , \qquad \vec{E} = -\frac{1}{c}\,\partial_t\vec{A} - \vec{\partial}\phi\ , \tag{1.38}$$

defined up to the *gauge* ambiguity,

$$\phi \to \phi - \frac{1}{c}\partial_t f\ , \qquad \vec{A} \to \vec{A} + \vec{\partial}f\ , \tag{1.39}$$

with $f(t,\vec{x})$ an arbitrary smooth function. Fixing this ambiguity (partially) by imposing the Lorenz condition [4]

$$\frac{1}{c}\,\partial_t\,\phi + \vec{\partial}\,\vec{A} = 0\ , \tag{1.40}$$

the first pair of Maxwell equations can be rewritten as wave equations

$$\begin{aligned} \left(\vec{\partial}^{\,2} - \frac{1}{c^2}\partial_t^{\,2}\right)\phi &= -\rho_e\ , \\ \left(\vec{\partial}^{\,2} - \frac{1}{c^2}\partial_t^{\,2}\right)\vec{A} &= -\frac{\vec{J}_e}{c}\ , \end{aligned} \tag{1.41}$$

[4] Lorenz the *danish* instead of Lorentz the *dutch*.

which reveals the interpretation of c as the speed of electromagnetic waves. These waves also provide an electromagnetic theory of optics, since oscillating solutions of the electric and magnetic fields exist even outside charged matter.

The interaction between electromagnetic fields and charged particles is determined by the Lorentz force law

$$m_p \frac{d\vec{v}_p}{dt} = e_p \left(\vec{E} + \frac{\vec{v}_p}{c} \times \vec{B} \right) . \tag{1.42}$$

which in turn derives from the Lagrangian

$$\sum_p \tfrac{1}{2}\, m_p\, \vec{v}_p^{\,2} - \sum_p e_p \left(\phi(t, \vec{x}_p) - \frac{\vec{v}_p}{c}\, \vec{A}(t, \vec{x}_p) \right) , \tag{1.43}$$

The second term may be rewritten in terms of currents and charge densities to obtain the action principle behind Maxwell's equations (1.35),

$$S_{\text{Maxwell--Lorentz}} = \tfrac{1}{2} \int dt\, d^3x \, \left(\vec{E}^{\,2} - \vec{B}^{\,2} \right) - \frac{1}{c} \int dt\, d^3x \, \left(c\, \rho_e\, \phi - \vec{J}_e \cdot \vec{A} \right) , \tag{1.44}$$

with $\vec{E}$ and $\vec{B}$ implicitly written in terms of the potentials $\phi, \vec{A}$ via eq. (1.38). Notice that the potential formalism is very useful in writing the interactions with charges, and essential in the Lagrangian formalism.

The main problem of principle with this theory is the interpretation of the coupling constant c as a velocity of propagation of electromagnetic waves. Immediately the question arises: velocity with respect to what particular frame? Indeed, a Galilean transformation $\vec{x} \to \vec{x} + \vec{v}\, t$ cannot possibly be a symmetry of the Maxwell equations, because such a transformation would change the velocity of light to $\vec{c} - \vec{v}$. Hence, if we insist in adhering to a mechanical interpretation of electromagnetism (ether), it seems that Maxwell equations are only valid in the particular frame that sits at rest with respect to the ether.

In this situation, it becomes interesting to measure the velocity of the Earth with respect to the ether. The famous experiments of Michelson–Morley failed to reveal any such drift velocity. At the same time, stelar aberration and other optical data disfavored the possibility that the Earth could be dragging the ether along with its motion. So, there was a real problem that was tackled by Lorentz and others, with a number of *ad hoc* hypotheses.

Despite the fact that Lorentz had developed most of the right formulas of special relativity, it is notable that he was unable to interpret them correctly. Poincaré was also close to the solution, as he proposed that the Galilean group of transformations should be replaced by a different group and, in doing so, giving privilege to the electromagnetic part of the total Lagrangian. One would need to replace S_{Newton} with something different that would be compatible with the strange behavior of light. One could say that the Lorentz–Poincaré combination was bound to find the right answer sooner or later... but the first complete solution was given by Einstein in his famous 1905 work.

1.2.3 Einstein

Einstein's starting point was to take for granted the Galilean principle of relativity, i.e. that motion at constant velocity could not have absolute meaning, and at the same time assume as a feature of nature that the velocity of light is a constant, independent of the frame of reference.

Heuristically, one could argue that one simply needs the existence of a maximum velocity of propagation of signals. If one assumes the existence of this limiting velocity, the principle of relativity requires it to be the same in all frames in relative uniform motion, because otherwise one could assign an intrinsic velocity to a frame of reference by quoting the maximal velocity of signals in that frame. From this point of view, one could have special relativity even if light was not propagating right at the limiting velocity. For simplicity, let us assume in any case that c, the light's velocity *in vacuo*, is the maximal one. Then, the principle of relativity says that there must exist a group of transformations that leaves the physics invariant and that preserves the condition of uniform rectilinear motion. Such a transformation,

$$(t,\vec{x}\,) \to (t',\vec{x}\,'\,) = \widetilde{L}\,(t,\vec{x}\,) \tag{1.45}$$

must send straight trajectories $\vec{x}(t)$ into straight trajectories $\vec{x}'\,(t')$, and so it must be a *linear* map. Furthermore, the invariance of light's velocity means that

$$\vec{x}^{\,2} - c^2 t^2 = 0 = \vec{x}^{\,\prime 2} - c^2 t^{\prime 2} \,. \tag{1.46}$$

This equation, together with the linearity property, implies that $\vec{x}^{\,2} - c^2t^2$ is invariant up to a constant factor,

$$\vec{x}^{\,2} - c^2 t^2 \to f(\vec{V}\,)\,(\vec{x}^{\,2} - c^2 t^{\,2})\,, \tag{1.47}$$

where $f(\vec{V}\,)$ can only depend on the relative velocity of the frame. Spatial isotropy further imposes the condition that f must be a function of the modulus $|\vec{V}\,| = V$. Introducing now two boosts with velocities $\vec{V}_1$ and $\vec{V}_2$, and relative velocity $\vec{V}_{12}$, we find, upon performing two iterated transformations,

$$f(V_2) = f(V_1) f(V_{12})\,, \quad \text{equivalently} \quad f(V_{12}) = \frac{f(V_2)}{f(V_1)}\,, \tag{1.48}$$

but V_{12} depends on the relative angle between $\vec{V}_1$ and $\vec{V}_2$, unlike the right hand side of the previous equation. Hence, we conclude that f does not depend on $\vec{V}$ and in fact $f = 1$. We thus define the Lorentz group as those linear transformations that leave the quadratic form

$$c^2 t^2 - \vec{x}^{\,2} = \text{invariant}\,. \tag{1.49}$$

This group is called $O(1,3)$. When supplemented by the spacetime translations it becomes the Poincaré group, a ten-parameter group which contracts to the Galilean group in the limit $V/c \to 0$. Factoring out the subgroup of $SO(3)$ rotations, we can align the boost velocity in a given direction, say x, and we are left with a two-dimensional problem of linear transformations leaving the so-called *interval*

$$c^2 t^2 - x^2 = I = \text{invariant}\,. \tag{1.50}$$

From now on, we choose units so that $c = 1$. Then, $-I = x^2 - t^2$ is the analytic continuation, under $t \to it$, of the standard Euclidean interval $I_E = x^2 + t^2$, invariant under rotations on the (t,x) plane by angle θ, and space or time inversions. The corresponding transformations for the quadratic form at hand can be obtained by rotating $\theta \to i\psi$. Leading to

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh\psi & \sinh\psi \\ \sinh\psi & \cosh\psi \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}\,. \tag{1.51}$$

The relative velocity is obtained by monitoring the transformation of the point $x = 0$, leading to $\tanh\psi = x'/t' = V$. Hence, we have the usual transformations

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & \gamma V \\ \gamma V & \gamma \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}, \tag{1.52}$$

with $\gamma = (1 - V^2)^{-1/2}$.

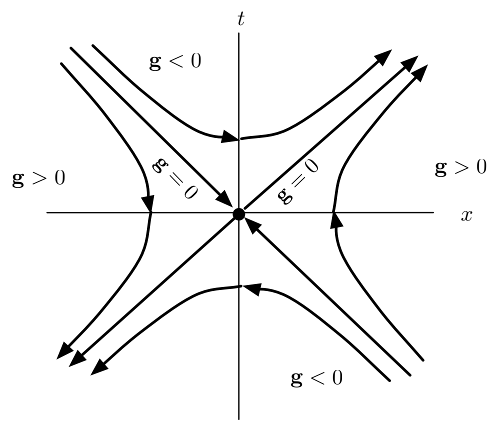

Figure 1.1: *The orbits of the Lorentz group in the (t, x) plane form a set of hyperbolae, defined by $\mathbf{g} = -I = -t^2 + x^2 =$ constant. Orbits in the spacelike region $\mathbf{g} > 0$ intersect the $t = 0$ axis, whereas orbits in the timelike region, $\mathbf{g} < 0$ intersect the $x = 0$ axis and preserve the sign of the times. The hyperbolae degenerate to straight lines in the light-like region, $\mathbf{g} = 0$ given by all the points connected by light rays from the origin.*

The geometrical interpretation of special relativity was given by Minkowski in 1908. We can think of $\mathbf{R}^{1+3} \equiv \mathbf{R} \times \mathbf{R}^3$ as the space-time, where the first factor is 'time' and the second factor is 'space'. We have the set of 'events' $(t, \vec{x}\,) \in \mathbf{R}^{1+3}$, equipped with a metric

$$\mathbf{g}(\Delta t, \Delta\vec{x}) = \Delta\vec{x}^{\,2} - \Delta t^{\,2} \tag{1.53}$$

between pairs of events.[5] This generalizes the standard Euclidean metric of $\mathbf{R}^3$, which computes the rotationally invariant length-squared of the vector $\Delta\vec{x}$, by the addition of the term $-\Delta t^2$ which makes the Minkowski metric non-positive definite. Lorentz transformations introduce motions on $\mathbf{R}^{1+3}$ that preserve this metric, inducing a causal structure. To explain the physical meaning of the metric, we notice that any point in $\mathbf{R}^{1+3}$ must fall into one of three Lorentz-invariant sets, depending on the sign of $\mathbf{g}$.

If $\mathbf{g} < 0$, there exists a frame in which $\Delta\vec{x}^{\,2} = 0$ and the events 'occur' at the same point in space. In this case we say that the interval between such events is 'timelike' because in this rest frame, the interval can be written as $\mathbf{g} = -\Delta\tau^2$, where τ is the time as measured by a clock in that rest frame, called 'proper time'.

On the other hand, for $\mathbf{g} > 0$ there is a frame where $\Delta t^2 = 0$, and the events can be considered as 'simultaneous'. In this situation we speak of 'spacelike' separated events. In the special frame of simultaneity the metric equals the standard Euclidean metric $\mathbf{g} = \Delta\vec{x}^{\,2}$.

[5] A common notation for infinitesimal intervals is $\mathbf{g}(dt, d\vec{x}\,) \equiv ds^2 = -dt^2 + d\vec{x}^{\,2}$.

Finally, the events with $\mathbf{g} = 0$ are said to be 'lightlike' separated and correspond to those that can be connected by light rays. Generally speaking, the physical content of the metric is that $\sqrt{\pm\mathbf{g}}$ can be interpreted as either a standard time lapse or a standard spatial distance.

Problem: Twin paradox

Given two timelike separated events, A and B, prove that any traveler making the journey from A to B will maximize her/his travel time by going at constant speed. Show that the subjective travel time can be made arbitrarily short if we tolerate arbitrary accelerations (Langevin's twin paradox).

It turns out that the sign of Δt is Lorentz invariant for timelike intervals, making it possible to establish causal relationships between such events. In particular, the velocity of any inertial motion between timelike events, $\vec{v} = \Delta\vec{x}/\Delta t$, is subluminal, $|\vec{v}\,| < 1$. On the other hand, the ordering of times for spacelike events is *not* Lorentz invariant, so that in that case we cannot agree to a Lorentz-invariant notion of causal influence. Effective velocities in that case would be superluminal. Hence, we conclude that only sub-luminal velocities can correspond to causal processes.

Free particle dynamics

With the simple elements introduced so far we are ready to study the relativistic dynamics of free particles. Let us consider a causal trajectory of a particle of mass m, specified as a function $\vec{x}(t)$ in such a way that the velocity $\vec{v} = d\vec{x}/dt$ is sub-luminal at all times, $|\vec{v}\,| < 1$. Following the general rules of Lagrangian dynamics, we define an action

$$S_{\rm P} = \int dt\, L_{\rm P}(\vec{x}(t), \vec{v}(t))$$

with the sole requirement that it gives the correct equations of motion for a free particle, i.e. $\vec{v} =$ constant, and that it be Lorentz invariant. To find the right action, let us imagine that we jump on the particle itself, parametrizing the trajectory in the rest frame of the particle. The time variable on this frame is, by definition, the proper time τ, and the state of the particle is that of 'rest' at all proper times. Using the general policy of defining the Lagrangian as 'kinetic energy minus potential energy' from elementary mechanics, we can write

$$S_{\rm P} = -C \int d\tau \;, \tag{1.54}$$

where C is a constant specifying the 'potential energy' of the particle at rest. From the definition of proper time we have $d\tau^2 = dt^2 - d\vec{x}^{\,2}$, where $(t, \vec{x})$ are coordinates in another arbitrary Lorentz frame, and we learn that (1.54) is invariant under Lorentz transformations. Written in an arbitrary frame we thus have

$$S_{\rm P} = -C \int dt\, \sqrt{1 - \vec{v}^{\,2}} \;, \tag{1.55}$$

and the value of the constant C can be obtained by expanding the Lagrangian at low velocities and matching to the Newtonian form: $-C\sqrt{1-\vec{v}^{\,2}} \approx -C + \frac{1}{2}\,C\,\vec{v}^{\,2} \approx -C + \frac{1}{2}\,m\,\vec{v}^{\,2}$, which determines $C = m$ and the final form

$$S_{\rm P} = -m \int d\tau = -m \int dt \sqrt{1 - \vec{v}^{\,2}} \ , \tag{1.56}$$

of the free particle action in Special Relativity. From here, we get the free equation of motion, $d\vec{v}/dt = 0$ and the relativistic versions of the conserved quantities in the motion, the momentum and the energy

$$\vec{p} = \frac{\partial L_{\rm P}}{\partial \vec{v}} = m\,\vec{v}\,\gamma \ , \qquad E = \vec{v} \cdot \vec{p} - L_{\rm P} = m\,\gamma = \sqrt{\vec{p}^{\,2} + m^2} \ , \tag{1.57}$$

where $\gamma = (1 - \vec{v}^{\,2})^{-1/2}$. Notice that Lorentz invariance of the dynamics leads naturally to the notion of a potential energy at zero velocity, given by the rest mass, or the famous $E = mc^2$ if the conventional units were restored. Writing the last of (1.57) equations in the form $E^2 - \vec{p}^{\,2} = m^2$, we notice that the four-vector $(E, \vec{p})$ behaves under Lorentz transformations just as the coordinates $(t, \vec{x}\,)$.

1.3 Relativistic dynamics

The procedure outlined in the previous section can be repeated for more complicated systems. In postulating dynamical principles based on Lorentz symmetry, we must always start by guessing a Lorentz-invariant action. Further constraints are brought by the principles of locality and the matching to known limits, such as the Newtonian approximation.

1.3.1 The Lorentz group

In order to proceed with this program, it is useful to introduce some powerful notation to help in the search of Lorentz-invariant combinations of physical quantities.

The Lorentz group $O(1,3)$ was defined as the set of linear maps of $\mathbf{R}^{1+3}$ onto itself with the condition of keeping the Minkowski interval invariant. Working in differential notation

$$ds^2 = -d\tau^2 = -dt^2 + d\vec{x}^{\,2} \tag{1.58}$$

is left invariant. In the notation $(t, \vec{x}\,) = (x^0, \vec{x}\,) = (x^a)$, with $a = 0, 1, 2, 3$, we have

$$x^a \to x'^{\,a} = \sum_b L^a{}_b\, x^b \equiv L^a{}_b\, x^b \;, \tag{1.59}$$

where we have used Einstein's convention of implicit summation of repeated indices in top-down pairs. [6] Then, the Minkowski metric η_{ab},

$$ds^2 = \eta_{ab}\, dx^a\, dx^b \;, \qquad (\eta_{ab}) = \mathrm{diag}\,(-1,1,1,1) \tag{1.60}$$

is left invariant by the Lorentz matrices,

$$\eta_{ab}\, L^a{}_c\, L^b{}_d = \eta_{cd} \tag{1.61}$$

or, in matrix notation $\eta = L^t\, \eta\, L$. Inverting this relation and noting that $\eta = \eta^{-1}$ we have $\eta = L^{-1}\eta(L^{-1})^t$ or, in index notation

$$\eta_{ab}\, (L^{-1})_c{}^a\, (L^{-1})_d{}^b = \eta_{cd} \;. \tag{1.62}$$

Setting $a = b = 0$ in (1.61) we have

$$-1 = \eta_{00} = L^a{}_0\, L^b{}_0\, \eta_{ab} = -(L^0{}_0)^2 + \sum_i (L^i{}_0)^2 \;, \tag{1.63}$$

so that $(L^0{}_0)^2 \geq 1$ and one cannot change continuously the sign of $L^0{}_0$. Similarly, from $\det\eta = \det\eta\,(\det L)^2$ we find that $\det L = \pm 1$ and we cannot smoothly change the sign of the determinant either. This divides the set of L matrices into four connected components according to the signs of $L^0{}_0$ and $\det L$. Proper Lorentz transformations have both positive and are continuously connected to the identity. We can reach the other components by the action of $\mathcal{P}$ and $\mathcal{T}$ on the proper transformations, where $\mathcal{P}$ stands for parity, $(t, \vec{x}) \to (t, -\vec{x})$, and $\mathcal{T}$ is time reversal, $(t, \vec{x}) \to (-t, \vec{x})$.

[6] In the following, we will use Einstein's summation convention for any set of four-dimensional indices, keeping the explicit sum notation for vector-like three-dimensional indices.

Upon analytic continuation $(t, x) \to (it, x)$ the Lorentzian metric becomes Euclidean $ds^2 = d(it)^2 + d\vec{x}^{\,2}$. The resulting connected group is $SO(4)$, which is isomorphic to $SU(2)_L \times SU(2)_R$. The operation of analytic continuation, being continuous, should not change the discrete structure of group representations. This means that all representations of the Lorentz group can be constructed as the pairs (j_L, j_R), with $j_{L,R}$ the standard spins of $SU(2)$ irreducible representations. This form is very useful to describe spinor degrees of freedom, although most of the applications in classical physics can be exhausted by the so-called *tensor* representations, which appear in the tensor products of vector representations.

We thus conclude that a relativistic model is given by a Lorentz-invariant Lagrangian defined over Minkowski spacetime, i.e. the pair $(\mathbf{R}^{3+1}, \eta)$, with dynamical variables in some representations of $O(3,1)$.

Tensor representations

The defining (vector) representation of the Lorentz group corresponds to any quantity that transforms as the coordinates:

$$U^a \to L^a{}_b \, U^b \; . \tag{1.64}$$

Any such quantity is called a 'contravariant' vector. Given two contravariant vectors, U, V, we can form a two-index object $T^{ab} = U^a V^b$ by 'tensor' product. It transforms as

$$T^{ab} \to L^a{}_c L^b{}_c \, T^{cd} \; .$$

Other combinations with the same transformation law are linear combinations of tensor products, $U^a V^b + X^a Y^b + \ldots$. Hence, any set of quantities transforming as above may be defined as a second-rank contravariant tensor, and the generalization to arbitrary rank is obvious.

A different transformation law is followed by the differential operators

$$\partial_a \equiv \frac{\partial}{\partial x^a} \; .$$

If we view $x'^{\,a}$ as functions of x^a, we have

$$\frac{\partial x'^{\,a}}{\partial x^b} = L^a{}_b \; , \qquad \frac{\partial x^a}{\partial x'^{\,b}} = (L^{-1})_b{}^a \; . \tag{1.65}$$

Using the chain rule,

$$\partial'{}_a = \frac{\partial x^b}{\partial x'^{\,a}} \, \partial_b = (L^{-1})_a{}^b \, \partial_b \; . \tag{1.66}$$

Such transformation law, with the inverse Lorentz matrix L^{-1}, is conventionally called 'covariant'. General covariant tensors can be defined following the previous steps of their contravariant cousins, by tensor product and addition. For example, a rank-two covariant tensor transforms as

$$A_{ab} \to (L^{-1})_a{}^c \, (L^{-1})_b{}^d \, A_{cd} \tag{1.67}$$

and so on. Mixed tensors with p contravariant and q covariant indices can be constructed in straightforward fashion. They can be given an intrinsic definition by regarding the collections of indexed numbers as components in an abstract basis formed by the contravariant differentials dx^a and the covariant derivative operators ∂_b, i.e. we require the formal expression

$$T = \sum_{a_1, \ldots, b_1, \ldots} T^{a_1 \ldots a_p}{}_{b_1 \ldots b_q} \, dx^{b_1} \otimes \cdots \otimes dx^{b_q} \otimes \partial_{a_1} \otimes \cdots \otimes \partial_{a_p} \tag{1.68}$$

to be invariant under a change of basis

$$dx^a \to L^a{}_b\, dx^b\ , \qquad \partial_a \to (L^{-1})_a{}^b\, \partial_b\ . \tag{1.69}$$

The transformation laws of the components follow suit.

This construction also shows that any pair of indices that are identified and summed up with Einstein's convention become irrelevant as far as the transformation rules is concerned. That is, the 'contracted' product $U^a V_a$ of a contravariant and a covariant vector is a Lorentz invariant, whereas the contraction $V_a T^{ab}$ transforms as a contravariant vector, according to the free index b. We can also construct invariant differential operators, such as

$$\partial^2 \equiv \partial_a \partial^a = \eta^{ab} \partial_a \partial_b = -\partial_t^2 + \vec{\partial}^{\,2}\ , \tag{1.70}$$

usually called Laplacian or d'Alembertian.

Invariant tensors

The task of constructing Lorentz invariants is greatly aided by the definition of *invariant* tensors. These are tensors that retain their form under Lorentz transformations. The most obvious case is the Kronecker delta δ^a_b with upper-and-lower indices. It transforms as an invariant mixed tensor of rank two:

$$\delta^a_b = L^a{}_c\, (L^{-1})_b{}^d\, \delta^c_d\ . \tag{1.71}$$

Another important invariant tensor is the metric itself. According to (1.62) we have

$$\eta_{ab} (L^{-1})_c{}^a\, (L^{-1})_d{}^b = \eta_{cd}\ , \tag{1.72}$$

as corresponds to the transformation law of a rank-two covariant tensor. It is convenient to define the inverse metric (equal to itself) with upper indices

$$(\eta^{ab}) = \mathrm{diag}\,(-1, 1, 1, 1)\ . \tag{1.73}$$

Hence, $\eta^{ab}\eta_{bc} = \delta^a_c$, and given the tensor character of η_{ab} and δ^a_b, it follows that η^{ab} is an invariant contravariant tensor of rank two.

The invariant tensors constructed from the metric are especially useful because they can be used to construct covariant tensors out of contravariant tensors and viceversa. By the rules already stated it is very easy to check that, given a contravariant tensor, the contraction

$$\eta_{ab}\, T^{bc\cdots} = T_a{}^{c\cdots} \tag{1.74}$$

transforms as a mixed tensor with the indices as indicated. We say that the metric can be used to 'lower' an index. Analogously, we can use the upper-index version to 'raise' covariant indices into contravariant ones.

A final interesting object is the completely antisymmetric Levi–Civita symbol. We define it as the signature of the permutation $\sigma : (0, 1, 2, 3) \to (a = \sigma(0),\, b = \sigma(1),\, c = \sigma(2),\, d = \sigma(3))$, i.e.

$$\epsilon_{0123} = 1\ , \qquad \epsilon_{\sigma(0)\sigma(1)\sigma(2)\sigma(3)} = \mathrm{sign}\,(\sigma)\ . \tag{1.75}$$

The identity

$$\epsilon_{efgh}\, L_a{}^e L_b{}^f L_c{}^g L_d{}^h = \det(L)\, \epsilon_{abcd} \tag{1.76}$$

implies that the Levi–Civita symbol is a true invariant tensor for proper Lorentz transformations, and flips its sign under parity or time reversal. We can also define associated objects by raising indices with the inverse metric η^{ab}. In particular, the completely contravariant version is also completely antisymmetric with

$$\epsilon^{0123} = \eta^{0a}\eta^{1b}\eta^{2c}\eta^{3d}\,\epsilon_{abcd} = \det\left(\eta^{-1}\right)\epsilon_{0123} = -1\ . \tag{1.77}$$

A useful formula for contractions of arbitrary d-dimensional Levi–Civita tensors is

$$\epsilon^{c_1\cdots c_n a_{n+1}\cdots a_d}\,\epsilon_{c_1\cdots c_n b_{n+1}\cdots b_d} = (-1)^{\rm sig}\, n!\,\det\left(\delta^{a_j}{}_{b_k}\right) = (-1)^{\rm sig}\, n!(d-n)!\,\delta^{a_{n+1}}_{[b_{n+1}}\cdots\delta^{a_d}_{b_d]}\ , \tag{1.78}$$

where $(-1)^{\rm sig}$ is the signature, $+1$ for Euclidean metrics and -1 for Lorentzian ones, and the square brackets stand for the complete anti-symmetrization of indices.

Spinors

In known examples in nature, spinor representations are associated to fermionic fields. These are fundamentally more 'quantum' that their bosonic counterparts. By the spin-statistics theorem of quantum field theory, we know that the bosonic fields come in representations of integer Lorentz spin, i.e of tensor type. For this reason we will not dwell on spinor representations in these notes, except for a quick review of standard notation.

The simplest spinor representation is carried by the so-called Dirac spinor, ψ, which transforms in the representation $(1/2,0)\oplus(0,1/2)$, of complex dimension four. To construct the Lorentz action in this representation, one usually starts from the four-dimensional Dirac matrices γ^a, satisfying the Clifford algebra,

$$\{\gamma^a,\gamma^b\} = -2\eta^{ab}\ ,$$

so that

$$\sigma^{ab} = \frac{i}{4}[\gamma^a,\gamma^b]$$

generate the $SO(1,3)$ Lie algebra. Then, the spinor representation is given by $\psi \to \mathcal{D}(L)\,\psi$, where $\mathcal{D}(L)$ is the four-dimensional matrix

$$\mathcal{D}(L) = \exp\left(\frac{i}{2}\sum_{a,b}\theta_{ab}\sigma^{ab}\right)\ ,$$

and the antisymmetric matrix of parameters θ_{ab} is defined in terms of the Lorentz transformation matrix L, by the relation $L=\exp(\theta)$. The irreducible components are reached acting with the projectors $P_\pm = \frac{1}{2}(1\pm\gamma_5)$, with $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. The resulting two-component spinors $\psi_\pm = P_\pm\psi$ are called left- and right-handed Weyl spinors. The two projectors are mapped into one another by parity. Hence, insisting on parity as a good symmetry forces the use of the four-component Dirac spinors.

1.3.2 Relativistic particles revisited

We can now recast the particle dynamics in a more covariant language. Let $x^a(\tau)$ be the trajectory of a particle in proper-time parametrization. The four-velocity defined as

$$u^a = \frac{dx^a}{d\tau}$$

is a contravariant four-vector under the Lorentz group, and the free equation of motion may be written as

$$\frac{d^2x^a}{d\tau^2} = \frac{du^a}{d\tau} = 0 \ .$$

Conserved energy and momentum can also be assembled into a contravariant four-vector, called the four-momentum $p^a = mu^a$, where $(p^0, \vec{p}) = (E, \vec{p})$. The so-called 'dispersion' or 'mass-shell' relation $E^2 - \vec{p}^{\,2} = m^2$ may be written in condensed form as $p^2 + m^2 = 0$. [7] Equivalently, the four-velocity is constrained to satisfy $u^2 = u_a u^a = -1$.

The free particle action over a trajectory γ can be written in the following variety of equivalent forms:

$$S_{\rm P} = -m \int_\gamma d\tau = -m \int_\gamma dt \frac{d\tau}{dt} = -m \int_\gamma dt \sqrt{1 - \vec{v}^{\,2}} = -m \int_\gamma dt \sqrt{-\eta_{ab} \frac{dx^a}{dt}\frac{dx^b}{dt}} \ . \qquad (1.79)$$

In particular, the second form shows that the coordinate time t is just an integration parameter which might be redefined at will. In many situations, the freedom to reparametrize the time variable does offer some rewards. So we consider the action in arbitrary parametrization

$$S_{\rm P} = -m \int d\tau = -m \int d\sigma \, \frac{d\tau}{d\sigma} = -m \int d\sigma \sqrt{-\eta_{ab} \frac{dx^a}{d\sigma}\frac{dx^b}{d\sigma}} \ . \qquad (1.80)$$

There is a useful trick that simplifies the action by linearizing it and, at the same time, permitting a smooth massless limit, a convenient fact for the discussion of light propagation.

Let us consider the action

$$S_{\rm P}[x^a, e] = \tfrac{1}{2} \int d\sigma \left(\frac{1}{e(\sigma)} \eta_{ab} \frac{dx^a}{d\sigma}\frac{dx^b}{d\sigma} - m^2 \, e(\sigma) \right) \ , \qquad (1.81)$$

where the function $e(\sigma)$ is considered as an independent variable. Its equation of motion $\delta S_P / \delta e = 0$ leads to

$$\frac{dx^a}{d\sigma}\frac{dx_a}{d\sigma} = -m^2 \, e(\sigma)^2 \ . \qquad (1.82)$$

Form this equation we can solve for $e(\sigma)$ and substitute back into (1.81), obtaining the original action (1.80). In this way, we show that both actions are physically equivalent.

The action (1.81) is invariant under reparametrizations of the path (just as (1.80) was) provided we let $e(\sigma)$ transform in such a way that $e(\sigma)d\sigma$ is left invariant. Worldline parameters for which $e(\sigma) =$ constant are called *affine*. Choosing $e(\sigma) = 1$ and taking the limit $m \to 0$ produces the equations of motion of a massless particle,

$$\frac{d^2x^a}{d\sigma^2} = 0 \ , \qquad \frac{dx^a}{d\sigma}\frac{dx_a}{d\sigma} = 0 \ .$$

The first equation says that the trajectory is 'straight' in the σ parametrization, and the second equation says that it lies on the light cone.

[7] Experimental measurements of dispersion relations, $p^2 + m^2 = 0$, provide the most stringent tests of Lorentz symmetry, down to the 10^{-20} level.

For massive particles, it is convenient to chose the proper time as the path parameter, $\sigma = \tau$, so that (1.82) sets $e(\tau) = 1/m$. This means that we can simply use the linearized version of the action

$$S_{\rm P} = \tfrac{1}{2}\, m \int d\tau \, \left(\eta_{ab} \frac{dx^a}{d\tau}\frac{dx^b}{d\tau} - 1 \right) \;, \tag{1.83}$$

which is actually the most *naive* generalization of the Newtonian action. The resulting equations of motion, $d^2x^a/d\tau^2 = 0$, must be supplemented by the field equation of $e(\tau)$, which was lost when setting it to a constant in the action. This equation of motion is simply the mass-shell condition $u^2 = -1$, or equivalently $p^2 + m^2 = 0$. The energy of the particle in a 'laboratory' frame is $E = p^0 = -\eta_{00}\, p^0$, so that the Lorentz-invariant expression for the energy measured by an observer at four-velocity u^a is

$$E_u(p) = -\eta_{ab}\, p^a u^b = -p_a u^a \;. \tag{1.84}$$

Relativistic forces on particles

Introducing interactions in a relativistic theory of particles is a rather non-trivial issue, given the fact that a limit to the speed of information transfer is built into the very fabric of special relativity. This means that standard generalizations of many-particle interaction Lagrangians from Newtonian mechanics are bound to be problematic, since specifying potentials involving particles located at different points in spacetime is a sort of 'action at a distance'.

There are two basic solutions to this problem. The first solution is to take a phenomenological attitude and consider *contact* interactions in spacetime. In this view, particles are free except for sharply defined collisions at spacetime points, whose physical effect is to change the energy-momentum of each particle $p^a_{\rm in} \rightarrow p^a_{\rm out}$ at the collision point. The physical requirement that a system of self-interacting particles still behaves like an effective isolated system when discussed globally (Newton's third law) implies that energy-momentum should be conserved locally at every collision event:

$$\sum_{\text{colliding particles}} p^a_{\rm in} = \sum_{\text{colliding particles}} p^a_{\rm out} \;. \tag{1.85}$$

A clear shortcoming of this prescription is the arbitrariness of the detailed interaction law, since (1.85) does not determine the precise values of outgoing momenta, given the incoming ones. A rather more satisfactory solution to the interaction problem is to follow the blueprint of Maxwell's theory and prescribe the particle interactions to be *mediated* by fields. Fields are assumed to satisfy *local* relativistic equations, ensuring that the solutions are waves traveling at most at the speed of light, thus automatically incorporating the required retardation effects between particles. Since the interaction of a fixed background field configuration with a given particle must be locally specified at the particle trajectory, we can make contact with the formal Lagrangian framework by describing such interaction with a generalized relativistic potential. Thus, we generalize the free particle action as

$$S_{\rm P} = -m \int d\tau - \int d\tau \, \mathcal{V}\left(x(\tau), \dot{x}(\tau)\right) \;, \tag{1.86}$$

where we have allowed for the potential to depend on the particle's four velocity as well as the particle coordinate. In order to find the equations of motion implied by this action we must be careful to use a general parametrization of the path, i.e. we write (1.86) in the form

$$S_{\rm P} = -m \int d\sigma \frac{d\tau}{d\sigma} \left[1 + \mathcal{V}\left(x(\sigma), \frac{d\sigma}{d\tau}\frac{dx}{d\sigma}\right)\right] , \tag{1.87}$$

with $d\tau/d\sigma = \sqrt{-\eta_{ab} \frac{dx^a}{d\sigma} \frac{dx^b}{d\sigma}}$.

Varying this action under fluctuations of the trajectory function $x^a(\sigma) \to x^a(\sigma) + \delta x^a(\sigma)$ and requiring it to be stationary we obtain the equations of motion in the form of a relativistic Newton's law

$$m \frac{d^2 x^a}{d\tau^2} = F^a . \tag{1.88}$$

where the generalized force four-vector F^a depends on the potential $\mathcal{V}$ in a very complicated and non-linear way. Even for the case of a potential without dependence on four-velocities, $\mathcal{V}_0 = q_0 \phi(x)$, we find the non-trivial velocity-dependent force

$$F_a = -m\, \partial_a \Phi - m\, \dot{x}_a\, \dot{x}^b\, \partial_b \Phi , \tag{1.89}$$

with

$$\Phi = \log\left(1 + \frac{q_0}{m}\phi\right) . \tag{1.90}$$

The result is *very different* from the naive guess for the relativistic generalization of the Newtonian force, which would be $-q_0 \partial_a \phi$.

We can generalize such purely scalar coupling by expanding the general potential $\mathcal{V}(x, \dot{x})$ in powers of four-velocities, obtaining a set of couplings of the particle's worldline to generalized fields transforming in symmetric tensor representations of the Lorentz group:

$$S_{\rm P} = -m \int d\tau - q_0 \int d\tau\, \phi(x(\tau)) - q_1 \int d\tau\, \dot{x}^a \phi_a(x(\tau)) - \tfrac{1}{2}\, q_2 \int d\tau\, \dot{x}^a \dot{x}^b\, \phi_{ab}(x(\tau)) + \dots \tag{1.91}$$

where we have separated conventional numerical coefficients, q_i, called 'charges' according to $q_n \phi_{a_1 \dots a_n} = \dot{\partial}_{a_1} \dots \dot{\partial}_{a_n} \mathcal{V}$, and we use the notation $\dot{\partial}_a \equiv \frac{\partial}{\partial \dot{x}^a}$. . These terms can be interpreted as the interaction of the particle with an *external* scalar field ϕ, a vector field ϕ_a, a symmetric tensor field ϕ_{ab}, etc. In general, only symmetric tensors can be coupled in this way, unless the particle is decorated with some other degrees of freedom (such as for example a spin four-vector S^a).

The generic relativistic force obtained in this way is extremely non-linear in the external potentials, except for the case of a world-line potential linear in the four-velocities, i.e. $\mathcal{V}(x, \dot{x}) = q_1\, \dot{x}^a\, \phi_a$. In this case, the interaction potential has an intrinsic meaning as a purely geometrical line integral over the trajectory:

$$\int_\gamma d\tau\ \dot{x}^a \phi_a = \int_\gamma dx^a \phi_a ,$$

and the relativistic force is linear in velocities,

$$F_a = -q_1\, \Phi_{ab}\, \dot{x}^b , \tag{1.92}$$

and *also linear* in potentials $\Phi_{ab} \equiv \partial_a \phi_b - \partial_b \phi_a$. In fact, it is only for the coupling to vector fields that we find such simple force laws. The general case is similar to the situation with scalar fields (1.89) in the sense that we are led to extremely non-linear force laws. We shall see later that the vector-field force corresponds to the special case of electromagnetic theory.

The coupling of a system of particles to external fields can be summarized in field-theory Lagrangians by defining appropriate currents. For scalar couplings we have a scalar 'density'

$$J_{\rm S}(x) = \sum_p \int d\tau_p \, q_p \, \delta^{(4)}(x - x(\tau_p)) \ , \tag{1.93}$$

which is defined so that [8]

$$\int d^4x \, J_{\rm S}(x) \, \phi(x) = \sum_p q_p \int d\tau_p \, \phi(x(\tau_p)) \ .$$

Vector fields are always coupled via standard vector currents

$$J^a_{\rm V}(x) = \sum_p \int d\tau_p \, q_p \, \dot{x}^a \, \delta^{(4)}(x - x(\tau_p)). \tag{1.94}$$

Vector currents are always associated to conservation laws. To see this, we pass to the non-covariant representation by performing the substitution $d\tau_p = dt \, d\tau_p/dt$ in the proper time integrals and using the temporal component of the delta function $\delta(t - t(\tau_p))$ to compute the integral over t. The result is the usual expression for the currents

$$J^0(t, \vec{x}\,) = \sum_p q_p \, \delta^{(3)}(\vec{x} - \vec{x}_p(t)) \ , \qquad \vec{J}(t, \vec{x}\,) = \sum_p q_p \vec{v}_p \, \delta^{(3)}(\vec{x} - \vec{x}_p(t)) \ . \tag{1.95}$$

The covariant conservation law $\partial_a J^a = 0$ translates then into the usual $\partial_t J^0 + \vec{\partial} \vec{J} = 0$.

In an analogous fashion, we may consider higher tensorial generalizations. The most interesting one is the two-index symmetric object

$$J^{ab}_{\rm T}(x) = \sum_p \int d\tau_p \, q_p \, \dot{x}^a \, \dot{x}^b \, \delta^{(4)}(x - x(\tau_p)) \ , \tag{1.96}$$

which actually contains the scalar density above in the trace component, since $\dot{x}^a \dot{x}_a = -1$ implies $\eta_{ab} J^{ab}_{\rm T} = -J_{\rm S}$. The symmetric tensor (1.96) acquires an important physical interpretation when $q_p = m_p$, the rest mass of the particles.

The local energy-momentum tensor

The second order tensor

$$T^{ab}(x) = \sum_p \int d\tau_p \; m_p \, \dot{x}^a_p \, \dot{x}^b_p \, \delta^{(4)}(x - x(\tau_p)) \tag{1.97}$$

is called the energy-momentum tensor. According to the previous subsection, it measures the response of the system of particles to the introduction of an external field transforming as a

[8] The four-dimensional delta function $\delta^{(4)}(x - x(\tau)) = \delta(t - t(\tau)) \delta^{(3)}(\vec{x} - \vec{x}(\tau))$ is Lorentz-invariant up to a sign of the transformation's determinant.

symmetric tensor of rank two, and coupling proportionally to the mass of the particles. Using repeatedly the relation

$$\int d\tau F(x(\tau),\dot{x}(\tau))\, \frac{dt}{d\tau}\, \delta^{(4)}(x-x(\tau)) = F(x(t),\dot{x}(t))\, \delta^{(3)}(\vec{x}-\vec{x}_p) \tag{1.98}$$

we find that

$$T^{00} = \sum_p E_p\, \delta^{(3)}(\vec{x}-\vec{x}_p) \tag{1.99}$$

is the energy density of the particle system. In turn,

$$T^{i0} = \sum_p E_p\, v_p^i\, \delta^{(3)}(\vec{x}-\vec{x}_p) = \sum_p p_p^i\, \delta^{(3)}(\vec{x}-\vec{x}_p)\ , \tag{1.100}$$

which we can interpret either as the energy current, or as the momentum density. On the other hand

$$T^{ij} = \sum_p p_p^i\, v_p^j\, \delta^{(3)}(\vec{x}-\vec{x}_p) = \sum_p p_p^j\, v_p^i\, \delta^{(3)}(\vec{x}-\vec{x}_p) \tag{1.101}$$

is either the flux of i-th momentum in the j-the direction, or viceversa.

Since T^{ab} characterizes the density and flow of energy, it should be associated to a local conservation law, i.e. to a current that satisfies a 'continuity' equation $\partial_a J^a = 0$, just like the electromagnetic current. The natural current to be associated with the energy-momentum tensor is the four-momentum flux as measured by a local observer with four-velocity u^a,

$$\mathcal{P}_u^a = -u_b\, T^{ab}\ . \tag{1.102}$$

To justify this expression, notice that it reduces to the correct T^{0a} for an observer at rest and, being a tensor equation, it is valid in all Lorentz frames. We define the local current at all points by giving a family of fiducial observers with four-velocity field u^a. If these observers are stationary with respect to one another, all u^a are parallel, and $\partial_a u^b = 0$. Under these conditions, we obtain conservation of the local energy-momentum flux, $\partial_a \mathcal{P}_u^a = 0$, if the energy-momentum tensor satisfies $\partial_a T^{ab} = 0$.

A direct computation for the particle system yields

$$\partial_a T^{ab} = \sum_p m_p \int d\tau_p\, \dot{x}_p^a\, \dot{x}_p^b\, \partial_a \delta^{(4)}(x-x(\tau_p))\ . \tag{1.103}$$

Acting on the delta function $\dot{x}^a \partial_a\, \delta^{(4)}(x-x(\tau)) = -\dot{x}^a \partial/\partial x^a(\tau)\, \delta^{4)}(x-x(\tau)) = -d/d\tau\, \delta^{(4)}(x - x(\tau))$. Hence

$$\partial_a T^{ab} = -\sum_p m_p \int d\tau_p\, \frac{d}{d\tau_p}\left(\dot{x}_p^b\, \delta^{(4)}(x-x(\tau_p))\right) + \sum_p m_p \int d\tau_p\, \ddot{x}_p^b\, \delta^{(4)}(x-x(\tau_p))\ . \tag{1.104}$$

The first term is supported on the endpoints and thus vanishes, whereas the last term can be written

$$\partial_a T^{ab} = \sum_p \int d\tau_p\, \frac{dp_p^b}{d\tau_p}\, \delta^{(4)}(x-x(\tau_p)) = \sum_p \frac{dp_p^b}{dt}\, \delta^{(3)}(\vec{x}-\vec{x}_p) \tag{1.105}$$

as the density of momentum *non-conservation.* Hence, the *local* energy-momentum conservation requires that particles are free $\ddot{x}^a = 0$, or perhaps that their interactions are localized in spacetime (in this case, $\sum_{\text{in}} p = \sum_{\text{out}} p$ at each interaction point). We conclude that, up to regions of measure zero in spacetime, the local conservation of the energy momentum tensor is equivalent to the equation of motion in a system of free particles. This property will be recurrent in all well-defined matter systems. The local conservation of the energy-momentum tensor is of paramount importance in relativistic theories, since it embodies the basic requirements of locality and Lorentz symmetry.

Angular momentum and spin

The local conservation of the energy-momentum tensor allows us to define the conserved four-momentum of a system

$$P^a = \int_{\mathbf{R}^3} T^{0a} \ , \tag{1.106}$$

which behaves as the conserved four-momentum of a particle. In particular, we can define the total mass $M = \sqrt{-P^2}$ and the global four-velocity $U^a = P^a/M$. Analogously, we can define the total orbital angular momentum with respect to the origin as

$$L^{ab} = X^a P^b - X^b P^a \ , \tag{1.107}$$

where $X^a = (t, \vec{X})$ is the "center of mass" trajectory, where

$$\vec{X} = \frac{1}{M} \int_{\mathbf{R}^3} \vec{x}\, T^{00} \ .$$

The intrinsic angular momentum, or *spin*, is defined as the angular momentum with respect to the center of mass

$$S^{ij} = \int_{\mathbf{R}^3} \Big((x^i - X^i)\, T^{0j} - (x^j - X^j)\, T^{0i} \Big) \ , \tag{1.108}$$

which differs from the total angular momentum with respec to the origin

$$J^{ij} = \int_{\mathbf{R}^3} \left(x^i T^{0j} - x^j T^{0i} \right) \ ,$$

by

$$J^{ij} = L^{ij} + S^{ij} \ .$$

In vector notation, $S^i = \frac{1}{2} \sum_{jk} \epsilon^{ijk} S^{jk}$, i.e. $S^{jk} = \sum_i S^i \epsilon^{ijk}$. We can generalize this to a four-vector by stating that $S^0 = 0$ in the rest frame of the centre of mass. Hence, a Lorentz-invariant equation that reduces to these in the rest frame is

$$S_a = \tfrac{1}{2}\, \epsilon_{abcd} S^{bc} U^d \ . \tag{1.109}$$

The vanishing of the temporal component in the rest frame translates into $U_a S^a = 0$.

The local tensor object

$$M^{abc} \equiv x^a T^{bc} - x^b T^{ac} \tag{1.110}$$

is conserved provided the energy-momentum tensor is *symmetric*, $\partial_a M^{abc} = T^{bc} - T^{cb} = 0$, and defines an angular momentum density $\mathcal{J}_u^{ab} = -u_c M^{cab}$ associated to a spacelike surface orthogonal to the field of four-velocities u^a. On a $t =$ constant surface it defines the conserved angular-momentum tensor

$$J^{ab} = -J^{ba} = \int_{\mathbf{R}^3} M^{0ab} = \int_{\mathbf{R}^3} (x^a T^{b0} - x^b T^{a0}) \ , \tag{1.111}$$

whose spatial components give J^{ij} above. We can project the spin four-vector out of J^{ab} by the formula (Pauli–Lubanski)

$$S_a = \tfrac{1}{2}\,\epsilon_{abcd} J^{bc} U^d \ , \tag{1.112}$$

since the orbital part L^{ab} drops out from this expression.

Conservation of P^a, J^a and S^a implies that

$$\frac{dU^a}{d\tau_{\rm CM}} = \frac{dS^a}{d\tau_{\rm CM}} = 0 \ , \tag{1.113}$$

where $\tau_{\rm CM}$ is the proper time associated to the trajectory of the center of mass.

Fluids

It is interesting for applications to take the fluid limit. This is a particle system with very short-ranged collisions, so that a comoving observer sees the system as characterized by an energy density, a pressure, and perhaps gradients of the net velocity field, after one averages over the short-distance and short-time collisions. The fluid is called *perfect* if the comoving observer sees the fluid as isotropic, characterized solely by the density and the pressure:

$$\rho = \left\langle \sum_p E_p\, \delta^{(3)}(\vec{x}-\vec{x}_p) \right\rangle_{\rm fluid} \ , \qquad p = \frac{1}{3}\sum_i \left\langle \sum_p p_p^i\, v_p^i\, \delta^{(3)}(\vec{x}-\vec{x}_p) \right\rangle_{\rm fluid} \ . \tag{1.114}$$

Hence, in the fluid limit, we have $T^{00} = \rho$ and $T^{ij} = p\,\delta^{ij}$ in the comoving frame. [9] This determines the compact expression

$$T^{ab}(x)|_{\rm fluid} = p(x)\,\eta^{ab} + \left(p(x)+\rho(x)\right)\, U^a(x)\, U^b(x) \tag{1.115}$$

in a general inertial frame. Here, U^a is the (spacetime dependent) field of four-velocities of the fluid. The conservation equation $\partial_a T^{ab} = 0$ implies, in the non-relativistic limit, $p \ll \rho$, $|\vec{v}\,|dp/dt \ll |\vec{\partial}\,p|$, the two Euler equations for the dynamics of a perfect fluid [10]

$$\partial_t \rho + \vec{\partial}(\rho\,\vec{v}\,) = 0 \ , \qquad \rho\left(\partial_t \vec{v} + (\vec{v}\cdot\vec{\partial}\,)\vec{v}\right) = -\vec{\partial}\,p \ . \tag{1.116}$$

1.3.3 Relativistic fields

Field theory model building is a highly nontrivial endeavor, plagued with subtleties already at the level of classical physics. A local Lagrangian of fields Ψ in arbitrary representations of the Lorentz group may be organized by its dependence on derivatives $\mathcal{L}(\Psi, \partial_a\Psi, \partial_a\partial_b\Psi, \dots)$. The corresponding equations of motion are obtained by requiring the action to be stationary under the variations $\Psi \to \Psi + \delta\Psi$. After repeated application of integration by parts and neglecting boundary terms one finds

$$0 = -\frac{\partial\mathcal{L}}{\partial\Psi} + \partial_a \frac{\partial\mathcal{L}}{\partial(\partial_a\Psi)} - \partial_a\partial_b \frac{\partial\mathcal{L}}{\partial(\partial_a\partial_b\Psi)} + \dots \ , \tag{1.117}$$

[9] The fluid averages are defined as $\langle A\rangle_{\rm fluid} = \Delta V^{-1}\int_{\Delta V} A$ for a small volume element ΔV, still large enough to contain a macroscopic number of particles. Using kinetic theory, one can argue that $p_p^i v_p^i$ is the momentum transfer per unit area and time (i.e. the contribution to pressure) exerted by a particle over a surface orthogonal to the ith direction.

[10] **Exercise:** Check it.

where the signs alternate according to the number of derivatives acting on Ψ. In most applications, one can restrict to Lagrangians depending polynomially upon one single derivative of the field, $\partial_a \Psi$ (perhaps after repeated application of partial integrations). In this case we obtain the more standard form of the Euler–Lagrange equations, which reduces to the first two terms in (1.117). Terms in the Lagrangian with more than two derivatives must be safely considered as perturbative corrections in the sense of the local expansion, and their effect is only to be trusted when they are not dominant.

A better Noether

We now pause to introduce a new derivation of Noether's theorem for the case of field theories, which stands out for its generality. Let $\mathcal{L}[\Psi]$ be a general functional of $\Psi(x)$ with polynomial dependence on ∂_a, but otherwise generic. We assume that, under a transformation $\delta_\epsilon \Psi = \epsilon\, \xi[\Psi]$ with a constant ϵ, the Lagrangian density transforms as a total derivative $\delta_\epsilon \mathcal{L} = \epsilon \partial_a f^a_\xi[\Psi]$.

Consider now a *more general* transformation where the symmetry parameter is given a functional dependence on spacetime coordinates, i.e. $\delta\Psi = \epsilon(x)\xi[\Psi]$. The variation of the Lagrangian density can be expanded in derivatives of $\epsilon(x)$ as

$$\delta\mathcal{L} = \epsilon K[\Psi] + \partial_a \epsilon\, K^a[\Psi] + \partial_a \partial_b \epsilon\, K^{ab}[\Psi] + \cdots$$

Restricting to a constant $\epsilon(x)$, we know that only the first term survives, and the symmetry condition implies then that $K[\Psi] = \partial_a f^a_\xi[\Psi]$ is a total derivative. In any case, inserting this expansion into the variation of the action $S = \int \mathcal{L}[\Psi]$ we obtain, after integrating by parts

$$\delta S = -\int \epsilon\, \partial_a J^a_\xi$$

where

$$J^a_\xi = -f^a_\xi[\Psi] + K^a[\Psi] - \partial_b K^{ab}[\Psi] + \cdots \tag{1.118}$$

Since the variation $\delta\Psi = \epsilon(x)\xi[\Psi]$ is a particular case of a general variation of the fields, on a configuration that extremizes the action we conclude that this current is conserved, $\partial_a J^a_\xi = 0$. Hence, in (1.118) we have a constructive definition of the Noether current, valid in an arbitrary derivative expansion. Should it be possible to write the dependence on derivatives in $\mathcal{L}[\Psi]$ as some function of $\partial_a \Psi$ (perhaps up to total derivatives), we would obtain the previous result, since $K^a = \partial\mathcal{L}/\partial(\partial_a\Psi)$ in this case.

In what follows, we list the most common types of field theories, as established by their pedagogical features or their importance in the actual description of nature.

Scalars

The prototype relativistic field theory with minimal derivative content is the scalar model

$$\mathcal{L} = -\tfrac{1}{2}\, \eta^{ab} \partial_a \phi\, \partial_b \phi - V(\phi)\ . \tag{1.119}$$

The first term is the only Lorentz-invariant combination of two derivatives of the field, whose detailed structure

$$-\tfrac{1}{2}\, (\partial\phi)^2 = \tfrac{1}{2}\, (\partial_t \phi)^2 - \tfrac{1}{2}\, (\vec{\partial}\phi)^2\ , \tag{1.120}$$

shows that the time derivative is the strict 'kinetic' term and the spatial derivative is really part of the 'potential energy'. Up to a total derivative, we also have

$$\mathcal{L} = \tfrac{1}{2}\, \phi\, \partial^2 \phi - V(\phi) + \partial(\ldots) \,, \tag{1.121}$$

which leads to the field equation:

$$\partial^2 \phi - V'(\phi) = 0 \,. \tag{1.122}$$

One usually separates the linear and quadratic parts of the potential $V(\phi) = J\phi + \frac{1}{2}\, m^2 \phi^2 + \ldots$ to interpret them as the coupling to a scalar 'source' and a 'mass' term. A nonlinear generalization of (1.119) involving several scalar fields ϕ^I is the so-called nonlinear sigma-model, with many applications in particle physics and condensed matter physics

$$\mathcal{L} = -\tfrac{1}{2} \sum_{IJ} G_{IJ}(\phi)\, \partial_a \phi^I\, \partial^a \phi^J - V(\phi) \,. \tag{1.123}$$

We return now to the basic Klein–Gordon model (1.119) and investigate its solutions. Start with the massless free field, $V(\phi) = 0$. The solution of the field equation $\partial^2 \phi = 0$ is given by a linear superposition of plane waves $\phi_k(x) = \widetilde{\phi}_k \exp(-ikx) + \text{c.c.}$ of wave vector $\vec{k}$ and frequency $\omega = k^0 = |\vec{k}\,|$. These waves are defined on all of $\mathbf{R}^{1+3}$, traveling at the speed of light from the infinite past to the infinite future.

The wave vector k^a is orthogonal to the constant-phase surfaces $k^a x_a = \text{constant}$. Consider the integral curves $x^a(\sigma)$ that solve the linear differential equation $dx^a/d\sigma = k^a$ and have, by definition, tangent vector k^a. These curves are orthogonal to the constant-phase surfaces and can be interpreted as trajectories of zero-mass particles, by the condition $k^2 = (dx^a/d\sigma)(dx_a/d\sigma) = 0$. The frequency measured in a rest frame is $\omega = k^0$, so that the frequency measured by an observer with four-velocity u^a is

$$\omega = -u_a k^a \,. \tag{1.124}$$

We notice the analogy with the formula for the energy of a particle, $E = -p_a u^a$. In the quantum theory, $E = \hbar\omega$ and $\vec{p} = \hbar\vec{k}$ for a photon, so that the particle 'model' for light propagation is no longer an analogy but a strict fact. [11] For the purposes of special and general relativity, we can formally treat the propagation of localized, massless wave packets as the zero-mass limit of particle propagation with the operator replacement

$$m \frac{d}{d\tau} \to \frac{d}{d\sigma} \,,$$

where σ is an affine parameter along the 'light' ray.

The next level of complication is the case of an external 'source' $V(\phi) = J$, with field equation $\partial^2 \phi = J$, solved formally as

$$\phi = \phi_{\text{wave}} + \frac{1}{\partial^2} J \,,$$

where the first term is a solution of the massless wave equation *in vacuo*, $\partial^2 \phi_{\text{wave}} = 0$, and the second term is a formal representation of a particular solution with structure similar to the Newtonian potential, except for the replacement of the operator $\vec{\partial}^2$ by the operator $\partial^2 =$

[11] Although we coach this discussion for the case of scalar fields and scalar particles, the propagation properties of massless scalar waves generalize readily to higher spin cases, including electromagnetic waves and the 'photon' particles.

$-\partial_t^2 + \vec{\partial}^2$, which takes the Poisson equation into the Klein–Gordon equation. Given a purely static solution of the Poisson equation $\vec{\partial}^2 \phi = J$,

$$\phi_{\text{static}}(\vec{x}\,) = \frac{1}{\vec{\partial}^2} J = -\int \frac{d^3y}{4\pi} \frac{J(\vec{y}\,)}{|\vec{x} - \vec{y}\,|} \ ,$$

with no time dependence, we can obtain a solution of the relativistic equation by introducing time-dependence through the trick of the retarded potential, i.e. evaluating the source at the retarded time $t - |\vec{x} - \vec{y}\,|$, to represent the fact that disturbances on the field ϕ travel at the speed of light, as vacuum waves.

$$\phi(t, \vec{x}\,)_{\text{retarded}} = -\int \frac{d^3y}{4\pi} \frac{J(t - |\vec{x} - \vec{y}\,|, \vec{y}\,)}{|\vec{x} - \vec{y}\,|} \ . \tag{1.125}$$

We can now use this same intuition to interpret the general solution. For an arbitrary potential we may rewrite (1.122) as

$$\phi(t, \vec{x}\,)_{\text{retarded}} = \frac{1}{\partial^2} V'(\phi) = -\int \frac{d^3y}{4\pi} \frac{V'[\phi(t - |\vec{x} - \vec{y}\,|, \vec{y}\,)]}{|\vec{x} - \vec{y}\,|} \ . \tag{1.126}$$

Rather than a general solution, this is an integral equation which we may solve iteratively by redefining the source order by order as a function of the approximate solution at lower orders. In particular, each insertion of a term in $V'(\phi)$ represents a 'kick' of the free waves that transport the interaction at the speed of light. In this way, we can understand how successive kicks due to a mass term $V'(\phi) \sim m^2 \phi$ have the collective effect of reducing the speed of propagation from that of light to a superposition of various speeds below light, as if the waves were a superposition of particles with dispersion $E = \sqrt{\vec{p}^{\,2} + m^2}$.

Spinors

Fields in spinor representations of the Lorentz group are also very important in nature (all fermions) but the appropriate Lagrangians require the Dirac formalism and are, strictly speaking, part of the quantum version of the theory. Here, we shall simply quote for completeness the case of the basic free Dirac field. A correct relativistic propagation is ensured if we write

$$(\partial^2 - m^2)\psi = 0$$

as the field equation. However, using the properties of Dirac matrices, one can see that $\partial^2 - m^2 = (i\gamma^a \partial_a + m)(i\gamma^a \partial_a - m)$. Hence, the relativistic equation on spinors with the most general set of solutions is the so-called Dirac equation

$$(i\gamma^a \partial_a - m)\psi = 0,$$

which follows from the Lagrangian

$$\mathcal{L}_{\text{Dirac}} = \bar{\psi}(i\gamma^a \partial_a - m)\psi \ ,$$

where $\bar{\psi} = \psi^\dagger \gamma^0$ is the Lorentz-conjugated representation. The most salient feature of this Lagrangian is its first order character in derivatives, as opposed to the bosonic counterparts.

Vector fields and gauge redundancy

Higher-spin tensor fields always lead to subtleties. Consider, for example, a free vector field ϕ^a of mass m, with equation of motion

$$(\partial^2 - m^2)\,\phi^a = 0\ , \tag{1.127}$$

consisting of four copies of the Klein–Gordon equation. A candidate Lorentz-invariant Lagrangian with the correct equations of motion would be

$$\mathcal{L}[\,\phi^a\,]_{\rm KG} = \tfrac{1}{2}\,\eta_{ab}\,\phi^a\,\partial^2\,\phi^b - \tfrac{1}{2}\,m^2\,\eta_{ab}\,\phi^a\phi^b \tag{1.128}$$

as a direct generalization of the scalar Klein–Gordon Lagrangian. However, the fact that η_{ab} is *not* positive definite leads to problems, since the whole Lagrangian of the ϕ^0 field has the wrong sign. This field may then store arbitrarily large amounts of negative energy, and any interaction term would render the dynamics unstable. So, we face an interesting dilemma: the field equation following from this Lagrangian seems to be correct and yet the energy density appears to be ill-defined.

We cannot simply drop the offending component ϕ^0, because that would spoil the Lorentz symmetry. However, for a plane-wave solution in momentum space, $\phi^a = \widetilde{\phi}^{\,a} e^{ipx}$,

$$(p^2 + m^2)\widetilde{\phi}^a = 0\ ,$$

we may impose the Lorentz-invariant condition of transversality between the polarization vector $\widetilde{\phi}^a$ and the propagation vector p^a, i.e.

$$p_a\,\widetilde{\phi}^{\,a} = 0\ . \tag{1.129}$$

This condition brings $\widetilde{\phi}^0$ to vanish in the rest frame $(p^a) = (m, 0, 0, 0)$. Therefore, we conclude that such 'transverse waves' satisfying (1.129) will not have energy stability problems.

This suggests that we add the transversality condition $\partial_a \phi^a = 0$ as an extra equation, together with the Klein–Gordon equation (1.127). In fact, in this case we may redefine (1.127) by any multiple of $\partial_b\,\partial_a \phi^a$, and the particular choice

$$\Big(\eta_{ab}(\partial^2 - m^2) - \partial_a\partial_b\Big)\,\phi^b = 0 \tag{1.130}$$

does enforce the transversality condition automatically, provided $m \neq 0$. To see this, we take its divergence to find $m^2\partial_a\phi^a = 0$. The Lagrangian associated to the equation of motion (1.130) is

$$\mathcal{L} = -\tfrac{1}{4}\Phi_{ab}\,\Phi^{ab} - \tfrac{1}{2}\,m^2\,\phi_a\,\phi^a\ , \tag{1.131}$$

where we have defined $\Phi_{ab} = \partial_a\phi_a - \partial_b\phi_a$. We conclude that (1.131) is the consistent Lagrangian of a free vector field, describing the propagation of *three* polarization degrees of freedom.

More subtleties lurk in the zero-mass limit $m \to 0$. In this case, the 'rest frame' of a plane wave does not exist, as it propagates at the speed of light. It is possible to reach the standard null frame $(p^a) = (\omega, 0, 0, \omega)$, but then the transversality condition only enforces $\widetilde{\phi}^0 = \widetilde{\phi}^3$, which is not enough to cancel out the offending time-like component ϕ^0. On the other hand, if (1.131) has no energy balance problems, the same should be true of the $m^2 \to 0$ limit.

There is an interesting escape out of this paradox. Suppose we *declare* by hand that $\widetilde{\phi}^{\,0} = 0$ on the standard null frame. This means that, because of the transversality constraint, we are actually demanding the wave polarization to be transverse in the spatial sense, $\sum_i \widetilde{\phi}_i p^i = \widetilde{\phi}_0 = 0$, with only two polarization degrees of freedom remaining. The problem with this prescription is of course the violence to Lorentz invariance, since the Lorentz transformation of such a polarization vector will in general produce longitudinal and time-like components. However, whatever the Lorentz transformation does, it must preserve the transversality constraint $p^a \widetilde{\phi}_a = 0$, which precisely for $p^2 = 0$ has a *redundancy*: if $\widetilde{\phi}^a$ is transverse, so is $\widetilde{\phi}^a + \tilde{f}\, p^a$, with $\tilde{f}$ an arbitrary constant. This means that $\widetilde{\phi}_a$ will shift by a term proportional to p_a under Lorentz transformations.

The way to remove the offending degree of freedom in a Lorentz-invariant fashion is simply to declare *all* configurations related by such shifts as *physically equivalent*, in the sense that all physical observables should be exactly invariant under $\widetilde{\phi}^a \to \widetilde{\phi}^a + \tilde{f}\, p^a$, or its position-space counterpart

$$\phi_a \longrightarrow \phi_a + \partial_a f \; . \tag{1.132}$$

We may then use the degree of freedom in $\tilde{f}$ to remove $\widetilde{\phi}_0$ in the standard null frame. In doing so, we remove $\widetilde{\phi}_3$ too, as they are linked by the transversality constraint. Therefore, we end up with a consistent Lorentz-invariant theory of massless vector fields with only two physical degrees of freedom. We can find the Lagrangian of this theory by working with the explicitly transverse fields obtained by the linear projection

$$\widetilde{\phi}^a_{\rm T} = \widetilde{P}^a{}_b\, \widetilde{\phi}^b \equiv \left(\delta^a{}_b - \frac{p^a\, p_b}{p^2} \right) \widetilde{\phi}^b \; , \tag{1.133}$$

which translates into a *non-local* projector in position space,

$$\phi^a_{\rm T} = P^a{}_b\, \phi^b \equiv \left(\delta^a_b - \frac{\partial^a\, \partial_b}{\partial^2} \right) \phi^b . \tag{1.134}$$

Notice that such transverse fields are automatically gauge-invariant since

$$P^a{}_b\, \phi^b = P^a{}_b (\phi^b + \partial^b f) \; . \tag{1.135}$$

Evaluating then the massless Klein–Gordon Lagrangian on transverse gauge-invariant fields we find the Lagrangian

$$\tfrac{1}{2}\, \eta_{ab}\, \phi^a_{\rm T}\, \partial^2\, \phi^b_{\rm T} = -\tfrac{1}{4}\, \Phi_{ab}\, \Phi^{ab} \; , \tag{1.136}$$

corresponding to the massless limit of (1.131). It is important to keep in mind, however, that the massless vector model is assumed to be defined with the built-in redundancy under $\phi_a \to \phi_a + \partial_a f$, for an arbitrary function $f(x)$, and in particular it has one less degree of freedom than the naive massless limit of the massive model. So, even if (1.136) is the smooth $m^2 \to 0$ limit of (1.131) at the level of Lagrangians, there is a discontinuous jump in degrees of freedom as we go to the massless limit.

The gauge redundancy is very useful in constructing dynamical laws for massless vector fields. For example, a linear coupling of the form

$$\mathcal{L}_J = \phi_a\, J^a$$

is bound to break gauge invariance unless we demand the conservation of the 'current': $\partial_a J^a = 0$. Should the current *not* be conserved, so that gauge invariance is broken, then the theory with this coupling would simply describe three degrees of freedom instead of two.

The gauge redundancy is often called a 'local symmetry', a rather misleading name, since its real role is to ensure that the theory describes the minimal number of degrees of freedom (two for a vector field) even after Lorentz-invariant couplings are specified.

Problem: Gauge is not a symmetry

Consider the massive vector field model with Lagrangian

$$\mathcal{L}_{\text{massive}} = -\tfrac{1}{4}\,\Phi_{ab}\,\Phi^{ab} - \tfrac{1}{2}\,m^2\,\phi_a\,\phi^a\ . \tag{1.137}$$

Show that it is *not* gauge invariant. Define the so-called Stueckelberg model, which couples a massless vector field and a massless scalar field φ with Lagrangian

$$\mathcal{L}[\,\phi_a,\varphi\,] = -\tfrac{1}{4}\Phi_{ab}\,\Phi^{ab} - \tfrac{1}{2}\,\eta^{ab}(m\phi_a + \partial_a\varphi)(m\phi_b + \partial_b\varphi)\ . \tag{1.138}$$

Show that the Stueckelberg model is gauge-invariant under the transformations

$$\phi_a(x) \to \phi_a(x) + \partial_a f(x)\ , \qquad \varphi(x) \to \varphi(x) - f(x)/m\ .$$

Show that the Stueckelberg model is equivalent to (1.137) by an appropriate fixing of the gauge redundancy.

Hence, we find that a model with three degrees of freedom and no gauge invariance is equivalent to a gauge-invariant model with two degrees of freedom, plus one explicit degree of freedom, so that gauge 'symmetry' is not real... just a matter of bookkeeping conventions in describing the true degrees of freedom.

'Integrate out' the scalar field φ in (1.138), by solving its equation of motion and substituting back into the action. Show that the resulting equivalent Lagrangian is

$$\mathcal{L}_{\text{non local}} = -\tfrac{1}{4}\,\Phi_{ab}\,\Phi^{ab} + \tfrac{1}{4}\,m^2\,\Phi_{ab}\frac{1}{\partial^2}\,\Phi^{ab}\ . \tag{1.139}$$

That is, a massless vector gauge theory, corrected by a non-local term proportional to the mass squared. Notice that *it is* gauge invariant. The non-local term is the result of the existence of more degrees of freedom than just the two massless polarizations. This non-local model is actually the result of imposing the gauge invariance 'by hand' in the naive Klein–Gordon Lagrangian, i.e. show that

$$\mathcal{L}_{\text{non local}} = \mathcal{L}[\,\phi^a\,]_{\text{TKG}} = \tfrac{1}{2}\,\eta_{ab}\,\phi^a_{\text{T}}\,\partial^2\,\phi^b_{\text{T}} - \tfrac{1}{2}\,m^2\,\eta_{ab}\,\phi^a_{\text{T}}\phi^b_{\text{T}}\ .$$

Hence, insisting on gauge invariance for a *massive* vector theory requires either a non-local model or a local model with an extra degree of freedom. This is natural, since we know that a massive vector field must have three polarization degrees of freedom, forcing a gauge-invariant description with two polarizations requires the introduction of the third polarization as an explicit field degree of freedom, the so-called Stueckelberg field, a precursor of the Higgs field.

Maxwell, the Lorentz-invariant way

We now reproduce the Lagrangian structure of massless vector fields from the direct physical construction of Maxwell's theory. One can exhibit the Lorentz covariance of Maxwell's equations by defining the electromagnetic tensor F_{ab}, via

$$F^{0i} = E^i\ , \qquad F_{ij} = \sum_k \epsilon_{ijk} B_k$$

in terms of the electric and magnetic field vectors $\vec{E}, \vec{B}$. Defining also the current four-vector $J^a = (\rho_e, \vec{J}_e)$, the equations can be cast into the system

$$\partial_a F^{ab} = -J^b \ , \qquad \epsilon^{abcd}\partial_b F_{cd} = 0 \ , \tag{1.140}$$

which are manifestly Lorentz-covariant. Thus, we confirm Poincaré's dictum that the Maxwell system remains untouched in the new relativistic formalism, whereas the Newtonian particle dynamics had to yield. Curiously, it was Maxwell the first to write down a relativistic equation in the mid XIX century!

In order to derive these equations from a Lagrangian, we introduce the potential A^a by solving the second Maxwell equation via the Poincaré lemma: locally, an antisymmetric tensor with vanishing antisymmetrized divergence can always be written as [12]

$$F_{ab} = \partial_a A_b - \partial_b A_a \tag{1.141}$$

in terms of some vector field A^a, which always comes with a built-in redundancy, i.e. a gauge ambiguity by the redefinition $A^a \to A^a + \partial^a f$. The rest of Maxwell's equations then follow upon varying the Maxwell Lagrangian

$$\mathcal{L}_{\text{Maxwell}} = -\tfrac{1}{4} F_{ab}F^{ab} + J_a A^a \tag{1.142}$$

with respect to A^a. As explained before, the coupling to the electric current does not spoil gauge invariance (up to total derivatives) provided the current is indeed conserved, $\partial_a J^a = 0$.

A convenient Lorentz-invariant convention to fix partially the gauge redundancy is the Lorenz condition $\partial_a A^a = 0$, whose main virtue is the simplification of the Maxwell equations, simply collapsing them to four Klein–Gordon equations

$$\partial^2 A^a = -J^a \ , \tag{1.143}$$

whose standard solution was discussed before

$$A^a(t, \vec{x}) = -\frac{1}{\partial^2}\, J^a = \int \frac{d^3y}{4\pi} \frac{J^a(t - |\vec{x} - \vec{y}\,|, \vec{y}\,)}{|\vec{x} - \vec{y}\,|} \ , \tag{1.144}$$

for fields that vanish at infinity and with causal (retarded) boundary conditions. This relation accounts for most of classical electrodynamics.

Free electromagnetic waves can be added to A^a as solutions of the homogeneous equation with $J^a = 0$. By the linearity property of Maxwell's equations, any such solution is a superposition of plane waves of the form

$$A^a(x) = \tilde{A}^a(k)\, e^{ikx} + \text{c.c.} \ ,$$

where $\tilde{A}^a(k)$ is now a constant vector and $k\,x = \eta_{ab}k^a x^b = -\omega\, t + \vec{k}\cdot\vec{x}$. The four-vector k^a includes the wave vector and the frequency. Maxwell equations and the gauge condition impose $k^a \tilde{A}_a(k) = 0$ and $k^2 = 0$. By further use of the gauge redundancy one can set $\tilde{A}^0(k) = 0 = \sum_i k^i \tilde{A}^i(k)$, leaving fully transverse waves with two polarization degrees of freedom.

The coupling to charges, given by the linear term in the field-theory action $\int d^4x\, J_a A^a$, with the *conserved* electromagnetic current

$$J^a(x) = \sum_p e_p \frac{dx^a}{d\tau}\, \delta^{(4)}(x - x_p) \ ,$$

[12] An explicit proof is obtained by the following ansatz: $A^a(x) = -\int_0^1 d\lambda\, \lambda\, F^{ab}(\lambda x)\, x_b$.

produces a correction to the free particle Lagrangian of the form

$$S_{\rm P} = -\sum_p m_p \int d\tau_p + \sum_p e_p \int d\tau_p A_a \frac{dx^a}{d\tau_p} \; , \tag{1.145}$$

which in turn leads to the so-called Lorentz force in its fully relativistic incarnation

$$m_p \, \frac{d^2 x^a}{d\tau^2} = e_p \, F^a_{\ \ b} \, \frac{dx^b}{d\tau} \; . \tag{1.146}$$

Problem: Parity violating photons?

Consider adding a Lorentz-invariant term to the Lagrangian of the Maxwell theory of the form

$$\mathcal{L}_\vartheta = \vartheta \, \widetilde{F}_{ab} F^{ab}$$

where

$$\widetilde{F}_{ab} = \tfrac{1}{2} \, \epsilon_{abcd} F^{cd}$$

is the dual tensor. Check that the dual tensor has the roles of electric and magnetic fields reversed. Write the Lagrangian $\mathcal{L}_\vartheta$ in terms of electric and magnetic fields and study its behavior under parity and time reversal. Why is this Lagrangian not usually considered as a correction to Maxwell theory?

Higher spin generalizations

The construction of field theories in higher tensor representations reproduces some of the problems already present in the vector case. For example, a naive action for a massless two-index tensor field ϕ_{ab} of the form

$$\mathcal{L}[\,\phi_{ab}\,]_{\rm KG} = \tfrac{1}{2} \, \phi_{ab} \, \partial^2 \, \phi^{ab}$$

has stability issues of similar nature, and may be solved in a similar fashion by introducing gauge redundancies. In particular, for the antisymmetric case, $b_{cd} = -b_{dc}$, we introduce a gauge symmetry $b_{cd} \to b_{cd} + \partial_{[c} f_{d]}$, whereas the symmetric case, $h_{ab} = h_{(ab)}$, requires a corresponding 'symmetric' symmetry $h_{ab} \to h_{ab} + \partial_{(a} f_{b)}$.

The resulting gauge-invariant action for the antisymmetric case is a simple generalization of the Maxwell construction:

$$\mathcal{L}[\,b_{cd}\,] = -\frac{1}{12} H_{abc} H^{abc} \; , \qquad H_{abc} \equiv 6 \, \partial_{[a} b_{bc]} \; .$$

which gives a theory with a single degree of polarization (equivalent to a pseudoscalar, also called *axion*).

The symmetric case is more complicated, and more interesting for the theory of gravity. There are four independent Lorentz-invariant structures at the quadratic, two-derivative level:

$$\partial_a h_{bc} \, \partial^a h^{bc} \; , \qquad \partial_a h^{ab} \, \partial^c h_{cb} \; , \qquad \partial_a h^b_b \, \partial^a h^c_c \; , \qquad \partial_a h^c_c \, \partial_b h^{ab} \; ,$$

up to total derivatives. Imposing now gauge invariance one finds a unique answer up to overall normalization, the so-called Fierz–Pauli Lagrangian

$$\mathcal{L}[\,h_{ab}\,]_{\rm FP} = -\tfrac{1}{4} \partial_a h_{bc} \, \partial^a h^{bc} + \tfrac{1}{4} \partial_a h^c_c \, \partial^a h^c_c - \tfrac{1}{2} \, \partial_a h^c_c \, \partial_b h^{ab} + \tfrac{1}{2} \, \partial_a h^{ab} \, \partial^c h_{bc} \; , \tag{1.147}$$

which will turn out to be the linearized approximation to Einstein's theory!

1.3.4 Energy-momentum tensor in field theories

Having defined the energy-momentum tensor of a particle system with contact interactions, we can also monitor the local exchanges of energy and momentum in the more physical case where the interactions are mediated by fields. Consider for example a system of charged particles whose electromagnetic interactions are governed by Lorentz forces

$$m_p\,\ddot{x}^a_p = e_p\,F^a{}_b\,\dot{x}^b_p\ . \tag{1.148}$$

Then, the energy-momentum tensor of the particle system is not conserved alone,

$$\partial_b T^{ab}_{\text{particles}} = F^a{}_b\,J^b\ . \tag{1.149}$$

as some energy flows into the field degrees of freedom. Thus, local energy conservation requires that we add the intrinsic energy-momentum tensor of the electromagnetic field, canceling the source term in (1.149). We know from elementary discussions of electrodynamics that the energy density of the electromagnetic field is given by $\frac{1}{2}(\vec{E}^2+\vec{B}^2)$, whereas the momentum density (the Poynting vector) is $\vec{E}\times\vec{B}$. These two quantities fix the values of T^{00} and T^{0i}. Hence we may try an ansatz

$$c_1\,F^{ac}F_c{}^b + c_2\,\eta^{ab}\,F_{cd}F^{cd}$$

depending on the two independent quadratic combinations of F_{ab} which are allowed by Lorentz symmetry and the antisymmetry of the electromagnetic tensor. Imposing now the condition

$$\partial_b T^{ab}_{\text{Maxwell}} = -F^{ab}\,J_b$$

given the equations of motion $\partial_a F^{ab} = -J^b$ we fix the constants $c_1 = -1$ and $c_s = -1/4$ to obtain the Maxwell energy-momentum tensor

$$T^{ab}_{\text{Maxwell}} = -F^{ac}F_c{}^b - \tfrac{1}{4}\eta^{ab}\,F_{cd}F^{cd}\ . \tag{1.150}$$

Alternatively, we can derive (1.150) from Noether's theorem, by recalling that energy and momentum are ultimately defined as Noether charges for the symmetry under Minkowski space-time translations. Applying the field-theoretical version of Noether's theorem to a translational symmetry in Minkowski space: $x^a \to x^a + \epsilon^a$ we find

$$\mathcal{L}[\Psi(x+a)] - \mathcal{L}[\Psi(x)] = \delta\mathcal{L} + \mathcal{O}(\epsilon^2) = \epsilon^a\partial_a\mathcal{L} + \mathcal{O}(\epsilon^2)\ ,$$

so that the Lagrangian is only invariant up to a total derivative. For a translation in the a-th direction we have $\partial_a\mathcal{L}[\Psi] = \partial_b f^b_{\xi_a}$, with $f^b_{\xi_a} = \delta^b_a\mathcal{L}$. The operator $\xi_a[\Psi]$ that generates the translation on the fields is found from

$$\delta\Psi = \epsilon^a\partial_a\Psi \equiv \epsilon^a\xi_a[\Psi]\ ,$$

and the resulting Noether current for a Lagrangian with at most quadratic dependence on derivatives is

$$(\mathcal{J}_a)^b = \xi_a[\Psi]\frac{\partial\mathcal{L}}{\partial(\partial_b\Psi)} - f^b_{\xi_a} = \partial_a\Psi\frac{\partial\mathcal{L}}{\partial(\partial_b\Psi)} - \delta^b_a\mathcal{L}\ . \tag{1.151}$$

We thus define the energy-momentum tensor as

$$T^{ab} = -\eta^{ac}\left[(\mathcal{J}_c)^b + \partial_e\,\Delta^{be}_c\right]\ , \tag{1.152}$$

where $\Delta[\Psi]$ is assumed to be a tensor *local* in the fields and antisymmetric in the upper indices: $\Delta_c^{be} = -\Delta_c^{eb}$. It is called the 'Noether ambiguity', and it is related to the fact that, just as Lagrangians are not completely determined by the equations of motion, local currents are not completely determined by the requirements of symmetry and conservation. Indeed, the Noether ambiguity does not change the local conservation of the current or the global charge obtained by integrating the time-like component over all space. The main use of the Noether ambiguity is its role in restoring the symmetry of the energy-momentum tensor T_{ab}, even if the set of currents $(\mathcal{J}_a)^b$ as defined by (1.151) may exhibit no particular symmetry properties in the indices a and b.

The simplest example is the case of a scalar field with Lagrangian

$$\mathcal{L}_\phi = -\tfrac{1}{2}\,(\partial\phi)^2 - V(\phi)\ ,$$

whose energy-momentum tensor is given by

$$T_{ab} = \partial_a\phi\,\partial_b\phi + \eta_{ab}\,\mathcal{L}_\phi\ , \tag{1.153}$$

coming out symmetric without any need for an improvement term. The situation is slightly different for the case of the Maxwell field, with Lagrangian

$$\mathcal{L}_{\text{Maxwell}} = -\tfrac{1}{4}\,F_{ab}F^{ab}\ .$$

The naive energy-momentum tensor for $\delta_a A_b = \partial_a A_b$ turns out to be given by

$$-\partial_a A_c\,F^c{}_b + \eta_{ab}\,\mathcal{L}_{\text{Maxwell}}\ ,$$

which is neither symmetric nor gauge-invariant. We can fix this by adding the improvement term $\partial_c(A_a F^c{}_b)$, which is a Noether ambiguity. Using then the equations of motion, $\partial_c F^{ca} = 0$, one obtains the final form of the energy momentum tensor

$$T^{ab}_{\text{Maxwell}} = -F^{ac}F_c{}^b + \eta^{ab}\,\mathcal{L}_{\text{Maxwell}}\ . \tag{1.154}$$

The improvement term is equivalent to the removal of the gauge ambiguity by the definition of an appropriate gauge-invariant translation of the Maxwell field. Since $\delta_a A_b = \partial_a A_b$ is not gauge invariant, we can 'improve' it by a simultaneous gauge transformation

$$\bar{\delta}_a A_b \equiv \delta_a A_b + \delta'_a A_b$$

with $\delta' A_b \equiv \epsilon^a \delta'_a A_b = \partial_b(-\epsilon^a A_a)$ an appropriate field-dependent gauge transformation. The new gauge-invariant translation is $\bar{\delta}_a A_b = F_{ab}$ which, if used in the Noether definition (1.152) without improvement term, yields the right gauge-invariant energy-momentum tensor (1.154).

The energy-momentum tensor of the Maxwell field has the important property of being *traceless*, $T^a{}_a = 0$. For the massive vector theory this trace is instead proportional to the photon mass:

$$T^{ab}_{\text{massive}} = T^{ab}_{\text{Maxwell}} - \tfrac{1}{2}\,m^2\,\eta^{ab}\,A_c\,A^c\ , \qquad \eta_{cd}\,T^{cd}_{\text{massive}} = -2m^2\,A_a\,A^a\ .$$

The trace of the energy-momentum tensor is in general sensitive to the mass terms in the problem. For example, for a system of particles, we already noticed that the trace of the energy-momentum tensor is controlled by the scalar density of 'rest mass':

$$T^a{}_a\Big|_{\text{particles}} = -J_m = -\sum_p \int d\tau_p\, m_p\, \delta^{(4)}(x - x(\tau_p))\ .$$

More generally, the trace of the energy-momentum tensor controls the behavior of the theory under *scale transformations* of the form $x \to \lambda\, x$. The associated Noether current is $J^a_{\rm scale} = x_b\, T^{ab}$, so that its local conservation follows from that of the energy-momentum tensor, plus the traceless condition $T^a{}_a = 0$. This means that a massless scalar field theory is *not* scale invariant except in two dimensions. [13]

Problem: Energy-momentum conservation and equations of motion

Show that the local conservation of the energy-momentum tensor, $\partial_a T^{ab} = 0$, implies the equations of motion for both scalar and electromagnetic fields, provided certain genericity conditions are met. Namely the equations of motion are satisfied except at points where $\partial_a \phi = 0$ in the scalar field case. For Maxwell fields one must require that the matrix F^a_b be invertible.

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[13] **Exercise:** Prove the statements in this paragraph.

Chapter 2

The Principle of Equivalence

It is certainly tempting to apply the previous machinery of relativistic field theory to the problem of gravitation. That is, to postulate some tensorial potential that generalizes the Newtonian gravitational potential.

It looks natural to follow the blueprint of Maxwell's theory by postulating a vector-like 'gravitational potential' g_a with field strength $G_{ab} = \partial_a g_b - \partial_b g_a$ and a time-like component proportional to the Newtonian potential $g^0 \sim \phi_{\rm N}$. A Lagrangian for such a theory would read

$$\mathcal{L}_{\rm G-vector} = -\frac{1}{16\pi G}\, G_{ab}\, G^{ab} - g_a\, J^a_m \ ,$$

where J^a_m is the current of mass. In particular, its time component is exactly the rest-mass density,

$$J^0_m = \sum_p m_p\, \delta^{(3)}(\vec{x} - \vec{x}_p(t))$$

for a system of free particles. So this theory has the peculiarity that gravity only couples universally to rest masses, rather than energies. This would imply, for example, that a Helium atom would gravitate considerably less than a Bottom quark, since very little of the He mass comes from quark 'rest' mass. In any case, a more urgent problem of the vector theory is the fact that equal-sign masses should repel one another. This can be momentarily fixed by changing the sign of the Newton constant, $G \to -G$, but then one finds the gravitational energy density to be *negative definite* [1]

$$T^{00}_{\rm grav} = -\frac{1}{8\pi |G|}\left(\vec{E}^2_G + \vec{B}^2_G\right) < 0 \ .$$

A much better attempt at a relativistic generalization of Newton's theory would promote $\phi_{\rm N}$ to a fully fledged Lorentz scalar field $g(x)$ with Lagrangian

$$\mathcal{L}_{\rm G-scalar} = -\frac{1}{8\pi G}(\partial g)^2 - g\, J_m \ ,$$

where now J_m is the Lorentz-invariant mass 'density'. For a system of free particles we have

$$J_m = \sum_p \int d\tau_p\, m_p\, \delta^{(4)}(x - x_p(\tau_p)) \ .$$

[1] These problems were known to Maxwell himself, who tried to construct such a theory of gravitation.

In fact, as shown in the discussion after (1.96), such density is essentially the trace of the energy-momentum tensor: $J_m = -T^a_a$. This means that such scalar gravity field, while leading to attractive forces, fails to couple to electromagnetic fields and for a fluid of density ρ and pressure p, it couples to the net combination $\rho - 3p$, which vanishes for ultrarelativistic fluids (radiation). Therefore, while the scalar gravity theory is logically consistent, and was indeed seriously considered by many authors, including the likes of Poincaré, Abraham, Nördstrom and Einstein, it actually fails to incorporate the gravitation of 'internal energies' in a satisfactory way.

If we insist that gravity should couple to total energy, rather than rest mass or a combination of energy and pressure, we are naturally led to use the energy-momentum tensor in the role of 'matter current', since it is T_{ab} the object that encapsulates the local conservation of energy in SR. In this way, one is naturally led to a gravitational potential of tensor nature, i.e. a minimal coupling of the form

$$S_{\rm int} \sim \int d^4x \; h_{ab}\, T^{ab} \,, \tag{2.1}$$

between T_{ab} and a symmetry rank-two gravitational potential h_{ab}. The local conservation of the energy-momentum tensor, $\partial_a T^{ab} = 0$, implies that this Lagrangian interaction is 'gauge invariant' under the transformations

$$h_{ab} \to h_{ab} + \partial_a f_b + \partial_b f_a \,, \tag{2.2}$$

with $f_a(x)$ an arbitrary four-vector function. Hence, we are led to the development of a symmetric two-index generalization of Maxwell's theory, the Fierz–Pauli theory, coupled to the energy-momentum tensor through (2.1).

However, such a program quickly becomes treacherous. The reason is that any field coupling to energy must necessarily couple to itself, resulting in a nonlinear theory. The gist of the argument is as follows. The Fierz–Pauli Lagrangian contributes a term to the total energy-momentum tensor quadratic in the h-field, i.e. $T_{(h)} \sim (\partial h)(\partial h)$, suppressing the detailed index structure. Since h_{ab} must couple to all forms of energy, including the energy contained in the h-field itself, we are led to write (2.1) in the 'improved' form

$$S_{\rm int} \sim \int h_{ab}(T^{ab}_{\rm matter} + T^{ab}_{(h)}) \,,$$

which includes a cubic interaction term in the h_{ab} field. In turn, taking this term into account in the computation of the h-field energy momentum tensor, we will obtain a similar cubic term in $T^{ab}_{(h)}$, which induces a quartic term when fed into the interaction term, and so on... Hence, we are led to an iterative construction of a full non-linear theory. Feynman and others were able to show that this iterative construction indeed converges to Einstein's theory of general relativity. We shall return to this issue later on, but for the time being we will be following another route.

2.1 From Galileo to Einstein

Einstein chose to develop the theory of gravity on top of the Equivalence Principle (EP) between gravitation and inertia. A principle (going back to the tales of Galileo at the Pisa tower) stating, in its weakest version, that the gravitational 'charge' of particles is universally proportional to its inertial mass.

When a Lagrangian shows some constraint in its couplings to a great accuracy (recall Eötvös measurements), it is natural to try representing this constraint as the result of some symmetry. In the case at hand, this is possible in a rather interesting way. The Newtonian equation of motion for a particle of mass m on some gravitational potential is

$$m \frac{d\vec{v}}{dt} = -m\,\vec{\partial}\,\phi_{\rm N} \;, \tag{2.3}$$

and the EP simply means that m drops from this equation, so that trajectories are intrinsically determined by initial conditions, independently of the particular mass considered. This suggest that these trajectories have an *intrinsic* geometrical meaning. Consider, for example, a constant field with uniform gravity acceleration $\vec{g}$,

$$\frac{d^2\vec{x}}{dt^2} = \vec{g} \;. \tag{2.4}$$

Now, in a reference frame in free fall,

$$\vec{\xi} = \vec{x} - \tfrac{1}{2}\,\vec{g}\,t^2 \tag{2.5}$$

we have $d^2\vec{\xi}/dt^2 = 0$, recovering the familiar feature that the gravitational field disappears in free fall. However, this was ultimately possible due to the particle mass dropping from the equations. Hence, we can represent the equivalence principle as a symmetry statement by saying that all systems in free fall are inertial, in the sense that they feel no gravitational forces.

Conversely, a constant acceleration of magnitude $\vec{g}$ is indistinguishable from a constant gravitational field of the same intensity. In general, this simulation of gravitational fields by accelerations is only possible locally. We shall refer to such gravitational fields that can disappear globally by entering free fall as 'fictitious' gravitational fields. In 'true' ones, one can still eliminate the gravitational field, but only locally. If the gravitational acceleration at the spacetime point P is given by $\vec{g}_P = -\vec{\partial}\phi_{\rm N}(P)$, the frame transformation $\vec{\xi} = \vec{x} - \frac{1}{2}\,\vec{g}_P\,t^2$ gives

$$\frac{d^2\vec{\xi}}{dt^2} = \vec{g}(\vec{x},t) - \vec{g}_P \;, \tag{2.6}$$

so that the acceleration vanishes at the point P. Away from the free-fall point P, there is a residual 'tidal' acceleration, proportional to the first derivatives of the gravitational force, i.e. the second derivative of the potential. We can define the 'tidal tensor' for static fields as

$$R_{ij} \equiv \partial_i \partial_j \phi_{\rm N} \;, \tag{2.7}$$

which provides a local diagnostic of whether a gravitational field is 'true' or 'fictitious', namely the tidal tensor of a *static* field vanishes in a region of space if and only if the field is fictitious. to see this, notice that $R_{ij} = 0$ on a finite domain implies that $\phi_{\rm N}$ is linear in the spatial coordinates, $\phi_{\rm N}(\vec{x}) = a + \vec{b}\cdot\vec{x}$. In this case the gravitational force is $\vec{g}(\vec{x}) = -\vec{b}$, a uniform field, which may be removed by the free-fall frame $\vec{\xi} = \vec{x} + \frac{1}{2}\,\vec{b}\,t^2$.

In the relativistic theory, the EP admits the same formulation: *gravitational fields can be locally eliminated by entering free fall*, except that this time it is Special Relativity (SR), rather than Newtonian particle mechanics, what is assumed to be valid at the freely falling frames. There are at least two novel relativistic effects with far-reaching consequences in relation to the EP. The first is the existence of a novel kind of time dilation effects, associated to gravitational fields, and second is the occurrence of non-euclidean geometry. Both effects can be illustrated in a simple heuristic way by studying appropriate 'fictitious' gravitational fields.

The relativistic elevator and Rindler space

We saw that, according to the EP, uniform and constant gravitational fields can be manufactured by constructing an elevator undergoing constant acceleration. In the relativistic theory, such an elevator would reach eventually the speed of light, so it is not obvious what is the relativistic concept associated to 'constant acceleration'. Given a timelike trajectory $x(\tau)$, with four-velocity

$$u^a = \frac{dx^a}{d\tau} \;, \qquad u^2 = -1 \;, \tag{2.8}$$

we define the acceleration four-vector by

$$a^b = \frac{du^b}{d\tau} \;, \qquad a^2 = a_b\, a^b \;. \tag{2.9}$$

Taking the derivative of $u^2 = -1$ we find that acceleration and velocity are orthogonal $a^b\, u_b = 0$. In an inertial frame x'^a momentarily at rest with respect to the accelerated trajectory we have $u = (1, \vec{0})$ and the orthogonality implies $a = (0, \vec{a}')$, $\vec{a}' \cdot \vec{a}' = a^2$. Hence, we shall define constant acceleration as $a^2 = g^2$ with g a pure number, constant in time.

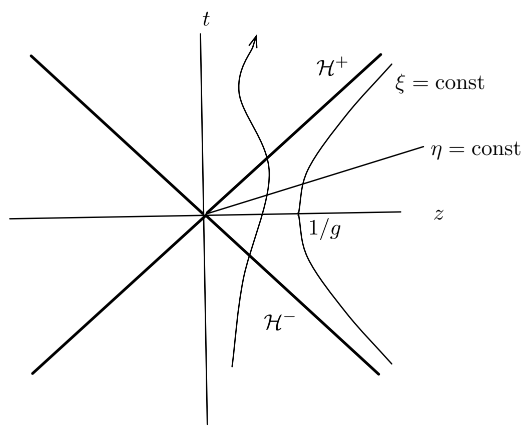

Figure 2.1: *The so-called Rindler wedge is the region* $0 < |t| < z$. *Uniformly accelerated observers follow the* $\xi =$ constant *hyperbolic trajectories, which degenerate into the past and future particle horizons* $\mathcal{H}^\pm$. *We also show the timelike trajectory of another observer. The part of her history to the future of* $\mathcal{H}^+$ *is completely inaccesible to the accelerated observer, whereas her crossing of the horizon takes an infinite subjective time for him.*

Now we go back to the 'laboratory' frame and take an accelerated rectilinear motion in the z-direction, $\vec{a} = d^2\vec{x}/d\tau^2 = (0, 0, a^z)$. We take $a^z > 0$ by convention. From the three algebraic equations

$$u^2 = -1 \;, \qquad u \cdot a = 0 \;, \qquad a^2 = g^2$$

we derive

$$a^0 = \frac{du^0}{d\tau} = g\, u^z \;, \qquad a^z = \frac{du^z}{d\tau} = g\, u^0 \;. \tag{2.10}$$

The solution of these ordinary differential equations is

$$(u^0 \pm u^z) = (u^0 \pm u^z)(0)\; e^{\pm g\tau} \;.$$

Choosing now $u^0(0) = 1$ and $u^z(0) = 0$ (zero velocity at $\tau = 0$) we obtain the parametric form of the trajectory,

$$t - t(0) = \frac{1}{g} \sinh(g\tau) , \qquad z - z(0) = \frac{1}{g} (\cosh(g\tau) - 1) , \tag{2.11}$$

and choosing the origin or coordinates so that $t(0) = 0, z(0) = 1/g$ we see that 'constantly accelerated' trajectories lie on the hyperbolae

$$z^2 - t^2 = \frac{1}{g^2} . \tag{2.12}$$

Now consider a family of accelerated observers running on hyperbolae $\xi =$ constant, according to the change of coordinates

$$z \pm t = \frac{1}{g} e^{g\xi} e^{\pm g\eta} . \tag{2.13}$$

Each such observer feels a constant acceleration $g_\xi = g e^{-g\xi}$. If we arrange all of them to pass through $t = 0$ simultaneously in the local frame of any one of them, then they will always stay on the lines $\eta =$ constant. Any such line is related to the $t = 0$ line by a Lorentz transformation, so that the relative rest condition of the accelerated observers is maintained in time. Hence, the constructed family of accelerated observers furnishes a 'relativistic elevator'.

The proper time of each observer can be computed from the relation

$$t(\tau_\xi) = \frac{1}{g} e^{g\xi} \sinh(g\eta(\tau_\xi)) = \frac{1}{g_\xi} \sinh(g_\xi \tau_\xi)$$

which results in $g\eta(\tau_\xi) = g_\xi \tau_\xi$, or

$$\tau_\xi = \eta\, e^{g\xi} . \tag{2.14}$$

Using this result, we find the ratio of proper times at different local accelerations given by

$$\frac{\Delta\tau(\xi)}{\Delta\tau(\xi')} = e^{g(\xi - \xi')} = \frac{g_{\xi'}}{g_\xi} . \tag{2.15}$$

According to the EP, this means that clocks run slower the more intense the gravitational field. This effect is one of the classic predictions of GR, called 'gravitational red shift'.

The space (η, ξ) with arbitrary real values of the coordinates is called *Rindler space*. It covers the wedge $0 < |t| < z$ of Minkowski space, and represents the part of it accesible to accelerated observers moving with positive uniform acceleration in the z-direction. Locally, Rindler space has identical geometry to Minkowski space, although the metric in Rindler coordinates reads

$$ds^2 = -dt^2 + dz^2 + \ldots = e^{2g\xi}(-d\eta^2 + d\xi^2) + \ldots \tag{2.16}$$

where the dots stand for the spectator directions. We see that the two-dimensional section where the accelerations take place amounts to a local conformal transformation of lenght and time standards.

The future boundary of the Rindler wedge $\mathcal{H}^+$ is called the future *horizon*. Beyond this horizon, we find all the events that are causally disconnected from any accelerated observer in Rindler space. An observer at rest at $z > 1/g$ will eventually be passed by our accelerated elevator and by the time her worldline passes through the curve $z = t$, any subsequent signal that might emit will never reach the elevator. At the same time, the observer on the elevator will never see her cross the horizon, her motions being constantly elongated in time by an infinite time dilation factor.

There is an analogous notion of 'past horizon', $\mathcal{H}^-$, which is defined by the time-reversal of the future horizon. Such notions of causal horizons feature prominently in the theory of black holes and also on the global properties of cosmological spacetimes.

Rotation and non-euclidean geometry

Consider two frames of reference in SR. K is inertial and K' rotates at angular velocity Ω. Imagine setting up an operational measure of π, i.e. we draw a circumference of radius R and measure its length. The ratio being 2π in the inertial system K. An observer on K' would use measuring rods that, according to K, suffer Lorentz contraction when laid along the circumference, but remain intact when measuring the radius. Hence, the observer on K' will measure a larger ratio of circumference to radius, by a factor of the Lorentz contraction $\sqrt{1-v^2} = \sqrt{1-\Omega^2 R^2}$. This leads to a 'measurement' of π as

$$\pi_{K'} = \frac{\pi}{\sqrt{1-\Omega^2 R^2}} > \pi \ . \tag{2.17}$$

This is characteristic of spaces of *negative* curvature. According to the EP, there is no local difference between accelerated frames and true gravitational fields, the two differing only at the level of global properties. Hence we should expect non-euclidean geometry to arise in true gravitational fields as a general rule. This simple *gedanken experiment* was pivotal in Einstein's own heuristic path to GR.

2.2 Riemannian geometry and the Equivalence Principle

We are now ready to use the EP as a guiding principle in our construction of the relativistic theory of gravitation. Let us consider a spacetime X^{3+1} with a gravitational field (G-field), and send a local probe in the form of a particle of mass m, moving under the sole influence of gravity. We shall assume the 'probe' approximation, consisting in neglecting the back-reaction of the probe particle on the background G-field. Denoting $M_{\rm G-field}$ the effective mass generating the gravitational field, the probe approximation is expected to hold in the limit $m \ll M_{\rm G-field}$.

We characterize the equivalence principle in a pragmatic way as follows: Around any point $P \in X^{3+1}$, we can erect a free-fall frame through P, denoted ξ^a_P, with the property that special relativity is locally valid in a small neighborhood of P. The whole spacetime X^{3+1} can be covered by local free-fall frames, which we denote as *frefos*, to indicate that they are associated to freely falling observers. We can also refer all phenomena to an external frame, x^α, which we denote as *fido*, to signify the fact that it may be associated to a single fiducial observer. Notice that no restrictions are placed on the properties of the *fido*, and no canonical choice of *frefos* throughout spacetime exists in general, if only because any *frefo* can be rotated by a Lorentz transformation and remain a valid *frefo*, and this can be done independently at each point in spacetime.

2.2.1 Particle probes

Let us consider a freely-falling particle with trajectory γ, parametrized in the *fido* frame as the functions $x^\alpha(\sigma)$. We can partition the path as $\gamma = \cup_P \gamma_P$, with γ_P small 'pieces' around points P in the path, each of them parametrized in the *frefos* by the functions $\xi^a_P(\sigma)$. We then use the EP to define the particle action applying SR on each piece, and subsequently taking the limit of a fine partition:

$$S[\gamma] = \lim_{\{P\}\to\gamma} \sum_P \Delta S[\gamma_P] \;, \tag{2.18}$$

where

$$\Delta S[\gamma_P] \approx -m \Delta\tau_P = -m\Delta\sigma_P \sqrt{-\eta_{ab} \frac{d\xi^a_P}{d\sigma}\frac{d\xi^b_P}{d\sigma}} \;, \tag{2.19}$$

which gives in the limit

$$S[\gamma] = -m\int_\gamma d\tau = -m\int_\gamma d\sigma \sqrt{-\eta_{ab}\frac{d\xi^a_{P(\sigma)}}{d\sigma}\frac{d\xi^b_{P(\sigma)}}{d\sigma}} \;. \tag{2.20}$$

The metric

Hence the EP establishes the particle action as proportional to the elapsed proper time, just as in ordinary SR, and provides a way to compute it, given a family of *frefos* along γ. This parametrization serves the purpose of *defining* the action through the EP, but it is not particularly 'user-frendly', since it contains a large amount of redundant information, related to the arbitrary choice of *frefos* at each point in γ. A more convenient procedure is to relate the action to the *fido* parametrization in terms of the functions $x^\alpha(\sigma)$. We can do this by using the chain rule to obtain

$$\frac{d\xi^a_P}{d\sigma} = \frac{\partial \xi^a_P}{\partial x^\alpha}\frac{dx^\alpha}{d\sigma} \;. \tag{2.21}$$

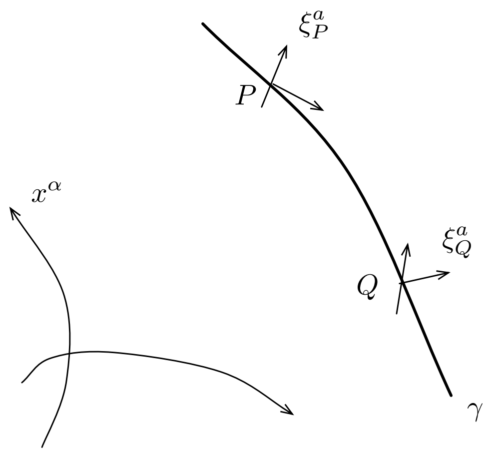

Figure 2.2: *A particle path γ with two frefos at points P and Q and an external fido frame x^α.*

Defining the *tetrads* as the matrix of partial derivatives

$$e^a_\alpha(P) \equiv \frac{\partial \xi^a_P}{\partial x^\alpha}\Big|_P \,, \tag{2.22}$$

we may write

$$\eta_{ab} \frac{d\xi^a_{P(\sigma)}}{d\sigma} \frac{d\xi^b_{P(\sigma)}}{d\sigma} = g_{\alpha\beta}(x(\sigma)) \, \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma}$$

with the so-called *metric* defined point by point as a sort of 'Lorentz square' of the tetrads:

$$g_{\alpha\beta}(P) = \eta_{ab} \, e^a_\alpha(P) \, e^b_\beta(P) \,. \tag{2.23}$$

Hence, we finally obtain the particle action in *fido* parametrization:

$$S[\gamma] = -m \int_\gamma d\tau = -m \int_\gamma d\sigma \sqrt{-g_{\alpha\beta}(x(\sigma)) \, \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma}} \,. \tag{2.24}$$

We conclude that all the information about the gravitational field, as far as particle probes is concerned, is encoded in the ten metric functions $g_{\alpha\beta}$, and the G-field can be given a geometrical interpretation as determining proper times and proper distances according to the formal expression

$$ds^2 = -d\tau^2 = g_{\alpha\beta} \, dx^\alpha dx^\beta \,,$$

which characterizes the notion of a *Riemannian manifold* in the mathematical literature. The 'manifold' character is ensured by the fact that we did not specify any particular *fido* frame, so if we change to a new frame $y^\mu = y^\mu(x)$, the metric changes according to the map

$$g^{(y)}_{\mu\nu}(y) = \frac{\partial x^\alpha}{\partial y^\mu} \frac{\partial x^\beta}{\partial y^\nu} \, g^{(x)}_{\alpha\beta}(x) \,, \tag{2.25}$$

i.e. as a generalized second-rank tensor.

The connection

We can actually go further and derive the equation of motion stemming from (2.24) by demanding the action to be a local minimum with respect to variations of the trajectory $x^\alpha(\sigma) \to x^\alpha(\sigma) + \delta x^\alpha(\sigma)$. We find

$$0 = \delta S_{\rm P} = \int \frac{d\sigma}{2\sqrt{-g_{\alpha\beta}\,\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma}}}\,\delta\left(g_{\alpha\beta}\,\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma}\right) \ .$$

We may now take advantage of the reparametrization invariance to restore the proper time parameter $\sigma \to \tau$ to remove the square root in the denominator and write

$$0 = \int d\tau \left(\tfrac{1}{2}\,\partial_\mu g_{\alpha\beta}\,\dot{x}^\alpha \dot{x}^\beta\,\delta x^\mu + g_{\alpha\beta}\,\dot{x}^\alpha \frac{d}{d\tau}\delta x^\beta\right) \ .$$

Reshuffling the indices, integrating by parts and multiplying the resulting equation of motion by the inverse of the metric matrix we obtain the final equation of motion

$$0 = \frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \ , \tag{2.26}$$

where the Γ coefficients are called the Christoffel symbols and determined in terms of first derivatives of the metric:

$$\Gamma^\alpha_{\mu\nu} = \tfrac{1}{2}\,g^{\alpha\beta}\left(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}\right) \ , \tag{2.27}$$

with $g^{\alpha\beta} = (g^{-1})^{\alpha\beta}$ a standard notation for the inverse matrix of metric functions.

Equation (2.27) allows us to give a more precise specification of the EP, in terms of the Riemannian metric. Since this equation is valid in any frame, it is in particular valid in the *frefo* frame around P, given by the coordinates ξ^a_P. However, the EP implies that such a equation should be given by the SR one, i.e.

$$\frac{d^2\xi^a_P}{d\tau^2}\bigg|_P = 0 \ , \tag{2.28}$$

and we conclude that the *frefo* at P is defined as such coordinate system on which the metric, transformed according to (2.25), satisfies

$$g^{(\xi)}_{ab}(P) = \eta_{ab} \ , \qquad \partial_c\, g^{(\xi)}_{ab}(P) = 0 \ , \tag{2.29}$$

the second condition being equivalent to $\Gamma^{(\xi)\,a}_{\ \ bc}(P) = 0$.

It is interesting to rederive (2.26) directly from (2.28) since it gives an alternative perspective on the Christoffel symbols. Using

$$\frac{d\xi^a_P}{d\tau} = \frac{\partial \xi^a_P}{\partial x^\alpha}\frac{dx^\alpha}{d\tau} \ , \qquad \frac{d}{d\tau} = \frac{dx^\mu}{d\tau}\frac{\partial}{\partial x^\mu} \tag{2.30}$$

we can rewrite (2.28) as

$$0 = \frac{\partial \xi^a_P}{\partial x^\alpha}\left(\frac{d^2x^\alpha}{d\tau^2} + \frac{\partial x^\alpha}{\partial \xi^c_P}\frac{\partial^2 \xi^c_P}{\partial x^\mu \partial x^\nu}\,\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right) \ ,$$

implying that the Christoffel symbols can be written alternatively in terms of the *frefo* data at P as determined by the second derivatives of the free-fall coordinates:

$$\Gamma^{\alpha}_{\mu\nu}(P) = \left(\frac{\partial x^{\alpha}}{\partial \xi^{a}_{P}} \frac{\partial^{2}\xi^{a}_{P}}{\partial x^{\mu}\partial x^{\nu}}\right)\Big|_{P} = e^{\alpha}_{a}(P)\, \frac{\partial^{2}\xi^{a}_{P}}{\partial x^{\mu}\partial x^{\nu}}\Big|_{P} \,. \tag{2.31}$$

Written in this form, the 'force' coefficients are known as the *affine connection*, and the above expression in terms of the metric derivatives is also referred to as the Levi–Civita connection.[2] Both notions of connection are equivalent in our presentation of the gravitational theory, although they may be different in more general gravitational theories.

Local versus global flatness

The condition (2.29) gives a mathematically precise statement of the EP. It is interesting to ask whether the required *frefo* can always be introduced at an arbitrary point, for a *generic* $g_{\alpha\beta}$, or some restriction must be put on the metric of spacetime. To settle this question, consider constructing the frame ξ^{a}_{P} in a Taylor expansion in terms of the *fido* frame x^{α}. Without loss of generality we may define the additive normalization of the coordinates so that $x^{\alpha}(P) = \xi^{a}_{P}(P) = 0$. Then we can expand the function $x^{\alpha}(\xi^{a}_{P})$ as

$$x^{\alpha} = A^{\alpha}_{a}\,\xi^{a} + \frac{1}{2}\,B^{\alpha}_{ab}\,\xi^{a}\,\xi^{b} + \frac{1}{6}C^{\alpha}_{abc}\,\xi^{a}\,\xi^{b}\,\xi^{c} + \mathcal{O}(\xi^{4}) \,,$$

where the A, B, C are collections of constant coefficients arising from the Taylor expansion. Using now (2.25) we find for the metric in the *frefo* frame

$$g_{ab}(P) = A^{\alpha}_{a}\,A^{\beta}_{b}\,g_{\alpha\beta}(P) \,,$$

so that $g_{ab}(P) = \eta_{ab}$ by simply choosing the A^{α}_{a} to diagonalize and rescale appropriately the original metric matrix at point P. There is in fact an ambiguity by multiplication of A^{α}_{a} by a Lorentz transformation, since η_{ab} is invariant under those. Computing now the Christoffel symbols in the ξ-frame, using for example (2.31), we find

$$\Gamma^{a}_{bc}(P) = (A^{-1})^{a}_{\alpha}\,A^{\beta}_{b}\,A^{\gamma}_{c}\,\Gamma^{\alpha}_{\beta\gamma}(P) + (A^{-1})^{a}_{\alpha}\,B^{\alpha}_{bc} \,,$$

so that $\Gamma^{a}_{bc}(P) = 0$ can be readily arranged by setting

$$B^{\alpha}_{bc} = -A^{\beta}_{b}\,A^{\gamma}_{c}\,\Gamma^{\alpha}_{\beta\gamma}(P) \,.$$

Hence, we conclude that the EP, as given by (2.29), can always be reached in an appropriate *frefo*. On the other hand, the second derivatives of the metric at P, or equivalently the first derivatives of the connection, will be determined by the 80 independent coefficients C^{α}_{abc}. All in all there are 100 independent components in the second derivatives of the metric $\partial_{\alpha}\partial_{\beta}g_{\mu\nu}(P)$, so that we cannot remove them by a clever choice of *frefo*. The offset of 20 components gives the degrees of freedom of the *curvature* in a four-dimensional spacetime. The argument just given here is known in the mathematical literature as the 'local flatness theorem', since it specifies that any Riemannian space is 'flat' to first order in a local expansion, with 'intrinsic' curvature effects arising only in second order in this local expansion.

[2] Notice that, in (2.31) we evaluate the right-hand side at P only after the second derivative has been taken. In particular, the second derivative term is *not equal* to the first derivative of the tetrad field.

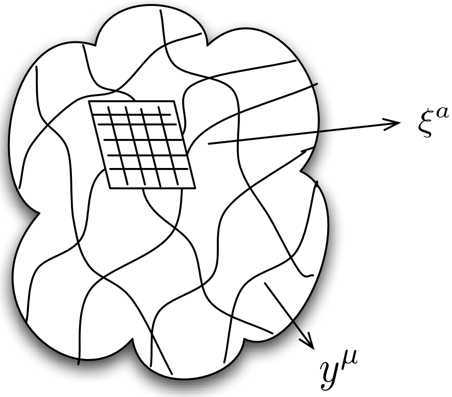

Figure 2.3: *A general gravitational field is characterized by a general metric, reducing to Minkowski space only locally around a point, parametrized by a frame* ξ^a*. This is the geometrical set up for a space with Riemannian geometry.*

Fictitious gravitational fields are precisely those for which the second derivatives of the metric vanish identically in a finite region, once an appropriate *frefo* has ben chosen. It follows that for these fields there is a *frefo* for which the metric equals the Minkowski metric in the whole region. In other words, fictitious fields are nothing but Minkowski space, perhaps parametrized in a general *fido* frame.

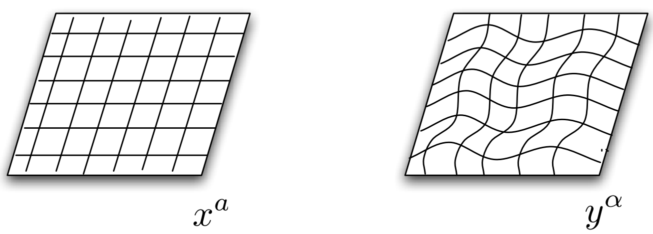

Figure 2.4: *An arbitrary 'fictitious' field is just Minkowski spacetime (left) disguised in a general system of curvilinear and/or accelerated coordinates (right).*

2.2.2 Gravitational forces

The structure of the particle equation of motion (2.26) presents gravitational forces as proportional to Γ, which appear unified with 'inertial' forces, in agreement with the spirit of the EP:

$$\frac{d^2x^\mu}{d\tau^2} = F_g^\mu = -\Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau};.$$

Furthermore, the metric functions acquire the interpretation of 'gravitational potentials', since their first derivatives determine the 'forces'. We can make these intuitions more precise by taking the weak-field limit and further the Newtonian limit.

Weak fields

In keeping with the idea that gravitational physics amounts to replacing the Minkowski metric η_{ab} by a general Riemannian metric $g_{\alpha\beta}$ we may study the case where gravitational effects are small, so that we write $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$, with $|h_{\alpha\beta}| \ll 1$, a condition that requires restricting the reference frames, since we may induce large components of the metric by simply using a frame with large accelerations. Hence, we assume that an 'almost' *frefo* frame extends over the region of interest and we only allow further coordinate transformations that maintain the 'small field' property. Writing those transformations as $x^\alpha \to x'^\alpha = x^\alpha - \varepsilon^\alpha(x)$, the metric transformations (2.25) imply

$$\eta_{\alpha\beta} + h_{\alpha\beta}(x) \to \eta_{\alpha\beta} + h'_{\alpha\beta}(x') = \frac{\partial x^\gamma}{\partial x'^\alpha}\frac{\partial x^\delta}{\partial x'^\beta}\left(\eta_{\gamma\delta} + h_{\gamma\delta}(x)\right) .$$

Upon explicit computation to leading order in ε^α one finds that the *functional* shift in $h_{\alpha\beta}$ is given by

$$h'_{\alpha\beta}(x) - h_{\alpha\beta}(x) = \partial_\alpha \varepsilon_\beta(x) + \partial_\beta \varepsilon_\alpha(x) + \mathcal{O}(\varepsilon^2) . \qquad (2.32)$$

Therefore, the condition on the allowed 'weak-diffeomorphisms' is $|\partial_\alpha \varepsilon_\beta| \ll 1$. Incidentally, the weak-diffeormorphisms act on the weak gravitational field $h_{\alpha\beta}$ as a gauge symmetry for a Lorentz-covariant symmetric tensor field. Expanding now the particle action to leading order in $h_{\alpha\beta}$ one finds

$$-m\int_\gamma d\sigma\sqrt{-(\eta_{\alpha\beta}+h_{\alpha\beta})\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma}} = S[\gamma]_{\text{Minkowski}} + \tfrac{1}{2}\, m\int d\tau\, h_{\alpha\beta}\,\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} + \mathcal{O}(h^2) , \qquad (2.33)$$

where τ in the last term is defined as the proper time with respect to the Minkowski metric. The term linear in $h_{\alpha\beta}$ can be rewritten as

$$\tfrac{1}{2}\int d^4x\, h_{\alpha\beta}\, T^{\alpha\beta} ,$$

where $T^{\alpha\beta} = m\int d\tau \dot{x}^\alpha\, \dot{x}^\beta\, \delta^{(4)}(x - x(\tau))$ is the standard definition of the particle's energy momentum tensor in SR (cf. (2.1)). Hence, we conclude that the leading perturbative coupling of gravity, as dictated by the EP, conforms to the general arguments based on SR. However, the EP is much more powerful, giving a precise resummation of powers of $h_{\alpha\beta}$ in the matter-gravity coupling.

Newtonian limit

We now sharpen the interpretation of Γ as gravitational forces by taking the non-relativistic limit of the equation of motion (2.26). It is useful to transform it from the proper-time parametrization to coordinate time parametrization. Using the chain rule one finds

$$\frac{d^2x^\alpha}{dt^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} = h(t)\frac{dx^\alpha}{dt} , \qquad (2.34)$$

with

$$h(t) = -\frac{d^2t/d\tau^2}{(dt/d\tau)^2} \; .$$

We may consider now the spatial components of (2.34) in the weak-field regime, for *stationary* fields, $\partial_t h_{\alpha\beta} \ll \partial_i h_{\alpha\beta}$, and in the non-relativistic limit, $v^i = dx^i/dt \ll 1$. Keeping terms up to first order in velocities we obtain

$$\frac{dv_i}{dt} \approx \tfrac{1}{2}\, \partial_i\, h_{00} - \sum_j v_j (\partial_j h_{0i} - \partial_i h_{0j}) \; . \tag{2.35}$$

The first term is the standard gradient gravitational force for a field with potential

$$\phi_{\rm N} \approx -\tfrac{1}{2}\, h_{00} \; , \tag{2.36}$$

whereas the second term has the form of a Coriolis force. To see this, notice that we may define the effective 'angular velocity' vector

$$\vec{\Omega} = \tfrac{1}{2}\, \vec{\partial} \times \vec{h} \; , \tag{2.37}$$

where $\vec{h} = (h_{0i})$. The effective Newton equation takes then the form

$$\frac{d\vec{v}}{dt} \approx -\vec{\partial}\, \phi_{\rm N} + 2\, \vec{v} \times \vec{\Omega} \; , \tag{2.38}$$

which contains the potential and Coriolis terms. Hence, we confirm that the equivalence principle *unifies* what in Newtonian theory is regarded as separate 'inertial' and 'gravitational' forces.

2.2.3 Diff tensor calculus

As stated above, the present construction of the metric field, based on the EP, produces in a natural way a redundant description with respect to arbitrary *fido* reparametrizations, $x \to y(x)$, i.e. the so-called diffeomorphisms of the spacetime onto itself, Diff(X^{3+1}). The action on the metric is such that it is formally a tensor in the sense

$$g_{\alpha\beta}\, dx^\alpha \otimes dx^\beta = \text{Diff} \;\; \text{invariant} \; , \tag{2.39}$$

where

$$dx^\alpha = \frac{\partial x^\alpha}{\partial y^\mu}\, dy^\mu \; ,$$

and the matrix of partial derivatives is a general invertible real matrix, i.e. a member of the $GL(4, \mathbf{R})$ group, specified independently point by point on the manifold X^{3+1}. These properties of the metric are actually inherited from similar properties of the tetrads e^a_α, which behave as mixed tensors with respect to the Lorentz group (latin index) and the Diff group (greek index). It follows that the obvious operations of raising and lowering indices can be defined in terms of the respective metrics, maintaining the corresponding tensorial properties:

$$e_{a\alpha} \equiv \eta_{ab}\, e^b_\alpha \; , \qquad e^{a\alpha} \equiv g^{\alpha\beta} e^a_\beta \; , \qquad e^\alpha_a \equiv \eta_{ab}\, g^{\alpha\beta}\, e^b_\beta \; ,$$

where $g^{\alpha\beta} g_{\beta\gamma} = \delta^\alpha_\gamma$ and the 'inverse tetrad' satisfies $e^\alpha_a e^a_\beta = \delta^\alpha_\beta$ and $e^\alpha_a e^b_\alpha = \delta^b_a$.

The notion of Diff tensor introduced for the metric in (2.39) can be generalized to any Lorentz tensor, $T^{a\dots}_{b\dots}$, defined point by point through a family of *frefos* covering the spacetime manifold. Namely, the object

$$T^{\alpha_1\dots\alpha_p}{}_{\beta_1\dots\beta_q} \equiv e^{\alpha_1}_{a_1}\cdots e^{\alpha_p}_{a_p}\, e^{b_1}_{\beta_1}\cdots e^{b_q}_{\beta_q}\, T^{a_1\dots a_p}{}_{b_1\dots b_q}\ , \tag{2.40}$$

behaves as a Diff tensor according to the positions of the respective indices:

$$T = T^{\alpha_1\dots\alpha_p}{}_{\beta_1\dots\beta_q}\, dx^{\beta_1}\otimes\dots\otimes dx^{\beta_q}\otimes\partial_{\alpha_1}\otimes\dots\otimes\partial_{\alpha_p} = \text{Diff invariant}\ . \tag{2.41}$$

The metric itself is a particular example of this rule, as well as the particle four-velocity in the *fido* frame

$$u^\alpha \equiv \frac{dx^\alpha}{d\tau} = e^\alpha_a\, u^a_P = e^\alpha_a\, \frac{d\xi^a_P}{d\tau}\ . \tag{2.42}$$

Covariant derivatives

Using the second identity in (2.30) and this definition of the four-velocity we may rewrite the particle equation of motion (2.26) as

$$\nabla_u\, u^\alpha = 0\ , \tag{2.43}$$

where we define the *covariant derivative* in the direction u^α by the relation

$$\nabla_u V^\mu \equiv u^\alpha \nabla_\alpha V^\mu \equiv u^\alpha\left(\partial_\alpha V^\mu + \Gamma^\mu_{\alpha\beta}V^\beta\right)\ . \tag{2.44}$$

The so-defined covariant derivative behaves as a Diff vector, when acting on Diff vectors. In fact, it conforms to the general definition of Diff tensors (2.40) since

$$\nabla_\mu V^\nu = e^a_\mu\, e^\nu_b\, \partial_a\, V^b\ , \tag{2.45}$$

as can be easily concluded by explicit computation, using the *affine* form of the connection coefficients (2.31). With the same effort, one can define the covariant derivative acting on a ‘covariant’ vector field:

$$\nabla_\mu U_\nu \equiv e^a_\mu\, e^b_\nu\, \partial_a\, U_b\ ,$$

which becomes, upon explicit computation, equal to

$$\nabla_\mu U_\nu = \partial_\mu U_\nu - \Gamma^\alpha_{\mu\nu} U_\alpha\ . \tag{2.46}$$

Entirely analogous manipulations show that the covariant derivative acting on a general tensor takes the explicit form

$$\nabla_\mu T^{\alpha\dots}{}_{\beta\dots} = \partial_\mu T^{\alpha\dots}{}_{\beta\dots} + \Gamma^\alpha_{\mu\rho}\, T^{\rho\dots}{}_{\beta\dots} + \dots - \Gamma^\rho_{\mu\beta}\, T^{\alpha\dots}{}_{\rho\dots} - \dots\ . \tag{2.47}$$

Hence, ‘contravariant’ indices involve a positive Γ-correction to the ordinary derivative, whereas ‘covariant’ indices involve a negative Γ-correction.

A further crucial property of the covariant derivative is the so-called metric compatibility, again obtained by explicit calculation from (2.47):

$$\nabla_\mu\, g_{\alpha\beta} = 0\ , \tag{2.48}$$

which ensures that the operation of raising and lowering indices with the Riemannian metric commutes with the covariant derivative operator.

Densities

An interesting variation of the Diff-tensor construction above occurs when considering the behavior of the Levi–Civita 'almost tensor" ϵ_{abcd} under the map from *frefos* to *fidos*. In particular,

$$\epsilon_{abcd}\, e^a_\alpha\, e^b_\beta\, e^c_\gamma\, e^d_\delta \tag{2.49}$$

is completely antisymmetric in the indices $\alpha, \beta, \gamma, \delta$. Defining $\epsilon_{\alpha\beta\gamma\delta}$ as the completely antisymmetric object with $\epsilon_{0123} = 1$, we have

$$\epsilon_{abcd}\, e^a_\alpha\, e^b_\beta\, e^c_\gamma\, e^d_\delta = \epsilon_{\alpha\beta\gamma\delta}\, \epsilon_{abcd}\, e^a_0\, e^b_1\, e^c_2\, e^d_3 = \epsilon_{\alpha\beta\gamma\delta} \det(e) \; .$$

Since $g \equiv \det(g) = \det(e)^2 \det(\eta) = -|\det(e)|^2$ we can write $\det(e) = (-g)^{1/2}$ and

$$\sqrt{-g}\; \epsilon_{\alpha\beta\gamma\delta} = \epsilon_{abcd}\, e^a_\alpha\, e^b_\beta\, e^c_\gamma\, e^d_\delta$$

defines a rank-four covariant tensor,

$$\sqrt{-g}\; \epsilon_{\alpha\beta\gamma\delta}\; dx^\alpha \otimes dx^\beta \otimes dx^\gamma \otimes dx^\delta = \text{Diff invariant} \; . \tag{2.50}$$

Analogously, we can define the contravariant object $\epsilon^{\alpha\beta\gamma\delta}$ by the usual raising of indices with the inverse metric $g^{\alpha\beta}$, and obtain an associated tensor

$$\frac{\epsilon^{\alpha\beta\gamma\delta}}{\sqrt{-g}}\; \partial_\alpha \otimes \partial_\beta \otimes \partial_\gamma \otimes \partial_\delta = \text{Diff invariant} \; . \tag{2.51}$$

In some references, collections of quantities that behave as tensors up to a power of $\sqrt{-g}$ are called *tensor densities*. One notable example is the delta function $\delta^{(4)}(x - x_0)$, which transforms as a scalar density. In particular $(-g)^{-1/2}\delta^{(4)}(x - x_0)$ is a true scalar.

2.2.4 Minimal coupling

We are ready to use the formal machinery introduced so far to couple a generic Lorentz-invariant Lagrangian $\mathcal{L}(\Psi^{a\cdots}{}_{b\ldots}, \partial_a, \ldots)$ to an external gravitational field characterized by a metric field $g_{\alpha\beta}$. The strategy is identical to the one that worked before for free particles. We start by covering the spacetime X^{3+1} with *frefos* and specify the action in terms of the limit of a fine partition:

$$S = \lim_{\{P\} \to X^{3+1}} \sum_P \Delta_P S \; , \tag{2.52}$$

with

$$\Delta_P S \approx (\Delta^4 \xi_P)\, \mathcal{L}(\, \Psi^{a\cdots}{}_{b\ldots},\, \partial/\partial\xi^a_P, \ldots)\, |_P \tag{2.53}$$

the element of action in a *frefo*. We now convert to the reference *fido* with the translation of Lorentz tensorial structures to Diff tensorial structures. As for the volume measure we have

$$(\Delta^4 \xi_P) \approx (\Delta^4 x) \left| \det\left(\frac{\partial \xi_P}{\partial x}\right) \right| = \Delta^4 x\; |\det(e)| = \Delta^4 x\, \sqrt{-\det(g_{\alpha\beta})} \; ,$$

or, in the more familiar differential form $d^4\xi = d^4x\, \sqrt{-g}$. Hence, the coupling to gravity is reduced to a mere exercise in notation:

$$S[\Psi, g_{\alpha\beta}\,] = \int_{X^{3+1}} d^4x\, \sqrt{-g}\; \mathcal{L}(\Psi^{\alpha\cdots}{}_{\beta\ldots}, \nabla_\alpha, \ldots) \; . \tag{2.54}$$

Namely, we replace Lorentz tensors by their Diff-tensor forms using the map in (2.40), we replace the ordinary derivatives by the covariant derivatives, and we integrate the Lagrangian over spacetime with a Diff-invariant measure.

Examples of the procedure include the Lagrangian of a scalar field,

$$S_\phi = -\int d^4x\sqrt{-g}\left(\tfrac{1}{2}\, g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + V(\phi)\right)\ , \tag{2.55}$$

with field equation

$$\nabla^2\phi = V'(\phi)\ ,$$

or the Maxwell theory in a gravitational field,

$$S_{\mathrm{Maxwell}} = -\int d^4x\sqrt{-g}\left(\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - J_\mu A^\mu\right)\ , \tag{2.56}$$

where $A_\mu = e_\mu^a A_a$, $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$, $F^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta}$. The field equations

$$\nabla_\mu F^{\mu\nu} = -J^\nu$$

may also be written as [3]

$$\partial_\mu(\sqrt{-g}F^{\mu\nu}) = -\sqrt{-g}\ J^\nu\ .$$

We have already seen the minimal coupling algorithm in action in deriving the point particle dynamics, which we quote here again in its more general version, as a covariant form of (1.81):

$$S_{\mathrm{P}}(x^a, e) = \frac{1}{2}\int d\sigma\left(\frac{1}{e(\sigma)}g_{\alpha\beta}\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma} - m^2\, e(\sigma)\right) \tag{2.57}$$

with the Minkowski equation of motion $du^a/d\tau = 0$ transformed into $u^\beta\nabla_\beta\, u^\alpha = 0$. In an analogous fashion, the free equation for the spin degree of freedom of a particle, $dS^a/d\tau = 0$, generalizes to $u^\alpha\nabla_\alpha S^\beta = 0$ in a general gravitational field. In components, we have

$$\frac{dS^\alpha}{d\tau} + \Gamma^\alpha_{\beta\gamma}u^\beta S^\gamma = 0\ , \tag{2.58}$$

an equation describing the precession of gyroscopes in a gravitational field.

Energy-momentum tensor

We can now repeat the weak-gravity analysis which was already done for the case of point particles. Setting $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ and keeping the linear term in $h_{\alpha\beta}$ one finds

$$S[\Psi, \eta + h] = S[\Psi, \eta] + \frac{1}{2}\int d^4x\, h_{\alpha\beta}\left(2\frac{\partial\mathcal{L}}{\partial h_{\alpha\beta}} + \eta^{\alpha\beta}\mathcal{L}\right)\Bigg|_{h=0} + \mathcal{O}(h^2)\ , \tag{2.59}$$

where we have used $\sqrt{-\det(\eta+h)} = 1 + \frac{1}{2}\,\eta^{\alpha\beta}h_{\alpha\beta} + \mathcal{O}(h^2)$, easily proved in the eigenvalue basis. The quantity in parenthesis in (2.59) measures the *linear response* of the matter system to the

[3] **Exercise:** show that for an antisymmetric tensor field $A_{\mu\nu\ldots} = A_{[\mu\nu\ldots]}$ we have $\nabla_{[\alpha}A_{\mu\nu\ldots]} = \partial_{[\alpha}A_{\mu\nu\ldots]}$, and furthermore $\nabla_\alpha A^{\alpha\mu\cdots} = \frac{1}{\sqrt{-g}}\partial_\alpha\left(\sqrt{-g}\, A^{\alpha\mu\cdots}\right)$. Show that the useful expression for the scalar Laplacian $\nabla^2\phi = \frac{1}{\sqrt{-g}}\partial_\alpha\left(\sqrt{-g}g^{\alpha\beta}\partial_\beta\,\phi\right)$ follows as a corollary.

perturbation by a gravitational field. By explicit computation we see that it coincides with the energy-momentum tensor of the matter system in all the usual cases involving point particles, scalars and electromagnetic fields. Rather than quoting these results for the case of weak fields, we can actually generalize the result for the case of linear response in an arbitrary gravitational field, i.e. the behavior under a perturbation $g_{\alpha\beta} \to g_{\alpha\beta} + \delta g_{\alpha\beta}$. In this case we have

$$S[\Psi, g + \delta g] = S[\Psi, g] + \frac{1}{2} \int d^4x \sqrt{-g}\, \delta g_{\alpha\beta}\, T^{\alpha\beta} + \mathcal{O}(\delta g^2) \,, \tag{2.60}$$

where the symmetric tensor $T^{\alpha\beta}$ is given by

$$T^{\alpha\beta} = 2\frac{\partial \mathcal{L}}{\partial g_{\alpha\beta}} + g^{\alpha\beta}\mathcal{L} \,. \tag{2.61}$$

In obtaining this equation we used $\sqrt{-(g+\delta g)} = \sqrt{-g} + \delta\sqrt{-g} = \sqrt{-g} + \frac{1}{2}\sqrt{-g}g^{\alpha\beta}\delta g_{\alpha\beta} + \mathcal{O}(\delta g^2)$, which is again easily proved in the eigenvalue basis. Another useful identity which is proved in a similar way is $\delta g^{\alpha\beta} = -g^{\alpha\mu}g^{\beta\nu}\delta g_{\mu\nu}$. Using all these results, it is easy to show that, for a system of particles, scalars and electromagnetic fields, we get a total linear-response function

$$T^{\alpha\beta} = T^{\alpha\beta}_{\text{particles}} + T^{\alpha\beta}_{\text{scalar}} + T^{\alpha\beta}_{\text{Maxwell}} \,,$$

where

$$\begin{aligned}
T^{\alpha\beta}_{\text{particles}} &= \sum_p m_p \frac{dx^\alpha}{d\tau_p}\frac{dx^\beta}{d\tau_p}\frac{\delta^{(4)}(x - x_p)}{\sqrt{-g}} \\
T^{\alpha\beta}_{\text{scalar}} &= \partial^\alpha\phi\, \partial^\beta\phi - g^{\alpha\beta}\left(\tfrac{1}{2}(\partial\phi)^2 + V(\phi)\right) \\
T^{\alpha\beta}_{\text{Maxwell}} &= -F^{\alpha\gamma}F_\gamma^{\ \beta} - \tfrac{1}{4}g^{\alpha\beta}F^{\mu\nu}F_{\mu\nu} \,.
\end{aligned} \tag{2.62}$$

The answer is just the Diff-tensor forms of the Noetherian energy-momentum tensors for each of these systems. Hence, we conclude that the energy-momentum tensor measures the response of the matter system to a general gravitational perturbation in an arbitrary gravitational field.

Limitations of minimal coupling. General covariance

The minimal coupling algorithm described in the previous sections follows from the EP and allows us to write down the effects of gravitation for an arbitrary system. The question thus arises, why is it called *minimal*?.

The reason is that there could be higher order terms in the Lagrangian, depending on second derivatives of the metric tensor or higher, that vanish in Minkowski space (hence are invisible in SR) but do not vanish locally in a *frefo* (recall that the EP only requires the first derivatives of the metric to vanish in free fall). In other words, there could be different scalar Lagrangians with the same Minkowskian limit. This phenomenon occurs for example when considering powers of covariant derivatives of tensors.

Let V_ρ by a covariant tensor field and consider the double covariant derivative $\nabla_\mu\nabla_\nu V_\rho$. This is rank-three covariant tensor by construction. Its flat-space limit, obtained setting $g = \eta$, is the Lorentz tensor $\partial_a\partial_b V_c$. This Lorentz tensor is obviously symmetric in the indices a, b. However, the original general tensor is not always symmetric in the indices μ, ν. Noting that $\nabla \sim \partial + \Gamma$, we see that this tensor contains terms of the form $\partial^2 V$ that are indeed symmetric in μ, ν, and terms of the form $\Gamma\partial V$ and $\Gamma\Gamma V$, that vanish in a local free fall frame, but there are also terms of the form $(\partial\Gamma)V$ that *do not* vanish in a free fall frame, and are not necessarily symmetric in μ, ν. These terms do vanish in a free fall frame for a fictitious gravitational field, but not in a general one.

Hence, any term in the Lagrangian involving higher than two covariant derivatives of tensors, with some antisymmetry property, will not be generated by the minimal coupling algorithm. Such terms are perfectly possible and do not violate the strictly formulated EP, because the latter specifies that free fall removes gravitational effects up to second derivatives of the metric (except in fictitious fields, where we remove the gravitational effects to all orders). All these terms involve higher powers of derivatives in the Lagrangian, and thus are suppressed by inverse powers of the spatial range of variation of the gravitational field.

In order to cope with these subtleties, one usually extends the method of minimal coupling to the so-called 'principle of general covariance', which states that one should write down all possible terms in the Lagrangian that are scalars under general coordinate transformations and that reduce to Lorentz invariant terms in the flat space limit $g \to \eta$. This may include terms that effectively reduce to vanishing Lorentz tensors, such as the antisymmetric second covariant derivative of tensor fields. In keeping with the strategy of the 'locality principle', all such terms should be organized in a long-distance expansion and their effects only show up at sufficiently small scales.

Spin connection

We have left out of the minimal coupling rules the case of spinors. This was deliberate, as coupling spinor fields to gravity requires special work. The simplicity of the minimal coupling prescription is based on the deceptively 'trivial' replacement of latin (Lorentz) indices by greek (Diff) indices. This is ultimately possible because the point-by-point transformation matrices of Diff tensors belong to the $GL(4, \mathbf{R})$ group and, the Lorentz group being a subgroup, we can always get a representation of the Lorentz group by restricting a representation of the general linear group. Alas, the converse is not true, as there are representations of $O(3,1)$ that *cannot* be extended into representations of $GL(4, \mathbf{R})$. These are precisely the spinorial representations, used to describe fermions in nature.

Even if the simple minimal coupling algorithm does not work in an obvious way, there is no problem of principle with applying the EP to say, the Dirac Lagrangian. In fact, the solution requires focusing not on the metric field, but rather on the seemingly more cumbersome tetrad fields, defined at each point as a 'Lorentzian square root' of the metric:

$$g_{\mu\nu}(P) = e^a_\mu(P)\, e^b_\nu(P)\, \eta_{ab} \;.$$

According to this definition, the tetrads are four four-vectors that vary continuously through the spacetime manifold. However, given one particular choice of tetrads e^a_μ, the Lorentz-rotated one, $L^a{}_b e^b_\mu$, is equally good, so that we have an ambiguity by a Lorentz transformation *at each point* in spacetime. Since the free fall frames do not extend globally, there is no canonical way of choosing a set of Lorentz transformations, i.e. we have a true redundancy, a *gauge symmetry.* [4]

[4] In fact, the same ambiguity is present in Minkowski spacetime, i.e. for fictitious gravitational fields. The difference is that in Minkowski space there is a *special* choice of tetrads where a *single* Minkowskian frame covers all spacetime.

In order to ensure invariance under this redundancy, we introduce a gauge field in the usual fashion, called the *spin connection*. Given a vector field $V^a = e^a_\mu V^\mu$, its derivative $\partial_\mu V^a$ is not Lorentz-covariant, but the combination

$$D_\mu V^a = \partial_\mu V^a + \omega_\mu{}^a{}_b V^b \tag{2.63}$$

is Lorentz-covariant, $D_\mu V^a \to L^a_b D_\mu V^b$, provided the gauge field transforms as

$$\omega_\mu \to L\left(\omega_\mu + \partial_\mu\right) L^{-1} \,, \tag{2.64}$$

in matrix notation. The spin connection is the price we pay for our insistence on using the local Lorentz frames at each point, and it should not introduce any new independent degrees of freedom. The explicit choice

$$\omega_\mu{}^a{}_b \equiv e^\nu_b \, \nabla_\mu \, e^a_\nu \equiv e^\nu_b \left(\partial_\mu e^a_\nu - \Gamma^\alpha_{\mu\nu} e^a_\alpha\right) \tag{2.65}$$

satisfies all the transformation rules. Conversely this definition implies that the Christoffel symbols can be calculated in terms of the tetrad derivatives as [5]

$$\Gamma^\alpha_{\mu\nu} = e^\alpha_a \, D_\mu \, e^a_\nu \equiv e^\alpha_a \left(\partial_\mu \, e^a_\nu + \omega_\mu{}^b{}_a \, e^b_\nu\right) \tag{2.66}$$

in a general gravitational field. In this way, the two representations of a tensor in terms of 'curved' or 'flat' indices have ∇_μ and D_μ as the respective covariant derivative operators:

$$D_\mu V^a = e^a_\alpha \nabla_\mu V^\alpha \,. \tag{2.67}$$

This relation expresses the fact that the spin connection does not introduce new degrees of freedom and could be used to 'define' ω_μ.

The utility of the spin connection is that it can be used to define covariant derivatives for any fields specified in terms of Lorentz components. In particular, spinor fields. The Dirac operator in the presence of gravitation is defined as

$$\not{D} \equiv \gamma^\mu D_\mu = \gamma^\mu \left(\partial_\mu + \tfrac{1}{2}\, \omega^{ab}_\mu \sigma_{ab}\right) \,, \tag{2.68}$$

where $\gamma^\mu = e^\mu_a \gamma^a$ is a basis of curved-space Dirac matrices satisfying the generalized Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}$.

[5] Notice the subtle difference between this formula and (2.31). Actually, the spin connection measures precisely to what extent the *second* derivative in (2.31) fails to be the *first* derivative of the tetrad.

2.3 An interlude: Riemannian geometry

We have argued that the equivalence principle, together with some locality assumptions, sets the mathematical arena of a gravitational theory to be given by Riemannian geometry. In the mathematical literature, spaces with a positive-definite metric determining local distances

$$d\ell^2 = \sum_{ij=1}^{d} g_{ij}\, dy^i\, dy^j \tag{2.69}$$

are called d-dimensional Riemannian manifolds. The meaning of this expression is that any curve γ, parametrized by $y^i(\sigma)$, has length

$$\ell(\gamma) = \int_\gamma d\sigma \sqrt{\sum_{ij} g_{ij} \frac{dy^i}{d\sigma}\frac{dy^j}{d\sigma}} \ . \tag{2.70}$$

An intuitive interpretation of a Riemannian metric can be given by visualizing the manifold $\mathcal{M}$ as a curved submanifold embedded in a higher-dimensional flat space $\mathbf{R}^D$, with a sufficiently large $D > d$. Let X^I denote cartesian coordinates on $\mathbf{R}^D$ and let the submanifold $\mathcal{M}$ of interest be parametrized by functions $X^J(y^j)$. At a given point on $\mathcal{M}$ we can consider the tangent hyperplane, generated by the vectors of components

$$E_i^I = \frac{\partial X^I}{\partial y^i} \ ,$$

with scalar product

$$E_i \cdot E_j = \sum_{I=1}^{D} E_i^I E_j^I \ .$$

The length of a curve $X^I(\sigma)$ in $\mathbf{R}^D$ is given by

$$\ell(\gamma) = \int_\gamma d\sigma \sqrt{\sum_I \frac{dX^I}{d\sigma}\frac{dX^I}{d\sigma}} \ . \tag{2.71}$$

However, when the curve lies inside $\mathcal{M}$ we can also describe it by functions $y^i(\sigma)$, then

$$\frac{dX^I}{d\sigma} = \sum_j \frac{dy^j}{d\sigma}\frac{\partial X^I}{\partial y^j} = \sum_j E_j^I \frac{dy^j}{d\sigma}$$

and the formula (2.70) follows with the metric "induced" from the embedding:

$$g_{ij}^{(\text{induced})} = \sum_{I=1}^{D} E_i^I E_j^I \ . \tag{2.72}$$

The starting point of Riemannian geometry is the abstraction of the embedding space. Once the intrinsic metric g_{ij} is defined, Riemann concentrates on those properties that follow from the metric alone. [6]

[6] Incidentally, a famous theorem by the even more famous John Nash shows that any given Riemannian metric g_{ij} can be induced by embedding the manifold $\mathcal{M}$ in $\mathbf{R}^D$, for a sufficiently large dimension D.

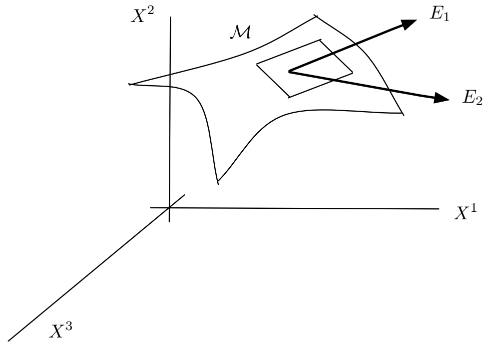

Figure 2.5: *A two-dimensional surface embedded in* $\mathbf{R}^3$*. The vectors* E_1 *and* E_2 *give a basis of a local tangent space. The induced metric is given by* $g_{ij} = E_i \cdot E_j$*.*

The appropriate generalization for GR is that of a Riemannian space with Lorentzian signature, meaning that the eigenvalues of the metric in a special (free fall) frame, have signs $(- + ++)$. Hence, we see explicitly how the EP embodies the geometrical interpretation of GR. From this standpoint, we could derive GR by simply providing a physical interpretation of each element of Riemannian geometry, such as the notions of connection, torsion, curvature, isometries, etc.

In the mathematical literature, the free-fall equation of motion goes under the name of 'geodesic equation'. This refers to the related problem of finding 'smallest length' paths between two points in a curved manifold. If our manifold is equipped with a metric $d\ell^2 = \sum_{ij} g_{ij}\, dy^i dy^j$, the length along a given path can be obtained by (2.70). Minima of this functional define the locally shortest paths, i.e. the geodesics. But we see that the mathematical problem is identical to ours, except for the use of positive definite metrics, rather than Lorentzian ones. Accordingly, the local geodesic equation has the form

$$\sum_i T^i \nabla_i T^j = 0 \,, \tag{2.73}$$

where $\nabla \sim \partial + \Gamma$ in a completely analogous fashion and $T^i = dy^i/d\ell$ is the tangent vector to the geodesic curve, parametrized by the proper length ℓ. In particular, the explicit formula for the Christoffel symbols in terms of the first derivatives of the metric is exactly the same. So, we can adopt the geometry language and say that the motion in a gravitational field is equivalent to the motion along geodesics of a curved spacetime manifold. In our case, the geodesics are not 'minimal length' paths, but rather 'maximum proper-time' ones.

This construction can be further generalized to define an induced metric for any submanifold Σ of $\mathcal{M}$, which of course will also be a submanifold of $\mathbf{R}^D$. Iterating the procedure gives the induced metric on Σ

$$g^{(\Sigma)}_{uv} = \sum_I \frac{\partial X^I}{\partial \sigma^u}\frac{\partial X^I}{\partial \sigma^v} = \sum_{ij} g_{ij}\frac{\partial y^i}{\partial \sigma^u}\frac{\partial y^j}{\partial \sigma^v} \,, \tag{2.74}$$

where σ^u label coordinates on Σ which can be specified by embedding functions $y^j(\sigma)$ into $\mathcal{M}$ or $X^I(x(\sigma))$ into $\mathbf{R}^D$.

It is very important that no confusion arises between the four-dimensional spacetime geometry, characterized by the Lorentzian metric $g_{\mu\nu}$, and the metric on spatial submanifolds. In our previous example of relatively rotating frames, a non-Euclidean *spatial* geometry was operationally demonstrated in the rotating frame. However, the four-dimensional *spacetime* geometry remains just Minkowski space expressed in a rotating frame.

Parallel transport, holonomy and curvature

In general, if a vector V^i is defined at each point of a curve $x^j(\sigma)$ with tangent vector $T^i = dx^i/d\sigma$, we can define the covariant derivative along the curve as

$$\frac{\nabla V^j}{d\sigma} \equiv \nabla_T V^i \equiv \sum_j T^j \nabla_j V^i \ . \tag{2.75}$$

If this covariant derivative vanishes, one says that the vector V is being defined by *parallel transport* along the curve $x^i(\sigma)$. The justification for this name comes from the fact that any two vectors V and W, carried by parallel transport along the curve, $\nabla_T V = \nabla_T W = 0$, satisfy

$$\frac{d}{d\sigma}\left(\sum_{ij} g_{ij} V^i W^j\right) = \sum_{ijk} T^k \nabla_k (g_{ij} V^i W^j) = 0 \ ,$$

as a consequence of the metric compatibility property, $\nabla_k g_{ij} = 0$. Hence, the local metric definition of angles and norms is preserved under parallel transport. [7] We can write more explicitly the parallel-transport equation as

$$\frac{dV^i}{d\sigma} + \sum_j (\Gamma_T)^i_j V^j = 0 \ ,$$

where we define the matrix $(\Gamma_T)^i_j \equiv \sum_k T^k \Gamma^i_{jk}$. This equation can be solved to give the vector at some point Q, parallel-transported from some other point P, along the curve γ_{PQ}, in terms of a formal path-ordered product

$$V(Q) = \mathcal{P} \exp\left(-\int_{\gamma_{PQ}} \Gamma_T\right) V(P) \ . \tag{2.76}$$

For an infinitesimal displacement we have $V(Q) = (1 - \Delta\sigma \Gamma_T) V(P)$, so that the path-ordered product can be defined as the formal limiting product over an infinitesimal partition of the path

$$\mathcal{P} \exp\left(-\int_{\gamma_{PQ}} \Gamma_T\right) = \lim_{\Delta\sigma_s \to 0} \prod_s e^{-\Delta\sigma_s \Gamma_T(s)} \ .$$

The concept of parallel transport can be used to define a notion of curvature. Consider a closed curve γ_P, based at P, given by the function $x^i(\sigma)$ that starts and returns to a point P, i.e. $x^i(0) = x^i(2\pi)$, with σ an angular variable. The corresponding closed-path parallel transporter

$$U[\gamma_P] = \mathcal{P} \exp\left(-\oint_{\gamma_P} \Gamma_T\right)$$

[7] Notice that a geodesic is defined as a proper-length parametrized curve whose tangent vector is 'carried forward' by parallel transport along itself, $\nabla_T T = 0$.

depends both on the curve and the point P, and is called *holonomy*. Since parallel transport preserves norms and angles, $U(\gamma_P)$ is some matrix in the $SO(d)$ group, where d is the dimension of the Riemannian manifold. [8] In Minkowski space, or a 'flat' space in general, the holonomy of any curve is the identity matrix. In this case, the parallel transport of vectors between two arbitrary points is independent of the path joining them. A local measure of the deviation from flatness, i.e. a mathematical definition of 'curvature', is obtained by considering the limit of very small closed curves.

Let us make one little 'square' composed of four curves intersecting in pairs, using the parameters of these curves as coordinates (x, y) of a two-dimensional submanifold with size of order ϵ. We consider tangent vector fields X and Y whose only nonvanishing components are $X^x = Y^y = 1$ respectively. Denoting $X^i = \partial z^i/\partial x$ and $Y^i = \partial z^i/\partial y$ the components in an arbitrary coordinate system, we have the following identity

$$[X,Y]^i \equiv \sum_j (X^j \partial_j Y^i - Y^j \partial_j X^i) = 0 \; , \tag{2.77}$$

where the so-defined vector field $[X,Y]$ is known as the *Lie bracket* of X and Y. Hence, when two vector fields are used to coordinate a two-dimensional submanifold, their Lie bracket vanishes.[9]

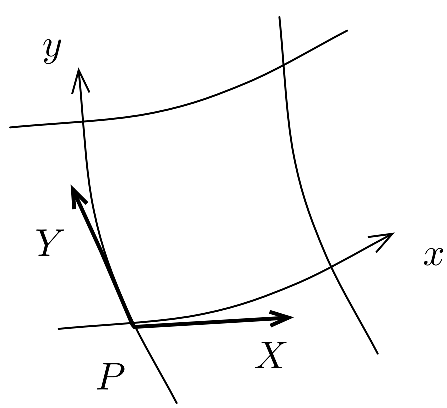

Figure 2.6: *A quadrilateral round-trip at P with direction vectors X,Y and holonomy $U(\gamma_P) = U_{-\epsilon Y} U'_{-\epsilon X} U'_{\epsilon Y} U_{\epsilon X}$.*

The holonomy around this quadrilateral round trip is given by

$$U(\gamma_P) = U_{-\epsilon Y} U'_{-\epsilon X} U'_{\epsilon Y} U_{\epsilon X}$$

where we have, with quadratic precision in ϵ,

$$U_{\epsilon X} = e^{-\epsilon \Gamma_X} = 1 - \epsilon\, \Gamma_X + \tfrac{1}{2}\, \epsilon^2\, (\Gamma_X)^2 + O(\epsilon^3)$$

with a similar definition for $U_{\epsilon Y}$. The primed transporters are computed from the displaced versions of the connection

$$\Gamma'_Y = \Gamma_Y + \epsilon\, \partial_X \Gamma_Y + O(\epsilon^2) \; , \qquad \Gamma'_X = \Gamma_X + \epsilon\, \partial_Y \Gamma_X + O(\epsilon^2) \; ,$$

[8] In four-dimensional Lorentzian signature the holonomy is a matrix in the Lorentz group $O(3,1)$.
[9] The converse statement is also true and it is a particular case of a theorem by Frobenius.

with the notation $\partial_X = \sum_i X^i \partial_i$ and analogously for Y. Upon explicit evaluation of the holonomy we find the first non-trivial term occurring at order ϵ^2 and given by

$$U(\gamma_P) = 1 - \epsilon^2 \left(\partial_X \Gamma_Y - \partial_Y \Gamma_X - \Gamma_X \Gamma_Y + \Gamma_Y \Gamma_X\right) + O(\epsilon^3) \ .$$

Evaluating now the derivatives in index notation and using $[X, Y] = 0$ one finds

$$U(\gamma_P)^i_j = \delta^i_j - \epsilon^2 \sum_{k,l} X^k \, Y^l \, R_{klj}{}^i + O(\epsilon^3) \ , \tag{2.78}$$

where the coefficients

$$R_{klj}{}^i = -\partial_k \Gamma^i_{lj} - \sum_n \Gamma^i_{kn} \Gamma^n_{lj} - (k \leftrightarrow l) \tag{2.79}$$

define the Riemann tensor. Despite its explicit definition in terms of non-tensor objects, it is a Diff tensor with 20 independent components in four dimensions. [10] Equivalently, it can be represented in terms of the spin connection via the mixed Diff-Lorentz tensor $R_{kl}{}^{ab} = \sum_{ij} e^a_i e^{bj} \, R_{klj}{}^i$,

$$R_{kl}^{\ ab} = -\partial_k \omega^{ab}_l + \partial_l \omega^{ab}_k - [\omega_k, \omega_l]^{ab} \ , \tag{2.80}$$

thus taking the form of a standard non-abelian field strength of Yang–Mills type, associated to the local $O(1,3)$ gauge group.

The Riemann tensor has the purely geometrical interpretation of characterizing completely the local curvature: a Riemannian space is flat, i.e. metrically $\mathbf{R}^d$, if and only if the Riemann tensor vanishes identically.

For submanifolds Σ there is an important distinction between *intrinsic* and *extrinsic* curvature. The intrinsic curvature is the Riemannian curvature of the induced metric $g^{(\Sigma)}_{uv}$, whereas the extrinsic curvature defines how the submanifold is 'twisted' inside the ambient manifold. When Σ is a codimension-one submanifold the 'twisting' is determined by the normal unit vector to Σ, which we denote by n^i. The bending of Σ is then related to the failure of n^i to follow parallel transport when moved around Σ. Hence we define the extrinsic curvature as the symmetric covariant derivative

$$K_{ij} \equiv \nabla_{(i} n_{j)} = \tfrac{1}{2} \left(\nabla_i n_j + \nabla_j n_i\right) . \tag{2.81}$$

The symmetrization of the covariant derivative can be obviated in the definition of the extrinsic curvature provided $\nabla_i n_j$ is computed by extending n^i away from Σ as a geodesic normal vector, i.e. satisfying $n^i \nabla_i n^j = 0$.

Non-null codimension-one submanifolds (also called hypersurfaces) admit some useful specific formulae. In particular, the two-index object $\Pi_{ij} = g_{ij} - n_i n_j$ is a projector that gives the induced metric when restricted to the tangent space of Σ, i.e. in intrinsic coordinates for Σ one has $g^{(\Sigma)}_{uv} = \Pi^{(\Sigma)}_{uv}$. For *spacelike* hypersurfaces in Lorentzian signature, one has $n_\mu n^\mu = -1$, and the projector reads $\Pi_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$.

[10] An equivalent, if more elegant expression with manifest tensorial form is $\left([\nabla_X, \nabla_Y] - \nabla_{[X,Y]}\right) V^i = -\sum_{klj} X^k Y^l R_{klj}{}^i \, V^j$, valid even for the case that $[X, Y] \neq 0$.

Non-metrical connections and derivatives

The concept of connection (covariant derivative) is actually more primitive than the notion of metric. One can define ∇_μ axiomatically and from there one can reach the concepts of parallel transport, holonomy and curvature. The general covariant derivatives are still of the form $\nabla \sim \partial + \Gamma$, but now the connection coefficients $\Gamma^\alpha_{\mu\nu}$ need not be symmetric in the lower indices, with the antisymmetric part $T^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\nu\mu}$ defining the *torsion*. One can check explicitly that this is always a tensor.

Even in the presence of a metric, it is possible to have non-standard connections, in the sense that they might not be metric compatible, i.e. $\nabla g \neq 0$, and/or they might have torsion. However, demanding the connection to be symmetric (torsion free) and metric compatible, $\nabla g = 0$, fixes it completely to be the so-called Levi–Civita connection, defined by the Christoffel symbols. In practice, one can always write an arbitrary connection as the sum of the Levi–Civita connection and a remainder. This remainder may depend on torsion terms and/or contain the degrees of freedom that violate the metric compatibility, and it can be regarded as a non-minimal coupling in covariant derivatives. The important fact about this remainder is that it is always a tensor. In this way, we may still use the standard Riemannian language for physics applications, with the non-standard connection degrees of freedom interpreted as a type of exotic tensor 'matter' that does not follow the minimal coupling prescription.

This freedom in the use of non-standard connections shows up in modern generalizations of GR, in the context of supergravity theories. Historically, this was also the arena for various forgotten alternative theories, tried by the 'founding fathers'. In particular, Einstein contemplated connections with torsion in some of his attempts at constructing a 'unified theory' of gravity and electromagnetism. Even earlier, Weyl considered connections satisfying

$$\nabla_i g_{jk} = A_i \, g_{jk}$$

for some vector field A_i. In general these connections do not preserve the length of a vector under parallel transport, although they do preserve the locally measured angles:

$$\cos(V, W) \equiv \frac{V \cdot W}{\sqrt{|V|^2 |W|^2}} \,,$$

where $V \cdot W \equiv \sum_{ij} g_{ij} V^i W^j$, and $|V|^2 = V \cdot V$. In Weyl's theory, the vector field A_i was again related to electromagnetism. [11]

[11] **Exercise:** Check that the Weyl connection preserves angles.

Chapter 3

Dynamics of the gravitational field

With the formal machinery introduced so far, we are able to solve for the dynamics of arbitrary matter systems in a prescribed gravitational field, as specified by some metric tensor $g_{\alpha\beta}$. In this so-called *probe approximation*, the spacetime metric is considered as a fixed external background, and the back-reaction of the energy-momentum of matter fields on the metric is assumed to be small on the length scales of interest. In particular, the classic experimental tests of GR are analyzed within this approximation, once the metric outside a poin-tlike source is given. The remaining part of the problem is the specification of the dynamics of the gravitational field itself, i.e. the Lagrangian of the metric tensor.

3.1 Einstein's law

Since $g_{\alpha\beta}$ plays the role of a gravitational potential, we seek the relativistic analog of the Newtonian action

$$S_{\text{Newton}} = \frac{1}{8\pi G} \int dt\, d^3x\ \phi_{\text{N}}\, \vec{\partial}^2\, \phi_{\text{N}}$$

in a scalar Lagrangian linear in second derivatives of the metric, i.e.

$$S = \int d^4x \sqrt{-g}\ \left(\mathcal{L}_g(g_{\alpha\beta}) + \mathcal{L}_m(\Psi, \nabla_\alpha)\right)\ , \tag{3.1}$$

with $\mathcal{L}_m$ the Lagrangian of generic matter degrees of freedom Ψ, minimally coupled to $g_{\mu\nu}$ as in the previous sections, and $\mathcal{L}_g \sim \partial^2 g$ is the purely gravitational Lagrangian, a Diff scalar. The obvious choice $\nabla^2 g$ vanishes because of the metric compatibility of the affine connection. A hint at the solution is provided by the previous discussion on the limits of minimal coupling, where it was argued that the commutator of covariant derivatives can be used to diagnose the non-triviality of the gravitational field. Here, we shall argue this point on physical grounds.

3.1.1 Tidal forces

The basic property of a fictitious gravitational field is the existence of a global free-fall frame that 'transforms away' gravity globally. In the free-fall frame, initially parallel trajectories stay so in time, with vanishing relative acceleration. On the other hand, in a real gravitational field, the EP only removes the gravitational field at a point, leaving a residual relative acceleration between nearby particle trajectories. The corresponding residual forces are called 'tidal forces'.

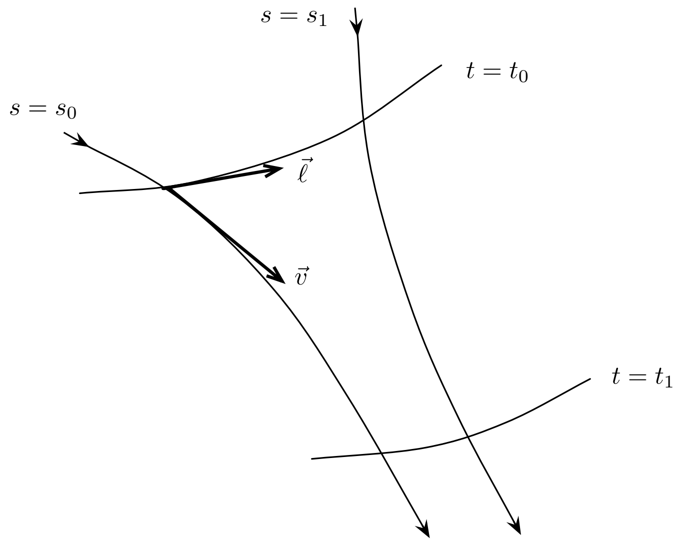

Figure 3.1: *Family of free-fall trajectories parametrized by global time and fiducial parameter s. The relative separation of the falling points is given by $\Delta s\, d\vec{x}/ds$.*

We can easily discuss tidal forces in the Newtonian theory by considering a family of free-falling particles along a curve $\vec{x}(s)$ at time $t = 0$. We also assume that the initial velocities of these particles varies continuously along the family, i.e. the function

$$\vec{v}(s, 0) = \frac{d\vec{x}}{dt}(s, 0)$$

is a continuous vector function of s. Now, we let them develop in time, so that the set of trajectories span a two-dimensional surface coordinated by (s, t). The relative separation at time t of two particles with parametric distance Δs is

$$\Delta\vec{x} = \Delta s\, \frac{d\vec{x}}{ds}(s, t) \;,$$

to first order in Δs. Denoting the transverse separation vector $\vec{\ell} = d\vec{x}/ds$, the rate of separation of the falling trajectories is $\Delta s\, d\vec{\ell}/dt$ and the relative acceleration $\Delta s\, d^2\vec{\ell}/dt^2$. Using now the equation of motion

$$\frac{d^2\vec{x}}{dt^2}(s, t) = -\vec{\partial}\,\phi_{\rm N}(\vec{x}(s, t))$$

we obtain the final expression for the tidal accelerations,

$$a^i_{\rm tidal} = \frac{d^2\ell_i}{dt^2} = -(\vec{\ell}\cdot\vec{\partial})\,\partial_i\phi_{\rm N} = -\sum_j R_{ij}\ell_j \;, \tag{3.2}$$

which indeed depend on the second derivatives of the gravitational potential and reproduces our previously introduced tidal tensor $R_{ij} = \partial_i\partial_j\phi_{\rm N}$.

In the relativistic case, we consider an entirely analogous situation, i.e. a family of free-falling geodesics $x^\mu(\tau, s)$, parametrized by the geodesic label, s, and proper time τ. The separation and velocity four-vectors are defined by

$$\ell^\mu = \frac{dx^\mu}{ds} \,, \qquad u^\nu = \frac{dx^\nu}{d\tau} \,. \tag{3.3}$$

We can now define covariant derivatives along parametrized curves as

$$\frac{\nabla}{ds} \equiv \nabla_\ell \equiv \ell^\mu \nabla_\mu \,, \qquad \frac{\nabla}{d\tau} \equiv \nabla_u \equiv u^\mu \nabla_\mu \,. \tag{3.4}$$

With this definition, the free-fall equation of motion may be rewritten as

$$\frac{du^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta = \nabla_u \, u^\mu = u^\alpha \nabla_\alpha u^\mu = 0 \,. \tag{3.5}$$

Using these formulas, one can explicitly write

$$\nabla_u \, \ell^\alpha = u^\mu \nabla_\mu \ell^\alpha = \frac{d\ell^\alpha}{d\tau} + \Gamma^\alpha_{\mu\nu} u^\mu \ell^\nu = \frac{d^2 x^\alpha}{d\tau ds} + \Gamma^\alpha_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{ds} \,.$$

Since this expression is symmetric in s, τ, it is actually equal to

$$\frac{d^2 x^\alpha}{ds d\tau} + \Gamma^\alpha_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{d\tau} = \ell^\mu \nabla_\mu u^\alpha \,.$$

Hence, we have proven

$$\nabla_u \, \ell^\alpha = u^\mu \nabla_\mu \ell^\alpha = \ell^\mu \nabla_\mu u^\alpha = \nabla_\ell \, u^\alpha \tag{3.6}$$

With these preliminaries, we are ready to define the relative acceleration of neighbouring geodesics as

$$a^\mu_{\rm tidal} = (\nabla_u)^2 \, \ell^\mu = u^\alpha \nabla_\alpha (u^\beta \nabla_\beta \, \ell^\mu) = u^\alpha \nabla_\alpha (\ell^\beta \nabla_\beta \, u^\mu) \,, \tag{3.7}$$

where we have used (3.6) in the last equality. Manipulating this expression we find

$$a^\alpha_{\rm tidal} = u^\mu (\nabla_\mu \ell^\nu) \nabla_\nu u^\alpha + u^\mu \ell^\nu \nabla_\nu \nabla_\mu u^\alpha + u^\mu \ell^\nu \, [\nabla_\mu, \nabla_\nu \,] \, u^\alpha \,. \tag{3.8}$$

Using again (3.6) in the first term and the geodesic condition $u^\alpha \nabla_\alpha u^\beta = 0$ we finally obtain

$$a^\alpha_{\rm tidal} = u^\mu \ell^\nu \, [\, \nabla_\mu, \nabla_\nu \,] \, u^\alpha \,. \tag{3.9}$$

Thus, we have proven that the physical distinction between real and fictitious gravitational fields can be locally parametrized by the commutator of covariant derivatives acting on vector fields. Such an object defines the relativistic version of the tidal tensor, i.e. the so-called Riemann tensor.

3.1.2 The Riemann tensor

The commutator $[\nabla_\mu, \nabla_\nu \,] \, V^\alpha$ defines a tensor object that vanishes in a fictitious gravitational field (i.e. a reparametrized Minkowski space) because its flat-space version satisfies $[\partial_a, \partial_b] V^c = 0$ indentically. Upon direct evaluation, we find

$$[\nabla_\mu, \nabla_\nu \,] \, V^\alpha = -R_{\mu\nu\rho}{}^\alpha \, V^\rho \,, \tag{3.10}$$

where

$$R_{\mu\nu\rho}{}^{\alpha} = -\partial_\mu \Gamma^\alpha_{\nu\rho} - \Gamma^\alpha_{\mu\sigma}\Gamma^\sigma_{\nu\rho} + (\mu \leftrightarrow \nu) \tag{3.11}$$

is the Riemann tensor. As expected, it contains second derivatives of the metric that need not vanish anywhere in a real gravitational field, even in a free-fall frame. The explicit calculation above can be repeated for the case of a covariant vector field,

$$[\nabla_\mu, \nabla_\nu]\, V_\alpha = R_{\mu\nu\alpha}{}^{\rho}\, V_\rho \;, \tag{3.12}$$

and for a general tensor

$$[\nabla_\mu, \nabla_\nu]\, T^{\alpha\ldots}{}_{\beta\ldots} = -R_{\mu\nu\rho}{}^{\alpha}\, T^{\rho\ldots}{}_{\beta\ldots} - \ldots + R_{\mu\nu\beta}{}^{\rho}\, T^{\alpha\ldots}{}_{\rho\ldots} + \ldots \;. \tag{3.13}$$

This means that an arbitrary power of covariant derivatives can be reduced to symmetrized covariant derivatives (which disappear in a free-fall frame) plus terms involving powers of the Riemann tensor.

Mathematically, we say that a manifold is *curved* when the Riemann tensor does not vanish. Conversely, if the Riemann tensor vanishes identically inside a finite region of spacetime, there exists a frame in which the metric is exactly Minkowskian, $g = \eta$, throughout that region.

The Riemann tensor has in principle 256 components, which are reduced to 20 independent ones due to a large set of symmetry properties: [1]

$$R_{(\mu\nu)\rho}{}^{\alpha} = 0 \;, \qquad R_{[\mu\nu\rho]}{}^{\alpha} = 0 \;. \tag{3.14}$$

(we denote $[\ldots]$ the antisymmetrization operation and $(\ldots)$ the symmetrization operation). For the completely covariant form $R_{\mu\nu\rho\sigma} = g_{\sigma\alpha}R_{\mu\nu\rho}{}^{\alpha}$, we have

$$R_{\mu\nu(\rho\sigma)} = 0 \;. \tag{3.15}$$

It follows from these three identities that $R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$. Finally, the Riemann tensor satisfies a set of differential relations, called the Bianchi indentities.

$$\nabla_{[\mu} R_{\nu\rho]\sigma}{}^{\alpha} = 0 \;. \tag{3.16}$$

To prove these, we start from

$$[\nabla_\mu, \nabla_\nu]\, \nabla_\rho\, w_\sigma = R_{\mu\nu\rho}{}^{\alpha}\nabla_\alpha w_\sigma + R_{\mu\nu\rho}{}^{\alpha}\nabla_\sigma w_\alpha \;. \tag{3.17}$$

On the other hand

$$\nabla_\mu\, [\nabla_\nu, \nabla_\rho]\, w_\sigma = \nabla_\mu (R_{\mu\nu\rho}{}^{\alpha} w_\alpha) = w_\alpha \nabla_\mu R_{\nu\rho\sigma}{}^{\alpha} + R_{\nu\rho\sigma}{}^{\alpha}\nabla_\mu w_\alpha \;.$$

Antisymmetrizing in μ, ν, ρ in both equations, the left hand side becomes the same. Equality of the right hand sides implies

$$R_{[\mu\nu\rho]}{}^{\alpha}\nabla_\alpha w_\sigma + R_{[\mu\nu(\sigma)}{}^{\alpha}\nabla_{\rho]} w_\alpha = w_\alpha \nabla_{[\mu} R_{\nu\rho]\sigma}{}^{\alpha} + R_{[\nu\rho(\sigma)}{}^{\alpha}\nabla_{\mu]} w_\alpha \;,$$

where the inserted parenthesis indicate that the index in question is *not* antisymmetrized. The first term on the left hand side of this equation vanishes by the symmetry properties of the

[1] **Exercise:** Prove all these.

Riemann tensor, whereas the second terms on both sides cancel one another, and we obtain, for all w_α,

$$w_\alpha \nabla_{[\mu} R_{\nu\rho]\sigma}{}^{\alpha} = 0 \ , \tag{3.18}$$

which yields the Bianchi identities. [2]

The symmetry properties imply that the Riemann tensor produces only one independent second rank tensor under contraction, the so called Ricci tensor

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu} = R_{\nu\mu} \ , \tag{3.19}$$

and a *unique* scalar, the Ricci scalar

$$R = g^{\mu\nu} R_{\mu\nu} \ . \tag{3.20}$$

Contracting the Bianchi identities we obtain their projection over the Ricci tensors, [3]

$$\nabla^\mu G_{\mu\nu} = 0 \ , \tag{3.21}$$

where we have defined the Einstein tensor as

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}\, g_{\mu\nu} R \ . \tag{3.22}$$

Another interesting object is the Weyl tensor, defined by

$$C_{\mu\nu}{}^{\rho\sigma} \equiv R_{\mu\nu}{}^{\rho\sigma} - 2R_{[\mu}{}^{[\rho}\delta_{\nu]}{}^{\sigma]} + \tfrac{1}{3}\, R\, \delta_{[\mu}{}^{\rho}\delta_{\nu]}{}^{\sigma} \ . \tag{3.23}$$

It can be interpreted as the part of the Riemann tensor that is not controlled by the Ricci component. Its main application is as a criterion for conformal equivalence of metrics, namely two metrics related by a local rescaling $g_{\mu\nu}(x) \to \Omega^2(x) g_{\mu\nu}(x)$ have the same Weyl tensor.

3.1.3 The Hilbert Lagrangian

Comparing the Newtonian and relativistic equations for the tidal forces, we see that we can make a correspondence inspired in the non-relativistic limit,

$$R_{\mu j\nu}{}^{i}\, u^\mu u^\nu \approx R_{0j0}{}^{i} \approx \partial_j \partial^i \phi_{\mathrm{N}} \ , \tag{3.24}$$

so that we are essentially forced to choose as the Lagrangian for the gravitational field the unique scalar constructed from the Riemann tensor, i.e. the Ricci scalar. This leads to the Hilbert action [4]

$$S_{\mathrm{H}} = \frac{1}{2\kappa^2} \int d^4x\, \sqrt{-g}\, R \ , \tag{3.25}$$

where κ^2 is proportional to G, the precise constant of proportionality to be determined below.

In fact, there is a superficially more important term that can be added to the purely gravitational Lagrangian: an additive renormalization of R by a constant. It is more important because it does not contain derivatives and thus dominates the long distance physics. This is the famous

[2] In more mathematical terms, the Bianchi identities are nothing but the Jacobi identity for the covariant derivative operator: $0 = [\nabla_\mu, [\nabla_\nu, \nabla_\rho]] +$ cyclic permutations.

[3] **Exercise:** Prove it.

[4] In fact, Hilbert obtained the correct field equations from this action at the same time, if not before, Einstein himself.

cosmological constant, Einstein's 'biggest blunder', by his own admission and, in hindsight, one of his major discoveries!

The complete Lagrangian of gravity and matter then reads,

$$S[g, \Psi] = \int d^4x \sqrt{-g} \left(\frac{1}{2\kappa^2}(R - 2\Lambda) + \mathcal{L}_m(\Psi, \nabla) \right) . \tag{3.26}$$

3.1.4 Einstein's equations

We are now ready to derive Einstein's equations for the gravitational field, by simply stabilizing the variation of the Lagrangian with respect to the metric $g_{\mu\nu}$. We first consider the variation of the gravitational Lagrangian. It contains three terms,

$$\delta(\sqrt{-g}\,(R - 2\Lambda)) = \delta(\sqrt{-g})(R - 2\Lambda) + \sqrt{-g}\,\delta g^{\mu\nu} R_{\mu\nu} + \sqrt{-g}\, g^{\mu\nu} \delta R_{\mu\nu} \,.$$

Using the relations $\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g} g^{\mu\nu}\delta g_{\mu\nu}$ and $\delta g^{\mu\nu} = -g^{\mu\alpha} g^{\nu\beta} \delta g_{\alpha\beta}$ we can evaluate the first two terms. As for the last one we have

$$\delta R_{\mu\nu} = -\partial_\mu \delta\Gamma^\alpha_{\alpha\nu} + \partial_\alpha \delta\Gamma^\alpha_{\mu\nu} + O(\Gamma\delta\Gamma) \,.$$

The terms proportional to $\Gamma\delta\Gamma$ vanish on a free fall frame at a given point. Now, although the connection Γ is not a tensor, the differece between two connections is a tensor, since the offending inhomogeous terms vanish. So we can find the complete expression by covariantizing the previous one,

$$\delta R_{\mu\nu} = -\nabla_\mu \delta\Gamma^\alpha_{\alpha\nu} + \nabla_\alpha \delta\Gamma^\alpha_{\mu\nu} \,. \tag{3.27}$$

The same type of reasoning shows that

$$\delta\Gamma^\alpha_{\mu\nu} = \tfrac{1}{2}\, g^{\alpha\sigma} \left(\nabla_\mu \delta g_{\sigma\nu} + \nabla_\nu \delta g_{\sigma\mu} - \nabla_\sigma \delta g_{\mu\nu} \right) \,. \tag{3.28}$$

So, the final result is that $g^{\mu\nu}\delta R_{\mu\nu} = \nabla_\mu K^\mu$ with K^μ a four-vector field linear in covariant derivatives of $\delta g_{\mu\nu}$. Since this variation contributes a total derivative, it is tempting to neglect it on the grounds that the metric is fixed on the boundary of spacetime, in accord with common practice in the Lagrangian formalism. There is a subtlety though, in that K^μ depends on $\delta g_{\mu\nu}$ through its covariant derivative, and the condition that these vanish is not quite the same as the vanishing of $\delta g_{\alpha\beta}$. In fact, as remarked by Hawking and Gibbons, one can add a boundary term to Hilbert's action in such a way that those terms are cancelled out, and one is left with a standard variational principle. [5] The resulting equations are identical to those obtained by a naive procedure in which one simply neglects the K^μ term, so that we will not worry about this issue at this point.

Collecting all terms, the variation of the gravitational action is

$$\delta S_{\mathrm{H}} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left(-R^{\mu\nu} + \tfrac{1}{2}\, g^{\mu\nu}(R - 2\Lambda) \right) \delta g_{\mu\nu} \,. \tag{3.29}$$

Next we turn to the matter sector. Here, the first order variation defines the *energy-momentum tensor*, according to (2.60) and (2.61), through

$$\delta S_m = \int d^4x \sqrt{-g}\, \tfrac{1}{2}\, T^{\mu\nu} \delta g_{\mu\nu} \,. \tag{3.30}$$

[5] The Gibbons–Hawking term is given by $S_{\mathrm{GH}} = \frac{1}{\kappa^2} \oint K$, where the integral is defined over the boundary, and K is the trace of the extrinsic curvature associated to the induced metric.

In terms of the Lagrangian,

$$T^{\mu\nu} = 2\frac{\partial \mathcal{L}_m}{\partial g_{\mu\nu}} + g^{\mu\nu}\,\mathcal{L}_m \; . \tag{3.31}$$

Finally, we can put all the terms together and write down Einstein's equations as

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}\,g_{\mu\nu}(R - 2\Lambda) = \kappa^2 T_{\mu\nu} \; , \tag{3.32}$$

a set of ten non-linear second-order partial differential equations. The non-linearity, which persists even for vacuum solutions, makes it highly non-trivial to find exact solutions. Flat Minkowski space $g_{\alpha\beta} = \eta_{\alpha\beta}$ is of course a solution with $T_{\mu\nu} = \Lambda = 0$. The vacuum equations are obtained by setting to zero the matter sector and tracing the equation to obtain $R - 2(R - 2\Lambda) = -R + 4\Lambda = 0$. Hence,

$$R_{\mu\nu} = \Lambda g_{\mu\nu} \tag{3.33}$$

gives the vacuum equations. Manifolds solving the vacuum Einstein equations are called *Einstein manifolds*. For $\Lambda = 0$ we have the so-called *Ricci-flat manifolds*.

The cosmological constant term is equivalent to a vacuum energy on the matter energy-momentum tensor. For example, if the matter sector contains a scalar field with energy-momentum tensor

$$T^{(\phi)}_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\tfrac{1}{2}\,(\partial\phi)^2 + V(\phi)\right) \; ,$$

we see that the shift $V(\phi) \to V(\phi) + \Lambda/\kappa^2$ in the origin of potential energies introduces the cosmological constant if it was zero on the left hand side of Einstein's equations. Alternatively, for a perfect fluid ansatz,

$$T^{(\text{fluid})}_{\mu\nu} = (p + \rho)\,U_\mu U_\nu + p\,g_{\mu\nu} \; ,$$

the cosmological constant is equivalent to a matter component with the equation of state $p = -\rho = -\Lambda$, so that we may fold Λ into the non-gravitational Lagrangian as a matter of convention and write the equations as [6]

$$G_{\mu\nu} = \kappa^2\,T_{\mu\nu} \; . \tag{3.34}$$

It remains to determine the absolute normalization of the action, i.e. the relation between κ and ordinary Newton's constant G. One way of fixing this normalization will be explained in the next section, using the matching of the Newtonian potential to the leading linearized solution of (3.34). Here we will offer an alternative argument which refers directly to the physical interpretation of the Riemann tensor as the relativistic version of the tidal force tensor. We begin by rewriting (3.34) in the equivalent form

$$R_{\mu\nu} = \kappa^2\,(T_{\mu\nu} - \tfrac{1}{2}\,g_{\mu\nu}T) \; , \tag{3.35}$$

where $T = g^{\alpha\beta}T_{\alpha\beta}$ is the trace of the energy-momentum tensor. From the non-relativistic limit of the tidal equation (3.24) we learn that

$$\vec{\partial}^{\,2}\phi_{\rm N} \approx R_{00} = \kappa^2(T_{00} + \tfrac{1}{2}\,T) \; , \tag{3.36}$$

[6] This innocent-looking statement causes the so-called 'cosmological constant problem'; the astonishing fact that the measured value of Λ, given by $G\,\Lambda \sim 10^{-120}$, is so incredibly small, despite the multitude of independent contributions to Λ from the matter sector, each of them potentially much larger than the experimental bound.

where we have used $g_{\mu\nu} \approx \eta_{\mu\nu}$. Assuming that the source is given by a non-relativistic system of slow particles, we have $T_{00} \approx \rho_m$ and $T = -T_{00} + \sum_i T_{ii} \approx -T_{00}$, so that the right-hand side of (3.36) is $\frac{\kappa^2}{2}\,\rho_m$, which agrees with the Poisson equation for

$$\kappa^2 = 8\pi G\ . \tag{3.37}$$

Einstein's equations can be viewed as a set of partial differential equations determining the metric $g_{\mu\nu}$ on a spatial surface at time t, given initial data on a spatial surface at time $t = t_0$. The initial data are 'positions' $g_{\mu\nu}(t_0)$, and 'velocities' $\partial_t g_{\mu\nu}(t_0)$. Apparently, the system of ten equations just determines the ten components of the metric, but this is actually deceptive, since there are four degrees of freedom not determined by the equations, related to Diff covariance. Indeed, only the six spatial equations

$$G_{ij} = 8\pi G\, T_{ij}$$

contain second derivatives in time and are true 'dynamical equations'. The remaining four,

$$G^{0\mu} = 8\pi G\, T^{0\mu} \tag{3.38}$$

only contain up to first derivatives in time of $g_{\mu\nu}$. This follows from the Bianchi identities, $\nabla_\mu G^{\mu\nu} = 0$, which can be written as

$$\nabla_0 G^{0\mu} = -\sum_i \nabla_i G^{\mu i}\ .$$

The right hand side contains derivatives up to second order in time, thus $G^{0\mu}$ can only contain up to *first* time derivatives of the metric. The conclusion is that the four equations (3.38) must be imposed as constraints on initial data. This remark is very useful in practice, when faced with the task of solving Einstein's equations. In many situations with high symmetry, it is useful to evaluate the $G_{00} = 8\pi G\, T_{00}$ equation first, since it often expresses "constants of the motion" in the combined gravity/matter system.

Energy-momentum 'local' conservation

The Bianchi identity on the Einstein tensor $\nabla_\mu G^{\mu\nu} = 0$ implies, via the Einstein equations, a covariant conservation law for the energy-momentum tensor of all non-gravitational degrees of freedom,

$$\nabla_\mu\, T^{\mu\nu} = 0\ . \tag{3.39}$$

We can derive this relation, as well as the Bianchi identity, from the invariance of the action under reparametrizations. Since S_m is manifestly invariant under coordinate transformations: $x \to x'(x)$, we have for infinitesimal ones $x' = x - \xi(x) + O(\xi^2)$,

$$\delta_\xi S_m = \int d^4x \sqrt{-g}\, \tfrac{1}{2}\, T^{\mu\nu}\, \delta_\xi g_{\mu\nu} = 0\ , \tag{3.40}$$

with $\delta_\xi g_{\mu\nu}$ the variation of the metric *functions* under the reparametrization, i.e.

$$g'_{\mu\nu}(x) - g_{\mu\nu}(x) \equiv \delta_\xi g_{\mu\nu}(x) + O(\xi^2)\ . \tag{3.41}$$

Using

$$g'_{\mu\nu}(x) = g'_{\mu\nu}(x' + \xi) = g'_{\mu\nu}(x') + \xi^\alpha \partial_\alpha g'_{\mu\nu}(x') + O(\xi^2)$$

and the tensor law

$$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^{\,\mu}} \frac{\partial x^\beta}{\partial x'^{\,\nu}} \, g_{\alpha\beta}(x)$$

to linear order in ξ, we find

$$\delta_\xi g_{\mu\nu} \equiv \pounds_\xi g_{\mu\nu} = \xi^\alpha \partial_\alpha g_{\mu\nu} + g_{\mu\alpha} \partial_\nu \xi^\alpha + g_{\nu\alpha} \partial_\mu \xi^\alpha = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \ . \tag{3.42}$$

This is referred to as the Lie derivative of the metric in the mathematical literature. [7] Introducing this back into the variation of the action we obtain, upon integration by parts,

$$0 = \int d^4x \sqrt{-g} \, \nabla_\mu T^{\mu\nu} \xi_\nu \tag{3.43}$$

for all ξ_ν, which leads to the covariant conservation relation. Expanding the covariant derivative,

$$\partial_\mu T^{\mu\nu} = -2\Gamma^\mu_{\mu\alpha} T^{\alpha\nu} \tag{3.44}$$

we see that local conservation of matter energy-momentum is spoiled by the 'gravitational forces' proportional to the connection coefficients. We can say that energy and momentum is transferred back and forth between matter and the gravitational field. This looks similar to analogous situations in SR where one has to include energy densities for all fields in the problem in order to gain strict conservation. In this case, it would be natural to search for a local energy-momentum tensor for the gravitational field alone, so that, when added to $T_{\mu\nu}$ the sum would be locally conserved. In fact, it turns out that such an object exists, but it cannot be defined as a tensor, i.e. it does not have 'intrinsic' properties. A particularly striking manifestation of this is the fact that, on a freely falling frame through a point P, the gravitational field and whatever its energy density might be, dissapears locally. All we are left with on this frame is the SR equation

$$\partial_a T^{ab}\Big|_P = 0 \ .$$

Hence, purely gravitational energy density is a fundamentally global construct. It cannot be localized in the usual sense.

It remains the interesting question of what conditions must be imposed on a spacetime so that it supports local conservation of the matter energy-momentum alone, i.e. which spacetimes behave in this respect like Minkowski space? Recall that any conservation law must be phrased in terms of a locally conserved current. In SR, one defines the local *density* of four-momentum as $J^a_u = -u_b T^{ab}$ for a field of local observers with four-velocity u^a. If all four-velocities are parallel, so that the test observes are at relative rest, we have $\partial_a u_b = 0$ and the so-defined current is conserved $\partial_a J^a_u = 0$. The analog in curved spacetime is the covariant conservation $\nabla_\mu J^\mu_\xi = 0$ for

$$J^\mu_\xi = -T^{\mu\nu} \xi_\nu \ , \tag{3.45}$$

[7] Repeating this for a general tensor, we obtain the tensorial generalization of Lie derivatives along the four-vector ξ as

$$\pounds_\xi T^{\mu\cdots}{}_{\nu\cdots} = \xi^\alpha \partial_\alpha T^{\mu\cdots}{}_{\nu\cdots} + \partial_\nu \xi^\alpha \, T^{\mu\cdots}{}_{\alpha\cdots} + \cdots - \partial_\alpha \xi^\mu \, T^{\alpha\cdots}{}_{\nu\cdots} - \cdots$$

When the torsion vanishes, we can replace all ordinary derivatives by covariant derivatives.

defined in terms of some some vector field ξ^μ of local observer four-velocities. Now, given $\nabla_\mu T^{\mu\nu} = 0$, the current J^μ_ξ is conserved if and only if the vector field satisfies $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$, i.e. the metric has vanishing Lie derivative in the ξ direction. In this case, we say that the integral curves of ξ define an *isometry* of the metric. Hence, for spacetimes with special symmetries, local conservation of matter energy-momentum still holds.

3.2 Weak gravitational fields

We consider now the linearized approximation of Einstein's equations with $\Lambda = 0$, valid for the physical situation of weak gravitational fields in a Minkowski background or more generally for weak gravitational fields in any space-time whose radius of curvature is much larger than the length scales of interest. In the course of this analysis, we will make contact with the Newtonian theory, define global notions of mass and angular momentum in asymptotically flat spaces, and obtain an elementary discussion of gravitational radiation.

The basic situation under consideration is that of metrics of the form

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \ , \tag{3.46}$$

with $h_{\alpha\beta}$ 'small'. The appropriate notion of 'smallness' must be qualified because of the basic ambiguity by reparametrizations. Even flat Minkowski space can show a metric tensor that has 'large' components, simply because it was written in a frame with large accelerations. Hence, we shall split the metric as above, as a Minkowskian part plus a perturbation, adopting a *preferred* Minkowskian frame in which the flat part of the metric is the standard Minkowski one η_{ab}. After this preferred frame x^a has been chosen, only Lorentz transformations on the x^a are allowed, leaving the background metric η_{ab} invariant. The action of a general infinitesimal reparametrization on this preferred Minkowskian frame, $x^a \to x^a - \xi^a(x)$, induces a change of the metric *functions*

$$g_{ab}(x) \to g_{ab}(x) + \partial_a\xi_b(x) + \partial_b\xi_a(x) + O(\xi^2) \ , \tag{3.47}$$

which can be reinterpreted as a redefinition of h_{ab}, rather than a reparametrization of the Minkowskian coordinates x^a. Therefore, the linearized gravitational field tensor h_{ab} inherits a *gauge redundancy*

$$h_{ab} \to h_{ab} + \partial_a\xi_b + \partial_b\xi_a \tag{3.48}$$

as the linearized approximation of the general coordinate invariance of the underlying theory.

We can now linearize Einstein's equations, starting from

$$\Gamma^c_{ab} = \Gamma^c_{ab}{}^{(1)} + O(h^2) = \tfrac{1}{2}\,\eta^{cd}\left(\partial_a h_{bd} + \partial_b h_{ad} - \partial_d h_{ab}\right) + O(h^2) \ . \tag{3.49}$$

In the following, it will be convenient to raise and lower Lorentz indices with the Lorentz metric η_{ab}. The Ricci tensor linearizes as

$$R_{ab} = R^{(1)}_{ab} + O(h^2) = -\partial_a\Gamma^c_{bc}{}^{(1)} + \partial_c\Gamma^c_{ab}{}^{(1)} + O(h^2) \ , \tag{3.50}$$

and the Einstein tensor is given by [8]

$$G_{ab} = G^{(1)}_{ab} + O(h^2) = \tfrac{1}{2}\left(-\partial^2\gamma_{ab} + \partial_a\partial^c\gamma_{cb} + \partial_b\partial^c\gamma_{ca} - \eta_{ab}\partial^c\partial^d\gamma_{cd}\right) + O(h^2) \tag{3.51}$$

in terms of the related field

$$\gamma_{ab} \equiv h_{ab} - \tfrac{1}{2}\,\eta_{ab}h \ , \qquad h \equiv \eta^{ab}h_{ab} \ . \tag{3.52}$$

The Einstein tensor simplifies considerably by choosing an analog of the Lorentz gauge in electrodynamics, i.e.

$$\partial^a\gamma_{ab} = 0 \ . \tag{3.53}$$

[8]**Exercise:** Verify this form.

Under these conditions, Einstein's equations boil down to

$$-\partial^2\gamma_{ab} = 16\pi G\, T_{ab} \;, \tag{3.54}$$

which takes the form of a standard wave equation, with an equally standard solution in terms of retarded potentials

$$\gamma_{ab}(x) = 4G \int \frac{T_{ab}(t - |\vec{x} - \vec{y}\,|,\, \vec{y}\,)}{|\vec{x} - \vec{y}\,|}\, d^3y \tag{3.55}$$

for fields that vanish at infinity. Fields that do not vanish at infinity can be incorporated by adding solutions of the homogeneous vacuum equation, $\partial^2\gamma_{ab} = 0$, i.e. gravitational waves propagating in *vacuo* at the speed of light.

Assuming that the energy-momentum of the sources is time-independent and non-relativistic, i.e. velocities of massive objects are small: $|T^{00}| \gg |T^{0i}| \gg |T^{ij}|$, we approximate the solution of (3.55) by $\gamma_{ij} \approx 0 \approx \gamma_{0i}$, and the dominant component is

$$\gamma_{00}(\vec{x}) \approx 4G \int d^3y\, \frac{T_{00}(\vec{y}\,)}{|\vec{y} - \vec{x}\,|} \;. \tag{3.56}$$

Since $T_{00} \approx \rho_m$ for non-relativistic sources, the matter *mass* density, we have a direct relation to the Newtonian potential ϕ_{N},

$$\gamma_{00} \approx -4\,\phi_{\mathrm{N}} \;. \tag{3.57}$$

Using now $\gamma = -h = -\gamma_{00}$ and $h_{ab} = \gamma_{ab} - \frac{1}{2}\,\eta_{ab}\gamma$ we obtain for the original metric perturbation

$$h_{00} \approx \tfrac{1}{2}\,\gamma_{00} \;, \qquad h_{0i} \approx 0 \;, \qquad h_{ij} \approx \delta_{ij} h_{00} \;,$$

which gives back the known asymptotic form of the metric perturbation,

$$h_{00} \approx -2\phi_{\mathrm{N}} = \frac{2GM}{|\vec{x}\,|} \;, \qquad M \equiv \int d^3x\, \rho_m(\vec{x}\,) \;,$$

previously derived from the non-relativistic limit of the geodesic equation. Hence, we have found the following asymptotic form of the metric as sourced by a non-relativistic system of mass M:

$$ds^2 = -\left(1 - \frac{2GM}{r} + \ldots\right) dt^2 + \left(1 + \frac{2GM}{r} + \ldots\right) d\vec{x}^{\,2} \;, \tag{3.58}$$

valid for $r = |\vec{x}\,| \gg 2GM$.

3.2.1 Systematic weak-field expansion

In a strict sense, equation (3.55) is inconsistent with any non-trivial self-gravity effects. The linearized theory coupled to the matter energy-momentum tensor satisfies $\partial^a\gamma_{ab} = 0 = \partial^a T_{ab}$. But this implies that particles contributing to T_{ab} necessarily follow geodesics of the zeroth-order metric η_{ab}, instead of the perturbed one $\eta_{ab} + h_{ab}$. We can remedy this problem by a suitable generalization of (3.55) to an *exact* equation equivalent to Einstein's equations.

Under the basic decomposition $g = \eta + h$, we write the full Einstein tensor as

$$G_{ab}(\eta + h) = G^{(1)}_{ab}(h) + G'_{ab}(\eta + h) \;, \tag{3.59}$$

where G'_{ab} is simply the deviation of the exact Einstein tensor from the linear approximation. By definition, the small field expansion of G'_{ab} starts at *quadratic* order in h_{ab}. Defining now

$$t_{ab}(h) \equiv -\frac{1}{8\pi G} G'_{ab}(\eta + h) \tag{3.60}$$

we see that Einstein's equations can be exactly rewritten as

$$G^{(1)}_{ab} = 8\pi G\, \mathcal{T}_{ab} \;, \tag{3.61}$$

where

$$\mathcal{T}_{ab} \equiv T_{ab} + t_{ab} \tag{3.62}$$

is a sort of *total* energy momentum tensor of matter and gravitation. This definition has t_{ab} obviously playing the role of the energy-momentum 'tensor' of the gravitational field, and is completely tied to the particular background flat spacetime and the family of coordinate systems in which its metric takes the Minkowskian simple form. The quotations here refer to the fact that t_{ab} is not a true tensor with respect to general diffeomorphisms of the full spacetime. It is only a Lorentz tensor with respect to the flat background spacetime. One should not become mystified about this fact, since the whole perturbative construction singles out the Minkowski metric η_{ab} as special, and the equations are written in manifestly non-covariant form.

Now, in the situation in which the complete solution for γ_{ab} vanishes at infinity, the equation (3.55) still holds exactly with the replacement of T_{ab} everywhere by $\mathcal{T}_{ab}$

$$\gamma_{ab}(x) = 4G \int d^3y \, \frac{\mathcal{T}_{ab}(t - |\vec{y} - \vec{x}\,|, t)}{|\vec{x} - \vec{y}\,|} \;. \tag{3.63}$$

Notice that now $\mathcal{T}_{ab}$ depends on γ_{ab}, so that this is an integral equation for γ_{ab}, containing exactly the same information as Einstein's equations (plus the condition that γ_{ab} vanishes at infinity, preserving the asymptotic flatness). The systematics of the weak-field expansion can be obtained from the iterative solution of this integral equation. Namely, the n-th order term in the evaluation of h_{ab} via eq. (3.63) uses the $(n-1)$-th order evaluation of the gravitational stress-energy tensor t_{ab}, plus the matter energy-momentum tensor calculated in the $(n-1)$-th order background metric, after solving the matter equations. In particular, this construction only requires that γ_{ab} is a weak field at large distances, since (3.63) is valid for arbitrarily strong fields which still preserve the condition of asymptotic flatness. Hence, the iterative weak-field solution of (3.63) can be understood as a gradual reconstruction of the field 'from outside in'.

The standard local conservation law in Minkowski space also holds *exactly* in terms of the complete tensor:

$$\partial_a \mathcal{T}^{ab} = 0 \;, \tag{3.64}$$

as a result of the exact transversality of the linearized Einstein tensor, $\partial^a G^{(1)}_{ab} = 0$.

This iterative scheme shows precisely how the Lorentz-invariant theory of a symmetric tensor field (the Fierz–Pauli scheme) must be corrected by nonlinear terms in order to achieve consistency with energy-momentum conservation (3.64). Hence, in hindsight we have found that Einstein's theory is the minimal consistent theory of a symmetric tensor field.

Sometimes the statement is made that this formalism has problems with gauge invariance, since the leading order (quadratic in γ_{ab}) gravitational tensor $t^{(2)}_{ab}$, as well as its higher-order cousins, are *not* gauge invariant under the linearized transformation $h_{ab} \to h_{ab} + \partial_a \xi_b + \partial_b \xi_a$.

In fact, it turns out that the complete *exact* $\mathcal{T}_{ab}$ must be gauge-invariant under linearized gauge transformations when evaluated on a solution of Einstein's equations (off shell, the gauge group is the full non-linear Diff group).

For sufficiently localized systems, with T_{ab} of compact support, and in the absence of radiation, one finds that $\gamma_{ab} = O(1/r)$ at spatial infinity. Then $\mathcal{T}_{ab} = O(1/r^4)$ and its integral over a spatial section converges. In this situation, we can derive a useful formula (due to Arnowitt, Deser and Misner) for the total energy in asymptotically flat spacetimes. In the conformal gauge $\partial^a \gamma_{ab} = 0$, the exact Einstein equations (3.61) can be written as $-\partial^2 \gamma_{ab} = 16\pi G\, \mathcal{T}_{ab}$, so that

$$E_{\rm ADM} = \int_{\mathbf{R}^3} \mathcal{T}_{00} = -\frac{1}{16\pi G} \int_{\mathbf{R}^3} \partial^2 \gamma_{00} \ . \tag{3.65}$$

By repeated use of the transvesality condition, $\partial^a \gamma_{ab} = 0$, we can eliminate the time derivatives to obtain [9]

$$E_{\rm ADM} = \frac{1}{16\pi G} \sum_{i,j} \oint_{\mathbf{S}^2_\infty} dS^i \left(\partial_j \gamma_{ij} - \partial_i \gamma_{00}\right) = \frac{1}{16\pi G} \sum_{i,j} \oint_{\mathbf{S}^2_\infty} dS^i \left(\partial_j h_{ij} - \partial_i h_{jj}\right) \ . \tag{3.66}$$

This formula shows that the total energy in stationary, asymptotically flat spacetimes only depends on the asymptotic properties of the metric.[10] This result can be generalized to obtain similar expressions for the total momentum and angular momentum of localized self-gravitating systems. These results are presented in the next section using a different method.

Long-distance metric components for isolated sources

We have seen that the leading Newtonian approximation to (3.55) yields a long-distance metric field $h_{00} \sim 2GM/r$. The long-distance expansion can be systematized by performing a *multipole expansion* of the integral, i.e. we consider sources with effectively 'compact' support over a region of average size $\ell \ll |\vec{x}\,|$ and expand

$$\frac{1}{|\vec{x} - \vec{y}\,|} = \frac{1}{r} + \frac{\vec{x}\cdot\vec{y}}{r^3} + \dots \ ,$$

with $r \equiv |\vec{x}\,|$. Monopole terms are of $O(1/r)$, dipole terms of $O(1/r^2)$ and so on. This expansion can be applied to all components on the metric field γ_{ab}. For simplicity, we shall continue with our assumption that T_{ab} is time-independent, although this hypothesis can be lifted by a more detailed treatment of the multipole expansion.

For time-independent T_{ab}, the conservation equation $\partial_a T^{ab} = 0$ implies $\sum_k \partial_k T_{ak} = 0$. This leads to some useful indentities for the spatial integrals of the energy-momentum tensor. Denoting $\int d^3y \equiv \int_{\mathbf{R}^3}$, we have

$$\int_{\mathbf{R}^3} T_{aj} = \int_{\mathbf{R}^3} \sum_k \partial_k \left(T_{ka} x_j\right) = 0 \ , \tag{3.67}$$

where we have neglected the boundary terms in the last step, under the assumption that T_{ab} has compact support. Setting $a = 0$ in this equation we obtain that the total momentum

[9] **Exercise**: Complete the calculations.

[10] An important theorem by Shoen and Yau establishes that $E_{\rm ADM} \geq 0$ for any smooth, asymptotically flat, solution of Einstein's equations, provided $T_{\mu\nu}$ satisfies appropriate positivity conditions.

vanishes $P_i = \int_{\mathbf{R}^3} T_{0i} = 0$. With $a = i$, we see that the volume average of the stresses T_{ij} also vanishes. This implies that the monopole approximations to γ_{0i} and γ_{ij} both vanish, and these terms are dominated by the dipole contribution. Since $h_{0i} = \gamma_{0i}$ and $h_{ij} = \gamma_{ij} - \frac{1}{2}\,\delta_{ij}\gamma$, with $\gamma = -\gamma_{00} + \text{dipole terms}$, we see that h_{ij} is dominated by the monopole contribution to γ_{00}.

Hence, in order to capture the leading behaviour of the metric perturbation, it suffices to consider the monopole approximation to γ_{00} and the dipole approximation to γ_{0i}. In these conditions, the monopole and dipole approximations to γ_{0i}:

$$h_{0i}(\vec{x}\,) = \gamma_{0i}(\vec{x}\,) = \frac{4G}{r}\int_{\mathbf{R}^3} T_{0i} + \frac{4G}{r^3}\sum_j x_j \int_{\mathbf{R}^3} y_j\, T_{0i} + \dots\ .$$

The first monopole term, proportional to the total momentum of the system, vanishes for time-independent sources. More generally, it can be eliminated by adjusting the frame to be at rest with respect to the sources. Then, as already stated above, the leading contribution to h_{0i} is the dipole term. We can simplify it by using again the time-independence condition to write

$$0 = \int_{\mathbf{R}^3} y_i y_j \sum_k \partial_k T_{0k} = -\int_{\mathbf{R}^3} (y_j\, T_{0i} + y_i\, T_{0j})\ ,$$

so that the dipole term is purely antisymmetric in the indices i, j. Finally, using that the angular momentum is defined as

$$J_k \equiv \tfrac{1}{2}\sum_{ij} \epsilon_{ijk} J_{ij} \equiv -\tfrac{1}{2}\sum_{ij}\epsilon_{ijk}\int_{\mathbf{R}^3}(y_i T_{0j} - y_j T_{0i})\ , \tag{3.68}$$

(the minus sign is there because we have lowered the time component of the energy-momentum tensor) we see that we can read off the angular momentum of the source system from the $O(1/r^2)$ tail of the off-diagonal metric components,

$$h_{0i}(\vec{x}\,) = \frac{2G}{r^3}\sum_{j,k}\epsilon_{ijk} x_j J_k\ . \tag{3.69}$$

Hence, in *stationary situations*, the long-distance behaviour of the asymptotically flat spacetime is controlled by the total mass and angular momentum of the sources 'inside' by the following $r \to \infty$ asymptotics:

$$ds^2 \to -\left(1 - \frac{2GM}{r} + \dots\right)dt^2 + \left(\frac{4G}{r^3}(\vec{x}\times\vec{J}\,) + \dots\right)\cdot d\vec{x}\,dt + \left(1 + \frac{2GM}{r} + \dots\right)d\vec{x}^{\,2}\ . \tag{3.70}$$

Following our general discussion above, these formulas still hold in the presence of strong self-gravity effects, with M and $\vec{J}$ calculated in terms of the total energy-momentum pseudotensor $\mathcal{T}_{ab}$.

The asymptotic form (3.70) gives the leading deviation from SR in computing the effects of weak gravitational fields on test particle probes, such as the case of GR effects on the solar system. The particular forms of the metric functions are universal predictions of Einstein's theory which control the leading effects of gravitational time delays and gravitational bending of light. It is interesting, however, that the classic test of Mercury's perihelion advance requires going to order $(GM/r)^2$ in the weak-field expansion for g_{00}, which will be determined in the following chapter to be

$$g_{00} = -1 + \frac{2GM}{r} - \frac{2G^2M^2}{r^2} + \dots$$

To see this, consider a metric of the form

$$ds^2 = -(1+2\phi+\alpha\phi^2)dt^2 + 2\vec{g}\cdot d\vec{x}\,dt + (1-2\phi)\,d\vec{x}^{\,2}\ , \tag{3.71}$$

with α some constant, and expand the particle Lagrangian up to next-to-leading order in the non-relativistic approximation, and to leading order in $\vec{g}$. Since ϕ is, to leading order, the Newtonian potential $\phi \approx -GM/r$, we have $v^2 \sim \phi$ in typical bound orbital motions. Therefore, keeping terms to $\mathcal{O}(v^4)$ requires that we also keep the $\mathcal{O}(\phi^2)$ and $\mathcal{O}(v^2\phi)$ terms:

$$L = -m\sqrt{-g_{ab}\frac{dx^a}{dt}\frac{dx^b}{dt}} \approx -m+\tfrac{1}{2}\,m\,\vec{v}^{\,2}+\tfrac{1}{8}m\,\vec{v}^{\,4}-m\,\phi+\tfrac{1}{2}\,m\,(1-2\alpha)\phi^2-\tfrac{3}{2}\,m\,\phi\,\vec{v}^{\,2}+m\,\vec{g}\cdot\vec{v}+\dots\ , \tag{3.72}$$

so that the $\mathcal{O}(\phi^2)$ term in the time-time component g_{00} of the metric (3.71) does contribute to next-to-leading order to the relativistic corrections. Notice, however, that a similar term in the spatial components of the metric only contributes to order $\phi^2\vec{v}^{\,2}$.

The $\mathcal{O}(\vec{v}^{\,4})$ and $\mathcal{O}(\phi^2)$ terms are respectively the leading relativistic corrections to the kinetic and potential energy of the particle. On the other hand, the $\mathcal{O}(\phi\,\vec{v}^{\,2})$ and $\mathcal{O}(\vec{g}\cdot\vec{v}\,)$ terms represent qualitatively new forces that depend on velocities. The first of these is responsible for the so-called 'geodetic' precession of gyros (see the problem below) in orbit, whereas the second one is a genuine 'magnetic' coupling, to be discussed in the next section.

In the analysis of precision experimental tests, one usually parametrizes the deviation from Einstein's theory by the so-called post-Newtonian parameters which represent the metric *ansatz*

$$ds^2 \to -\left(1-\frac{2GM}{r}+(1+\bar{\beta}\,)\frac{2G^2M^2}{r^2}+\dots\right)dt^2+\left(1+(1+\bar{\gamma})\frac{2GM}{r}+\dots\right)d\vec{x}^{\,2}+\dots\ , \tag{3.73}$$

where we neglect the usually very small off-diagonal term at this level and have reverted to more standard notation in which $\alpha = 1+\bar{\beta}$. Solar-system precision tests currently bound $\bar{\gamma}, \bar{\beta}$ to be at most of order 10^{-5}.

Gravitomagnetism

The off-diagonal terms in (3.70) and (3.71) have a characteristic effect on test particles. As we have seen in (2.38), its effect can always be reinterpreted as inducing a 'Coriolis-like' force with effective angular velocity $\frac{1}{2}\,\vec{\partial}\times\vec{g}\,$. More precisely, we have 'Coriolis field', since the effective angular velocity has a non-trivial space dependence. In another suggestive analogy, a term $\delta L = m\,\vec{g}\cdot\vec{v}$ in the effective particle Lagrangian has the same form as a coupling of a charged particle to an electromagnetic vector potential. Hence, we may regard such coupling as 'gravitomagnetic'. Using the explicit form of $\vec{g}$ we can write the effective interaction in the form of a 'magnetic moment' correction:

$$\delta L = m\,\vec{g}\cdot\vec{v} = -\vec{\mu}_J\cdot\vec{L}\ , \qquad \vec{\mu}_J \equiv -\frac{2G}{r^3}\vec{J}\ . \tag{3.74}$$

As a result, we find the so-called Lense–Thirring precession effect on the orbital momentum, as a direct analog of the Larmor precession in a magnetic field. To derive it, notice that a correction $\delta L = m\vec{g}\cdot\vec{v}$ to the Lagrangian produces, to leading order in $\vec{g}$, a correction $\delta H = -\vec{p}\cdot\vec{g}$ to the

Hamiltonian. Using then the Hamilton equations one obtains

$$\frac{d\vec{L}}{dt} = \frac{\partial H}{\partial \vec{p}} \times \vec{p} - \vec{x} \times \frac{\partial H}{\partial \vec{x}} = \frac{2G}{r^3} \left(-(\vec{x} \times \vec{J}\,) \times \vec{p} + \vec{x} \times (\vec{J} \times \vec{p}) \right) \,,$$

which further reduces to

$$\frac{d\vec{L}}{dt} = \vec{\mu}_J \times \vec{L} \,. \tag{3.75}$$

We find that orbital planes undergo a precession induced by a spin-orbit coupling which is entirely analogous to the one operating between the nuclear spin and the electron's orbital motion in atoms. This effect is also responsible for a small precession of perihelia of the same order of magnitude. When applied to the motion of gyroscopes, we obtain the same physics described by the spin free-fall equation (2.58).

Problem: Orbital precession

Consider the secular effect of the sun's spin on the orbit of a planet. In addition to the Larmor-like precession of the orbital plane obtained in (3.75), one can derive a precession of perihelia within the plane of the orbit, by studying the evolution of the Runge–Lenz vector

$$\vec{\mathcal{A}} = \frac{1}{m} \vec{p} \times \vec{L} - \frac{GMm}{r} \vec{x} \,,$$

which is accidentally conserved under the purely newtonian Kepler motion, staying constant in the orbital plane and directed towards the perihelion. Therefore, the two vectors $\vec{L}$ and $\vec{\mathcal{A}}$ together determine the overall orientation of the orbit.

Compute the time derivative of the Runge–Lenz vector to obtain

$$\frac{d\vec{\mathcal{A}}}{dt} = \frac{2G}{r^3} \vec{J} \times \vec{\mathcal{A}} + \frac{6G}{mr^5} (\vec{J} \cdot \vec{L})(\vec{x} \times \vec{L}) \,.$$

The first term gives the Larmor-like precession. Upon further averaging over a full revolution, show that

$$\left\langle \frac{d\vec{V}}{dt} \right\rangle_{\text{secular}} = \vec{\Omega}_{\text{LT}} \times \vec{V} \,,$$

where $\vec{V}$ is either $\vec{L}$ or $\vec{\mathcal{A}}$, and the Lense–Thirring angular velocity is given by

$$\vec{\Omega}_{\text{LT}} = \frac{2G|\vec{J}\,|}{a^3(1-e^2)^{3/2}} \left(\vec{n}_J - 3\, \vec{n}_L (\vec{n}_J \cdot \vec{n}_L) \right) \,,$$

where a is the semimajor axis of the ellipse and e its eccentricity, while $\vec{n}_J$ and $\vec{n}_L$ are unit vectors in the $\vec{J}$ and $\vec{L}$ directions, respectively.

We find that the full orbit as a whole precesses around the Lense–Thirring vector $\vec{\Omega}_{\text{LT}}$.

Problem: Precession of orbiting gyroscopes

Consider a spherical gyroscope in orbit. Starting from the term $-\frac{3}{2} m\phi\, \vec{v}^{\,2}$ in (3.72), integrate it over the particles forming the gyroscope to obtain an effective Lagrangian

$$\delta L_{\text{eff}} \approx \frac{3GM}{2} \int d^3z\, \rho(\vec{z}\,) \frac{(\vec{v}_0 + \vec{\omega} \times \vec{z}\,)^2}{|\vec{r}_0 + \vec{z}\,|} \,.$$

where $\vec{v}_0$ and $\vec{r}_0$ are the velocity and position of the gyro's centre of mass, with mass density $\rho(\vec{z}\,)$, and $\vec{\omega}$ is its angular velocity. We further consider the cross-term proportional to $\vec{v}_0 \cdot (\vec{\omega} \times \vec{z}\,)$ and use the dipole approximation

$$\frac{1}{|\vec{r}_0 + \vec{z}\,|} \approx \frac{1}{r_0} - \frac{\vec{z} \cdot \vec{r}_0}{r_0^3} + \dots \; .$$

Calculate the integral using

$$\int d^3z\, \rho(\vec{z}\,)\, z_i z_j = \tfrac{1}{2}\, I\, \delta_{ij} \; ,$$

with I the gyro's moment of inertia, so that its spin is $\vec{S} = I\,\vec{\omega}$. Then, prove that the effective Lagrangian reduces to

$$\delta L_{\text{geo}} \approx -\vec{\Omega}_{\text{geo}} \cdot \vec{S} \; ,$$

where the *geodetic* frequency is given by

$$\vec{\Omega}_{\text{geo}} \approx \frac{3GM}{2r_0^3} (\vec{r}_0 \times \vec{v}_0\,) \; ,$$

and induces the so-called *geodetic* precession of the top's spin

$$\frac{d\vec{S}}{dt} = \vec{\Omega}_{\text{geo}} \times \vec{S} \; .$$

There is also a *hyperfine* effect (also referred to as Lense–Thirring effect), coupling directly the gyro's spin $\vec{S}$ to the spin of the gravitational field, $\vec{J}$, induced by the "magnetic" term $2G\vec{v} \cdot (\vec{x} \times \vec{J})/r^3$, i.e.

$$\delta L = 2G \int d^3z\, \rho(\vec{z}\,)\, \frac{\vec{J} \cdot ((\vec{v}_0 + \vec{\omega} \times \vec{z}\,) \times \vec{r}\,)}{r^3} \; ,$$

with $\vec{r} = \vec{r}_0 + \vec{z}$. Perform this integral in the dipole approximation to obtain

$$\delta L_{\text{hyper}} \approx -\vec{\Omega}_{\text{hyper}} \cdot \vec{S} \; ,$$

where

$$\vec{\Omega}_{\text{hyper}} = \frac{3G\vec{r}_0(\vec{r}_0 \cdot \vec{J}\,)}{r_0^5} - \frac{G\vec{J}}{r_0^3} \; .$$

The geodetic precession for the case of satelites around Earth is in the ball park of 10 seconds of arc per year. The hyperfine precession is smaller by a factor

$$\frac{|\vec{\Omega}_{\text{hyper}}|}{|\vec{\Omega}_{\text{geo}}|} \sim 6.5 \times 10^{-3} \; .$$

3.2.2 Gravitational radiation

We consider now situations where the sources have a non-trivial time dependence, leading to *radiation*-type gravitational fields. We first assume that the energy-momentum distribution of the sources has compact support, and the radiation is analyzed *in vacuo*, far away from such sources. This means that the radiation field γ_{ab} solves the free wave equation far away from the sources, together with the transversality condition:

$$\partial^2 \gamma_{ab}^{(\text{w})} = \partial^a \gamma_{ab}^{(\text{w})} = 0 \; . \tag{3.76}$$

The gauge symmetry (3.48) acts on γ_{ab} as

$$\gamma_{ab} \to \gamma_{ab} + \partial_a \xi_b + \partial_b \xi_a - \eta_{ab}\, \partial^c \xi_c \ , \tag{3.77}$$

and the transversality condition $\partial^a \gamma_{ab} = 0$ is preserved by all transformations of the form (3.77) with harmonic vector fields $\partial^2 \xi_a = 0$. Hence, out of the six degrees of freedom left by (3.76) we may still eliminate four, leaving just two non-redundant degrees of freedom in γ_{ab}. We can easily demonstrate this in momentum space, since an arbitrary solution to (3.76) is a superposition of plane waves $\gamma_{ab}(p) \propto \exp(ipx) + \text{c.c.}$ with $p^2 = 0$. Choosing the frame so that $p = (\omega, 0, 0, \omega)$ the transversality conditions reduce to

$$p^a \gamma_{ab} = 0 = \omega(\gamma_{0b} + \gamma_{3b}) \ . \tag{3.78}$$

The four components ξ_b can be used to set $h_{0a} = 0$ which, together with (3.78) may be used to enforce the vanishing of all components of γ_{ab} and h_{ab} except for the two-dimensional submatrix with components along the (12)-plane, subject to the extra condition of vanishing trace. We refer to such a standard form of the fluctuating metric tensor as the 'transverse-traceless gauge, $h^{\rm TT}_{ab} = \gamma^{\rm TT}_{ab}$.

Defining $\varepsilon_\pm = \widetilde{h}_{11} \mp i\widetilde{h}_{12}$, a rotation of angle θ on the (12) plane transforms them as

$$\varepsilon_\pm \to e^{\pm 2i\theta} \varepsilon_\pm \ . \tag{3.79}$$

In this situation, we say that the plane waves have *helicity* ± 2. This is what is meant by referring to gravitational waves as spin-two excitations.

Gravitational waves are truly physical, capable of transferring energy to any detector sensitive to gravitational fields. For example, the nonrelativistic limit of the geodesic deviation equation yields [11]

$$\frac{d^2 \ell_j}{dt^2} \approx \sum_k \ell_k R_{k00j} \approx \sum_k \tfrac{1}{2}\, \frac{d^2 h^{\rm TT}_{jk}}{dt^2}\, \ell_k \ , \tag{3.80}$$

so that gravitational waves induce non-zero curvature and produce accelerations on the relative positions ℓ_j of test particles. Since metric perturbation is transverse and traceless in this gauge, the matrix h_{jk} can be written as a linear combination of the Pauli matrices σ_3 and σ_1. The component proportional to σ_3 is called the (+) wave polarization, whereas the component proportional to σ_1 is usually denoted ($\times$) and corresponds to the same polarization after a 45-degree rotation.

Luminosity in gravitational radiation

We can estimate the energetics of radiation emission in terms of some generic properties of the radiating system. We begin by noticing that the purely spatial components (those remaining in the TT gauge) are given in terms of the source's energy-momentum tensor by the leading-order expression [12]

$$\gamma_{ij}(t, \vec{x}) = 4G \int d^3 y \frac{T_{ij}(t - |\vec{x} - \vec{y}|, \vec{y}\,)}{|\vec{x} - \vec{y}\,|} \ . \tag{3.81}$$

[11] **Exercise:** Check this formula.

[12] **Exercise:** Show that the Fourier components of non-zero frequency with at least one time index, $\tilde{\gamma}_{0b}(\omega) \equiv \int dt\, e^{i\omega t} \gamma_{0b}(t)$, are determined in terms of the purely spatial ones, $\tilde{\gamma}_{ij}(\omega)$, as a consequence of the transversality constraint $\partial^a \gamma_{ab} = 0$.

Evaluating this integral in the monopole approximation we find

$$\gamma_{ij}(x) \approx \frac{4G}{r} \int_{\mathbf{R}^3} T_{ij}\Big|_{\text{retarded}} \tag{3.82}$$

with $r \equiv |\vec{x}\,|$ and the retarded time given by $t_r = t - r$. In this expression, we are neglecting the effects of the dipole corrections to the retarded time in (3.81), i.e. the terms linear in $\vec{y}$ in

$$t - |\vec{x} - \vec{y}| = t - r + \frac{\vec{x} \cdot \vec{y}}{r} + \mathcal{O}(\vec{y}^{\,2}\,)$$

These terms arise in the Taylor expansion of the energy-momentum tensor as

$$T_{ij}(t - |\vec{x} - \vec{y}\,|, \vec{y}\,) = T_{ij}(t_r, \vec{y}) + \frac{\vec{x} \cdot \vec{y}}{r}\, \partial_t\, T_{ij}(t_r, \vec{y}\,) + \dots$$

The time derivative of T_{ij} determines the frequency of the radiation, so that this term is negligible in the 'small emitter limit', $\ell/\lambda \ll 1$, namely the radiation has long wavelength compared to the size ℓ of the source.

Using now the conservation equation $\partial^b T_{ba} = -\partial_t T_{0a} + \sum_k \partial_k T_{ka} = 0$ and repeated integration by parts we can obtain the identity

$$\int_{\mathbf{R}^3} T_{ij} = \tfrac{1}{2}\, \partial_t^2 \int_{\mathbf{R}^3} T_{00}\; y_i\, y_j \ ,$$

where we neglect all surface integrals on account of the compact support of the sources' energy-momentum. Hence we finally obtain

$$\gamma_{ij}(t, \vec{x}\,) = \frac{2G}{r}\, \frac{d^2 \mathcal{Q}_{ij}}{dt^2}\Big|_{\text{retarded}} \ , \tag{3.83}$$

where the derivatives are evaluated at the 'point-like' retarded time $t_r = t - r$ and we have defined

$$\mathcal{Q}_{ij} \equiv \int_{\mathbf{R}^3} T_{00}\, y_i\, y_j \ , \tag{3.84}$$

the quadrupole moment of the source's energy density. We thus conclude that gravitational waves in the low-frequency approximation, $\omega\ell \ll 1$, are generated by the quadrupole mass distribution, unlike electromagnetic waves, which are generated by charge dipole moments. This fact is largely responsible for the smallness of gravitational radiation. Since the quadrupole moment is of order $\mathcal{Q} \sim M\ell^2$ for a system of mass M and size ℓ, the metric perturbation at large distances is of order

$$\gamma \sim \frac{GMv^2}{r} \ , \tag{3.85}$$

so that we have the usual relativistic suppression of order v^2 with respect to the static Newtonian term.

The energy carried out by such gravitational waves can be captured by measuring the flux through a large sphere at large distances from the source. In particular, the rate of energy loss is given by the *luminosity*

$$L_G = \frac{dE_{\text{rad}}}{dt} = \int_{\mathbf{S}^2_r} \sum_i n_i t^{0i} \ , \tag{3.86}$$

where $n_i = x_i/r$ is the unit radial vector on the large sphere of radius r.

To estimate (3.86) we consider first a TT wave traveling in the x^3 direction, i.e. depending on just two functions $\gamma_+(t-x^3)$ and $\gamma_\times(t-x^3)$. By direct calculation one finds that the flux in the 3-direction is given by [13]

$$t^{03} = \frac{1}{16\pi G}\left((\partial_t\gamma_+)^2 + (\partial_t\gamma_\times)^2\right) . \tag{3.87}$$

For the particular wave we are considering, we can write the same expression in the more symmetric fashion

$$t^{03} = \frac{1}{32\pi G}\sum_{ij}\partial_t\gamma_{ij}^{\rm TT}\,\partial_t\gamma_{ij}^{\rm TT} . \tag{3.88}$$

In taking the average over all directions it is useful to remove the transversality constraint, since this obviously depends on the direction. The no-trace condition can still be enforced in a rotationally-invariant way by considering the trace-free matrix $\gamma_{ij}^{(s)} \equiv \gamma_{ij} - \frac{1}{3}\delta_{ij}\delta^{kl}\widetilde{\gamma}_{kl}$ with five degrees of freedom. Hence, by substituting $\gamma^{(s)}$ in place of $\gamma^{\rm TT}$ in (3.88) we implement the averaging, up to a factor of 2/5, i.e.

$$\left\langle \sum_i n_i t^{0i}\right\rangle_{\rm directions} \approx \frac{2}{5}\cdot\frac{1}{32\pi G}\sum_{ij}\partial_t\gamma_{ij}^{(s)}\,\partial_t\gamma_{ij}^{(s)} . \tag{3.89}$$

Finally, using the long-distance expression for the radiation field (3.83) and integrating over the $\mathbf{S}_r^2$ sphere we find for the luminosity

$$L_G = \frac{G}{5}\sum_{i,j}\left\langle\frac{d^3\mathcal{Q}_{ij}^{(s)}}{dt^3}\right\rangle_{\rm retarded}^2 , \tag{3.90}$$

where $\mathcal{Q}_{ij}^{(s)} = \mathcal{Q}_{ij} - \frac{1}{3}\delta_{ij}\delta^{kl}\mathcal{Q}_{kl}$ is the traceless quadrupole moment of the matter distribution. The order of magnitude of this for power radiated at a typical frequency $\omega \sim v/\ell$ by a system of typical masses M is

$$L_G \sim GM^2\ell^4\frac{v^6}{\ell^6} \sim \frac{G^4M^5}{\ell^5} \sim M\frac{v^8}{\ell} , \tag{3.91}$$

A useful characterization of amount of energy loss in gravitational waves is the following. Given the luminosity power $L_G = dE_{\rm rad}/dt$, we can consider the amount of energy lost in the time that a single wave crest is emitted, $\omega^{-1} \sim \ell/v$, i.e. $L_G\,\ell/v$ and compare it with the gravitational self-energy, which is of order $|E_G| \sim GM^2/\ell$. Then, we find

$$\frac{\omega^{-1}\cdot L_G}{|E_G|} \sim v^5 \sim \phi^{5/2} , \tag{3.92}$$

where ϕ is the average value of the gravitational potential. If the system is not too relativistic, most of the self-energy loss can be interpreted as the 'decay' of typical orbits,

$$L_G = \frac{dE_{\rm rad}}{dt} = -\frac{dE_G}{dt} \approx -\frac{GM^2}{\ell^2}\frac{d\ell}{dt} .$$

[13]**Exercise:** Check this formula.

So the system 'shrinks' by gravitational wave emission at a rate

$$\frac{d\ell}{dt} \approx \frac{L_G \cdot \ell}{E_G} \sim -v^6 \; . \tag{3.93}$$

Equivalently, the rate of period decay for internal motions is $dT/dt \sim -v^5$. The hierarchy of gravitational field intensities in nature ranges from $\phi_{\rm bh} \sim GM/\ell \sim 1$ in the case of black holes, to $\phi_{\rm pulsar} \sim 10^{-1}$, or $\phi_{\rm white\,dwarf} \sim 10^{-4}$. In the solar system, $\phi_\odot \sim 10^{-6}$ and $\phi_\oplus \sim 10^{-9}$. Hence, we really need highly relativistic systems and intrinsically strong gravitational fields in order to have significant energy loss by gravitational radiation. The famous binary pulsar PSR1913+16 studied by Hulse and Taylor has provided such a system for decades now, fulfilling the most impressive test of Einstein's theory so far (down to the 10^{-3} level in a strong field regime). In the near future, optical interferometers on Earth and space such as LIGO and LISA should be able to detect gravitational radiation outbursts from violent, yet very remote phenomena, such as neutron star and/or black hole collisions.

Problem: Gravitational radiation from a binary pulsar

Compute the power emitted in gravitational radiation by a binary system formed by stars of equal mass M orbiting at relative distance R apart. Use this result to determine the rate at which the orbit shrinks by emission of gravitational radiation.

Chapter 4

Exact solutions

Exact solutions of Einstein's equations with a definite physical interpretation are hard to come by. In a sense, any metric can be regarded as a solution, provided we 'invent' an exotic energy-momentum tensor that fulfills the right hand side of the equations. In fact, the Bianchi identity ensures that such energy-momentum tensor is at least conserved. However, it will have unphysical features, such as locally measurable negative energy density. If one postulates that $T_{\mu\nu}$ is computed from some standard matter model of particles, fluids or field theories, then no general methods exist to find exact solutions of the gravitational equations.

One can simplify matters by looking for solutions with some prescribed symmetry that is interesting for physical reasons. For example, in the context of cosmological models, we seek metrics with the properties of *homogeneity* and *isotropy* which appear to be satisfied by the large-scale structure of the observable universe. For problems involving motion in the solar system, we are interested in metrics with axial and/or spherical symmetry.

In general, the diagnosis of special symmetries of the spacetime geometry is a non-trivial issue, because the metric tensor contains a lot of redundant information, changing its form under reparametrizations. Hence, we first discuss *isometries* in general, and then we concentrate on particular cases of physical interest either in cosmology or in the physics of the solar system.

Isometries

We use again the action functional of pointlike particles as a physical probe of the spacetime structure. Let us consider the linearized version of the action

$$S_{\rm P} = \tfrac{1}{2}\, m \int d\tau \, (g_{\mu\nu}\, \dot{x}^\mu \, \dot{x}^\nu - 1) \; . \tag{4.1}$$

We identify symmetries of the geometry as symmetries of the particle action for arbitrary trajectories. Hence, let us consider a *motion* in the spacetime manifold specified by a set of curves generated by a vector field ξ. As a function of the parameter λ, we have

$$\frac{dx^\mu}{d\lambda} = \xi^\mu(x) \; , \tag{4.2}$$

as the equation determining the integral curves. Infinitesimally, we induce a motion by $x^\mu \to x^\mu + \lambda\, \xi^\mu(x)$. What is the condition for these curves to be *orbits* of the action of some symmetry group? We give a physical definition of an *isometry* as those vector-field flows that leave the particle action invariant, i.e. these transformations leave the metric 'rigid' as far as particle propagation concerns. Since the action itself in invariant under reparametrizations, this criterion is independent of any coordinate system that we might use.

The variation of the particle action under the transformation $\delta_\lambda x^\mu = \lambda \xi^\mu$ is

$$\delta_\lambda S_{\rm P} = \lambda \int d\tau \, (\pounds_\xi \, g_{\mu\nu}) \, \dot{x}^\mu \, \dot{x}^\nu \; , \tag{4.3}$$

where the integrand is controlled by the *Lie derivative* of the metric

$$\pounds_\xi \, g_{\mu\nu} \equiv \xi^\alpha \partial_\alpha g_{\mu\nu} + \partial_\mu \xi^\alpha g_{\alpha\nu} + \partial_\nu \xi^\alpha g_{\alpha\mu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \; . \tag{4.4}$$

Hence, the condition on the metric for the action to be invariant if that ξ^α is a *Killing vector*:

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0 \; . \tag{4.5}$$

In this case, we can define a conserved quantity along physical trajectories using the standard formula for the Noether charge,

$$Q_\xi = -\frac{\partial \mathcal{L}_{\rm P}}{\partial \dot{x}^\mu} \, \xi^\mu = -m \, g_{\mu\nu} \, \dot{x}^\mu \, \xi^\nu = -p_\mu \xi^\mu \; . \tag{4.6}$$

The constancy of $Q_\xi = -p_\mu \xi^\mu$ along timelike geodesics generalizes to the case of null geodesics, where $Q_\xi = -k \cdot \xi$ and k^μ is the vector tangent to the light ray trajectory. It fact, it generalizes to any geodesic with tangent T^μ, even spacelike ones, for $Q_\xi = T_\mu \xi^\mu$ is covariantly constant,

$$\nabla_T (\xi \cdot T) = T^\alpha \nabla_\alpha (\xi_\mu T^\mu) = T^\alpha T^\mu \nabla_\alpha \xi_\mu + \xi_\mu T^\alpha \nabla_\alpha T^\mu = 0 \; , \tag{4.7}$$

the first term vanishing because of the Killing condition and the second because of the geodesic condition.

If we adapt our coordinate system to lie along the integral curves of some Killing vector, then we can choose $\xi^\alpha = \delta^{\alpha\lambda}$ as the components of the Killing vector. In this case, the Killing equation for the metric reduces to

$$\frac{\partial g_{\mu\nu}}{\partial \lambda} = 0 \; . \tag{4.8}$$

In other words, in adapted coordinates, the metric is independent of the symmetric coordinate. For example, a metric is called *stationary* if it has a timelike Killing vector. Choosing the time coordinate adapted to this Killing vector, we have $\xi = (1,0,0,0)$ in this frame, and $\partial_t g_{\mu\nu} = 0$, hence

$$ds^2_{\rm stationary} = g_{00}(\vec{x}\,) \, dt^2 + 2 \sum_i g_{0i}(\vec{x}\,) \, dx^i \, dt + \sum_{ij} g_{ij}(\vec{x}\,) \, dx^i \, dx^j \; . \tag{4.9}$$

Furthermore, a stationary metric is called *static* if it is invariant under time reversal $t \to -t$. This leads to

$$ds^2_{\rm static} = g_{00}(\vec{x}\,) \, dt^2 + \sum_{ij} g_{ij}(\vec{x}\,) \, dx^i \, dx^j \; . \tag{4.10}$$

If a metric has more than one Killing vector, one can consider the algebra of transformations that they form under commutation, where

$$[\delta_\xi \, , \, \delta_\eta \,] = \delta_{[\xi,\eta\,]} \qquad [\xi \, , \, \eta\,]^\mu = \xi^\alpha \partial_\alpha \eta^\mu - \eta^\alpha \partial_\alpha \xi^\mu \; . \tag{4.11}$$

This commutator algebra is very transparent if we adopt the notation favoured by mathematicians, i.e. vector fields in general can be regarded as differential operators acting on arbitrary functions,

$$\xi \equiv \xi^\alpha \frac{\partial}{\partial x^\alpha} \ .$$

In this notation, a coordinate λ adapted to the Killing ξ corresponds to $\xi = \partial/\partial\lambda$. Hence, if a coordinate system can be defined as adapted to two Killing vectors simultaneously, it is clear that their commutator vanishes. The converse is also true, i.e. the necessary and sufficient condition for the integral curves of two vector fields to coordinate a two-dimensional submanifold is that they commute.[1] In our case, this means that non-abelian isometry groups do not give such simple rules as abelian ones for the metric components. In general, for non-abelian isometries, one must list all possible invariants of the coordinates and the differentials dx^a and restrict the metric accordingly. For example, for our static spatial sections, if we let the rotation group $SO(3)$ act on $\vec{x} \in \mathbf{R}^3$ in the standard fashion, the basic invariants are $\vec{x}^{\,2}$, $d\vec{x}^{\,2}$ and $\vec{x} \cdot d\vec{x}$. So the general *ansatz* for a static, spherically symmetric metric is

$$ds^2 = -F(\rho)dt^2 + D(\rho)\,(\vec{x} \cdot d\vec{x}\,)^2 + C(\rho)\,d\vec{x}^{\,2} \ , \tag{4.12}$$

where $\rho^2 = \vec{x}^{\,2}$ and $F(\rho), D(\rho), C(\rho)$ are general functions. Picking spherical coordinates on $\mathbf{R}^3$ we have $d\vec{x}^{\,2} = d\rho^2 + \rho^2\,d\Omega^2$ and $(\vec{x} \cdot d\vec{x}\,)^2 = \rho^2 d\rho^2$, so that combining all terms we end up with a form

$$ds^2 = -F(\rho)dt^2 + G(\rho)d\rho^2 + H(\rho)\,d\Omega^2 \ , \tag{4.13}$$

where $d\Omega^2 = d\ell^2_{\mathbf{S}^2} = d\theta^2 + \sin^2\theta\,d\phi^2$. We may now change coordinates to $r^2 = H(\rho)$ so that the area of fixed radius $\mathbf{S}^2$ spheres is $4\pi r^2$. So the final form of the static and $SO(3)$-symmetric *ansatz* is

$$ds^2 = -A(r)\,dt^2 + B(r)\,dr^2 + r^2\,d\Omega^2 \ . \tag{4.14}$$

Upon direct calculation, the components of the Ricci tensor for this metric read

$$\begin{aligned}
R_{tt} &= \frac{A''}{2B} - \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{A'}{rB} \\
R_{rr} &= -\frac{A''}{2A} + \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{B'}{rB} \\
R_{\theta\theta} &= 1 - \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right) - \frac{1}{B} \\
R_{\phi\phi} &= \sin^2\theta\,R_{\theta\theta} \ ,
\end{aligned} \tag{4.15}$$

where the primes denote derivative with respect to r.

The fastest way of obtaining this result is to compute the Christoffel symbols by variation of the particle action in the metric *ansatz*:

$$S_{\rm P} = \tfrac{1}{2}\,m \int d\tau\,(g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - 1) = \tfrac{1}{2}\,m \int d\tau\left(-A(r)\,\dot{t}^2 + B(r)\,\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2) - 1\right) \ , \tag{4.16}$$

and match to the geodesic equation. This produces naturally all the non-vanishing Christoffel components. Then, one must compute explicitly the Ricci tensor from these connection coefficients.

[1] A theorem by Frobenius states that the integral curves of set of vector fields $\xi^{(a)}$ belong to a submanifold if and only if their Lie algebra closes on themselves $[\xi^{(a)}, \xi^{(b)}] = \sum_c f^{ab}_c\,\xi^{(c)}$. Moreover, these integral curves are good coordinates when they form an abelian subalgebra.

Spherically symmetric vacuum solutions

The ansatz (4.14) obtains the Minkowski metric in polar coordinates

$$ds^2 = -dt^2 + dr^2 + r^2\, d\Omega^2$$

for $A = B = 1$. This is the trivial vacuum solution with vanishing cosmological constant $\Lambda = 0$. It is not too difficult to generalize the Minkowski solution to vacuum spacetimes with positive or negative cosmological constant. Since Λ defines a length scale, let us denote $|\Lambda| = 3/R^2$, where R is the radius at which the effects of the cosmological constant are felt.

Inserting the ansatz (4.14) in the vacuum Einstein equations, $R_{\mu\nu} = \Lambda\, g_{\mu\nu}$, we find

$$\frac{R_{rr}}{B} + \frac{R_{tt}}{A} = \frac{1}{rB}\left(\frac{A'}{A} + \frac{B'}{B}\right) = 0\ ,$$

which implies $A(r)B(r) = $ constant. Since $A(0) = B(0) = 1$ to get back Minkowski spacetime at short distances, we find $A(r) = 1/B(r)$. Inserting this result now into $R_{\theta\theta} = \Lambda g_{\theta\theta} = r^2\Lambda$ we obtain

$$(rA)' = 1 - \Lambda r^2\ ,$$

which, together with the boundary condition at $r = 0$, is solved by $A(r) = 1 - \Lambda r^2/3$. These metrics

$$ds^2_\Lambda = -dt^2\left(1 - \tfrac{1}{3}\Lambda r^2\right) + \frac{dr^2}{1 - \frac{1}{3}\Lambda r^2} + r^2\, d\Omega^2 \tag{4.17}$$

are known as de Sitter spacetime (dS) for $\Lambda > 0$ and Anti de Sitter spacetime (AdS) for $\Lambda < 0$.

AdS is a smooth non-compact spacetime of constant negative curvature. Defining a new radial variable ρ as $r = R\sinh(\rho/R)$, we learn that the spatial sections at constant t have 3-metric

$$ds^2_{\mathbf{H}^3} = d\rho^2 + R^2\,\sinh^2(\rho/R)\, d\Omega^2\ ,$$

which is the three-dimensional generalization of the famous Lobachewski pseudosphere, a space in which volumes enclosed by large spheres scale like the surface area.

The dS metric is singular at $r_s = R = \sqrt{3/\Lambda}$, but this is only a singularity of the particular frame used to write down the metric. Clearly, the curvature stays constant throughout all dS spacetime, and one can explicitly show that the dS metric can be analytically extended beyond $r = R$. To see this, define

$$z_0 = \sqrt{R^2 - r^2}\ \sinh(t/R)\ , \qquad z_1 = \sqrt{R^2 - r^2}\ \cosh(t/R)\ ,$$

and let $z_2^2 + z_3^2 + z_4^2 = r^2$ parametrize an $\mathbf{S}^2$ of radius r embedded in $\mathbf{R}^3$. Then, it is easy to see that dS spacetime is the hyperboloid

$$-z_0^2 + z_1^2 + z_2^2 + z_3^2 + z_4^2 = R^2$$

embedded in five-dimensional $\mathbf{R}^5$ with Minkowski metric

$$ds^2_{\mathbf{R}^5} = -dz_0^2 + dz_1^2 + dz_2^2 + dz_3^2 + dz_4^2\ .$$

Hence, we see that dS is a homogenous spacetime without singularities. There is another intrinsic representation of the dS metric that covers the entire manifold. Setting

$$z_0 = R\,\sinh(t'/R)\ , \qquad z_k = R\,\cosh(t'/R)\, N_k\ , k = 1, 2, 3, 4\ ,$$

where N_k is a unit vector in $\mathbf{R}^4$, parametrizing an $\mathbf{S}^3$ of unit radius, we have

$$ds^2_{\rm dS} = -dt'^{\,2} + R^2 \, \cosh^2(t'/R) \, d\ell^2_{\mathbf{S}^3} \, . \tag{4.18}$$

In this form, the dS metric looks very different from (4.17). It consists of a 3-sphere that contracts from the infinite past to a radius R and then re-expands again into the future with an asymptotically exponential rate. Hence, dS space is a *cosmological* spacetime, something clearly surprising if given only the (4.17) static form. [2]

Problem: May "The Force" be with you

Study the radial geodesics of massive test particles in the vacua with nonvanishing cosmological constant. Show that Λ behaves as a repulsive cosmic potential for dS and as a "confining" cosmic potential for AdS.

---

[2]**Exercise:** Verify the changes of variables leading to the different forms of the dS metric outlined in this section.

4.1 Cosmological solutions

Let us reconsider the de Sitter spacetime in global coordinates

$$ds^2_{\rm dS} = -dt^2 + R^2 \cosh^2(t/R)\, d\ell^2_{\mathbf{S}^3} \,. \tag{4.19}$$

this is a prototype of a spacetime whose spatial sections are *homogeneous* and *isotropic*. Homogeneity means that the spatial sections (here 3-spheres of radius $R\cosh(t/R)$) have an isometry relating any two points, i.e. that no point has a special character. Isotropy means that around any point, there is no preferred direction. Since these are the two properties that characterize the observable universe on a very large scale, one usually discusses cosmological models assuming these restrictions from the outset. Thus, one considers the natural generalization of (4.19) that retains the homogeneity and isotropy properties, resulting in the so-called Friedmann–Robertson–Walker (FRW) spacetimes,

$$ds^2 = -dt^2 + R(t)^2\, d\ell^2_{\mathbf{K}} \,, \tag{4.20}$$

where $d\ell^2_{\mathbf{K}}$ stands for the spatial metric of a three-manifold that satisfies the criteria of homogeneity and isotropy. The overall size of this manifold is controlled by the function $R(t)$ which is to be determined by Einstein's equations. It can be shown that the spatial section of constant curvature and homogeneity can only be one of three options: a 3-sphere in the case of positive curvature, a 3-hyperboloid in the case of negative curvature, and flat $\mathbf{R}^3$ for zero curvature. We can write the three types of metric at once by the following polar parametrization

$$d\ell^2_{\mathbf{K}} = \frac{dr^2}{1 - K\, r^2} + r^2\, d\Omega^2 \,, \tag{4.21}$$

where $K = 1$ for the positive curvature $\mathbf{S}^3$, $K = 0$ for flat $\mathbf{R}^3$, and $K = -1$ for the 3-hyperboloid $\mathbf{H}^3$ of negative curvature. Homogeneity and isotropy of $\mathbf{R}^3$ is obvious. The other two 3-manifolds of constant curvature can be obtained by the equations for the 3-surface

$$Kx_1^2 + x_2^2 + x_3^2 + x_4^2 = K \,, \tag{4.22}$$

embedded in $\mathbf{R}^4$ with metric $ds^2_{\mathbf{R}^4} = Kdx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$. This confirms the homogeneity of $\mathbf{S}^3$ and $\mathbf{H}^3$.

The case $K = 1$ yields model universes like dS, in that the spatial sections are compact, whereas the models $K = 0, -1$ correspond to non-compact (so-called *open*) universes. What particular model is realized in nature, as well as the precise features of the 'size' function $R(t)$ depends on the Einstein equations.

It is important to notice that our 'FRW parametrization', given by the metrics (4.20) and (4.21), may suffer from some ambiguities when it comes to global issues. A famous example is that of the 'steady state metric', [3] one of the presentations of de Sitter spacetime:

$$ds^2 = -d\bar{t}^2 + e^{2\bar{t}/R} d\vec{y}^{\,2} \,, \tag{4.23}$$

where $\vec{y} \in \mathbf{R}^3$. This is a particular case of FRW with $K = 0$. However, there is change of variables revealing that it is identical to the $z_0 + z_4 \geq 0$ domain of the complete dS manifold. Hence, a FRW manifold with $K = 0$ can be a 'patch' of another FRW manifold with $K = 1$.

[3]The *steady state* universe of Gold, Hoyle and Bondi was once a fashionable model of cosmology, now long forgotten since the discovery of the cosmic microwave background radiation.

Problem: The many faces of de Sitter

Find proper domains of the global de Sitter manifold and appropriate coordinate systems that make them into FRW cosmologies with $K = 0$ (flat de Sitter or 'steady-state' metric) and $K = -1$ (open de Sitter).

4.1.1 The standard cosmological model

A good physical model of the universe in different epochs of its history is given in terms of three components: pressureless matter, or 'dust', and a cosmological constant as a dominant component today, and radiation as a dominant component in the past. All situations can be modeled as a perfect fluid with energy momentum tensor $T_{00} = \rho$ and $T_{ij} = p g_{ij}$, with $p = 0$ for matter, $p = \rho/3$ for radiation, and $p = -\rho$ for vacuum energy (cosmological constant). We can summarize the three components by the equation of state $p = w\rho$, with three values of w. If the components or 'species' are approximately non-interacting, we can write $\rho = \sum_s \rho_s$ and $p_s = w_s \rho_s$ for each component.

Then, the Einstein's equations can be reduced to the time-time component (so-called Friedmann equation)

$$\left(\frac{dR}{dt}\right)^2 + K = \frac{8\pi G}{3}\,\rho\,R^2\;, \tag{4.24}$$

and the space-space components

$$\frac{d^2R}{dt^2} = -\frac{4\pi G}{3}(\rho + 3p)\,R\;. \tag{4.25}$$

The second equation can be traded by that of local energy conservation, also equivalent to the Bianchi identities $\nabla_\mu T^{\mu\nu} = 0$, or

$$\frac{d}{dt}(\rho R^3) = -p\,\frac{d}{dt}(R^3)\;. \tag{4.26}$$

This equation is valid for each component separately, provided they are non-interacting. Using $p_s = w_s \rho_s$ we can solve it as $\rho_s \propto R^{-3(1+w_s)}$. The Friedmann equation is usually written in terms of the Hubble parameter $H \equiv R^{-1}\,dR/dt$, as

$$H^2 = -\frac{K}{R^2} + \frac{8\pi G}{3}\rho\;. \tag{4.27}$$

Another useful definition is the so-called density parameter

$$\Omega_s = \frac{8\pi G \rho_s}{3H^2}\;, \qquad \Omega = \sum_s \Omega_s\;, \tag{4.28}$$

so that $\Omega > 1$ corresponds to $K = +1$, $\Omega < 1$ corresponds to $K = -1$, whereas $\Omega = 1$ deals a flat universe with $K = 0$. At present, Ω is believed to be well approximated by the sum of three components:

$$\Omega_0 = \Omega_{\rm b} + \Omega_{\rm dm} + \Omega_{\rm de}\;, \tag{4.29}$$

where $\Omega_{\mathrm{b}} \approx 0.04$ is the component in baryonic matter, $\Omega_{\mathrm{dm}} \approx 0.24$ is the dark matter component and $\Omega_{\mathrm{de}} \approx 0.73$ is the 'cosmological constant' component. The big surprise of the last decade is the fact that Ω_{de} is actually dominant. Hence, $\Omega_0 \approx 1$ within experimental errors, and the universe appears to have flat, $K = 0$ spatial sections.

Inserting the solution of the energy conservation equation $\rho_s = \rho_{s,0}(R_0/R)^{3(1+w_s)}$ into the Friedmann equation, we may recast it in the form

$$\frac{1}{2}\left(\frac{da}{dt}\right)^2 + V_{\mathrm{eff}}(a) = 0 \ . \tag{4.30}$$

where $a = R/R_0$, the quantities with zero subscript denoting the "present" values. The Friedmann equation is thus formally equivalent to a mechanical zero-energy motion of a particle in a potential

$$V_{\mathrm{eff}}(a) = \frac{K}{2R_0^2} - \sum_s \frac{4\pi G\rho_{s,0}}{3}\frac{1}{a^{1+3w_s}} = \frac{H_0^2}{2}\left(\Omega_0 - 1 - \sum_s \frac{\Omega_{s,0}}{a^{1+3w_s}}\right) \ . \tag{4.31}$$

The regions where $V_{\mathrm{eff}} < 0$ determine the range of $R(t)$. In general, if one considers all three types of 'species' to be present with non-zero values (positive energy density of nonrelativistic matter and radiation and positive or negative vacuum energy), one sees that radiation always dominates near $R = 0$, whereas the cosmological constant controls the large R behaviour, the 'dust' and 'curvature' components only shaping the transient effects at intermediate sizes. For $\Lambda < 0$, the effective potential V_{eff} is monotonically increasing and the universe always bounces between a *bang* and a *crunch*, quite independently of the sign of K and the ordinary matter content. For $\Lambda > 0$, the potential V_{eff} has always a "Λ" shape (a single local maximum) and the qualitative behaviour depends on whether the potential is positive, zero or negative at the maximum. This is controled by the actual values of the relative energy densities, as well as the sign of K. If $\max(V_{\mathrm{eff}}) > 0$, one possibility is that the universe expands to a maximum size and bounces back to a crunch, but another possibility is that it starts contracting from $R = \infty$ in the infinite past down to a minimum size and then reexpands again, a deformation of the dS solution. The famous static Einstein universe that started modern cosmology in 1917 corresponds to $\max(V_{\mathrm{eff}}) = 0$ and it is clearly unstable. Finally, if $K \leq 0$ (and still $\Lambda > 0$) then $\max(V_{\mathrm{eff}}) < 0$ and the universe expands forever from zero size to infinite size. This picture may have some important variations if we remove (with fine tunning) the two asymptotically dominating species (radiation at small R and/or vacuum energy at large R). [4]

A useful physical parameter that can be directly related to observations is the *deceleration* parameter

$$q \equiv -\frac{R\, d^2R/dt^2}{(dR/dt)^2} = \frac{1+3\bar{w}}{2}\,\Omega \ , \tag{4.32}$$

where we have used (4.25) and defined the average 'speed of sound' $\bar{w} = p/\rho$. For positive values of Ω, the expansion of the universe will accelerate if $\bar{w} < -1/3$, and will decelerate otherwise. Splitting in terms of species, we have

$$q = \sum_s \frac{1+3w_s}{2}\Omega_s \ . \tag{4.33}$$

[4]**Exercise:** Study the details of the qualitative motion in the effective potential (4.31) for various combinations of matter, radiation and vacuum domination, and interpret the resulting cosmological models.

At present, $q_0 \approx \frac{1}{2}\,\Omega_{\rm matter} - \Omega_{\rm de}$, and the measurement of q_0 supports our determination of $\Omega_{\rm de}$ as a dominant component.

The connection with observation depends on the phenomenon of cosmological *red shift*. There are many ways of obtaining this very basic relation. Here we just give a shorcut based on the observation that (4.20) is conformally related to a static spacetime, i.e. defining a new time variable $\eta = \int^t dt'/R(t')$ we can write any Robertson–Walker spacetime as

$$ds^2 = R(\eta)^2 \left(-d\eta^2 + d\ell_{\mathbf{K}}^2\right) \,.$$

Hence, frequencies $\omega \sim d/dt$ are related to frequencies measured in the static spacetime $d/d\eta$ by a factor of $R(t)$, which is the only overall time-dependent scale of the spacetime. An equivalent argument in terms of particle paths is the following: if two photons are sent from point P_1 to point P_2 with a conformal time separation $\Delta\eta_1$ at emission, they arrive at P_2 with the same conformal time separation, $\Delta\eta_2 = \Delta\eta_1$, since the null geodesics can be computed in the *static* metric $d\tilde{s}^2 = -d\eta^2 + d\ell_{\mathbf{K}}^2$. Then the corresponding proper times are related by

$$\frac{\Delta\tau_1}{\Delta\tau_2} = \frac{R(\eta_1)}{R(\eta_2)} \,.$$

Thus, we have the homogeneity property $\lambda(t) \sim R(t)$ for the wavelength of any electromagnetic wave that solves Maxwell's equations in the Robertson–Walker metric. Defining the redshift coefficient z in terms of the relative wavelenght increase between emission and reception:

$$z \equiv \frac{\lambda_0 - \lambda_e}{\lambda_e} = \frac{\lambda_0}{\lambda_e} - 1 = \frac{R_0}{R_e} - 1 \,. \tag{4.34}$$

Furthermore, using the linear approximation of $R(t)$ for nearby galaxies, $R(t_0) \approx R_e + \frac{dR}{dt}(t_0 - t_e)$. Since the proper distance to nearby galaxies is given by $D \approx t_0 - t_e$, we find the approximate law

$$z \approx H_0\, D \,, \qquad H_0 \approx 0.24\,{\rm Gpc}^{-1} \,, \tag{4.35}$$

originally tested by Hubble when the expansion of the universe was discovered back in 1929 (in the units used commonly by astronomers, $H_0 \approx 73$ Km s^{-1} Mpc^{-1}). A more careful measurement of z for far away galaxies allows us to determine H_0 (to linear order) and q_0 (to first order in deceleration), determining the type of FRW model that describes our universe.

Problem: Hubble's law

Derive a more precise version of Hubble's law that is sensitive to the deceleration parameter, q_0, in terms of the *luminosity distance* $D_L^2 = L/4\pi F$, where L is the absolute luminosity of a source and F is the flux of light at reception. Using the definition of luminosity distance and the geodesic equation for photons in a FRW model, obtain the relation

$$D_L = H_0^{-1}\left(z + \tfrac{1}{2}\,(1-q_0) z^2 + \ldots\right) \,,$$

which determines H_0 and q_0 from the directly measurable z and D_L.

Horizon and flatness problems

FRW models starting with a *bang* at $t = 0$ pose the following question: is it possible that not all of $\mathbf{K}$ comes into causal contact in the time lapsed since the bang? This of course depends on the time passed since $t = 0$, the rate of expansion and the geometry of $\mathbf{K}$. The trajectories of light rays satisfy $d\ell/d\eta = \pm 1$, where ℓ is a proper distance in $\mathbf{K}$ and η is the conformal time defined in the previous subsection. Hence, the causal structure is essentially determined by that of an effective two-dimensional Minkowski spacetime parametrized by (η, ℓ). There are horizons if the past-directed light cones from a point P do not cover the entire $\mathbf{K}$ before the 'initial' time η_* is reached.

Since

$$\eta - \eta_* = \int_0^t \frac{dt'}{R(t')} , \tag{4.36}$$

such 'particle horizons' will never occur if $\eta_* = -\infty$, i.e. if this integral diverges at the lower limit. Conversely, if $\eta_* > -\infty$ (such as all FRW models that are dominated by radiation or dust near $t = 0$), then there is a horizon if $\eta_P - \eta_* < \ell_{\max}(\mathbf{K})$, where $\ell_{\max}(\mathbf{K})$ is the maximum proper length covered by a light ray in $\mathbf{K}$ (this is only relevant for compact $\mathbf{K}$). There is a global notion of horizon, associated to the whole history of a particle worldline, called 'event horizon', defined as the particle horizon corresponding to the asymptotically large time limit.

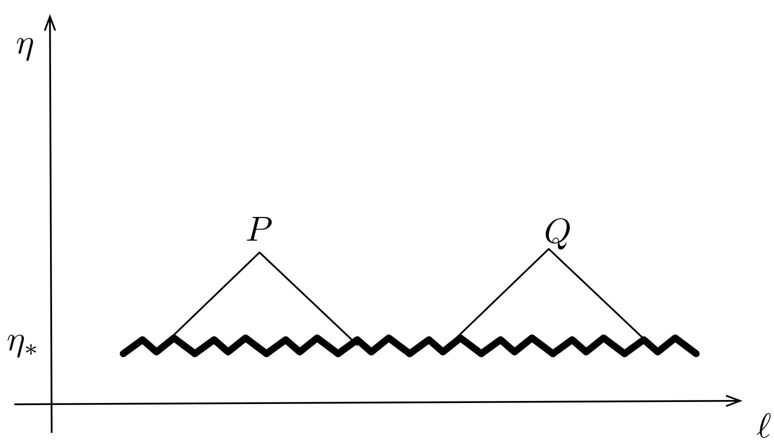

Figure 4.1: *Local conditions at points P and Q could not be causally connected in the time elapsed since the big bang.*

The behavior of (4.36) at the lower limit can be elucidated by rewriting it in the form

$$\eta - \eta_* = \frac{1}{R_0} \int_0^a \frac{da'}{a'\, da'/dt} = \frac{1}{R_0} \int_0^a \frac{da'}{a' \sqrt{-2V_{\rm eff}(a')}} .$$

Hence, η_* converges if $|V_{\rm eff}(a)|$ decreases as $a \to 0$ (recall that $V_{\rm eff}$ is always negative for the actual solutions). Examining the dependence of $V_{\rm eff}$ on a we see that horizons will be present for $\bar{w} > -1/3$, i.e. for any cosmology that starts decelerating right from the bang. On the other hand, for cosmologies that start out (marginally) accelerating, $\bar{w} \leq -1/3$, the bang occurs at minus infinity in conformal time, and there are no horizons.

Our model universe as we have described it so far, with radiation domination at early times, does have horizons. This poses the question of why we see an homogeneous universe after all, since no causal processes could achieve this homogeneity by standard local interactions. An interesting way out of this conundrum is the idea of *inflation*, consisting on the possibility that a phase of large accelerated expansion exists for a finite amount of time after the bang. Such a phase gives a large contribution to (4.36) and thus pushes η_* to more negative values. This means that a phase of inflation can be used to remove horizons.

FRW models with flat spatial sections correspond to $\Omega = 1$ at all times. We can ask how natural is this condition as an asymptotic behavior as $t \to 0$ or $t \to \infty$ in expanding models with a bang. From the previous equations we have

$$\Omega = 1 + \frac{K}{H^2 R^2} \,. \tag{4.37}$$

Using the definition of H we have $H^2 R^2 = (dR/dt)^2 = R_0^2 (da/dt)^2 = -2R_0^2 V_{\rm eff}(a)$. Hence we find

$$\Omega - 1 = \frac{K}{2R_0^2 |V_{\rm eff}(a)|} \,. \tag{4.38}$$

Since the monotonicity of $V_{\rm eff}$ is associated with the accelerating/decelerating character of the universe, we see that accelerating phases ($\bar{w} < -1/3$ or increasing $|V_{\rm eff}|$) drive Ω to unity, whereas decelerating phases ($\bar{w} > -1/3$ or decreasing $|V_{\rm eff}|$) drive Ω *away* from unity.

Our universe has been dominated by radiation and dust till relatively recently, i.e. it has been decelerating for most of its history. Hence, the fact that Ω_0 is so close to unity implies that it was much closer at early times. In the radiation era, the value of Ω was thus extremely *fine-tuned*, an initial condition without a clear explanation within the standard cosmological model. Inflation provides an interesting perspective on this so called *flatness problem*. Since accelerated universes have $\Omega = 1$ as a future attractor, an early phase of inflation is also able to dynamically tune $\Omega \approx 1$, setting the initial conditions of the radiation era.

4.2 The Schwarzschild solution

The Schwarzschild metric is the unique asymptotically flat vacuum solution with $SO(3)$ symmetry and static. It is therefore the appropriate metric for the exterior of a spherically symmetric and time independent matter distribution. In fact, it turns out that the static condition is redundant, as proved by Birkhoff, all vacuum solutions of $SO(3)$ isometry group are static (there is no purely monopole gravitational radiation). This result is entirely analogous to its electromagnetic cousin.

Plugging the *ansatz* (4.14) into the vacuum Einstein equation $R_{\mu\nu} = 0$ one finds

$$\frac{R_{rr}}{B} + \frac{R_{tt}}{A} = \frac{1}{rB}\left(\frac{A'}{A} + \frac{B'}{B}\right) = 0 \ , \tag{4.39}$$

where primes stand for radial derivatives. From here, we obtain $A(r)B(r) = \text{constant}$, and requiring that the metric approaches Minkowski at infinity, we get $A(r)B(r) = 1$. Now, it is enough to concentrate on the $R_{\theta\theta}$ equation, that reads

$$R_{\theta\theta} = 1 - A'(r)r - A(r) = 0 \ ,$$

with solution $rA(r) = r + \text{constant}$. Since we must approach $g_{tt} \to -1 + 2GM/r + \ldots$ at large radius, this is enough to fix the complete metric as

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2\, d\Omega^2 \ , \tag{4.40}$$

where M accounts for the total energy contained in the spacetime. [5]

Problem: Birkhoff Theorem

Show that the time-independence of the metric *ansatz* for the Schwarzschild solution is redundant, namely *vacuum* solutions with $SO(3)$ symmetry are necessarily static.

The metric components are singular at the so-called Schwarzschild radius $R_s = 2GM$. For standard astrophysical bodies, such as stars and planets, one has $R_s \approx 3(M/M_\odot)$ Km, so that this singularity is irrelevant in many applications, since the Schwarzschild solution must be matched to some 'interior' solution at the surface of the body. Any object for which the vacuum Schwarschild solution would apply down to $r = R_s$ is called a *black hole*.

The behavior of the $r = R_s$ surface is similar to the analogous surface in de Sitter space, the singularity being just an artifact of the particular coordinate system. A freely falling observer is only sensitive to the tidal forces, which are controlled by the Riemann tensor. By direct calculation, it is possible to check that all the non-vanishing components of the Riemann tensor in the vicinity of $r = R_s$ are of order $1/R_s^2$. Therefore, tidal accelerations have the expected value inferred by extrapolation from Newtonian scaling: for an object of length ℓ, $a(\ell)_{\text{tidal}} \sim GM\ell/R_s^3 \sim \ell/R_s^2$. This shows that an infalling observer will not notice any singular behavior

[5] It is also possible to generalize the asyptotically flat Schwarschild solution to analogous solutions with dS or AdS asymptotics. The *naive* answer obtained in the limit $(GM)^2\Lambda \ll 1$ turns out to be exact, i.e. $A = B^{-1} = 1 - \frac{1}{3}\Lambda r^2 - \frac{2GM}{r}$.

upon crossing the surface $r = R_s$. This crossing occurs in finite proper time. The proper time to radially fall from some initial radius $r_0 > R_s$ down to R_s is

$$\Delta\tau = \int_{R_s}^{r_0} \frac{dr}{\sqrt{\frac{R_s}{r} - \frac{R_s}{r_0}}} < \infty \ .$$

Hence, the proper time for radial fall is finite, as well as the tidal forces at $r = R_s$, and there is no physical singularity.

The singularity of the metric is a just a pathology of the specific coordinate system used. This frame is adapted to observers at fixed radial position, that see the metric as static and spherically symmetric. However, the physical status of these observers is increasingly singular as we approach the Schwarzschild radius R_s. To see this, let us change coordinates to a radial variable that measures proper distance from R_s, i.e.

$$\rho = \int_{R_s}^{r} dr' \sqrt{g_{rr}(r')} \ .$$

In these coordinates, the Schwarzschild metric takes the form

$$ds^2 = -\left(1 - R_s/r(\rho)\right) dt^2 + d\rho^2 + r(\rho)^2 \, d\Omega^2 \ .$$

Near the critical sphere, $r \sim R_s$ we have $\rho \approx 2\sqrt{R_s(r - R_s)}$ and the metric

$$ds^2 \approx -\rho^2 \left(\frac{dt}{2R_s}\right)^2 + d\rho^2 + r(\rho)^2 \, d\Omega^2 \ .$$

Hence, static observers at fixed Ω, or radially infalling observers, are only sensitive to the two dimensional section

$$ds_2^2 = -\rho^2 dt^2/4R_s^2 + d\rho^2 \ .$$

If we further transform to $\rho = 2R_s \exp(\xi/2R_s)$ we end up with

$$ds_2^2 = e^{\xi/R_s} \left(-dt^2 + d\xi^2\right) \ ,$$

which is nothing but the metric of Rindler space. Hence, observers at fixed r coordinate, or fixed ξ coordinate, are locally accelerated with respect to freely falling ones with a proper acceleration equal to

$$g_\xi = \frac{e^{-\xi/2R_s}}{2R_s} \ , \tag{4.41}$$

which diverges in the $\xi \to -\infty$ limit or, equivalently $r \to R_s$. It takes infinite forces to stay put at fixed r position just on top of the critical surface. This shows that it is the static coordinate system that becomes unphysical, rather than the spacetime itself. From the appearance of Rindler space in our treatment, we see that the critical surface $r = R_s$ is entirely analogous to the $\xi = -\infty$ surface of Rindler space, namely the event horizon.

4.2.1 The classic tests

We now discuss the three classic tests of general relativity. We demonstrate the gravitational red shift effect, the advance of perihelia and the bending of light in the background of the Schwarzschild solution. Today, the precision measurements based on these effects confirm the theory down to the 10^{-5} level.

Gravitational redshift

In a general spacetime with metric $g_{\alpha\beta}$ the locally measured frequency of light depends on the point of measurement, as well as the state of motion of the observer. An invariant definition is given by the expression (1.124)

$$\omega = -k_\mu u^\mu \ , \tag{4.42}$$

where k_μ is the wave vector of the light ray and u^μ is the four-velocity of the observer. Hence, in order to relate the measured frequency at two events, one typically needs to solve for the null geodesic of the light path between them. Things are greatly simplified if the observers's trajectories are tangent to a Killing vector. In general, given a Killing vector ξ and a tangent vector T^a to some geodesic curve, the contraction $\xi_\mu T^\mu$ is constant along the geodesic (c.f. (4.7)). Applying this lemma to a pair of static observers with four-velocity parallel to the temporal Killing vector of Schwarzschild, $\xi = \partial/\partial t$, we find that $\xi_\alpha k^\alpha =$ constant along the light path and therefore is the same at emission and reception. The four-velocity of static observers is normalized by defintion $u^\alpha u_\alpha = -1$, so

$$u^\alpha = \frac{\xi^\alpha}{\sqrt{-\xi_\mu \xi^\mu}} \ . \tag{4.43}$$

From $(\xi \cdot k)_{\text{emission}} = (\xi \cdot k)_{\text{reception}}$ we deduce

$$\frac{\omega_1}{\omega_2} = \frac{(-\xi^2)_2^{1/2}}{(-\xi^2)_1^{1/2}} = \sqrt{\frac{1 - 2GM/r_2}{1 - 2GM/r_1}} \ . \tag{4.44}$$

Hence, light redshifts as it climbs through the gravitational field.

Problem: Feynman's clock

Feynman shows up in your office playing with his clock, repeatedly throwing it upwards into the air. Then, he asks you if the clock is running faster or slower than it should (according to the apocryphal Feynman gospels, a true story).

More specifically, consider two identical clocks A and B. Clock A remains at rest at the top of a building, while B is taken on a variety of round trips, after both clocks have been synchronized. When B returns, we compare the clocks again. Decide whether B comes retarded or advanced with respect to A, in the following cases (you must argue your answer, of course):

(1) We tie B to a balloon and lift it at some fixed height. After a while we bring it back.

(2) B goes down to the first floor and, after some time, returns to the top.

(3) B is thrown up into the air and compared to A when it falls back (Feynman's question).

(4) B is lowered to the bottom of a deep mine, and taken back up after some time.

(5) We drill a tunnel across the whole planet, going through the centre of the Earth. We drop B and compare it with A when it comes back.

(6) B is taken on board of a plane and compared to A after a round trip around the globe in eastward direction.

(7) Same as before, but now the plane goes westward.

Planetary motion

Planets will be approximated as pointlike particles of mass m following free-fall geodesics in the Schwarzschild metric. In strict analogy with Kepler's problem, this system can be completely integrated once the symmetries are taken into account. As a consequence of the spherical symmetry, particle orbits lie on a plane, which can be regarded as equatorial. Restricted to this plane, the spacetime enjoys two commuting Killing vectors: the temporal one $\xi = \partial/\partial t$ and the polar one $\eta = \partial/\partial\phi$. The associated Noether charges are the energy and angular momentum

$$E = -p_\alpha \xi^\alpha = m(1 - 2GM/r)\,\dot{t}\;, \qquad L = p_\alpha \eta^\alpha = mr^2\,\dot{\phi}\;. \tag{4.45}$$

Defining the energy and angular momentum per unit mass $\varepsilon \equiv E/m$, $\ell = L/m$, we have, from the dispersion relation $p_\alpha p^\alpha + m^2 = 0$, or $u^2 = -1$,

$$-1 = -\frac{\varepsilon^2}{1 - 2GM/r} + \frac{\dot{r}^2}{1 - 2GM/r} + \frac{\ell^2}{r^2}\;. \tag{4.46}$$

This equation is equivalent to the classical one-dimensional motion of a unit-mass particle of energy per unit mass equal to $\varepsilon_{\text{eff}} = (\varepsilon^2 - 1)/2$, subject to the effective potential

$$V_{\text{eff}} = -\frac{GM}{r} + \frac{\ell^2}{2r^2} - \frac{GM\ell^2}{r^3}\;. \tag{4.47}$$

So that $\dot{r}$ can be solved from

$$\frac{1}{2}\,\dot{r}^{\,2} + V_{\text{eff}}(r) = \varepsilon_{\text{eff}}\;. \tag{4.48}$$

Compared to the Newtonian problem, the crucial new feature of this potential is the last term $-GM\ell^2/r^3$ which dominates over the Newtonian one at small enough radii, and is negative definite, representing a clear tendency for the orbits to be destabilized towards small radius. The centrifugal barrier is no longer infinite. In fact, it is not present at all if $\ell^2 < 12(GM)^2$. In this case, any particle on a trapped orbit, $\varepsilon^2 < 1$ will fall to the Schwarzschild radius.[6]

For $\ell^2 > 12(GM)^2$ the potential has a stable minimum at R_+ and an ustable maximum at R_-, with

$$R_\pm = \frac{\ell^2 \pm \sqrt{\ell^4 - 12(\ell GM)^2}}{2GM}\;. \tag{4.49}$$

Thus, stable circular orbits exist at R_+, which is restricted to be greater than $6GM$. Unstable circular orbits exist at R_- which is restricted to $3GM < R_- < 6GM$. Any particle in a circular orbit has angular momentum per unit mass

$$\ell = \sqrt{\frac{GMR_\pm^2}{R_\pm - 3GM}}\;, \tag{4.50}$$

and energy per unit mass equal to (obtained setting $\dot{r} = 0$),

$$\varepsilon = \frac{R_\pm - 2GM}{\sqrt{R_\pm(R_\pm - 3GM)}}\;. \tag{4.51}$$

[6]Notice that the effective energy ε_{eff} is nothing but the Newtonian energy per unit mass in the nonrelativistic limit.

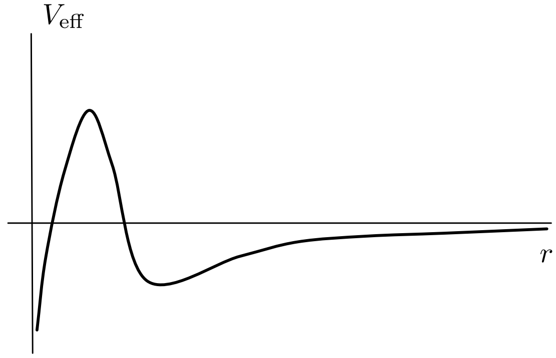

Figure 4.2: *Effective potential for massive particles with $\ell^2 > 12(GM)^2$. For $\ell^2 < 12(GM)^2$ the centrifugal barrier is gone.*

For $3GM < R_- < 4GM$, ε ranges between ∞ and unity, so that any particle in unstable circular orbits with $R_- < 4GM$ will escape to infinity if perturbed outwards.

Regarding the stable circular orbits, the lowest possible one sits at $R_+ = 6GM$, and its binding energy per unit mass is

$$\varepsilon_b = 1 - \varepsilon = 1 - \sqrt{8/9} \approx 0.06 \; . \tag{4.52}$$

Hence, particles in circular orbits at very large radius, with $\varepsilon \approx 1$, will steadily lose energy and angular momentum by gravitational radiation emission. Gradually, they will spiral down to the last possible circular orbit at $R_+ = 6GM$, and then fall rapidly to the Schwarzschild radius. In the quasistatic process of gravitational radiation emission, up to six per cent of the initial energy is radiated, a quantity which is not negligible. So, we see that gravitational radiation, being cummulative, can have significant effects over time, even if the radiation power is so small.

The parametric equation for the orbits can be obtained by writing $\dot{r} = \sqrt{2(\varepsilon_{\text{eff}} - V_{\text{eff}})}$ and dividing by $\dot{\phi} = L/mr^2$ to obtain

$$\Delta\phi = \int_{r_i}^{r_f} \frac{d\phi}{dr} dr = \int_{r_i}^{r_f} \frac{L dr}{\sqrt{2m^2 r^4 (\varepsilon_{\text{eff}} - V_{\text{eff}})}} \; . \tag{4.53}$$

Setting r_i and r_f at the turning points, which solve $V_{\text{eff}}(r_\pm) = \varepsilon_{\text{eff}}$, computes the perihelion advance for closed orbits. We can derive a simple approximation to the orbital precession in the case of nearly circular orbits (small eccentricity). For a nearly circular orbit, we can study the harmonic approximation to the potential near R_+, and we find a characteristic frequency

$$\omega_r^2 = V''_{\text{eff}}(R_+) = \frac{GM(R_+ - 6GM)}{R_+^3(R_+ - 3GM)} \; . \tag{4.54}$$

On the other hand, the angular frequency per revolution in the polar angle ϕ is

$$\omega_\phi^2 = \frac{\ell^2}{R_+^4} = \frac{GM}{R_+^2(R_+ - 3GM)} \; . \tag{4.55}$$

The inequality of these two frequencies translates in a precession of the orbit. The precession frequency being simply the difference

$$\omega_p = \omega_\phi - \omega_r = \omega_\phi \left(1 - \sqrt{1 - 6GM/R_+}\right) . \tag{4.56}$$

To lowest non-vanishing order for large R_+, we find

$$\omega_p \approx \frac{3(GM)^{3/2}}{R_+^{5/2}} , \tag{4.57}$$

or, in terms of the angular advance per period $T_{\rm orb} \approx 2\pi R_+ \sqrt{R_+/GM}$:

$$\Delta\phi_p \approx \frac{6\pi GM}{R_+} . \tag{4.58}$$

A direct evaluation of (4.53) for arbitrary orbits yields the general result

$$\Delta\phi_p \approx \frac{6\pi GM}{(1-e^2)a} , \tag{4.59}$$

in terms of the eccentricity e and the semimajor axis of the ellipse a. This gives the famous 43 seconds of arc per century for Mercury. For the equally famous binary pulsar of Hulse and Taylor, the precession is about 4 degrees per year!

It is interesting to notice that, despite being a leading correction of order $\mathcal{O}(\phi) = \mathcal{O}(GM/r)$ to the Newtonian theory, it is only visible if terms of $\mathcal{O}(\phi^2)$ are kept in the weak-field expansion of the Schwarzschild metric. In this sense, the orbital precession gives a more crucial test of the Einstein theory than the gravitational redshift and the bending of light rays, despite being effects of the same 'size'.

Gravitational lensing

We discuss light (photon) propagation taking the $m \to 0$ limit of the particle propagation. In this limit the proper time becomes a bad parameter and we need to re-parametrize the trajectory in terms of an affine parameter or in terms of coordinate time. For the purposes of describing photon orbits we can avoid such choices by focusing on the parametric equation (4.53)

$$\frac{d\phi}{dr} = \left[\frac{2m^2 r^4}{L^2}\left(\varepsilon_{\rm eff} - V_{\rm eff}\right)\right]^{-1/2}$$

which does admit a non-singular $m \to 0$ limit. In this limit the photon angular momentum is given by $L_\gamma = |\vec{p}_\gamma|\, b = E_\gamma\, b$, where b is the impact parameter for unbounded orbits. The resulting photon parametric equation is

$$\frac{d\phi}{dr} = \left[\frac{r^4}{b^2} - r(r - 2GM)\right]^{-1/2} . \tag{4.60}$$

The expression in brackets vanishes at the turning points r_0 of the orbit, which are then localized at the largest positive root of the equation

$$r_0^3 - b^2(r_0 - 2GM) = 0 . \tag{4.61}$$

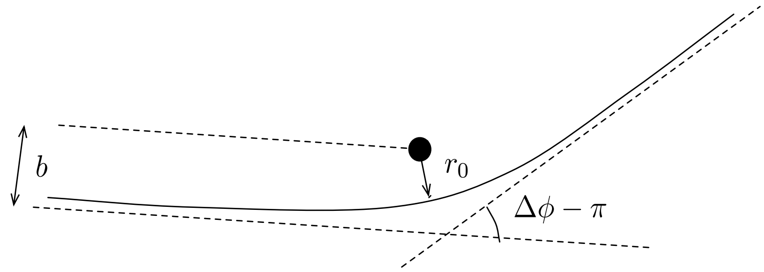

Figure 4.3: *Gravitational bending of a light ray grazing at a distance r_0 from a central field.*

Such a root only exists for $b \geq b_c = 3^{3/2}GM$, which corresponds to the smallest possible impact parameter such that the photon is not captured, giving the capture cross section

$$\sigma_{\text{capture}} = \pi b_c^2 = 27\pi(GM)^2 \ . \tag{4.62}$$

It is interesting that the critical value of the impact parameter corresponds to a turning point $r_0 = 3GM$, which happens to be degenerate, in the sense that the photon admits a circular (albeit unstable) orbit at this radius.

The light that is not captured will scatter off the gravitational field. The deflection of the light ray is defined by

$$\delta\phi = \Delta\phi - \pi \ , \tag{4.63}$$

with $\Delta\phi = \phi(\sigma = +\infty) - \phi(\sigma = -\infty)$.

So, the total $\Delta\phi$ is twice the amount from the turning point r_0 out to infinity:

$$\Delta\phi = 2\int_{r_0}^{\infty} \frac{dr}{\sqrt{r^4 b^{-2} - r(r - 2GM)}} \ . \tag{4.64}$$

To estimate this, we change variables to $u = r_0/r$ and eliminate b via (4.61) to obtain

$$\Delta\phi = 2\int_0^1 \frac{du}{\sqrt{1 - u^2 - \frac{2GM}{r_0}(1 - u^3)}} \ . \tag{4.65}$$

Expanding to leading order in GM/r_0 we finally obtain

$$\Delta\phi = \pi + \frac{4GM}{r_0} + \ldots = \pi + \frac{4GM}{b} + \ldots \ , \tag{4.66}$$

since the difference between r_0 and b is already of order GM/r_0. Hence, the deflection of light is, to leading order

$$\delta\phi \approx \frac{4GM}{b} \ . \tag{4.67}$$

For starlight grazing the Sun, one obtains the famous 1.75 seconds of arc that were measured by Eddington in 1919 and that started Einstein's life as a public figure.

Problem: Dragging Langevin

Planet X is bound to an extremal Kerr black hole of mass M in a circular orbit, lying at the equatorial edge of the Ergosphere. Two twin inhabitants of this Planet are separated by sending one of them into a forward-oriented circular trajectory along the equator of the Ergosphere's edge. The runaway twin rides a fast spaceship capable of thrusting to a relativistic factor $\gamma = (1 - v^2)^{-1/2} = 1/\varepsilon$. The second twin only parts on an interception course 'one year' later, using an entirely identical ship and following also a circular trajectory along the equatorial edge of the Ergosphere.

Find the fastest interception trajectory, the location of the meeting point (where and when) and the age difference of the twins after their reunion.

Hints:

- The maximal velocity of the ship, or equivalently the maximal relativistic gamma factor $\gamma = 1/\varepsilon$, is the manufacturer's specification, measured with respect to an idealized inertial frame in Minkowski space-time.
- Above, 'one year' means of course one revolution of Planet X around the extremal Kerr black hole.
- It is suggested to first find the angular velocity of any circular orbit around a Kerr black hole, and then particularize the result to the concrete case of this problem.
- Solve the problem first for the idealized case of a light-fast ship with $\varepsilon = 0$, and later perturb this result for finite ε (leading approximation for small ε is enough). Approximate all trajectories as constant-velocity motions, neglecting the sharp periods of acceleration of the spaceships.