Preprint typeset in JHEP style - HYPER VERSION

IFT-UAM/CSIC-08

Low energy windows in Quantum Gravity

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Abstract:

Contents

1. Introduction

Those are lectures on a topic that nobody understands fully. Not only that, but also it is not the first time I lecture on the failures of the scientific community to come to terms with the topic. The only exuse I can offer (and a poor one as it stands) is that I am not the only physicist to be fascinated by the challenge of understanding quantum mechanics in the presence of gravitation. The practical utility of the answer to this question will not presumably be great (although how would be know for sure beforehand?) but it is a matter of principle, and, as such, we hope to better understand both quantum mechanics and gravitation if we are able to clarify the issue.

The mass scale associated to this problem just by sheer dimensional analysis is Planck's mass,

$$m_p \sim G^{-1/2} \sim 10^{19} GeV \tag{1.1}$$

If we remember that 1GeV is the rough scale of hadronic physics (the mass and Comptom wavelength of a proton, for example), this means that quantum gravity effects will only be apparent when we are able to explore an energy region roughly 10^{19} times bigger (or an scale distance correspondingly smaller; these two statements are supposed to be equivalent owing to Heisenberg's principle).

In a bottom-up approach there is a working low energy effective theory, and quantum effects in gravity can be reliably computed for energies much smaller than Planck mass. There are two caveats to this. First of all, we do not understand why the oberved cosmological constant is so small: the natural value from the low energy point of view ought to be much bigger. We will have unfortunately nothing new to add to this problem. The second point is that one has to rethink again the lore of effective theories in the presence of horizons. We shall comment on that in due time.

There is not a universal consensus even on the most promising avenues of research from the opposite top-down viewpoint. Many people think that strings [76] are the best buy (I sort of agree with this); but it is true that after more than two decades of intense effort nothing substantial has come out of them. Others [61] try to quantize directly the Einstein-Hilbert lagrangian, something that is at variance with our experience in effective field theories. But it is also true that as we have already remarked, the smallish value of the observed cosmological constant also cries out of the standard effective theories lore.

These lectures are quite idiosyncratic in that I only talk on topics of with I think I understand something. It is my purpose to keep them quite broad minded and general, and I do not want to repeat the standard textbook stuff already summarized in many excellent books and review articles, many of them highly opinionated. It is hoped that at least some new ideas can be rescued from them.

2. Quantum effects in an external (fixed) gravitational field

2.1 The Unruh effect

Before entering the subject matter as such (of which no much is known) let us dwell for a while in a quantum effect due to the non-inertial character of the observer. By the equivalence principle, this ought to be related to a gravitational field. We are talking of the Unruh effect that although was discovered after Hawking predicted the black hole thermal emission, is in fact logically simpler and independent.

Let us consider the trajectory of an accelerated observer in two dimensional flat space

$$\begin{aligned} t &= \frac{1}{a} sinh\, a\tau \\ x &= \frac{1}{a} cosh\, a\tau \end{aligned} \tag{2.1}$$

This is such that the four-velocity is given by

$$u = \left(cosh\, a\tau \;,\; sinh\, a\tau\right) \tag{2.2}$$

normalized to

$$u^2 = 1 \tag{2.3}$$

and the acceleration

$$\dot{u} \equiv a\left(sinh\, a\,\tau \;,\; cosh\, a\tau\right) \tag{2.4}$$

obeys

$$\begin{aligned} a^2 &= -1 \\ a.u &= 0 \end{aligned} \tag{2.5}$$

In *comoving* coordinates, id est, adapted to the four-velocity,

$$u = \frac{\partial}{\partial \xi^0} \tag{2.6}$$

the worldline of the accelerated observer is

$$\begin{aligned} \xi^0(\tau) &= \tau \\ \xi^1(\tau) &= 0 \end{aligned} \tag{2.7}$$

In general

$$\begin{aligned} t &= \frac{e^{a\xi^1}}{a}\, sinh\, a\xi^0 \\ x &= \frac{e^{a\xi^1}}{a} cosh\, a\xi^0 \end{aligned} \tag{2.8}$$

so that the value of the coordinate ξ^1 tells us which hyperbola we are talking about

$$t^2 - x^2 = -\frac{e^{2a\xi^1}}{a^2} \tag{2.9}$$

In terms of these coordinates the Minkowski metric reads

$$ds^2 = dt^2 - dx^2 = e^{2a\xi^1}\left(\left(d\xi^0\right)^2 - \left(d\xi^1\right)^2\right) \tag{2.10}$$

When

$$\begin{aligned} -\infty \leq \xi^0 \leq \infty \\ -\infty \leq \xi^1 \leq \infty \end{aligned} \tag{2.11}$$

only one quarter of the original Minkowski space has been covered, namely the one corresponding to

$$|t| \leq x \tag{2.12}$$

This is called *Rindler's wedge* or *Rindler space.* The lightcone plays the role of the *event horizon.*

Let us now consider an scalar field

$$S = \frac{1}{2}\int dt \wedge dx \left(\left(\frac{\partial\phi}{\partial t}\right)^2 - \left(\frac{\partial\phi}{\partial x}\right)^2\right) = \frac{1}{2}\int d\xi^0 \wedge d\xi^1 \left(\left(\frac{\partial\phi}{\partial\xi^0}\right)^2 - \left(\frac{\partial\phi}{\partial\xi^1}\right)^2\right) \tag{2.13}$$

We can use lightcone coordinates

$$x_{\pm} \equiv t \pm x \tag{2.14}$$

as well as

$$X\pm \equiv \xi^0 \pm \xi^1 \tag{2.15}$$

The full solution of the classical equations of motion

$$\frac{\partial^2}{\partial x^+ \partial x^-}\phi = \frac{\partial^2}{\partial X^+ \partial X^-}\phi = 0 \tag{2.16}$$

is a combination of rightmoving, positive frequency modes such as

$$f_R^+(\omega) \equiv e^{-i\omega x^-} = e^{-i\omega(t-x)} \tag{2.17}$$

and their complex conjugates, wich are negative energy left movers.

It is worthwile to stop a while to think on the reason why we say that it is rightmoving. It is because

$$\hat{k} f_R^+ = -i\frac{\partial}{\partial x} f_R^+ = \omega f_R^+$$

The reason why we say that it also enjoys positive frequency is because

$$\hat{H} f_R^+ = i \frac{\partial}{\partial t} f_R^+ = \omega f_R^+.$$

The plane waves

$$g_L^+(x^+) \equiv e^{-i\omega x^+} \tag{2.18}$$

are left-moving, positive energy solutions.

The general classical solution can be expanded in a sum of a Fourier series for the left movers and a corresponding series for the right movers. We split the series in f_R, f_R^*, g_L, g_L^* considering that

$$0 \leq \omega \leq \infty \tag{2.19}$$

We could as well suppress the complex conjugate basis functions and integrate from

$$-\infty \leq \omega \leq \infty \tag{2.20}$$

$$\phi = \int_0^\infty \frac{d\omega}{\sqrt{4\pi\omega}} \left(\left(a_R^-(\omega) e^{-i\omega x^-} + a_R^+(\omega) e^{i\omega x^-} \right) + \left(a_L^-(\omega) e^{-i\omega x^+} + a_L^+(\omega) e^{i\omega x^+} \right) \right) \tag{2.21}$$

We could also say the corresponding solutions

$$F_R^+(\Omega) \equiv e^{-i\Omega X^-} \tag{2.22}$$

are right-moving positive frequency with respect to the new space and time coordinates (ξ^0, ξ^1)

The relationship between the two light cone coordinates is given by:

$$\begin{aligned} x^- &= -\frac{1}{a} e^{-aX^-} \\ x^+ &= \frac{1}{a} e^{aX^+} \end{aligned} \tag{2.23}$$

We then have a different expansion

$$\phi = \int_0^\infty \frac{d\Omega}{\sqrt{4\pi\Omega}} \left(\left(b_R^-(\Omega) e^{-i\Omega X^-} + b_R^+(\Omega) e^{i\Omega X^-} \right) + \left(b_L^-(\Omega) e^{-i\Omega X^+} + b_L^+(\Omega) e^{i\Omega X^+} \right) \right) \tag{2.24}$$

We are then tempted to write the field operator

$$\begin{aligned} \hat{\phi} &= \int_0^\infty \frac{d\omega}{\sqrt{4\pi\omega}} \left(\left(\hat{a}_R(\omega) e^{-i\omega x^-} + \hat{a}_R^+(\omega) e^{i\omega x^-} \right) + \left(\hat{a}_L(\omega) e^{-i\omega x^+} + \hat{a}_L^+(\omega) e^{i\omega x^+} \right) \right) = \\ &\int_0^\infty \frac{d\Omega}{\sqrt{4\pi\Omega}} \left(\left(\hat{b}_R(\Omega) e^{-i\Omega X^-} + \hat{b}_R^+(\Omega) e^{i\Omega X^-} \right) + \left(\hat{b}_L(\Omega) e^{-i\Omega X^+} + \hat{b}_L^+(\Omega) e^{i\Omega X^+} \right) \right) \end{aligned} \tag{2.25}$$

where the operators obey canonical commutation relations

$$\begin{aligned} [\hat{a}(\omega)_R, \hat{a}^+(\omega')_R] &= \delta(\omega - \omega') \\ [\hat{b}(\Omega)_R, \hat{b}^+(\Omega')_R] &= \delta(\Omega - \Omega') \end{aligned} \tag{2.26}$$

and so on.

- We now *define* the Minkowski vacuum state by the condition

$$\hat{a}_R(\omega)|0_M\rangle = 0 \tag{2.27}$$

It is clear that this is the vacuum whose excitations would measure an inertial observer. The Rindler vacuum instead will be defined by

$$\hat{b}_R(\omega)|0_R\rangle = 0 \tag{2.28}$$

and this is the ground state for excitations measured by the accelerated observer.

- Assuming that the Minkowski vacuum is a physical state, the Rindler state requires an infinite energy to be prepared: It can be checked from the expansions that

$$\langle 0|T_{x^-x^-}|0\rangle \sim \langle 0_M|\frac{\partial\hat{\phi}}{\partial x^-}\frac{\partial\hat{\phi}}{\partial x^-}|0_M\rangle = \langle 0_R|\frac{\partial\hat{\phi}}{\partial X^-}\frac{\partial\hat{\phi}}{\partial X^-}|0_R\rangle \tag{2.29}$$

This yields

$$\langle 0_R|\frac{\partial\hat{\phi}}{\partial x^-}\frac{\partial\hat{\phi}}{\partial x^-}|0_R\rangle = \left(\frac{\partial X^-}{\partial x^-}\right)^2\langle 0_R|\frac{\partial\hat{\phi}}{\partial X^-}\frac{\partial\hat{\phi}}{\partial X^-}|0_R\rangle = \frac{1}{a^2\left(x^-\right)^2}\langle 0_R|\frac{\partial\hat{\phi}}{\partial X^-}\frac{\partial\hat{\phi}}{\partial X^-}|0_R\rangle \tag{2.30}$$

which is expected to diverge at the future horizon $x^- = 0$.

In a completely analogous way we would have shown that

$$\langle 0|T_{x^+x^+}|0\rangle \tag{2.31}$$

are expected to diverge at the past horizon, $x^+ = 0$.

- It is clear that we can Fourier expand one set of modes in terms of the other:

$$F_R^+(\Omega) = e^{-i\Omega X^-} = \int_{-\infty}^{\infty} d\omega\rho(\omega)e^{-i\omega x^-} = \int_0^{\infty} d\omega\left(\rho(\omega)f_R^+(\omega) + \rho(-\omega)f_R^*(\omega)\right) \tag{2.32}$$

with

$$\rho(\omega) = \int\frac{dx^-}{2\pi}e^{-i\Omega X^-}e^{i\omega x^-} = \int\frac{dx^-}{2\pi}e^{i\omega x^-}\left(-ax^-\right)^{\frac{i\Omega}{a}} =$$
$$\frac{i}{2\pi\omega}\left(\frac{a}{i\omega}\right)^{i\frac{\Omega}{a}}\Gamma\left(1+i\frac{\Omega}{a}\right) = -\frac{1}{2\pi\omega}e^{\frac{\pi\Omega}{2a}}e^{i\frac{\Omega}{a}log\,\frac{a}{\omega}}\Gamma\left(i\frac{\Omega}{a}\right) \tag{2.33}$$

We also have

$$f_R(\omega) = e^{-i\omega x^-} = \int_{-\infty}^{\infty} d\Omega\,\gamma(\Omega)e^{-i\Omega X^-} = \int_0^{\infty} d\Omega\left(\gamma(\Omega)\,F_R(\Omega) + \gamma(-\Omega)\,F_R^*(\Omega)\right) =$$
$$\int_0^{\infty} d\Omega\sqrt{\frac{\omega}{\Omega}}\left(\alpha(\Omega)F_R - \beta^*(\Omega)F_R^*\right) \tag{2.34}$$

where this last notation has been introduced with an eye for the Bogoliubov transformation that will appear in a moment, and

$$\gamma(\Omega) = \int_{-\infty}^{\infty} \frac{dX^-}{2\pi} e^{-i\omega x^-} e^{i\Omega X^-} = -\int_{-\infty}^{0} \frac{dx^-}{2\pi} \frac{1}{ax^-} e^{-i\omega x^-} (-ax^-)^{\frac{-i\Omega}{a}} =$$
$$\int_0^{\infty} \frac{dy}{2\pi} \frac{1}{ay} e^{i\omega y} (ay)^{-i\frac{\Omega}{a}} = \int_0^{\infty} \frac{dt}{2\pi a} \frac{1}{t} e^{-t} \left(\frac{ita}{\omega}\right)^{-i\frac{\Omega}{a}} =$$
$$\frac{1}{2\pi a} e^{\frac{\pi\Omega}{2a}} e^{-i\frac{\Omega}{a} \log \frac{a}{\omega}} \Gamma\left(-i\frac{\Omega}{a}\right) \tag{2.35}$$

This clearly implies that

$$|\alpha(\Omega)|^2 = e^{\frac{2\pi\Omega}{a}} |\beta(\Omega)|^2 \tag{2.36}$$

- There is a Bogolyubov transformation relating both sets of creation and destruction operators. Symbolically, the change of basis we have just done yields

$$\phi \sim \sum \hat{a}_R (\alpha F - \beta^* F^*) + \hat{a}_R^+ (\alpha^* F^* - \beta F) + left =$$
$$\sum \hat{b}_R F + \hat{b}_R^+ F^* + left \tag{2.37}$$

In gory detail,

$$\hat{b}_R(\Omega) = \int_0^{\infty} d\omega \left(\alpha_{\Omega\omega} \hat{a}_R(\omega) - \beta_{\Omega\omega} \hat{a}_R^+(\omega)\right) \tag{2.38}$$

The canonical commutation relations do imply that (suppressing carets over operators from now on)

$$\left[\int_0^{\infty} d\omega_1 \left(\alpha_{\Omega_1,\omega_1} a(\omega_1) - \beta_{\Omega_1,\omega_1} a^+(\omega_1)\right), \int d\omega_2 \left(\alpha^*_{\Omega_2,\omega_2} a(\omega_2) - \beta^*_{\Omega_2,\omega_2} a^+(\omega_2)\right)\right] = \delta(\Omega_1 - \Omega_2) =$$
$$\int d\omega \left(\alpha_{\Omega_1\omega} \alpha^*_{\Omega_2\omega} - \beta_{\Omega_1\omega} \beta^*_{\Omega_2\omega}\right) \tag{2.39}$$

which is a normalization condition for Bogoliubov's coefficients. It implies, in particular, that

$$\int d\omega \left(|\alpha_{\Omega\omega}|^2 - |\beta_{\Omega\omega}|^2\right) = \delta(0) = \int d\omega \left(e^{\frac{2\pi\Omega}{a}} - 1\right) |\beta_{\Omega\omega}|^2 \tag{2.40}$$

The expectation value of b-particles in the Minkowski vacuum will be

$$\langle 0_M | N_\Omega \equiv b_\Omega^+ b_\Omega | 0_M \rangle =$$
$$\langle 0_M | \int d\omega_1 \left(\alpha^*_{\Omega\omega_1} a_\omega^+ - \beta_{\Omega\omega_1} a_\omega\right) \int d\omega_2 \left(\alpha_{\Omega\omega_2} a_\omega - \beta_{\Omega\omega_2} a_\omega^+\right) | 0_M \rangle = \int d\omega |\beta_{\Omega\omega}|^2 =$$
$$\frac{1}{e^{\frac{2\pi\Omega}{a}} - 1} \delta(0) \sim \frac{1}{e^{\frac{2\pi\Omega}{a}} - 1} V \tag{2.41}$$

where V has to be interpreted as the volume of space. These massless particles detected by the accelerated oberved in the Minkowski vacuum obey the Bose-Einstein distribution at a temperature

$$T = \frac{a}{2\pi} \tag{2.42}$$

This is the *Unruh temperature*. In order to get to a temperature of

$$T = 1 \sim 10^{-16} erg \sim 10^{-10} MeV \tag{2.43}$$

and given the fact that the gravitational acceleration at earth is

$$g \sim 10 ms^{-2} \sim 10^{-29} MeV \tag{2.44}$$

the corresponding acceleration necessary to raise the temperature a miserable degree is

$$a \sim 10^{19} g \tag{2.45}$$

The possibillity of its detection in storage rings has been advanced by Bell and Leinaas [**?**]. More recently, a proposal was put forward by Chen and Tajima [17] of detecting Unruh radiation with the help of ultra-intense lasers. It seems however that we have to wait somewhat before getting experimental confirmation of such an effect.

2.2 The Kawking effect.

2.3 Physics in maximally symmetric spaces.

The (mathematically) simplest non-flat space-times are those of constant curvature, traditionally knowm as de Sitter or anti de Sitter. It seems that they are in some sense the most natural candidates for the *vacuum* of quantum gravity. We will comment on that in due time. For the time being, we shall explore some aspects of quantum physics in thsose spaces.

It is of interest to understand their relationship with ordinary spheres. We shall study for a while flat spaces with arbitrary signature.

Some flat metrics in $\mathbb{R}^{n+1}$ will be considered, namely, for arbitrary $\pm$ signs, denoted by $\epsilon_M = \pm 1$,

$$ds^2 = \sum_{A=0}^{n} \epsilon_A dx_A^2 \equiv \eta_{AB} dx^A dx^B \tag{2.46}$$

Given a metric, there is a corresponding algebra

$$[M_{AB}, M_{CD}] = i\left(\eta_{BC} M_{AD} - \eta_{AC} M_{BD} - \eta_{BD} M_{AC} + \eta_{AD} M_{BC}\right) \tag{2.47}$$

This algebra is a real form of the complex algebra $SO(n)$, including the de Sitter group, dS(n) as well as the anti-de Sitter group, AdS(n), and also its euclidean versions EdS(n) and EAdS(n).The aim of this report is to put together some notes on it.

These algebras make many appearances in physics. One of the most important ones is as the conformal group. The conformal group $C(m,n)$ of $\mathbb{R}_{m+n}$ endowed with a Minkowski-like metric with m-times and n-spaces is just $SO(m+1, n+1)$ ($SO(4,2)$ in the four-dimensional case.)

Casimir operators are given by

$$C_p \equiv tr\, M^p = \sum M_{A_1 B_1} \ldots M_{A_p A_1} \tag{2.48}$$

of which only those with even p are nonvanishing, owing to antisymmetry.

When $n = 2m \in 2\mathbb{Z}$, there is in addition the Levi-Civita invariant:

$$E_m = \epsilon_{A_1 B_1 \ldots A_m B_m} M_{A_1 B_1} \ldots M_{A_m B_m} \tag{2.49}$$

which is such that

$$E_m^2 = C_{2m} \tag{2.50}$$

and distinguishes both chiralities.

2.3.1 Gamma matrices

Gamma matrices are defined through the Clifford algebra

$$\{\gamma_M, \gamma_N\} = 2\eta_{MN} \tag{2.51}$$

(which implies $\gamma_M^2 = \epsilon_M$). A particular representation of the group of rotations SO(n) can always be obtained from the representation of the euclidean Clifford algebra

$$\{\Gamma_M, \Gamma_N\} = 2\delta_{MN} \tag{2.52}$$

by hermitian matrices, for example the tensor product of n sigma matrices:

$$\begin{aligned}
&\Gamma_1 \equiv \sigma_2 \otimes \sigma_3 \ldots \otimes \sigma_3\\
&\Gamma_2 \equiv \sigma_1 \otimes \sigma_3 \ldots \otimes \sigma_3\\
&\ldots\\
&\Gamma_{2n-1} \equiv \sigma_2 \otimes \sigma_2 \ldots \otimes \sigma_2\\
&\Gamma_{2n} \equiv \sigma_1 \otimes \sigma_1 \ldots \otimes \sigma_1\\
&\Gamma_{2n+1} \equiv \sigma_3 \otimes \sigma_3 \ldots \otimes \sigma_3
\end{aligned} \tag{2.53}$$

by

$$\gamma_M = i\Gamma_M \tag{2.54}$$

whenever $\epsilon_M = -1$ and by

$$\gamma_M = \Gamma_M \tag{2.55}$$

otherwise. This implies the hermiticity assignment

$$\gamma_M^+ = \epsilon_M \gamma_M \tag{2.56}$$

The matrices

$$\sigma_{MN} \equiv \frac{1}{2}[\gamma_M, \gamma_N] \tag{2.57}$$

are such that

$$[\sigma_{AB}, \gamma_C] = 2\eta_{CB}\gamma_A - 2\eta_{AC}\gamma_B \tag{2.58}$$

as well as

$$\gamma_A\gamma_B = \eta_{AB} + \sigma_{AB} \tag{2.59}$$

and the algebra

$$[\sigma_{AB}, \sigma_{CD}] = 2\left(\eta_{BD}\sigma_{CA} - \sigma_{CB}\eta_{AD} + \eta_{CB}\sigma_{AD} - \eta_{AC}\sigma_{BD}\right) \tag{2.60}$$

This means that $\frac{i}{2}\sigma_{AB}$ is a matrix representation of (2.47).

$$D_s\left(M_{AB}\right) = \frac{1}{2}\sigma_{AB} = \frac{i}{4}[\gamma_A, \gamma_B] \tag{2.61}$$

Not all generators are hermitic, however (except in the euclidean case). Actually:

$$M_{AB}^+ = \epsilon_A\epsilon_B M_{AB} \tag{2.62}$$

Non hermitic generators mean that the (finite dimensional) representation is not unitary, and can be more or less associated to noncompact directions in the group manifold.

2.4 Constant curvature spaces $C^{\pm}_{\epsilon_M}$ with $SO(s, n+1-s)$ isometry.

All the spaces we are going to be interested at can be obtained by analytic continuation from either the sphere S_n,

$$\sum_{A=0}^{n} X_A^2 \equiv \delta_{AB} dX^A dX^B = l^2 \tag{2.63}$$

on a flat space with metric

$$ds^2 = \delta_{AB} dX^A dX^B \tag{2.64}$$

or else the real projective space, $\mathbb{RP}^n = S_n/\mathbb{Z}_2$, where the antipodal mapping

$$\mathbb{Z}_2 : X^A \rightarrow -X^A \tag{2.65}$$

The sphere is then the universal covering space of the projective plane, and $\pi_1(\mathbb{RP}_n) = \mathbb{Z}_2$ Functions on the projective plane are given by even functions on the sphere

$$f(X^A) = f(-X^A) \tag{2.66}$$

The projective plane is non-orientable for even values of n; but it is orientable for odd values of n. For example, $\mathbb{RP}_1 \sim S_1$.

Let us focus on the fundamental hyperquadrics of flat space, which are well known to be the only hypersurfaces of constant curvature of flat space ([32]).

The metric induced on the surface

$$\sum_{A=0}^{n} \epsilon_A X_A^2 \equiv \eta_{AB} X^A X^B = \pm L^2 \tag{2.67}$$

by the imbedding on the (n+1)-dimensional flat space M_{n+1} with metric

$$ds^2 = \sum_{A=0}^{n} \epsilon_A dX_A^2 \equiv \eta_{AB} dX^A dX^B \tag{2.68}$$

If we group together all minus signs in the metric, then Wolf's notation is $\mathbb{R}^{n+1}_s$ for the space whose metric enjoys exactly s minus signs (which Wolf takes as times, but for us they are spaces).

Furthermore, the pseudoriemannian spheres and hyperbolic spaces are given by

$$\begin{aligned} S^n_s &\equiv X \in \mathbb{R}^{n+1}_s \quad \& \quad \eta_{AB} X^A X^B = L^2 \\ H^n_s &\equiv X \in \mathbb{R}^{n+1}_{s+1} \quad \& \quad \eta_{AB} X^A X^B = -L^2 \end{aligned} \tag{2.69}$$

They are both n-dimensional pseudoriemannian manifolds with signature $(n-s, s)$. We shall be mainly interested in the Lorentzian case $s = n-1$. Useful diffeomorphisms are

- $S^n_s \to \mathbb{R}^s \times S^{n-s}$, i.e.

$$X \to \left(X^1, \ldots, X^s; \frac{1}{\sqrt{L^2 + \sum_{i=1}^s X_i^2}} \left(X^{s+1}, \ldots, X^{n+1}\right)\right)$$

- $H^n_s \to S^s \times \mathbb{R}^{n-s}$, i.e.

$$X \to \left(\frac{1}{\sqrt{L^2 + \sum_{i=s+2}^{n+1} X_i^2}} \left(X^1, \ldots, X^{s+1}\right); X^{s+2}, \ldots, X^{n+1}\right)$$

- $S^n_s \to H^n_{n-s}$,i.e.

$$X \to \left(X^{s+1}, \ldots, X^{n+1}; X^1, \ldots, X^s\right)$$

The universal (i.e. simply connected) pseudo-riemannian coverings are $\tilde{S}^n_{n-1} \neq S^n_{n-1}$ and $\tilde{H}^n_{n-1} = H^n_{n-1}$.

It is a fact of life that the complete connected manifolds M^n_s of constant curvature are those isometric to a quotient $\tilde{S}^n_s/\Gamma$ or else $\tilde{H}^n_s/\Gamma$, where Γ is a group of isometries acting freely and properly discontinuously. [1]

Sometimes the coordinates X^A themselves (obeying (2.67)) will be used; in those cases they will be referred to as *Weierstrass coordinates*.

Please note that the situations $(\epsilon_M, +)$ and $(-\epsilon_M, -)$ are such that spacetime metric only changes by a global sign, the Christoffels invariant as well as the Riemann tensor $R^\alpha{}_{\beta\gamma\delta}$ and the Ricci tensor $R_{\beta\delta} \equiv R^\alpha{}_{\beta\alpha\delta}$. The curvature scalar $R \equiv g^{\alpha\beta} R_{\alpha\beta}$ then changes sign. This spaces will be labeled

$$C^{\pm}_{\epsilon_M} \tag{2.71}$$

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[1] A manifold is *homogeneous* if the full group of isometries is *transitive* (that is, that $\forall x, y \in M, \exists g \in G, g(x) = y$).

For the manifold to be *symmetric* the Riemann tensor has got to be covariantly constant.

On the other hand, a group action is *free* (i.e., without *fixed points*) if $\forall x \in M$, the little group $\{g \in G, g(x) = x\}$ is trivial.

Finally, the action is *properly discontinuous* if $\forall x \in M$ there is a neighborhood U such that $\{\gamma \in \Gamma, \gamma(U) \text{ meets } U\}$ is finite.

A *covering* $p : S \to T$ is a continous map of connected, locally arcwise connected spaces.

The covering is *universal* if S is simply connected.

The *deck transformations* $h \in \Gamma(S/T)$ are those homeomorphisms $h : S \to S$ such that

$$p \circ h = p \tag{2.70}$$

A *Clifford translation* is an isometry f such that $d\,(x, f\,(x))$ is a constant. The Clifford translations of Euclidean space $\mathbb{R}^n$ are just the ordinary translations. The only Clifford translation of hyperbolic space $\mathbb{R}^n$ is the identity. If $p : N \to M$ is a riemannian covering, Γ is the group of deck transformations, and M is homogeneous, then every element of Γ is a Clifford translation.

The mother of all these spaces is the n-sphere, for which all $\epsilon_A = 1$ and the sign in the second member is plus as well.

One of the purposes of our research is to study the extent to which physical quantities are determined on the spaces $C^{\pm}_{\epsilon_M}$ by analytic continuation from the sphere.

2.5 The general complete connected homogeneous Lorentz manifolds of constant curvature M^n_{n-1}

More generally, complete connected homogeneous Lorentz manifolds of constant curvature $M^n_{s=n-1}$ can be fully classified ([93]). In order to understand it we need some preliminaries.

First of all, recall that every compact subgroup of $O(s, n+1-s)$ is conjugate to a subgroup of $O(s) \times O(n+1-s)$.[2]

There is a map between the isometry group of the universal covering of S^n_{n-1}, which we dub $\tilde{S}^n_{n-1}$ (the isometry group is denoted by $I\left(\tilde{S}^n_{n-1}\right)$) and the isometry group of S^n_{n-1} itself, which we denote by $I\left(S^n_{n-1}\right)$, to wit, if we represent the projection by

$$\pi : \tilde{S}^n_{n-1} \longrightarrow S^n_{n-1}$$

then the mapping is defined by:

$$f : I\left(\tilde{S}^n_{n-1}\right) \rightarrow I\left(S^n_{n-1}\right) \equiv O(n-1, 2)$$

through

$$f(g)\pi(\tilde{x}) = \pi(g\tilde{x})$$

The kernel of this map is the group D of deck transformations.

We shall now define some convenient subgroups of isometries.

---

[2]Let us remind the reader that the *center* of a Lie algebra is the set of elements Z such that $[Z, X] = 0, \forall X \in \underline{G}$.

We define the *lower central series* $\mathcal{D}_k\underline{G}$ inductively by

$$\mathcal{D}_1\underline{G} = [\underline{G}, \underline{G}]$$

and

$$\mathcal{D}_k\underline{G} = [\underline{G}, \mathcal{D}_{k-1}\underline{G}]$$

We also define the *derived series* $\mathcal{D}^k\underline{G}$ inductively by

$$\mathcal{D}^1\underline{G} = [\underline{G}, \underline{G}]$$

and

$$\mathcal{D}^k\underline{G} = [\mathcal{D}^{k-1}\underline{G}, \mathcal{D}^{k-1}\underline{G}]$$

A Lie algebra is called *nilpotent* if $\exists k, \mathcal{D}_k\underline{G} = 0$.

A Lie algebra is called *solvable* if $\exists k, \mathcal{D}^k\underline{G} = 0$.

A Lie algebra is *semisimple* if there is no nonzero solvable ideal.

The maximal solvable ideal is called the *radical*, $Rad(\underline{G})$.

A Lie algebra is called *reductive* if its radical is equal to its center.

- $$\tilde{A}_S \equiv f^{-1}(\mathbb{Z}_2) \subset I(\tilde{S}^N_{n-1})$$

- Let us now restrict to the case n odd. we define the matrix

$$J \equiv i\sigma_2 \otimes 1_{\frac{n+1}{2}} = \begin{pmatrix} i\sigma_2 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & i\sigma_2 \end{pmatrix} \in GL(n+1, \mathbb{R}) \tag{2.72}$$

(this is such that $J^2 = -1$) as well as a rotation $R(\theta) \in O(n-1,2)$ defined by

$$R(\theta) \equiv cos\,\theta\, 1 + sin\,\theta\, J$$

Indeed

$$R^T(\theta)\eta R(\theta) = \begin{pmatrix} -i\sigma_2 & & & \\ & \cdot & & \\ & & -i\sigma_2 & \\ & & & i\sigma_2 \end{pmatrix} \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & \cdot & \\ & & & 1 \end{pmatrix} \begin{pmatrix} i\sigma_2 & & & \\ & i\sigma_2 & & \\ & & \cdot & \\ & & & i\sigma_2 \end{pmatrix} = \eta$$

They obviously close into a one-dimensional abelian subgroup:

$$R\left(\theta_1\right) R\left(\theta_2\right) = R\left(\theta_1 + \theta_2\right)$$

Finally, we define for n odd,

$$A_Z = \{R\left(\theta\right) \in O\left(n-1,2\right)\}$$

and

$$\tilde{A}_Z \equiv f^{-1}\left(A_Z\right) \subset I\left(\tilde{S}^n_{n-1}\right)$$

- Now we define the parabolic translations $T_p(\theta)$,

$$T_p\left(\theta\right) \equiv \begin{pmatrix} 1 - i\sigma_2\theta & 0 & -i\sigma_2\theta \\ 0 & 1_{n-3} & 0 \\ i\sigma_2\theta & 0 & 1 + i\sigma_2\theta \end{pmatrix} \in O\left(n-1,2\right) \tag{2.73}$$

As a matter of fact,

$$T_p^T(\theta)\eta T_p(\theta) = \begin{pmatrix} 1 + i\sigma_2\theta & 0 & -i\sigma_2\theta \\ 0 & 1_{n-3} & 0 \\ i\sigma_2\theta & 0 & 1 - i\sigma_2\theta \end{pmatrix} \begin{pmatrix} -1_2 & & \\ & -1_{n-3} & \\ & & 1_2 \end{pmatrix} \begin{pmatrix} -1 + i\sigma_2\theta & 0 & i\sigma_2\theta \\ 0 & -1_{n-3} & 0 \\ i\sigma_2\theta & 0 & 1 + i\sigma_2\theta \end{pmatrix}$$
$$= \eta$$

It is also plain that they also close into a one-dimensional abelian subgroup

$$T_p(\theta_1)\, T_p(\theta_2) = T_p(\theta_1 + \theta_2)$$

Then,

$$A_P \equiv \{\pm T_p(\theta) \in O(n-1,2)\}$$

and

$$\tilde{A}_P = f^{-1}(A_P) \subset I(\tilde{S}^n_{n-1})$$

- When $n = 4$ we would like to consider the hyperbolic rotations

$$R_h(\theta) \equiv \begin{pmatrix} \cosh\theta\, 1_2 & \sinh\theta\, 1_2 \\ \sinh\theta\, 1_2 & \cosh\theta\, 1_2 \end{pmatrix} \in O(2,2) \tag{2.74}$$

Now define

$$A_H \equiv \{\pm R_h(\theta) \in O(2,2)$$
$$\tilde{A}_H \equiv f^{-1}(A_H) \subset I\left(\tilde{S}^3_2\right)$$

Let us now recall that the defining equation for $S^n_{n-1} \subset \mathbb{R}^{n+1}_{n-1}$ reads

$$X_n^2 + X_{n-1}^2 = L^2 + \sum_{i=1}^{n-1} X_i^2.$$

A generator of the fundamental group $\pi_1\left(S^n_{n-1}\right)$ is then given by

$$\sigma : [0,1] \to \sigma(t) \equiv (0 \ldots 0; L\sin 2\pi t, L\cos 2\pi t)$$

It is a fact that $\tilde{S}^n_{n-1}/\Gamma$ is homogeneous if and only if Γ is conjugate to a discrete subgroup of $\tilde{A}_S$, $\tilde{A}_Z$(for n odd) or $\tilde{A}_H$ (for $n = 3$).

Let $\tilde{\mathcal{S}}$ denote the family of all isometry classes of pseudo-spherical spaceforms $\tilde{S}^n_{n-1}/\Gamma$, $\Gamma \subset \tilde{A}_S$.

Let $\tilde{\mathcal{Z}}$ denote the family of all isometry classes of pseudo-spherical spaceforms $\tilde{S}^n_{n-1}/\Gamma$, $\Gamma \subset \tilde{A}_Z$.

Let $\tilde{\mathcal{P}}$ denote the family of all isometry classes of pseudo-spherical spaceforms $\tilde{S}^n_{n-1}/\Gamma$, $\Gamma \subset \tilde{A}_P$ which are not contained in $\tilde{\mathcal{S}}$.

Let $\tilde{\mathcal{H}}$ denote the family of all isometry classes of pseudo-spherical spaceforms $\tilde{S}^n_{n-1}/\Gamma$, $\Gamma \subset \tilde{A}_H$ which are not contained in $\tilde{\mathcal{S}}$.

To summarize (Wolf)

- The zero curvature manifolds are isometric to $\mathbb{R}^n_s/\Gamma$, where Γ is a discrete group of translations.
- For positive curvature there are several possibilities:
  - ◇ $M^n_s \in \tilde{\mathcal{S}}$, which means that it is a covering of $S^n_{n-1}/\mathbb{Z}_2$.
  - ◇ $M^{2p+1}_s \in \tilde{\mathcal{Z}}$.
  - ◇ $M^{n\geq 3}_s \in \tilde{\mathcal{P}}$.
  - ◇ $M^3_s \in \tilde{\mathcal{H}}$.
- For negative curvature, the manifold is isometric to H^n_{n-1} or to $H^n_{n-1}/\mathbb{Z}_2$

[3]

2.5.1 Stereographic coordinates for the sphere S_n

- Let us perform an stereographic projection of the first $n-1$ coordinates from the south pole, $X^n = -l$, and represent the projected coordinates in $\mathbb{R}^n$ by small face letters:

$$x^\mu_S \equiv \frac{2l}{X^n + l} X^\mu \equiv \Omega^{-1} X^\mu \tag{2.76}$$

The defining equation

$$\sum_0^{n-1} X_A^2 = l^2 \tag{2.77}$$

[3]The *normalizer* of H in G is the set of all $g \in G$ such that

$$gHg^{-1} = H$$

The *centralizer* of H in G is the set of all $g \in G$ such that

$$ghg^{-1} = h, \forall h \in H$$

It is plain then the centralizer is contained in the normalizer.

Let us call $I(M)$ the group of all isometries, and $I(M)_x$ the subgroup preserving x. There is also a local construction, starting from isometries of neighborhoods of a given point $x \in M$, leaving x itself fixed. If we identify two such maps if they agree in a neighborhood of x, we get a group H_x, the *group of local isometries at x.*

M is called *isotropic* at x if $I(M)_x$ is transitive on the unit sphere in M_x. This can be proved to be equivalent to the manifold being *two-point homogeneous*, i.e.,that the isometry group $I(M)$ is transitive on equidistant pairs of points.

It is a fact that a locally isotropic manifold is locally symmetric.

The signature of the Grassmann manifolds

$$O(s, t \equiv n-s)/O(s_1, t_1 = n_1 - s_1) \times O(s_2 = s - s_1, t_2 = n - n_1 - s + s_1) \tag{2.75}$$

is

$$s_1 t_2 + s_2 t_1$$

then implies

$$X^n = l\frac{1 - \frac{x_S^2}{4l^2}}{1 + \frac{x_S^2}{4l^2}} \tag{2.78}$$

so that

$$\Omega_S \equiv \frac{X^n + l}{2l} = \frac{1}{1 + \frac{x_S^2}{4l^2}} \tag{2.79}$$

Then

$$X^n = (2\Omega_S - 1)l \tag{2.80}$$

and

$$ds^2 = \sum_0^n dX_A^2 = \Omega_S^2 \delta_{ij} dx_S^i dx_S^j \tag{2.81}$$

That is, they are Riemannian coordinates in the sense of Eisenhart [32]. These coordinates can be defined for all constant curvature spaces. The Riemann tensor is given by:

$$R_{\mu\nu\rho\sigma} = \frac{R}{n(n-1)}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right) \tag{2.82}$$

and the Ricci tensor

$$R_{\mu\nu} = \frac{1}{n} R g_{\mu\nu} \tag{2.83}$$

where

$$R = \frac{n(n-1)}{l^2} \tag{2.84}$$

and the curvature

$$K_0 = \frac{1}{l^2} \tag{2.85}$$

Besides [32]

$$\nabla_\mu \nabla_\nu X^A = -\frac{X^A}{l^2} g_{\mu\nu} \tag{2.86}$$

These coordinates are singular at the South pole itself. They cover the whole sphere but for the south pole.

We could have projected from the north pole instead, that is,

$$x_N^\mu \equiv \frac{2l}{X^n - l} X^\mu \tag{2.87}$$

Let us call x_S and x_N the two sets of coordinates. They are enough to cover the manifold.

In the intersection of the two local systems (that is, the sphere without the two poles) consistency demand that

$$\begin{aligned} X^n &= (2\Omega_N + 1)l = (2\Omega_S - 1) \\ X^\mu &= \Omega_N x_N^\mu = \Omega_S x_S^\mu \end{aligned} \tag{2.88}$$

so that the change of coordinates is given by an inversion

$$x_S^\mu = -4l^2 \frac{x_N^\mu}{x_N^2} \tag{2.89}$$

and

$$\frac{x_S^2}{4l^2} = \frac{4l^2}{x_N^2} \tag{2.90}$$

Besides

$$\Omega_N = -\frac{1}{1 + \frac{x_N^2}{4l^2}} \tag{2.91}$$

and

$$\Omega_N = \Omega_S - 1 \tag{2.92}$$

It is then plain that

$$ds_N^2 = ds_S^2 \tag{2.93}$$

- The equations of the geodesics in Weierstrass coordinates are [32]:

$$X^A = X_0^A \, cos\, \frac{s}{l} + N_0^A l \, sin \frac{s}{l} \tag{2.94}$$

so that the distance in the enveloping space between two points whose geodesic distance is s is given by:

$$D(X, X_0)^2 \equiv \sum \left(X^A - X_0^A\right)^2 = 4l^2 \, sin^2 \, \frac{s}{2l} \tag{2.95}$$

The euclidean distance between two points $D(X, Y)$ translates into

$$D(x, y)^2 = \sum \left(\Omega(x) x^\mu - \Omega(y) y^\mu\right)^2 + 4l^2 \left(\Omega(x) - \Omega(y)\right)^2 \tag{2.96}$$

It is a fact that

$$\delta_{AB} X^A Y^B = -\frac{1}{2} \left(\vec{x} - \vec{y}\right)^2 \Omega_x \Omega_y \tag{2.97}$$

Thus

$$D(X, Y)^2 \equiv \sum \left(X^A - Y^A\right)^2 = 2l^2 + \left(\vec{x} - \vec{y}\right)^2 \Omega_x \Omega_y \tag{2.98}$$

It is also evident that

$$D(x_N, y_N) = D(x_S, y_S) \tag{2.99}$$

- The simplest possible example is just the ordinary circle, S^1, embedded in $\mathbb{R}^2$, which will be represented by the two coordinates (x, y). The south pole stereographic projection is defined through

$$\xi_S \equiv \frac{2l}{y + l} x \tag{2.100}$$

the correspondence with the general notation is then

$$\begin{aligned} X^\mu &\to x \\ X^n &\to y \\ x^\mu &\to \xi \end{aligned} \tag{2.101}$$

The metric reads

$$ds^2 = \frac{1}{1+\frac{\xi^2}{4l^2}} d\xi^2 = l^2 d\theta^2 \tag{2.102}$$

This means that the stereographic coordinate is related to the polar angle in a direct way:

$$\xi = 2l\, tg\, \frac{\theta}{2} \tag{2.103}$$

which means that

$$\begin{aligned} y &= l\frac{1-\frac{\xi^2}{4l^2}}{1+\frac{\xi^2}{4l^2}} = l\, cos\, \theta \\ \Omega &\equiv \frac{y+l}{2l} = \frac{1}{1+\frac{\xi^2}{4l^2}} = \frac{1+cos\, \theta}{2} \\ x &= \Omega\xi = l\, sin\, \theta \end{aligned} \tag{2.104}$$

The other neighborhood is covered by

$$\xi_N \equiv \frac{2l}{y-l} x \tag{2.105}$$

Clearly,

$$\xi_N = -\frac{4l^2}{\xi_S} = -2l\, cot\, \frac{\theta}{2} \tag{2.106}$$

It is plain that we could have written

$$\xi_N = 2l\, tg\, \frac{\theta_N}{2} \tag{2.107}$$

with

$$\theta_N = \theta_S + \frac{\pi}{2} \tag{2.108}$$

It is useful to remember that

$$\begin{aligned} \frac{\partial\xi}{\partial\theta} &= l\left(1+\frac{\xi^2}{4l^2}\right) = \frac{2l}{1+cos\, \theta} = \frac{l}{cos^2\frac{\theta}{2}} \\ \frac{\partial\theta}{\partial x} &= -\frac{cos^3\, \theta}{l\, sin^2\, \theta} \\ \frac{\partial\theta}{\partial y} &= \frac{cos^2\, \theta}{l\, sin\, \theta} \end{aligned} \tag{2.109}$$

It is clear that the angular momentum reads

$$L \equiv x\partial_y - y\partial_x = \left(\frac{2ly}{y+l} + \frac{2lx^2}{(y+l)^2}\right)\partial_\xi = \frac{l}{\Omega}\partial_\xi \tag{2.110}$$

(remembering that

$$\begin{aligned}\frac{\partial\xi}{\partial x} &= \frac{1}{\Omega}\\ \frac{\partial\xi}{\partial y} &= -\frac{1}{2l\Omega^2}x\end{aligned} \tag{2.111}$$

- The generators of the $SO(n+1)$ group are in the coordinates of $\mathbb{R}^{n+1}$

$$L_{AB} \equiv X^A\partial_B - X^B\partial_A \tag{2.112}$$

Functions defined on the sphere obey

$$\frac{\partial f}{\partial r} = 0 = X^A\frac{\partial}{\partial X^A}f \tag{2.113}$$

This means that

$$\frac{\partial}{\partial X^n} = -\frac{X^\mu}{X^n}\frac{\partial}{\partial X^\mu} \tag{2.114}$$

$$\begin{aligned}\frac{\partial x^\mu}{\partial X^\rho} &= \frac{1}{\Omega}\delta^\mu_\rho\\ \frac{\partial x^\mu}{\partial X^n} &= -\frac{X^\mu}{2l\Omega^2}\end{aligned} \tag{2.115}$$

(stereographic projection is defined outside the sphere as well, so that the result does not fulfill (2.114)) as well as (on the sphere itself)

$$\begin{aligned}\partial_\rho X^\mu &= \Omega\delta^\mu_\rho - \frac{\Omega^2}{2l^2}x^\mu x^\rho\\ \partial_\rho X^n &= 2l\partial_\rho\Omega = -\frac{\Omega^2}{l}x^\rho\end{aligned} \tag{2.116}$$

(indeed, $X^\mu\partial_\rho X^\mu + X^n\partial_\rho X^n = 0$).

We then obtain [4]

$$\frac{\partial}{\partial X^n} = -\frac{1}{2l\Omega}x^\mu\partial_\mu \tag{2.120}$$

[4] One could have argued otherwise: acting on functions defined on the sphere,

$$\frac{\partial}{\partial X^\mu} = \frac{\partial x^\rho}{\partial X^\mu}\partial_\rho = \frac{1}{\Omega}\partial_\mu \tag{2.117}$$

$\therefore$

$$X^\mu\frac{\partial}{\partial X^\mu} = x^\mu\partial_\mu \tag{2.118}$$

conveying the fact that

$$\frac{\partial}{\partial X^n} = -\frac{1}{X^n}x^\mu\partial_\mu \tag{2.119}$$

We can then write

$$L_{\mu\nu} = x^\mu \partial_\nu - x^\nu \partial_\mu \tag{2.121}$$

as well as (remembering that $X^n = (2\Omega - 1)\, l$)

$$L_{n\mu} = X^n \frac{\partial}{\partial X^\mu} - X^\mu \frac{\partial}{\partial X^n} = \frac{X^n}{\Omega}\partial_\mu + \frac{\Omega x^\mu}{2l\Omega} x.\partial =$$
$$\frac{l}{\Omega}\partial_\mu + \frac{1}{2l} x^\sigma L_{\mu\sigma} \tag{2.122}$$

2.5.2 Coordinates for $C^{\pm}_{\epsilon_M}$

- Let us choose coordinates in such a way that in the defining equation

$$\sum_{A=0}^{n} \epsilon_A X_A^2 \equiv \eta_{AB} dX^A dX^B = \pm l^2 \tag{2.123}$$

on a flat space with metric

$$ds^2 = \eta_{AB} dX^A dX^B \tag{2.124}$$

then

$$\epsilon_n = \pm 1 \tag{2.125}$$

This can always be achieved, by reshuffling the coordinates if necessary, because when the sign in the second member is negative, at least one of the coordinates has got to be an space.

We then define the south pole stereographic projection for $\mu = 0 \ldots n-1$, as

$$x_S^\mu \equiv \frac{2l}{X_n + l} X^\mu \equiv \frac{X^\mu}{\Omega} \tag{2.126}$$

The equation of the surface then leads to

$$X_n = l \frac{1 \mp \frac{x_S^2}{4l^2}}{1 \pm \frac{x_S^2}{4l^2}} \tag{2.127}$$

where

$$x_S^2 \equiv \sum_{\mu=0}^{n-1} \epsilon_\mu \left(x_S^\mu\right)^2 \tag{2.128}$$

besides,

$$\Omega_S = \frac{1}{1 \pm \frac{x_S^2}{4l^2}} \tag{2.129}$$

It is worth noticing that when the sign in the second member is negative, then

$$\frac{x_S^2}{4l^2} = \frac{X_n - l}{X_n + l} \tag{2.130}$$

(The range of X_n is now $-\infty \leq X_n \leq \infty$) but the range covered by the stereographic projection is $-l \leq X_n \leq \infty$. In this whole interval $\frac{X_n - l}{X_n + l} \leq 1$ and $X_n \to \infty$ when $x_S^2 \to 4l^2$.

The metric in these coordinates is conformally flat:

$$ds^2 = \Omega_S^2 \eta_{\mu\nu} dx_S^\mu dx_S^\nu \tag{2.131}$$

Please remark that when $x^2 = 0$ then $X_n = l$

We could have done projection from the North pole (for that we need than $x^2 \neq 0$; that is $X_n \neq l$): Uniqueness of the definition of X_n needs

$$2\Omega_N + 1 = 2\Omega_S - 1 \tag{2.132}$$

and uniqueness of the definition of X^μ

$$x_N^\mu = \frac{\Omega_S}{\Omega_N} x_S^\mu = \mp \frac{4l^2}{x_S^2} x_S^\mu \tag{2.133}$$

The antipodal $\mathbb{Z}_2$ map $X^A \to -X^A$ is equivalent to a change of the reference pole in stereographic coordinates

$$x_N^\mu \leftrightarrow x_S^\mu \tag{2.134}$$

$\therefore$ Only functions on the sphere which are invariant under the exchange of north and south pole stereographic coordinates are well defined on the projective plane, $\mathbb{RP}_n$.

- Working out the derivatives

$$\begin{aligned} \frac{\partial x_S^\mu}{\partial X^\rho} &= \frac{1}{\Omega_S} \delta_\rho^\mu \\ \frac{x_S^\mu}{\partial X_n} &= -\frac{x_S^\mu}{2l\Omega_S} \\ \frac{\partial}{\partial X_n} &= -\frac{1}{2l\Omega_S} x_S . \partial_S \end{aligned} \tag{2.135}$$

The Lorentz generators read

$$\begin{aligned} L_{\mu\nu} &\equiv \epsilon_\mu X^\mu \frac{\partial}{\partial X^\nu} - \epsilon_\nu X^\nu \frac{\partial}{\partial X^\mu} = \epsilon_\mu x_S^\mu \partial_\nu^S - \epsilon_\nu x_S^\nu \partial_\mu^S \\ L_{n\mu} &= -\frac{l}{\Omega_S} \partial_\mu^S - \frac{x_S^\sigma}{2l} L_{\mu\sigma} \end{aligned} \tag{2.136}$$

But this last expression appears to be valid only when the plus sign is chosen in the second member of the defining equation (2.123).

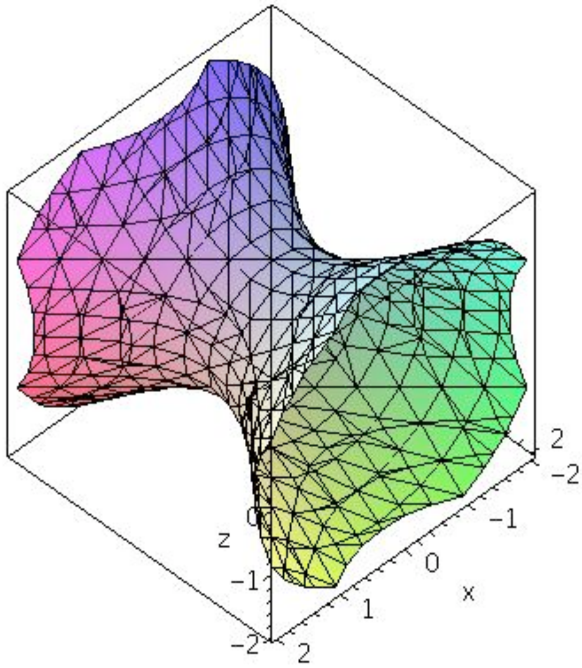

Figure 1: A pictorial representation of Anti de Sitter ($X_0^2 + X_1^2 = l^2 + \vec{X}^2$ in $\mathbb{R}^n$).

2.5.3 Analytic continuation from the sphere or the projective plane.

The metric on S^{n-1} is

$$ds^2_{n-1} = d\theta^2_{n-1} + sin^2\,\theta_{n-1} d\theta^2_{n-2} + \ldots + sin^2\,\theta_{n-1}\, sin^2\,\theta_{n-2} \ldots sin^2\,\theta_2 d\theta_1^2 \qquad (2.137)$$

This corresponds to the surface

$$r = 1 \qquad (2.138)$$

expressed in polar coordinates in $\mathbb{R}^n$

$$\begin{aligned} X_n &= r cos\,\theta_{n-1} \\ X_{n-1} &= r sin\,\theta_{n-1}\, cos\,\theta_{n-2} \\ &\ldots \\ X_2 &= r sin\,\theta_{n-1}\, sin\,\theta_{n-2} \ldots cos\,\theta_1 \\ X_1 &= r sin\,\theta_{n-1}\, sin\,\theta_{n-2} \ldots sin\,\theta_1 \end{aligned} \qquad (2.139)$$

The range of the different angles is

$$\begin{aligned} 0 &\leq \theta_{n-1} \leq \pi \\ 0 &\leq \theta_{n-2} \leq \pi \\ &\ldots \\ 0 &\leq \theta_1 \leq 2\pi \end{aligned} \qquad (2.140)$$

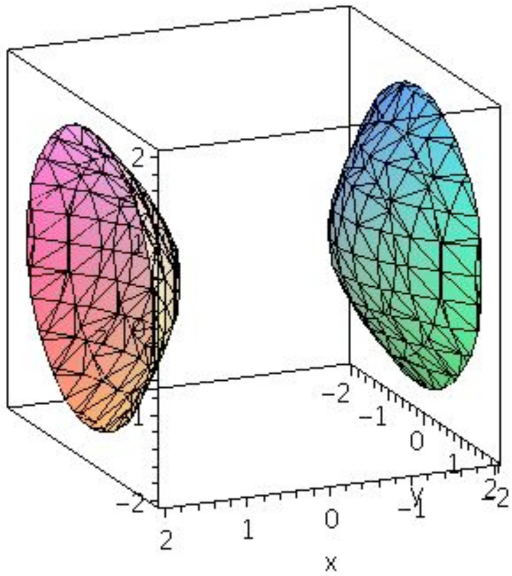

Figure 2: A pictorial representation of Euclidean Anti de Sitter (or Euclidean de Sitter) $(X_0^2 - X_1^2 = l^2 + \vec{X}^2 \text{ in } \mathbb{R}^n)$.

(This coincides with [69]). The antipodal mapping is given by:

$$\begin{aligned}
&\theta_{n-1} \to \pi - \theta_{n-1} \\
&\theta_{n-2} \to \pi - \theta_{n-2} \\
&\ldots \\
&\theta_1 \to \pi + \theta_1
\end{aligned} \tag{2.141}$$

This restricts in fact the range of the angular variables in the projective space to half its natural range

$$\begin{aligned}
&0 \leq \theta_{n-1} \leq \pi/2 \\
&0 \leq \theta_{n-2} \leq \pi/2 \\
&\ldots \\
&0 \leq \theta_1 \leq \pi
\end{aligned} \tag{2.142}$$

- $\mathbf{S_n \to EAdS_n}$

  We shall continue

$$\begin{aligned}
&X_n \to X_0 \\
&\vec{X} \to i\vec{X}
\end{aligned}$$

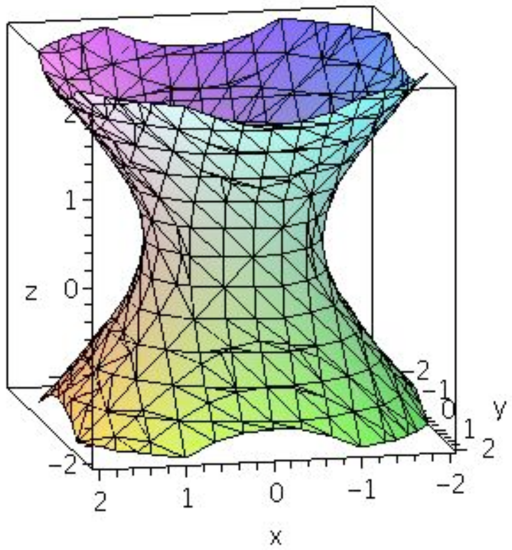

Figure 3: A pictorial representation of de Sitter ($X_0^2 - X_1^2 = -l^2 + \vec{X}^2$) in $\mathbb{R}^n$).

and

$$\theta_{n-1} \to i\theta_{n-1} \tag{2.143}$$

This leads to

$$ds^2 = -l^2 \left(d\theta_{n-1}^2 + sinh^2\,\theta_{n-1} d\Omega_{n-2}^2\right) \tag{2.144}$$

- $\mathbf{S_n \to AdS_n}$

  Here we have to do

$$\begin{aligned} &X_n \to X_0 \\ &X_{n-1} \to X_n \\ &\vec{X} \to i\vec{X} \end{aligned} \tag{2.145}$$

as well as

$$\theta_{n-1} \to i\theta_{n-1} \tag{2.146}$$

yielding

$$ds^2 = l^2 \left(d\theta_{n-1}^2 - sin^2\,\theta_{n-1} \left(d\theta_{n-2}^2 + sinh^2\,\theta_{n-2} d\Omega_{n-3}^2\right)\right) \tag{2.147}$$

- $\mathbf{S_n \to dS_n}$

This is different insofar as it corresponds to $EAdS_n$ with *imaginary radius*. The changes here are

$$\begin{aligned} X_n &\to X_0 \\ \vec{X} &\to i\vec{X} \end{aligned} \tag{2.148}$$

as well as

$$\theta_{n-1} \to i\left(\theta_{n-1} + \pi/2\right) \tag{2.149}$$

leading to

$$ds^2 = l^2\left(d\theta_{n-1}^2 - cosh^2\,\theta_{n-1} d\Omega_{n-2}^2\right) \tag{2.150}$$

2.5.4 Poincaré

A generalization of Poincaré's metric for the half-plane can easily be obtained by introducing the horospheric coordinates [5]. It will always be assumed that $\epsilon_0 = +1$, that is that X^0 is a time, and also that $\epsilon_n = -1$, that is X^n is a space, in our conventions. Otherwise (like in the case of the sphere S_n) it it not possible to construct these coordinates.

$$\begin{aligned} \frac{l}{z} &\equiv X^- \\ y^i &\equiv zX^i \end{aligned} \tag{2.151}$$

where

$$x^- \equiv X^n - X^0 \tag{2.152}$$

$1 \leq i, j \ldots \leq n-1$. The promised generalization of the Poincaré metric is:

$$ds^2 = \frac{\sum_1^{n-1} \epsilon_i dy_i^2 \mp l^2 dz^2}{z^2} \tag{2.153}$$

where the signs are correlated with the ones defined in (2.67), and the surfaces $z = const$ are sometimes called *horospheres*. This form of the metric is *conformally flat* in a manifest way.

The curvature scalar is given by:

$$R = \pm\frac{n(n-1)}{l^2} \tag{2.154}$$

For any constant curvature space,

$$\begin{aligned} R_{\mu\nu} &= \frac{R}{n} g_{\mu\nu} \\ R_{\mu\nu\rho\sigma} &= \frac{R}{n(n-1)}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\rho}g_{\nu\sigma}\right) \end{aligned} \tag{2.155}$$

In our case this yields

$$R_{\mu\nu} = \pm \frac{n-1}{l^2} g_{\mu\nu}$$
$$R_{\mu\nu\rho\sigma} = \pm \frac{1}{l^2} \left(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\rho} g_{\nu\sigma} \right) \tag{2.156}$$

Please note that the curvature only depends on the sign on the second member, and not on the signs ϵ_A themselves.

It is clear, on the other hand, that the isometry group of the corresponding manifold is one of the real forms of the complex algebra $SO(n+1)$. The Killing vector fields are explicitly given (no sum in the definition) by

$$L_{AB} \equiv \epsilon_A x^A \partial_B - \epsilon_B x^B \partial_A \equiv x_A \partial_B - x_B \partial_A \tag{2.157}$$

The square of the corresponding Killing vector is

$$L^2 = \epsilon_B x_A^2 + \epsilon_A x_B^2 \tag{2.158}$$

To be specific, when the metric is given by:

$$ds^2 = \frac{\sum^{n-1} \delta_{ij} dy^i dy^j \mp l^2 dz^2}{z^2} \tag{2.159}$$

i.,e., $C^{\mp}_{1^n,-1}$, then the isometry group is $SO(n,1)$.

- This is the case for what could be called **euclidean de Sitter**, $EdS_n = H_0^n \equiv C^{-}_{1^n,-1}$, which in our conventions has got all coordinates timelike, and *negative* [5] curvature. This is the version of Lobatchevsky upper half plane used by Witten [?] to analyze the AdS/CFT correspondence. Witten refers to ot as ”euclidean AdS”.

  The metric of EdS_n in Poincar’e coordinates reads:

$$ds^2_{EdS_n} = \frac{\sum^{n-1} \delta_{ij} dy^i dy^j + l^2 dz^2}{z^2} \tag{2.161}$$

- The related situation where

$$ds^2 = \frac{-\sum^{n-1} \delta_{ij} dy^i dy^j \mp l^2 dz^2}{z^2} \tag{2.162}$$

[5] We use the Landau-Lifshitz Spacelike conventions (LLSC) [50] and we define the Cosmological Constant in such a way that for a space of constant curvature,

$$R_{\mu\nu} = -\frac{2}{d-2} \lambda g_{\mu\nu} \tag{2.160}$$

i.e., $C^{\pm}_{1,-1^n}$ enjoys $SO(1,n)$ as isometry group, and includes **de Sitter space,** dS_n when z is a timelike coordinate, $dS_n = H^n_{n-1} \equiv C^{-}_{1,-1^n}$. Its metric reads

$$ds^2_{dS_n} = \frac{-\sum^{n-1} \delta_{ij} dy^i dy^j + l^2 dz^2}{z^2} \tag{2.163}$$

In our conventions de Sitter has negative curvature, but positive cosmological constant. Globally, dS_n is given by:

$$x_0^2 - x_1^2 - \ldots - x_n^2 = -l^2 \tag{2.164}$$

The square of the Killing vectors M_{0a} (candidates to be timelike) are

$$M^2_{0a} = x_a^2 - x_0^2 = \sum_{b \neq a} x_b^2 - l^2 \tag{2.165}$$

so they are timelike only outside the *horizon* defined as

$$H_{0a} \equiv \sum_{b \neq a} x_b^2 = l^2 \tag{2.166}$$

For example, the horizon corresponding to H_{0n} is

$$\sum y_i^2 = l^2 z^2 \tag{2.167}$$

This means that de Sitter space, dS_n is not globally static.

- What one would want to call **Euclidean anti de Sitter** , $EAdS_n = S^n_n \equiv C^{+}_{1,-1^n}$, has got all its coordinates spacelike, and *positive* curvature. To be specific

$$ds^2_{EAdS_n} = \frac{-\sum^{n-1} \delta_{ij} dy^i dy^j - l^2 dz^2}{z^2} \tag{2.168}$$

Pleate note that the metric is just the one corresponding to EdS_n, with a change of sign. This explains the change of sign in the scalar curvature.

Globally,

$$x_0^2 - x_1^2 - \ldots - x_n^2 = l^2 \tag{2.169}$$

(That is, de Sitter with imaginary radius).

- Finally, when the metric is given by

$$ds^2 = \frac{\sum^{n-1} \eta_{ij} dy^i dy^j \mp l^2 dz^2}{z^2} \tag{2.170}$$

(where as usual, $\eta_{ij} \equiv diag(1, (-1)^{n-2})$), i.e. $C^{\pm}_{1^2,-1^{n-1}}$ then the isometry group is $SO(2, n-1)$. This includes the regular **Anti de Sitter**, $AdS_n = S^n_{n-1} \equiv$

$C^+_{1^2,-1^{n-1}}$ when the z coordinate is spacelike. For us AdS_n has positive curvature and negative cosmological constant.

$$ds^2_{AdS_n} = \frac{\sum^{n-1} \eta_{ij} dy^i dy^j - l^2 dz^2}{z^2} \tag{2.171}$$

Globally, AdS_n is

$$x_0^2 + x_1^2 - x_2^2 - \ldots - x_n^2 = l^2 \tag{2.172}$$

In this case there is a globally defined timelike Killing vector field, namely M_{01} Indeed, $M_{01}^2 = x_0^2 + x_1^2 = l^2 + \sum_{a \neq 1} x_a^2$ is everywhere positive. This means that anti de Sitter space, AdS_n is globally static, as opposed to de Sitter.

2.5.5 Conformal structure

- $\mathbf{dS_n}$

  The four-dimensional de Sitter space can be globally coordinatized by

$$\begin{aligned} X^0 &= l\, sinh\, \tau \\ X^i &= l\, n^i\, cosh\, \tau \;\; (i = 1 \ldots n) \end{aligned} \tag{2.173}$$

  where $\sum_{i=1}^{i=n} n_i^2 = 1$ and $-\infty \leq \tau \leq \infty$. This gives

$$ds^2 = l^2 \left(d\tau^2 - cosh^2\, \tau\, d\Omega^2_{n-1}\right) \tag{2.174}$$

  A further change of coordinates, namely $cos\, T = \frac{1}{cosh\, \tau}$ where $-\pi/2 \leq T \leq \pi/2$ yields

$$ds^2 = \frac{l^2}{cos^2\, T} \left(dT^2 - d\Omega^2_{n-1}\right) \tag{2.175}$$

  which is conformal to a piece of $\mathbb{R} \times S^{n-1}$, which is the Einstein static universe, the template used by Hawking and Ellis [47] to study conformal structure. The piece is a slab in the timelike direction, but otherwise including the full three-sphere at each time. The fact that conformal infinity is spacelike means that there are both particle and event horizons.

- $\mathbf{AdS_n}$

  The fact that in this case there are two times suggests:

$$\begin{aligned} X^0 &= l \frac{cos\, \tau}{cos\, \rho} \\ X^4 &= l \frac{sin\, \tau}{cos\, \rho} \\ X^i &= l\, n^i\, tg\, \rho \;\; (i = 1 \ldots n-1) \end{aligned} \tag{2.176}$$

where $\sum_{i=1}^{i=n-1} n_i^2 = 1$ and $-\pi \leq \tau \leq \pi$, $0 \leq \rho \leq \pi/2$. The space is again conformal to a piece of half Einstein' s static universe:

$$ds^2 = \frac{l^2}{cos^2\rho}\left(d\tau^2 - d\rho^2 - sin^2\rho d\Omega_{n-2}^2\right) = \frac{l^2}{cos^2\rho}\left(d\tau^2 - d\Omega_{n-1}^2\right) \tag{2.177}$$

If we want to eliminate the closed timelike lines, one can consider the covering space $-\infty \leq \tau \leq \infty$. The slab of $\mathbb{R} \times S^{n-1}$ to which AdS_n is conformal to includes now the full timelike direction, but only an hemisphere at each particular time. Null and spacelike infinity can be considered as the timelike surfaces $\rho = 0$ and $\rho = \pi/2$. This implies that there are no Cauchy surfaces.

- **EAdS$_\mathbf{n}$**

  We write

$$\begin{aligned} X^\mu &= l n^\mu \, sinh \, \tau \\ X^n &= l cosh \, \tau \end{aligned} \tag{2.178}$$

with $\sum_{\mu=0}^{n-1} \epsilon_\mu n_\mu^2 = 1$, so that the metric reads

$$ds^2 = l^2 \left(d\tau^2 + sinh \, \tau^2 d\Omega_{n-1}^2\right) \tag{2.179}$$

The change of variables

$$e^T = th \, \tau/2 \tag{2.180}$$

yields

$$ds^2 = l^2 \frac{e^{2T}}{1 - e^{2T}} \left(dT^2 + d\Omega_{n-1}^2\right) \tag{2.181}$$

(the other half of the global space would be covered by another copy of the above metric).

In this metric, $X_n \geq X_0$ always, which means that in Poincaré coordinates $z \geq 0$, and $z \to 0$ when $\tau \to \infty$, which is equivalent to $T \to \infty$, and represents the boundary of the space, a S_{n-1} sphere.

2.5.6 The Poincare patch

- **dS$_\mathbf{n}$**

  In this case it is clear that

$$z \leq 0 \tag{2.182}$$

always, and $z \to -\infty$ as $\tau \to \infty$.

- **$\mathbf{AdS_n}$**

  It is clear that the region $z \geq 0$ corresponds to the patch

$$\pi/4 \leq \tau \leq \pi \tag{2.183}$$

and the region $0 \geq z$ to

$$-\pi \leq \tau \leq -3\pi/4 \tag{2.184}$$

The region

$$z = 0 \tag{2.185}$$

is dubbed the *boundary* (of the Poincaré patch) of AdS and corresponds to

$$\rho = \pi/2 \tag{2.186}$$

Finally

$$z = \infty \tag{2.187}$$

is usually called the *horizon* and corresponds to $X^n = X^0$, that is,

$$\tau = \pi/4 \tag{2.188}$$

or else

$$\tau = -3\pi/4 \tag{2.189}$$

(assuming $\rho \neq \pi/2$).

When $\rho = \pi/2 - \epsilon$ and $\tau = \pi/4 \pm \delta$,

$$z = \pm\frac{\sqrt{2}}{2}\frac{\epsilon}{\delta} \tag{2.190}$$

and the limit depends on how the limit point $\epsilon = \delta = 0$ is reached.

The same thing happens when $\rho = \pi/2 - \epsilon$ and $\tau = -3\pi/4 \pm \delta$,

$$z = \mp\frac{\sqrt{2}}{2}\frac{\epsilon}{\delta} \tag{2.191}$$

2.6 The group theoretical approach

2.6.1 Contractions

Consider as given the algebra

$$[M_{AB}, M_{CD}] = i\left(\eta_{BC}M_{AD} - \eta_{AC}M_{BD} - \eta_{BD}M_{AC} + \eta_{AD}M_{BC}\right) \tag{2.192}$$

corresponding to one of the real forms of $SO(n+1)$, say $SO(p, n+1-p)$.

It is possible to contract in a timelike coordinate, say x^0, in the following manner. Let the coordinates different from x^0 be numbered by $a = 1, \ldots, n+1$. The algebra splits as:

$$\begin{aligned}
&[M_{ab}, M_{cd}] = i\left(\eta_{bc}M_{ad} - \eta_{ac}M_{bd} - \eta_{bd}M_{ac} + \eta_{ad}M_{bc}\right)\\
&[M_{a0}, M_{cd}] = i\left(0 - \eta_{ac}M_{0d} - 0 + \eta_{ad}M_{0c}\right)\\
&[M_{a0}, M_{c0}] = i\left(0 - 0 - M_{ac} + 0\right)
\end{aligned} \tag{2.193}$$

The generators are now redefined (and given dimension one)

$$P_a \equiv \frac{M_{a0}}{R} \tag{2.194}$$

In the limit $R \to \infty$, the algebra contracts Inönü-Wigner to $ISO(p-1, n+1-p)$, where P_a play now the rôle of the translations:

$$\begin{aligned}
&[M_{ab}, M_{cd}] = i\left(\eta_{bc}M_{ad} - \eta_{ac}M_{bd} - \eta_{bd}M_{ac} + \eta_{ad}M_{bc}\right)\\
&[P_a, M_{cd}] = i\left(\eta_{ac}P_d - \eta_{ad}P_c\right)\\
&[P_a, P_c] = 0
\end{aligned} \tag{2.195}$$

Given a set of gamma matrices associated to the metric η_{ab}, our previous results imply that $M_{ab} \equiv i\sigma_{ab}$ and $P_a \equiv \gamma_a$ yield a representation of the reduced algebra.

An example of this reduction is the one from four-dimensional AdS, $SO(2,3)$ to the Poincaré group, $ISO(1,3)$.

The reduction along an spacelike direction (say x^n) is completely analogous:

$$\begin{aligned}
&[M_{ab}, M_{cd}] = i\left(\eta_{bc}M_{ad} - \eta_{ac}M_{bd} - \eta_{bd}M_{ac} + \eta_{ad}M_{bc}\right)\\
&[M_{an}, M_{cd}] = i\left(0 - \eta_{ac}M_{nd} - 0 + \eta_{ad}M_{nc}\right)\\
&[M_{an}, M_{cn}] = i\left(0 - 0 + M_{ac} + 0\right)
\end{aligned} \tag{2.196}$$

The generators are now redefined

$$P_a \equiv \frac{M_{an}}{R} \tag{2.197}$$

resulting in

$$\begin{aligned}
&[M_{ab}, M_{cd}] = i\left(\eta_{bc}M_{ad} - \eta_{ac}M_{bd} - \eta_{bd}M_{ac} + \eta_{ad}M_{bc}\right)\\
&[P_a, M_{cd}] = i\left(\eta_{ac}P_d - \eta_{ad}P_c\right)\\
&[P_a, P_c] = 0
\end{aligned} \tag{2.198}$$

In the limit $R \to \infty$, the algebra contracts Inönü-Wigner to $ISO(p, n-p)$, where P_a play again the rôle of the translations:

An example of this reduction is the one from four-dimensional AdS, $SO(1,4)$ to the Poincaré group, $ISO(1,3)$.

2.6.2 Casimirs, laplacians and Green's functions

The Laplacian (d' Alembert) operator in $\mathbb{R}^{n+1}$ is

$$\Delta_{n+1} \equiv \eta^{AB}\partial_A\partial_B \tag{2.199}$$

[6]

[6] We include here some curious and potentially useful facts. The flat metric in $\mathbb{R}^{n+1}$ can be written in horospheric coordinates, provided the radius l is also kept as another coordinate:

$$ds^2 = -dx^+dx^- + \sum \epsilon_i dx_i^2 = \sum \epsilon_i \frac{dy_i^2}{z^2} \mp \frac{l^2}{z^2}dz^2 - \left(\pm 1 - \sum \epsilon_i \frac{y_i^2}{l^2 z^2}\right) dl^2 - \sum 2\frac{\epsilon_i y_i}{l z^2} dy_i dl \tag{2.200}$$

It can be easily computed that

$$\sqrt{|g|} = \frac{l}{z^n} \tag{2.201}$$

This form of the metric is not convenient however, because the off-diagonal terms cause eventually problems when reducing it to the surface $l = const$. If new coordinates are introduced:

$$\xi_i \equiv \log \frac{y_i}{l} \tag{2.202}$$

the metric reads

$$ds^2 = \sum \epsilon_i \frac{l^2}{z^2} e^{2\xi_i} d\xi_i^2 \mp \frac{l^2}{z^2} dz^2 \mp dl^2 \tag{2.203}$$

The laplacian on scalars is then

$$\Delta_{n+1} = \frac{1}{\sqrt{|g|}} \left(\partial_\mu g^{\mu\nu} \sqrt{|g|} \partial_\nu\right) \tag{2.204}$$

yielding

$$\Delta_{n+1} = \sum \epsilon_i \frac{z^2}{l^2} e^{-2\xi_i} \left(\partial_i^2 - \partial_i\right) \mp \frac{2-n}{l^2} z\partial_z \mp \frac{z^2}{l^2}\partial_z^2 \mp \frac{n}{l}\partial_l \mp \partial_l^2 \tag{2.205}$$

which reduces in the old coordinates y_i to

$$\Delta_{n+1} = \sum \epsilon_i z^2 \frac{\partial^2}{\partial y_i^2} \mp \frac{2-n}{l^2} z\partial_z \mp \frac{z^2}{l^2}\partial_z^2 \mp \frac{n}{l}\partial_l \mp \partial_l^2 \tag{2.206}$$

In the former coordinates the inverse metric reads:

$$\begin{aligned} g^{ll} &= \mp 1 \\ g^{li} &= \mp \frac{y_i}{l} \\ g^{zz} &= \mp \frac{z^2}{l^2} \\ g^{ij} &= \epsilon_i z^2 \delta_{ij} \mp \frac{y_i y_j}{l^2} \end{aligned} \tag{2.207}$$

yielding

$$\begin{aligned} &\Delta_{n+1}^{nd} = \mp \frac{n}{l}\frac{\partial}{\partial l} \mp \frac{\partial^2}{\partial l^2} \mp \sum \frac{2y_i}{l}\frac{\partial^2}{\partial l \partial y_i} \mp \frac{2-n}{l^2} z \frac{\partial}{\partial z} \mp \frac{z^2}{l^2}\frac{\partial^2}{\partial z^2} + \\ &\sum \epsilon_i z^2 \frac{\partial^2}{\partial y_i^2} - \frac{n}{l^2}\sum y_j \frac{\partial}{\partial y_j} - \sum \frac{y_i y_j}{l^2}\frac{\partial^2}{\partial y_i \partial y_j} \end{aligned} \tag{2.208}$$

The same operator in $C^{\pm}_{\epsilon_M}$ is

On the other hand, the quadratic Casimir of the group:

$$C_2 \equiv -\frac{1}{2} M^{AB} M_{AB} \tag{2.211}$$

can be written in the form

$$C_2 = -x^A \partial^B \left(x_A \partial_B - x_B \partial_A \right) = (-1+n) x^A \partial_A + \left(-x^2 \eta^{AB} + x^A x^B \right) \partial_A \partial_B =$$
$$n x^A \partial_A + \left(\mp l^2 \eta^{AB} - x^A x^B \right) \partial_A \partial_B = x^A \partial_A \left(x^B \partial_B + n - 1 \right) \mp l^2 \Delta_{n+1} \tag{2.212}$$

The situation can be clarified (following [91]) by normalizing to the space with unit radius, using

$$x^A = l y^A \tag{2.213}$$

where it is understood that

$$\eta_{AB} y^A y^B = \pm 1 \tag{2.214}$$

This gives

$$dx^A = dl y^A + l dy^A \tag{2.215}$$

and

$$ds^2 = \pm dl^2 + l^2 g_{\mu\nu} dx^\mu dx^\nu \tag{2.216}$$

where the metric in the unit radius space is

$$g_{\mu\nu} \equiv \eta_{AB} \partial_\mu y^A \partial_\nu y^B \tag{2.217}$$

Using the formula for the laplacian on scalars,

$$\Delta_{n+1} = \frac{1}{\sqrt{|g|}} \left(\partial_A g^{AB} \sqrt{|g|} \partial_B \right) \tag{2.218}$$

and the fact that $\sqrt{|g|}_{n+1} = l^n \sqrt{|g|}_n$ it is found that

$$\Delta_{n+1} = \pm \frac{n}{l} \partial_l \pm \partial_l^2 + \frac{1}{l^2} \Delta_n \tag{2.219}$$

It is a fact that

$$l \partial_l = x^A \partial_A \tag{2.220}$$

yielding

$$C_2 = -l^2 \partial_l^2 - n l \partial_l \pm l^2 \Delta_{n+1} = \pm \Delta_n \tag{2.221}$$

$$\Delta_n \equiv -\frac{2-n}{l^2} z \frac{\partial}{\partial z} - \frac{z^2}{l^2} \frac{\partial^2}{\partial z^2} + z^2 \sum_i \epsilon_i \frac{\partial^2}{\partial y_i^2} \tag{2.209}$$

There is a simple relationship:

$$\Delta_{n+1} = \mp \frac{n}{l} \partial_l \mp \partial_l^2 + \Delta_n \tag{2.210}$$

2.6.3 More on the four dimensional Lorentz group

Let us recall the full Poincaré $ISO(1,3)$ algebra. It has in it three rotations, $J_i \equiv \frac{1}{2}\epsilon_{ijk}M_{jk}$, three boosts, $K_i \equiv M_{i0}$ as well as four translations, $P_\mu \equiv (H, P_i)$, and the algebra reads

$$\begin{aligned}
&[J_i, J_j] = i\epsilon_{ijk}J_k \\
&[K_i, K_j] = -i\epsilon_{ijk}J_k \\
&[J_i, K_j] = i\epsilon_{ijk}K_k \\
&[K_i, P_j] = iH\delta_{ij} \\
&[K_i, H] = iP_i
\end{aligned} \tag{2.222}$$

The Lorentz subalgebra $SO(1,3)$ is isomorphic to the algebra of $SU(2) \times SU(2)$, since if we define $\sigma_i^{\pm} \equiv J_i \pm iK_i$

$$[\sigma_i^{\pm}, \sigma_j^{\pm}] = i\epsilon_{ijk}\sigma_k^{\pm} \tag{2.223}$$

$$[\sigma_i^{+}, \sigma_j^{-}] = 0 \tag{2.224}$$

Accordingly the finite dimensional, non unitary irreps are labeled by a couple of half integers, (j_1, j_2). Unitary representations of non-compact groups are always infinite dimensional ([94]).

2.6.4 $SL\,(2, \mathbb{C})$

$$\widetilde{x} \equiv x^0 + \vec{x}\vec{\sigma} = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} \tag{2.225}$$

Lorentz transformations:

$$\widetilde{x}' \equiv M\widetilde{x}M^+\,, M \in SL\,(2, \mathbb{C}) \tag{2.226}$$

$$\begin{aligned}
x^0 &= \frac{1}{2}tr\,\widetilde{x} \\
x^i &= \frac{1}{2}tr\,\widetilde{x}\sigma^i
\end{aligned} \tag{2.227}$$

This means that $M \in SU(2)$ corresponds to an space rotation, because then $MM^+ = 1$,so that x^0 is unaffected. It is plain that the two-dimensional matrices corresponding to finite rotations are:

$$U\,(R\,(J_1)) = \begin{pmatrix} cos\,\frac{\alpha}{2} & isin\,\frac{\alpha}{2} \\ isin\frac{\alpha}{2} & cos\,\frac{\alpha}{2} \end{pmatrix} \tag{2.228}$$

This matrix is determined by the condition

$$tr\,\widetilde{x}\sigma^1 = tr\,U\widetilde{x}U^+\sigma^1 \tag{2.229}$$

$$U\left(R\left(J_2\right)\right)=\begin{pmatrix} cos\,\frac{\alpha}{2} & sin\,\frac{\alpha}{2} \\ -sin\frac{\alpha}{2} & cos\,\frac{\alpha}{2} \end{pmatrix} \tag{2.230}$$

$$U\left(R\left(J_3\right)\right)=\begin{pmatrix} e^{i\frac{\alpha}{2}} & 0 \\ 0 & e^{-i\frac{\alpha}{2}} \end{pmatrix} \tag{2.231}$$

Boosts along the first axis correspond to

$$\begin{aligned} tr\,\widetilde{x}\sigma^2 &= tr\,M\widetilde{x}M^+\sigma^2 \\ tr\,\widetilde{x}\sigma^3 &= tr\,M\widetilde{x}M^+\sigma^3 \end{aligned} \tag{2.232}$$

$$M\left(B\left(K_1\right)\right)=-\begin{pmatrix} \sqrt{\frac{1+\gamma}{2}} & \sqrt{\frac{\gamma-1}{2}} \\ \sqrt{\frac{\gamma-1}{2}} & \sqrt{\frac{1+\gamma}{2}} \end{pmatrix} \tag{2.233}$$

where, as usual,

$$\gamma\equiv\left(1-v^2\right)^{-1/2} \tag{2.234}$$

$$M\left(B\left(K_2\right)\right)=\begin{pmatrix} \sqrt{\frac{1+\gamma}{2}} & i\sqrt{\frac{\gamma-1}{2}} \\ -i\sqrt{\frac{\gamma-1}{2}} & \sqrt{\frac{1+\gamma}{2}} \end{pmatrix} \tag{2.235}$$

$$M\left(B\left(K_3\right)\right)=-\begin{pmatrix} \left(\frac{1-v}{1+v}\right)^{1/4} & 0 \\ 0 & \left(\frac{1-v}{1+v}\right)^{-1/4} \end{pmatrix} \tag{2.236}$$

This identification conveys a mapping of generators:

$$\begin{aligned} J_i &= \sigma_i \\ K_i &= i\sigma_i \end{aligned} \tag{2.237}$$

In general

$$M(z)\equiv e^{iz_i\sigma_i}=cos\,z+isin\,z\frac{\vec{z}\vec{\sigma}}{z} \tag{2.238}$$

where

$$z\equiv\sqrt{\sum z_i^2} \tag{2.239}$$

For an arbitrary Lorentz transformation, the real and imaginary part can be made explicit:

$$\vec{z}=\vec{j}+i\vec{k} \tag{2.240}$$

It is a fact of life that

$$\begin{aligned} M(z) = cos\,u\,cosh\,v-\left(\frac{cos\,u\,sinh\,v}{u^2+v^2}\left(u\vec{j}+v\vec{k}\right)+\frac{sin\,u\,cosh\,v}{u^2+v^2}\left(u\vec{k}-v\vec{j}\right)\right)\vec{\sigma}- \\ i\left(sin\,u\,sinh\,v+\left(\frac{cos\,u\,sinh\,v}{u^2+v^2}\left(u\vec{k}-v\vec{j}\right)-\frac{sin\,u\,cosh\,v}{u^2+v^2}\left(u\vec{j}+v\vec{k}\right)\right)\vec{\sigma}\right) \end{aligned} \tag{2.241}$$

with

$$u\sqrt{2} \equiv \sqrt{\sqrt{\left(\vec{j}^2 - \vec{k}^2\right)^2 + 4\left(\vec{j}.\vec{k}\right)^2} + \vec{j}^2 - \vec{k}^2}$$

$$v\sqrt{2} \equiv \sqrt{\sqrt{\left(\vec{j}^2 - \vec{k}^2\right)^2 + 4\left(\vec{j}.\vec{k}\right)^2} - \vec{j}^2 + \vec{k}^2} \qquad (2.242)$$

2.6.5 Unitary representations of the four dimensional Poincaré group

- **Massive** When the first Casimir operator

$$P^2 \equiv m^2 \qquad (2.243)$$

does not vanish, $m \neq 0$, there is a Lorentz transformation so that

$$p^\mu = L(p)^\mu{}_\nu\,(mu^\nu) \qquad (2.244)$$

where the timelike unitary vector u is defined by:

$$u^\nu \equiv (1, \vec{0}) \qquad (2.245)$$

This means that all timelike vectors of the same length are related by a Lorentz transformation. The transformation $L(p)$ is not defined in an unique manner. Given one $L(p)$, clearly

$$L(p)W(u) \qquad (2.246)$$

produces the same effect, as long as

$$Wu = u \qquad (2.247)$$

i.e., belongs to the *little group*, or *stabilizer* of the vector u, which we shall denote by S_u. The fact that the little group is nontrivial is the basis of the whole construction by Wigner of the induced representations. In the present case, it is easy to check that the little group is compact, namely the full set of ordinary three-dimensional rotations,

$$S_u = SO(3) \qquad (2.248)$$

In the Hilbert space of states, on which we want the unitary representation to act we shall diagonalize the momentum P^μ as well as the square of the Pauli-Lubansky vector, W^2:

$$P^\mu|p', i\rangle = (p')^\mu|p', i\rangle \qquad (2.249)$$

Labels will be chosen so that the states transform as unitary irreducible representations of the little group:

$$U(W)|k, a\rangle = \sum_b D(W)_{ab}|k, b\rangle \qquad (2.250)$$

The labels can then be chosen as $-j \leq a \equiv j_3 \leq j$, with $2j \in \mathbb{Z}$

Following Wigner [**?**], we are going to choose a particular Wigner boost amongst the whole set that maps mu into p, in some canonical way. This transformation is named $L_c(p)$. Furthermore

$$|p, a\rangle \equiv U(L_c(p))|(m, \vec{0}), a\rangle \tag{2.251}$$

Where $U(g)$ is the unitary representative of the group element g, and it is an (as yet unknown) operator acting in the physical Hilbert space (usually a Fock space). The extra labels on the left are by definition the same as on the right.

It is now the case that under an arbitrary Lorentz transformation, L,

$$U(L)|p, a\rangle = U(LL_c(p))|(m, \vec{0}), a\rangle \tag{2.252}$$

Clearly,

$$LL_c(p) = L_c(q)W(L, p) \tag{2.253}$$

where the vector q is such that

$$q \equiv Lp \tag{2.254}$$

It follows that

$$U(L)|p, a\rangle = U\left(L_c(q)\right) U\left(W(L, p)\right)|(m, \vec{0}), a\rangle = \sum_b U(L_c(q)) D_{ab}\left(W(L, p)\right)|(m, \vec{0}), b\rangle = \sum_b D_{ab}\left(W(L, p)\right)|Lp, b\rangle \tag{2.255}$$

- **Massless**

The fiducial null momentum will be chosen as

$$k = (1, 0, 0, 1) \tag{2.256}$$

Let us begin by considering an abelian subalgebra isomorphic to the two-dimensional translation algebra:

$$T(2) = (T_1 \equiv K_1 - J_2, T_2 \equiv K_2 + J_1) \tag{2.257}$$

The finite translations will be denoted by $\Delta_{1,2}(a_{1,2})$. A long time ago, Wigner showed that by adding J_3 to the above subalgebra, one obtained the algebra of the little group of a massless particle, which was isomorphic to the algebra of the euclidean two-dimensional group, $E(2) = ISO(2)$:

$$S_k = E(2) \equiv (T_1, T_2, J_3) \tag{2.258}$$

the semidirect sum of the translations T_A, $(A = 1, 2)$ and the rotations $R(\theta)$ generated by $J_3 \in SO(2)$. A simple calculation shows that

$$\begin{aligned} R(\theta)\Delta_1(a_1)R^{-1}(\theta) &= \Delta_1\left(a_1 cos\,\theta + a_2 sin\,\theta\right) \\ R(\theta)\Delta_2(a_2)R^{-1}(\theta) &= \Delta_2\left(-a_1 sin\,\theta + a_2 cos\,\theta\right) \end{aligned} \tag{2.259}$$

This means that given eigenstates of the translations, i.e.

$$T_{1,2}|\psi\rangle = a_{1,2}|\psi\rangle \tag{2.260}$$

the state

$$U\left(R(\theta)\right)|\psi\rangle \tag{2.261}$$

is another eigenstate with eigenvalues $(a_1 cos\,\theta + a_2 sin\,\theta, -a_1 sin\,\theta + a_2 cos\,\theta)$. The only way to avoid this degeneracy (unobserved in the physical world) is to postulate that

$$T_{1,2}|\psi\rangle = 0 \tag{2.262}$$

Physical states are then characterized by the eigenvalue of J_3, which is the *helicity* (the projection of the angular momentum in the direction of the motion)

$$J_3|\psi\rangle = h|\psi\rangle \tag{2.263}$$

The helicity has got to satisfy

$$e^{4\pi h} = 1 \tag{2.264}$$

States with oposite helicity are related by parity.

2.6.6 dS(4)

Another way of represent the four-dimensional de Sitter group is as follows. Let us split the generators into four *boosts* and six *rotations*:

$$\begin{aligned} &M_{0I} \equiv K_I \\ &M_{IJ} \end{aligned} \tag{2.265}$$

where the five-dimensional spatial indices run from $I, J, \ldots = 1, 2, 3, 4$. Next define, for the four-dimensional spatial indices $i, j, \ldots = 1, 2, 3$

$$\begin{aligned} M_{ij} &= \epsilon_{ijk} L_k \\ M_{4i} &= N_i \end{aligned} \tag{2.266}$$

The commutators read

$$\begin{aligned} &[K_4, K_i] = -iN_i \\ &[K_i, K_j] = -i\epsilon_{ijk} L_k \\ &[K_4, L_i] = 0 \\ &[K_4, N_i] = -iP_i \\ &[K_i, L_j] = i\epsilon_{ijk} K_k \\ &[K_i, N_j] = i\delta_{ij} K_4 \end{aligned} \tag{2.267}$$

as well as

$$
\begin{aligned}
[L_i, L_j] &= i\epsilon_{ijk} L_k \\
[N_i, N_j] &= i\epsilon_{ijk} L_k \\
[L_i, N_j] &= i\epsilon_{ijk} N_k
\end{aligned}
\tag{2.268}
$$

so that if we define

$$
J_i^{\pm} \equiv L_i \pm N_i \tag{2.269}
$$

there are two commuting $SO(3)$ algebras:

$$
\begin{aligned}
[J_i^+, J_j^+] &= i\epsilon_{ijk} J_k^+ \\
[J_i^-, J_j^-] &= i\epsilon_{ijk} J_k^- \\
[J_i^+, J_j^-] &= 0
\end{aligned}
\tag{2.270}
$$

(Of course this is a simple consequence of the isomorphism $SO(4) \sim SO(3) \times SO(3)$, and the fact that $SO(4)$ is a subgroup of the de Sitter group).

It is plain to verify that the little group of a null vector is now the euclidean three-dimensional group $E(3)$, generated by the six elements

$$
\begin{aligned}
[L_i, L_j] &= i\epsilon_{ijk} L_k \\
[T_i, T_j] &= 0 \\
[L_i, T_j] &= i\epsilon_{ijk} T_k
\end{aligned}
\tag{2.271}
$$

where

$$
T_i \equiv K_i + N_i \tag{2.272}
$$

and the group that takes a null vector into a multiple of itself is none other than $SIM(3)$, where the Lie algebra is augmented with the new generator K_4, and

$$
\begin{aligned}
[K_4, L_i] &= 0 \\
[K_4, T_i] &= -iT_i
\end{aligned}
\tag{2.273}
$$

2.6.7 AdS(4)

Let us call x^0 and x^4 the two times, so that the metric $\eta = diag\,(1, -1, -1, -1, 1)$. We shall define the *hamiltonian* as the hermitian operator

$$
H \equiv M_{40} \tag{2.274}
$$

because it reduces to the minkowskian hamiltonian in an Inönü-Wigner contraction. We shall also consider the six ladder operators

$$
M_i^{\pm} \equiv M_{0i} \pm iM_{4i} \tag{2.275}
$$

that obey $\left(M_i^+\right)^+ = M_i^-$ as well as

$$[H, M_i^\pm] = \pm M_i^\pm$$
$$[M_i^\pm, M_j^\pm] = 0$$
$$[M_i^+, M_j^-] = 2\left(H\delta_{ij} - iM_{ij}\right) \tag{2.276}$$

It is obvious that they raise and lower the energy of the states.

Together with M_{ij}, that generate $SO(3)$, H (that generates $SO(2)$) constitute the maximal compact subgroup, $SO(3) \times SO(2)$.

On the other hand, the Casimir reads:

$$C_2 = -\frac{1}{2}M^{AB}M_{AB} = -H^2 - J^2 + \frac{1}{2}\sum_i \{M_i^+, M_i^-\} = -H\left(H - (n-1)\right) - J^2 + \sum_i M_i^+ M_i^- \tag{2.277}$$

where

$$J^2 \equiv \frac{1}{2}M^{ij}M_{ij} \tag{2.278}$$

Let us asume that there is a lowest weight state, in the representation, that is

$$H|E_0, s\rangle = E_0|E_0, s\rangle$$
$$J^2|E_0, s\rangle = s(s+1)|E_0, s\rangle$$
$$M_i^-|E_0, s\rangle = 0 \tag{2.279}$$

Then, on this representation and in four dimensions the Casimir reads (it can be computed on the lowest weight state)

$$C_2 = -E_0\left(E_0 - 3\right) - s(s+1) \tag{2.280}$$

This value is constant on all states of a given representation.

Several bounds can be easily extracted, following [91]

- When $s \geq 1$ there is in general a state with $E = E_0 + 1$ but $j = s - 1$. Then

  $$C_2 = \langle E_0 + 1, s - 1|C_2|E_0 + 1 s - 1\rangle =$$
  $$-(E_0 + 1)(E_0 - 2) - s(s-1) + \sum |M_i^-|E_0 + 1, s - 1\rangle|^2 = -E_0(E_0 - 3) - s(s+1)$$

  This implies that

  $$E_0 \geq s + 1 \tag{2.281}$$

  In the limiting case, $|M_i^-|E_0{+}1, s{-}1\rangle = 0$, so that $|E_0{+}1, s{-}1\rangle$ is itself a ground state, which decouples along with its descendants. This is interpreted as due to a gauge symmetry, corresponding to a *massless multiplet*. The corresponding casimir is

  $$C_2 = -2\left(s^2 - 1\right) \tag{2.282}$$

- Let us now consider a state with $j = s$ and some unknown energy E. The casimir reads

$$-C_2 = E_0(E_0-3)+s(s-1) = E(E-3)+s(s-1)-\sum_i |M_i^-|E,s\rangle|^2 \quad (2.283)$$

- For $s = 0$ the first excited state with $s = 0$ has got $E = E_0 + 2$, because $0 \otimes 1 = 1$ and $1 \otimes 1 = 2 \oplus 0$, yielding $4E_0 - 2 \geq 0$.

- The limiting case $E_0 = 1/2$ is the famous Dirac' s *singleton*, with only one state for a given value of the spin, and casimir $-C_2 = -5/4$

- For $s = 1/2$, the first excited state with $s = 1/2$ has $E = E_0 + 1$, because $1/2 \otimes 1 = 3/2 \oplus 1/2$, yielding $E_0 - 1 \geq 0$.

- The limiting case, $E_0 = 1$ is again a singleton, also with $-C_2 = -5/4$

2.6.8 Oscillators

This useful technique yields all unitary irreducible representations in a simple manner. The only drawback is that it cannot be worked out for arbitrary dimension. Each case has its own specific characteristics. We shall do it here for the four dimensional case.

We shall assume , following [91] a certain number,p, of mutually commuting pairs of bosonic creation and annihilation operators, transforming as doublets under the compact subgroup $SO(2) \times SO(3) \subset SO(3,2)$, $(a_i(r), b^i(r))$, $r = 1 \ldots p$, and i is the doublet index for $SU(2) \sim SO(3)$. There is another annihilation operator c_i when we need an odd number os oscillators. The total number of oscillators will then be n=2p or n=2p+1. In this paragraph the number of spacetime dimensions is four, so that no confusion should arise. We define $a^i \equiv a_i^+$, $b^i \equiv b_i^+$, $c^i \equiv c_i^+$, and a dot product as $a^i.a_j \equiv \sum_{r=1}^p a^i(r)a_j(r)$. The basic commutation relations are:

$$\begin{aligned}
&[a_i(r), a^j(s)] = \delta_i^j \delta_{rs} \\
&[b_i(r), b^j(s)] = \delta_i^j \delta_{rs} \\
&[c_i, c^j] = \delta_i^j
\end{aligned} \quad (2.284)$$

Then the four operators

$$\begin{aligned}
&U_j^i \equiv a^i.a_j + b_j.b^i + \frac{1}{2}\left(c^i c_j + c_j c^i\right) = a^i.a_j + b_j.b^i + c^i c_j + \frac{1}{2}\delta_j^i = \\
&a^i.a_j + b_j.b^i + c_j c^i - \frac{1}{2}\delta_j^i = \\
&a^i.a_j + b^i.b_j + c^i c_j + \delta_j^i
\end{aligned} \quad (2.285)$$

obey $\left(U_j^i\right)^+ = U_i^j$ as well as

$$[U_j^i, U_l^k] = \delta_j^k U_l^i - \delta_l^i U_j^k \quad (2.286)$$

and

$$Q \equiv \frac{1}{2}U_i^i = \frac{1}{2}(N+n) \tag{2.287}$$

as well as

$$\begin{aligned}
[Q, U_l^k] &= 0 \\
[Q, a^i] &= \frac{1}{2}a^i \\
[Q, a_i] &= -\frac{1}{2}a_i
\end{aligned} \tag{2.288}$$

This means that they are generators of $SU(2)$, (which we would like to identify with M_{ij}). Here N is the total oscillator level corresponding to the a oscillators, the b oscillators, and, in its case, to the c oscillators. (We will eventually identify Q with H).

To be specific,

$$\begin{aligned}
T_1 &\equiv \frac{1}{2}\left(U_2^1 + U_1^2\right) = \frac{1}{2} tr\, \sigma_1 U \\
T_2 &\equiv -\frac{i}{2}\left(U_2^1 - U_1^2\right) = \frac{1}{2} tr\, \sigma_2 U \\
T_3 &\equiv \frac{1}{2}\left(U_1^1 - U_2^2\right) = \frac{1}{2} tr\, \sigma_3 U
\end{aligned} \tag{2.289}$$

obey the $SU(2)$ algebra

$$[T_i, T_j] = i\epsilon_{ijk} T_k \tag{2.290}$$

and, besides,

$$[Q, T_i] = 0 \tag{2.291}$$

Le us now construct the buiding blocks for $M_i^{\pm}$:

$$S^{ij} = (S_{ij})^+ = a^i.b^j + a^j.b^i + c^i c^j \tag{2.292}$$

It is a fact that

$$\begin{aligned}
[Q, S^{ij}] &= S^{ij} \\
[Q, S_{ij}] &= -S_{ij} \\
[S^{ij}, S^{kl}] &= [S_{ij}, S_{kl}] = 0 \\
[S^{ij}, S_{kl}] &= \delta_k^i U_l^j + \delta_l^i U_k^j + \delta_k^j U_l^i + \delta_l^j U_k^i
\end{aligned} \tag{2.293}$$

We can then identify

$$\begin{aligned}
M_1^+ &\equiv \frac{1}{2\sqrt{2}} \sigma_{ab}^3 S^{ab} \\
M_2^+ &\equiv \frac{1}{2\sqrt{2}} i\delta_{ab} S^{ab} \\
M_3^+ &\equiv \frac{1}{2\sqrt{2}} \left(-\sigma_{ab}^1\right) S^{ab}
\end{aligned} \tag{2.294}$$

and correspondingly

$$
\begin{aligned}
M_1^- &\equiv \frac{1}{2\sqrt{2}}\sigma_3^{ab} S_{ab} \\
M_2^- &\equiv \frac{1}{2\sqrt{2}}\left(-i\delta^{ab}\right) S_{ab} \\
M_3^- &\equiv \frac{1}{2\sqrt{2}}\left(-\sigma_1^{ab}\right) S_{ab}
\end{aligned}
\tag{2.295}
$$

which obey

$$
[M_i^+, M_j^-] = 2H\delta_{ij} - 2iM_{ij} \tag{2.296}
$$

In order to construct a representation, we start with a vacuum state such that

$$
S_{ij}|\Omega\rangle = 0 \tag{2.297}
$$

The simplest possibility is just to take the Fock vacuum

$$
|\Omega\rangle = |0\rangle \tag{2.298}
$$

But it is also possible to take

$$
|\Omega\rangle = a^i(r_1)a^j(r_2)b^k(r_3)b^l(r_4)\ldots|0\rangle \tag{2.299}
$$

as long as they do not include a pair $a^i(r)b^j(r) + a^j(r)b^i(r)$, (which are the building blocks out of which the operators S_{ij} are constructed).

Let us examine the simplest cases in detail:

- **n=1. One oscillator only**.

  Then $Q = \frac{c_1^+ c_1 + c_2^+ c_2 + 1}{2} \equiv \frac{N_1+N_2+1}{2}$ and $S_{ij} = c_i^+ c_j^+$. Besides,

$$
\begin{aligned}
T_1 &= \frac{1}{2}\left(c_1^+ c_2 + c_2^+ c_1\right) \\
T_2 &= -\frac{i}{2}\left(c_1^+ c_2 - c_2^+ c_1\right) \\
T_3 &= \frac{1}{2}\left(c_1^+ c_1 - c_2^+ c_2\right)
\end{aligned}
\tag{2.300}
$$

  and the Casimir

$$
\vec{T}^2 = \frac{1}{4}\left(N_1 + N_2\right)\left(N_1 + N_2 + 2\right) \tag{2.301}
$$

  There are two possibilities for the vacuum state.

- $|\Omega\rangle = |0\rangle$ This corresponds to $Q = \frac{1}{2}$ and $s = 0$.

  The states are of the type

$$
c_1^+ c_1^+ |0\rangle \tag{2.302}
$$

  with $\vec{T}^2 = 2$ (j=1) etc. In general there will be an even number of creation operators. This is then the $s = 0$ *singleton* representation.

- $|\Omega\rangle = c_1^+|0\rangle$ This corresponds to $Q=1$ and $\vec{T}^2 = 3/4$, so that s=1/2. This is the $s=1/2$ singleton.

- **n=2 (p=1) Two oscillators**

  In this case,

$$
\begin{aligned}
T_1 &= \frac{1}{2}\left(a_1^+ a_2 + a_2^+ a_1 + b_1^+ b_2 + b_2^+ b_1\right) \\
T_2 &= -\frac{i}{2}\left(a_1^+ a_2 - a_2^+ a_1 + b_1^+ b_2 - b_2^+ b_1\right) \\
T_3 &= \frac{1}{2}\left(a_1^+ a_1 - a_2^+ a_2 + b_1^+ b_1 - b_2^+ b_2\right) \qquad (2.303)
\end{aligned}
$$

  and the Casimir

$$
\begin{aligned}
&\vec{T}^2 = \frac{1}{4}\left(\left(N_1^a + N_2^a\right)\left(N_1^a + N_2^a + 2\right) + \left(N_1^b + N_2^b\right)\left(N_1^b + N_2^b + 2\right)\right) + \\
&\frac{1}{2}\left(N_1^a - N_2^a\right)\left(N_1^b - N_2^b\right) + \frac{1}{2}\left(a_1^+ a_2 b_2^+ b_1 + a_2^+ a_1 b_1^+ b_2\right) \qquad (2.304)
\end{aligned}
$$

  The are, again, several possibillities for the vacuum:

- $|\Omega\rangle = |0\rangle$. This has $Q=1$ and $s=0$, and corresponds to the massless $s=0$ representation .

- $|\Omega\rangle = a_1^+|0\rangle$ This has got $Q=3/2$ and $s=1/2$, and yields the massless $s=1/2$ representation.

- $|\Omega\rangle = \left(a_1^+\right)^m|0\rangle$ This has $Q = m/2+1$ and $s = m/2$.

- $|\Omega\rangle = \left(a_1^+ b_2^+ - a_2^+ b_1^+\right)|0\rangle$ This has $Q=2$, and

$$
\begin{aligned}
&\left(N_1^a + N_1^b\right)|\Omega\rangle = \left(N_2^a + N_2^b\right)|\Omega\rangle = |\Omega\rangle \\
&\left(N_1^a - N_2^a\right)|\Omega\rangle = -|\Omega\rangle \\
&\left(a_1^+ a_2 b_2^+ b_1 + a_2^+ a_1 b_1^+ b_2\right)|\Omega\rangle = -|\Omega\rangle \qquad (2.305)
\end{aligned}
$$

  ao that altogether, $\vec{T}^2 = 0$. This is then the second massless $s=0$ representation.

- **Finally** $n \geq 3$ **More than two oscillators**

  More than two oscillators yield massive representations.

2.7 σ-models

Let us consider the two-dimensional sigma-model with target space $C^{\pm}_{\epsilon_N}$. The action is given by:

$$S = \frac{1}{4\pi\alpha'} \int \sqrt{|\gamma|} d^2\xi \gamma^{ab} \partial_a X^\mu \partial_b X^\nu g_{\mu\nu}(X) \tag{2.306}$$

The beta function of the coupling is

$$\beta_{\mu\nu} = \alpha' R_{\mu\nu} + \frac{(\alpha')^2}{2} R_{\mu\alpha\beta\gamma} R_\nu^{\alpha\beta\gamma} + O((\alpha')^3) \tag{2.307}$$

and using

$$\begin{aligned} R_{\mu\nu} &= \pm \frac{n-1}{l^2} g_{\mu\nu} \\ R_{\mu\nu\rho\sigma} &= \pm \frac{1}{l^2} \left(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\rho} g_{\nu\sigma} \right) \end{aligned} \tag{2.308}$$

they reduce to

$$\beta_{\mu\nu} = \pm \alpha' \frac{n-1}{l^2} g_{\mu\nu} + (\alpha')^2 \frac{n-1}{l^4} g_{\mu\nu} + O((\alpha')^3) \tag{2.309}$$

This means that de Sitter space is one loop asymptotically free whereas anti de Sitter is not.

3. The linear regime. Fierz-Pauli and beyond.

3.1 The unitarity road to consistent lagrangians

Let us start with the well-known analysis which leads eventually to the Fierz-Pauli lagrangian for a free massless spin two particle (cf. [87],[69]). A simple road is as follows: the quadratic part of the lagrangian is the inverse of the propagator, and the propagator is related to the possible polarizations. There are five of those in the massive spin two case, which can be represented as $\epsilon^A_{\mu\nu}$ $A = 1\ldots 5$, with

$$
\begin{aligned}
&\epsilon^A_{\mu\nu} = \epsilon^A_{\nu\mu}\\
&k^\mu \epsilon^A_{\mu\nu} = 0\\
&\eta^{\mu\nu}\epsilon^A_{\mu\nu} = 0
\end{aligned}
\tag{3.1}
$$

We can expand the momentum space [7] propagator in terms of the basic tensors k^μ and the off-shell transverse projection operator $\eta^T_{\mu\nu} \equiv \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}$ as

$$
\begin{aligned}
&D_{\mu\nu\lambda\sigma} \equiv \sum_A \epsilon^A_{\mu\nu}\epsilon^A_{\lambda\sigma} = c_1 \eta^T_{\mu\nu}\eta^T_{\lambda\sigma} + c_2 \eta^T_{\mu\nu} k_\lambda k_\sigma + k_\mu k_\nu \eta^T_{\lambda\sigma}\\
&+c_3(\eta^T_{\mu\lambda}\eta^T_{\nu\sigma} + \eta^T_{\mu\sigma}\eta^T_{\nu\lambda}) + c_4(k_\mu k_\sigma \eta^T_{\nu\lambda} + k_\mu k_\lambda \eta^T_{\nu\sigma} +\\
&k_\nu k_\sigma \eta^T_{\mu\lambda} + k_\nu k_\lambda \eta^T_{\mu\sigma} + c_5 k_\mu k_\nu k_\lambda k_\sigma
\end{aligned}
\tag{3.2}
$$

Imposing off-shell transversality and tracelessness we get uniquely

$$
D_{\mu\nu\lambda\sigma} = c_1 \left(\eta^T_{\mu\nu}\eta^T_{\lambda\sigma} - \frac{3}{2}(\eta^T_{\mu\lambda}\eta^T_{\nu\sigma} + \eta^T_{\mu\sigma}\eta^T_{\nu\lambda})\right)
\tag{3.3}
$$

Acting on conserved currents, we can drop the superscript T.

In order to find the lagrangian, we have to compute the propagator by imposing transversality on shell only. Otherwise there are unwanted degeneracies. This amounts to change the projector in (3.3) $\eta^T_{\mu\nu}$ for a quantity $\eta^{TOS}_{\mu\nu} \equiv \eta_{\mu\nu} - \frac{k_\mu k_\nu}{m^2}$, which behaves as a projector on shell only:

$$
\begin{aligned}
&\eta^{TOS}_{\mu\nu} k^\nu = k_\mu \frac{m^2 - k^2}{m^2}\\
&\eta^{TOS}_{\mu\nu}\eta^{\mu\nu} = 3 + \frac{m^2 - k^2}{m^2}\\
&\eta^{TOS}_{\mu\nu}(\eta^{TOS})^{\nu\rho} = \eta^{TOS\ \rho}_{\mu} + \frac{k^2 - m^2}{m^4} k_\mu k^\rho
\end{aligned}
\tag{3.4}
$$

[7] Both position and momentum space notation will be used for convenience. Although most formulas will be written in arbitrary dimension, most of the polarization reasoning is implicitly four-dimensional.

What remains is

$$D^m_{\mu\nu\lambda\sigma} = c_1 \left(\eta^{TOS}_{\mu\nu} \eta^{TOS}_{\lambda\sigma} - \frac{3}{2} (\eta^{TOS}_{\mu\lambda} \eta^{TOS}_{\nu\sigma} + \eta^{TOS}_{\mu\sigma} \eta^{TOS}_{\nu\lambda}) \right) \tag{3.5}$$

The lagrangian is then found by computing the inverse.

$$(K^m)_{\mu\nu\alpha\beta} (D^m)^{\alpha\beta}{}_{\lambda\delta} = \frac{1}{2} (\eta_{\mu\lambda}\eta_{\nu\delta} + \eta_{\mu\delta}\eta_{\lambda\nu}) \tag{3.6}$$

The conventional normalization corresponds to

$$c_1 = -\frac{4}{3} \frac{1}{k^2 - m^2} \tag{3.7}$$

and yields

$$\begin{aligned} (K^m)_{\mu\nu\rho\sigma} = & \frac{k^2 - m^2}{8} (\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - 2\eta_{\mu\nu}\eta_{\rho\sigma}) \\ -\frac{1}{8} & (k_\mu k_\rho \eta_{\nu\sigma} + k_\nu k_\sigma \eta_{\mu\rho} + k_\mu k_\sigma \eta_{\nu\rho} + k_\nu k_\rho \eta_{\mu\sigma} - 2 k_\mu k_\nu \eta_{\rho\sigma} - 2 k_\rho k_\sigma \eta_{\mu\nu}) \end{aligned} \tag{3.8}$$

which corresponds to the Fierz-Pauli lagrangian

$$L_{FP} = \frac{1}{4} \partial_\mu h^{\nu\rho} \partial^\mu h_{\nu\rho} - \frac{1}{2} \partial_\mu h^{\nu\rho} \partial^\nu h_{\mu\rho} + \frac{1}{2} \partial_\mu h \partial^\sigma h_{\sigma\mu} - \frac{1}{4} \partial_\mu h \partial^\mu h - \frac{m^2}{4} (h_{\alpha\beta} h^{\alpha\beta} - h^2) \tag{3.9}$$

where $h \equiv \eta^{\mu\nu} h_{\mu\nu}$.

It follows that

$$k^\nu K^m_{\mu\nu\rho\sigma} h^{\rho\sigma} = -2m^2 (k^\rho h_{\rho\mu} - k_\mu h) \tag{3.10}$$

so that necessarily,

$$k^2 h = k_\rho k_\sigma h^{\rho\sigma} \tag{3.11}$$

The trace gives:

$$\eta^{\mu\nu} K^m_{\mu\nu\rho\sigma} h^{\rho\sigma} = -2(1-n) m^2 h \tag{3.12}$$

which in turn implies that

$$h = k_\mu k_\nu h^{\mu\nu} = 0 \tag{3.13}$$

and using (7.43),

$$k^\mu h_{\mu\nu} = 0 \tag{3.14}$$

so that the field obeys the Klein-Gordon equation

$$(\Box + m^2) h_{\mu\nu} = 0 \tag{3.15}$$

It can be shown ([?]) that this particular mass term is the only one which is compatible with unitarity.

3.2 The massless limit.

The massless limit is singular. Three polarizations can be written as

$$k_\mu u_\nu + k_\nu u_\mu \tag{3.16}$$

with $k.u = 0$. Namely, in an obvious notation, $(e_{(a)} \equiv \partial_a$, etc)

$$\begin{aligned}
&k \otimes k\\
&k \otimes e_{(1)} + e_{(1)} \otimes k\\
&k \otimes e_{(2)} + e_{(2)} \otimes k
\end{aligned} \tag{3.17}$$

The remaining two are

$$\begin{aligned}
\epsilon_1 &\equiv e_{(1)} \otimes e_{(2)} + e_{(2)} \otimes e_{(1)}\\
\epsilon_2 &\equiv e_{(1)} \otimes e_{(1)} - e_{(2)} \otimes e_{(2)}
\end{aligned} \tag{3.18}$$

and under the little group, they transform into the other three (cf.[**?**]).

This means that exactly the same type of reasoning that gives rise to the abelian gauge invariance yields the unimodular theory of Einstein, which is invariant under area preserving diffs only:

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \tag{3.19}$$

with

$$\partial_\mu \xi^\mu = 0 \tag{3.20}$$

Once we implement this symmetry (with or without the unimodularity condition (3.20)), then there is a gauge in which the massless Fierz-Pauli propagator is defined up to a constant as:

$$D^{GF}_{\mu\nu\rho\sigma} = c_2(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma}) \tag{3.21}$$

And then, it is a simple matter to show that, acting on conserved currents,

$$D^{GF}_{\mu\nu\rho\sigma} = D^m_{\mu\nu\rho\sigma} + \frac{c_1}{2}\eta_{\mu\nu}\eta_{\rho\sigma} \tag{3.22}$$

which means that there is an extra admixture of spin $s = 0$ in the massless case.

The conventional normalization corresponds to

$$c_2 = \frac{4}{k^2} \tag{3.23}$$

and yields

$$K^{GF}_{\mu\nu\rho\sigma} = \frac{k^2}{8}(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma}) \tag{3.24}$$

This corresponds to the massless Fierz-Pauli lagrangian with the harmonic gauge condition

$$L_{GF} = \frac{1}{2}(\partial_\nu h_\mu{}^\nu - \frac{1}{2}\partial_\mu h)^2 \tag{3.25}$$

that is

$$L_0 = \frac{1}{4}(\partial_\mu h_{\alpha\beta})^2 - \frac{1}{8}(\partial_\mu h)^2 \tag{3.26}$$

3.3 Unimodular lagrangians

If we implement the restricted gauge symmetry only, a simpler lagrangian exists:

$$L_u = \frac{1}{4}(\partial_\mu h_{\alpha\beta})^2 - \frac{1}{2}\partial_\mu h_{\alpha\beta}\partial^\alpha h^{\mu\beta} \tag{3.27}$$

although the full Fierz-Pauli lagrangian L_{FP} is obviously still invariant under the restricted symmetry. This is exactly the same thing that would have been gotten by putting $h = 0$ in the Fierz-Pauli lagrangian, that is

$$\begin{aligned}(K_u)_{\mu\nu\rho\sigma} = &\frac{k^2}{8}(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}) + \\ -&\frac{1}{8}(k_\mu k_\rho \eta_{\nu\sigma} + k_\rho k_\nu \eta_{\mu\sigma} + k_\sigma k_\nu \eta_{\mu\rho} + k_\sigma k_\mu \eta_{\nu\rho})\end{aligned} \tag{3.28}$$

Let us now construct a massive unimodular theory. In order to do that, we postulate the most general mass term, say

$$-\frac{m^2}{8}(2h_{\mu\nu}h^{\mu\nu} - rh^2) \tag{3.29}$$

where r is an arbitrary constant (which for the full Fierz-Pauli theory happens to take the value $r = 2$). The posited full kinetic operator is then

$$\begin{aligned}(K_u^m)_{\mu\nu\rho\sigma} = &\frac{k^2 - m^2}{8}(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}) + \\ r&\frac{m^2}{8}\eta_{\mu\nu}\eta_{\rho\sigma} - \frac{1}{8}(k_\mu k_\rho \eta_{\nu\sigma} + k_\rho k_\nu \eta_{\mu\sigma} + k_\sigma k_\nu \eta_{\mu\rho} + k_\sigma k_\mu \eta_{\nu\rho})\end{aligned} \tag{3.30}$$

The corresponding equation of motion is:

$$(K_u^m h)_{\mu\nu} = \frac{k^2 - m^2}{4}h_{\mu\nu} + \frac{rm^2}{8}h\eta_{\mu\nu} - \frac{1}{4}(k_\mu k_\rho h_\nu{}^\rho + k_\nu k_\rho h_\mu{}^\rho) \tag{3.31}$$

Computing again the transverse part of the equation of motion:

$$(K_u^m.h)_{\mu\nu} = \frac{k^2 - m^2}{4}h_{\mu\nu} + r\frac{m^2}{8}h\eta_{\mu\nu} - \frac{1}{4}(k_\mu k^\rho h_{\nu\rho} + k_\nu k^\rho h_{\mu\rho}) \tag{3.32}$$

$$k^\mu k^\nu (K_u^m)_{\mu\nu\rho\sigma}h^{\rho\sigma} = -(k^2 + m^2)k_\rho k_\sigma h^{\rho\sigma} + r\frac{m^2 k^2}{2}h = 0 \tag{3.33}$$

and the trace:

$$\eta^{\mu\nu}(K_u^m)_{\mu\nu\rho\sigma}h^{\rho\sigma} = (k^2 - m^2 + \frac{n}{2}rm^2)h - 2k_\rho k_\sigma h^{\rho\sigma} = 0 \tag{3.34}$$

This two conditions enforce

$$h = k_\rho k_\sigma h^{\rho\sigma} = 0 \tag{3.35}$$

as long as $r > 0$. Even when $r = 0$ they do enforce full transversality, although tracelessness is then only guaranteed off shell

$$(k^2 - m^2)h = 0 \tag{3.36}$$

The conclusion of this analysis is that the unimodular theory becomes massive with a mass term of the Fierz-Pauli type.

3.4 Propagators

Logically, our attention should now turn to a discussion of the unimodular massive propagator. The fact is that, for the minimal model (3.27), supplemented by a mass term such as the one in [3.29], there is no propagator, because this lagrangian is singular. This is perhaps somewhat of a surprise, because there is no known gauge symmetry when the mass is nonvanishing, but it is nevertheless true. Actually, the situation is as follows: there is a particular mode, proportional to

$$(\eta_u^T)_{\rho\sigma} \equiv (k^2 + m^2 - r\frac{m^2}{2})\eta_{\rho\sigma} - (k^2 + m^2 - r\frac{n}{2}m^2)\frac{k_\rho k_\sigma}{k^2} \tag{3.37}$$

such that

$$(K_u^m \eta_u^T)_{\mu\nu} \tag{3.38}$$

is transverse, i.e.

$$(K_u^m \eta_u^T)_{\mu\nu} k^\mu = 0 \tag{3.39}$$

Although this is not a zero mode *sensu stricto*, it is enough to make the lagrangian singular. The situation is somewhat strange. Nevertheless, we already know, because of the argument of the polarizations at the beginning of the present section, that the correct lagrangian for massive spin 2 is the Fierz-Pauli one, (3.9). On the other hand, we know that the model (3.27 3.29) is the minimal one which can be extended to *exactly* the Fierz-Pauli one while keeping only the restricted gauge symmetry in the massless case.

While it would be interesting to further study the minimal theory, we shall therefore confine our attention from now on to the Fierz-Pauli lagrangian.

Let us begin our discussion with the most general Lorentz invariant local lagrangian for a free massless symmetric tensor field $h_{\mu\nu}$,

$$\mathcal{L} = \mathcal{L}^I + \beta\ \mathcal{L}^{II} + a\ \mathcal{L}^{III} + b\ \mathcal{L}^{IV}, \tag{3.40}$$

where we have introduced

$$\mathcal{L}^{I} = \frac{1}{4}\,\partial_\mu h^{\nu\rho}\partial^\mu h_{\nu\rho}, \quad \mathcal{L}^{II} = -\frac{1}{2}\,\partial_\mu h^{\mu\rho}\partial_\nu h^{\nu}_{\rho},$$
$$\mathcal{L}^{III} = \frac{1}{2}\,\partial^\mu h \partial^\rho h_{\mu\rho}, \quad \mathcal{L}^{IV} = -\frac{1}{4}\,\partial_\mu h \partial^\mu h. \tag{3.41}$$

The first term is strictly needed for the propagation of spin two particles, and we give it the conventional normalization. Before proceeding to the dynamical analysis, which will be done in Subsection 2.4, it will be useful to consider the possible symmetries of (3.40) according to the values of β, a and b.

3.5 Intermediate states

Let us consider (cf. for example, [?]) the free energy in the presence of arbitrary conserved sources. This quantity is an exceedingly useful one to consider, because in summarizes in a very simple way the physical content of the theory. We shall assume two spatially disconnected sources: $T_{\alpha\beta} \equiv (T_1)_{\alpha\beta}\delta^{(3)}(\vec{x}-\vec{x}_1) + (T_2)_{\alpha\beta}\delta^{(3)}(\vec{x}-\vec{x}_2)$, with

$$\partial_\alpha (T_1)^{\alpha\beta} = \partial_\alpha (T_2)^{\alpha\beta} = 0 \tag{3.42}$$

Keeping only the term bilinear in the sources, assumed to act for a total time interval $\int dx^0 \equiv T$, one easily gets:

$$W = -\frac{2}{3}T\int d^3k \frac{1}{\vec{k}^2+m^2} e^{i\vec{k}(\vec{x}-\vec{y})} E_{12} \tag{3.43}$$

Starting with the massive Fierz-Pauli theory, the answer stemming from (3.5) is

$$E_{12} = (tr\, T_1 tr\, T_2 - 3tr\, T_1 T_2) \tag{3.44}$$

In the massless case, the Fierz-Pauli interacion energy in the harmonic gauge is proportional instead to

$$E_{12} \equiv \frac{1}{2}\left(2trT_1T_2 - (tr\, T_1)(tr\, T_2)\right) \tag{3.45}$$

Even forgetting about the coefficients, there is a mismatch of $3/2$ in the term $tr\, T_1 T_2$; this is the famous van Dam-Veltman discontinuity ([?]), which indicates that there is some sort of non smoothness in the massless limit.

In full [8] detail:

$$\begin{aligned} E_{12} = \; & \frac{1}{2}T_1^{00}(T_2^{00}+T_2^{11}+T_2^{22}+T_2^{33}) + \frac{1}{2}T_1^{11}(T_2^{00}+T_2^{11}-T_2^{22}-T_2^{33}) + \\ & \frac{1}{2}T_1^{22}(T_2^{00}-T_2^{11}+T_2^{22}-T_2^{33}) + \frac{1}{2}T_1^{33}(T_2^{00}-T_2^{11}-T_2^{22}+T_2^{33}) + \\ & 2\left(T_1^{12}T_2^{12}+T_1^{13}T_2^{13}+T_1^{23}T_2^{23}-T_1^{01}T_2^{01}-T_1^{02}T_2^{02}-T_1^{03}T_2^{03}\right) \end{aligned} \tag{3.48}$$

[8]t The resulting expression can be further simplified using current conservation:

$$\begin{aligned} T^{00} &= \frac{\kappa}{\omega}T^{03} = \frac{\kappa^2}{\omega^2}T^{33} \\ T^{0i} &= \frac{\kappa}{\omega}T^{3i} \end{aligned} \tag{3.46}$$

In order to identify possible off-shell intermediate states in the massless Fierz-Pauli theory, it is useful to transform the expression (3.48) (that is, before using current conservation) into the suggestive form proposed by Dicus and Willenbrock [?].

$$\begin{aligned}E_{12} = \; & \frac{1}{2}(T_1^{11} - T_1^{22})(T_2^{11} - T_2^{22}) + 2T_1^{12}T_2^{12} + \\ & 2\left(T_1^{13}T_2^{13} + T_1^{23}T_2^{23} - T_1^{01}T_2^{01} - T_1^{02}T_2^{02}\right) * \\ & \frac{1}{6}[2(T_1^{00} - T_1^{33}) + T_1^{11} + T_1^{22}][2(T_2^{00} - T_2^{33}) + T_2^{11} + T_2^{22}] \\ & -\frac{1}{6}[-T_1^{00} + T_1^{11} + T_1^{22} + T_1^{33}][-T_2^{00} + T_2^{11} + T_2^{22} + T_2^{33}]\end{aligned} \tag{3.49}$$

This can be easily checked: in order for the coefficient of T_1^{00} in (3.49) to be the same as the one in (3.48) we have to add a term $T_1^{00}T_2^{33}$, and also if we want the coefficient of T_1^{33} in (3.49) to be the same as in (3.48) we have to add another term $T_1^{33}T_2^{00}$. But in order for the coefficients of T_1^{03} to match, we have to add $-2T_1^{03}T_2^{03}$, which exactly cancel owing to the conservation of the energy momentum tensor.

This expansion can be spelled down physically as follows. Let us introduce a real basis of polarizations in the generic case as

$$\begin{aligned}\epsilon_3 &= e_{(0)} \otimes e_{(0)} - e_{(1)} \otimes e_{(1)} \\ \epsilon_4 &= e_{(0)} \otimes e_{(1)} + e_{(1)} \otimes e_{(0)} \\ \epsilon_5 &= e_{(0)} \otimes e_{(2)} + e_{(2)} \otimes e_{(0)}\end{aligned} \tag{3.50}$$

Then the second line of (3.49) is proportional to:

$$T_1^{\mu\nu}(2\epsilon_4 + \epsilon_5)_{\mu\nu}(2\epsilon_4 + \epsilon_5)_{\rho\sigma}T_2^{\rho\sigma} \tag{3.51}$$

and the third one to

$$T_1^{\mu\nu}(\epsilon_2 + 2e_3)_{\mu\nu}(\epsilon_2 + 2e_3)_{\rho\sigma}T_2^{\rho\sigma} \tag{3.52}$$

whereas the last one is a spin zero contribution

$$T_1^{\mu\nu}(\epsilon_1 + \epsilon_2 + e_3)_{\mu\nu}(\epsilon_1 + \epsilon_2 + e_3)_{\rho\sigma}T_2^{\rho\sigma} \tag{3.53}$$

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getting

$$\begin{aligned}E_{12} = \; & \frac{1}{2}(T_1^{11} - T_1^{22})(T_2^{11} - T_2^{22}) + 2T_1^{12}T_2^{12} \\ & +\frac{m^2}{2\omega^2}(T_1^{11} + T_1^{22})T_2^{33} + 2\frac{m^2}{\omega^2}(T_1^{13}T_2^{13} + T_1^{23}T_2^{23}) \\ & +\frac{1}{2}T_1^{33}\left(\frac{m^4}{\omega^4}T_2^{33} - \frac{m^2}{\omega^2}(T_2^{11} + T_2^{22})\right)\end{aligned} \tag{3.47}$$

This clearly shows that in the massless limit only the two polarizations in (3.18) contribute (cf. [?][?]) to this physical observable.

which obviously does not correspond to spin two, but is nevertheless neccessary to cancel the contribution of the unphysical polarizations in the massless case. So that not only are off-shell spin zero components allowed by the theory as intermediate states, but as has been pointed out by Dicus and Willenbrok, they are actually neccessary for consistency. The appearance of these components was first pointed out in [**?**].

Coming back to our main theme, a natural question is how can we experimentally discriminate between both theories? There is an easy answer, namely that graviton scattering amplitudes are expected to be different in detail. But unfortunately, graviton scattering data do not abound.

A most interesting, and perhaps feasible experiment would be to weigh the vacuum energy, i.e. Casimir energy. Indeed, under the restricted variations in (**??**) which we have labelled $\delta^t g_{\mu\nu}$, the vacuum energy does not affect [9] the equations of motion.

A related point is the following. Granting that the two Einstein theories are indeed different at the quantum level, the most important physical question is whether this improves or otherwise reformulates in some way the problem of the cosmological constant. Interesting suggestions in this direction have been made by [**?**] and [**?**], although no compelling model exists yet.

3.6 TDiff and enhanced symmetries.

Under a general transformation of the fields $h_{\mu\nu} \mapsto h_{\mu\nu} + \delta h_{\mu\nu}$, and up to total derivatives, we have

$$\begin{aligned} \delta\mathcal{L}^I &= -\frac{1}{2}\delta h_{\mu\nu}\Box h^{\mu\nu}, \\ \delta\mathcal{L}^{II} &= \delta h_{\mu\nu}\partial^\rho\partial^{(\mu}h^{\nu)}_\rho, \\ \delta\mathcal{L}^{III} &= -\frac{1}{2}\Big(\delta h\partial^\mu\partial^\nu h_{\mu\nu} + \delta h_{\mu\nu}\partial^\mu\partial^\nu h\Big), \\ \delta\mathcal{L}^{IV} &= \frac{1}{2}\delta h\Box h. \end{aligned} \tag{3.54}$$

It follows that the combination [6]

$$\mathcal{L}_A \equiv \mathcal{L}^I + \mathcal{L}^{II} \tag{3.55}$$

is invariant under restricted gauge transformations

$$\delta h_{\mu\nu} = 2\partial_{(\mu}\xi_{\nu)}, \tag{3.56}$$

with

$$\partial_\mu\xi^\mu = 0. \tag{3.57}$$

[9] This point has been developed in discussions with Tomás Ortín.

Since $\mathcal{L}^{III}$ and $\mathcal{L}^{IV}$ are (separately) invariant under this symmetry, the most general TDiff invariant Lagrangian has $\beta = 1$, and arbitrary coefficients a and b:

$$\mathcal{L}_{\text{TDiff}} \equiv \mathcal{L}_A + a\ \mathcal{L}^{III} + b\ \mathcal{L}^{IV}. \tag{3.58}$$

An enhanced symmetry can be obtained by adjusting a and b appropriately. For instance, $a = b = 1$ corresponds to the Fierz-Pauli Lagrangian [**?**], which is invariant under full diffeomorphisms (Diff), where the condition (3.57) is dropped. In fact, a one parameter family of Lagrangians can be obtained from the Fierz-Pauli one through non-derivative field redefinitions,

$$h_{\mu\nu} \mapsto h_{\mu\nu} + \lambda h \eta_{\mu\nu}, \qquad (\lambda \neq -1/n) \tag{3.59}$$

where n is the space-time dimension and the condition $\lambda \neq -1/n$ is necessary for the transformation to be invertible. Under this redefinition, the parameters in the Lagrangian (3.58) change as

$$a \mapsto a + \lambda\,(an - 2)\,, \quad b \mapsto b + 2\lambda(nb - a - 1) + \lambda^2(bn^2 - n(2a+1) + 2). \tag{3.60}$$

Starting from $a = b = 1$, the new parameters are related by

$$b = \frac{1 - 2a + (n-1)a^2}{(n-2)}. \tag{3.61}$$

It follows that Lagrangians where this relation is satisfied are equivalent to Fierz-Pauli, with the exception of the case $a = 2/n$, which cannot be reached from $a = 1$ with $\lambda \neq -1/n$.

A second possibility is to enhance TDiff with an additional Weyl symmetry,

$$\delta h_{\mu\nu} = \frac{2}{n}\phi\eta_{\mu\nu}, \tag{3.62}$$

by which the action becomes independent of the trace. In the generic transverse Lagrangian $\mathcal{L}_{\text{TDiff}}[h_{\mu\nu}]$ of Eq. (3.58), replace $h_{\mu\nu}$ with the traceless part

$$h_{\mu\nu} \mapsto \hat{h}_{\mu\nu} \equiv h_{\mu\nu} - (h/n)\eta_{\mu\nu}. \tag{3.63}$$

This is formally analogous to (3.59) with $\lambda = -1/n$, but cannot be interpreted as a field redefinition. As such, it would be singular, because the trace h cannot be recovered from $\hat{h}_{\mu\nu}$. The resulting Lagrangian

$$\mathcal{L}_{\text{WTDiff}}[h_{\mu\nu}] \equiv \mathcal{L}_{\text{TDiff}}[\hat{h}_{\mu\nu}], \tag{3.64}$$

is still invariant under TDiff [the replacement (3.63) does not change the coefficients in front of the terms $\mathcal{L}^I$ and $\mathcal{L}^{II}$]. Moreover, it is invariant under (3.62), since $\hat{h}_{\mu\nu}$

is. Using (3.60) with $\lambda = -1/n$, we immediately find that this "WTDiff" symmetry corresponds to Lagrangian parameters

$$a = \frac{2}{n}, \qquad b = \frac{n+2}{n^2}. \tag{3.65}$$

This is the exceptional case mentioned at the end of the previous paragraph. Note that the densitized metric $\tilde{g}_{\mu\nu} = g^{-1/n} g_{\mu\nu} \approx \eta_{\mu\nu} + \hat{h}_{\mu\nu}$ enjoys the property that $\tilde{g} = 1$. This is the starting point for the non-linear generalization of the WTDiff invariant theory, which is discussed in Subsection 2.5.

It is easy to show that Diff and WTDiff exhaust all possible enhancements of TDiff for a Lagrangian of the form (3.40) (and that, in fact, these are its largest possible gauge symmetry groups). Note first, that the variation of $\mathcal{L}^I$ involves a term $\Box h^{\mu\nu}$. For arbitrary $h_{\mu\nu}$, this will only cancel against other terms in (3.54) provided that the transformation is of the form

$$\delta h_{\mu\nu} = 2\partial_{(\mu}\xi_{\nu)} + \frac{2\phi}{n}\eta_{\mu\nu}, \tag{3.66}$$

for some ξ^μ and ϕ. The vector can be decomposed as

$$\xi_\mu = \eta_\mu + \partial_\mu \psi \tag{3.67}$$

where $\partial_\mu \eta^\mu = 0$. Using (3.54) we readily find

$$\begin{aligned} \delta\mathcal{L} &= \eta_\nu(\beta - 1)\Box(\partial_\mu h^{\mu\nu}) \\ &+ \frac{\psi}{2}\left[(b-a)\Box^2 h + (2\beta - a - 1)\Box(\partial_\mu\partial_\nu h^{\mu\nu})\right] \\ &+ \frac{\phi}{n}\left[(bn - a - 1)\Box h + (2\beta - na)\partial_\mu\partial_\nu h^{\mu\nu}\right]. \end{aligned} \tag{3.68}$$

TDiff corresponds to taking $\beta = 1$, with arbitrary transverse η^μ and with $\phi = \psi = 0$. This symmetry can be enhanced with nonvanishing ϕ and ψ satisfying the relation

$$n(a-1)\Box\psi = 2(2 - an)\phi, \tag{3.69}$$

provided that

$$b = \frac{1 - 2a + (n-1)a^2}{(n-2)}. \tag{3.70}$$

Eq. (3.69) ensures the cancellation of the terms with $\partial_\mu\partial_\nu h^{\mu\nu}$, and Eq. (3.70) eliminates terms containing the trace h. Eq. (3.70) agrees with (3.61), and therefore the Lagrangian with the enhanced symmetry is equivalent to Fierz-Pauli, unless $a = 2/n$, which corresponds to WTDiff[10].

[10] Incidentally, it may be noted that for $n = 2$ both possibilities coincide, since in this case the symmetry of the Fierz-Pauli Lagrangian is full diffeomorphisms plus Weyl transformations.

3.7 Comparing Diff and WTDiff

Let us briefly consider the differences between the two enhanced symmetry groups. A first question is whether the Fierz-Pauli theory $\mathcal{L}_{\mathrm{Diff}}$ is classically equivalent to $\mathcal{L}_{\mathrm{WTDiff}}$. Since Diff includes TDiff, we can use (3.64) to obtain

$$\frac{\delta \mathcal{S}_{\mathrm{WTDiff}}[h]}{\delta h_{\mu\nu}} = \frac{\delta \mathcal{S}_{\mathrm{Diff}}[\hat{h}]}{\delta \hat{h}_{\rho\sigma}} \left(\delta^{\mu}_{(\rho}\delta^{\nu}_{\sigma)} - \frac{1}{n}\eta_{\rho\sigma}\eta^{\mu\nu} \right). \tag{3.71}$$

Hence, the WTDiff equations of motion are traceless

$$\frac{\delta \mathcal{S}_{\mathrm{WTDiff}}[h]}{\delta h_{\mu\nu}} \eta_{\mu\nu} \equiv 0.$$

In the WTDiff theory, the trace of h can be changed arbitrarily by a Weyl transformation, and we can always go to the gauge where $h = 0$. Likewise, in the familiar Diff theory we can choose a gauge where $h = 0$. Then, $h_{\mu\nu} = \hat{h}_{\mu\nu}$, and the WTDiff equations of motion (e.o.m.) are just the traceless part of the Fierz-Pauli e.o.m. Differentiating Eq. (3.71) with respect to x^{μ} and using the Bianchi identity

$$\partial_{\rho} \left(\frac{\delta \mathcal{S}_{\mathrm{Diff}}[h]}{\delta h_{\rho\sigma}} \right) = 0,$$

one easily finds that $\delta \mathcal{S}_{\mathrm{WTDiff}}[h]/\delta h_{\mu\nu} = 0$ implies

$$\frac{\delta \mathcal{S}_{\mathrm{Diff}}[h]}{\delta h_{\rho\sigma}} \eta_{\rho\sigma} = \Lambda.$$

Hence, the trace of the Fierz-Pauli e.o.m. is also recovered from the WTDiff e.o.m. (in the gauge $h = 0$), up to an arbitrary integration constant Λ which plays the role of a cosmological constant[11] . Thus, the two theories are closely related, but they are not quite the same.

Let us now consider the relation between the corresponding symmetry groups. Acting infinitesimally on $h_{\mu\nu}$ they give

$$\delta^{D} h_{\mu\nu} = 2\partial_{(\mu}\xi_{\nu)} = 2\partial_{(\mu}\eta_{\nu)} + \partial_{\mu}\partial_{\nu}\psi \tag{3.72}$$

$$\delta^{WTD} h_{\mu\nu} = 2\partial_{(\mu}\bar{\eta}_{\nu)} + \frac{2}{n}\phi\eta_{\mu\nu} \tag{3.73}$$

where $\partial_{\mu}\eta^{\mu} = \partial_{\mu}\bar{\eta}^{\mu} = 0$. In (3.72) we have decomposed $\xi_{\nu} = \eta_{\nu} + \partial_{\nu}\psi$ into transverse and longitudinal part. The intersection of Diff and WTDiff can be found by equating (3.72) and (3.73)

$$2\partial_{(\mu}\eta_{\nu)} + \partial_{\mu}\partial_{\nu}\psi = 2\partial_{(\mu}\bar{\eta}_{\nu)} + \frac{2}{n}\phi\eta_{\mu\nu}. \tag{3.74}$$

[11]Here we assume $\Lambda = O(h)$.

Taking the trace, we have

$$\Box\psi = 2\phi. \tag{3.75}$$

The divergence of (3.74) now yields

$$\Box(\bar{\eta}_\mu - \eta_\mu) = \frac{n-1}{n}\Box\partial_\mu\psi. \tag{3.76}$$

Taking the divergence once more, we have

$$\Box\phi = 0. \tag{3.77}$$

Taking the derivative of (3.76) with respect to ν, symmetrizing with respect to μ and ν, and using (3.74) and (3.75), we have $(n-2)\partial_\mu\partial_\nu\Box\psi = 0$. For $n \neq 2$ this implies $\partial_\mu\partial_\nu\phi = 0$, *i.e.*

$$\phi = b_\mu x^\mu + c,$$

where b_μ and c are constants. Hence, not every Weyl transformation belongs to Diff, since only the ϕ's which are linear in x^μ qualify as such. Conversely, the subset of Diff which can be expressed as Weyl transformations are the solutions of the conformal Killing equation for the Minkowski metric [**?**],

$$\partial_{(\mu}\xi^{CD}_{\nu)} = \frac{1}{n}\phi\eta_{\mu\nu}, \tag{3.78}$$

where $\phi = \partial^\rho\xi^{CD}_\rho$ (and, as shown above, ϕ has to be a linear function of x^μ). These solutions generate the so called conformal group, which we may denote by CDiff. In conclusion, the enhanced symmetry groups Diff and WTDiff are not subsets of each other. Rather, their intersection is the set of TDiff plus CDiff.

3.8 Traceless Fierz-Pauli and WTDiff

- An alternative route to the WTDiff invariant theory is to try and construct a Lagrangian which will yield the traceless part of Einstein's equations.

  It is clear, however, that we can only obtain traceless equations of motion from a Lagrangian which is invariant under Weyl transformations. If the e.o.m. are traceless, then $\delta S = 0$ for variations of the form for $\delta h_{\mu\nu} \propto \eta_{\mu\nu}$. This symmetry is not included in Diff, and therefore the traceless part of Einstein's equations cannot be recovered from this Lagrangian in every gauge. Rather, we should look for a Lagrangian which will yield the traceless part of Einstein's equations in *some* gauge.

  Let us consider the Diff e.o.m. in momentum space

$$\frac{\delta\mathcal{S}_{\text{Diff}}[h]}{\delta h_{\rho\sigma}} = D^{\rho\sigma\mu\nu}_{\text{Diff}} h_{\mu\nu}, \tag{3.79}$$

where

$$8D^{\mu\nu\rho\sigma}_{\rm Diff} = k^2\left(\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} - 2\eta^{\mu\nu}\eta^{\rho\sigma}\right) -$$
$$\left(k^\mu k^\rho\eta^{\nu\sigma} + k^\nu k^\sigma\eta^{\mu\rho} + k^\mu k^\sigma\eta^{\nu\rho} + k^\nu k^\rho\eta^{\mu\sigma} - 2k^\mu k^\nu\eta^{\rho\sigma} - 2k^\rho k^\sigma\eta^{\mu\nu}\right) \tag{3.80}$$

We can also define the traces

$$\text{tr}\, D^{\rho\sigma}_{\rm Diff} = \eta_{\mu\nu}D^{\mu\nu\rho\sigma}_{\rm Diff} = \frac{n-2}{4}\left(k_\rho k_\sigma - k^2\eta_{\rho\sigma}\right),$$
$$\text{tr}\ \text{tr}\, D_{\rm Diff} = \eta_{\mu\nu}\eta_{\rho\sigma}D^{\rho\sigma\mu\nu}_{\rm Diff} = -\frac{(n-1)(n-2)}{4}k^2. \tag{3.81}$$

The traceless part of the $D^{\rho\sigma\mu\nu}_{\rm Diff}$,

$$8\left(D^t_{\rm Diff}\right)^{\mu\nu\rho\sigma} = 8\left(D^{\mu\nu\rho\sigma}_{\rm Diff} - \frac{1}{n}\eta^{\mu\nu}\text{tr}\ D^{\rho\sigma}_{\rm Diff}\right) =$$
$$k^2\left(\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} + 2\left(\frac{n-2}{n} - 1\right)2\eta^{\mu\nu}\eta^{\rho\sigma}\right) -$$
$$\left(k^\mu k^\rho\eta^{\nu\sigma} + k^\nu k^\sigma\eta^{\mu\rho} + k^\mu k^\sigma\eta^{\nu\rho} + k^\nu k^\rho\eta^{\mu\sigma} - 2k^\mu k^\nu\eta^{\rho\sigma} + 2\left(\frac{n-2}{n} - 1\right)k^\rho k^\sigma\eta^{\mu\nu}\right).$$

cannot be derived from a Lagrangian for any dimension $n \neq 2$ as it is not symmetric in the indices $(\rho\sigma)$ vs. $(\mu\nu)$. Nevertheless, we can still define traceless symmetric Lagrangians. One might think of substituting $\eta^{\mu\nu}$ in the previous expression by tr $D^{\mu\nu}_{\rm Diff}$, and dividing by its trace. However, this would be non-local.

- Some people [69] define

$$8\left(\hat{D}_{\rm Diff}\right)^{\mu\nu\rho\sigma} = 8\left(D^{\mu\nu\rho\sigma}_{\rm Diff} - \frac{1}{n-2}\eta^{\mu\nu}\text{tr}\ D^{\rho\sigma}_{\rm Diff}\right) =$$
$$k^2\left(\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\right) - \left(k^\mu k^\rho\eta^{\nu\sigma} + k^\nu k^\sigma\eta^{\mu\rho} + k^\mu k^\sigma\eta^{\nu\rho} + k^\nu k^\rho\eta^{\mu\sigma} - 2k^\mu k^\nu\eta^{\rho\sigma}\right).$$

- A very important property is the transverse character. Actually, we shall prove that

$$D_{\mu\nu}(h) = \frac{1}{4}\left(k^2\left(h_{\mu\nu} - h\eta_{\mu\nu}\right) - \left(k_\mu k^\lambda h_{\lambda\nu} + k_\nu k^\lambda h_{\mu\lambda} - k_\mu k_\nu h - \eta_{\mu\nu}k^\lambda k^\sigma k_{\lambda\sigma}\right)\right) =$$
$$2k^\lambda\eta_{\mu\nu\lambda} = 2\partial^\lambda\eta_{\mu\nu\lambda} \tag{3.82}$$

The tensor η is not uniquely defined [69]. Let us simply show just one possibility

$$\eta^{\mu\nu\lambda}_{AD} = -\partial_\sigma K^{\mu\lambda\nu\sigma} \tag{3.83}$$

where the *superpotential* is given by

$$K_{\mu\nu\rho\sigma} \equiv \frac{1}{2}\left(\eta_{\mu\sigma}\bar{h}_{\nu\rho} + \eta_{\nu\rho}\bar{h}_{\mu\sigma} - \eta_{\mu\rho}\bar{h}_{\nu\sigma} - \eta_{\nu\sigma}\bar{h}_{\mu\rho}\right) =$$
$$\frac{1}{2}\left(\left(\eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\nu\rho}\eta_{\mu\sigma}\right)h + \eta_{\mu\sigma}h_{\nu\rho} + \eta_{\nu\rho}h_{\mu\sigma} - \eta_{\mu\rho}h_{\nu\sigma} - \eta_{\nu\sigma}h_{\mu\rho}\right)$$

enjoys exactly the same set of symmetries as the Riemann tensor, and is defined in terms of the convenient variable

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu} \tag{3.84}$$

Indeed

$$k^\sigma K_{\mu\nu\rho\sigma} = \frac{1}{2}\Big((\eta_{\mu\rho} k_\nu - k_\mu \eta_{\nu\rho})\, h + k_\mu h_{\nu\rho} + \eta_{\nu\rho} k^\sigma h_{\mu\sigma} - \eta_{\mu\rho} k^\sigma h_{\nu\sigma} - k_\nu h_{\mu\rho} \Big) \tag{3.85}$$

and

$$k^\nu \eta^{AD}_{\mu\rho\nu} = k^\nu k^\sigma K_{\mu\nu\rho\sigma} = \frac{1}{2}\Big((k^2 \eta_{\mu\rho} - k_\mu k_\rho)\, h + k_\mu k^\nu h_{\nu\rho} + k_\rho k^\sigma h_{\mu\sigma} - \eta_{\mu\rho} k^\nu k^\sigma h_{\nu\sigma} - k^2 h_{\mu\rho} \Big) = -2 D_{\mu\rho\nu\sigma} h^{\nu\sigma} \tag{3.86}$$

- For a local Lagrangian which is still invariant under TDiff, we must restrict to deformations which correspond to changes in the parameters a and b in (3.40). The most general symmetric Lagrangian with these properties is of the form

$$D^{\mu\nu\rho\sigma}_{t\text{Diff}} \equiv D^{\mu\nu\rho\sigma}_{\text{Diff}} - \eta^{\mu\nu} D^{\rho\sigma} - D^{\mu\nu} \eta^{\rho\sigma}, \tag{3.87}$$

  with $D_{\rho\sigma}$ a symmetric operator at most quadratic in the momentum. Asking that the result be traceless leads to:

$$M^{\mu\nu} = \frac{1}{n} \left(\text{tr } D^{\mu\nu}_{\text{Diff}} - (\text{tr } M) \eta^{\mu\nu} \right), \tag{3.88}$$

  which implies

$$\text{tr } M = \frac{1}{2n} \text{tr tr } D_{\text{Diff}}. \tag{3.89}$$

  Therefore

$$M^{\mu\nu} = \frac{1}{n} \left(\text{tr } D^{\mu\nu}_{\text{Diff}} - \frac{1}{2n} (\text{tr tr } D_{\text{Diff}}) \eta^{\mu\nu} \right), \tag{3.90}$$

  and we can write

$$\begin{aligned} 8 D^{\mu\nu\rho\sigma}_{t\text{Diff}} = \; & k^2 (\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}) - (k_\mu k_\rho \eta_{\nu\sigma} + k_\nu k_\sigma \eta_{\mu\rho} + k_\mu k_\sigma \eta_{\nu\rho} + k_\nu k_\rho \eta_{\mu\sigma}) \\ & - \frac{2(n+2)}{n^2} k^2 \eta_{\mu\nu}\eta_{\rho\sigma} + \frac{4}{n} (k_\mu k_\nu \eta_{\rho\sigma} + k_\rho k_\sigma \eta_{\mu\nu}). \end{aligned} \tag{3.91}$$

  Moving back to the position space, this corresponds to the WTDiff Lagrangian, i.e. the case $a = \frac{2}{n}$ and $b = \frac{n+2}{n^2}$ in (3.58). As shown before, this yields the traceless part of the Fierz-Pauli e.o.m. in the gauge $h = 0$.

A similar analysis could be done for the massive case. However, as we shall see in the next section, the corresponding Lagrangian has a ghost.

3.9 Dynamical analysis of the general massless Lagrangian.

The little group argument mentioned in the introduction indicates that the quantum theory is not unitary unless the Lagrangian is invariant under TDiff. In fact, in the absence of TDiff symmetry the Hamiltonian is unbounded below. This leads to pathologies such as classical instabilities or the existence of ghosts.

To show this, as well as to analyze the physical degrees of freedom of the general massless theory (3.40), it is very convenient to use the "cosmological" decomposition in terms of scalars, vectors, and tensors under spatial rotations $SO(3)$ (see e.g. [?]),

$$\begin{aligned} h_{00} &= A \\ h_{0i} &= \partial_i B + V_i \\ h_{ij} &= \psi\delta_{ij} + \partial_i\partial_j E + 2\partial_{(i}F_{j)} + t_{ij} \end{aligned} \tag{3.92}$$

where $\partial^i F_i = \partial^i V_i = \partial^i t_{ij} = t^i_i = 0$. The point of this decomposition is that in the linearized theory the scalars (A, B, ψ, E), vectors (V_i, F_i) and tensors (t_{ij}) decouple from each other. Also, we can easily identify the physical degrees of freedom without having to fix a gauge (see Appendix A).

The tensors t_{ij} only contribute to $\mathcal{L}^I$, and one readily finds

$$^{(t)}\mathcal{L} = -\frac{1}{4}t^{ij}\Box t_{ij} \tag{3.93}$$

The vectors contribute both to $\mathcal{L}^I$ and $\mathcal{L}^{II}$. Working in Fourier space for the spatial coordinates and after some straightforward algebra, we have

$$^{(v)}\mathcal{L} = \frac{1}{2}\kappa^2\left(V^i - \dot{F}^i\right)^2 + \frac{1}{2}(\beta-1)\left(\kappa^2F^i + \dot{V}^i\right)^2. \tag{3.94}$$

For $\beta = 1$, corresponding to TDiff symmetry, there are no derivatives of V^i in the Lagrangian. Variation with respect to V^i leads to the constraint $V^i - \dot{F}^i = 0$, which upon substitution in (**??**) shows that there is no vector dynamics.

Other values of β lead to pathologies. The Hamiltonian is given by

$$^{(v)}\mathcal{H} = \frac{(\Pi_F + \kappa^2 V)^2}{2\kappa^2} - \frac{[\Pi_V + (1-\beta)\kappa^2 F]^2}{2(1-\beta)} + \frac{(1-\beta)\kappa^4F^2}{2} - \frac{\kappa^2V^2}{2}, \tag{3.95}$$

where the momenta are given by $\Pi_F = \kappa^2\left(\dot{F} - V\right)$ and $\Pi_V = (\beta-1)\left(\kappa^2F + \dot{V}\right)$, and we have suppressed the index i in the vectors F and V. Because of the alternating signs in Eq. (3.95), the Hamiltonian is not bounded below. Generically this leads to a classical instability. The momenta satisfy the equations $\dot{\Pi}_F = \kappa^2\Pi_V$ and $\dot{\Pi}_V = -\Pi_F$. These have the general oscillatory solution

$$|\kappa|\Pi_V + \mathrm{i}\ \Pi_F = C\exp\mathrm{i}(|\kappa|t + \phi_0),$$

where C and ϕ_0 are real integration constants. On the other hand, V and F satisfy

$$\ddot{V} + \kappa^2 V = \frac{-\beta}{(\beta - 1)} \Pi_F, \tag{3.96}$$

$$\ddot{F} + \kappa^2 F = \frac{\beta}{(\beta - 1)} \Pi_V. \tag{3.97}$$

For $\beta \neq 0$ these are equations for forced oscillators. For large times, the homogeneous solution becomes irrelevant and we have

$$V + \mathrm{i}|\kappa| F \sim \left(\frac{\beta C t}{(\beta - 1)|\kappa|} \right) \exp \mathrm{i}(|\kappa| t + \phi_0),$$

whose amplitude grows without bound, linearly with time. This classical instability is not present for $\beta = 0$. However, in this case F and V decouple and we have

$$^{(v)}\mathcal{L}_{\beta=0} = \frac{1}{2}\kappa^2 (\partial_\mu F^i)^2 - \frac{1}{2}(\partial_\mu V^i)^2,$$

so V_i are ghosts.

Hence, the only case where the vector Lagrangian is not problematic is $\beta = 1$, corresponding to invariance under TDiff. The scalar Lagrangian is then given by[12]

$$\begin{aligned} ^{(s)}\mathcal{L}_{\text{TDiff}} = &\frac{1}{4}\left[(\partial_\mu A)^2 - 2\kappa^2 (\partial_\mu B)^2 + N(\partial_\mu \psi)^2 - 2\kappa^2 \partial_\mu \psi \partial^\mu E + \kappa^4 (\partial_\mu E)^2\right] \\ &- \frac{1}{2}\left[(\dot{A} + \kappa^2 B)^2 - \kappa^2 \dot{B}^2 - \kappa^2 \psi^2 + 2\kappa^4 E\psi - \kappa^6 E^2 + 2\kappa^2 \dot{B}(\psi - \kappa^2 E)\right] \\ &+ \frac{a}{2}\left[(\dot{A} - N\dot{\psi} + \kappa^2 \dot{E})(\dot{A} + \kappa^2 B) - \kappa^2 (A - N\psi + \kappa^2 E)(\dot{B} - \psi + \kappa^2 E)\right] \\ &- \frac{b}{4}\left[\partial_\mu (A - N\psi + \kappa^2 E)\right]^2, \end{aligned} \tag{3.98}$$

where $N = n - 1$ is the dimension of space. It is easy to check that B is a Lagrange multiplier, leading to the constraint

$$(N - 1)\psi = (a - 1)h, \tag{3.99}$$

where $h = A - N\psi + \kappa^2 E$ is the trace of the metric perturbation. Substituting this back into the scalar action (3.98) we readily find

$$^{(s)}\mathcal{L}_{\text{TDiff}} = -\frac{\Delta b}{4}(\partial_\mu h)^2, \tag{3.100}$$

where

$$\Delta b \equiv b - \frac{1 - 2a + (n-1)a^2}{n - 2}. \tag{3.101}$$

Hence, the scalar sector contains a single physical degree of freedom, proportional to the trace. Whether this scalar is a ghost or not is determined by the parameters a and b. For $b = (1-2a+(n-1)a^2)/(n-2)$, corresponding to the enhanced symmetries which we studied in the previous subsection, the scalar sector disappears completely, and we are just left with the tensor modes.

[12]The equivalent expression in terms of gauge invariant combinations is given in Appendix A.

4. Conceptual issues in quantum gravity. The diffeomorphism group

All theories of gravity we are going to be interested at are invariant under (a subgroup of) the group of all diffeomorphisms of the spacetime manifold (perhaps keeping invariant some boundary conditions).

We shall represent diffeomorphisms (diffs) without any loss of generality as a *local translation*:

$$y = x + \xi(x) \equiv T_{\xi(x)} x \tag{4.1}$$

so that

$$\delta x \equiv y - x \equiv \xi(x) \tag{4.2}$$

and the jacobian must enjoy a nonvanishing determinant

$$det \left(\delta^\alpha_\beta + \partial_\beta \xi^\alpha\right) \neq 0 \tag{4.3}$$

The group law is mapping composition:

$$\eta \circ \xi \equiv x \rightarrow x + \xi(x) + \eta(x + \xi(x)) \tag{4.4}$$

The inverse diff

$$x = y + \xi^{-1}(y) \tag{4.5}$$

(this is just the definition of $\xi^{-1}(y)$). In order to compute it, we start from

$$x + \xi(x) = y = x - \xi^{-1}(y) \tag{4.6}$$

so that

$$\xi\left(y + \xi^{-1}(y)\right) + \xi^{-1}(y) \equiv T\left(\xi^{-1}(y)\right)\xi(y) + \xi^{-1}(y) = 0 \tag{4.7}$$

as well as

$$\xi(x) + \xi^{-1}\left(x + \xi(x)\right) = 0 \tag{4.8}$$

This means that, at least formally,

$$\xi^{(-1)\mu}(x) = -T^{-1}(\xi)\xi^\mu = -\xi^\mu + \xi^\lambda\partial_\lambda\xi^\mu + \xi^\lambda\partial_\lambda\left(\xi^\sigma\partial_\sigma\xi^\mu\right) + \frac{1}{2}\xi^\lambda\xi^\sigma\partial_\lambda\partial_\sigma\xi^\mu + O(\xi^4) \tag{4.9}$$

where we [13] *define* the differential operator that translates the argument of a function as the corresponding Taylor series:

$$T(\xi) \equiv \sum \frac{1}{n!}\xi^{a_1}\ldots\xi^{a_n}\partial_{a_1}\ldots\partial_{a_n} \tag{4.12}$$

[13] This fact can be checked in the analytic case by expanding

$$\xi^\mu \equiv \sum E^\mu{}_{\alpha_1\ldots\alpha_n} x^{\alpha_1}\ldots x^{\alpha_n} \tag{4.10}$$

and assuming all constant coefficients $E^\mu{}_{\alpha_1\ldots\alpha_n}$ to be of the same order ϵ. One finds by equating the formal power series that

$$\xi^{(-1)\mu} = \left(-E^\mu + E^\mu_\nu E^\nu\right) + \left(-E^\mu_\nu + E^\mu_\rho E^\rho_\nu + 2E^\mu_{\rho\nu}E^\rho\right)x^\nu + O(\epsilon^2) \tag{4.11}$$

In spite of the name of *translations*, those operators do not commute in general:

$$[T(\xi), T(\eta)] = T([\xi, \eta]) + \ldots \tag{4.13}$$

By definition,

$$T^{-1}(\xi) = T(\xi^{-1}) = 1 - \left(\xi^\alpha - \xi^\beta \partial_\beta \xi^a\right) \partial_\alpha + O(\xi^2) \tag{4.14}$$

This means that BCH formulas (*confer* [9]) are in principle not valid for these generalized exponentials.

Acting now with a second diff

$$z = y + \eta(y) \tag{4.15}$$

the composition of the two is still another diff:

$$z = x + \xi(x) + \eta(x + \xi(x)) = x + \xi(x) + T(\xi)\eta(x) \tag{4.16}$$

Clearly

$$z^a = x^a + \xi^a(x) + \eta^a(x) + \partial_b \eta^a \xi^b + O(\xi^2) \tag{4.17}$$

so that, to linear order

$$\eta \circ \xi = \eta + \xi + (\xi.\partial)\eta \tag{4.18}$$

and

$$[\eta, \xi] = \pounds(\xi)\eta \tag{4.19}$$

The group $Diff(M)$ is not locally compact for $n > 1$. We shall call *large* diffs those that are not continuously connected to the identity. The set of all small diffs is denoted as $Diff_0(M)$

The component group is the *mapping class group* (MMG).

$$MCG(M) = Diff(M)/Diff_0(M) \tag{4.20}$$

If we call $Diff^1_x(M)$ the little group (stabilizer) of x, then

$$M \sim Diff(M)/Diff^1_x(M) \tag{4.21}$$

If the manifold is endowed with a measure, say $m(M)$, than there is a natural subgroup of $Diff(M)$, namely the subgroup of all diffs which preserve the given measure, $Diff(M, m)$.

The linear subgroup of $Diff(M)$ is $GL(n)$ and the corresponding linear subgroup of $TDiff(M)$ is $SL(n)$. General mathematical references are [85][54].

4.1 Coordinates in $Diff(M)$

Given a vector field, $\xi(x)$, there is a one parameter subgroup of diffs generated by it, i.e., More precisely, a one-parameter subgroup is defined by the system of ordinary differential equations associated to the vector field $\xi(x)$:

$$\frac{dx^\alpha}{dt} = \xi^\alpha \left(x^\beta(t)\right) \tag{4.22}$$

The integral curves are

$$x^{\alpha\prime} = f^\alpha \left(t, x^a\right) \tag{4.23}$$

with initial conditions such that

$$x^\alpha = f^\alpha \left(0, x^\alpha\right) \tag{4.24}$$

Then the mapping

$$f_t : x_0 \equiv x \rightarrow x_t \equiv x' \tag{4.25}$$

is a local group of diffs.

Expanding in powers of t, it is easily discovered that

$$x^{\alpha\prime} = x^\alpha + t\xi^\alpha + \frac{t^2}{2}\xi^\beta\partial_\beta\xi^\alpha + \frac{t^3}{6}\left(\xi^\beta\partial_\beta\xi^\gamma\partial_\gamma\xi^\alpha + \xi^\beta\xi^\gamma\partial_\beta\partial_\gamma\xi^\alpha\right) + O(t^4) \tag{4.26}$$

Nevertheless it is well-known that there are in general diffs (even arbitrarily close to the identity) that do not lie on one-parameter subgroups. An explicit example is the C^∞ diff in $\mathbb{C}$

$$z \rightarrow e^{\frac{2\pi i}{N}} z + \alpha\, z^{N+1} \tag{4.27}$$

The statement is that this diff does not lie on a one-parameter subgroup of diffs

$$\xi(t) : \mathbb{C} \rightarrow \mathbb{C} \tag{4.28}$$

with $\xi(0) = 1$. The result is essentially contained in previous work by Sternberg [75]. Freifeld's proof [40] consists in an explicit analysis of the expansion

$$\xi(t, z, \bar{z}) = \sum_{m,n=0}^{\infty} a_{m.n}(t) z^m \bar{z}^n \tag{4.29}$$

An even simpler example in the circle S^1, put forward by Milnor [64] is

$$\theta \rightarrow \theta + \frac{\pi}{N} + \epsilon\, sin^2\left(N\theta\right) = \theta + \frac{\pi}{N} + \epsilon\frac{1 - cos\, 2N\theta}{2} \equiv f_M(\theta) \tag{4.30}$$

with $0 < \epsilon < \frac{1}{N}$, in such a way that

$$\frac{df_M}{d\theta} = 1 + \epsilon N\, sin\, 2N\theta \neq 0. \tag{4.31}$$

The claim is that there is no vector ξ such that

$$f(\theta) = Exp(\xi) \tag{4.32}$$

although there is a representation in terms of several exponentials:

$$f(\theta) = Exp(\xi_1) \circ \ldots \circ Exp(\xi_k) \tag{4.33}$$

The proof of this statement is really simple. There is always a point with period $2n$, namely the origin $\theta = 0$. If we iterate f_M we get

$$0 \rightarrow \pi/n \rightarrow 2\pi/n \rightarrow \ldots (2n)\pi/n \tag{4.34}$$

On the other hand, no other point $0 < \theta_0 < \pi/n$ is congruent with its iterated image, because, $\theta_1 \equiv f_M(\theta_0)$ satisfies

$$\theta_0 + \pi/n < \theta_1 < 2\pi/n \tag{4.35}$$

and

$$\theta_1 + \pi/n < \theta_2 < 2\pi/n \tag{4.36}$$

and so on. Now Milnor shows that there is no f such that

$$f_m = f \circ f \tag{4.37}$$

This is stronger than we claimed, because if $f_M = Exp(v)$ then of course $f_M = Exp(v/2) \circ Exp(v/2)$. Noe a very simple diagrammatic analysis shows that if an arbitrary function f has orbits of even period, say $2m$ then $f \circ f$ gets two orbits of half-period, m. If f has orbits of odd period, $2p+1$, then there is also an orbit of $f \circ f$ with the same odd period. The point is that this shows that the number of orbits of even period for any function of the type $f \circ f$ must be even.

Now we just saw that f_M has *one* orbit of period $2n$, namely the one corresponding to $\theta = 0$.

It is also curious no remark that there are unimodular diffs such that the generating vector is not transverse, i.e. in $\mathbb{R}^2$

$$\begin{aligned} x &\rightarrow -e^y \\ y &\rightarrow x e^{-y} \end{aligned} \tag{4.38}$$

and $\partial_a \xi^a = -2 - x e^{-y} \neq 0$.

The reason seems to be that it is not smoothly connected with the identity.

Even for diffs which are connected to the identity, the divergenceless condition only holds to first order. Let us consider, for example,

$$\begin{aligned} x &\rightarrow e^x - 1 \\ y &\rightarrow y e^{-x} \end{aligned} \tag{4.39}$$

Then

$$\xi = \big(e^x - 1 - x, y(e^{-x} - 1)\big) \tag{4.40}$$

and

$$\partial_i \xi^i = e^x + e^{-x} - 2 = 0 + O(x^2) \tag{4.41}$$

4.2 Integration over $Diff_x^1$

We can in principle write

$$\xi^\mu = \xi_T^\mu + \xi_L^\mu \tag{4.42}$$

where

$$\xi_T^\mu \equiv \xi^\mu - \partial^\mu \Box^{-1} \partial_\alpha \xi^\alpha$$

and

$$\xi_L^\mu \equiv \partial^\mu \Box^{-1} \partial_\alpha \xi^\alpha$$

It is plain that

$$\partial_\alpha \xi_T^\alpha = 0$$

This decomposition is unique to the extent that the inverse of the laplacian is unique. This will need some boundary conditions in general.

Transverse diffeomorphisms for a subgroup, that leaves invariant the Lebesgue measure on $\mathbb{R}^n$.

Some geometric properties, in particular the sectional curvature of the subgroup TDiff in the case of the torus have been considered by Arnold [9].

4.3 Observables in quantum gravity.

What is the meaning of background independence? Is is the same thing as to say that quantum gravity is got to be a topological theory, such as Chern-Simons theories?

4.4 Fields

The active interpretation of a diff

$$y = f(x) \tag{4.43}$$

leads to

$$(f^*\phi)(x) \equiv \phi^\xi(x) \equiv \phi(y) \tag{4.44}$$

This means that

$$\phi^\xi(x) = \phi\left(x + \xi(x)\right) = T(\xi)\phi(x) \tag{4.45}$$

Lagrangian densities in QFT are usually scalars. In the linear approximation,

$$\delta\phi = \xi^a \partial_a \phi \tag{4.46}$$

A vector field obeys

$$(f_* V)^\mu(y) = V^\alpha(x)\partial_\alpha f^\mu \tag{4.47}$$

and a one-form

$$(f^*\omega)_\mu(x) \equiv \omega^\xi_\mu(x) = \omega_\alpha(y)\partial_\mu f^\alpha \tag{4.48}$$

The metric transforms as a covariant tensor:

$$g^\xi_{\alpha\beta}(x) = \frac{\partial f^\gamma}{\partial x^\alpha}\frac{\partial f^\delta}{\partial x^\beta} g_{\gamma\delta}(y) \tag{4.49}$$

and the determinant transform as

$$g^\xi(x) = J(y/x)^2 g(y) \tag{4.50}$$

where

$$J(y/x) \equiv det\, \frac{\partial f^\gamma}{\partial x^\alpha} = J(x/y)^{-1} \tag{4.51}$$

The jacobian can be expanded:

$$\begin{gathered} J(x/y) = det\,\left(\delta^a_b - \partial_b \xi^a\right) = e^{tr\, log\left(\delta^a_b - \partial_b \xi^a\right)} = \\ 1 - \partial_a \xi^a + \frac{1}{2}\left(\partial_a \xi^a\right)^2 - \frac{1}{2}\partial_a \xi^b \partial_b \xi^a + O(\xi^3) \end{gathered} \tag{4.52}$$

The subgroup of those diffs that enjoy unit determinant, [14] dubbed TDiff in a previous paper of ours, correspond to “transverse vectors”

$$\partial_a \xi^a = 0 \tag{4.53}$$

We shall reserve the name *unimodular* for exactly those diffs, and not for the ones that leave the metric volume element invariant (cf. later on in this paper).

[14] This is called by Arnold, SDiff.

Nevertheless it has already been remarked that there are in general (that is, in more than one dimension) diffs (even arbitrarily close to the identity) that do not lie on one-parameter subgroups.

When the manifold is endowed with a metric (which did not play any role until now), its determinant is usually considered to transform (more on this later) as:

$$\delta g = -2\partial_a \xi^a g \tag{4.54}$$

so that

$$\delta\sqrt{|g|}\phi = -\partial_a\left(\sqrt{|g|}\phi\xi^a\right) = -\nabla_a\sqrt{|g|}\left(\phi\xi^a\right) \tag{4.55}$$

which is the origin of the usual recipe to build diff- invariant actions: *the lagrangian is a scalar times the square root of the determinant of the metric tensor.*

4.5 Fake gauge theories

It is well known (cf [38]) that *any* theory can be made gauge invariant by means of a sort of group averaging.

Assume a lagrangian

$$L(\phi, \partial\phi) \tag{4.56}$$

which includes matter fields that transform under a certain representation of a group G, $g \in G$

$$\phi^g \equiv D(g)\phi \tag{4.57}$$

Let us now perform a local transformation $U(x) \in G$, so that the lagrangian reads

$$L\left(U(x)\phi(x), \partial_a\left(U\phi\right)\right) \tag{4.58}$$

Now

$$\partial_a\left(U\phi\right) = \partial_a U\phi + U\partial_a\phi = U\left(\partial_a + U^{-1}\partial_a U\right)\phi \equiv U\left(\partial_a + A_a U\right)\phi \equiv UD_a\phi \tag{4.59}$$

where we have introduced the “fake gauge field”

$$A_a \equiv U^{-1}\partial_a U \tag{4.60}$$

The resulting theory is obviously invariant under

$$\begin{aligned} &U \rightarrow UV^{-1}(x) \\ &\phi \rightarrow V(x)\phi \end{aligned} \tag{4.61}$$

which leaves the combination $U\phi$ invariant. The transformation of the “fake” field U is fixed by this requirement of *redundancy*. The original theory is then recovered in the *unitary gauge*, $U = 1$.

The construct A_a transforms as a true gauge field:

$$A_a \rightarrow V\left(A_a + \partial_a\right)V^{-1} \tag{4.62}$$

4.6 Fake Diff

It is also well known that any theory can be put in a covariant (diff invariant) language.

For example, if we have a theory invariant under the subgroup of diffs with unit determinant, namely, TDiff, such as

$$S = \int d^4x\, \Phi(x) \tag{4.63}$$

We can formally write the transformed lagrangian

$$L = T(V)\Phi \equiv \sum \frac{1}{n!} V^{a_1} \ldots V^{a_n} \partial_{a_1} \ldots \partial_{a_n} \Phi \tag{4.64}$$

which is invariant under

$$\begin{aligned} &\Phi \rightarrow T(\eta)\Phi \\ &T(V) \rightarrow T(V) T^{-1}(\eta) \end{aligned} \tag{4.65}$$

This yields for the first few terms of the part linear in η:

$$V^a_\eta = V^a - \eta^a + V^c \partial_c \eta^a - \frac{1}{2} V^c V^d \partial_c \partial_d \eta^a + O(\partial^3 \eta) \tag{4.66}$$

An amusing thing is that, contrasting with the compensator mechanism that was proposed in [5], the equation of motion of the new field V^a is

$$\frac{\delta S}{\delta V^a} = \sum \frac{1}{(n-1)!} V^{a_2} \ldots V^{a_n} \partial_a \partial_{a_2} \ldots \partial_{a_n} \Phi = 0 \tag{4.67}$$

which is verified by

$$\Phi = constant \tag{4.68}$$

whereas in the compensator mechanism the analogous equation did imply

$$\Phi = 0 \tag{4.69}$$

All this is a bit formal. To be specific, let us write

$$S[\Phi, V] \equiv \int d^4x \Phi(x + V(x)) = \int d^4y J(x/y) \Phi(y) = \int d^4y \frac{1}{|det\ (\delta^\mu_\rho + \partial_\rho V^\mu)\ |} \Phi(y) \tag{4.70}$$

where

$$y \equiv x + V(x) \tag{4.71}$$

Under a further diff,

$$z = y + \xi \tag{4.72}$$

$$S[\Phi, V] = \int d^4z J(y/z) J(x/y) \Phi(z) = \int d^4z J(x/z) \Phi(z) =$$
$$\int d^4y \frac{1}{|det\ (\delta^\mu_\rho + \partial_\rho V^\mu + \partial_\rho \xi^\mu)\,|} \Phi(y) \tag{4.73}$$

so that the action is invariant if we define

$$V_\xi = V + \xi \tag{4.74}$$

In particular, if we demand that

$$\int d^4x \frac{1}{|det\ \left(\delta^\mu_\rho + \partial_\rho V^\mu_\xi\right)|} \Phi_\xi(x) = \int d^4x \frac{1}{|det\ (\delta^\mu_\rho + \partial_\rho V^\mu)\,|} \Phi(x) \tag{4.75}$$

we are led to

$$\frac{1}{|det\ \left(\delta^\mu_\rho + \partial_\rho V^\mu_\xi\right)|} = J(x/y) \frac{1}{|det\ (\delta^\mu_\rho + \partial_\rho V^\mu)\,|} \tag{4.76}$$

that is

$$|det\ \left(\delta^\mu_\rho + \partial_\rho V^\mu_\xi\right)| = J(y/x) |det\ \left(\delta^\mu_\rho + \partial_\rho V^\mu|\right)| \tag{4.77}$$

whose linear part is

$$1 + \partial_\mu V^\mu_\xi = (1 + \partial_\alpha \xi^\alpha)(1 + \partial_\beta V^\beta) \tag{4.78}$$

leading again to

$$V_\xi = V + \xi \tag{4.79}$$

5. The effective field theory approach

Let us now summarize the things that can be learned from the bottom-up approach. This is solid knowledge (with some caveats). What happens is that almost all the important questions we hope to understand with quantum gravity, such as the resolution of singularities etc, shy away from the region of applicability of the low energy effective field theory approach. Nevertheless this seems to us the only ladder we can step on to try and get a higher view of the complications of the subject.

The coupling constant of general relativity, κ, has mass dimension $[\kappa] = -1$. The Planch mass is defined as

$$M_p \equiv \frac{1}{\kappa} \tag{5.1}$$

If we assume that the fundamental symmetry of gravity is Diff(M) invariance, then according to the Wilsonian wisdom the most general lagrangian that includes gravitation assuming that new physics appears at a scale Λ can be written as:

$$\begin{aligned} S = \int d^4x\sqrt{g}\Big(&c_0\Lambda^4 + c_1\Lambda^2\bar{R} + c_2\bar{R}^2 + \frac{c_3}{\Lambda^2}\bar{R}^4 + \ldots \\ &+\frac{1}{2}\bar{\nabla}_\mu\phi\bar{\nabla}^\mu\phi + c_4\bar{R}\phi^2 + \frac{c_5}{\Lambda^2}\bar{R}^2\phi^2 + \frac{c_6}{\Lambda^2}\bar{R}^{\mu\nu}\bar{\nabla}_\mu\phi\bar{\nabla}_\nu\phi + \ldots\Big) \end{aligned} \tag{5.2}$$

where $\bar{R}^n$ represents some trace of the n-th power of the background Riemann tensor, and c_n are dimensionless constants.

Now, experiment tells us that

$$c_1\Lambda^2 = M_p^2 \tag{5.3}$$

so that, barring very small or else very big values for the constant c_0, this means that

$$\Lambda \sim M_p \tag{5.4}$$

which in turn make unavoidable the prediction that the cosmological constant should be of order M_p^4 unless c_0 is finely tuned to 60 decimal places or so.

The experimental fact that the value of the cosmological constant is instead of the same order of magnitude as the Hubble constant

$$\lambda_{obs} \sim H_0 \sim 10^{-60}M_p \tag{5.5}$$

is probably an indication of some subtleties still to be understood in the Wilsonian approach in the presence of gravity.

The contribution of an irrelevant operator of dimension N to a process with characteristic energy scale E is then up to logs, of the order

$$\left(\frac{E}{M_p}\right)^N \tag{5.6}$$

Much effort has been devoted, for example, to analyze theories quadratic in the curvature [?]. If one takes them seriously as fundamental theories, the graviton propagator is quartic in the momenta, schematically

$$\frac{1}{k^4 - M_p^2 k^2} \tag{5.7}$$

This generically improves the ultraviolet behavior (as a matter of fact, some of these theories are renormalizable), but the have problems with unitarity, because the quartic propagator can be written

$$\frac{1}{k^4 - M_p^2 k^2} = -\frac{1}{M_p^2}\left(\frac{1}{k^2} - \frac{1}{k^2 - M_p^2}\right) \tag{5.8}$$

where the residue of the pole of the second term has the wrong sign.

In the static limit this would predict a correction to the Newtonian potential of the form

$$V(r) = -G m_1 m_2 \left(\frac{1}{r} - e^{-M_p r}\frac{1}{r}\right) \tag{5.9}$$

Donoghue [24] claims that or low energies, $\frac{E}{M_p} \rightarrow 0$, the Yukawa piece just has support on the origin, and all we have the right to claim [15] is that

$$V(r) = -G m_1 m_2 \left(\frac{1}{r} + C\delta^{(3)}(\vec{x})\right) \tag{5.10}$$

(where C is a constant which can be determined in a precise manner from the parameters of the low-energy effective lagrangian).

The effective field theory description is fine as long as the energy involved is much smaller than the Planck mass. When this is not the case anymore, all operators in the effective lagrangian are equally important, including the matter content. There is no high energy limit in which gravity is decoupled from the other interactions.

The opposite point of view, namely that it makes sense to quantize just the Einstein-Hilbert action, in isolation with the rest of interactions in the Universe, which is the one held by the Loop Quantum Gravity community is at variance with all this effective lagrangians experience. This is one of the strongest arguments for studying supergravity and superstrings.

5.1 Vierbeins

Let us introduce tetrads (that is, orthonormal frames) following Weyl ([90])

[15]The identity

$$\lim_{\epsilon\rightarrow 0} e^{-\frac{r}{\epsilon}} = 4\pi\epsilon^2 r\delta^{(3)}(\vec{x})$$

is used to that effect.

$$g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu \tag{5.11}$$

Let us show, by working out in detail a two dimensional example, that not all tetrads corresponding to a given metric are related by a Lorentz transformation. The flat metric will be euclidean, i.e.,

$$\delta_{ab} e^a{}_\mu e^b{}_\nu = g_{\mu\nu} \tag{5.12}$$

We shall first determine covariant components in terms of contravariant ones.

$$\begin{aligned} e_{\underline{1}}{}^1 e_{\underline{2}\,1} + e_{\underline{1}}{}^2 e_{\underline{2}2} = 0 \\ e_{\underline{2}}{}^1 e_{\underline{1}1} + e_{\underline{2}}{}^2 e_{\underline{1}2} = 0 \end{aligned} \tag{5.13}$$

This gives

$$\begin{aligned} e_{\underline{2}1} &= -\frac{e_{\underline{1}}{}^2}{e_{\underline{1}}{}^1} e_{\underline{2}2} \\ e_{\underline{1}1} &= -\frac{e_{\underline{2}}{}^2}{e_{\underline{2}}{}^1} e_{\underline{1}2} \end{aligned} \tag{5.14}$$

Let us now impose

$$\begin{aligned} e_{\underline{1}}{}^1 e_{\underline{1}1} + e_{\underline{1}}{}^2 e_{\underline{1}2} = 1 \\ e_{\underline{2}}{}^1 e_{\underline{2}1} + e_{\underline{2}}{}^2 e_{\underline{2}2} = 1 \end{aligned} \tag{5.15}$$

leading to

$$\begin{aligned} e_{\underline{1}2} &= -e e_{\underline{2}}{}^1 \\ e_{\underline{2}2} &= e e_{\underline{1}}{}^1 \\ e_{\underline{1}1} &= e e_{\underline{2}}{}^2 \\ e_{\underline{2}1} &= -e e_{\underline{1}}{}^2 \end{aligned} \tag{5.16}$$

where

$$e^{-1} \equiv \left(e_{\underline{1}}{}^1 e_{\underline{2}}{}^2 - e_{\underline{2}}{}^1 e_{\underline{1}}{}^2 \right) \tag{5.17}$$

which is exactly Eisenhart's result, expressing in a very explicit way covariant components in terms of contravariant ones.

Now we impose that

$$e^\mu_a = g_{\mu\nu} e_a{}^\nu \tag{5.18}$$

This leaves arbitrary the components $e_{\underline{1}}{}^1$ and $e_{\underline{1}}{}^2$, whereas:

$$\begin{aligned} e_{\underline{2}}{}^1 - \frac{1}{e} \left(g_{21} e_{\underline{1}}{}^1 + g_{22} e_{\underline{1}}{}^2 \right) \\ e_{\underline{2}}{}^2 = \frac{1}{e} \left(g_{11} e_{\underline{1}}{}^1 + g_{12} e_{\underline{1}}{}^2 \right) \end{aligned} \tag{5.19}$$

Finally, we still have to impose:

$$
\begin{aligned}
g_{11} &= \left(e_{\underline{1}1}\right)^2 + \left(e_{\underline{2}1}\right)^2 \\
g_{12} &= e_{\underline{1}1} e_{\underline{1}2} + e_{\underline{2}1} e_{\underline{2}2} \\
g_{22} &= \left(e_{\underline{1}2}\right)^2 + \left(e_{\underline{2}2}\right)^2
\end{aligned}
\tag{5.20}
$$

Perhaps surprisingly, the three conditions above are fulfilled provided

$$
g_{11} \left(e_{\underline{1}}{}^1\right)^2 + 2 g_{12} e_{\underline{1}}{}^1 e_{\underline{1}}{}^2 + g_{22} \left(e_{\underline{1}}{}^2\right)^2 = 1 \tag{5.21}
$$

which in turn is satisfied (provided $g_{11} \neq 0$) as long as

$$
e_{\underline{1}}{}^1 = \frac{-g_{12} e_{\underline{1}}{}^2 \pm \sqrt{g_{11} - e^2 \left(e_{\underline{1}}{}^2\right)^2}}{g_{11}} \tag{5.22}
$$

so that we can always choose $e_{\underline{1}}{}^2 = 0$, leading to

$$
\begin{aligned}
e_{\underline{1}}{}^2 &= 0 \\
e_{\underline{1}}{}^1 &= \frac{1}{\sqrt{g_{11}}} \\
e_{\underline{2}}{}^1 &= -\frac{g_{21}}{\sqrt{g_{11}}} \\
e_{\underline{2}}{}^2 &= \sqrt{g_{11}} \\
e_{\underline{2}1} &= 0 \\
e_{\underline{2}2} &= \frac{e}{\sqrt{g_{11}}} \\
e_{\underline{1}1} &= \sqrt{g_{11}} \\
e_{\underline{1}2} &= \frac{g_{21}}{\sqrt{g_{11}}}
\end{aligned}
\tag{5.23}
$$

The general tetrad in its Lorentz orbit is

$$
\begin{aligned}
e_{\underline{1}}{}^1 &= \frac{cosh\,\chi - sinh\,\chi\, g_{21}}{\sqrt{g_{11}}} \\
e_{\underline{1}}{}^2 &= sinh\,\chi\, g_{11} \\
e_{\underline{2}}{}^1 &= \frac{sinh\,\chi - cosh\,\chi\, g_{21}}{\sqrt{g_{11}}} \\
e_{\underline{2}}{}^2 &= cosh\,\chi \sqrt{g_{11}}
\end{aligned}
\tag{5.24}
$$

Another possible choice in (5.22) is

$$
e_{\underline{1}}{}^2 = \frac{\sqrt{g_{11}}}{\sqrt{g}} \tag{5.25}
$$

In order for this solution to be in the orbit, it is necessary that

$$sinh\,\chi = \frac{1}{\sqrt{g}} \tag{5.26}$$

Then

$$e_{\underline{1}}{}^{2} = -\frac{\sqrt{g_{11}}}{\sqrt{g}} g_{12} \tag{5.27}$$

ought to equal

$$\frac{\sqrt{1+g} - g_{21}}{\sqrt{g g_{11}}} \tag{5.28}$$

i.e.

$$g_{11} g_{22} = \sqrt{1+g} - g_{21} \tag{5.29}$$

which can easily be shown to be false, for example, for a diagonal metric

Latin (which we will denote following B. Zumino as flat or Lorentz) indices are raised and lowered using the Minkowski metric η_{ab}, whereas greek (curved or Einstein) indices are raised and lowered using the spacetime metric $g_{\alpha\beta}$. The reason is that if we define the inverse tetrad through

$$e^{a}{}_{\mu} E^{\mu}{}_{b} = \delta^{a}_{b} \tag{5.30}$$

then by multiplying with $e_{a\rho} \equiv \eta_{ac} e^{c}{}_{\rho}$

$$E_{\rho b} \equiv g_{\rho\sigma} E^{\sigma}{}_{b} = e_{b\rho} \tag{5.31}$$

Changing indices in (5.30) leads to:

$$g_{\alpha\beta} e_{a}{}^{\alpha} e_{b}{}^{\beta} = \eta_{ab} \tag{5.32}$$

Tetrads are defined up to a local Lorentz transformation $L \in O(1,3)$:

$$(e')^{a}{}_{\mu} \equiv L^{a}{}_{b}(x) e^{b}{}_{\mu} \tag{5.33}$$

where the defining (*fundamental*) representation is such that

$$\eta_{cd} L^{c}{}_{a}(x) L^{d}{}_{b}(x) = \eta_{ab} \tag{5.34}$$

This implies that

$$L_{da} L^{db} = \delta^{b}_{a} \tag{5.35}$$

that is, defining a matrix $L \equiv (L_{ab})$

$$(L^{T})^{-1} = \eta^{-1} L \eta^{-1} \tag{5.36}$$

For any field that transforms under any representation $D(L)$ of the Lorentz group (the tetrad in particular, that transforms as the vector representation, $(1/2, 1/2)$)

$$\phi \rightarrow \phi' \equiv D(L)\phi \tag{5.37}$$

a *covariant derivative* $\nabla_\mu \equiv \partial_\mu + \omega_\mu$ can be defined such that

$$(\nabla_\mu \phi)' = D(L)\nabla_\mu \phi \tag{5.38}$$

This condition implies that the Lorentz (or spin) connection transforms as:

$$\omega'_\mu = D_L \omega_\mu D_L^{-1} - \partial_\mu D_L D_L^{-1} \tag{5.39}$$

The spin conection thus defined is endowed with indices on the representation D:

$$\omega_\mu \equiv \omega^u_{\mu\ v} = \omega^{ab}_\mu D(T_{ab})^u{}_v \tag{5.40}$$

Where T_{ab} is a local basis of the Lie algebra of the Lorentz group where we label the six generators of $SO(1,3)$ by two antisymmetric four-dimensional indices: $T^{ab} = -T^{ba}$, In this basis the commutators read

$$[T_{cd}, T_{ab}] = i\eta_{da}T_{cb} - i\eta_{ac}T_{db} - i\eta_{db}T_{ca} + i\eta_{cb}T_{da} \tag{5.41}$$

The linearized approximation corresponds to

$$D(L)^u{}_v = \delta^u{}_v + il^{ab}D(T_{ab})^u{}_v \tag{5.42}$$

This yields

$$\begin{aligned}
&(\omega')^{ab}_\mu D(T_{ab})^u{}_v = D^u_{L\ w}\omega^{ab}_\mu D(T_{ab})^{wz} D^L{}_{vz} - \partial_\mu D_L{}^{uw}.D^L{}_{vw} = \\
&\omega^{ab}_\mu D(T_{ab})^u{}_v + il^{cd}D(T_{cd})^u{}_w \omega^{ab}_\mu D(T_{ab})^{wv} + \omega^{ab}_\mu D(T_{ab})^{uz} il^{cd} D(T_{cd})_{vz} \\
&-i\partial_\mu l^{ab} D(T_{ab})^u{}_v
\end{aligned} \tag{5.43}$$

which using the commutator algebra as well as the antisymmetry of l_{ab} and ω^{ab}_μ in the Lorentz indices can be reduced to

$$\delta\omega^{ab}_\mu \equiv (\omega')^{ab}_\mu - \omega^{ab}_\mu = 4l_c{}^a\omega^{cb}_\mu - i\partial_\mu l^{ab} \tag{5.44}$$

In the fundamental (vector) representation the finite form reads

$$\{(x^a)' = L^{ab}x_c\} \Rightarrow \{x^a = x'_b L^{ba}\} \tag{5.45}$$

so that, in an obvious notation,

$$(\omega_{ab})'_\mu = L_{ac}\omega^{cd}_\mu L_{da} - \partial_\mu L_{ac} L_b{}^c \tag{5.46}$$

In terms of the one-forms

$$\omega^{ab} \equiv \omega^{ab}_\mu dx^\mu \tag{5.47}$$

the transformation rule of the Lorentz connection reads

$$\omega' = L\omega L^{-1} - \partial_\mu L L^{-1} \tag{5.48}$$

Einstein and Lorentz indices convey different symmetries in general. In the particular case when we demand that Lorentz and Einstein indices are fully equivalent, we are assuming the *tetrad postulate*, namely that the doubly covariant derivative of the tetrad vanishes

$$\partial_\nu e^a{}_\mu + (\omega_\nu)^a{}_b e^b{}_\mu - \Gamma^\lambda_{\nu\mu} e^a{}_\lambda = 0 \tag{5.49}$$

This fully determines the Lorentz connection in terms of the Christoffel symbols:

$$(\omega_\nu)^a{}_c = -e_c{}^\mu \partial_\nu e^a{}_\mu + \Gamma^\lambda_{\nu\mu} e_c{}^\mu e^a{}_\lambda \tag{5.50}$$

This connection is torsionless:

$$T \equiv de + \omega \wedge e = 0 \tag{5.51}$$

The field strength (curvature tensor) is a two-form given by

$$F \equiv d\omega + \omega \wedge \omega \tag{5.52}$$

that is

$$F^{ab} = d\omega^{ab} + \omega^{ac} \wedge \omega^b_c \tag{5.53}$$

It so happens that the Riemann-Christoffel tensor is given by

$$R_{\mu\nu\rho\sigma} = F^{ab}_{\mu\nu} e_{a\rho} e_{b\sigma} \tag{5.54}$$

5.2 The background field technique

The usual definition of effective action is

$$W \equiv -log \int \mathcal{D}\phi \, e^{-S[\phi]-\int J\phi} \tag{5.55}$$

We define

$$\phi_c = \langle \phi \rangle_J \equiv \frac{\delta W}{\delta J} \tag{5.56}$$

(please remark that $[\phi_c] = 1$ because $\frac{\delta J(x)}{\delta J(y)} = \delta^{(n)}(x-y)$) and perform a Legendre transform in order to reach the efective action

$$\Gamma(\phi_c) = W(J) - \int J\phi_c \tag{5.57}$$

in such a way that

$$J = -\frac{\delta \Gamma}{\delta \phi_c} \tag{5.58}$$

which conveys the fact that the action is stationary in the absence of sources.

In the background field technique we define

$$W_B(J, \bar{\phi}) \equiv -log \int \mathcal{D}\phi \, e^{-S[\bar{\phi}+\phi]-\int J\phi} \tag{5.59}$$

Following 't Hooft [78] (*confer* [1]) we only introduce sources for the quantum fields

Shifting integration variables in the functional integral, this means that

$$W_B(J, \bar{\phi}) = W(J) - \int J\bar{\phi} \tag{5.60}$$

The classical field is now

$$\phi_c^B = \frac{\delta W_B}{\delta J} = \phi_c - \bar{\phi} \tag{5.61}$$

and the effective action

$$\Gamma_B(\phi_c^B, \bar{\phi}) = W_B(J, \bar{\phi}) - \int J\phi_c^B = W - \int J\bar{\phi} - \int J(\phi_c - \bar{\phi}) = \Gamma(\phi_c) \tag{5.62}$$

To say it otherwise,

$$\Gamma_B(\phi_c^B, \bar{\phi}) = \Gamma(\phi_c = \bar{\phi} + \phi_c^B) \tag{5.63}$$

or else,

$$\Gamma(\phi_c) = \Gamma_B(\phi_c - \bar{\phi}, \bar{\phi}) \tag{5.64}$$

That is that we are usually interested in

$$\Gamma_B(\phi_c^B = 0, \bar{\phi}) = \Gamma(\phi_c = \bar{\phi}) \tag{5.65}$$

Let us write the scalar euclidean action as

$$S=\int d^n x \frac{1}{2}\phi_x\left(-\Box+m^2\right)\phi_x+V(\phi) \tag{5.66}$$

In the one loop approximation,

$$W_B^{(1)}=\bar{S}+\frac{1}{2}log\, det\, \delta^2 S(\bar{\phi})-\frac{1}{2}\int J\left(\delta^{\bar{2}} S(\bar{\phi})\right)^{-1} J \tag{5.67}$$

and

$$\Gamma_B^{(1)}=\bar{S}+\frac{1}{2}log\, det\, \delta^2 S(\bar{\phi})+\frac{1}{2}\int \phi_c^B\left(\delta^{\bar{2}} S(\bar{\phi})\right)^{-1} \phi_c^B \tag{5.68}$$

The equation of motion for the background field is

$$\left(-\Box+m^2+V'(\bar{\phi})\right)\bar{\phi}=0 \tag{5.69}$$

The one loop operator is given by

$$\delta^2 S(\bar{\phi})=-\Box+m^2+V''(\bar{\phi}) \tag{5.70}$$

All this is to be compared with Weinberg's formula ([89] Vol.II,p.68) for the ordinary (not background) efective action:

$$e^{-\Gamma(\bar{\phi})}\equiv\int_{1PI}\mathcal{D}\phi\, e^{-S(\bar{\phi}+\phi)} \tag{5.71}$$

Another important property of the effctive action is that the full quantum equations of motion (Schwinger-Dyson) are equivalent to the *classical* (tree approximation) of the equations of motion of the effective action, *id est*

$$\frac{\delta\Gamma(\phi_c)}{\delta\phi_c}=0 \tag{5.72}$$

In the one-loop approximation, the effective action is then given by:

$$\Gamma^{(1)}(\bar{\phi})=\bar{S}(\bar{\phi})+\frac{1}{2}log\, det\,\left(-\Box+m^2+V''(\bar{\phi})\right) \tag{5.73}$$

Our purpose in life is to compute the effective action. The determinant can be computed with the help of the zeta function, and this in turn in terms of the heat kernel $K(\tau)\equiv e^{-\tau\bar{\Delta}^{-1}}$

$$\Gamma\equiv-\frac{1}{2}\frac{d}{ds}\frac{1}{\Gamma(s)}\int_0^\infty d\tau\,\tau^{s-1}\,tr\int d^n x\, K(\tau|x,x)\bigg|_{s=0} \tag{5.74}$$

The heat kernel itself can be obtained through the Barvinsky-Vilkovisky expansion which, unlike the Schwinger-de Witt one, is uniform in proper time (*confer* the book [65]). We start from the exact solution of the free case in flat space,

$$K_0^m(\tau|x,y)\equiv\langle x|e^{-\tau(-\Box+m^2)}|y\rangle=\int\frac{d^np}{(2\pi)^n}d\tau e^{-\tau(p^2+m^2)+ip(x-y)}=\frac{1}{(4\pi)^{n/2}}e^{-\frac{(x-y)^2}{2\tau}-m^2\tau} \tag{5.75}$$

and make the ansatz

$$K=K_0+K_0Q \tag{5.76}$$

The heat equation

$$\frac{d}{d\tau}K=-(-\Box-m^2-\bar{M}^2)K \tag{5.77}$$

(where

$$\bar{M}^2\equiv V''(\bar{\phi})\big) \tag{5.78}$$

eventually leads to

$$K_0\frac{dQ}{d\tau}=-\bar{M}^2K_0 \tag{5.79}$$

whose solution is

$$Q(\tau)=-\int_0^\tau dsK_0^{-1}(s)\bar{M}^2K_0(s) \tag{5.80}$$

This means that

$$K_1(\tau)=-K_0(\tau)\int_0^\tau dsK_0^{-1}(s)\bar{M}^2K_0(s)=-\int_0^\tau dsK_0(t-s)\bar{M}^2K_0(s) \tag{5.81}$$

Let us be explicit

$$\begin{aligned}&K_1(\tau|x,y)=-\int d^nu\int_0^\tau dsK_0(t-s|x,u)\bar{M}^2(u)K_0(s|u,y)\equiv\\&\int d^nu\bar{M}^2(u)\mathcal{K}_1(\tau|x,y;u)=\int d^nk\bar{M}^2(k)\mathcal{K}_1(\tau|x,y;k)\end{aligned} \tag{5.82}$$

with

$$\begin{aligned}&\mathcal{K}_1(\tau|x,y)\equiv-\int_0^\tau dsK_0(\tau-s|x,u)K_0(s|u,y)=\\&-\int_0^\tau ds\frac{d^np}{(2\pi)^n}\frac{d^nk}{(2\pi)^n}e^{-(\tau-s)(p^2+m^2)+ip(x-u)}e^{-s(k^2+m^2)+ik(u-y)}=\\&\frac{d^np}{(2\pi)^n}\frac{d^nk}{(2\pi)^n}\frac{1-e^{\tau(p^2-k^2)}}{p^2-k^2}e^{-\tau(p^2+m^2)+ip(x-u)+ik(u-y)}\end{aligned} \tag{5.83}$$

This yields

$$\mathcal{K}_1(\tau|x,y;q)=\int\frac{d^nk}{(2\pi)^n}\frac{1-e^{\tau((k+q)^2-k^2)}}{(k+q)^2-k^2}e^{-\tau((k+q)^2+m^2)+ix(k+q)-iky} \tag{5.84}$$

so that

$$\int d^n x \mathcal{K}_1(\tau|x,x;q) = \int \frac{d^n k}{(2\pi)^n}(-\tau)e^{-\tau(k^2+m^2)}\delta^{(n)}(q) \tag{5.85}$$

Only the constant zero mode of $\bar{M}^2$ contributes to first order.

Putting all this together leads eventually to

$$\begin{aligned}
&\Gamma(\bar{M}^2) = -\frac{1}{2}\frac{d}{ds}\frac{1}{\Gamma(s)}\int_0^\infty d\tau\,\tau^{s-1}\,tr\int\frac{d^n k}{(2\pi)^n}(-\tau)e^{-\tau(k^2+m^2)}\bar{M}^2(q=0)\Big|_{s=0} = \\
&-\frac{1}{2}\int\frac{d^n k}{(2\pi)^n}\frac{d}{ds}\frac{1}{\Gamma(s)}\frac{-\Gamma(s+1)}{(k^2+m^2)^{1+s}}\bar{M}^2(q=0)\Big|_{s=0} = \\
&-\frac{1}{2}\int\frac{d^n k}{(2\pi)^n}\frac{d}{ds}\frac{-s}{(k^2+m^2)^{1+s}}\bar{M}^2(q=0)\Big|_{s=0} = \\
&-\frac{1}{2}\int\frac{d^n k}{(2\pi)^n}\frac{-1}{k^2+m^2}\bar{M}^2(q=0) = 0
\end{aligned} \tag{5.86}$$

where the last integral has been evaluated in dimensional regularization.

xhe recurrence relationship reads:

$$K_n(\tau) = K_0(\tau)\int_0^\tau ds K_0(s)\bar{M}^2 K_{n-1}(s) \tag{5.87}$$

As a matter of fact

$$\begin{aligned}
&log\,det\,(-\Box+m^2+\bar{M}^2) = tr\,log\,(-\Box+m^2)\left(1+(-\Box+m^2)^{-1}\bar{M}^2\right) = \\
&tr\,log\,(-\Box+m^2) + tr\sum_{m=1}^\infty\frac{(-1)^{m+1}}{m}\left((-\Box+m^2)^{-1}\bar{M}^2\right)^m = \\
&C+\sum_{m=1}^\infty\frac{(-1)^{m+1}}{m}\int_{x_1\ldots x_m}(-\Box+m^2)^{-1}_{x_1x_2}\bar{M}^2_{x_2}(-\Box+m^2)^{-1}_{x_2x_3}\bar{M}^2_{x_3}\ldots(-\Box+m^2)^{-1}_{x_mx_1}\bar{M}^2_{x_1} = \\
&C+\sum_{m=1}^\infty\frac{(-1)^{m+1}}{m}\int_{x_1\ldots x_m}\int_{p_1\ldots p_{2m}}e^{ip_1(x_1-x_2)}e^{ip_2x_2}\frac{\bar{M}^2_{p_2}}{p_1^2+m^2}\ldots e^{ip_{2m-1}(x_m-x_1)}e^{ip_{2m}x_1}\frac{\bar{M}^2_{p_{2m}}}{p_{2m-1}^2+m^2} = \\
&C+\sum_{m=1}^\infty\frac{(-1)^{m+1}}{m}\int_{p_1\ldots p_{2m}}\delta(p_1+p_{2m-1}-p_{2m})\delta(-p_1+p_2+p_3)\ldots\frac{\bar{M}^2_{p_2}}{p_1^2+m^2}\cdots\frac{\bar{M}^2_{p_{2m}}}{p_{2m-1}^2+m^2} = \\
&C+\sum_{m=1}^\infty\frac{(-1)^{m+1}}{m}\int_{p_2\ldots p_{2m}}\delta(p_2+p_4+\ldots+p_{2m})\bar{M}_{p_2}\ldots\bar{M}_{p_{2m}}\mathcal{D}^{(m)}(p_2\ldots p_{2m})
\end{aligned} \tag{5.88}$$

where

$$\begin{aligned}
&(-\Box+m^2)^{-1}_{xy} \equiv \int\frac{d^n p}{(2\pi)^n}e^{ip(x-y)}\frac{1}{p^2+m^2} \\
&\bar{M}^2_x \equiv \int\frac{d^n p}{(2\pi)^n}e^{ipx}\bar{M}^2_p
\end{aligned} \tag{5.89}$$

The nontrivial piece of the determinant is

$$\mathcal{D}^{(m)}(p_2 \ldots p_{2m}) \equiv \int_{p_1 \ldots p_{2m-1}} \delta(p_1+p_{2m-1}-p_{2m})\delta(-p_1+p_2+p_3)\ldots \frac{1}{p_1^2+m^2}\cdots\frac{1}{p_{2m-1}^2+m^2} \tag{5.90}$$

There are m Dirac deltas, of which $m-1$ are efficient in killing a momentum integration. Given the fact that there were previously m of those, there is one momentum integration left, that is, all those diagrams are one-loop ones.

The final expression for $\mathcal{D}^{(m)}$ is

$$\mathcal{D}^{(m)}(p_2 \ldots p_{2m}) \equiv \int \frac{d^n p}{(2\pi)^n}\frac{1}{p^2+m^2}\frac{1}{(p-p_2)^2+m^2}\cdots\frac{1}{(p-p_2-p_4-\ldots-p_{2m-2})^2+m^2} \tag{5.91}$$

In $d=4$ dimensions, the first two terms are divergent (although the term $m=1$ is taken to be zero in dimensional regularization), and the rest are given by finite integrals.

The *effective potential* corresponds to the coefficient to the zero mode,i.e.

$$\bar{M}_p^2 = (2\pi)^n\,\delta^{(n)}(p)\,\bar{M}^2\left(\bar{\phi}\right) \tag{5.92}$$

We have

$$\begin{aligned} V_{eff} &= C + \sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}\left((2\pi)^n\,\bar{M}\right)^m \int \frac{d^n p}{(2\pi)^n}\left(\frac{1}{p^2+m^2}\right)^m = \\ & C + \int \frac{d^n p}{(2\pi)^n}\, log\left(1+(2\pi)^n\,\bar{M}\frac{1}{p^2+m^2}\right) \end{aligned} \tag{5.93}$$

We have

$$\begin{aligned} V_{eff} &= C + \sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}\left((2\pi)^n\,\bar{M}\right)^m \int \frac{d^n p}{(2\pi)^n}\left(\frac{1}{p^2+m^2}\right)^m = \\ & C + \int \frac{d^n p}{(2\pi)^n}\, log\left(1+(2\pi)^n\,\bar{M}^2\frac{1}{p^2+m^2}\right) \end{aligned} \tag{5.94}$$

This is similar to the formula by Iliopoulos et al [?]. At any rate, it is much easier to use the zeta-function approach to get, in four dimensions:

$$V_{eff} = \frac{1}{2}m^2\bar{\phi}^2 + V\left(\bar{\phi}\right) + \frac{\left(m^2+\bar{M}(\bar{\phi})^2\right)^2}{64\pi^2}\left(log\,\frac{m^2+\bar{M}(\bar{\phi})^2}{\mu^2}-3/2\right) \tag{5.95}$$

If we follow Coleman and Weinberg and define the coupling constant in the massless ϕ^4_4 theory as

$$\lambda \equiv \left.\frac{d^4 V_{eff}(\bar{\phi})}{d\bar{\phi}^4}\right|_{\bar{\phi}=M} \tag{5.96}$$

we get [18]

$$V_{eff} = \frac{1}{2}m^2\bar{\phi}^2 + \lambda\frac{\bar{\phi}^4}{24} + \frac{\lambda^2\bar{\phi}^2}{256\pi^2}\left(log\,\frac{\bar{\phi}^2}{M^2}-25/6\right) \tag{5.97}$$

6. The ζ-function approach

Given an operator M such that

$$M\phi_n = \lambda_n \phi_n \tag{6.1}$$

we define by analogy with Riemann's ζ-function

$$\zeta_R(s) \equiv \sum_{n=1}^{\infty} n^{-s} \tag{6.2}$$

(which can be analytically continued so that

$$\left.\begin{aligned} \zeta_R(0) &= -\frac{1}{2} \\ \left.\frac{d\zeta_R}{ds}\right|_{s=0} &= -\frac{1}{2} log\, 2\pi \end{aligned}\right) \tag{6.3}$$

the ζ-function associated with the operator M, namely,

$$\zeta(s) \equiv \sum_{n=0}^{\infty} \lambda_n^{-s} \tag{6.4}$$

so that

$$log\, det\, M \equiv -\left.\frac{d\zeta}{ds}\right|_{s=0} \tag{6.5}$$

It is also useful to define the *heat kernel* operator

$$K(\tau) \equiv e^{-\tau M} \equiv \sum_n e^{-\lambda_n \tau} |\phi_n\rangle\langle\phi_n| \tag{6.6}$$

in such a way that

$$tr\, K(\tau) = \sum_n e^{-\lambda_n \tau} \tag{6.7}$$

It is a fact of life that

$$\frac{1}{\Gamma(s)} \int_0^\infty d\tau\, \tau^{s-1}\, tr\, K(\tau) = \sum_{n=0}^{\infty} \lambda_n^{-s} \equiv \zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty d\tau\, \tau^{s-1}\, tr \int \sqrt{|g|} d^n x\, K(x,x;\tau) \tag{6.8}$$

On the other hand, it is formally true at the operator level, that

$$\frac{\partial K}{\partial \tau} = -M\, K(\tau) \tag{6.9}$$

which is a heat equation of sorts. This can be written as

$$\frac{\partial K(x,x',\tau)}{\partial \tau} = -\int d^n z\, \langle x|M|z\rangle\, K(z,x',\tau) \tag{6.10}$$

The fact that at the operator level $K(\tau = 0) = 1$ means that

$$K(x, y, \tau = 0) = \delta^n(x - y) \tag{6.11}$$

For the Laplace operator in flat space, which is the starting point in all perturbative calculations,

$$\mu^2 M = -\sum_{i=1}^{n}\left(\frac{\partial}{\partial x^i}\right)^2 + m^2 \tag{6.12}$$

We have introduced an arbitrary mass parameter, μ, to make the eigenvalues dimensionless. One finds

$$K(x, y; \tau) = \mu^n\,(4\pi\tau)^{-n/2}\,e^{-\frac{\mu^2(x-y)^2}{4\tau}-\frac{m^2}{\mu^2}\tau} \tag{6.13}$$

This leads inmediatly to

$$\zeta(s) = \mu^n V\left(\frac{m^2}{4\pi\mu^2}\right)^{n/2-s}\frac{\Gamma(s-n/2)}{\Gamma(s)} = \mu^n V\left(\frac{m^2}{4\pi\mu^2}\right)^{n/2-s}\frac{1}{(s-1)(s-2)\ldots(s-n/2)} \tag{6.14}$$

where

$$V \equiv \int d^n x \tag{6.15}$$

and we have assumed that $n \in 2\mathbb{Z}$. The corresponding derivative is then

$$\frac{d\zeta(s)}{ds} = (4\pi)^{-n/2}\frac{V m^n}{(s-1)(s-2)\ldots(s-n/2)}\left(-log\,\frac{m^2}{\mu^2} - \frac{1}{s-n/2} - \frac{1}{s-(n/2-1)} - \ldots - \frac{1}{s-1}\right) \tag{6.16}$$

This means that for any even dimension,

$$\frac{1}{2}log\,det\,M = -\frac{1}{2}\left.\frac{d\zeta(s)}{ds}\right|_{s=0} = (4\pi)^{-n/2}\frac{V m^n}{(n/2)!}\left(log\,\frac{m^2}{\mu^2} - \left(1 + \frac{1}{2} + \ldots + \frac{1}{n/2}\right)\right) \tag{6.17}$$

In $n = 4$ dimensions, in particular, this yields

$$\frac{1}{2}log\,det\,M = \frac{V m^4}{32\pi^2}\left(log\,\frac{m^2}{\mu^2} - 3/2\right) \tag{6.18}$$

Al desarrollar

$$W^a_\mu = \bar{A}^a_\mu + A^a_\mu \tag{6.19}$$

el tensor campo reza:

$$\begin{aligned} F^a_{\mu\nu} = \bar{F}^a_{\mu\nu} + \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + c_{abc}(\bar{A}^b_\mu A^c_\nu + \bar{A}^c_\nu A^b_\nu) + g c_{abc} A^b_\mu A^c_\nu = \\ \bar{F}^a_{\mu\nu} + \bar{\nabla}_\mu A^a_\nu - \bar{\nabla}_\nu A^a_\mu + g c_{abc} A^b_\mu A^c_\nu \end{aligned} \tag{6.20}$$

con lo que teniendo en cuenta que la integral de una derivada covariante total se anula, así como el uso de la fórmula (??) para reducir (bajo la integral) el término

$$(\bar{\nabla}_\mu A_\nu)_a (\bar{\nabla}^\nu A^\mu)_a = (\bar{\nabla}_\nu A_\nu)^2 - g c_{abc} \bar{F}^a_{\mu\nu} A^b_\mu A^c_\nu \tag{6.21}$$

la acción de Yang Mills se puede escribir

$$\begin{aligned} S = -\frac{1}{4g^2} \int d^4x \left[\bar{F}^a_{\mu\nu} + \bar{\nabla}_\mu A^a_\nu - \bar{\nabla}_\nu A^a_\mu + g c_{abc} A^b_\mu A^c_\nu\right]^2 = \\ [(\bar{F}^a_{\mu\nu})^2 + 2(\bar{\nabla}_\mu A^a_\nu)^2 - 2(\bar{\nabla}_\mu A^\mu_a)^2 + \\ 4g c_{abc} \bar{F}^a_{\mu\nu} A^b_\mu A^c_\nu + 2g \bar{\nabla}_\mu A^a_\nu c_{abc} A^{b\,\mu} A^{c\,\nu} + g^2 c_{abc} c_{auv} A^b_\mu A^c_\nu A^{u\,\mu} A^{v\,\nu}] \end{aligned} \tag{6.22}$$

Para cálculos a un lazo es suficiente con considerar los términos de segundo orden en las fluctuaciones cuánticas:

$$S = -\frac{1}{4g^2} \int d^4x \left[(\bar{F}^a_{\mu\nu})^2 + 2(\bar{\nabla}_\mu A^a_\nu)^2 - 2(\bar{\nabla}_\mu A^\mu_a)^2 + 4g c_{abc} \bar{F}^a_{\mu\nu} A^b_\mu A^c_\nu\right] \tag{6.23}$$

Escogeremos como término que viola la simetría (??)

$$L_{gf} = -\frac{1}{2\alpha} (\bar{\nabla}_\mu A^\mu_a)^2 \equiv -F^2_a \tag{6.24}$$

de forma que

$$\frac{\delta F_a}{\delta \omega_b} = \frac{1}{g\sqrt{2}} \bar{\nabla}_\mu \nabla^\mu \delta_{ab} \tag{6.25}$$

y el término adicional de los fantasmas reza:

$$L_{gh} = -\frac{1}{g\sqrt{2}} \bar{\nabla}_\mu \bar{c}^a \nabla^\mu c_a \tag{6.26}$$

Hasta orden un lazo, los resultados en CF se pueden expresar en términos de un determinante que, sin embargo, es difícil de calcular en general (más tarde hablaremos más de este tema). La manera de trabajar con CF a más de un lazo consiste en completar el desarrollo de la acción, y calcular diagramas 1PI con patas externas terminando en los CF.Una vez calculadas las funciones de Green para CF general, se puede escoger un gauge para el CF, que no tiene por qué coincidir con el gauge usado para la integración sobre los campos cuánticos. De hecho este último paso sólo es necesario para poder definir propagadores del CF, que a su vez se usan para conectar los trozos 1PI, y constuir de esta manera las funciones de Green conexas, que determinan la matriz S usando LSZ. Los detalles de estos cálculos se pueden encontrar en el curso de Abbott en Cracow en 1981 ([**?**]).

De esta forma el propagador gauge será en términos del campo de fondo (CF),

$$\Delta^{ab}_{\mu\nu} = -i\delta_{ab}[\frac{\eta_{\mu\nu}}{p^2+i\epsilon} + (\alpha-1)\frac{p_\mu p_\nu}{(p^2+i\epsilon)^2}] \tag{6.27}$$

Y el propagador de los fantasmas:

$$D_{ab} = i\delta_{ab}\frac{1}{p^2+i\epsilon} \tag{6.28}$$

El vértice gauge/dos fantasmas será

$$V_{\bar{c}c\bar{A}} = gC_{abc}(p+q)^\mu \tag{6.29}$$

cuando el boson gauge sea de fondo, y

$$V_{\bar{c}cA} = gC_{abc}p^\mu \tag{6.30}$$

cuando el bosón gauge sea cuántico.

El vértice a tres gluones cuánticos:

$$V_{AAA} = gC_{abc}[\eta_{\mu\lambda}(p-r)_\nu + \eta_{n\lambda}(r-q)_\mu + \eta_{\mu\nu}(q-p)_\lambda] \tag{6.31}$$

Y a dos gluones cuánticos y uno de fondo

$$V_{\bar{A}AA} = gC_{abc}[\eta\mu\lambda(p-r-\frac{1}{\alpha}q)_\nu + \eta_{n\lambda}(r-q)_\mu + \eta_{\mu\nu}(q-p+\frac{1}{\alpha}r)_\lambda] \tag{6.32}$$

El vértice con dos fantasmas, un bosón gauge de fondo y un boson gauge cuántico es

$$V_{cc\bar{A}A} = -ig^2 C_{ace}C_{edb}\eta_{\mu\nu} \tag{6.33}$$

Y si los dos bosones gauge son de fondo:

$$V_{cc\bar{A}\bar{A}} = -ig^2[C_{ace}C_{edb} - C_{ade}C_{ecb}]\eta_{\mu\nu} \tag{6.34}$$

Y finalmente el vértice a cuatro gluones cuánticos, o bien dos cuánticos y dos de fondo (que son idénticos), será:

$$
\begin{aligned}
&V_{AAAA} = -ig^2[C_{abe}C_{ecd}(\eta_{\mu\lambda}\eta_{\nu\rho} - \eta_{\mu\rho}\eta_{\nu\lambda}) \\
&+C_{ade}C_{ebc}(\eta_{\mu\nu}\eta_{\lambda\rho} - \eta_{\mu\lambda}\eta_{\nu\rho}) \\
&+C_{ace}C_{ebd}(\eta_{\mu\nu}\eta_{\lambda\rho} - \eta_{\mu\rho}\eta_{\nu\lambda})]
\end{aligned} \tag{6.35}
$$

Mientras que si dos de los bosones gauge son de fonde el resultado es:

$$
\begin{aligned}
&V_{\bar{A}\bar{A}AA} = -ig^2[C_{abe}C_{ecd}(\eta_{\mu\lambda}\eta_{\nu\rho} - \eta_{\mu\rho}\eta_{\nu\lambda} + \frac{1}{\alpha}\eta_{\mu\nu}\eta_{\lambda\rho}) \\
&+C_{ade}C_{ebc}(\eta_{\mu\nu}\eta_{\lambda\rho} - \eta_{\mu\lambda}\eta_{\nu\rho} - \frac{1}{\alpha}\eta_{\mu\rho}\eta_{\nu\lambda}) \\
&+C_{ace}C_{ebd}(\eta_{\mu\nu}\eta_{\lambda\rho} - \eta_{\mu\rho}\eta_{\nu\lambda})]
\end{aligned} \tag{6.36}
$$

La acción efectiva CF será:

$$
\Gamma_{CF}(\bar{A}, < A >_{CF}) \equiv W_{CF}(\bar{A}.A) - \int J. < A >_{CF} \tag{6.37}
$$

donde

$$
< A >_{CF} = \frac{\delta W_{CF}}{\delta J} \tag{6.38}
$$

y es en general diferente del campo de fondo, $\bar{A}$.

- Ejercicio. Demostrar que

$$
Z_{CF}(\bar{A}, J) = Z(J)e^{-i\int J.\bar{A}} \tag{6.39}
$$

- Solución. Basta con efectuar una traslacion de la variable de integración en la integral de camino.

- Demostrar también que

$$
< A >_{CF} = < A > -\bar{A} \tag{6.40}
$$

Usando los resultados del ejercicio anterior se demustra la relación básica

$$
\Gamma_{CF}(\bar{A}, < A >_{CF}) = \Gamma(\bar{A} + < A >_{CF}) \tag{6.41}
$$

de donde se sigue una expresión para la acción efectiva ordeinaria en términos de la acción efectiva CF:

$$
\Gamma_{CF}(\bar{A}, < A >_{CF} = 0) = \Gamma(\bar{A}) \tag{6.42}
$$

La contribución de los campos gauge se puede empaquetar de la forma (5.22) con

$$(W^\mu)_{ab\rho\sigma} = -(\bar{A}^{\mu(adj)})_{ab}\eta_{\rho\sigma} \tag{6.43}$$

de forma que

$$\phi = -2\bar{F}^{(adj)} \tag{6.44}$$

y lo que llamábamos

$$F = -\bar{F}_{\mu\nu}\eta_{\rho\sigma} \tag{6.45}$$

Esto quiere decir que el contratérmino proveniente de los gampos gauge es

$$\Delta L = \frac{1}{8\pi^2\epsilon}(-4\frac{C_G}{4} + \frac{C_G}{24})\bar{F}^a_{\mu\nu}\bar{F}^{\mu\nu}_a \tag{6.46}$$

donde el factor -1 viene de la $tr\phi^2$ y el factor $1/24$ de la trF^2. En total queda

$$\Delta L = \frac{5}{6}\frac{1}{8\pi^2\epsilon}C_G\bar{F}^a_{\mu\nu}\bar{F}^{\mu\nu}_a \tag{6.47}$$

La contribución de los fantasmas corresponde a campos complejos, con:

$$\mathcal{Y} = \bar{F}^{(adj)} \tag{6.48}$$

de forma que

$$\Delta L_{gh} = \frac{1}{12}\frac{1}{8\pi^2\epsilon}C_G\bar{F}^a_{\mu\nu}\bar{F}^{\mu\nu}_a \tag{6.49}$$

te resultado implica que

$$\frac{1}{4g_B^2} = \frac{1}{4g_R^2} + \frac{11}{12}\frac{C_G}{8\pi^2\epsilon} \tag{6.50}$$

o lo que es lo mismo,

$$g_B = g_R(1 - \frac{11C_G g_R^2}{48\pi^2\epsilon}) \tag{6.51}$$

Aparentemente

$$\beta = (g_R\frac{d}{dg_R} - 1)a_1 = -\frac{11C_G}{24\pi^2}g_R^3 \tag{6.52}$$

Sin embargo el valor exacto es exactamente la mitad, ya que la constante de Yang-Mills

$$g \sim \mu^{(4-n)/2} \equiv (\mu')^{n-4} \tag{6.53}$$

de forma que

$$\frac{d\mu}{\mu} = 2\frac{d\mu'}{\mu'} \tag{6.54}$$

y

$$\beta(g) = -\frac{11C_G}{48\pi^2}g^3 \tag{6.55}$$

Este resultado implica que los gluones se comportan como partículas libres a muy cortas distancias. La contrapartida de este fenómeno es que a largas distancias, en el límite infrarrojo, la teoría está fuertemente acoplada, y se cree que los estados asintóticos no son los que aparecen en el lagrangiano libre, sino estados ligados de gluones, en combinaciones singletes frente al grupo gauge (que llamaremos genéricamente*gluebolas*). Ninguno de estos estados tiene masa cero, además. Exiiste una energía mínima necesaria para producir estas bolas.La teoría es difícil de estudiar analíticamente en este régimen, al que de momento sólo se tiene acceso mediante el estudio en el retículo.

6.1 Efficient computation of determinants

Como hemos visto repetidamente, el cálculo de la acción efectiva a un lazo es equivalente al cómputo de un determinante de un cierto operador. Exiisten maneras eficientes de efectuar esos cálculos, utilizando técnicas introducidas por Schwinger y de Witt ([20]).

Consideremos la integral ordinaria

$$I(\lambda) \equiv \int_0^\infty \frac{dx}{x} e^{-ix\lambda} \tag{6.56}$$

Puede parecer a primera vista que la integral es independiente de λ, ya que si hacemos el cambio $z \equiv x\lambda$ desaparece toda dependencia expl'icita con λ. Esto es una ilusión, sin embargo, ya que al ser la integral divergente en el l´ı mite inferior tenemos que definirla mediante un proceso l´ı mite; por ejemplo

$$I(\lambda) \equiv \lim_{\epsilon \to 0} I(\epsilon, \lambda) \equiv \lim_{\epsilon \to 0} \int_\epsilon^\infty \frac{dx}{x} e^{-ix\lambda} \tag{6.57}$$

Ahora es fácil de ver que

$$\lim_{\epsilon \to 0} \frac{\partial I(\epsilon, \lambda)}{\partial \lambda} = -\frac{1}{\lambda} \tag{6.58}$$

Esto demuestra que

$$I(\lambda) = -\log \lambda + C \tag{6.59}$$

donde C es una constante que resulta ser divergente.

Ahora bien, si tuviéramos un operador Δ diagonalizable con autovalores discretos, $\{\lambda_n\}$ entonces es claro que

$$\log \det \Delta = tr \log \Delta \equiv \sum_n \log \lambda_n \tag{6.60}$$

El procedimiento matemático para definir el determinante de un operador consiste casi siempre en efectuar continuación anal´ı tica a partir de una situación donde el espectro es de este estilo (por ejemplo, el espacio eucl´ı deo con condiciones de contorno periódicas).

Nos vemos entonces conducidos a la definición

$$\log \det \Delta \equiv -\int \frac{d\tau}{\tau} tr e^{-i\tau\Delta} \tag{6.61}$$

Fijémonos en el operador

$$K(\tau) \equiv e^{-i\tau\Delta} \tag{6.62}$$

En la práctica trabajaremos con una representación definida por lo que vulgarmente se conoce como *ecuación del calor* (aunque es realmente una continuación anal´ı tica de ella) y que supondremos definida en dimensión arbitraria, n:

$$(i\frac{\partial}{\partial \tau} - \Delta_x) K(x, y; \tau) = 0 \tag{6.63}$$

ecuación que resolveremos con la condición inicial

$$K(x, y, 0) = \delta^{(n)}(x - y) \tag{6.64}$$

y a cuya solución llamaremos por abuso del lenguaje *núcleo del calor*. Los operadores que vamos a considerar son todos de la forma (que es mucho más general de lo que parece)

$$\Delta \equiv D^{\mu} D_{\mu} + Y \tag{6.65}$$

con

$$D_{\mu} \equiv \partial_{\mu} + \phi_{\mu} \tag{6.66}$$

En el caso $X = Y = 0$ la solución expĺı cita de la ecuación del calor es:

$$K_0(x, y; \tau) = \frac{i}{(4\pi i\tau)^{n/2}} e^{-i\frac{\sigma^2}{4\tau}} \tag{6.67}$$

donde el cuadrado de la distancia geodésica viene dado por:

$$\sigma^2 \equiv (x - y)^2 \tag{6.68}$$

Como hemos visto anteriormente, las divergencias ultravioletas vienen dadas por el ĺı mite inferior de la integral, el cual a su vez está dominado por el desarrollo de (la parte diagonal del) núcleo del calor para tiempos pequeños, llamado de Schwinger-de Witt

$$K(\tau) = K_0(\tau) \sum_{p=0} a_p(x, y)(i\tau)^p \tag{6.69}$$

donde por consistencia

$$a_0(x, x) = 1 \tag{6.70}$$

Representaremos con mayúsculas a la parte diagonal de los coeficientes integrada a todo el espacio:

$$A_n \equiv \int \sqrt{g} d^n x a_n(x, x) \tag{6.71}$$

donde g es el determinante de la métrica definida en el espacio. de forma que

$$A_0 = vol \tag{6.72}$$

Podemos entonces definir la integral del determinante de la siguiente manera:

$$\log\det\Delta \equiv -\int \frac{d\tau}{\tau} K(\tau) \equiv -lim_{s\to 0} \int_0^{\infty} \frac{d\tau}{\tau} \frac{i}{(4\pi i\tau)^{n/2}} \sum_{p=0} (i\tau)^p tr a_p e^{-i\frac{\sigma^2}{4\tau}} \tag{6.73}$$

es decir

$$\log\det\Delta = -\sum_p \frac{(-1)^p \sigma^{2p-n} i}{4^p i^n \pi^{\nu/2}} tr\, a_p \Gamma(n/2 - p) \tag{6.74}$$

en $n = 4$ dimensiones cuando $p = 0$ hay un término que diverge como $\frac{1}{\sigma^4}$ y es proporcional a a_0 y, por consiguiente, independiente del operador concreto que estemos considerando, Δ; el siguiente término es independiente de σ y viene dado cuando $n = 4 - \epsilon$ por

$$\frac{i}{8\pi^2\epsilon} a_2 \tag{6.75}$$

ya que sólo aparecen coeficientes pares en el desarrollo de Schwinger-de Witt. Todos los demás términos desaparecen al tomar el lí mite cuando $\sigma \to 0$. Desde este punto de vista, calcular el determinante es equivalente a determinar el coeficiente a_2 en el desarrollo.

Para ello, procedamos iterativamente. Substituyendo el desarrollo de Schwinger-de Witt en la ecuación del calor, obtenemos al orden más bajo (τ^{-1})

$$\sigma.Da_0 = 0 \tag{6.76}$$

y genéricamente

$$\sigma.Da_{p+1} + \Delta a_p + (p+1)a_{p+1} = 0 \tag{6.77}$$

Derivando la primera ecuación covariantemente

$$D_\lambda(\sigma^\mu D_\mu a_0) = 0 \tag{6.78}$$

se deduce

$$[D_\mu a_0] = 0 \tag{6.79}$$

donde

$$[A] \equiv lim_{\sigma \to 0} A \tag{6.80}$$

Derivando una segunda vez se obtiene

$$[(D_\mu D_\nu + D_\nu D_\mu)a_0] = 0 \tag{6.81}$$

de donde

$$[D^2 a_0] = 0 \tag{6.82}$$

y definiendo

$$W_{\mu\nu} \equiv [D_\mu, D_\nu] \tag{6.83}$$

se obtiene

$$[D_\mu D_\nu a_o] = \frac{1}{2}[([D_\mu, D_\nu] + \{D_\mu D_\nu\})a_o] = \frac{1}{2} W_{\mu\nu} \tag{6.84}$$

donde también se ha utilizado el hecho de que

$$[a_0] = 1 \tag{6.85}$$

Tomando $p = 0$ en (6.77)

$$-a_1 = \Delta a_0 + \sigma - Da_1 \tag{6.86}$$

lo cual inmediatamdente implica que

$$[a_1] = -[\Delta a_0] = -Y \tag{6.87}$$

Por otra parte, tomando $p = 1$ en (6.77)

$$-2a_2 = \Delta a_1 + \sigma.Da_2 \tag{6.88}$$

de donde

$$[a_2] = -\frac{1}{2}[\Delta a_1] \tag{6.89}$$

Es decir, que todo nuestro problema es calcular el segundo miembro. Para ello derivamos otra vez la expresión correspondiente a $p = 0$:

$$-D_\mu a_1 = D_\mu a_0 + D_\mu a_1 + \sigma^\lambda D_\mu D_\lambda a_1 \tag{6.90}$$

lo que implica

$$[\Delta a_1] = [D^2 a_1] + [Y a_1] = -\frac{1}{3}[D^2 D^2 a_0] - Y^2 - \frac{1}{3}D^2 Y \tag{6.91}$$

Ahora bien, derivando tres veces la expresión (6.76) se obtiene:

$$[(D_\delta D_\sigma D_\rho D_\mu + D_\delta D_\sigma D_\mu D_\rho + D_\delta D_\rho D_\mu D_\sigma + D_\sigma D_\rho D_\mu D_\delta + s^\lambda D_\delta D_\sigma D_\rho D_\mu D_\lambda)a_0] = 0 \tag{6.92}$$

Contrayendo con $\eta^{\delta\sigma}\eta^{\rho\mu}$

$$[(D^2 D^2 + D^\mu D^2 D_\mu)a_0] = 0 \tag{6.93}$$

Y contrayendo con $\eta^{\delta\rho}\eta^{\sigma\mu}$

$$[(D^\mu D^\nu D_\mu D_\nu)a_0] = 0 \tag{6.94}$$

Ahora bien,

$$[(D^\sigma D^\mu D_\mu D_\sigma)a_0] = [(D^\mu D^\sigma D_\mu D_\sigma + \frac{1}{2}W^{\sigma\mu}D_\mu D_\sigma)a_0] = -\frac{1}{2}W^2 \tag{6.95}$$

es decir, que

$$[D^2 D^2 a_0] = \frac{1}{2}W^2 \tag{6.96}$$

y

$$[a_2] = -\frac{1}{2}[\Delta a_1] = \frac{1}{6}[D^2 D^2 a_0] + \frac{1}{2}Y^2 + \frac{1}{6}D^2 Y = \frac{1}{12}W^2 + \frac{1}{2}Y^2 + \frac{1}{6}D^2 Y \tag{6.97}$$

de forma que

$$logdet\Delta = -\frac{2}{\epsilon}\frac{i}{(4\pi)^2}\int d^n\phi tr(\frac{1}{12}W^{\mu\nu}W_{\mu\nu} + \frac{1}{2}Y^2) \tag{6.98}$$

(el término en D^2Y desaparece al integrar siempre que no haya fronteras).

El cálculo a un lazo que efectuamos diagramáticamente en la sección anterior se reduce al cálculo del determinante del operador que actúa sobre los campos gauge, además del del operador que actúa sobre los fantasmas. El primero es (las constantes multiplicativas son irrelevantes):

$$\Delta_{bc}^{gauge\mu\nu} \equiv \bar{\nabla}^2 \delta_{bc} \eta^{\mu\nu} + 2g\bar{F}_{bc}^{\mu\nu(ad)} \tag{6.99}$$

en tanto que el operador que actúa sobre los fantasmas es:

$$\Delta_{fant} = \bar{\nabla}_\mu \partial^\mu \delta_{ab} = \bar{\nabla}_\mu \bar{\nabla}^\mu \delta_{ab} \tag{6.100}$$

(donde la igualdad es debida precisamente a la condición gauge de fondo), y

$$\bar{\nabla}_{ab\mu} \equiv \partial_\mu \delta_{ab} + g\bar{A}_{\mu ab}^{(adj)} \tag{6.101}$$

Para el operador gauge tenemos $Y = 2g\bar{F}_{\mu\nu}^{(adj)}$, en tanto que para el operador de los fantasmas $Y = 0$. Para ambos, $\phi = g\bar{A}_{\mu ab}^{(adj)}$.

El resultado de utilizar la fórmula (6.98) es

$$\Gamma = -\frac{1}{2}\log\det\Delta^{gauge} + \log\det\Delta^{fant} = \frac{22C_2(G)}{12}\frac{1}{16\pi^2\epsilon}\int d^4\phi F_{\mu\nu}^a F_a^{\mu\nu} \tag{6.102}$$

lo cual conduce de nuevo a

$$\frac{1}{g_0^2} - \frac{22C_2(G)}{48\pi^2\epsilon} = \frac{1}{g^2} \tag{6.103}$$

6.2 The first estimate for the back-reaction

It is often said that the back reaction of quantum fields in a classical spacetime is given by the effective equations

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = \frac{8\pi G}{c^4} \langle 0 | T_{\alpha\beta}(\phi) | 0 \rangle \tag{6.104}$$

The status of this has been somewhat clarified by Barvinsky and Nesterov [11]. Let us elaborate on it from a slightly different viewpoint. We shall start from the partition function of the world,

$$Z(J, j) \equiv \int \mathcal{D}g \mathcal{D}\psi e^{-S(g,\psi) - \int J.g - j.\psi} \tag{6.105}$$

where g is the gravitational field (forgetting indices for the time being) and ψ represents the matter content.

The true equations of motion for the gravitational field are

$$0 = \int \mathcal{D}g \mathcal{D}\psi \frac{\delta}{\delta g(x)} e^{-S(g,\psi) - \int J.g - j.\psi} = \int \mathcal{D}g \mathcal{D}\psi \left(\frac{\delta S}{\delta g(x)} - J(x) \right) e^{-S(g,\psi) - \int J.g - j.\psi} \tag{6.106}$$

We now expand in Taylor series around a background such that

$$\left. \frac{\delta S(g,\psi)}{\delta \psi} \right|_{\psi = \bar{\psi}} = j \tag{6.107}$$

(it is plain that $\bar{\psi} = \bar{\psi}(j)$).

$$0 = \int \mathcal{D}g \mathcal{D}\psi \left(\frac{\delta S}{\delta g(x)}(g, \bar{\psi}) + \right.$$
$$\left. \int_{y,z} \frac{\delta^3 S}{\delta g(x) \delta\psi_y \delta\psi_z}(g, \bar{\psi}) \psi_y \psi_z + O(\psi^3) - J(x) \right) e^{-S(g,\psi) - \int J.g - j.\psi} \tag{6.108}$$

(The linear term is absent because

$$\frac{\delta^2 S}{\delta g(x) \delta\psi(y)}(g, \bar{\psi}) = \frac{\delta j}{\delta g} = 0) \tag{6.109}$$

The equation defining the matter propagator in presence of an arbitrary background is

$$\int_y K_{xy}(g)\, G_{yz}(g) = \delta_{xz} \tag{6.110}$$

where

$$K_{xy}(g) \equiv \frac{\delta^2 S}{\delta\psi_x \delta\psi_y}(g, \bar{\psi}) \tag{6.111}$$

and, besides, at the one-loop approximation,

$$\int \mathcal{D}\psi\, \psi_x \psi_y\, e^{-S(g,\psi)-j\psi} = \left(G^{(1)}_{xy}(g) - \int_u G^{(1)}_{xu}(g) j_u \int_v G^{(1)}_{yv}(g) j_v \right)$$
$$e^{-\frac{1}{2}\int_{xy} j_x G^{(1)}_{xy}(g) j_y} + O(\lambda) \tag{6.112}$$

On the other hand, let us define

$$e^{-S_{eff}(g)} \equiv \int \mathcal{D}\psi e^{-S(g,\psi)} \tag{6.113}$$

By expanding again in Taylor series,

$$S(g,\psi) - \int j\psi = S(g,\bar{\psi}) + \frac{1}{2}\int_{x,y} K_{x,y}(g,\bar{\psi})\psi_x\psi_y + O(\psi^3) \tag{6.114}$$

At one loop

$$S^{(1)}_{eff}(g,\bar{\psi}) = -\frac{1}{2} log\, det\, K(g.\bar{\psi}) \tag{6.115}$$

in such a way that

$$\frac{\delta S^{(1)}_{eff}(g,\bar{\psi})}{\delta g} = -\frac{1}{2} tr\, K^{-1} \frac{\delta K(g)}{\delta g} = -\frac{1}{2} tr\, G^{(1)} \frac{\delta^3 S(g)}{\delta g \delta\psi\delta\psi} \tag{6.116}$$

The equations of motion in this approximation then stand as

$$0 = \int \mathcal{D}g \left(\frac{\delta S}{\delta g_{\alpha\beta}(x)}(g,\bar{\psi}) + \frac{\delta S^{(1)}_{eff}(g)}{\delta g_{\alpha\beta}} - J_{\alpha\beta}(x) \right) e^{-S(g,\bar{\psi}) - \int J_{\mu\nu} g^{\mu\nu}} \tag{6.117}$$

It is somewhat difficult to assess the physical domain of validity of the different approximations made so far.

6.3 The lowest order quantum corrections.

Let us consider the perturbative expansion around a background in more detail. For an arbitrary function (of interest in the TDiff invariant setting), and denoting

$$\begin{aligned}
&\bar{g}^{\alpha\beta}h_{\alpha\beta} \equiv h\\
&\bar{g}^{\alpha\beta}(h^2)_{\alpha\beta} \equiv tr\, h^2
\end{aligned} \tag{6.118}$$

$$f\left(|g|\right) = f\left(|\bar{g}|\right) + \kappa f'\left(|\bar{g}|\right).|\bar{g}|h + \frac{\kappa^2}{2}\left(\left(f'\left(|\bar{g}|\right)|\bar{g}| + f''\left(|\bar{g}|\right)|\bar{g}|^2\right)h^2 - |\bar{g}|f'\left(|\bar{g}|\right) tr\, h^2\right) \tag{6.119}$$

In particular,

$$\sqrt{|g|} = \sqrt{\bar{g}}\left(1 + \frac{1}{2}\kappa h + \frac{1}{2}\kappa^2\left(\frac{1}{4}h^2 - \frac{1}{2}h^{\alpha\beta}h_{\alpha\beta}\right)\right) \tag{6.120}$$

The Christoffel symbols expand in the following way:

$$\Gamma^{\mu}_{\nu\rho} \equiv \sum_n \kappa^n \overset{(n)}{\Gamma}{}^{\mu}_{\nu\rho} \tag{6.121}$$

that is:

$$\begin{aligned}
&\overset{(0)}{\Gamma}{}^{\mu}_{\nu\rho} = \bar{\Gamma}^{\mu}_{\nu\rho}\\
&\overset{(1)}{\Gamma}{}^{\mu}_{\nu\rho} = -h^{\mu}_{\sigma}\bar{\Gamma}^{\sigma}_{\nu\rho} + \frac{1}{2}\bar{g}^{\mu\alpha}(-\partial_\alpha h_{\nu\rho} + \partial_\nu h_{\alpha\rho} + \partial_\rho h_{\alpha\nu})\\
&\overset{(2)}{\Gamma}{}^{\mu}_{\nu\rho} = -\frac{1}{2}h^{\mu\epsilon}(-\partial_\epsilon h_{\nu\rho} + \partial_\nu h_{\epsilon\rho} + \partial_\rho h_{\nu\epsilon}) + (h^2)^{\mu}_{\lambda}\bar{\Gamma}^{\lambda}_{\nu\rho}
\end{aligned} \tag{6.122}$$

Higher order terms can be written in a background-covariant way:

$$\begin{aligned}
&\overset{(1)}{\Gamma}{}^{\mu}_{\nu\rho} = \frac{1}{2}(-\bar{\nabla}^{\mu}h_{\nu\rho} + \bar{\nabla}_\nu h^{\mu}_{\rho} + \bar{\nabla}_\rho h^{\mu}_{\nu})\\
&\overset{(2)}{\Gamma}{}^{\mu}_{\nu\rho} = -\frac{1}{2}h^{\mu\epsilon}(-\bar{\nabla}_\epsilon h_{\nu\rho} + \bar{\nabla}_\nu h_{\epsilon\rho} + \bar{\nabla}_\rho h_{\nu\epsilon})
\end{aligned} \tag{6.123}$$

In particular,

$$\begin{aligned}
&\overset{(1)}{\Gamma}{}^{\mu}_{\mu\rho} = \frac{1}{2}\bar{\nabla}_\rho h\\
&\overset{(2)}{\Gamma}{}^{\mu}_{\mu\rho} = -\frac{1}{2}h^{\alpha\beta}\bar{\nabla}_\rho h_{\alpha\beta}
\end{aligned} \tag{6.124}$$

In order to expand the Einstein-Hilbert lagrangian we need to consider first the Ricci tensor:

$$R_{\mu\nu} \equiv \partial_\rho\Gamma^{\rho}_{\nu\mu} - \partial_\nu\Gamma^{\rho}_{\rho\mu} + \Gamma^{\lambda}_{\nu\mu}\Gamma^{\rho}_{\rho\lambda} - \Gamma^{\lambda}_{\rho\mu}\Gamma^{\rho}_{\nu\lambda} \tag{6.125}$$

To first order in κ we find:

$$\overset{(1)}{R}_{\mu\nu}=\bar{\nabla}_{\rho}\overset{(1)}{\Gamma}{}^{\rho}_{\nu\mu}-\bar{\nabla}_{\nu}\overset{(1)}{\Gamma}{}^{\rho}_{\rho\mu} \tag{6.126}$$

To be specific,

$$\overset{(1)}{R}_{\mu\nu}=\frac{1}{2}\left(-\bar{\nabla}_{\rho}\bar{\nabla}^{\rho}h_{\mu\nu}+\bar{\nabla}_{\rho}\bar{\nabla}_{\nu}h^{\rho}_{\mu}+\bar{\nabla}_{\rho}\bar{\nabla}_{\mu}h^{\rho}_{\nu}-\bar{\nabla}_{\nu}\bar{\nabla}_{\mu}h\right) \tag{6.127}$$

and the scalar of curvature,

$$\overset{(1)}{R}=\bar{g}^{\mu\nu}\overset{(1)}{R}_{\mu\nu}+\overset{(1)}{g}{}^{\mu\nu}\bar{R}_{\mu\nu}=\bar{\nabla}_{\rho}\bar{\nabla}_{\lambda}h^{\rho\lambda}-\bar{\nabla}_{\rho}\bar{\nabla}^{\rho}h-h^{\mu\nu}\bar{R}_{\mu\nu} \tag{6.128}$$

The Einstein-Hilbert action then reads to first order

$$S^{(1)}=-\frac{1}{2\kappa}\int d^{n}x\sqrt{|\bar{g}|}\left(\frac{1}{2}\left(\bar{R}+2\lambda\right)h-\bar{R}^{\mu\nu}h_{\mu\nu}\right) \tag{6.129}$$

This term vanishes whenever the background equations of motion hold

$$\bar{R}_{\mu\nu}=-\frac{2\lambda}{n-2}\bar{g}_{\mu\nu} \tag{6.130}$$

The Ricci tensor to second order in κ reads:

$$\overset{(2)}{R}_{\mu\nu}=\bar{\nabla}_{\rho}\overset{(2)}{\Gamma}{}^{\rho}_{\nu\mu}-\bar{\nabla}_{\nu}\overset{(2)}{\Gamma}{}^{\rho}_{\rho\mu}+\overset{(1)}{\Gamma}{}^{\lambda}_{\nu\mu}\overset{(1)}{\Gamma}{}^{\rho}_{\rho\lambda}-\overset{(1)}{\Gamma}{}^{\lambda}_{\rho\mu}\overset{(1)}{\Gamma}{}^{\rho}_{\nu\lambda} \tag{6.131}$$

in gory detail,

$$\begin{aligned}
&\overset{(2)}{R}_{\mu\nu}=-\frac{1}{2}\bar{\nabla}_{\rho}h^{\rho\lambda}\left(-\bar{\nabla}_{\lambda}h_{\nu\mu}+\bar{\nabla}_{\nu}h_{\lambda\mu}+\bar{\nabla}_{\mu}h_{\lambda\nu}\right)-\\
&\frac{1}{2}h^{\rho\lambda}\bar{\nabla}_{\rho}\left(-\bar{\nabla}_{\lambda}h_{\nu\mu}+\bar{\nabla}_{\nu}h_{\lambda\mu}+\bar{\nabla}_{\mu}h_{\lambda\nu}\right)\\
&+\frac{1}{2}\bar{\nabla}_{\nu}h^{\lambda\epsilon}\bar{\nabla}_{\mu}h_{\epsilon\lambda}+\frac{1}{2}h^{\lambda\epsilon}\bar{\nabla}_{\nu}\bar{\nabla}_{\mu}h_{\epsilon\lambda}+\\
&\frac{1}{4}\left(-\bar{\nabla}^{\lambda}h_{\nu\mu}+\bar{\nabla}_{\nu}h^{\lambda}_{\mu}+\bar{\nabla}_{\mu}h^{\lambda}_{\nu}\right)\bar{\nabla}_{\lambda}h-\\
&-\frac{1}{4}\left(-\bar{\nabla}^{\lambda}h_{\rho\mu}+\bar{\nabla}_{\rho}h^{\lambda}_{\mu}+\bar{\nabla}_{\mu}h^{\lambda}_{\rho}\right)\left(-\bar{\nabla}^{\rho}h_{\nu\lambda}+\bar{\nabla}_{\nu}h^{\rho}_{\lambda}+\bar{\nabla}_{\lambda}h^{\rho}_{\nu}\right)=\\
&\frac{1}{2}\bar{\nabla}_{\rho}h^{\rho\lambda}\bar{\nabla}_{\lambda}h_{\mu\nu}-\frac{1}{2}\bar{\nabla}_{\rho}h^{\rho\lambda}\bar{\nabla}_{\nu}h_{\lambda\mu}-\frac{1}{2}\bar{\nabla}_{\rho}h^{\rho\lambda}\bar{\nabla}_{\mu}h_{\lambda\nu}\\
&+\frac{1}{2}h^{\rho\lambda}\bar{\nabla}_{\rho}\bar{\nabla}_{\lambda}h_{\nu\mu}-\frac{1}{2}h^{\rho\lambda}\bar{\nabla}_{\rho}\bar{\nabla}_{\nu}h_{\lambda\mu}-\frac{1}{2}h^{\rho\lambda}\bar{\nabla}_{\rho}\bar{\nabla}_{\mu}h_{\lambda\nu}\\
&+\frac{1}{2}\bar{\nabla}_{\nu}h^{\lambda\epsilon}\bar{\nabla}_{\mu}h_{\epsilon\lambda}+\frac{1}{2}h^{\lambda\epsilon}\bar{\nabla}_{\nu}\bar{\nabla}_{\mu}h_{\epsilon\lambda}+\\
&-\frac{1}{4}\bar{\nabla}^{\lambda}h_{\nu\mu}\bar{\nabla}_{\lambda}h+\frac{1}{4}\bar{\nabla}_{\nu}h^{\lambda}_{\mu}\bar{\nabla}_{\lambda}h+\frac{1}{4}\bar{\nabla}_{\mu}h^{\lambda}_{\nu}\bar{\nabla}_{\lambda}h\\
&-\frac{1}{4}\bar{\nabla}^{\lambda}h_{\rho\mu}\bar{\nabla}^{\rho}h_{\nu\lambda}+\frac{1}{4}\bar{\nabla}^{\lambda}h_{\rho\mu}\bar{\nabla}_{\nu}h^{\rho}_{\lambda}+\frac{1}{4}\bar{\nabla}^{\lambda}h_{\rho\mu}\bar{\nabla}_{\lambda}h^{\rho}_{\nu}+\\
&+\frac{1}{4}\bar{\nabla}_{\rho}h^{\lambda}_{\mu}\bar{\nabla}^{\rho}h_{\nu\lambda}-\frac{1}{4}\bar{\nabla}_{\rho}h^{\lambda}_{\mu}\bar{\nabla}_{\nu}h^{\rho}_{\lambda}-\frac{1}{4}\bar{\nabla}_{\rho}h^{\lambda}_{\mu}\bar{\nabla}_{\lambda}h^{\rho}_{\nu}\\
&+\frac{1}{4}\bar{\nabla}_{\mu}h^{\lambda}_{\rho}\bar{\nabla}^{\rho}h_{\nu\lambda}-\frac{1}{4}\bar{\nabla}_{\mu}h^{\lambda}_{\rho}\bar{\nabla}_{\nu}h^{\rho}_{\lambda}-\frac{1}{4}\bar{\nabla}_{\mu}h^{\lambda}_{\rho}\bar{\nabla}_{\lambda}h^{\rho}_{\nu}
\end{aligned} \tag{6.132}$$

which implies

$$\begin{aligned}
\bar{g}^{\mu\nu} \overset{(2)}{R}_{\mu\nu} = & \frac{1}{2}\bar{\nabla}_\rho h^{\rho\lambda}\bar{\nabla}_\lambda h - \frac{1}{2}\bar{\nabla}_\rho h^{\rho\lambda}\bar{\nabla}^\mu h_{\lambda\mu} - \frac{1}{2}\bar{\nabla}_\rho h^{\rho\lambda}\bar{\nabla}^\mu h_{\lambda\mu} \\
& +\frac{1}{2}h^{\rho\lambda}\bar{\nabla}_\rho\bar{\nabla}_\lambda h - \frac{1}{2}h^{\rho\lambda}\bar{\nabla}_\rho\bar{\nabla}^\mu h_{\lambda\mu} - \frac{1}{2}h^{\rho\lambda}\bar{\nabla}_\rho\bar{\nabla}^\mu h_{\lambda\mu} \\
& +\frac{1}{2}\bar{\nabla}^\mu h^{\lambda\epsilon}\bar{\nabla}_\mu h_{\epsilon\lambda} + \frac{1}{2}h^{\lambda\epsilon}\bar{\nabla}^\mu\bar{\nabla}_\mu h_{\epsilon\lambda} + \\
& -\frac{1}{4}\bar{\nabla}^\lambda h\bar{\nabla}_\lambda h + \frac{1}{4}\bar{\nabla}^\mu h^\lambda_\mu\bar{\nabla}_\lambda h + \frac{1}{4}\bar{\nabla}^\mu h^\lambda_\mu\bar{\nabla}_\lambda h \\
& -\frac{1}{4}\bar{\nabla}^\lambda h^{\rho\mu}\bar{\nabla}_\rho h_{\mu\lambda} + \frac{1}{4}\bar{\nabla}^\lambda h_{\rho\mu}\bar{\nabla}^\mu h^\rho_\lambda + \frac{1}{4}\bar{\nabla}^\lambda h_{\rho\mu}\bar{\nabla}_\lambda h^{\rho\mu} + \\
& +\frac{1}{4}\bar{\nabla}_\rho h^{\lambda\mu}\bar{\nabla}^\rho h_{\mu\lambda} - \frac{1}{4}\bar{\nabla}_\rho h^\lambda_\mu\bar{\nabla}^\mu h^\rho_\lambda - \frac{1}{4}\bar{\nabla}_\rho h^\lambda_\mu\bar{\nabla}_\lambda h^{\rho\mu} \\
& +\frac{1}{4}\bar{\nabla}^\mu h^\lambda_\rho\bar{\nabla}^\rho h_{\mu\lambda} - \frac{1}{4}\bar{\nabla}_\mu h^\lambda_\rho\bar{\nabla}^\mu h^\rho_\lambda - \frac{1}{4}\bar{\nabla}_\mu h^\lambda_\rho\bar{\nabla}_\lambda h^{\rho\mu}
\end{aligned} \tag{6.133}$$

The scalar of curvature to second order reads:

$$\begin{aligned}
& \overset{(2)}{R} = \bar{g}^{\mu\nu} \overset{(2)}{R}_{\mu\nu} + \overset{(1)}{g}{}^{\mu\nu} \overset{(1)}{R}_{\mu\nu} + \overset{(2)}{g}{}^{\mu\nu}\bar{R}_{\mu\nu} = \\
& \bar{\nabla}_\rho h^{\rho\lambda}\bar{\nabla}_\lambda h - \bar{\nabla}_\rho h^{\rho\lambda}\bar{\nabla}_\delta h^\delta_\lambda + \frac{3}{4}\bar{\nabla}_\nu h^{\lambda\epsilon}\bar{\nabla}^\nu h_{\epsilon\lambda} - \frac{1}{4}\bar{\nabla}^\lambda h\bar{\nabla}_\lambda h \\
& +\frac{1}{2}h^{\alpha\beta}\bar{\nabla}_\alpha\bar{\nabla}_\beta h + \frac{1}{2}h^{\alpha\beta}\bar{\nabla}^\mu\bar{\nabla}_\mu h_{\alpha\beta} - h^{\alpha\beta}\bar{\nabla}_\alpha\bar{\nabla}^\mu h_{\mu\beta} - \frac{1}{2}\bar{\nabla}_\rho h_{\lambda\mu}\bar{\nabla}^\mu h^{\rho\lambda} \\
& -h^{\mu\nu}\frac{1}{2}\left(-\bar{\nabla}^2 h_{\mu\nu} + \bar{\nabla}_\rho\bar{\nabla}_\mu h^\rho_\nu + \bar{\nabla}_\rho\bar{\nabla}_\nu h^\rho_\mu - \bar{\nabla}_\nu\bar{\nabla}_\mu h\right) \\
& + \left(h^2\right)^{\mu\nu}\bar{R}_{\mu\nu}
\end{aligned} \tag{6.134}$$

The full action to second order (taking into account the product of terms of order in κ (2,0),(0,2) and (1,1)) reads

$$\begin{aligned}
& S^{(2)} \equiv -\frac{1}{2\kappa^2}\int d^n x\sqrt{|g|}\left.(R+2\lambda)\right|_{O(\kappa^3)} = -\frac{1}{2}\int d^n x\sqrt{|\bar{g}|}\left(\frac{\bar{R}+2\lambda}{2}\frac{1}{4}\left(h^2 - 2h^{\alpha\beta}h_{\alpha\beta}\right)\right. \\
& \bar{\nabla}_\rho h^{\rho\lambda}\bar{\nabla}_\lambda h - \bar{\nabla}_\rho h^{\rho\lambda}\bar{\nabla}_\delta h^\delta_\lambda + \frac{3}{4}\bar{\nabla}_\nu h^{\lambda\epsilon}\bar{\nabla}^\nu h_{\epsilon\lambda} - \frac{1}{4}\bar{\nabla}^\lambda h\bar{\nabla}_\lambda h \\
& +\frac{1}{2}h^{\alpha\beta}\bar{\nabla}_\alpha\bar{\nabla}_\beta h + \frac{1}{2}h^{\alpha\beta}\bar{\nabla}^\mu\bar{\nabla}_\mu h_{\alpha\beta} - h^{\alpha\beta}\bar{\nabla}_\alpha\bar{\nabla}^\mu h_{\mu\beta} - \frac{1}{2}\bar{\nabla}_\rho h_{\lambda\mu}\bar{\nabla}^\mu h^{\rho\lambda} \\
& -h^{\mu\nu}\frac{1}{2}\left(-\bar{\nabla}^2 h_{\mu\nu} + \bar{\nabla}_\rho\bar{\nabla}_\mu h^\rho_\nu + \bar{\nabla}_\rho\bar{\nabla}_\nu h^\rho_\mu - \bar{\nabla}_\nu\bar{\nabla}_\mu h\right) \\
& + \left(h^2\right)^{\mu\nu}\bar{R}_{\mu\nu} + \\
& \left.\frac{h}{2}\left(\bar{\nabla}_\rho\bar{\nabla}_\lambda h^{\rho\lambda} - \bar{\nabla}_\rho\bar{\nabla}^\rho h - h^{\mu\nu}\bar{R}_{\mu\nu}\right)\right)
\end{aligned} \tag{6.135}$$

Performing now the integrations by parts and substituting the background equations of motion leads to:

$$S^{(2)} = -\frac{1}{2}\int d^n x\sqrt{|\bar{g}|}\left(\frac{\lambda}{2(n-2)}\left(h^2 - 2h^{\alpha\beta}h_{\alpha\beta}\right)\right.$$
$$\left.-\frac{1}{2}\bar{\nabla}_\rho h^{\rho\lambda}\bar{\nabla}_\lambda h - \frac{1}{4}\bar{\nabla}_\mu h^{\alpha\beta}\bar{\nabla}^\mu h_{\alpha\beta} + \frac{1}{2}\bar{\nabla}_\rho h_{\lambda\mu}\bar{\nabla}^\mu h^{\rho\lambda} + \frac{1}{4}\bar{\nabla}^\lambda h\bar{\nabla}_\lambda h\right) \quad (6.136)$$

This action is sometimes referred to as the Fierz-Pauli action in an arbitrary background. Under an arbitrary variation

$$\delta S_{FP} = -\frac{1}{2}\int d^n x\sqrt{|\bar{g}|}\frac{1}{2}\left(\left(\frac{2\lambda}{n-2}\left(h\bar{g}^{\mu\nu} - 2h^{\mu\nu}\right) + \right.\right.$$
$$\left. +\bar{\nabla}_\alpha\bar{\nabla}_\beta h^{\alpha\beta}\bar{g}^{\mu\nu} + \bar{\nabla}_\mu\bar{\nabla}_\nu h + \bar{\nabla}_\alpha\bar{\nabla}^\alpha h^{\mu\nu} - \bar{\nabla}_\alpha\bar{\nabla}^\nu h^{\alpha\mu} - \bar{\nabla}_\alpha\bar{\nabla}^\mu h^{\nu\alpha} - \bar{\nabla}^\alpha\bar{\nabla}_\alpha h\bar{g}^{\mu\nu}\right)\delta h_{\mu\nu} +$$
$$\left.\bar{\nabla}_\mu\left(-\bar{\nabla}_\alpha h^{\alpha\mu}\delta h - \delta h^{\mu\alpha}\bar{\nabla}_\alpha h - \bar{\nabla}^\mu h^{\alpha\beta}\delta h_{\alpha\beta} + \bar{\nabla}^\alpha h^{\mu\beta}\delta h_{\alpha\beta} + \bar{\nabla}^\alpha h^{\beta\mu}\delta h_{\alpha\beta} + \bar{\nabla}^\mu h\delta h\right)\right) \equiv$$
$$-\frac{1}{2}\int d^n x\sqrt{|\bar{g}|}\left(\frac{\delta S}{\delta h_{\mu\nu}}\delta h_{\mu\nu} + \bar{\nabla}_\mu\left(L^{\mu\alpha\beta}\delta h_{\alpha\beta}\right)\right) \quad (6.137)$$

Let us denote

$$\frac{\delta S}{\delta h_{\mu\nu}} \equiv \bar{D}^{\mu\nu} \quad (6.138)$$

Particularizing to the gauge symmetry we get

$$\delta h_{\mu\nu} \equiv \bar{\nabla}_\mu\xi_\nu + \bar{\nabla}_\nu\xi_\mu \quad (6.139)$$

$$\delta S = 0 = -\frac{1}{2}\int d^n x\sqrt{|\bar{g}|}\left(\bar{D}^{\mu\nu}\left(\bar{\nabla}_\mu\xi_\nu + \bar{\nabla}_\nu\xi_\mu\right) + \bar{\nabla}_\mu\left(L^{\mu\alpha\beta}\left(\bar{\nabla}_\alpha\xi_\beta + \bar{\nabla}_\beta\xi_\alpha\right)\right)\right) =$$
$$-\frac{1}{2}\int d^n x\sqrt{|\bar{g}|}\left(-2\xi_\nu\bar{\nabla}_\mu\bar{D}^{\mu\nu}\right) + \bar{\nabla}_\mu\left(2L^{\mu\alpha\beta}\bar{\nabla}_\alpha\xi_\beta + 2\xi_\nu\bar{D}^{\mu\nu}\right) \quad (6.140)$$

As a consequence, there is the *gauge identity*

$$\bar{\nabla}_\mu\bar{D}^{\mu\nu} = 0 \quad (6.141)$$

and the off-shell conservation of the Noether current

$$J^\mu \equiv L^{\mu\alpha\beta}\bar{\nabla}_\alpha\xi_\beta + \xi_\nu\bar{D}^{\mu\nu} \quad (6.142)$$

Let us make this explicit (we follow [69] here) in terms of the superpotential in the particular case in which the parameter of the transformation is a background Killing vector

$$\bar{\nabla}_\mu\xi_\nu + \bar{\nabla}_\nu\xi_\mu = 0 \quad (6.143)$$

In this particular case

$$J^\mu = \xi_\nu\bar{D}^{\mu\nu} \quad (6.144)$$

and we want to show that

$$J^{\mu} = \bar{\nabla}_{\alpha} K^{\alpha\mu} \tag{6.145}$$

where $K^{(\alpha\mu)} = 0$. To be specific,

$$K^{\alpha\mu} = \bar{\nabla}_{\beta} K^{\mu\alpha\nu\beta} \xi_{\nu} - K^{\mu\beta\nu\alpha} \bar{\nabla}_{[\beta} \xi_{\nu]} \tag{6.146}$$

in terms of the *background superpotential*

$$K^{\mu\alpha\nu\beta} \equiv \frac{1}{2} \left(\bar{g}^{\mu\beta} \bar{h}^{\nu\alpha} + \bar{g}^{\nu\alpha} \bar{h}^{\mu\beta} - \bar{g}^{\mu\nu} \bar{h}^{\alpha\beta} - \bar{g}^{\alpha\beta} \bar{h}^{\mu\nu} \right) \tag{6.147}$$

To begin with, we shall write, following the classic work by Abbott and Deser,

$$\bar{D}_{\mu\nu} \equiv X_{\mu\nu} + Y_{\mu\nu} \tag{6.148}$$

where

$$2X_{\mu\nu} = -\frac{4\lambda}{n-2} \bar{h}_{\mu\nu} + \left(\bar{\nabla}_{\nu} \bar{\nabla}^{\lambda} - \bar{\nabla}^{\lambda} \bar{\nabla}_{\nu} \right) h_{\lambda\mu} = -\frac{4\lambda}{n-2} \bar{h}_{\mu\nu} + \bar{R}_{\mu\lambda\nu\sigma} h^{\lambda\sigma} + \bar{R}^{\lambda}{}_{\nu} h_{\mu\lambda} =$$
$$-\frac{2\lambda}{n-2} \bar{h}_{\mu\nu} + \bar{R}_{\mu\lambda\nu\sigma} \bar{h}^{\lambda\sigma} \tag{6.149}$$

(the background equations (BEM) $\bar{R}_{\mu\nu} = -\frac{2\lambda}{n-2} \bar{g}_{\mu\nu}$ have been used in the last step). Le us now compute

$$\bar{R}_{\nu\alpha\beta\sigma} K_{\mu}{}^{\alpha\beta\sigma} = \frac{1}{2} \left(\bar{R}_{\nu\alpha\beta\mu} \bar{h}^{\alpha\beta} - \bar{R}_{\nu\sigma} \bar{h}_{\mu}{}^{\sigma} - \bar{R}_{\nu\alpha\mu\sigma} \bar{h}^{\alpha\sigma} - \bar{R}_{\nu\beta} \bar{h}_{\mu}{}^{\beta} \right) =$$
$$\bar{R}_{\nu\alpha\beta\mu} \bar{h}^{\alpha\beta} + \frac{2\lambda}{n-2} \bar{h}_{\mu\nu} = -2X_{\mu\nu} \tag{6.150}$$

where BEM have been used again.

We are left with

$$2Y_{\mu\nu} = -\left(\bar{\nabla}_{\nu} \bar{\nabla}^{\lambda} - \bar{\nabla}^{\lambda} \bar{\nabla}_{\nu} \right) h_{\lambda\mu} + \bar{\nabla}_{\alpha} \bar{\nabla}_{\beta} h^{\alpha\beta} \bar{g}_{\mu\nu} + \bar{\nabla}_{\mu} \bar{\nabla}_{\nu} h + \bar{\nabla}_{\alpha} \bar{\nabla}^{\alpha} h_{\mu\nu} -$$
$$\bar{\nabla}_{\alpha} \bar{\nabla}_{\nu} h^{\alpha}{}_{\mu} - \bar{\nabla}_{\alpha} \bar{\nabla}_{\mu} h_{\nu}{}^{\alpha} - \bar{\nabla}^{\alpha} \bar{\nabla}_{a} h \bar{g}_{\mu\nu} =$$
$$-\bar{\nabla}_{\nu} \bar{\nabla}^{\lambda} h_{\lambda\mu} + \bar{\nabla}_{\alpha} \bar{\nabla}_{\beta} h^{\alpha\beta} \bar{g}_{\mu\nu} + \bar{\nabla}_{\mu} \bar{\nabla}_{\nu} h + \bar{\nabla}_{\alpha} \bar{\nabla}^{\alpha} h_{\mu\nu} - \bar{\nabla}_{\alpha} \bar{\nabla}_{\mu} h_{\nu}{}^{\alpha} - \bar{\nabla}^{\alpha} \bar{\nabla}_{a} h \bar{g}_{\mu\nu} \tag{6.151}$$

Let us now compute

$$\bar{\nabla}_{\alpha} \bar{\nabla}_{\beta} K^{\mu\alpha\nu\beta} = \frac{1}{2} \left(\bar{\nabla}_{\alpha} \bar{\nabla}^{\mu} h^{\alpha\nu} + \bar{\nabla}_{\nu} \bar{\nabla}_{\beta} h^{\mu\beta} - \bar{\nabla}_{\alpha} \bar{\nabla}_{\beta} h^{\alpha\beta} \bar{g}^{\mu\nu} - \bar{\nabla}_{\alpha} \bar{\nabla}^{\alpha} h^{\mu\nu} + \right.$$
$$\left. \bar{\nabla}_{\alpha} \bar{\nabla}^{\alpha} h \bar{g}^{\mu\nu} - \bar{\nabla}^{\nu} \bar{\nabla}^{\mu} h \right) = -Y^{\mu\nu} \tag{6.152}$$

We can now write

$$-Y^{\mu\nu} \xi_{\nu} = \bar{\nabla}_{\alpha} \bar{\nabla}_{\beta} K^{\mu\alpha\nu\beta} \xi_{\nu} = \bar{\nabla}_{\alpha} \left(\bar{\nabla}_{\beta} K^{\mu\alpha\nu\beta} \xi_{\nu} \right) - \bar{\nabla}_{\beta} K^{\mu\alpha\nu\beta} \bar{\nabla}_{\alpha} \xi_{\nu} =$$
$$\bar{\nabla}_{\alpha} \left(\bar{\nabla}_{\beta} K^{\mu\alpha\nu\beta} \xi_{\nu} - K^{\mu[\beta\nu]\alpha} \bar{\nabla}_{\beta} \xi_{\nu} \right) + K^{\mu[\beta\nu]\alpha} \bar{\nabla}_{\alpha} \bar{\nabla}_{\beta} \xi_{\nu} \tag{6.153}$$

Now we use the Ricci identity for Killing vectors

$$\bar{\nabla}_\alpha \bar{\nabla}_\beta \xi_\nu = -\xi^\lambda \bar{R}_{\beta\nu\alpha\lambda} \tag{6.154}$$

as well as the fact, easily checked by an explicit computation, that

$$K^{\mu[\beta\nu]\alpha} = \frac{1}{2} K^{\mu\alpha\nu\beta} \tag{6.155}$$

to get

$$-Y^{\mu\nu}\xi_\nu = \bar{\nabla}_\alpha \left(\bar{\nabla}_\beta K^{\mu\alpha\nu\beta}\xi_\nu - K^{\mu[\beta\nu]\alpha}\bar{\nabla}_\beta \xi_\nu \right) + X^{\mu\nu}\xi_\nu \tag{6.156}$$

This finally shows that

$$K^{\mu\alpha} = \bar{\nabla}_\beta K^{\mu\alpha\nu\beta}\xi_\nu - K^{\mu[\beta\nu]\alpha}\bar{\nabla}_\beta \xi_\nu \tag{6.157}$$

6.4 Reliable low energy results in quantum gravity

It is clear that whenever there is a divergence, there is loss of predictivity, and the corresponding coefficient in the effective lagrangian is an arbitrary constant to be determined by experiment. Finite nonlocal contributions can sometimes, however be interpreted as genuine predictions of the low-energy theory. Let us briefly point out some of those.

- The only non vanishing tree contribution to the graviton-graviton scattering in an helicity basis (neglecting the cosmological constant) has been already computed by deWitt [?] is

$$A^{(0)}(++;++) = i\frac{\kappa^2 s^3}{tu} \equiv i\frac{s^3}{M_p^2 tu} \tag{6.158}$$

  in terms of the usual Mandelstam variables

$$\begin{aligned} s &\equiv (p_1+p_2)^2 \\ t &\equiv (p_1-p_3)^2 \\ u &\equiv (p_1-p_4)^2 \end{aligned} \tag{6.159}$$

  The one-loop contribution has been determined by Dunbar-Norridge [28].

$$\begin{aligned} &A^{(1)}(++;--) = -i\frac{1}{30720\pi^2}\frac{s^2+t^2+u^2}{M_p^4} \\ &A^{(1)}(++;+-) = -\frac{1}{3}A^{(1)}(++;--) \\ &A^{(1)}(++;++) = \frac{1}{M_p^2}\frac{1}{4(4\pi)^{\frac{n}{2}}}\frac{\Gamma^2\left(\frac{n-2}{2}\right)\Gamma\left(\frac{6-n}{2}\right)}{\Gamma(n-3)}A^{(0)}(++;++)\,stu \\ &\left(\frac{4}{4-n}\left(\frac{log(-u)}{st}+\frac{log(-t)}{su}+\frac{log(-s)}{ut}\right)+\frac{1}{s^2}f\left(-\frac{t}{s},\frac{u}{s}\right)+\right. \\ &\left.2\left(\frac{log(-u)\,log(-s)}{su}+\frac{log(-t)\,log(-s)}{tu}+\frac{log(-t)\,log(-s)}{st}\right)\right) \end{aligned} \tag{6.160}$$

  where the dimensionless function is given by

$$\begin{aligned} &f\left(-\frac{t}{s},\frac{u}{s}\right) = \frac{(t+2u)(2t+u)(2t^4+2t^3u-t^2u^2+2tu^3+2u^4)}{s^6}\left(log^2\frac{t}{u}+\pi^2\right)+ \\ &\frac{(t-u)(341t^4+1609t^3u+2566t^2u^2+1609tu^3+341u^4)}{30s^5}log\frac{t}{u}+ \\ &\frac{1922t^4+9143t^3u+14622t^2u^2+9143tu^3+1922u^4}{180s^4} \end{aligned} \tag{6.161}$$

  There are several things remarkable about this result.

First of all it does not depend of the coefficients of the quadratic terms in the curvature in the effective action. This is due to the combination of the background equations of motion, which are

$$\bar{R}_{\mu\nu} = 0 \tag{6.162}$$

with the fact that in four dimensions, the Euler characteristic is given by

$$\chi = \frac{1}{16\pi^2} \int \sqrt{|g|} d^4x \left(\frac{1}{2} R^2 - 2 R_{\mu\nu} R^{\mu\nu} + \frac{1}{2} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \right) \tag{6.163}$$

which forces the term in Riemann squared also to vanish on shell.

The other thing is that it is divergent. This is to be interpreted as an infrared divergence, and on general grounds it is to be expected that this divergence is cancelled against the radiation of soft gravitons off external graviton lines (bremsstrahlung diagrams). This expectation has been confirmed in an explicit calculation by Donoghue and Torma [26]. They found an explicitly finite expression for the differential cross section

$$\frac{d\sigma}{\delta\Omega} = \frac{s^5}{512\pi^2 t^2 u^2 M_p^4} \left(1 + \frac{s}{16\pi^2 M_p^2} \left(log\frac{-t}{s} log\frac{-u}{s} + \frac{tu}{2s^2} f\left(\frac{-t}{s}, \frac{-u}{s} \right) - \left(\frac{t}{s} log\frac{-t}{s} + \frac{u}{s} log\frac{-u}{s} \right) \left(3log\,(2\pi)^2 + \gamma + log\frac{s}{\Lambda^2} + \frac{\sum \eta_i\eta_j \mathcal{F}^{(1)}(\gamma_{ij})}{\sum \eta_i\eta_j \mathcal{F}^{(0)}(\gamma_{ij})} \right) \right) \right)$$

The objects $\mathcal{F}$ are defined implicitly by the integral

$$\mathcal{F}^{(0)}(\gamma) + \frac{4-n}{2} \mathcal{F}^{(1)}(\gamma) + \ldots = \int d\Omega_{n-1} \frac{\left(cos\,\gamma_{ij} - cos\,\alpha_i cos\,\alpha_j \right)^2 - \frac{1}{2} sin^2\,\alpha_i sin^2\,\alpha_j}{\left(1 - cos\,\alpha_i \right)\left(1 - cos\,\alpha_j \right)} \tag{6.164}$$

Here γ_{ij} is the angle between the $n-1$ dimensional momenta of the hard gravitons; α_i is the angle between the ith hard and the soft gravitons; and η_i is $+1(-1)$ for incoming (outgoing) hard gravitons. Finally, $\Lambda << \sqrt{s}$ is an infrared cutoff.

It is remarkable that such a universal result exists in low energy quantum gravity.

- Quantum corrections to the gravitational potential. There are several ways to define in a precise way the concept of gravitational potential. Iwasaki [52] does this through an analysis of the bound state potential. Other possibility is to define it directly from the scattering amplitude [24]. At any rate there are both *classical* (id est, not involving $\hbar$) contributions, that go like $\frac{1}{r}$ and $\frac{1}{r^2}$, and come from the tree diagram (the dominant contribution) as well as from a piece of the triangle diagrams, and from another piece of the vertex correction; and fully quantum corrections, that go like $1/r^3$.

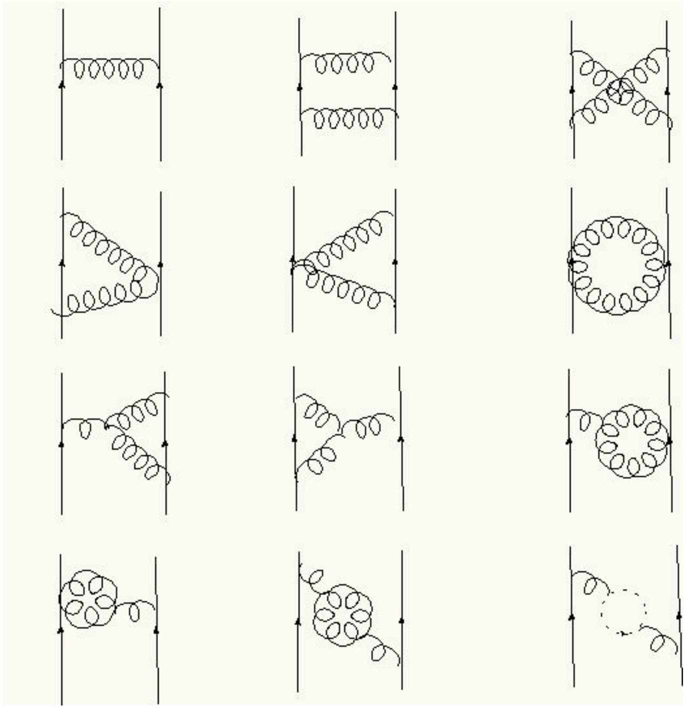

Figure 4: The set of diagrams contributing to the quantum corrections to the newtonian potential.

The full result coming from the sum of the non-analytic contributions of all diagrams in the picture is claimed by Bjerrum-Bohr and Donoghue [24] to be

$$V(r) = -\frac{Gm_1 m_2}{r}\left(1 + 3\frac{G\,(m_1 + m_2)}{r} + \frac{41}{10\pi}\frac{G\hbar}{r^2}\right) \tag{6.165}$$

There are some possible ambiguities coming from the freedom in defining the radial coordinate, and even the metric itself (confer [56][14]). It is however remarkable that the result as quoted does not depend on any of the coefficients of the terms higher in curvature in the low energy expansion and as such can be considered as a genuine low energy prediction of quantum gravity.

7. Transverse gravity: a case study

7.1 Classical equivalence of TDiff and scalar-tensor theories

The simple model with TDiff symmetry considered in [5], i.e.,

$$S=-\frac{1}{2\kappa^2}\int d^n x\sqrt{g}R+\int d^n x g^{\mu\nu}\partial_\mu\phi\partial_n\phi \tag{7.1}$$

leads to the following equations of motion

$$\begin{aligned} R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R&=\frac{2\kappa^2}{\sqrt{g}}\partial_\mu\phi\partial_\nu\phi\\ \partial_\mu\left(g^{\mu\nu}\partial_\nu\phi\right)&=0. \end{aligned} \tag{7.2}$$

It can be seen taking the covariant derivative of Einstein's equations and using the contracted Bianchi identity (as well as the e.o.m. of the scalar) that in order to achieve consistency the Lagrangian has to be a constant

$$\begin{aligned} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi&\equiv\Lambda\\ \partial_\mu\Lambda&=0 \end{aligned} \tag{7.3}$$

The action may also be written with the help of a Lagrange multiplier $\Lambda(x)$ as [?]

$$S=-\frac{1}{2\kappa^2}\int d^n x\sqrt{g}R+\int d^n x\sqrt{g}\chi^{-\frac{n(n-2)}{2}}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-\int d^n x\Lambda\left(\sqrt{g}\chi^{-\frac{n(n-2)}{2}}-1\right) \tag{7.4}$$

If we postulate that $g_{\mu\nu}$ transforms as a true metric and χ and Λ as scalars, then all the violation of Diff invariance has been transfered to the very last term. The equation of motion of the multiplier forces

$$g=\chi^{n(n-2)} \tag{7.5}$$

which of course reflects the lack of Diff invariance. The scalar and Einstein's e.o.m. read now

$$\begin{aligned} &\partial_\nu\left(\sqrt{g}\chi^{-\frac{n(n-2)}{2}}g^{\mu\nu}\partial_\mu\phi\right)=0\\ &R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=2\kappa^2\chi^{-\frac{n(n-2)}{2}}\left(\left(\partial_\mu\phi\partial_\nu\phi-\frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\right)+\frac{1}{2}g_{\mu\nu}\Lambda\right) \end{aligned} \tag{7.6}$$

while variation with respect to the auxiliary field χ gives

$$\Lambda=g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi. \tag{7.7}$$

Substitution into Einstein's equation along with the constraint (7.5) reproduces Einstein's equations in the previous form. Notice that also in this case the condition

$L_M = \Lambda = const.$ follows from consistency with the Bianchi identity. One could think that this result is particular of the case without kinetic energy term for the auxiliary field. This corresponds to taking the Einstein–Hilbert Lagrangian in the gravity part. If instead we had started from a Lagrangian with an arbitrary function of the determinant of the metric in front of the curvature scalar we would have arrived to

$$S = -\frac{1}{2\kappa^2}\int d^n x\sqrt{g}R + S_M + \int d^n x \Lambda \tag{7.8}$$

where the matter part is

$$S_M = \int d^n x\sqrt{g}\left[\frac{(n-1)(n-2)}{2\chi^2}g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi + \chi^{-\frac{n(n-2)}{2}}L_M\left[\chi,\phi,g_{\mu\nu}\right] - \chi^{-\frac{n(n-2)}{2}}\Lambda\right] \tag{7.9}$$

Suppose for a moment that $L_M = 0$, the multiplier continues to force (7.5) and varying the action with respect to the auxiliary field gives

$$\frac{(n-1)}{\chi^3}g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi + (n-1)\partial_\nu\left(\frac{1}{\chi^2}g^{\mu\nu}\partial_\mu\chi\right) - \frac{n}{2}\chi^{-\frac{n(n-2)+2}{2}}\Lambda = 0 \tag{7.10}$$

while Einstein's equations take the form

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa^2\left[\frac{(n-1)(n-2)}{\chi^2}\partial_\mu\chi\partial_\nu\chi - g_{\mu\nu}\frac{(n-1)(n-2)}{2\chi^2}g^{\alpha\beta}\partial_\alpha\chi\partial_\beta\chi + g_{\mu\nu}\chi^{-\frac{n(n-2)}{2}}\Lambda\right]. \tag{7.11}$$

Once again the Bianchi identity of the left hand side forces the consistency condition

$$0 = \nabla^\mu\left(\frac{1}{\sqrt{g}}\frac{\delta S_M}{\delta g^{\mu\nu}}\right) = \frac{1}{2}\chi^{-\frac{n(n-2)}{2}}\partial_\nu\Lambda \tag{7.12}$$

where we have used the equation of motion (7.10). A basic difference with the previous case is that now there is not a direct way to relate the constant Λ with the matter Lagrangian, instead it is related to the matter and the auxiliary fields through (7.10). Although we haven't considered a Lagrangian for matter, it is easy to convice oneself that nothing should change as long as L_M is a scalar. In fact a general argument can be given for Λ to be a constant [?]. Let us perform Diffeomorphism on the matter action, i.e., a change of coordinates in the active sense, keeping the volume element $d^n x$ unchanged. Since by hypothesis S_M is a scalar

$$\begin{aligned} 0 = \delta S_M &= \int d^n x\left[\frac{\delta S_M}{\delta\phi}\delta\phi + \frac{\delta S_M}{\delta\chi}\delta\chi + \frac{\delta S_M}{\delta\Lambda}\delta\Lambda + \frac{\delta S_M}{\delta g^{\mu\nu}}\delta g^{\mu\nu}\right] = \\ &= -\int d^n x\sqrt{g}\left(\nabla^\nu\left(\frac{1}{\sqrt{g}}\frac{\delta S_M}{\delta g^{\mu\nu}}\right) + \chi^{-\frac{n(n-2)}{2}}\partial_\nu\Lambda\right)\xi^\nu \end{aligned} \tag{7.13}$$

from which one reproduces (7.12) under the assumption that the fields verify their equations of motion[16].

[16] Verificar los cálculos, hay algún problema con la normalización y el signo.

It is in some sense natural to suspect of the hypothesis that our fields are scalars under general TDiff, since we have started from an action which is not. In fact, this assumption is inconsistent with the equation of motion of the multiplier (7.5). We can make all the reasonings automatically consistent using compensators [5]. Let $C(x)$ be a field (scalar density) such that the combination

$$g(x)C^2(x) \tag{7.14}$$

is invariant under an arbitrary diffeomorphism. Starting from a metric $g_{\mu\nu}$ one can define another metric

$$C^{-\frac{2}{n}}\hat{g}_{\mu\nu} = g^{-\frac{1}{n}}C^{-\frac{2}{n}}g_{\mu\nu} \tag{7.15}$$

whose determinant is $\hat{g} = 1$. Under TDiff, $\hat{g}_{\mu\nu}$ is a tensor and g a scalar. Therefore, the most general TDiff action takes the form

$$S = -\frac{1}{2\kappa^2}\int d^n x \chi^2\left[g,\phi_\omega\right]R\left[\hat{g}_{\mu\nu}\right] + \int d^n x L\left[g,\phi_\omega,\hat{g}_{\mu\nu}\right] \tag{7.16}$$

where ϕ_ω denotes a general collection of fields which may have some weight $\varnothing$ under an arbitrary Diff (not transverse). If one now performs such a Diff, the previous action takes the form

$$S = -\frac{1}{2\kappa^2}\int d^n x C^{-1}\chi^2\left[gC^2,\phi_\omega C^{-\varnothing}\right]R\left[\hat{g}_{\mu\nu}C^{-\frac{2}{n}}\right] + \int d^n x C^{-1}L\left[gC^2,\phi_\omega C^{-\varnothing},\hat{g}_{\mu\nu}C^{-\frac{2}{n}}\right] \tag{7.17}$$

which is invariant by construction. We may go to the Einstein frame

$$\begin{aligned} \bar{g}_{\mu\nu} &= \chi^{\frac{4}{n-2}}C^{-\frac{2}{n}}\hat{g}_{\mu\nu} \\ \bar{g} &= \chi^{\frac{4n}{(n-2)}}C^{-2}. \end{aligned} \tag{7.18}$$

This last constraint is implemented through a Lagrange multipler Λ. Finally the action reads

$$\begin{aligned} S &= -\frac{1}{2\kappa^2}\int d^n x\sqrt{\bar{g}}\bar{R} + S_M + \int d^n x \Lambda C^{-1} \\ S_M &= \int d^n x\sqrt{\bar{g}}\left[\frac{(n-1)(n-2)}{2\chi^2}\bar{g}^{\mu\nu}\partial_\mu\chi\partial_\nu\chi + \chi^{-\frac{2n}{(n-2)}}L\left[\chi,\phi_\omega C^{-\varnothing},\bar{g}_{\mu\nu}\right] - \chi^{-\frac{2n}{(n-2)}}\Lambda\right]. \end{aligned}$$

We have eliminated the combination gC^2 in favor of χ. Notice that this action is perfectly Diff invariant, so in principle there are no consistency problems. We recover the usual form of the action in the, so to say, unitary gauge $C = 1$. It is clear that if all ϕ are true scalars, the e.o.m. of the compensator forces the Lagrange multipler to vanish. In case the matter fields have some weight, there could be terms like for example

$$\frac{1}{2}\bar{g}^{\mu\nu}\partial_\mu(\phi_\omega C^{-\omega})\partial_\nu(\phi_\omega C^{-\omega}) \tag{7.19}$$

But redefining $\phi \equiv \phi_\omega C^{-\omega}$ we see that the e.o.m. for the compensator remains the same.

7.2 Abelian gauge invariance: transverse Fierz-Pauli symmetry.

A sometimes confusing issue is the following. The Fierz-Pauli (FP) symmetry is not exactly the linearization of full diffeomorphism invariance, which would have been

$$\delta h_{\alpha\beta} = \partial_\alpha \xi_\beta + \partial_\beta \xi_\alpha + \xi^\rho \partial_\rho h_{\alpha\beta} \tag{7.20}$$

insofar as the last term is absent. This issue is clearly explained in page 80 of Ortín's book [69].

Indeed, gauge fixing with the full FP symmetry is trivial, and e.g. harmonic gauge can be imposed:

$$\omega_\mu \equiv \partial^\lambda h_{\lambda\mu} - \frac{1}{2}\partial_\mu h = 0 \tag{7.21}$$

through a gauge fixing

$$L_{gf} = B^\mu \omega_\mu + \frac{\alpha}{2} B^\mu B_\mu \tag{7.22}$$

The ghost lagrangian is ,

$$L_{gh} = b^\rho \Box c_\rho \tag{7.23}$$

and the BRST transformations can be taken simply as:

$$\begin{aligned} s h_{\alpha\beta} &= \partial_\alpha c_\beta + \partial_\beta c_\alpha \\ s B_\mu &= 0 \\ s b_\mu &= -B_\mu \\ s c_\mu &= 0 \end{aligned} \tag{7.24}$$

Were we to implement the transverse part of the symmetry (TFP) only, the parameters are not arbitrary but rather

$$\partial_\alpha \xi^\alpha = 0 \tag{7.25}$$

This complicates matters in several ways. First of all, we cannot reach the full harmonic gauge. The best we can do is to impose, for example, the spatial piece, i.e.

$$\omega_i = 0 \tag{7.26}$$

or even better, [17] the three independent conditions:

$$\partial_\alpha \omega^{\alpha\beta} = 0 \tag{7.29}$$

[17] Another possibility would be to impose as a gauge condition the self-dual part of $d\omega$,i.e. .

$$\omega^+_{\alpha\beta} \equiv P^{+}_{\alpha\beta}{}^{\mu\nu} \omega_{\mu\nu} = 0 \tag{7.27}$$

where the projector on the space of self-dual forms, is given by

$$P^{+}_{\alpha\beta}{}^{\mu\nu} \equiv \frac{1}{2}\left(\delta^{\mu\nu}_{\alpha\beta} - i\epsilon_{\alpha\beta}{}^{\mu\nu}\right) \tag{7.28}$$

This cuts in half the number of independent components, so that it amounts to three independent conditions only.

where

$$\omega_{\mu\nu} \equiv \partial^\lambda \left(\partial_\mu h_{\lambda\nu} - \partial_\nu h_{\lambda\mu}\right) \tag{7.30}$$

Another interesting possibility would be to gauge fix three dimension four scalars. The problem is that there are only two whose variation is not a total derivative, namely,

$$\begin{aligned} \Phi_1 &\equiv \partial_\mu h_{\alpha\beta} \partial^\mu h^{\alpha\beta} \\ \Phi_2 &\equiv \partial_\mu h_{\alpha\beta} \partial^\alpha h^{\mu\beta} \end{aligned} \tag{7.31}$$

Independent ghosts are defined through

$$c^\mu \equiv \partial_\rho c_1^{\rho\mu} \tag{7.32}$$

where we have indicated as a subscript the ghost number. The antighosts will be treated momentarily

Those objects are transverse:

$$\partial_\rho c^\rho = \partial_\rho b^\rho = 0 \tag{7.33}$$

owing to the fact that the two-index ghosts are assumed to be completely antisymmetric (ghostly forms).

There is the apparent complication that the ghostly forms are only defined modulo total differentials:

$$\epsilon^{\mu\nu\alpha\beta} \partial_\alpha c_\beta^1 \tag{7.34}$$

(this is indeed the correct counting: $6 - (4-1) = 3$).

The gauge fixing is then

$$L_{gf} \equiv B^\alpha \partial^\rho \omega_{\alpha\rho} + \frac{\alpha}{2} B_\alpha^2 \tag{7.35}$$

The corresponding ghost lagrangian is

$$L_{gh} = -b_\alpha \Box^2 c^\alpha \tag{7.36}$$

which has got the drawback of being of fourth order in derivatives, which is irrelevant nevertheless, because it is independent of the gauge fields. [18] The corresponding BRST transformations are:

$$\begin{aligned} s h_{\alpha\beta} &= \partial_\alpha \partial^\mu c^1_{\mu\beta} + \partial_\beta \partial^\mu c^1_{\mu\alpha} \\ s B_\mu &= 0 \\ s b_\mu &= -B_\mu \\ s c^1_{\rho\mu} &= 0 \end{aligned} \tag{7.40}$$

[18] Were we to impose the self-dual form as a gauge condition, then the gauge fixing piece of the lagrangian can be taken as

$$L_{gf} = B^{\alpha\beta} \omega^+_{\alpha\beta} + \frac{\alpha}{2} B^2_{\alpha\beta} \tag{7.37}$$

where the fields $B_{\alpha\beta}$ represent the three components of a selfdual form; and the ghost lagrangian reads

$$L_{gh} = b^+_{\alpha\beta} P_+^{\alpha\beta\mu\nu} \Box \left(\partial_\mu c_\nu - \partial_\nu c_\mu\right) \tag{7.38}$$

7.3 The non-abelian case

Let us now use the convenient language of differential forms, indicating sometimes its degree by a subscript (this is trivially related to the ghost number):

$$c \equiv c_1 \equiv c_\mu dx^\mu \tag{7.41}$$

with the constraint (please refer to the Appendix for our notation on differential forms)

$$\delta c_1 = 0 \tag{7.42}$$

so that there is locally a ghostly two-form c_2 such that

$$c_1 = \delta c_2 \tag{7.43}$$

This means that

$$\delta s c_1 = 0 \tag{7.44}$$

Indeed, given that acting on the metric

$$s g_{\alpha\beta} = \pounds(c) g_{\alpha\beta} = c^\lambda \partial_\lambda g_{\alpha\beta} + g_{\alpha\mu}\partial_\beta c^\mu + g_{\beta\mu}\partial_\alpha c^\mu \tag{7.45}$$

nilpotency needs

$$s c_1 = -\frac{1}{2}\delta(c \wedge c) \tag{7.46}$$

that is

$$s c^\mu = \partial_\lambda \left(c^\lambda c^\mu\right) = c^\lambda \partial_\lambda c^\mu \tag{7.47}$$

which means that ghosts act geometrically as scalars from the BRST viewpoint[19]. It is clear that (assuming $[s, \delta] = 0$)

$$\begin{aligned} s^2 c_1 = -\frac{1}{2} s\delta\left(c_1 \wedge c_1\right) = -\frac{1}{2}\delta\left(s c_1 \wedge c_1 - c_1 \wedge s c_1\right) = \\ \frac{1}{4}\delta\left(\delta\left(c_1 \wedge c_1\right) \wedge c_1 - c_1 \wedge \delta\left(c_1 \wedge c_1\right)\right) = 0 \end{aligned} \tag{7.49}$$

It is more or less unavoidable also here that this piece of the lagrangian is of third order in derivatives. The BRST transformations are then

$$\begin{aligned} & s h_{\alpha\beta} = \partial_\alpha \partial^\mu c^1_{\mu\beta} + \partial_\beta \partial^\mu c^1_{\mu\alpha} \\ & s B_{\mu\nu} = 0 \\ & s b^+_{\mu\nu} = -B_{\mu\nu} \\ & s c^1_{\rho\mu} = 0 \end{aligned} \tag{7.39}$$

[19] Please note that owing to the odd Grassmann parity of the ghosts,

$$c \wedge c \neq 0 \tag{7.48}$$

Consistency of equations (7.43) and (7.46) demands

$$s\delta c_2 = \delta s c_2 = -\frac{1}{2}\delta\left(c \wedge c\right) \tag{7.50}$$

that is,

$$s c_2 = -\frac{1}{2} c \wedge c - \delta c_3 \tag{7.51}$$

This three form c_3 cannot be trivial, because using nilpotency again, this time on the ghost itself,

$$s^2 c_2 = 0 = s\left(-\frac{1}{2} c \wedge c\right) - s\delta c_3 \tag{7.52}$$

whereas

$$s\left(c \wedge c\right) = -\frac{1}{2}\delta\left(c \wedge c\right) \wedge c + \frac{1}{2} c \wedge \delta\left(c \wedge c\right) = -\delta\left(c \wedge c \wedge c\right) \tag{7.53}$$

conveying the fact that

$$s c_3 = \frac{1}{2} c \wedge c \wedge c - \delta c_4 \tag{7.54}$$

Once more, using nilpotency, and the fact that

$$s\left(c \wedge c \wedge c\right) = -\frac{3}{2}\left(c \wedge c \wedge c \wedge c\right) \tag{7.55}$$

yields

$$s^2 c_3 = 0 = -\frac{3}{2}\delta(c \wedge c \wedge c \wedge c) - s\delta c_4 \tag{7.56}$$

ao that, finally

$$s c_4 = \frac{3}{2} c \wedge c \wedge c \wedge c \tag{7.57}$$

and $s^2 = 0$ because there are no forms of dgree five in four dimensions.

So we need altogether 11 independent ghosts: 6 grassmann odd, ghost number one c_2, plus 4 Grasmann even, ghost number two c_3 plus one Grassmann odd, ghost number three, c_4.

For the antighosts the story is even simpler.We define the corresponding forms:

$$b_1 = \delta b_2 \tag{7.58}$$

and

$$\begin{aligned} s b_2 &= B_2 \\ s B_2 &= 0 \end{aligned} \tag{7.59}$$

The antighosts b are Grassmann odd, and enjoy ghost number -1, whereas B is Grassmann even and has vanishing ghost number.

This analysis coincides basically with the one performed earlier by Dragon and Kreuzer [29][57], which however employ a non covariant, less convenient language.

7.4 Gauge Fixing

The gauge fixing fermion has got to be a Lorentz scalar of ghost number -1. We can define the most general operator composed out of fields with zero ghost number:

$$H_{\alpha_1\alpha_2} = A_{\alpha_1\alpha_2\alpha_3\alpha_4} h^{\alpha_3\alpha_4} + B_{\alpha_1\alpha_2\alpha_3\alpha_4} B^{\alpha_3\alpha_4} + C_{a_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6} h^{\alpha_3\alpha_4} h^{\alpha_5\alpha_6} + \ldots \tag{7.60}$$

That is, the most general polynomial in the fields B and h. The most general composite operator with ghost number -1 is of the form:

$$G_{\alpha_1\alpha_2} \equiv K_{\alpha_1\alpha_2\alpha_3\alpha_4} b^{\alpha_3\alpha_4} + K_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6} b^{\alpha_3\alpha_4} b^{\alpha_3\alpha_4} c^{\alpha_5\alpha_6} + \ldots \tag{7.61}$$

so that the gauge fixing fermion is given by

$$\Psi \equiv G_{\mu\nu} H^{\mu\nu} \tag{7.62}$$

where the indices are raised and lowered with Minkowski's metric. The contribution to the lagrangian is

$$s\Psi = \left(K_{\alpha_1\alpha_2\alpha_3\alpha_4} B^{\alpha_3\alpha_4} + \ldots\right) H^{\alpha_1\alpha_2} - G_{\alpha_1\alpha_2} \left(A_{\alpha_1\alpha_2\alpha_3\alpha_4} \left(\nabla^{\alpha_3}\nabla_\lambda c^{\alpha_4\lambda} + \nabla^{\alpha_4}\nabla_\lambda c^{\alpha_3\lambda}\right) + \ldots\right) \tag{7.63}$$

7.5 Setup

Starting from a TDiff pure gravity term

$$S_g = -\frac{1}{2\kappa^2} \int d^n x\; g^a\; R \tag{7.64}$$

we can go to the Einstein frame through a conformal change of the metric (under the assumption $a \neq \frac{1}{2}$)

$$\begin{aligned} \bar{g}_{\mu\nu} &= \chi^2[g] g_{\mu\nu} \\ \chi^2 &= g^{\frac{2a-1}{n-2}} \end{aligned} \tag{7.65}$$

In this new frame the action takes the form

$$S_g = -\frac{1}{2\kappa^2} \int d^n x\; \sqrt{\bar{g}}\; \bar{R} + S_\chi + S_\Lambda \tag{7.66}$$

where S_χ is a kinetic term for the scalar

$$\begin{aligned} S_\chi &= \frac{1}{2\kappa^2}\, \frac{(n-1)(n-2)(2a-1)^2}{16(an-1)^2} \int d^n x \sqrt{\bar{g}}\; \frac{1}{\bar{g}^2}\, \bar{g}^{\mu\nu} \partial_\mu \bar{g} \partial_\nu \bar{g} \\ &= \frac{1}{2\kappa^2}\, (n-1)(n-2) \int d^n x \sqrt{\bar{g}}\; \frac{1}{\chi^2}\, \bar{g}^{\mu\nu} \partial_\mu \chi \partial_\nu \chi \end{aligned} \tag{7.67}$$

and we have introduced a Lagrange multipler Λ to fix the determinant of the metric

$$S_\Lambda = \frac{1}{2\kappa^2}\int d^n x\ \Lambda\ \left[\sqrt{\bar{g}}\,\chi^{-\frac{2(an-1)}{2a-1}} - 1\right]. \tag{7.68}$$

Notice that formal Diff invariance is only broken by this last term. If we now include a matter term

$$S_M = \int d^n x\, g^b\ L_M[g_{\mu\nu}, \phi] \tag{7.69}$$

after the conformal redefinition it takes the form

$$S_M = \int d^n x\ \sqrt{\bar{g}}\ \chi^{\frac{2n(b-a)+2(1-2b)}{2a-1}}\ L_M\left[\chi^{-2}\bar{g}_{\mu\nu}, \phi\right] \tag{7.70}$$

One should also include a kinetic term for the determinant of the metric since anyway it will be generated radiatively

$$\begin{aligned} S_k &= \frac{1}{2\kappa^2}\int d^n x\ g^c\ \left[\frac{1}{2}g^{\mu\nu}\partial_\mu g\partial_\nu g - V(g)\right] \\ &= \frac{1}{2\kappa^2}\int d^n x\ \sqrt{\bar{g}}\ \chi^{\frac{2n(c-a)+2(1-2c)}{2a-1}}\ \left[\frac{2(n-2)^2}{(2a-1)^2}\chi^{\frac{4(n-2)}{2a-1}}\bar{g}^{\mu\nu}\partial_\mu\chi\partial_\nu\chi - V(\chi(g))\right] \end{aligned} \tag{7.71}$$

7.6 Background TDiff

We can consider quantum fluctuations around a classical background, $\bar{g}_{\mu\nu}$. We parametrize the strength of the perturbation by a field of mass dimension one:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \kappa h_{\mu\nu} \tag{7.72}$$

where $\kappa^2 \equiv 8\pi G$, as usual. This expression involving the metric as a covariant tensor can be considered as an *exact* expansion; or rather as the definition itself of the perturbation to be considered; all other geometric expansions are then defined as formal series in the coupling constant κ.

If we introduce two other quantities.

$$h^{\mu\nu} \equiv \bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}h_{\alpha\beta} \tag{7.73}$$

and

$$h^\mu_\nu \equiv \bar{g}^{\mu\alpha}h_{\alpha\nu} \tag{7.74}$$

then,

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - \kappa h^{\mu\nu} + \kappa^2 (h^2)^{\mu\nu} + o(\kappa^3) \tag{7.75}$$

We shall denote for a general quantity:

$$A \equiv \sum_n \kappa^n \overset{(n)}{A} \tag{7.76}$$

The determinant of the metric expands as

$$g \equiv det\, g_{\mu\nu} = det\, \bar{g}_{\mu\lambda}\left(\delta^\lambda_\nu + \kappa \bar{g}^{\lambda\sigma} h_{\sigma\nu}\right) = \bar{g} e^{tr\, \log\left(\delta^\lambda_\nu + \kappa \bar{g}^{\lambda\sigma} h_{\sigma\nu}\right)} \tag{7.77}$$

namely,

$$g = \bar{g}\{1 + \kappa \bar{g}^{\alpha\beta} h_{\alpha\beta} + \frac{\kappa^2}{2}[(\bar{g}^{\alpha\beta} h_{\alpha\beta})^2 - \bar{g}^{\alpha\beta}(h^2)_{\alpha\beta})]\} \tag{7.78}$$

and

$$\sqrt{|g|} = \sqrt{|\bar{g}|}\{1 + \frac{\kappa}{2}\bar{g}^{\alpha\beta} h_{\alpha\beta} + \frac{\kappa^2}{8}[(\bar{g}^{\alpha\beta} h_{\alpha\beta})^2 - 2\bar{g}^{\alpha\beta}(h^2)_{\alpha\beta}]\} \tag{7.79}$$

Under an arbitrary diffeomorphism,

$$\delta g_{\mu\nu} = \pounds(\xi) g_{\mu\nu} = \xi^\alpha \partial_\alpha g_{\mu\nu} + \partial_\mu \xi^\alpha g_{\alpha\nu} + \partial_\nu \xi^\alpha g_{\mu\alpha} \tag{7.80}$$

The determinant of the metric transforms as

$$\delta|g| = |g| g^{\mu\nu} \delta g_{\mu\nu} \tag{7.81}$$

This is an exact formula in the linear diffeomorphism regime. The condition for the metric volume element to remain invariant, that is,

$$\delta|g| = 0 \tag{7.82}$$

is

$$\nabla_\mu \xi^\mu = 0 \tag{7.83}$$

which is equivalent to our unimodularity condition only in flat space.

The background field ansatz implies a further expansion in the coupling constant κ.

Backgroung gauge transformations correspond to

$$\begin{aligned} \delta\bar{g}_{\mu\nu} &= \pounds(\xi)\bar{g}_{\mu\nu} \\ \delta h_{\mu\nu} &= \pounds(\xi) h_{\mu\nu} \end{aligned} \tag{7.84}$$

The quantum gauge transformations read

$$\begin{aligned} \delta\bar{g}_{\mu\nu} &= 0 \\ \delta h_{\mu\nu} &= \frac{1}{\kappa}\pounds(\xi)\bar{g}_{\mu\nu} + \pounds(\xi) h_{\mu\nu} \end{aligned} \tag{7.85}$$

Those are the ones that must be gauge fixed.

Under those,

$$\bar{g}^{\alpha\beta}\delta h_{\alpha\beta} = \frac{1}{\kappa}\left(\xi^\rho \partial_\rho |\bar{g}| + 2\partial_\alpha \xi^a\right) + O(1) \tag{7.86}$$

i.e., even if the diffeomorphism is unimodular in our sense, $\partial_\alpha \xi^\alpha = 0$, there is a non-vanishing contribution to the variation of the determinant of the metric.

Suppose now that we have a TDiff action of the form

$$S_g = -\frac{1}{2\kappa^2} \int d^n x \, \sqrt{g^*} \, \left[f(g^*) \, R^* + 2 f_\lambda(g^*) \, \Lambda \right] \tag{7.87}$$

where f and f_λ are arbitrary functions of the determinant of th metric $g^* \equiv \det g^*_{\mu\nu}$, and the action is in general not Diff invariant. Moreover, under a Diff the action transforms to

$$S_g = -\frac{1}{2\kappa^2} \int d^n x \, \sqrt{g^*} \, \left[f(g^* C^2) \, R^* + 2 f_\lambda(g^* C^2) \, \Lambda \right] \tag{7.88}$$

where $C(x)$ is a compensator field, in particular the determinant of the Jacobian of the coordenates change, so that we can write in terms of the scalar field $\varphi^* \equiv g^* C^2$ a perfectly invariant action

$$S_g = -\frac{1}{2\kappa^2} \int d^n x \, \sqrt{g^*} \, \left[f(\varphi^*) \, R^* + 2 f_\lambda(\varphi^*) \, \Lambda \right] \tag{7.89}$$

To perform the computation is convenient to go to the Einstein frame, so we make a conformal transformation

$$\begin{aligned} g_{\mu\nu} &= \Omega^2 g^*_{\mu\nu} \\ g &= \Omega^{2n} g^* \\ \varphi &= g C^2 = \Omega^{2n} g^* C^2 = \Omega^{2n} \varphi^* \end{aligned} \tag{7.90}$$

If we choose the conformal factor as

$$\Omega^{n-2} = f(\varphi^*) = f(\Omega^{-2n} \varphi) \tag{7.91}$$

then in terms of the new metric the action takes the form

$$S_g = -\frac{1}{2\kappa^2} \int d^n x \, \sqrt{g} \, \left[R + 2 F_\lambda(\Omega) \, \Lambda \right] + \frac{(n-1)(n-2)}{2\kappa^2} \int d^n x \, \sqrt{g} \, \frac{1}{\Omega^2} \, g^{\mu\nu} \partial_\mu \Omega \partial_\nu \Omega \tag{7.92}$$

where we have made use of (7.91) in order to express f_λ in terms of Ω

$$\Omega^{-n} f_\lambda(\Omega^{-2n} \varphi(\Omega)) \equiv F_\lambda(\Omega) \tag{7.93}$$

A final redefinition of the scalar

$$\phi \equiv \sqrt{2(n-1)(n-2)} \, \ln \Omega \tag{7.94}$$

gives us the desired action

$$S_g = -\frac{1}{2\kappa^2} \int d^n x \, \sqrt{g} \, \left[R + 2 F_\lambda(\phi) \, \Lambda \right] + \frac{1}{2\kappa^2} \int d^n x \, \sqrt{g} \, \frac{1}{2} \, g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \tag{7.95}$$

Expanding the fields in a background and a fluctuation

$$\begin{aligned} g_{\mu\nu} &= \bar{g}_{\mu\nu} + \kappa h_{\mu\nu} \\ \phi &= \bar{\phi} + \kappa\phi \end{aligned} \tag{7.96}$$

and using

$$F_\lambda(\phi) = F_\lambda(\bar{\phi}) + \kappa F'_\lambda(\bar{\phi})\,\phi + \frac{\kappa^2}{2} F''_\lambda(\bar{\phi})\phi^2 + O(\kappa^3) \tag{7.97}$$

we find up to quadratic order the action

$$\begin{aligned} S_g = -\frac{1}{2\kappa^2} \int d^n x\, \sqrt{\bar{g}}\, \Bigg[&\bar{R} + 2\Lambda F_\lambda(\bar{\phi}) - \frac{1}{2}\bar{g}^{\mu\nu}\partial_\mu\bar{\phi}\partial_\nu\bar{\phi} + \\ &+\kappa\left(\frac{1}{2}h\bar{R} + \overset{(1)}{R} + \Lambda h F_\lambda(\bar{\phi}) + 2\Lambda F'_\lambda(\bar{\phi})\phi - \bar{g}^{\mu\nu}\partial_\mu\bar{\phi}\partial_\nu\phi - \frac{1}{4}\left(h\bar{g}^{\mu\nu} - 2h^{\mu\nu}\right)\partial_\mu\bar{\phi}\partial_\nu\bar{\phi}\right) + \\ &+\kappa^2\Bigg(\overset{(2)}{R} + \frac{1}{2}h\overset{(1)}{R} + \frac{\left(\bar{R} + 2\Lambda F_\lambda(\bar{\phi}) - \frac{1}{2}\bar{g}^{\mu\nu}\partial_\mu\bar{\phi}\partial_\nu\bar{\phi}\right)}{8}\left(h^2 - 2h_{\mu\nu}h^{\mu\nu}\right) + \Lambda h F'_\lambda(\bar{\phi})\phi + \\ &+\Lambda F''_\lambda(\bar{\phi})\phi^2 - \frac{1}{2}\bar{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}\left(2h^{\mu\nu} - h\bar{g}^{\mu\nu}\right)\partial_\mu\phi\partial_\nu\bar{\phi} + \frac{1}{4}\left(hh^{\mu\nu} - 2h^{\mu\alpha}h^{\nu}_{\alpha}\right)\partial_\mu\bar{\phi}\partial_\nu\bar{\phi}\Bigg)\Bigg] \end{aligned} \tag{7.98}$$

The term linear in the coupling cancels due to the background equations of motion, namely

$$\begin{aligned} \bar{\nabla}^2\bar{\phi} + 2\Lambda F'_\lambda(\bar{\phi}) &= 0 \\ \bar{R}_{\mu\nu} - \frac{1}{2}\bar{R}\bar{g}_{\mu\nu} - \Lambda F_\lambda(\bar{\phi})\bar{g}_{\mu\nu} - \frac{1}{2}\bar{\nabla}_\mu\bar{\phi}\bar{\nabla}_\nu\bar{\phi} + \frac{1}{4}\bar{g}_{\mu\nu}\bar{g}^{\alpha\beta}\bar{\nabla}_\alpha\bar{\phi}\bar{\nabla}_\beta\bar{\phi} &= 0 \end{aligned} \tag{7.99}$$

Using the known expansion for the scalar curvature (6.134) the quadratic order operator is

$$\begin{aligned} S_g = \frac{1}{2} \int d^n x\, \sqrt{\bar{g}}\, \Bigg[&h^{\alpha\beta}\Bigg(\frac{1}{4}\bar{g}_{\alpha\beta}\bar{g}_{\mu\nu}\bar{\nabla}^2 - \frac{1}{4}\bar{g}_{\alpha\mu}\bar{g}_{\beta\nu}\bar{\nabla}^2 + \frac{1}{2}\bar{g}_{\alpha\mu}\bar{\nabla}_\beta\bar{\nabla}_\nu - \frac{1}{2}\bar{g}_{\mu\nu}\bar{\nabla}_\alpha\bar{\nabla}_\beta \\ &+\frac{1}{2}\bar{g}_{\alpha\beta}\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\alpha\mu}\bar{R}_{\beta\nu} - \frac{1}{2}\bar{R}_{\alpha\mu\beta\nu} + \frac{1}{2}\bar{g}_{\alpha\mu}\partial_\beta\bar{\phi}\partial_\nu\bar{\phi} - \frac{1}{4}\bar{g}_{\alpha\beta}\partial_\mu\bar{\phi}\partial_\nu\bar{\phi} \\ &-\frac{\left(\bar{R} + 2\Lambda F_\lambda(\bar{\phi}) - \frac{1}{2}\bar{g}^{\rho\sigma}\partial_\rho\bar{\phi}\partial_\sigma\bar{\phi}\right)}{8}\left(\bar{g}_{\alpha\beta}\bar{g}_{\mu\nu} - 2\bar{g}_{\alpha\mu}\bar{g}_{\beta\nu}\right)\Bigg)h^{\mu\nu} \\ &+h^{\alpha\beta}\left(\frac{1}{2}\bar{g}_{\alpha\beta}\bar{g}^{\rho\sigma}\partial_\rho\bar{\phi}\partial_\sigma - \partial_\alpha\bar{\phi}\partial_\beta - \Lambda\bar{g}_{\alpha\beta}F'_\lambda(\bar{\phi})\right)\phi + \phi\left(-\frac{1}{2}\bar{\nabla}^2 - \Lambda F''_\lambda(\bar{\phi})\right)\phi\Bigg] \end{aligned} \tag{7.100}$$

At this stage the operator is very cumbersome, but we still have the freedom to fix the gauge in a way that simplifies the computationa lot. Taking the expresion

$$\chi_\nu = \bar{\nabla}^\mu h_{\mu\nu} - \frac{1}{2}\bar{\nabla}_\nu h - \phi\partial_\nu\bar{\phi} \tag{7.101}$$

we choose as gauge fixing term

$$S_{gf} = \frac{1}{2} \int d^n x \, \sqrt{\bar{g}} \, \frac{1}{2\xi} \, \bar{g}^{\mu\nu} \chi_\mu \chi_\nu \tag{7.102}$$

which after expanding can be expressed in the following form

$$\begin{aligned} S_{gf} = & \frac{1}{2} \int d^n x \, \sqrt{\bar{g}} \, \frac{1}{2\xi} \, \Big[h^{\alpha\beta} \left(\bar{g}_{\mu\nu} \bar{\nabla}_\alpha \bar{\nabla}_\beta - \bar{g}_{\alpha\mu} \bar{\nabla}_\beta \bar{\nabla}_\nu - \frac{1}{4} \bar{g}_{\alpha\beta} \bar{g}_{\mu\nu} \bar{\nabla}^2 \right) h^{\mu\nu} \\ & + 2 h^{\alpha\beta} \left(\partial_\alpha \bar{\phi} \partial_\beta + \bar{\nabla}_\alpha \bar{\nabla}_\beta \bar{\phi} - \frac{1}{2} \bar{g}_{\alpha\beta} \bar{g}^{\rho\sigma} \partial_\rho \bar{\phi} \partial_\sigma - \frac{1}{2} \bar{g}_{\alpha\beta} \bar{g}^{\rho\sigma} \bar{\nabla}_\rho \bar{\nabla}_\sigma \bar{\phi} \right) \phi \\ & + \phi \left(\bar{g}^{\alpha\beta} \partial_\alpha \bar{\phi} \partial_\beta \bar{\phi} \right) \phi \Big] \end{aligned} \tag{7.103}$$

Let us define the following tensor with the desired symmetry properties, i.e., symmetric in $(\mu\nu)$, $(\alpha\beta)$ and under the interchange $(\mu\nu) \leftrightarrow (\alpha\beta)$

$$\begin{aligned} C_{\alpha\beta\mu\nu} &= \frac{1}{4} \left(\bar{g}_{\alpha\mu} \bar{g}_{\beta\nu} + \bar{g}_{\alpha\nu} \bar{g}_{\beta\mu} - \bar{g}_{\alpha\beta} \bar{g}_{\mu\nu} \right) \\ C^{\alpha\beta\mu\nu} &= \bar{g}^{\alpha\mu} \bar{g}^{\beta\nu} + \bar{g}^{\alpha\nu} \bar{g}^{\beta\mu} - \frac{2}{n-2} \bar{g}^{\alpha\beta} \bar{g}^{\mu\nu} \\ \delta^{\alpha\beta}_{\mu\nu} &= \delta^{(\alpha}_\mu \delta^{\beta)}_\nu \end{aligned} \tag{7.104}$$

the full action can be written as

$$S_g + S_{gf} = \frac{1}{2} \int d^n x \, \sqrt{\bar{g}} \, \frac{1}{2} \, \left[h^{\alpha\beta} M_{\alpha\beta\mu\nu} h^{\mu\nu} + h^{\alpha\beta} D_{\alpha\beta} \phi + \phi E_{\mu\nu} h^{\mu\nu} + \phi F \phi \right] \tag{7.105}$$

where the operators are

$$\begin{aligned} M_{\alpha\beta\mu\nu} &= C_{\alpha\beta\rho\sigma} \left(-\delta^{\rho\sigma}_{\mu\nu} \bar{\nabla}^2 + \frac{1-\xi}{\xi} \bar{g}_{\mu\nu} \bar{\nabla}^{(\rho} \bar{\nabla}^{\sigma)} + \frac{2(\xi-1)}{\xi} \delta^{(\rho}_{(\mu} \bar{\nabla}^{\sigma)} \bar{\nabla}_{\nu)} + P^{\rho\sigma}_{\mu\nu} \right) \\ P^{\rho\sigma}_{\mu\nu} &= -2 \bar{R}^{(\rho}{}_\mu{}^{\sigma)}{}_\nu - 2 \delta^{(\rho}_{(\mu} \bar{R}^{\sigma)}_{\nu)} + \left(\bar{R} + 2\Lambda F_\lambda(\bar{\phi}) - \frac{1}{2} \bar{g}^{\alpha\beta} \partial_\alpha \bar{\phi} \partial_\beta \bar{\phi} \right) \delta^{\rho\sigma}_{\mu\nu} + \bar{g}^{\rho\sigma} \bar{R}_{\mu\nu} \\ & + \frac{2}{(n-2)} \bar{g}_{\mu\nu} \bar{R}^{\rho\sigma} - \frac{1}{(n-2)} \bar{g}_{\mu\nu} \bar{g}^{\rho\sigma} \bar{R} + 2 \delta^{(\rho}_{(\mu} \partial_{\nu)} \bar{\phi} \partial^{\sigma)} \bar{\phi} - \frac{1}{2} \bar{g}_{\mu\nu} \partial^\rho \bar{\phi} \partial^\sigma \bar{\phi} \\ & - \frac{1}{(n-2)} \bar{g}^{\rho\sigma} \partial_\mu \bar{\phi} \partial_\nu \bar{\phi} + \frac{1}{2(n-2)} \bar{g}_{\mu\nu} \bar{g}^{\rho\sigma} \partial_\lambda \bar{\phi} \partial^\lambda \bar{\phi} \\ D_{\alpha\beta} &= \frac{2(1-\xi)}{\xi} \, C_{\alpha\beta\rho\sigma} \, \bar{\nabla}^\rho \bar{\phi} \bar{\nabla}^\sigma + \frac{\xi+1}{\xi} \, C_{\alpha\beta\rho\sigma} \bar{\nabla}^\rho \bar{\nabla}^\sigma \bar{\phi} - \Lambda F'_\lambda(\bar{\phi}) \bar{g}_{\alpha\beta} \\ E_{\mu\nu} &= \frac{2(\xi-1)}{\xi} \, C_{\mu\nu\rho\sigma} \, \bar{\nabla}^\rho \bar{\phi} \bar{\nabla}^\sigma + \frac{\xi+1}{\xi} \, C_{\mu\nu\rho\sigma} \bar{\nabla}^\rho \bar{\nabla}^\sigma \bar{\phi} - \Lambda F'_\lambda(\bar{\phi}) \bar{g}_{\mu\nu} \\ F &= -\bar{\nabla}^2 - 2\Lambda F''_\lambda(\bar{\phi}) + \frac{1}{\xi} \bar{g}^{\rho\sigma} \partial_\rho \bar{\phi} \partial_\sigma \bar{\phi} \end{aligned} \tag{7.106}$$

in such a way that in terms of the combined field

$$\psi^A \equiv \begin{pmatrix} h^{\mu\nu} \\ \phi \end{pmatrix} \tag{7.107}$$

and in the minimal gauge, corresponding to $\xi = 1$, the operator

$$S = \frac{1}{2} \int d^n x \, \sqrt{\bar{g}} \, \frac{1}{2} \, \psi^A \Delta_{AB} \psi^B \tag{7.108}$$

is minimal, in the sense that it takes a Laplacian form

$$\Delta_{AB} = -g_{AB} \bar{\nabla}^2 + Y_{AB} \tag{7.109}$$

with the metric

$$g_{AB} = \begin{pmatrix} C_{\alpha\beta\mu\nu} & 0 \\ 0 & 1 \end{pmatrix} \tag{7.110}$$

the inverse metric

$$g^{AB} = \begin{pmatrix} C^{\alpha\beta\mu\nu} & 0 \\ 0 & 1 \end{pmatrix} \tag{7.111}$$

and the term without derivatives

$$Y_{AB} = \begin{pmatrix} C_{\alpha\beta\rho\sigma} P^{\rho\sigma}_{\mu\nu} & 2C_{\alpha\beta\rho\sigma} \bar{\nabla}^\rho \bar{\nabla}^\sigma \bar{\phi} - \Lambda F'_\lambda(\bar{\phi}) \bar{g}_{\alpha\beta} \\ 2C_{\mu\nu\rho\sigma} \bar{\nabla}^\rho \bar{\nabla}^\sigma \bar{\phi} - \Lambda F'_\lambda(\bar{\phi}) \bar{g}_{\mu\nu} & -2\Lambda F''_\lambda(\bar{\phi}) + \bar{g}^{\rho\sigma} \partial_\rho \bar{\phi} \partial_\sigma \bar{\phi} \end{pmatrix} \tag{7.112}$$

On the other hand, once we have an operator in the Laplacian form (7.109), the one-loop counterterm (supposing that we work in $n = 4$ dimensions) is given by the following coefficient in the heat kernel expansion

$$\begin{aligned} a_4 = \frac{1}{(4\pi)^{\frac{n}{2}}} \frac{1}{360} \int d^n x \sqrt{\bar{g}} \; \mathrm{tr} \; \big(180 Y^2 - 60 \bar{R} Y + 5 \bar{R}^2 - \\ -2 \bar{R}_{\mu\nu} \bar{R}^{\mu\nu} + 2 \bar{R}_{\mu\nu\rho\sigma} \bar{R}^{\mu\nu\rho\sigma} + 30 W_{\mu\nu} W^{\mu\nu} \big) \end{aligned} \tag{7.113}$$

and the field strength is defined through

$$[\bar{\nabla}_\mu, \bar{\nabla}_\nu] \psi^A = W^A_{B\mu\nu} \psi^B \tag{7.114}$$

Therefore, in order to find the counterterm we will need the following traces

$$
\begin{aligned}
\mathrm{tr}\ \mathbb{I} &= \delta^{\alpha\beta}_{\alpha\beta} + 1 = \frac{n(n+1)+2}{2} \\
\mathrm{tr}\ Y &= g^{AB} Y_{AB} = \delta^{\mu\nu}_{\alpha\beta} P^{\alpha\beta}_{\mu\nu} - 2\Lambda F''_\lambda(\bar\phi) + \bar g^{\rho\sigma}\partial_\rho\bar\phi\partial_\sigma\bar\phi \\
&= \frac{n(n+1)}{2}\left(\bar R + 2\Lambda F_\lambda(\bar\phi) - \frac{1}{2}\bar g^{\rho\sigma}\partial_\rho\bar\phi\partial_\sigma\bar\phi\right) - n\bar R + (n+2)\bar g^{\rho\sigma}\partial_\rho\bar\phi\partial_\sigma\bar\phi - 2\Lambda F''_\lambda(\bar\phi) \\
\mathrm{tr}\ Y^2 &= Y_{AB}\, g^{BC}\, Y_{CD}\, g^{DA} = P^{\alpha\beta}_{\mu\nu} P^{\mu\nu}_{\alpha\beta} + 2 D_{\alpha\beta} E_{\mu\nu} C^{\mu\nu\alpha\beta} + \left(2\Lambda F''_\lambda(\bar\phi) - \bar g^{\rho\sigma}\partial_\rho\bar\phi\partial_\sigma\bar\phi\right)^2 \\
&= 3\bar R_{\mu\nu\rho\sigma}\bar R^{\mu\nu\rho\sigma} + \frac{n^2-8n+4}{n-2}\bar R_{\mu\nu}\bar R^{\mu\nu} + \frac{n+2}{n-2}\bar R^2 - 2n\bar R\left(\bar R + 2\Lambda F_\lambda(\bar\phi) - \frac{1}{2}\bar g^{\rho\sigma}\partial_\rho\bar\phi\partial_\sigma\bar\phi\right) \\
&+ \frac{n\,(n+1)}{2}\left(\bar R + 2\Lambda F_\lambda(\bar\phi) - \frac{1}{2}\bar g^{\rho\sigma}\partial_\rho\bar\phi\partial_\sigma\bar\phi\right)^2 + 2\bar\nabla^2\bar\phi\bar\nabla^2\bar\phi - 8\Lambda F'_\lambda\,\bar\nabla^2\bar\phi \\
&- \frac{8n}{n-2}\Lambda^2(F'_\lambda(\bar\phi))^2 + \frac{n^2-5}{n-2}\left(\bar g^{\rho\sigma}\partial_\rho\bar\phi\partial_\sigma\bar\phi\right)^2 + \frac{n(4-n)(3n-8)-4(n-2)^2}{(n-2)^2}\bar R^{\mu\nu}\partial_\mu\bar\phi\partial_\nu\bar\phi \\
&- \frac{n^2+4n-16}{(n-2)^2}\,\bar R\,\bar g^{\rho\sigma}\partial_\rho\bar\phi\partial_\sigma\bar\phi + 2\,(n+1)\left(\bar R + 2\Lambda F_\lambda(\bar\phi) - \frac{1}{2}\bar g^{\rho\sigma}\partial_\rho\bar\phi\partial_\sigma\bar\phi\right)\bar g^{\gamma\lambda}\partial_\gamma\bar\phi\partial_\lambda\bar\phi \\
&+ \left(2\Lambda F''_\lambda(\bar\phi) - \bar g^{\rho\sigma}\partial_\rho\bar\phi\partial_\sigma\bar\phi\right)^2 \\
\mathrm{tr}\ W_{\mu\nu}W^{\mu\nu} &= -(n+2)\bar R_{\mu\nu\rho\sigma}\bar R^{\mu\nu\rho\sigma}
\end{aligned} \tag{7.11}
$$

Using the well known expression (7.113) of the fourth heat kernel coefficient one gets

$$
\begin{aligned}
a_4 = \frac{1}{(4\pi)^{\frac{n}{2}}}\,\frac{1}{360}\int d^n x\,\sqrt{\bar g}\,\Big\{&[542 + n(n+1) - 30(n+2)]\,\bar R_{\mu\nu\rho\sigma}\bar R^{\mu\nu\rho\sigma} \\
&+ \left[180\frac{n^2-8n+4}{n-2} - n(n+1) - 2\right]\bar R_{\mu\nu}\bar R^{\mu\nu} + \left[180\frac{n+2}{n-2} + 60n + \frac{5n(n+1)+10}{2}\right]\bar R^2 \\
&- 30n(n+13)\bar R\left(\bar R + 2\Lambda F_\lambda(\bar\phi) - \frac{1}{2}\bar g^{\rho\sigma}\partial_\rho\bar\phi\partial_\sigma\bar\phi\right) + 90n(n+1)\left(\bar R + 2\Lambda F_\lambda(\bar\phi) - \frac{1}{2}\bar g^{\rho\sigma}\partial_\rho\bar\phi\partial_\sigma\bar\phi\right)^2 \\
&+ 180\frac{n(4-n)(3n-8)-4(n-2)^2}{(n-2)^2}\,\bar R^{\mu\nu}\partial_\mu\bar\phi\partial_\nu\bar\phi - 60\left[3\frac{n^2+4n-16}{(n-2)^2} + (n+2)\right]\bar R\,\bar g^{\rho\sigma}\partial_\rho\bar\phi\partial_\sigma\bar\phi \\
&+ 360\,(n+1)\left(\bar R + 2\Lambda F_\lambda(\bar\phi) - \frac{1}{2}\bar g^{\rho\sigma}\partial_\rho\bar\phi\partial_\sigma\bar\phi\right)\bar g^{\gamma\lambda}\partial_\gamma\bar\phi\partial_\lambda\bar\phi + 180\frac{n^2+n-7}{n-2}\left(\bar g^{\rho\sigma}\partial_\rho\bar\phi\partial_\sigma\bar\phi\right)^2 \\
&+ 360\bar\nabla^2\bar\phi\bar\nabla^2\bar\phi + 120\Lambda\bar R F''_\lambda(\bar\phi) - 1440\Lambda F'_\lambda(\bar\phi)\bar\nabla^2\bar\phi - 720\Lambda F''_\lambda(\bar\phi)\bar g^{\rho\sigma}\partial_\rho\bar\phi\partial_\sigma\bar\phi \\
&- \frac{1440n}{n-2}\Lambda^2(F'_\lambda(\bar\phi))^2 + 720\Lambda^2(F''_\lambda(\bar\phi))^2\Big\}
\end{aligned} \tag{7.116}
$$

Remember that we need also the contribution coming from ghost loops. The gauge fixing term mantains background invariance, under which the background $\bar g_{\mu\nu}$ transforms as a metric and the fluctuation $h_{\mu\nu}$ as a tensor. On the other hand it has to break the quatum symmetry

$$\begin{aligned}
\delta\bar{g}_{\mu\nu} &= 0 \\
\delta h_{\mu\nu} &= \frac{2}{\kappa}\bar{\nabla}_{(\mu}\xi_{\nu)} + \mathcal{L}_\xi h_{\mu\nu} \\
\delta\bar{\phi} &= 0 \\
\delta\phi &= \frac{1}{\kappa}\xi^\mu\bar{\nabla}_\mu\left(\bar{\phi}+\kappa\phi\right)
\end{aligned} \tag{7.117}$$

The ghost Lagrangian is obtained performing a variation on the gauge fixing term

$$\delta\chi_\nu = \frac{1}{\kappa}\left(\bar{\nabla}^2\bar{g}_{\mu\nu} + \bar{R}_{\mu\nu} - \bar{\nabla}_\mu\bar{\phi}\bar{\nabla}_\nu\bar{\phi}\right)\xi^\mu \tag{7.118}$$

plus terms that give operators cubic in fluctuations and therefore are irrelevant at one loop (the ghosts are always quantum fields, they do not appear as external states). Then, as ghost Lagrangian we will take

$$S_{gh} = \frac{1}{2}\int d^n x\,\sqrt{\bar{g}}\,\frac{1}{2}V_\mu^*\left(-\bar{\nabla}^2\bar{g}^{\mu\nu} - \bar{R}^{\mu\nu} + \bar{\nabla}^\mu\bar{\phi}\bar{\nabla}^\nu\bar{\phi}\right)V_\nu \tag{7.119}$$

The relevant traces are

$$\begin{aligned}
\text{tr }\mathbb{I} &= n \\
\text{tr }Y &= -\bar{R} + \bar{g}^{\rho\sigma}\partial_\rho\bar{\phi}\partial_\sigma\bar{\phi} \\
\text{tr }Y^2 &= \bar{R}_{\mu\nu}\bar{R}^{\mu\nu} - 2\bar{R}^{\mu\nu}\partial_\mu\bar{\phi}\partial_\nu\bar{\phi} + \left(\bar{g}^{\rho\sigma}\partial_\rho\bar{\phi}\partial_\sigma\bar{\phi}\right)^2 \\
\text{tr }W_{\mu\nu}W^{\mu\nu} &= -\bar{R}_{\mu\nu\rho\sigma}\bar{R}^{\mu\nu\rho\sigma}
\end{aligned} \tag{7.120}$$

and the coefficient

$$\begin{aligned}
a_4^{gh} = \frac{1}{(4\pi)^{\frac{n}{2}}}\frac{1}{360}\int d^n x\,\sqrt{\bar{g}}\,&\Big\{[2n-30]\,\bar{R}_{\mu\nu\rho\sigma}\bar{R}^{\mu\nu\rho\sigma} + [180-2n]\,\bar{R}_{\mu\nu}\bar{R}^{\mu\nu} \\
&+ [60+5n]\,\bar{R}^2 - 360\bar{R}^{\mu\nu}\partial_\mu\bar{\phi}\partial_\nu\bar{\phi} - 60\bar{R}\bar{g}^{\rho\sigma}\partial_\rho\bar{\phi}\partial_\sigma\bar{\phi} + 180\left(\bar{g}^{\rho\sigma}\partial_\rho\bar{\phi}\partial_\sigma\bar{\phi}\right)^2\Big\}
\end{aligned} \tag{7.121}$$

Adding the two pieces together and particularizing to the physical dimension $n = 4$ one gets the one loop counterterm (notice the factor and the sign of the ghost contribution)

$$\begin{aligned}
\Delta S = \frac{1}{\epsilon}\left(a_4 - 2a_4^{gh}\right) = \frac{1}{\epsilon}\frac{1}{(4\pi)^2}\int d^4x\,\sqrt{\bar{g}}\,&\Big\{\frac{71}{60}\bar{R}_{\mu\nu\rho\sigma}\bar{R}^{\mu\nu\rho\sigma} - \frac{241}{60}\bar{R}_{\mu\nu}\bar{R}^{\mu\nu} + \frac{15}{8}\bar{R}^2 \\
&-\frac{17}{3}\bar{R}\left(\bar{R} + 2\Lambda F_\lambda(\bar{\phi}) - \frac{1}{2}\bar{g}^{\rho\sigma}\partial_\rho\bar{\phi}\partial_\sigma\bar{\phi}\right) + 5\left(\bar{R} + 2\Lambda F_\lambda(\bar{\phi}) - \frac{1}{2}\bar{g}^{\rho\sigma}\partial_\rho\bar{\phi}\partial_\sigma\bar{\phi}\right)^2 \\
&-\frac{8}{3}\bar{R}\,\bar{g}^{\rho\sigma}\partial_\rho\bar{\phi}\partial_\sigma\bar{\phi} + 5\left(\bar{R} + 2\Lambda F_\lambda(\bar{\phi}) - \frac{1}{2}\bar{g}^{\rho\sigma}\partial_\rho\bar{\phi}\partial_\sigma\bar{\phi}\right)\bar{g}^{\gamma\lambda}\partial_\gamma\bar{\phi}\partial_\lambda\bar{\phi} + \frac{9}{4}\left(\bar{g}^{\rho\sigma}\partial_\rho\bar{\phi}\partial_\sigma\bar{\phi}\right)^2 \\
&+\bar{\nabla}^2\bar{\phi}\bar{\nabla}^2\bar{\phi} + \frac{1}{3}\Lambda\bar{R}F_\lambda''(\bar{\phi}) - 4\Lambda F_\lambda'(\bar{\phi})\bar{\nabla}^2\bar{\phi} - 2\Lambda F_\lambda''(\bar{\phi})\bar{g}^{\rho\sigma}\partial_\rho\bar{\phi}\partial_\sigma\bar{\phi} - 8\Lambda^2(F_\lambda'(\bar{\phi}))^2 \\
&+2\Lambda^2(F_\lambda''(\bar{\phi}))^2\Big\}
\end{aligned} \tag{7.122}$$

In case that the cosmological constant vanishes the final result is

$$
\begin{aligned}
\Delta S =& \frac{1}{\epsilon}\frac{1}{(4\pi)^2}\frac{1}{360}\int d^4x\,\sqrt{\bar{g}}\,\left\{426\bar{R}_{\mu\nu\rho\sigma}\bar{R}^{\mu\nu\rho\sigma}-1446\bar{R}_{\mu\nu}\bar{R}^{\mu\nu}+435\bar{R}^2\right.\\
&\left.+60\bar{R}\,\bar{g}^{\rho\sigma}\partial_\rho\bar{\phi}\partial_\sigma\bar{\phi}+360\left(\bar{g}^{\rho\sigma}\partial_\rho\bar{\phi}\partial_\sigma\bar{\phi}\right)^2+360\bar{\nabla}^2\bar{\phi}\bar{\nabla}^2\bar{\phi}\right\}\\
=& \frac{1}{\epsilon}\frac{1}{(4\pi)^2}\int d^nx\,\sqrt{\bar{g}}\,\left\{\frac{43}{60}\bar{R}_{\mu\nu}\bar{R}^{\mu\nu}+\frac{1}{40}\bar{R}^2+\frac{1}{6}\bar{R}\,\bar{g}^{\rho\sigma}\partial_\rho\bar{\phi}\partial_\sigma\bar{\phi}+\left(\bar{g}^{\rho\sigma}\partial_\rho\bar{\phi}\partial_\sigma\bar{\phi}\right)^2\right.\\
&\left.+\bar{\nabla}^2\bar{\phi}\bar{\nabla}^2\bar{\phi}\right\}
\end{aligned}
\tag{7.123}
$$

which coincides with the result of 't Hooft and Veltman except for the last term. That term is however irrelevant in this case since it vanishes due to the background equations of motion. Using them the counterterm can be written in the form

$$
\Delta S = \frac{1}{\epsilon}\frac{1}{(4\pi)^2}\int d^4x\,\sqrt{\bar{g}}\,\frac{203}{40}\,\bar{R}^2 \tag{7.124}
$$

On the other hand, if we had considered pure gravity in the presence of a Cosmological Constant that would correspond in our notation to $F_\lambda(\bar{\phi})=1$ and $\bar{\phi}=0$. Nevertheless, in order to compare with the results present in the literature, we have to subtract from (7.122) the contribution from scalar loops, which is trivially

$$
\begin{aligned}
a_4^\phi =& \frac{1}{(4\pi)^{\frac{n}{2}}}\frac{1}{360}\int d^nx\,\sqrt{\bar{g}}\,\left\{2\bar{R}_{\mu\nu\rho\sigma}\bar{R}^{\mu\nu\rho\sigma}-2\bar{R}_{\mu\nu}\bar{R}^{\mu\nu}+5\bar{R}^2\right\}\\
&\frac{1}{(4\pi)^{\frac{n}{2}}}\int d^nx\,\sqrt{\bar{g}}\,\left\{\frac{1}{180}\bar{R}_{\mu\nu\rho\sigma}\bar{R}^{\mu\nu\rho\sigma}-\frac{1}{180}\bar{R}_{\mu\nu}\bar{R}^{\mu\nu}+\frac{1}{72}\bar{R}^2\right\}
\end{aligned}
\tag{7.125}
$$

in such a way that, after using the equations of motion and neglecting the topological invariant, the one-loop counterterm coincides with the well known result of Christensen and Duff [19]

$$
\Delta S = \frac{1}{\epsilon}\left(a_4-a_4^\phi-2a_4^{gh}\right)=\frac{1}{\epsilon}\frac{1}{(4\pi)^2}\frac{1}{180}\int d^nx\,\sqrt{\bar{g}}\,\left\{212\bar{R}_{\mu\nu\rho\sigma}\bar{R}^{\mu\nu\rho\sigma}-2088\,\Lambda^2\right\}
\tag{7.126}
$$

It is possible to use the background equations of motion (7.99) to simplify the final result (7.122). It is convenient to express the final result just in terms of the scalar, since we are interested in inverting the conformal transformation. The counterterm is then

$$
\begin{aligned}
\Delta S =& \frac{1}{\epsilon}\frac{1}{(4\pi)^2}\int d^4x\,\sqrt{\bar{g}}\,\left\{\frac{203}{160}\left(\bar{g}^{\rho\sigma}\partial_\rho\bar{\phi}\partial_\sigma\bar{\phi}\right)^2+\frac{57}{20}\Lambda\,F_\lambda(\bar{\phi})\bar{g}^{\rho\sigma}\partial_\rho\bar{\phi}\partial_\sigma\bar{\phi}\right.\\
&\left.-\frac{57}{5}\Lambda^2\left(F_\lambda(\bar{\phi})\right)^2+\frac{1}{3}\Lambda^2\left(F_\lambda'(\bar{\phi})\right)^2+2\,\Lambda^2\left(F_\lambda''(\bar{\phi})\right)^2-\frac{4}{3}\Lambda^2\,F_\lambda(\bar{\phi})\,F_\lambda''(\bar{\phi})\right\}
\end{aligned}
\tag{7.127}
$$

If we want to write the counterterm in the original frame we must undo the conformal tranformation, which is very easy once we have it in terms of the scalar. The scalar is related to the conformal factor through (7.94), and the conformal factor and the original function of the determinant of the metric verify (7.91). With this in mind, we can express the different contributions to the counterterm in terms of the functions appearing in (7.87). Taking into account the definition of the potential $F_\lambda(\bar\phi)$ given in (7.93) and supposing that $f(\varphi^*)$ is not a constant we get

$$
\begin{aligned}
F_\lambda(\varphi^*) &= [f(\varphi^*)]^{\frac{n}{2-n}}\ f_\lambda(\varphi^*) \\
F'_\lambda(\varphi^*) &= \frac{\partial F_\lambda}{\partial\phi}(\varphi^*) = \frac{\partial F_\lambda}{\partial\varphi^*}\frac{\partial\varphi^*}{\partial\Omega}\frac{\partial\Omega}{\partial\phi} \\
&= \frac{n-2}{\sqrt{2(n-1)(n-2)}}\, f^{\frac{2}{2-n}}\left[\frac{n}{2-n}\, f^{-1}\, f_\lambda + f'^{-1}\, f'_\lambda\right] \\
F''_\lambda(\varphi^*) &= \frac{\partial^2 F_\lambda}{\partial\phi^2}(\varphi^*) = \frac{\partial}{\partial\phi}\left(\frac{\partial F_\lambda}{\partial\phi}\right)(\varphi^*) = \frac{\partial}{\partial\varphi^*}\left(\frac{\partial F_\lambda}{\partial\phi}(\varphi^*)\right)\frac{\partial\varphi^*}{\partial\Omega}\frac{\partial\Omega}{\partial\phi} \\
&= \frac{n-2}{2(n-1)}\, f^{\frac{n-4}{n-2}}\left[\frac{n^2}{(n-2)^2}\, f^{-2}\, f_\lambda + \frac{n+2}{2-n}\, f^{-1}\, f'^{-1}\, f'_\lambda + f'^{-2}\, f''_\lambda - f'^{-3}\, f''\, f'_\lambda\right] \\
\bar g^{\rho\sigma}\partial_\rho\bar\phi\partial_\sigma\bar\phi &= \frac{2(n-1)}{n-2}\, f^{\frac{2(1-n)}{n-2}}\, f'^2\, g_*^{\mu\nu}\partial_\mu\varphi^*\partial_\nu\varphi^*
\end{aligned}
\tag{7.128}
$$

where the functions $f(\varphi^*)$ and $f_\lambda(\varphi^*)$ are given from the begining in terms of φ^* and we have denoted

$$
f'(\varphi^*) = \frac{\partial f(\varphi^*)}{\partial\varphi^*} \tag{7.129}
$$

and the others in a similar way. Finally, the counterterm of the theory (7.87) is

$$
\begin{aligned}
\Delta S = \frac{1}{\epsilon}\frac{1}{(4\pi)^2}\int d^4x\,\sqrt{g^*}\,\Big\{&\frac{1827}{160}\, f^{-4}\, f'^4\,(g_*^{\mu\nu}\partial_\mu\varphi^*\partial_\nu\varphi^*)^2 \\
&+\frac{171}{20}\Lambda\, f^{-3}\, f'^2\, f_\lambda\, g_*^{\mu\nu}\partial_\mu\varphi^*\partial_\nu\varphi^* - \frac{57}{5}\Lambda^2\, f^{-2}\, f_\lambda^2 + \frac{1}{9}\Lambda^2\left[f'^{-1}\, f'_\lambda - 2f^{-1}\, f_\lambda\right]^2 \\
&+\frac{2}{9}\Lambda^2\, f^2\left[4\, f^{-2}\, f_\lambda - 3f^{-1}\, f'^{-1}\, f'_\lambda + f'^{-2}\, f''_\lambda - f'^{-3}\, f''\, f'_\lambda\right] \\
&\times\left[2\, f^{-2}\, f_\lambda - 3f^{-1}\, f'^{-1}\, f'_\lambda + f'^{-2}\, f''_\lambda - f'^{-3}\, f''\, f'_\lambda\right]\Big\}
\end{aligned}
\tag{7.130}
$$

7.7 A slightly more general transverse action

Had we started from an action with a kinetic term for the determinant of the metric

$$
S = -\frac{1}{2\kappa^2}\int d^nx\,\sqrt{g^*}\,\left[f(g^*)R^* + 2f_\lambda(g^*)\Lambda - \frac{1}{2}f_\phi(g^*)g_*^{\mu\nu}\partial_\mu g^*\partial_\nu g^*\right] \tag{7.131}
$$

so that after an arbitrary change of coordenates

$$
S = -\frac{1}{2\kappa^2}\int d^nx\,\sqrt{g^*}\,\left[f(\varphi^*)R^* + 2f_\lambda(\varphi^*)\Lambda - \frac{1}{2}f_\phi(\varphi^*)g_*^{\mu\nu}\partial_\mu\varphi^*\partial_\nu\varphi^*\right] \tag{7.132}
$$

where the scalar field is $\varphi^* \equiv g^* C^2$. We should now go to the Einstein frame through a conformal transformation

$$
g_{\mu\nu} = \Omega^2 g^*_{\mu\nu} \tag{7.133}
$$

Choosing the conformal factor as[20]

$$\Omega^{n-2} = f(\varphi^*) \tag{7.134}$$

the action in the new frame takes the form

$$S = -\frac{1}{2\kappa^2} \int d^n x \ \sqrt{g} \ [R + 2F_\lambda(\Omega)\Lambda] + \frac{1}{2\kappa^2} \int d^n x \ \sqrt{g} \ \left[\frac{2(n-1)(n-2)}{\Omega^2} \right.$$
$$\left. +\Omega^{2-n} \, f_\phi \left(f^{-1}(\Omega^{n-2})\right) \left(\frac{\partial f^{-1}(\Omega)}{\partial \Omega}\right)^2 \right] \frac{1}{2} \, g^{\mu\nu} \partial_\mu \Omega \partial_\nu \Omega \tag{7.135}$$

Where we have defined

$$F_\lambda(\Omega) \equiv \Omega^{-n} \, f_\lambda \left(f^{-1}(\Omega^{n-2})\right) \tag{7.136}$$

A final redefinition of the scalar gives the desired action studied earlier

$$\left[\frac{2(n-1)(n-2)}{\Omega^2} + \Omega^{2-n} \, f_\phi \left(f^{-1}(\Omega^{n-2})\right) \left(\frac{\partial f^{-1}(\Omega^{n-2})}{\partial \Omega}\right)^2 \right] g^{\mu\nu} \partial_\mu \Omega \partial_\nu \Omega = g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \tag{7.137}$$

On the other hand one could have included a potential term

$$S_V = -\frac{1}{2\kappa^2} \int d^n x \ \sqrt{g^*} \, M^2 \, V(\varphi^*) \tag{7.138}$$

but it can be absorbed in the definition of $F_\lambda(\Omega)$, i.e.,

$$2\Lambda F_\lambda(\Omega) \equiv \Omega^{-n} \left(2\Lambda f_\lambda(f^{-1}(\Omega^{n-2}) + M^2 V(f^{-1}(\Omega^{n-2})\right) \tag{7.139}$$

so it does not include any interesting new issue and we won't consider it. In order to express the known counterterm in terms of these functions and the original variables we will need:

$$\begin{aligned}
F_\lambda(\varphi^*) &= f^{\frac{n}{2-n}}(\varphi^*) \, f_\lambda(\varphi^*) \\
F'_\lambda(\varphi^*) &= (n-2) \, f^{\frac{n+2}{4-2n}} \left[\frac{n}{2-n} \, f^{-1} \, f_\lambda + f'^{-1} \, f'_\lambda \right] \left[2(n-1)(n-2) \, f^{-1} + (n-2)^2 \, f'^{-2} \, f_\phi \right]^{-\frac{1}{2}} \\
F''_\lambda(\varphi^*) &= (n-2)^2 \, f^{\frac{2}{2-n}} \left[2(n-1)(n-2) \, f^{-1} + (n-2)^2 \, f'^{-2} \, f_\phi \right]^{-2} \left[\frac{2n^2(n-1)}{n-2} \, f^{-3} \, f_\lambda \right. \\
&\quad -2(n-1)(n+2) \, f^{-2} \, f'^{-1} \, f'_\lambda - 2(n-1)(n-2) \, f^{-1} \, f'^{-3} \, f'' \, f'_\lambda \\
&\quad +2(n-1)(n-2) \, f^{-1} \, f'^{-2} \, f''_\lambda + \frac{n(3n-2)}{2} \, f^{-2} \, f'^{-2} \, f_\lambda \, f_\phi \\
&\quad -\frac{(3n+2)(n-2)}{2} \, f^{-1} \, f'^{-3} \, f'_\lambda \, f_\phi + \frac{n(n-2)}{2} \, f^{-1} \, f'^{-3} \, f_\lambda \, f'_\phi \\
&\quad \left. -n(n-2) \, f^{-1} \, f'^{-4} \, f'' \, f_\lambda \, f_\phi + (n-2)^2 \, f'^{-4} \, f''_\lambda \, f_\phi - \frac{(n-2)^2}{2} \, f'^{-4} \, f'_\lambda \, f'_\phi \right] \\
\bar{g}^{\mu\nu} \partial_\mu \bar{\phi} \partial_\nu \bar{\phi} &= f^{\frac{2}{2-n}} \left[\frac{2(n-1)}{n-2} \, f^{-2} \, f'^2 + f^{-1} \, f_\phi \right] g_*^{\mu\nu} \partial_\mu \varphi^* \partial_\nu \varphi^*
\end{aligned} \tag{7.140}$$

[20] The following reasoning is not valid in case $f(\varphi^*) = constant$ since then (7.134) is not invertible to give $\varphi^* = f^{-1}(\Omega^{n-2})$.

So that the counterterm reads

$$
\begin{aligned}
\Delta S = \frac{1}{\epsilon}\frac{1}{(4\pi)^2} \int d^4x \, \sqrt{g^*} \, \Bigg\{ & \frac{203}{160} \left[3f^{-2}\, f'^2 + f^{-1} f_\phi\right]^2 (g_*^{\mu\nu}\partial_\mu\varphi^*\partial_\nu\varphi^*)^2 \\
& + \frac{57}{20}\, \Lambda \left[3f^{-3}\, f'^2\, f_\lambda + f^{-2}\, f_\lambda\, f_\phi\right] g_*^{\mu\nu}\partial_\mu\varphi^*\partial_\nu\varphi^* - \frac{57}{5}\, \Lambda^2\, f^{-2}\, f_\lambda^2 \\
& + \frac{1}{3}\, \Lambda^2 \left[f'^{-1}\, f'_\lambda - 2f^{-1}\, f_\lambda\right]^2 \left[3 + f\, f'^{-2}\, f_\phi\right]^{-1} + \frac{1}{2}\, \Lambda^2 \left[3f^{-1} + f'^{-2}\, f_\phi\right]^{-4} \\
& \times \big[24f^{-3}\, f_\lambda - 18f^{-2}\, f'^{-1}\, f'_\lambda - 6f^{-1}\, f'^{-3}\, f''\, f'_\lambda + 6f^{-1}\, f'^{-2}\, f''_\lambda + 10f^{-2}\, f'^{-2}\, f_\lambda\, f_\phi \\
& - 7f^{-1}\, f'^{-3}\, f'_\lambda\, f_\phi + 2f^{-1}\, f'^{-3}\, f_\lambda\, f'_\phi - 4f^{-1}\, f'^{-4}\, f''\, f_\lambda\, f_\phi + 2f'^{-4}\, f''_\lambda\, f_\phi - f'^{-4}\, f'_\lambda\, f'_\phi\big] \\
& \times \big[12f^{-3}\, f_\lambda - 18f^{-2}\, f'^{-1}\, f'_\lambda - 6f^{-1}\, f'^{-3}\, f''\, f'_\lambda + 6f^{-1}\, f'^{-2}\, f''_\lambda - 2f^{-2}\, f'^{-2}\, f_\lambda\, f_\phi \\
& - 7f^{-1}\, f'^{-3}\, f'_\lambda\, f_\phi + 2f^{-1}\, f'^{-3}\, f_\lambda\, f'_\phi - 4f^{-1}\, f'^{-4}\, f''\, f_\lambda\, f_\phi + 2f'^{-4}\, f''_\lambda\, f_\phi - f'^{-4}\, f'_\lambda\, f'_\phi \\
& - \frac{4}{3} f^{-1}\, f'^{-4}\, f_\lambda\, f_\phi^2 \big]\Bigg\}
\end{aligned} \tag{7.141}
$$

There are a couple of things to comment:

- In case $\Lambda = 0$ and the functions in front of the kinetic term and the Einstein–Hilbert term are $f = f_\phi = 1$ we recover the result of t'Hooft and Veltman

$$
\begin{aligned}
\Delta S &= \frac{1}{\epsilon}\frac{1}{(4\pi)^2} \int d^4x \, \sqrt{g^*} \, \frac{203}{160} \, (g_*^{\mu\nu}\partial_\mu\varphi^*\partial_\nu\varphi^*)^2 \\
&= \frac{1}{\epsilon}\frac{1}{(4\pi)^2} \int d^4x \, \sqrt{g^*} \, \frac{203}{40}\, R^2
\end{aligned} \tag{7.142}
$$

- Had we chosen the oposite function $f_\phi \to -f_\phi$ (does it correspond to a ghostly behaviour?) then the diferential equation

$$
3f^{-1}\, f'^2 - f_\phi = 0 \tag{7.143}
$$

  has a real solution and therefore there is another theory one-loop finite in case $\Lambda = 0$.

SOME INFORMATION ABOUT NONMINIMAL OPERATORS IS TO BE FOUND IN ANANTHANARAYAN [7].

Acknowledgments

I am grateful to John Donoghue, Jaume Garriga, David Toms and Enric Verdaguer for useful discussions. This work has been partially supported by the European Commission (HPRN-CT-200-00148) and by research programmes FPA2003-04597 and FPA2004-04582 (MEC, Spain) and DURSI 2005SGR00082.

8. Speculations on possible Ultraviolet completions of Einstein gravity.

A. General conventions

- The flat tangent metric is

$$\eta_{ab} \equiv diag\ (1, -1, -1, -1) \tag{A.1}$$

The Riemann tensor is

$$R^{\mu}{}_{\nu\alpha\beta} \equiv \partial_\alpha \Gamma^{\mu}_{\nu\beta} - \partial_\beta \Gamma^{\mu}_{\nu\alpha} + \Gamma^{\mu}_{\sigma\alpha}\Gamma^{\sigma}_{\nu\beta} - \Gamma^{\mu}_{\sigma\beta}\Gamma^{\sigma}_{\nu\alpha} \tag{A.2}$$

and we define the Riccci tensor as

$$R_{\mu\nu} \equiv R^{\lambda}{}_{\mu\lambda\nu} \tag{A.3}$$

Our conventions for the cosmological constant are such that for a constant curvature space

$$R_{\mu\nu} = -\frac{2}{d-2}\lambda g_{\mu\nu} \tag{A.4}$$

Then the ordinary de Sitter space has negative constant curvature, but enjoys *positive cosmological constant.*

The Einstein action is then defined as

$$S = -\frac{c^3}{2\kappa^2}\int \sqrt{|g|}\,(R + 2\lambda) + S_{matter} \tag{A.5}$$

with $\kappa^2 \equiv 8\pi G$.

- Background covariant derivatives can be integrated by parts:

$$\int d^4x \sqrt{|\bar{g}|}\bar{\nabla}_\mu L^\mu = \int d^4x \sqrt{|\bar{g}|}\frac{1}{\sqrt{|\bar{g}|}}\partial_\mu L^\mu = \int d^4x \partial_\mu L^\mu \tag{A.6}$$

Some commutators, which constitute the *Ricci identities*

$$\begin{aligned}
\left[\bar{\nabla}_\beta, \bar{\nabla}_\gamma\right] \omega_\rho &= \omega_\mu \bar{R}^{\mu}{}_{\rho\gamma\beta} \\
\left[\bar{\nabla}_\beta, \bar{\nabla}_\gamma\right] V^\rho &= -V^\mu \bar{R}^{\rho}{}_{\mu\gamma\beta} \\
\left[\bar{\nabla}_\beta, \bar{\nabla}_\gamma\right] h^{\alpha\beta} &= -h^{\lambda\beta}\bar{R}^{\alpha}{}_{\lambda\gamma\beta} + h^{\alpha\lambda}\bar{R}_{\lambda\gamma}
\end{aligned} \tag{A.7}$$

- Killing vector fields obey the Killing equation

$$\bar{\nabla}_\alpha \xi_\beta + \bar{\nabla}_\beta \xi_\alpha = 0 \tag{A.8}$$

The Ricci identities boil down in this case to

$$\bar{\nabla}_\mu \bar{\nabla}_\nu \xi_\alpha = -\xi^\lambda \bar{R}_{\nu\alpha\mu\lambda} \tag{A.9}$$

cf. [32].

- Let us recall our notation on differential forms. Given a p-form ([68]) written in a local chart as

$$\alpha \equiv \frac{1}{p!}\alpha_{\mu_1\ldots\mu_p} dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_p} \tag{A.10}$$

The components of the exterior differential are given by:

$$(d\alpha)_{\mu_0\mu_1\ldots\mu_p} \equiv \frac{1}{p!}\epsilon^{\lambda_0\lambda_1\ldots\lambda_p}_{\mu_0\ldots\mu_p}\partial_{\lambda_0}\alpha_{\lambda_1\ldots\lambda_p} \tag{A.11}$$

where the Kronecker tensor is given by the completely antisymmetric product pf Kronecker deltas.

$$\epsilon^{\lambda_0\lambda_1\ldots\lambda_p}_{\mu_0\ldots\mu_p} \equiv p!\delta^{\lambda_0}_{[\mu_0}\ldots\delta^{\lambda_p}_{\mu_p]} \tag{A.12}$$

Nilpotency follows easily

$$d^2 = 0 \tag{A.13}$$

Given a metric $g_{\mu\nu}$, the Hodge star operator maps p-forms into (n-p)-forms, with components

$$(\star\alpha)_{\mu_{p+1}\ldots\mu_n} \equiv \frac{1}{p!}\eta_{\mu_1\ldots\mu_n}\alpha^{\mu_1\ldots\mu_p} \tag{A.14}$$

This definition needs the components of the natural volume element n-form:

$$\eta_{\mu_1\ldots\mu_n} \equiv \sqrt{|g|}\epsilon^{1\ldots n}_{\mu_1\ldots\mu_n} \tag{A.15}$$

The co-differential is then given by:

$$\delta \equiv (-1)^p \star^{-1} d\star \tag{A.16}$$

and enjoys local components

$$(\delta\alpha)_{\rho_1\ldots\rho_{p-1}} = -\frac{1}{p!}\epsilon^{\mu_1\ldots\mu_p}_{\nu\rho_1\ldots\rho_{p-1}}\nabla^\nu\alpha_{\mu_1\ldots\mu_p} \tag{A.17}$$

so that also

$$\delta^2 = 0 \tag{A.18}$$

B. Some comments on the unitary gauge

Let us indeed examine the path integral

$$\int \mathcal{D}\phi \mathcal{D}g_{\mu\nu} e^{-S(\phi,g)} \tag{B.1}$$

which enjoys full $Diff$ invariance.

We first perform a point transformation, from the field variables $(g_{\mu\nu}, \phi)$ to the new variables $(g_{\mu\nu}, \phi = gC^2)$.We want to partial gauge fix it $Diff/TDiff$, such that the residual gauge invariance is $TDiff$. Formally we have

$$\mathcal{D}\phi \mathcal{D}g_{\mu\nu} = \mathcal{D}g_{\mu\nu} \mathcal{D}C \prod_x (2g(x)C(x)) \tag{B.2}$$

Therefore

$$\int \mathcal{D}\phi \mathcal{D}g_{\mu\nu} e^{-S(\phi,g)} = \int \mathcal{D}C \mathcal{D}g_{\mu\nu} e^{-S(\phi,g)+\int d^4x 2g(x)C(x)} \tag{B.3}$$

Anselmi [8] claims that in dimensional regularization all determinant of non-derivativa operators are regularized to unity. Then we go to the *unitary* gauge $C = 1$. Let us compute the Faddeev-Popov determinant, Δ. The gauge variation of the gauge condition is:

$$\delta C = -C\partial_\mu \xi^\mu + \xi^\mu \partial_\mu C \tag{B.4}$$

The important point is that this variation is independent of the metric, so that the determinant will also be independent of the metric, let is call it $\Delta(C)$. The path integral in the unitary gauge is then

$$\begin{aligned} &\int \mathcal{D}\phi \mathcal{D}g_{\mu\nu} e^{iS(\phi,g)} = \int \mathcal{D}C \mathcal{D}\delta(C-1)\Delta(C) g_{\mu\nu} e^{-S(\phi,g_{\mu\nu})+\int d^4x 2g(x)C(x)} = \\ &\int \mathcal{D}g_{\mu\nu} \Delta(1) g_{\mu\nu} e^{-S(g,g_{\mu\nu})+\int d^4x \, log\,(2g(x))} \end{aligned} \tag{B.5}$$

A somewhat similar point has been belavored in a related context by Fiol and Garriga [36].

Point canonical transformations in the path integral have been studied by Omote [67]. INFORMATION ON THE EQUIVALENCE THEOREM IS TO BE FOUND IN [53],[58],[2],[34].

C. Spherical harmonics

- The simplest way of getting eigenfunctions of the Laplace operator in the sphere is Helgason's (confer [?]). Consider the following harmonic polynomial in $\mathbb{R}^{n+1}$

$$f_{a,\lambda} \equiv (\vec{a}.\vec{x})^{\lambda} \tag{C.1}$$

with $\vec{a} \in \mathbb{C}$, $\vec{a}^2 = 0$.

Now we know that the full laplacian in $\mathbb{R}^{n+1}$ is

$$\Delta_{\mathbb{R}^{n+1}} = \frac{\partial^2}{\partial r^2} + \frac{n}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{S_n} \tag{C.2}$$

This yields

$$\Delta_{\mathbb{R}^{n+1}} f_{a,\lambda} = 0 = \frac{\lambda^2 + (n-1)\lambda}{r^2} f_{a,\lambda} + \frac{1}{r^2}\Delta_{S_n} f_{a,\lambda} \tag{C.3}$$

so that the eigenvalues of the Laplacian in the sphere S_n are

$$-\lambda(\lambda + n - 1) \tag{C.4}$$

It is more or less equivalent to start from traceless homogeneous polynomials

$$P \equiv \sum P_{(i_1 \ldots i_k)} x^{i_1} \ldots x^{i_k} \tag{C.5}$$

The number of such animals is the number of symmetric polynomials in n variables of degree λ minus the number of symmetric polynomials of degree $\lambda - 2$:

$$d(\lambda) = \binom{\lambda + n - 1}{\lambda} - \binom{\lambda + n - 3}{\lambda - 2} = \frac{(n + 2\lambda - 2)\,(\lambda + n - 3)!}{\lambda!\,(n-2)!} \tag{C.6}$$

- If we represent by μ an appropiate collection of indices, then we first build harmonic polynomials such that

$$\int_{S_n} d\Omega h^{*}_{\lambda'\mu'} h_{\lambda\mu} = \delta_{\lambda\lambda'}\delta_{\mu\mu'} r^{\lambda+\lambda'} \tag{C.7}$$

The hyperspherical harmonics are then defined by

$$h_{\lambda\mu} \equiv r^{\lambda} Y_{\lambda\mu} \tag{C.8}$$

and are normalized in such a way that

$$\int_{S_n} d\Omega Y^{*}_{\lambda'\mu'} Y_{\lambda\mu} = \delta_{\lambda\lambda'}\delta_{\mu\mu'} \tag{C.9}$$

- Gegenbauer polynomials are generalizations of Legendre polynomials, in the sense that

$$\frac{1}{|\vec{x}-\vec{x'}|^{n-2}} = \frac{1}{r_>^{n-2}\left(1+\left(\frac{r_<}{r_>}\right)^2 - 2\left(\frac{r_<}{r_>}\right)\hat{x}.\hat{x}'\right)^{\frac{n-2}{2}}} = \frac{1}{r_>^{n-2}}\sum_{\lambda=0}^{\infty}\left(\frac{r_<}{r_>}\right)^{\lambda} C_\lambda^{\frac{n-2}{2}}\left(\hat{x}.\hat{x}'\right) \tag{C.10}$$

Let us now prove the sum rule for hyperspherical harmonics. For concreteness, let us assume that

$$\begin{aligned} r &\equiv |\vec{x}_<| \\ r' &\equiv |\vec{x}_>| \end{aligned} \tag{C.11}$$

Then it is a fact of life that

$$\Delta\frac{1}{|\vec{x}-\vec{x}'|^{n-2}} = 0 = \sum_{\lambda=0}^{\infty}\frac{1}{(r')^{\lambda+n-2}}\Delta\left(r^\lambda C_\lambda^{\frac{n-2}{2}}\left(\hat{x}.\hat{x}'\right)\right) \tag{C.12}$$

Imposing term by term vanishing leads to

$$\left(\frac{1}{r^{n-1}}\frac{\partial}{\partial r}r^{n-1}\frac{\partial}{\partial r} - \frac{1}{r^2}\Delta_{S^{n-1}}\right)\left(r^\lambda C_\lambda^{\frac{n-2}{2}}\left(\hat{x}.\hat{x}'\right)\right) = 0 \tag{C.13}$$

which conveys the fact that

$$\Delta_{S^{n-1}}C_\lambda^{\frac{n-2}{2}}\left(\hat{x}.\hat{x}'\right) = \lambda\left(\lambda+n-2\right)C_\lambda^{\frac{n-2}{2}}\left(\hat{x}.\hat{x}'\right) \tag{C.14}$$

Since the hyperspherical harmonics are by assumption a complete set of eigenfunctions,

$$C_\lambda^{\frac{n-2}{2}}\left(\hat{x}.\hat{x}'\right) = \sum_\mu a_{\lambda\mu}\left(\vec{x}'\right)Y_{\lambda\mu}\left(\hat{x}\right) \tag{C.15}$$

where

$$a_{\lambda\mu}\left(\vec{x}'\right) = \int_{\hat{x}} C_\lambda^{\frac{n-2}{2}}\left(\hat{x}.\hat{x}'\right)Y^*_{\lambda\mu}\left(\hat{x}\right) = \frac{2(n-2)\pi^{n/2}}{\Gamma(n/2)\left(2\lambda+n-2\right)}Y^*_{\lambda\mu}\left(\hat{x}'\right) \tag{C.16}$$

This is related to the degeneracy $d(\lambda)$ of hyperspherical harmonics in the following way. Choosing $\hat{x}=\hat{x}'$, the sum rule leads to

$$C_\lambda^{\frac{n-2}{2}}\left(1\right) = K_\lambda\sum_\mu Y^*_{\lambda\mu}\left(\vec{x}'\right)Y_{\lambda\mu}\left(\hat{x}\right) \tag{C.17}$$

Integrating now over the unit sphere

$$C_\lambda^{\frac{n-2}{2}}\left(1\right)V(S_{n-1}) = K_\lambda\sum_\mu 1 = K_\lambda d(\lambda) \tag{C.18}$$

The result is

$$d(\lambda) = \frac{(n+2\lambda-2)\left(\lambda+n-3\right)!}{\lambda!\left(n-2\right)!} \tag{C.19}$$

- Let us now become more specific and perform some computations in gory detail. The metric on S^{n-1} is

$$ds^2_{n-1} = d\theta^2_{n-1} + sin^2\,\theta_{n-1} d\theta^2_{n-2} + \ldots + sin^2\,\theta_{n-1}\, sin^2\,\theta_{n-2} \ldots sin^2\,\theta_2 d\theta_1^2 \tag{C.20}$$

id est, in a recurrent form

$$\begin{aligned} ds_1^2 &= d\theta_1^2 \\ ds_n^2 &= d\theta_n^2 + sin^2\theta_n\, ds^2_{n-1} \end{aligned} \tag{C.21}$$

This corresponds to polar coordinates in $\mathbb{R}^n$

$$\begin{aligned} X_n &= cos\,\theta_{n-1} \\ X_{n-1} &= sin\,\theta_{n-1}\, cos\,\theta_{n-2} \\ &\ldots \\ X_2 &= sin\,\theta_{n-1}\, sin\,\theta_{n-2} \ldots cos\,\theta_1 \\ X_1 &= sin\,\theta_{n-1}\, sin\,\theta_{n-2} \ldots sin\,\theta_1 \end{aligned} \tag{C.22}$$

Spherical harmonics have been constructed quite explicitly by Higuchi [**?**], are such that

$$\Box_n Y_{l_n \ldots l_1}(\theta_n \ldots \theta_1) = -l_n(l_n + n - 1) Y_{l_n \ldots l_1}(\theta_n \ldots \theta_1) \tag{C.23}$$

They are orhonormal with respect to the induced riemannian measure

$$\delta\Omega_n \equiv \sqrt{|g|} d\theta_1 \wedge \ldots d\theta_n = d\theta_1 \ldots d\theta_n sin^{n-1}\theta_n\, sin^{n-2}\,\theta_{n-1} \ldots sin\,\theta_2 \tag{C.24}$$

which obeys

$$d\Omega_n = sin^{n-1}\theta_n d\theta_n d\Omega_{n-1} \tag{C.25}$$

and

$$V(S_{n-1}) = \int d\Omega_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)} \tag{C.26}$$

To be specific,

$$\int d\Omega_n Y_{l_n \ldots l_1}(\theta_n \ldots \theta_1) Y^*_{l'_n \ldots l'_1}(\theta_n \ldots \theta_1) = \delta_{l_n, l'_n} \ldots \delta_{l_n, l'_n} \tag{C.27}$$

- It is obvious that any function on the sphere can be expanded

$$\begin{aligned} f(\Omega) = \sum_{l_n \ldots l_1} C_{l_n \ldots l_1} Y_{l_n \ldots l_1}(\theta_n \ldots \theta_1) = \\ \sum_{l_n \ldots l_1} \int d\Omega' Y^*_{l_n \ldots l_1}(\theta'_n \ldots \theta'_1) f(\theta'_n \ldots \theta'_1) Y_{l_n \ldots l_1}(\theta_n \ldots \theta_1) \end{aligned}$$

which means

$$\sum_{l_n \ldots l_1} Y^*_{l_n \ldots l_1}(\theta'_n \ldots \theta'_1) Y_{l_n \ldots l_1}(\theta_n \ldots \theta_1) \equiv \delta(\Omega - \Omega') \tag{C.28}$$

where by definition

$$\int d\Omega' \delta(\Omega - \Omega') f(\theta') = f(\theta) \tag{C.29}$$

whence in a somewhat symbolic form,

$$\delta(\Omega - \Omega') = \delta(\theta'_1 - \theta_1) \ldots \delta(\theta'_n - \theta_n) sin^{-(n-1)} \theta'_n \, sin^{-(n-2)} \, \theta'_{n-1} \ldots sin^{-1} \, \theta'_2 \tag{C.30}$$

Now we can expand this function, as any other function, in series of Gegenbauer polynomials

$$\delta(\Omega - \Omega') = \sum_l d_l C^\nu_l(cos\, \theta_n) \tag{C.31}$$

Let us choose our reference frame in such a way that

$$\Omega . \Omega' \equiv cos\, \theta_n \tag{C.32}$$

id est, Ω' is pointing towards the North pole.

On functions constant on S_{n-1},

$$d\Omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} sin^{n-1}\, \theta_n \, d\theta_n \tag{C.33}$$

and, denoting $x \equiv cos\, \theta_n$

$$d\Omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \left(1 - x^2\right)^{\frac{n-2}{2}} dx \tag{C.34}$$

as well as

$$\delta(\Omega) = \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}} \delta(\theta_n) \frac{1}{sin^{n-1}\, \theta_n} = \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}} \delta(1 - x)(1 - x^2)^{\frac{2-n}{2}} \tag{C.35}$$

We can now integrate the two sides of the equation (C.31) against $C^\nu_{l'}(x)(1 - x)^{\nu - 1/2}$. The orthogonality property

$$\int_{-1}^{1} dx C^\nu_l(x) C^\nu_{l'}(x) (1 - x^2)^{\nu - 1/2} = \delta_{ll'} \frac{2^{1-2\nu} \pi \Gamma(l + 2\nu)}{l!(\nu + l)\Gamma(\nu)^2} \tag{C.36}$$

then implies

$$d_l \frac{2^{1-2\nu} \pi \Gamma(l + 2\nu)}{l!(\nu + l)\Gamma(\nu)^2} = \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}} \int_{-1}^{1} dx C^\nu_l(x)(1 - x^2)^{1-n/2} \delta(x - 1)(1 - x^2)^{\nu - 1/2} \tag{C.37}$$

The member of the right converges when $\nu = \frac{n-1}{2}$. Given in addition the fact that

$$C_l^{\nu}(1) = \frac{\Gamma(l+2\nu)}{l!\,\Gamma(2\nu)} \tag{C.38}$$

we can write

$$d_l = \frac{\Gamma(\frac{n}{2})(l+\frac{n-1}{2})\Gamma(\frac{n-1}{2})^2}{\Gamma(n-1)\pi^{\frac{n+1}{2}}2^{3-n}} = \frac{1}{V(S_n)}\frac{n-1+2l}{n-1} \tag{C.39}$$

(using $\Gamma(2x) = 2^{1-2x}\sqrt{\pi}\Gamma(x+\frac{1}{2})/\Gamma(x)$) as well as

$$\delta(\Omega - \Omega') = \sum_l \frac{1}{V(S_n)}\frac{n-1+2l}{n-1}C_l^{\frac{n-1}{2}}(cos\,\theta_n) \tag{C.40}$$

$$\sum_{l_n \ldots l_1} Y^*_{l_n \ldots l_1}(\theta'_n = 0 \ldots \theta'_1)Y_{l_n \ldots l_1}(\theta_n \ldots \theta_1) = \sum_l \frac{1}{V(S_n)}\frac{n-1+2l}{n-1}C_l^{\frac{n-1}{2}}(cos\,\theta_n) \tag{C.41}$$

If we employ the notation $l \equiv l_n$ and $\vec{m} \equiv (l_{n-1} \ldots l_1)$, then the preceding formula presumably means that

$$\sum_{\vec{m}} Y^*_{l \ldots \vec{m}}(\Omega_z)Y_{l \ldots \vec{m}}(\Omega) = \frac{1}{V(S_n)}\frac{n-1+2l}{n-1}C_l^{\frac{n-1}{2}}(cos\,\theta_n) \tag{C.42}$$

- We begin by defining some eigenfunctions of the differential operator:

$$D \equiv \frac{\partial^2}{\partial\theta^2} + (N-1)cot\,\theta\frac{\partial}{\partial\theta} - \frac{l\,(l+N-2)}{sin^2\,\theta} \tag{C.43}$$

such that

$$D\bar{P}^l_{NL}(\theta) = -L\,(L+N-1)\,\bar{P}^l_{NL}(\theta) \tag{C.44}$$

namely,

$$\bar{P}^l_{NL}(\theta) \equiv c^l_{NL}\,(sin\,\theta)^{-\frac{N-2}{2}}\,P^{-\left(l+\frac{N-2}{2}\right)}_{L+\frac{N-2}{2}}(cos\,\theta) \tag{C.45}$$

where $P^{\mu}_{\nu}(z)$ are Legendre functions , and the normalization is given by

$$c^l_{NL} \equiv \sqrt{\frac{2L+N-1}{2}\frac{(L+l+N-2)!}{(L-l)!}} \tag{C.46}$$

The differential equation that Legendre functions $P^{\mu}_{\nu}(z)$ are solutions of is given by

$$Lw(z) \equiv \left(1-z^2\right)\frac{d^2w}{dz^2} - 2z\frac{dw}{dz} + \left(\nu\,(\nu+1) - \frac{\mu^2}{1-z^2}\right)w = 0 \tag{C.47}$$

Changing variables $z = cos\theta$ this reads

$$\left(\frac{\partial^2}{\partial\theta^2} + cot\,\theta\frac{\partial}{\partial\theta} - \frac{\mu^2}{\sin^2\,\theta}\right) w\,(cos\,\theta) = -\nu\,(\nu+1)\,w\,(cos\,\theta) \tag{C.48}$$

and using this it is not difficult to actually prove the basic equation (C.44).

The harmonics themselves are given by:

$$Y_{l_N \ldots l_1}(\theta_N, \ldots, \theta_1) \equiv \prod_{n=2}^{N} \bar{P}^{l_{n-1}}_{n\,l_n}\,(\theta_n)\,\frac{1}{\sqrt{2\pi}} e^{il_1\theta_1} \tag{C.49}$$

- We can now employ the expansion (GR, 8.534)

$$e^{im\rho cos\,\phi} = 2^{\nu}\Gamma(\nu)\sum_{k=0}^{\infty}(\nu+k)i^k(m\rho)^{-\nu}J_{\nu+k}(m\rho)C^{\nu}_k(cos\,\phi) \tag{C.50}$$

and using our expansion of the Gegenbauer polynomials in terms of spherical harmonics,

$$\begin{aligned} e^{iz\Omega.\Omega'} &= 2^{n/2-1}\Gamma(n/2-1)\sum_{k=0}^{\infty}(n/2-1+k)i^k(z)^{-(n/2-1)}J_{n/2-1+k}(z) \\ &C_{k,n}\sum_{\vec{m}} Y^*_{k\ldots\vec{m}}(\Omega)Y_{k\ldots\vec{m}}(\Omega') \end{aligned} \tag{C.51}$$

where $C_{l,n}$ are apropiate constants.

Acknowledgments

This work has been partially supported by the European Commission (HPRN-CT-200-00148) and by FPA2006-05423 (DGI del MCyT, Spain) and Proyecto HEPHACOS ; P-ESP-00346 (CAM).

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