Quantum Field Theory.

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Functional integral.

In classical mechanics the trajectory is determined by the initial and final points. For example, for a free particle, the unique classical trajectory that at the instant of time $t = t_i$ is at the point $q = q_i$ and at the instant $t = t_f$ is at the point $q = q_f$ is

$$q(t) = q_i + \frac{q_i - q_f}{t_i - t_f}\,(t - t_i) \tag{1.1}$$

For an harmonic oscillator with the same boundary conditions the corresponding classical solution is

$$q(t) = \frac{q_f\ \sin\ \omega(t - t_i) + q_i\ \sin\ \omega(t - t_f)}{\sin\ \omega\ \ (t_i - t_f)} \tag{1.2}$$

In quantum mechanics there is no such thing as a unique trajectory with those boundary conditions. The only thing which is well defined is a transition amplitude between the initial and the final states, that is,

$$\langle q_f\ t_f | q_i\ t_i \rangle \tag{1.3}$$

This amplitude can be computed using Schrödinger's equation.

Dirac first suggested that may be one can think of quantum mechanics as if every possible trajectory (even non causal ones) have got some probability amplitude, and the total probality amplitude is in some sense a superposition of all those amplitudes, with some adecuate weights.

Schwinger had the insight that closure shold somewhat be implemented. Let us recall that given any hermitian operator with eigenkets $|m\rangle$, closure means that

$$\sum |m\rangle\langle m| = 1 \tag{1.4}$$

Then it must be the fact that

$$\langle f|i\rangle = \sum_{i_1 \ldots i_N} \langle f|i_1\rangle\langle i_1|i_2\rangle \ldots |i_{N-1}\rangle\langle i_{N-1}|i_N\rangle\langle i_N|i\rangle \tag{1.5}$$

It is natural to think that by choosing the intermediate states to somewhat interpolate between the initial and the final state there is some path (in an abstract space) conecting the initial and the final state, and that tha total amplitude must be in some sense the total sum of the contributions of those paths.

Feynman drawing on these ideas, postulated that the quantum mechanical transition amplitude from the state $i \equiv (q_i, t_i)$ to the state $f = (q_f, t_f)$ is given by the functional integral

$$K(f,i) \equiv \langle t_f q_f | t_i q_i \rangle = \int_i^f \mathcal{D}q(t) e^{\frac{i}{\hbar} S(f,i)} \tag{1.6}$$

Schroedinger states depend on time

$$i \frac{\partial}{\partial t} |\psi(t)\rangle = H |\psi(t)\rangle \tag{1.7}$$

Formally their time evolution is given by

$$|\psi(t)\rangle = e^{-iHt} |\psi(0)\rangle \tag{1.8}$$

Given an arbitrary Schroedinger operator (which is independent on time), say q its matrix elements are given by

$$\langle \psi(t) | q | \psi(t) \rangle = \langle \psi(0) | \; e^{iHt} \; q \; e^{-iHt} \; | \psi(0) \rangle \tag{1.9}$$

These expectation values are the only observables; any representation that leaves invariant those is a valid one. In Heisenberg representation, for example, states are time independent whereas

$$q_H(t) = \; e^{iHt} \; q_S(0) \; e^{-iHt} \tag{1.10}$$

where the Heisenberg and Schroedinger operators coincide at $t = 0$. Let us diagonalize it

$$q_S |q\rangle = q |q\rangle \tag{1.11}$$

Then the Heisenberg (time independent, then) state that diagonalizes $q_H(t)$ is given by

$$q_H(t) |q \; t\rangle \equiv q_H(t) \left(e^{iHt} \; |q\rangle \right) = q \left(e^{iHt} \; |q\rangle \right) = q |q \; t\rangle \tag{1.12}$$

Note the different sign with respect to Schroedinger states.

In quantum mechanics the measure $\mathcal{D}q$ is related to the Wiener measure. In QFT the measure is not understood. To quote Feynman himself in his famous Reviews of Modern Physics paper, *one feels as Cavalieri must have felt calculating the volume of a pyramid before the invention of calculus.* This lack of mathematical understanding of the measure hinders progress in the field. There is however some truth in the path integral. First of all it includes both classical physics in a limiting case as well as all perturbative results that can be reached by operator formalisms. Besides, Montecarlo computer simulations of the functional integral yield seemingly reasonable nonperturbative results.

In the classical limit $\hbar \rightarrow 0$ this integral is dominated by the stationary phase, which corresponds to the classical solution

$$\frac{\delta S}{\delta q} = 0 \tag{1.13}$$

Although we use for simplicity the QM language, we always have an eye on quantum fields $\phi(\vec{x}, t)$. They correspond to an infinite number of degrees of freedom $q_{\vec{x}}(t)$. Essentially, one degree of freedom per point of the three-dimensional space $\vec{x} \in \mathbb{R}^3$.

Let us give some details on Feynman's definition. Let us divide the total time interval $T \equiv t_f - t_i$ into N equal pieces of length $\epsilon = t_{i+1} - t_i$, in such a way that $t_0 \equiv t_i \ldots t_N \equiv t_f$ $(q_0 = q_i, q_N = q_f)$. Then

$$N\epsilon = T \tag{1.14}$$

Define

$$K(f, i) \equiv \lim_{N \rightarrow \infty} \frac{1}{A} \int_{-\infty}^{\infty} \frac{dq_1}{A} \ldots \frac{dq_{N-1}}{A} e^{\frac{i}{\hbar} S(f,i)} \tag{1.15}$$

The constant $A = \sqrt{\frac{2\pi i \hbar \epsilon}{m}}$ is determined in such a way that this limit exists for a free particle.

Let us check the *quantum mechanical composition law*

$$K(f, i) = \int K(f, n) d\; q_n \; K(n, i) \tag{1.16}$$

This is a basic quantum mechanical postulate stemming, as we pointed out earlier, from the competeness or closure of the eigenvalues of the position operator. It is sometimes dubbed the *cutting equation* or the *sewing equation.* The two integration variables will be q and $\tilde{q}$, such that $q_0 = q_i$ and $q_N = q_f$. Schematically, the different integrations are

$$\frac{1}{A} \frac{dq_1}{A} \ldots \frac{dq_{N-1}}{A} \; dq_n \; \frac{1}{A} \frac{d\tilde{q}_1}{A} \ldots \frac{d\tilde{q}_{N-1}}{A} \tag{1.17}$$

which is precisely what is necessary for the direct computation of $K(f, i)$ in $2N$ steps.

Let us work out the free particle in detail.

$$S = \frac{m}{2} \sum_{i=1}^{N} (q_i - q_{i-1})^2 \tag{1.18}$$

The first integral is just

$$K_1 = \int dq_1 e^{\frac{im}{2\hbar\epsilon}[(q_2-q_1)^2+(q_1-q_0)^2]} \tag{1.19}$$

which we define through analytic continuation from Gauss' famous integral which was already used when defining the functional measure

$$I \equiv \int_{-\infty}^{\infty} d\phi e^{-\lambda\phi^2} = \sqrt{\frac{\pi}{\lambda}} \tag{1.20}$$

The easiest way to show this is by squaring it.

$$I^2 = \int_{-\infty}^{\infty} d\phi_1 e^{-\lambda\phi_1^2} \int_{-\infty}^{\infty} d\phi_2 e^{-\lambda\phi_2^2} =$$
$$= \int_0^{\infty} d|\phi||\phi| \int_0^{2\pi} d\theta e^{-\lambda|\phi|^2} = 2\pi\frac{1}{2\lambda} = \frac{\pi}{\lambda} \tag{1.21}$$

$$\int_{-\infty}^{\infty} dq e^{-\lambda q^2} = \sqrt{\frac{\pi}{\lambda}} \tag{1.22}$$

($\lambda \in \mathbb{R}^+$), as well as

$$\int_{-\infty}^{\infty} dq\ e^{-\lambda q^2+bq} = e^{\frac{b^2}{4\lambda}} \sqrt{\frac{\pi}{\lambda}} \tag{1.23}$$

Then ($\lambda \equiv -\frac{im}{\hbar\epsilon}$; $b \equiv -\frac{im}{\hbar\epsilon}(q_2+q_0)$).

$$K_1 = \frac{1}{A^2} e^{i\frac{m}{2\hbar\epsilon}[q_2^2+q_0^2]} \sqrt{\frac{i\pi\hbar\epsilon}{m}} e^{-\frac{im}{4\hbar\epsilon}(q_2+q_0)^2} \tag{1.24}$$

which boils down to

$$K_1 = \sqrt{\frac{m}{2\pi i\hbar(2\epsilon)}} e^{\frac{im}{2\hbar(2\epsilon)}(q_2-q_0)^2} \tag{1.25}$$

Performing one step further

$$K_2 = \frac{1}{A}\int dq_2 K_1 e^{\frac{im}{2\hbar\epsilon}(q_3-q_2)^2} \tag{1.26}$$

After $N-1$ steps, we get

$$K(f,i) = \sqrt{\frac{m}{2\pi i\hbar N\epsilon}}\ e^{\frac{im}{2\hbar T}(q_f-q_i)^2} \tag{1.27}$$

(which does not depend on N)

This propagator obeys Schroedinger's equation

$$i\hbar\frac{\partial K(f,i)}{\partial t_f} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q_f^2}K(f,i) \tag{1.28}$$

Let us now explain the privileged setup of Euclidean time. Denote by

$$\phi_n(q) \equiv \langle q|n\rangle \tag{1.29}$$

the wave function of the energy eigenstate $|n\rangle$.

$$H|n\rangle \equiv E_n|n\rangle \tag{1.30}$$

in such a way that $\phi_0(q)$ corresponds to the ground state wave function. We are interested in the probability amplitude that if the system was in the ground state at time $t_i \sim -\infty$ in the distant past, it is still in this state at the time $t_f \sim +\infty$ in the distant future, in case an arbitrary external source

$$\int_{t_i}^{t_f} dt\ j(t)\ q(t) \tag{1.31}$$

has been added to the lagrangian between $t_1 \leq t \leq t_2$ where

$$t_i \leq t_1 \leq t_2 \leq t_f \tag{1.32}$$

Inserting a complete set of energy eigenstates we learn that

$$\langle q_f t_f|q_i t_i\rangle_j = \int dq_2\ dq_1\ \langle q_f t_f|q_2\ t_2\rangle\ \langle q_2 t_2|q_1\ t_1\rangle_j\ \langle q_1\ t_1|q_i\ t_i\rangle =$$
$$\sum_m \sum_n \int dq_2\ dq_1\ \langle q_f|e^{-iH(t_f-t_2)}|E_n\rangle\langle E_n|q_2 t_2\ \rangle\ \langle q_2 t_2|q_1\ t_1\rangle_j\ \langle q_1\ |E_m\rangle\langle E_m|e^{-iH(t_1-t_i)}|q_i\rangle$$

When both

$$\begin{aligned} t_f &= i\ \infty \\ t_i &\to -i\ \infty \end{aligned} \tag{1.33}$$

the lowest energy state $|0\rangle$ is selected.

$$\begin{aligned} &\langle q_f\ t_f|q_i\ t_i\rangle_j = e^{-iE_0 T}\ \phi_0\ (q_f)\phi_0^*\ (q_i)\ \times \\ &\times \int\ dq_1\ dq_2\ \phi_0^*\ (q_2)\ e^{iE_0 t_2}\ \langle q_2 t_2|q_1 t_1\rangle_j\ \phi_0(q_1)\ e^{-iE_0 t_1} \end{aligned} \tag{1.34}$$

It is convenient to choose the zero of energies so that

$$E_0 = 0 \tag{1.35}$$

The *vacuum persistence amplitude*, which we will denote by the name *partition function* using the statistical mechanical analogy, is then given by

$$\begin{aligned} \langle 0_+|0_-\rangle_j \equiv Z[j] &\equiv \frac{\langle q_f\ t_f|q_i\ t_i\rangle_j}{\phi_0\ (q_f)\phi_0^*\ (q_i)} = \\ &= \int\ dq\ dq'\ \phi_0^*\ (q')\ \ \langle q' t'|q t\rangle\ \phi_0(q) \end{aligned} \tag{1.36}$$

This expressions tell us how to compute vacuum persistence amplitudes; that is, it selects the vacuum as both the initial and final state.

This means that up to a j-independent constant (which is irrelevant in most QFT applications)

$$\langle 0_+|0_-\rangle \equiv Z[J] \equiv \int \mathcal{D}q\; e^{i\int_{i\infty}^{-i\infty} dt\;(L[q]+Jq)} \tag{1.37}$$

This is exactly what is gotten from the euclidean formulation, where from the very beginning

$$t \rightarrow \tau \equiv -i\;t \tag{1.38}$$

with the boundary condition that $q(\tau)$ approaches some constants (which could be zero) at $\tau = \pm\infty$. The euclidean integral selects automatically the vacuum persistence amplitude.

When we include an operator insertion what we get is the time-ordered product

$$\int \mathcal{D}q\;\mathcal{O}(t_a)\;\mathcal{O}(t_b)\;e^{iS} = \langle 0_+|T\;\mathcal{O}(t_a)\;\mathcal{O}(t_b)\;|0_-\rangle \tag{1.39}$$

This is obvious as a result of our practical definition of the path integral. It suffices to define a time mesh in such a way that both t_a and t_b are part of it.

1.1 Perturbation theory through gaussian integrals.

Let us reproduce the diagrammatic perturbation theory in terms of path integrals. The rules of the gaussian integral are equivalent to Wick's theorem on operator language. All this procedure is essentially algebraic, combinatoric even ; so that the formal limit to an infinite number of dof is trivial, with the obvious replacements

$$\begin{aligned}\sum_i &\rightarrow \sum_i \int d^3x\\ \delta_{ij} &\rightarrow \delta_{ij}\delta^3\;(x-y)\end{aligned} \tag{1.40}$$

Let us write nevertheless all results in the language of a finite number of degrees of freedom for the time being. Denote by ϕ the vector $\phi = (\phi_1 \ldots \phi_n) \in \mathbb{R}^n$. Then the Gaussian integral generalizes to

$$Z_0(J) \equiv \int d\phi\; e^{-\phi^T.M.\phi - J.\phi} = \pi^{n/2}(det\,M)^{-1/2}e^{\frac{1}{4}J^T.M^{-1}.J} \tag{1.41}$$

This integral allows us to obtain many others

$$\int d\phi\;\phi_i\;\phi_j\;e^{-\phi^T.M.\phi} = \frac{\partial}{\partial J_i}\frac{\partial}{\partial J_j}\;Z_0(J)|_{J=0} = \pi^{n/2}(det\,M)^{-1/2}\frac{1}{2}(M^{-1})_{ij}$$

or else

$$\int d\phi\ \phi_i\ \phi_j\ \phi_k\ \phi_l\ e^{-\phi^T.M.\phi} = \frac{\partial}{\partial J_i}\frac{\partial}{\partial J_j}\frac{\partial}{\partial J_k}\frac{\partial}{\partial J_l}\ Z(J)|_{J=0} = \pi^{n/2}(det\,M)^{-1/2}$$

$$\left\{\frac{1}{2}(M^{-1})_{ij}\frac{1}{2}(M^{-1})_{lk} + \frac{1}{2}(M^{-1})_{jl}\frac{1}{2}(M^{-1})_{ik} + \frac{1}{2}(M^{-1})_{jk}\frac{1}{2}(M^{-1})_{il}\right\}$$

Let us dubb as *propagator* the quantity

$$\Delta_{ij} \equiv \frac{1}{2}M^{-1}_{ij}). \tag{1.42}$$

The preceding integral then reads

$$\int d\phi\ \phi_{i_1}\dots\phi_{i_n}\ e^{-\phi^T.M.\phi} = \sum_{p\in C^n_2}\prod_{i_a\in p}\Delta_{i_a i_{p(a)}} \tag{1.43}$$

In the example there are $C^4_2 = 3$ different possible pairings.

In general we define the *Green function*

$$\langle 0_+\,|T\phi_{i_1}\dots\phi_{i_n}|\,0_-\rangle \equiv \int\ d\phi\ \phi_{i_1}\dots\phi_{i_n}e^{-\phi^T.M.\phi - V(\phi)} = e^{-V(\frac{\delta}{\delta J})}\ Z(J)|_{J=0} \equiv$$

$$e^{-V(\frac{\delta}{\delta J})}\int d\phi e^{-\phi^T.M.\phi - J.\phi}\ |_{J=0} \tag{1.44}$$

The time ordering operator, T, appears owing to the time dependence of the variables, which has not been explicitly indicated. The reader is invited to work out a few examples.

- Consider for example a cubic interaction

$$V(\phi) = \frac{g}{3!}\sum_i \phi_i^3 \tag{1.45}$$

It is plain that

$$Z(J) = \left(1 + \frac{g}{3!}\sum_i(-\partial_i)^3 + \frac{1}{2}\left(\frac{g}{3!}\right)^2\sum_i(-\partial_i)^3\sum_j(-\partial_j^3)\right)e^{\frac{1}{2}J\Delta J} \tag{1.46}$$

In an obvious notation

$$\begin{aligned}
&\partial_j \to \Delta J_j\\
&\partial_j^2 \to \Delta_{jj} + \Delta J_j^2\\
&\partial_j^3 \to 3\Delta_{jj}\Delta J_j + \Delta J_j^3\\
&\partial_i \to 3\Delta_{jj}\Delta J_j\Delta J_i + 3\Delta_{jj}\Delta_{ji} + 3\Delta J_j^2\Delta_{ji} + \Delta J_j^3\Delta J_i\\
&\partial_i^2 \to 6\Delta_{jj}\Delta_{ji}\Delta J_i + 3\Delta_{jj}\Delta J_i\Delta_{ii} + 3\Delta_{jj}\Delta J_j\Delta J_i^2 + 6\Delta J_j\Delta_{ji}^2 +\\
&+6\Delta J_j^2\Delta_{ji}\Delta J_i + \Delta J_j^3\Delta_{ii} + \Delta J_j^3\Delta J_i^2\\
&\partial_i^3 \to 9\Delta_{jj}\Delta_{ji}\Delta_{ii} + 9\Delta_{jj}\Delta_{ji}\Delta J_i^2 + 9\Delta_{jj}\Delta J_j\Delta_{ii} +\\
&+3\Delta_{jj}\Delta J_j\Delta J_i^3 + 6\Delta_{ji}^3 + 18\Delta J_j\Delta_{ji}^2\Delta J_i +\\
&9\Delta J_j^2\Delta_{ji}\Delta_{ii} + 9\Delta J_j^2\Delta_{ji}\Delta J_i^2 + 3\Delta J_j^3\Delta_{ii}\Delta J_i + \Delta J_i^3\Delta J_j^3
\end{aligned} \tag{1.47}$$

It is also plain that

$$\langle 0_+ | T\ \phi_k\ \phi_l\ | 0_- \rangle = \Delta_{kl} + \tfrac{1}{2}\tfrac{g^2}{36}\Big\{(18\Delta_{ij}^2\left(\Delta_{ik}\Delta_{jl} + \Delta_{il}\Delta_{jk}\right) +$$

$$+9\Delta_{ii}\Delta_{ij}\left(\Delta_{jk}\Delta_{jl} + \Delta_{jl}\Delta_{jk}\right) + 9\Delta_{jj}\Delta_{ij}\left(\Delta_{ik}\Delta_{il} + \Delta_{il}\Delta_{ik}\right) +$$

$$+6\Delta_{ii}\Delta_{jj}\left(\Delta_{ik}\Delta_{jl} + \Delta_{il}\Delta_{jk}\right) + \left(6\Delta_{ij}^3 + 9\Delta_{ij}\Delta_{ii}\Delta_{jj}\right)\Delta_{kl}\Big\}$$

We can divide all diagrams into two sets: connected and non-connected. Non-connected diagrams are by definition those can be divided in two by a line that does not cut any existing line of the diagram).

Consider the two-point function. To zeroth order in the coupling constant, this is just the propagator. To first order in g there is no diagram. To second order, there are two diagrams. The first one is a correction to the propagator, with a factor in front $\frac{g^2}{2}$; and the second one is a *tadpole*, also with a factor in front $\frac{g^2}{2}$.

- Consider the interaction denoted by ϕ^4_4.

$$V(\phi) = \frac{\lambda}{4!}\sum_i \phi_i^4 \tag{1.48}$$

Define

$$Z(J) \equiv \int d\phi e^{-\phi^T.M.\phi - V(\phi) - J.\phi} \tag{1.49}$$

and deduce

$$\begin{aligned}
&\frac{Z(J)}{Z_0(J)} = 1 + \frac{\lambda}{4!}\Big\{3\Delta_{ii}^2 + 6\Delta_{ii}(\Delta_{iu}J_u)^2 + (\Delta_{iu}J_u)^4\Big\}\\
&+\frac{1}{2}\left(\frac{\lambda}{4!}\right)^2\Big\{9\Delta_{ii}^2\Delta_{jj}^2 + 72\Delta_{ii}\Delta_{ij}^2\Delta_{jj} + 24\Delta_{ij}^4\\
&+(\Delta_{jv}J_v)^2[18\Delta_{ii}^2\Delta_{jj} + 72\Delta_{ii}\Delta_{ij}^2]\\
&+(\Delta_{iu}J_u)^2[18\Delta_{jj}^2\Delta_{ii} + 72\Delta_{jj}\Delta_{ij}^2]\\
&+\Delta_{iu}J_u\Delta_{jv}J_v[96\Delta_{ij}^3 + 144\Delta_{ii}\Delta_{ij}\Delta_{jj}]\\
&+3(\Delta_{iu}J_u)^4\Delta_{jj}^2 + 3(\Delta_{jv}J_v)^4\Delta_{ii}^2 + 48(\Delta_{jv}J_v)^3\Delta_{iu}J_u\Delta_{ii}\Delta_{jj} + 48(\Delta_{iv}J_v)^3\Delta_{ju}J_u\Delta_{ii}\Delta_{jj}\\
&+(\Delta_{iu}J_u)^2(\Delta_{jv}J_v)^2[36\Delta_{ii}\Delta_{jj} + 72\Delta_{ij}^2] + 6(\Delta_{iu}J_u)^2(\Delta_{jv}J_v)^2\Delta_{jj} + 6(\Delta_{ju}J_u)^2(\Delta_{iv}J_v)^2\\
&+16(\Delta_{iu}J_u)^3(\Delta_{jv}J_v)^3\Delta_{ij} + (\Delta_{iu}J_u)^4(\Delta_{jv}J_v)^4\Big\} + O(\lambda^3) \qquad (1.
\end{aligned}$$

This expression summarizes all Green functions perturbation theory up to $O(\lambda^2)$.

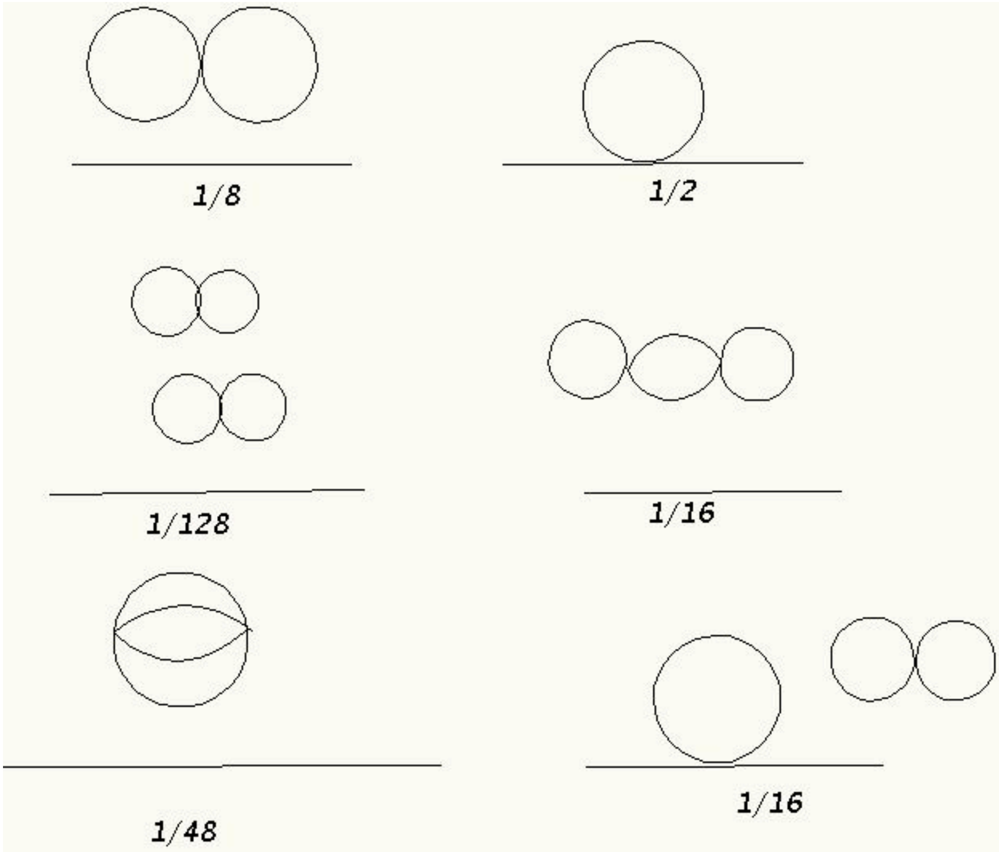

Figure 1.1: Vacuum diagrams

For example, the full two-point function reads

$$
\begin{aligned}
&\langle 0_+ | T\phi_k\phi_l | 0_- \rangle = \Delta_{kl} + \frac{\lambda}{4!}\Big\{3\Delta_{ii}^2\Delta_{kl} + 12\Delta_{ii}\Delta_{ik}\Delta_{il}\Big\} \\
&+\frac{1}{2}\left(\frac{\lambda}{4!}\right)^2\Big\{(9\Delta_{ii}^2\Delta_{jj}^2 + 72\Delta_{ii}\Delta_{ij}^2\Delta_{jj} + 24\Delta_{ij}^4)\Delta_{kl} \\
&+2\Delta_{jk}\Delta_{jl}(18\Delta_{ii}^2\Delta_{jj} + 72\Delta_{ii}\Delta_{ij}^2) + 2\Delta_{ik}\Delta_{il}(18\Delta_{jj}^2\Delta_{ii} + 72\Delta_{jj}\Delta_{ij}^2) \\
&+(\Delta_{ik}\Delta_{jl} + \Delta_{il}\Delta_{jk})(96\Delta_{ij}^3 + 144\Delta_{ii}\Delta_{ij}\Delta_{jj})\Big\} \qquad (1.51)
\end{aligned}
$$

In particular, terms linear in the coupling constant read

$$\langle 0_+ | T\phi_k\phi_l\phi_i\phi_i\phi_i\phi_i | 0_- \rangle \qquad (1.52)$$

From the $C_2^6 = 15$ possible pairings, 3 correspond to the combination

$$\Delta_{kl}\Delta_{ii} \qquad (1.53)$$

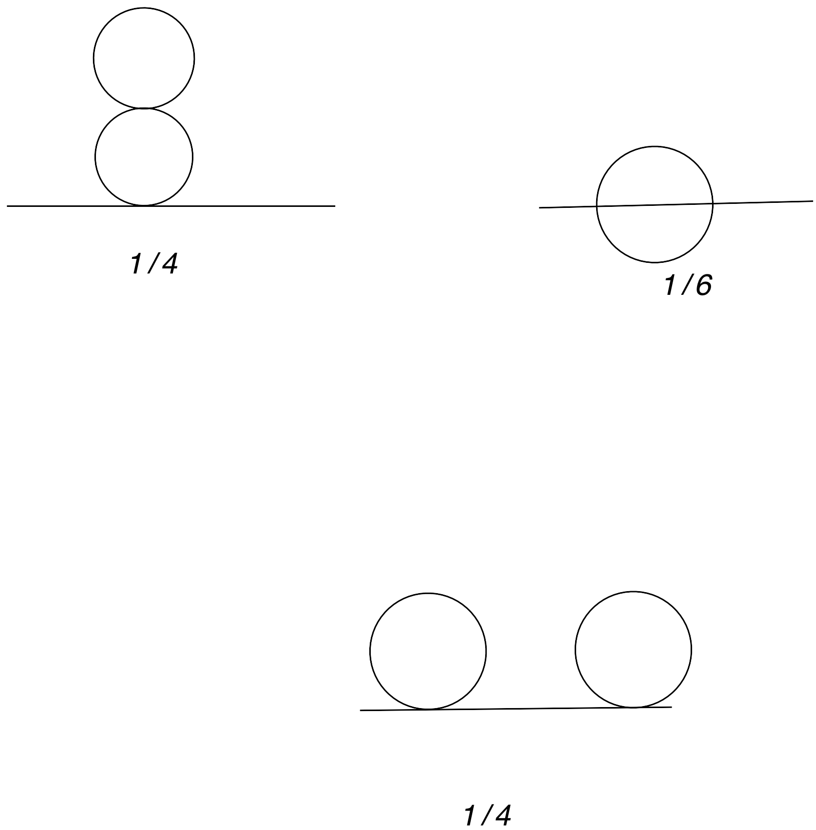

Figure 1.2: Connected contributions to the two-point function.

and the other 12, to the other possible combination

$$\Delta_{ik}\Delta_{il}\Delta_{ii} \tag{1.54}$$

Connected functions stem from

$$W[J] \equiv \log Z[J] \tag{1.55}$$

This object is often called the *free energy* by analogy with corresponding statistical-mechanics quantity

$$\langle 0_+ \,|T\phi_{i_1}\ldots\phi_{i_n}|\, 0_-\rangle_c \equiv \frac{\partial}{\partial J_{i_1}}\cdots\frac{\partial}{\partial J_{i_n}}\,\log Z(J)|_{J=0} \tag{1.56}$$

$$\begin{aligned} \partial W &= i\frac{\partial Z}{Z} \\ \partial^2 W &= i\left(\frac{\partial^2 Z}{Z} - \frac{\partial Z \partial Z}{Z^2}\right) \end{aligned} \tag{1.57}$$

In perturbation theory

$$Z = Z_0 + gZ_1 + g^2 Z_2 + O(g^3) \tag{1.58}$$

so that

$$\frac{1}{Z} = \frac{1}{Z_0} - \frac{Z_1}{Z_0^2}g + g^2\left(\frac{Z_1^2}{Z_0^2} - \frac{Z_2}{Z_0^2}\right) + O(g^3) \tag{1.59}$$

Normalize $Z_0 = 1$. Then

$$\frac{1}{Z} = 1 - Z_1 g + g^2\left(Z_1^2 - Z_2\right) + O(g^3) \tag{1.60}$$

as well as

$$\frac{1}{Z^2} = 1 - 2Z_1 g + g^2\left(3Z_1^2 - 2Z_2\right) + O(g^3) \tag{1.61}$$

which implies

$$\begin{aligned} &\langle 0_+ \,|\phi_i\phi_j|\, 0_-\rangle_c = \langle\phi_i\phi_j\rangle_0 - \langle\phi_i\rangle_0\langle\phi_j\rangle_0 \\ &\Big[-Z_1\langle\phi_i\phi_j\rangle_0 + \langle\phi_i\phi_j\rangle_1 + 2Z_1\langle\phi_i\rangle_0\langle\phi_j\rangle_0 - \langle\phi_i\rangle_0\langle\phi_j\rangle_1 - \langle\phi_i\rangle_1\langle\phi_j\rangle_0\Big]g + \\ &\Big[\langle\phi_i\phi_j\rangle_2 + \langle\phi_i\phi_j\rangle_0\left(Z_1^2 - Z_2\right) - \langle\phi_i\phi_j\rangle_1 Z_1 - \langle\phi_i\rangle_0\langle\phi_j\rangle_0\left(3Z_1^2 - 2Z_2\right) + \\ &2Z_1\left(\langle\phi_i\rangle_0\langle\phi_j\rangle_1 + \langle\phi_i\rangle_1\langle\phi_j\rangle_0\right) - \langle\phi_i\rangle_0\langle\phi_j\rangle_2 - \langle\phi_i\rangle_1\langle\phi_j\rangle_1 - \langle\phi_i\rangle_2\langle\phi_j\rangle_0\Big]g^2 \end{aligned}$$

It is amusing to check that all this is self-consistent. A formal proof is to be found in Hugh Osborn's lectures.

1.2 Berezin integral

The classical limit $\hbar \to 0$ of fermion fields does not exist unless we accept anticommuting c-numbers, that is, elements of a $\mathbb{Z}_2$ graded Grassmann algebra,

$$V = V_1 \oplus V_2 \tag{1.62}$$

The parity is denoted by $p \in \frac{\mathbb{Z}}{2\mathbb{Z}}$. Finite dimensional Grassmann algebras enjoy a set of N generators,

$$\{\psi_i, \psi_j\} = 0 \tag{1.63}$$

All of them are idempotent

$$\psi_i^2 = 0 \tag{1.64}$$

A representation of the algebra is given in terms of matrices as follows. Define

$$\sigma_\pm \equiv \frac{1}{\sqrt{2}} (\sigma_1 \pm i\sigma_2) \tag{1.65}$$

Then

$$\begin{aligned} &\sigma_\pm^2 = 0 \\ &\{s_\pm, \sigma_3\} = 0 \\ &\{\sigma_+, \sigma_-\} = 1 \end{aligned} \tag{1.66}$$

The representation of the n-dimensional Grassmann algebra is generated by the N matrices of dimension 2^N

$$\begin{aligned} &\psi_1 \equiv \sigma_+ \otimes \sigma_3 \otimes \ldots \otimes \sigma_3 \\ &\psi_2 \equiv \sigma_3 \otimes \sigma_+ \otimes \ldots \otimes \sigma_3 \\ &\psi_3 \equiv \sigma_3 \otimes \sigma_3 \otimes \sigma_+ \ldots \otimes \sigma_3 \\ &\ldots \\ &\psi_n \equiv \sigma_3 \otimes \sigma_3 \otimes \ldots \otimes \sigma_+ \end{aligned} \tag{1.67}$$

An arbitrary element of the algebra can be written as

$$\chi = \sum_{n<N} \sum_{i_1 \ldots i_n} c^{(n)}_{i_1 \ldots i_n} \psi_{i_1} \ldots \psi_{i_n} \tag{1.68}$$

(the coefficients $c^{(n)}_{i_1 \ldots i_N} \in \mathbb{C}$ are complex-valued antisymmetric tensors.)

Odd elements ($n \in 2\mathbb{Z}+1$) have fermionic character, and do anticommute with all other odd elements; whereas even ones have a bosonic character and commute with all other elements of the algebra, even or odd.

All analytic functions are just polynomials. For example, in a one dimensional algebra

$$f(\psi) = a + b\psi \tag{1.69}$$

And in two dimensions

$$f(\psi_1, \psi_2) = a + b_1\psi_1 + b_2\psi_2 + c\psi_1\psi_2 \tag{1.70}$$

Derivatives can be taken from the left or from the right, and they are in general different. Assuming $g(f) = 0$, for example

$$\frac{\partial_L}{\partial\psi_1} f = -b_1 + c\psi_2 \tag{1.71}$$

whereas

$$\frac{\partial_R}{\partial\psi_1} f = b_1 - c\psi_2 \tag{1.72}$$

The only translational invariant measure is Berezin's

$$\int d\psi\,(a + b\psi) \equiv b \tag{1.73}$$

Indeed

$$\int d\psi\,(a + b\,(\psi - \psi_0)) = \int d\psi\,(a + b\psi) \equiv b \tag{1.74}$$

In QFT we often encounter independent integration variables ψ_i y $\bar{\psi}_i$ (not related by complex conjugation)

$$\int d\bar{\psi} d\psi e^{-\bar{\psi}\lambda\psi} = \int d\bar{\psi} d\psi (1 - \bar{\psi}\lambda\psi) = \lambda \tag{1.75}$$

To belabor this point

$$\begin{aligned}
&\int d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2 \; e^{-\left(\bar{\psi}_1\psi_1 M_{11} + \bar{\psi}_1\psi_2 M_{12} + \bar{\psi}_2\psi_1 M_{21} + \bar{\psi}_2\psi_2 M_{22}\right)} = \\
&\int d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2 \left(M_{11} M_{22} \bar{\psi}_1\psi_1\bar{\psi}_2\psi_2 + M_{12} M_{21} \bar{\psi}_1\psi_2\bar{\psi}_2\psi_1\right) = \\
&M_{11}M_{22} - M_{12}M_{21} = det\, M
\end{aligned} \tag{1.76}$$

The gaussian integral is then defined as the determinant

$$\int d\bar{\psi} d\psi e^{-\bar{\psi}_i M^{ij} \psi_j} = \det M \tag{1.77}$$

Translational invariance allows for

$$Z_0\,(\eta, \bar{\eta}) \equiv \int d\bar{\psi} d\psi e^{-\bar{\psi}_i K^{ij} \psi_j - \bar{\eta}_i \psi_i - \bar{\psi}_l \eta_l} = \det K e^{\bar{\eta}_l K^{-1}_{lm} \eta_m} \tag{1.78}$$

The usual fermionic integration in QFT reads

$$V \equiv g\bar{\psi}_i N_{ij} \psi_j \tag{1.79}$$

N_{ij} is a matrix valued in different spaces (spin, flavor,color,etc)

We will be interested in Green functions of the type:

$$< 0_+ \left| T\psi_l\bar{\psi}_m \right| 0_- >\equiv \int d\bar{\psi}d\psi e^{-\bar{\psi}_i K^{ij}\psi_j - g\psi_i N^{il}\bar{\psi}_l}\psi_l\bar{\psi}_m \tag{1.80}$$

With $S \equiv K^{-1}$ we get

$$\begin{aligned}
&< 0_+ \left| T\psi_p\bar{\psi}_q \right| 0_- >\equiv S_{pq} + g(S_{pq}N_{ij}S_{ji} - S_{jq}S_{pi}N_{ij}) \\
&+\frac{g^2}{2}(S_{pq}S_{lk}N_{ij}S_{ji}N_{kl} - S_{pk}S_{lq}N_{ij}S_{ji}N_{kl} + S_{pk}S_{jq}S_{li}N_{ij}N_{kl} - S_{lk}S_{jq}S_{pi}N_{ij}N_{kl} \\
&+S_{lq}S_{jk}S_{pi}N_{ij}N_{kl} - S_{jk}S_{li}N_{ij}N_{kl}S_{pq})
\end{aligned} \tag{1.81}$$

1.3 Summary of functional integration.

- We just saw that

$$Z[J] = N\ e^{i\int d^4y\ V\left(\frac{\delta}{i\ \delta J(y)}\right)}\ Z_0[J] \tag{1.82}$$

Now we claim that this is equivalent to

$$Z[J] = N\ \left\{ e^{\frac{1}{2}\int d^4x_1 d^4x_2 \Delta(x_2-x_1)\frac{d}{\delta\phi(x_1)}\frac{\delta}{\delta\phi(x_2)}} \right\}\ e^{i\int d^4x\ (V(\phi)+J\phi)}\Big|_J = 0 \tag{1.83}$$

This stems from a generalization of the identity

$$F\left(\frac{\partial}{i\partial x_i}\right)\ G(x_i) = \left\{ G\left(\frac{\partial}{i\partial y_i}\right)\ F(y_j)\ \ e^{i\sum x_i y_i} \right\}\Big|_{y_k=0} \tag{1.84}$$

Let us prove it for plane waves

$$\begin{aligned}
G(x) &= e^{i\sum a_i x_i} \\
G(x) &\equiv e^{i\sum q_i x_i}
\end{aligned} \tag{1.85}$$

The first member gives

$$\begin{aligned}
&e^{a_i\partial_i}\ e^{iqx} = \left(1 + a_i\partial_i + \frac{1}{2}a_i a_j\partial_i\partial_j + \dots\right)\ e^{iqx} = \\
&= \left(1 + iqa - \frac{1}{2}(a.q)^2 + \dots\right)\ e^{iqx} = e^{iq(x+a)}
\end{aligned} \tag{1.86}$$

The second member, in turn, reads

$$e^{q\partial_y}\ \ e^{i(a+x)y}\Big|_{y=0} = e^{i(a+x)(q+y)}\Big|_{y=0} = e^{i(a+x)q} \quad QED. \tag{1.87}$$

The formula [1.83] clearly represents Feynman's perturbation series.

- Complex scalar fields

$$\phi \equiv \frac{\phi_1 + i\phi_2}{\sqrt{2}} \tag{1.88}$$

Assuming a symmetric $K_{ij} = K_{ji}$,

$$\int \mathcal{D}\phi \; e^{-\int d^n x \; \bar{\phi} \; K\phi} = \det \; K^{-1} \tag{1.89}$$

- *Berezinian or superdeterminant*,

$$\int \mathcal{D}\phi \; \mathcal{D}\psi \; \mathcal{D}\bar{\psi} \; e^{-\int d^n x \left\{ \bar{\phi} M\phi + \bar{\psi} K\psi + \bar{\psi} N_1 \phi + \bar{\phi} N_2 \psi \right\}} =$$

$$\int \mathcal{D}\phi \; \mathcal{D}\psi \; \mathcal{D}\bar{\psi} \; e^{-\int d^n x \left\{ \left(\bar{\phi} + \bar{\psi} N_1 M^{-1}\right) M \left(\phi + M^{-1} N_2 \psi\right) - \psi N_1 M^{-1} N_2 \psi + \bar{\psi} K \psi \psi \right\}} =$$

$$= \det \; M^{-1} \; \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \; e^{-\int d^n x \; \bar{\psi} \left(K - N_1 M^{-1} N_2\right)} = \det \; M^{-1} \det \; \left(K - N_1 M^{-1} N_2\right)$$

Denoting by

$$\mathcal{M} \equiv \begin{pmatrix} M & N_2 \\ N_1 & N_1 \end{pmatrix} \tag{1.90}$$

It is customary to call berezinian the quantity

$$\text{ber} \; \mathcal{M} \equiv \det \; M^{-1} \det \; \left(K - N_1 M^{-1} N_2\right) \tag{1.91}$$

- The only extra thing we need to postulate on the functional measure is that it must be invariant under translations, that is

$$\begin{aligned} \mathcal{D}\left(\psi + \psi_0\right) &= \mathcal{D}\psi \\ \mathcal{D}\left(\phi + \phi_0\right) &= \mathcal{D}\phi \end{aligned} \tag{1.92}$$

This we need to compute the action of external sources.

- Sometimes it is convenient to specify morte the measure. We can define

$$\mathcal{D}\phi \equiv \prod_x d\phi(x) \tag{1.93}$$

With this definition there is a divergent factor of $\delta^{(n)}(0)$ each time a point transformation is made.

$$\mathcal{D}\left[f\phi\right] \equiv \prod_x d\left[f(x)\phi(x)\right] = \det \left[f(x)\delta^{(n)}(x-y)\right] \; \mathcal{D}\phi =$$

$$e^{\text{tr} \; \log \; \left[f(x)\delta^{(n)}(x-y)\right]} \; \mathcal{D}\phi = e^{\delta^{(n)}(0) \int d^n x \;\; \log \;\; f(x)} \; \mathcal{D}\phi \tag{1.94}$$

- A very important property, which will be used repeateadly in the sequel is that the integral if a total derivative vanishes

$$\int \mathcal{D}\phi \frac{\delta}{\delta\phi} F[\phi] = 0 \tag{1.95}$$

- Under a change of functional variables

$$\begin{aligned} \mathcal{D}f(\phi) &= \det\left(\frac{\partial f}{\partial\phi}\right) \mathcal{D}\phi \\ \mathcal{D}g(\psi) &= \det{}^{-1}\left(\frac{\partial g}{\partial\psi}\right) \mathcal{D}\psi \end{aligned} \tag{1.96}$$

  The formula for the change of variables in fermionic integrals (inverse jacobian) is forced upon us by consistency, because already when λ is a constant

$$1 = \int d(\lambda\psi)\,(\lambda\psi) \tag{1.97}$$

- Unfortunately all this is formal, because all this determinants are divergent and must be regularized. This is, in some sense, the essence of quantum field theory.

2

Schwinger's action principle and the equations of motion.

From the functional integral Schwinger's action principle is more or less obvious. It states that under any smooth changes in the lagrangian

$$\delta\langle\phi_f\ t_f|\phi_i\ t_i\rangle = \int_{t_i}^{t_f} d^4x\ \langle\phi_f\ t_f\left|\frac{i}{\hbar}\delta L(x)\right|\phi_i\ t_i\rangle \tag{2.1}$$

Starting from Feynman's integral

$$\begin{aligned}\delta\langle\phi_f\ t_f|\phi_i\ t_i\rangle &= \delta\int\mathcal{D}\phi\ e^{i\int_i^f d^4x\ L} = \\ &= \int_{t_i}^{t_f} d^4x\ \langle\phi_f\ t_f\left|\frac{i}{\hbar}\delta L(x)\right|\phi_i\ t_i\rangle\end{aligned} \tag{2.2}$$

It should be clear that a logical possibility would also be to start from the action principle and then derive Feynman's integral as a consequence. In a sense, they are two formally equivalent formulations, one in terms of functional differentials, and the other in terms of functional integrals.

Either from the action principle or directly from the functional integral one can derive the equations of motion, sometimes called the Schwinger-Dyson equations. Those are an infinite set relating the two-point function to the four point function, and the four point function to the sixth order function, and so on. This hierarchy can be solved perturbatively, and it yields the diagrammatic Feynman series.

Choosing the ϕ^4_4 theory as an example, let us write the action as

$$S = \int d(vol)\left(\frac{1}{2}\ \phi(x)K\phi(x) - \frac{\lambda}{24}\phi(x)^4 - J(x)\phi(x)\right) \tag{2.3}$$

Here we have denoted the inverse propagator as

$$K \equiv \Box + m^2 \tag{2.4}$$

The starting point is the fact that the functional integral of a total derivative vanishes. This yields

$$\int \mathcal{D}\phi \ \frac{\delta}{\delta\phi(x_1)} \ e^{-S} = 0 \tag{2.5}$$

This is

$$K\langle\phi(x_1)\rangle_J - \frac{\lambda}{6}\langle\phi(x_1)^3\rangle_J - \langle J(x_1)\rangle_J = 0 \tag{2.6}$$

Let us now convolute with the propagator

$$\Delta_{xy} * K = \delta_{xy} \tag{2.7}$$

to get the formal solution. This an exact statement, holding formally form all values of λ and for arbitrary sources $J(x)$.

$$\langle\phi(x_1)\rangle_J - \frac{\lambda}{6}\Delta_{x_1x_2} * \langle\phi(x_2)^3\rangle_J - \Delta_{x_1x_2} * \langle J(x_2)\rangle_J = 0 \tag{2.8}$$

We can now functionally derive this extression with respect to $J(z)$. This yields

$$\langle T\phi(x_3)\phi(x_1)\rangle_J - \frac{\lambda}{6}\Delta_{x_1x_2} * \langle T\phi(x_3)\phi(x_2)^3\rangle_J - \Delta_{x_1x_2} * \langle J(x_2)\phi(x_3)\rangle_J - \Delta_{x_1x_3}\langle 1\rangle_J = 0 \tag{2.9}$$

This equation yields, as promised, the two-point function in terms of the four-point function. Functionally deriving twice more we get

$$\begin{aligned}&\langle T\phi(x_4)\phi(x_3)\phi(x_1)\rangle_J - \frac{\lambda}{6}\Delta_{x_1x_2} * \langle T\phi(x_4)\phi(x_3)\phi(x_2)^3\rangle_J - \\ &-\Delta_{x_1x_2} * \langle J(x_2)T\phi(x_4)\phi(x_3)\rangle_J - \Delta_{x_1x_2} * \delta_{x_2x_4}\langle\phi(x_3)\rangle_J - \Delta_{x_1x_3}\langle\phi(x_4)\rangle_J = 0\end{aligned} \tag{2.10}$$

as well as

$$\begin{aligned}&\langle T\phi(x_5)\phi(x_4)\phi(x_3)\phi(x_1)\rangle_J - \frac{\lambda}{6}\Delta_{x_1x_2} * \langle T\phi(x_5)\phi(x_4)\phi(x_3)\phi(x_2)^3\rangle_J - \\ &-\Delta_{x_1x_2} * \langle J(x_2)T\phi(x_5)\phi(x_4)\phi(x_3)\rangle_J - \Delta_{x_1x_2} * \langle\delta_{x_2x_5}T\phi(x_4)\phi(x_3)\rangle_J - \\ &-\Delta_{x_1x_2} * \delta_{x_2x_4}\langle T\phi(x_5)\phi(x_3)\rangle_J - \Delta_{x_1x_3}\langle T\phi(x_5)\phi(x_4)\rangle_J = 0\end{aligned} \tag{2.11}$$

Again those equations are exact equations: no approximations are involved in deriving them. We can now make contact with the perturbative expansion.

In the absence of sources

$$\langle T\phi(x_3)\phi(x_1)\rangle = \Delta\left(x_1 - x_3\right) + \frac{\lambda}{6}\int d^4x_2 \ \Delta(x_1 - x_2)\langle T\phi(x_3)\phi(x_2)^3\rangle \tag{2.12}$$

To $O(\lambda^2)$ all we need is the four-point function to $O(\lambda)$. This is easily obtained from the last equation

$$\begin{aligned}&\langle T\phi(x_5)\phi(x_4)\phi(x_3)\phi(x_1)\rangle = \Delta\left(x_1 - x_5\right)\Delta(x_4 - x_3) + \Delta\left(x_1 - x_4\right)\Delta(x_5 - x_3) + \\ &+\Delta\left(x_1 - x_3\right)\Delta\left(x_5 - x_4\right)\end{aligned} \tag{2.13}$$

which reproduces Wick's theorem. The tow point function reads

$$\langle T\phi(x_3)\phi(x_1)\rangle = \Delta\left(x_1 - x_3\right) + \frac{\lambda}{6}\int dx_2 \Delta\left(x_1 - x_2\right)) \, 3\Delta(0)\Delta\left(x_2 - x_3\right) \tag{2.14}$$

and physically represents the contribution of the tadpole.

This procedure can be easily extended to any order in perturbation theory as well as to more complicated theories, involving nontrivial spins. Let us compute now the $O(\lambda^2)$ contribution to the two point function. We need the sixth point function to $O(\lambda)$. Let us carry on

$$\begin{aligned}
&\langle T\phi(x_6)\phi(x_5)\phi(x_4)\phi(x_3)\phi(x_1)\rangle_J - \frac{\lambda}{6}\Delta_{x_1x_2} * \langle T\phi(x_6)\phi(x_5)\phi(x_4)\phi(x_3)\phi(x_2)^3\rangle_J - \\
&-\Delta_{x_1x_2} * \langle J(x_2)T\phi(x_6)\phi(x_5)\phi(x_4)\phi(x_3)\rangle_J - \Delta_{x_1x_2} * \langle \delta_{x_2x_5}T\phi(x_6)\phi(x_4)\phi(x_3)\rangle_J - \\
&-\Delta_{x_1x_2} * \langle \delta_{x_2x_6}T\phi(x_5)\phi(x_4)\phi(x_3)\rangle_J \\
&-\Delta_{x_1x_2} * \delta_{x_2x_4}\langle T\phi(x_6)\phi(x_5)\phi(x_3)\rangle_J - \Delta_{x_1x_3}\langle T\phi(x_6)\phi(x_5)\phi(x_4)\rangle_J = 0
\end{aligned} \tag{2.15}$$

and finally,

$$\begin{aligned}
&\langle T\phi(x_7)\phi(x_6)\phi(x_5)\phi(x_4)\phi(x_3)\phi(x_1)\rangle_J - \frac{\lambda}{6}\Delta_{x_1x_2} * \langle T\phi(x_7)\phi(x_6)\phi(x_5)\phi(x_4)\phi(x_3)\phi(x_2)^3\rangle_J - \\
&-\Delta_{x_1x_2} * \langle J(x_2)T\phi(x_7)\phi(x_6)\phi(x_5)\phi(x_4)\phi(x_3)\rangle_J - \Delta_{x_1x_2} * \langle \delta_{x_2x_7})T\phi(x_6)\phi(x_5)\phi(x_4)\phi(x_3)\rangle_J - \\
&-\Delta_{x_1x_2} * \langle \delta_{x_2x_5}T\phi(x_7)\phi(x_6)\phi(x_4)\phi(x_3)\rangle_J - \\
&-\Delta_{x_1x_2} * \langle \delta_{x_2x_6}T\phi(x_7)\phi(x_5)\phi(x_4)\phi(x_3)\rangle_J \\
&-\Delta_{x_1x_2} * \delta_{x_2x_4}\langle T\phi(x_7)\phi(x_6)\phi(x_5)\phi(x_3)\rangle_J - \Delta_{x_1x_3}\langle T\phi(x_7)\phi(x_6)\phi(x_5)\phi(x_4)\rangle_J = 0
\end{aligned} \tag{2.16}$$

When the sources vanish this yields to the lowest order, which is all we need, the sixth point function in terms of the four-point function.

$$\begin{aligned}
&\langle T\phi(x_7)\phi(x_6)\phi(x_5)\phi(x_4)\phi(x_3)\phi(x_1)\rangle = \Delta(x_1 - x_7)\langle T\phi(x_6)\phi(x_5)\phi(x_4)\phi(x_3)\rangle + \\
&+\Delta(x_1 - x_5)\langle T\phi(x_7)\phi(x_6)\phi(x_4)\phi(x_3)\rangle + \Delta(x_1 - x_6)\langle T\phi(x_7)\phi(x_5)\phi(x_4)\phi(x_3)\rangle \\
&+\Delta(x_1 - x_4)\langle T\phi(x_7)\phi(x_6)\phi(x_5)\phi(x_3)\rangle + \Delta(x_1 - x_3)\langle T\phi(x_7)\phi(x_6)\phi(x_5)\phi(x_4)\rangle
\end{aligned}$$

Plugging in our previous results,this reads

$$\begin{aligned}
&\langle T\phi(x_7)\phi(x_6)\phi(x_5)\phi(x_4)\phi(x_3)\phi(x_1)\rangle = \Delta_{17}\left(\Delta_{65}\Delta_{43} + \Delta_{64}\Delta_{53} + \Delta_{63}\Delta_{54}\right) + \\
&+\Delta_{15}\left(\Delta_{76}\Delta_{43} + \Delta_{74}\Delta_{63} + \Delta_{73}\Delta_{64}\right) + \Delta_{16}\left(\Delta_{75}\Delta_{43} + \Delta_{74}\Delta_{53} + \Delta_{73}\Delta_{54}\right) \\
&+\Delta_{14}\left(\Delta_{76}\Delta_{53} + \Delta_{75}\Delta_{63} + \Delta_{73}\Delta_{65}\right) + \Delta_{13}\left(\Delta_{76}\Delta_{54} + \Delta_{75}\Delta_{64} + \Delta_{74}\Delta_{65}\right)
\end{aligned}$$

The desired λ^2 contribution to $\langle T\phi(x_3)\phi(x_1)\rangle$ is

$$\begin{aligned}
&\frac{\lambda^2}{36}\int d^4x_2 d^4y \; \Delta(x_1 - x_2)\Delta(x_2 - y)\langle T\phi_3\phi_2^2\phi_y^3\rangle = \frac{\lambda^2}{36}\int d^4x_2 d^4y \; \Delta_{12}\Delta_{2y}\Big\{ \\
&\Delta_{y3}\left(\Delta_0\Delta_0 + \Delta_{2y}\Delta_{2y} + \Delta_{2y}\Delta_{2y}\right) + \Delta_{y2}\left(\Delta_{32}\Delta_0 + \Delta_{3y}\Delta_{2y} + \Delta_{3y}\Delta_{2y}\right) + \\
&+\Delta_{y2}\left(\Delta_{32}\Delta_0 + \Delta_{3y}\Delta_{2y} + \Delta_{3y}\Delta_{2y}\right) + \Delta_0\left(\Delta_{32}\Delta_{2y} + \Delta_{32}\Delta_{2y} + \Delta_{3y}\Delta_0\right) + \\
&+\Delta_0\left(\Delta_{32}\Delta_{2y} + \Delta_{32}\Delta_{2y} + \Delta_{3y}\Delta_0\right)\Big\}
\end{aligned} \tag{2.17}$$

2.1 The S-matrix in QFT.

The most important task in order to be able to extract testable predictions out of fundamental theories is to compite cross-sections. Those are known once S-matrix elements are known. There is a systematic recipe for computing these in terms of Feynman diagrams, namely the Feynman rules.

Let us recall that the in-states are defined so that

$$e^{-iHt}|in\rangle = e^{-iH_0 t}|in\rangle \tag{2.18}$$

The S-matrix is then given in the interaction representation by

$$S = \lim_{t''\to\infty} \lim_{t'\to-\infty} e^{iH_0 t''} e^{-iH(t''-t')} e^{-iH_0 t'} \tag{2.19}$$

2.2 Feynman Rules.

The simplest field is the spin zero scalar field. It corresponds to the Higgs field in the standard model. Its lagrangian reads

$$L = \frac{1}{2}(\partial_\mu \Phi \partial^\mu \Phi - m^2 \Phi^2) - \frac{g}{3!}\Phi^3 - \frac{\lambda}{4!}\Phi^4 \tag{2.20}$$

Reinstating c and $\hbar$, masses are really inverse Compton lengths $\frac{m^2 c^2}{\hbar^2}$. The quartic coupling constant λ is really $\frac{\lambda}{\hbar}$. Redefining

$$\Phi \equiv \hbar^{1/2}\tilde{\Phi} \tag{2.21}$$

the action reads

$$S = \hbar \int d^4x \left(\frac{1}{2}(\partial_\mu \tilde{\Phi} \partial^\mu \tilde{\Phi} - \frac{1}{l_c^2}\tilde{\Phi}^2) - \frac{g\hbar^{1/2}}{3!}\tilde{\Phi}^3 - \frac{\lambda}{4!}\tilde{\Phi}^4 \right) \tag{2.22}$$

Then this quartic coupling is dimensionless, $[\lambda] = 0$, and the cubic coupling has got dimensions of an inverse distance: $[g\hbar^{1/2}] = L^{-1}$.

Feynman boundary conditions (positive frequencies propagating towards the future; negative frequancies propagating towards the past) are equivalent to working in the euclidean fremework

$$x^0 \equiv -ix_4 \tag{2.23}$$

so that

$$iS = -S_E = -\int d^4x_E \left((\partial_\mu \phi)^2_E + m^2\phi^2 \right) \tag{2.24}$$

When Fourier transforming we shall continue in such a way that

$$p_0 = ip_4. \tag{2.25}$$

Instead of the discrete indices $i, j, k \ldots$,we have now four-dimensionsl coordinates, $x, y, z \ldots$.

$$\sum_{ij} -\phi_i M_{ij} \phi_j \to -\int d^4x d^4y \phi(x) M(x,y) \phi(y) \tag{2.26}$$

What is in the exponent of the path integral is just iS_{clas}. Then we are led to identify

$$M(x,y) = \frac{i}{2}(\Box + m^2)\delta^4(x-y) \tag{2.27}$$

Also

$$-\sum_i J_i \phi_i \to i \int d^4x J(x) \phi(x) \tag{2.28}$$

The bosonic propagator $\Delta \equiv \frac{1}{2} M^{-1}$ is defined through

$$\int d^4z M(x,z) \Delta(z,y) = 2\delta^4(x-y) \tag{2.29}$$

This can be solved in momentum space

$$\Delta(x-y) \equiv \int \frac{d^4p}{(2\pi)^4} e^{-ip.(x-y)} \Delta(p) \tag{2.30}$$

namely

$$\int \frac{d^4p}{(2\pi)^4} e^{-ip.(x-y)} \frac{i}{2} \left(-p^2 + m^2\right) \Delta(p) = \int \frac{d^4p}{(2\pi)^4} e^{-ip.(x-y)} \tag{2.31}$$

Feynman's boundary conditions (or else analytical continuation from euclidean space) lead to the contour defined by the limit $\epsilon \to 0^+$;

$$\Delta(p) = \frac{i}{p^2 - m^2 + i\epsilon} \tag{2.32}$$

Feynman's propagator is symmetric with respect to the exchange $p \to -p$, or what is the same thing, with respect to the exchange of the points x and y.

It is proportional to the Fourier transform of the time ordered two-point function

$$\Delta_{xy} \equiv \Delta(x-y) = \langle 0 | T\phi(x)\phi(y) | 0 \rangle \tag{2.33}$$

Let us compute it using the identity

$$\frac{i}{P + i\epsilon} = \int_0^\infty ds \; e^{isP} \tag{2.34}$$

$$\Delta(x_1 - x_2) = \int \frac{d^4k}{(2\pi)^4}\ e^{ik(x_1-x_2)}\ \int_0^\infty ds\ e^{is(k^2-m^2)} =$$
$$= \int_0^\infty ds\ e^{-ism^2} \int \frac{d^4k}{(2\pi^4}\ e^{is\left(k+\frac{(x_1-x_2)}{2s}\right)^2 - i\frac{(x_1-x_2)^2}{4s}} =$$
$$= \frac{1}{(2\pi)^4} \int_0^\infty ds\ e^{-ism^2} \left(\frac{\pi}{is}\right)^2 e^{-i\frac{(x_1-x_2)^2}{4s}} = \frac{1}{8\pi^2} \frac{2m}{\sqrt{(x_1-x_2)^2 - i\epsilon}} e^{i\frac{\pi}{2}} \times$$
$$\times K_1\left(-m\sqrt{(x_1-x_2)^2 - i\epsilon}\right) \tag{2.35}$$

The imaginary part appears as a condition of convergence of the integral

$$\int_0^\infty x^{\nu-1} e^{i\frac{\mu}{2}\left(x - \frac{\beta^2}{x}\right)}\ dx = 2\ \beta^\nu e^{\frac{i\nu\pi}{2}}\ K_{-\nu}(\beta\mu) \tag{2.36}$$

only when Im $\mu > 0$ and Im $(\beta^2\mu) < 0$.

In the massless limit the behavior

$$K_n(z) \sim \frac{2^{n-1}\ (n-1)!}{z^n} \tag{2.37}$$

implies

$$\Delta(x_1 - x_2) = -\frac{i}{4\pi^2} \frac{1}{(x_2 - x_1)^2 - i\epsilon} \tag{2.38}$$

Let us repeat the same calculation with the euclidean propagator.

$$\Delta_E(x_1 - x_2) = \int \frac{d^4k}{(2\pi)^4}\ e^{ik(x_2-x_1)}\ \int_0^\infty ds\ e^{-s(k_E^2+m^2)} =$$
$$= \frac{1}{16\pi^4} \int_0^\infty ds\ \left(\frac{\pi}{s}\right)^2 e^{-m^2 s - \frac{(x_2-x_1)^2}{4s}} = \frac{1}{4\pi^2} \frac{m}{\sqrt{(x_2-x_1)_E^2}}\ K_{-1}\left(m\sqrt{(x_2-x_1)_E^2}\right)$$

In the massless limit (Remember that $K_{-\nu}(z) = K_\nu(z)$).

$$\Delta_E(x_1 - x_2) = \frac{1}{4\pi^2\ (x_1 - x_2)_E^2} \tag{2.39}$$

Let us compute the first quantum corrections to this Green function in the scalar theory ϕ^4_4. All terms follow strictly the ones in [1.51].Please compare with the diagrams in the figure. The coefficients in front of each of them are combinatorial in nature and thera are elaborate formulas to compute them (cf. for example [**?**]). The safest way is however to go back to basics and expand the path integral directly.

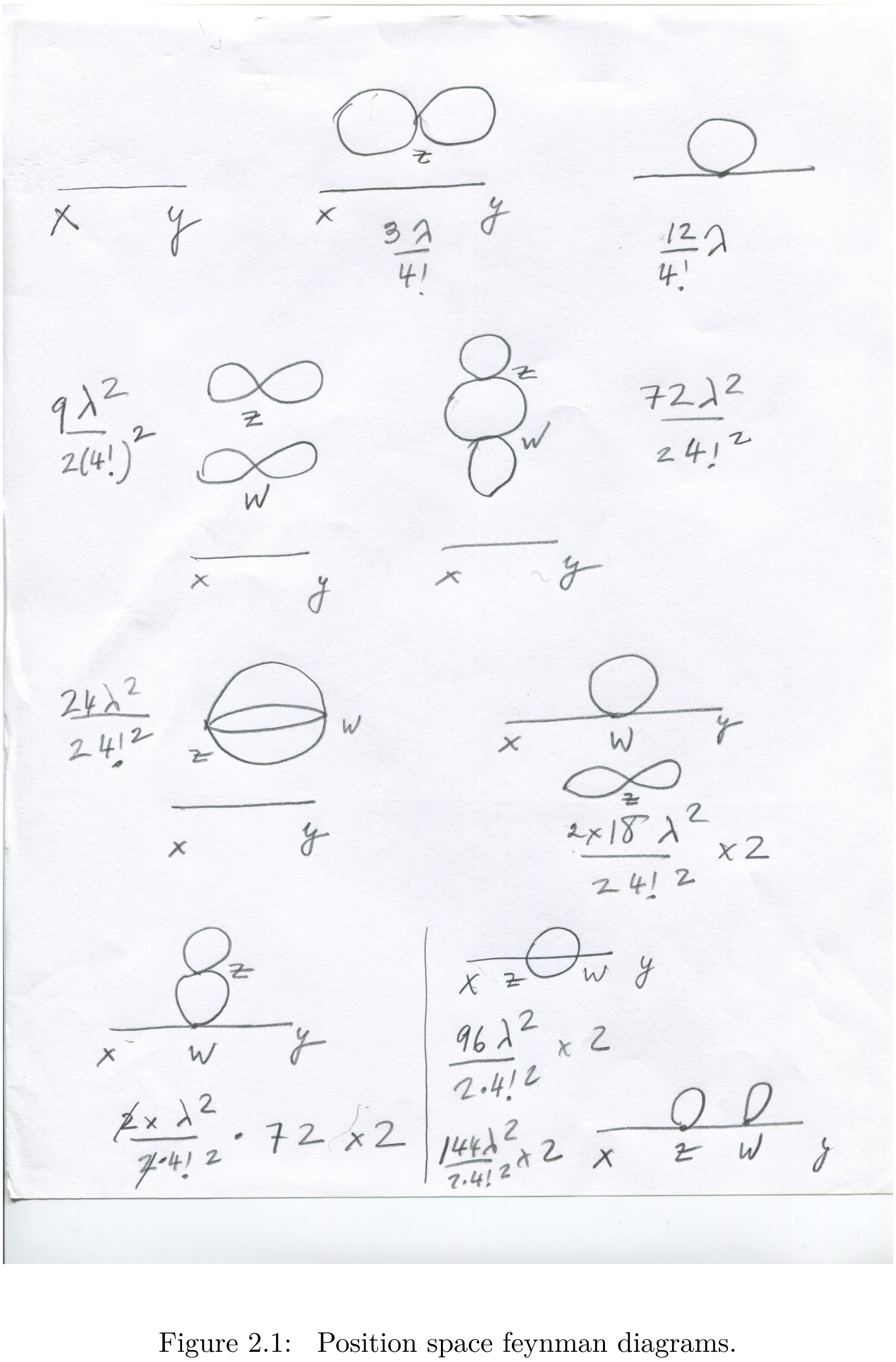

Figure 2.1: Position space feynman diagrams.

$$
\begin{aligned}
&\langle 0\,|T\phi(x)\phi(y)|\,0\rangle = \Delta(x-y) + \frac{\lambda}{4!}\int d^4x\Big\{3\Delta(z-z)^2\Delta(x-y) + \\
&+12\Delta(z-z)\Delta(z-x)\Delta(z-y)\Big\} \\
&+\frac{1}{2}\left(\frac{\lambda}{4!}\right)^2\int d^4z d^4w\Big\{\Big[9\Delta(z-z)^2\Delta(w-w)^2 + 72\Delta(z-z)\Delta(z-w)^2\Delta(w-w) + \\
&+24\Delta(z-w)^4\Big]\Delta(x-y) + 2\Delta(w-x)\Delta(w-y)\Big[18\Delta(z-z)^2\Delta(w-w) + \\
&+72\Delta(z-z)\Delta(z-w)^2\Big] + \\
&+2\Delta(z-x)\Delta(z-y)\Big[18\Delta(w-w)^2\Delta(z-z) + 72\Delta(w-w)\Delta(z-w)^2\Big] \\
&+(\Delta(z-x)\Delta(w-y) + \Delta(z-y)\Delta(w-x))\Big[96\Delta(z-w)^3 + \\
&+144\Delta(z-z)\Delta(z-w)\Delta(w-w)\Big]\Big\}
\end{aligned}
$$

Formally, the infinite volume of space-time is the quantity

$$V_x \equiv \int d^4x \tag{2.40}$$

It is usually covenient to work in momentum space. Let us denote

$$\int_p \equiv \int \frac{d^4p}{(2\pi)^4} \tag{2.41}$$

When integrating over space-time points we get a delta function in space-time enduring momentum conservation at each vertex, plus a global momentum conservation on external lines.

The momentum-space two-point-function then reads

$$
\begin{aligned}
&G(p) = \Delta(p) + \frac{\lambda}{4!}\Big\{3V_x\left(\int_q\Delta(q)\right)^2\Delta(p) + 12\int_q\Delta(q)\Delta(-p)\Delta(p)\Big\} \\
&+\frac{1}{2}\left(\frac{\lambda}{4!}\right)^2\Big\{\Big[9V_x^2\left(\int_q\Delta(q)\right)^2\left(\int_{q_1}\Delta(q_1)\right)^2 + 72V_x\left(\int_q\Delta(q)\right)^3 + \\
&+24V_x\int_{p_1p_2p_3}\Delta(p_1)\Delta(p_2)\Delta(p_3)\Delta(-p_1-p_2-p_3)\Big]\Delta(p) + \\
&4\Delta(p)\Delta(-p)\Big[18\left(\int_q\Delta(q)\right)^3V_x + 72\left(\int_q\Delta(q)\right)^2 + \\
&+48\int_{q_1q_2}\Delta(q_1)\Delta(q_2)\Delta(p-q_1-q_2)\Big] + 72\Delta(p)\left(\int_q\Delta(q)\right)^2\Big\}
\end{aligned} \tag{2.42}
$$

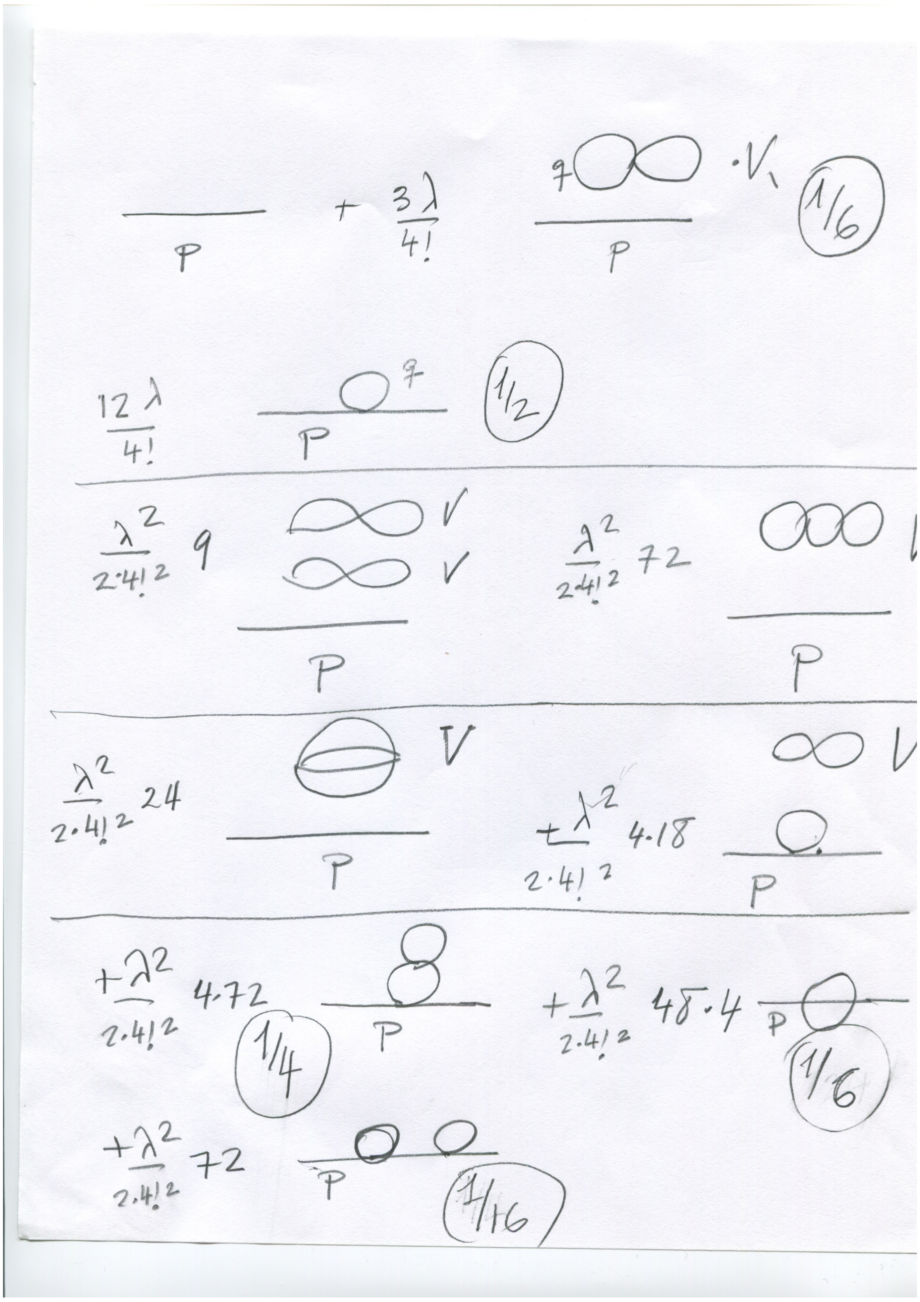

Figure 2.2: Momentum space feynman diagrams.

Please note that in a real scalar spinless theory

$$\Delta(p) = \Delta(-p). \tag{2.43}$$

The terms proportional to a power of the spacetime volume V_x are precisely the non-connected diagrams. The power affecting the volume is the number of vacuum bubbles in the diagram.

Let us now turn to abelian $U(1)$ gauge theories (quantum electrodynamics, QED). At at time it was considered as a model for all QFT; nowadays is considered as a rather special case. The lagrangian reads

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{2.44}$$

After integrating by parts

$$S = \int d^4x - \frac{1}{2} A^\mu \left(-\Box \eta_{\mu\nu} + \partial_\mu \partial_\nu\right) A^\nu \tag{2.45}$$

This means that

$$M_{\mu\nu}(x,y) = \left(-\Box \eta_{\mu\nu} + \partial_\mu \partial_\nu\right) \delta^4(x-y) \tag{2.46}$$

The operator M is singular, owing precisely to gauge invariance

$$\int d^4y M_{\mu\nu}(x,y) \partial^\nu \epsilon(y) = 0 \tag{2.47}$$

One solution to this problem is to chose a gauge condition, such as

$$\partial_\alpha A^\alpha = 0 \tag{2.48}$$

In order to do that, we add to the lagrangian an extra term

$$L_{gf} = \frac{1}{2\alpha} \left(\partial_\alpha A^\alpha\right)^2 \tag{2.49}$$

It is clear that when $\alpha \to \infty$ we implement the gauge condition formally

The new M operator reads

$$M_{\mu\nu}(x,y) = \left(-\Box \eta_{\mu\nu} + \left(1 - \frac{1}{\alpha}\right) \partial_\mu \partial_\nu\right) \delta^4(x-y) \tag{2.50}$$

There are good resons to think that the functional integral will be independent of α (this will be proved when studying the BRST approach to gauge fixing). Feynman actually favors

$$\alpha = 1 \tag{2.51}$$

which will be called somewaht symbolically *Feynman gauge.*

The photon propagator reads

$$\Delta_{\mu\nu} = \frac{\eta_{\mu\nu}}{p^2 + i\epsilon} \tag{2.52}$$

that is

$$\langle 0|TA_\mu(x)A_\nu(y)|0\rangle \tag{2.53}$$

The fermion-gauge coupling reads

$$L = \bar{\psi}(\gamma^\mu(i\partial_\mu - qA_\mu) - m)\psi \tag{2.54}$$

The fermion propagator

$$S(x-y) \equiv \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \frac{1}{\not{p} - m + i\epsilon} \tag{2.55}$$

in momentum space

$$S_{ab} = \left(\frac{1}{\gamma^\mu p_\mu - m + i\epsilon} \right)_{ab} \tag{2.56}$$

It is a fact

$$iS_{ab}(x-y) = \langle 0|T\psi_a(x)\bar{\psi}_b(y)|0\rangle \tag{2.57}$$

$S(x-y)$ represents the amplitude of propagation from the point y towards the point x,the, because of the conventional Fock's structure of Dirac's field ($\psi \sim (b, d^+)$) what propagates is electron towards the future or a positron toward the past (that is, negative charge towards the future). The arrow goes from $\bar{\psi}$ towards ψ and has nothing to do with four-momentum propagation. We represent the propagator with a continuous line and an arrow that goes from the point y to the point x. To change the arrow is the same as to change p to $-p$.

The photon-fermion-fermion vertex (the only one in QED)

$$iq\gamma^\mu_{ab} \tag{2.58}$$

The Yukawa coupling between scalar-fermion-fermion

$$g_y\bar{\psi}\phi\psi \tag{2.59}$$

with gets a vertex

$$ig_y \tag{2.60}$$

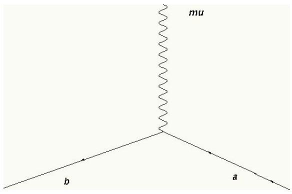

Figure 2.3: QED vertex

2.3 One-Particle-Irreducible Green functions

The basic building blocks out of which all Green functions can be reconstructed are not the connected functions, but rather, the one-particle irreducible (1PI) Green functions. Those are by definition the ones that can not be made disconnected by cutting one internal line. Their generating functional can be easily obtained from the free energy by a Legendre transform similar to the one used to define the hamiltonian out of the lagrangian Define

$$\phi_c \equiv \frac{\delta W}{\delta J} \tag{2.61}$$

Invert this to get the function $J(\phi_c)$. Then define the *effective action*

$$\Gamma[\phi_c] \equiv W[J] - \int d^n z J \phi_c \tag{2.62}$$

It is then plain that

$$\frac{\delta \Gamma}{\delta \phi_c(x)} = \int d^n y \frac{\delta W}{\delta J_y} \frac{\delta J_y}{\delta \phi_c(x)} - J(x) - \int d^n z \phi_c(z) \frac{\delta J}{\delta \phi_c(x)} = -J(x) \tag{2.63}$$

Connected functions in presence of sources are given by

$$G_n(x_1 \ldots x_n) \equiv \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} W[J] \tag{2.64}$$

It is plain that

$$G_n(x_1 \ldots x_n)|_{J=0} = G_n^c(x_1 \ldots x_n) \tag{2.65}$$

Define in an analogous way

$$\Gamma_n(x_1 \ldots x_n) \equiv \frac{\delta}{\delta \phi_1^c} \cdots \frac{\delta}{\delta \phi^c(x_n)} \Gamma(\phi_c) \tag{2.66}$$

It is claimed that 1PI functions are given by functional derivatives of the effective action.

$$\Gamma_n(x_1 \ldots x_n) = \Gamma_n(x_1 \ldots x_n)|_{\phi^c=0} \tag{2.67}$$

Remembering that

$$G_2(x, y) \equiv \frac{\delta^2 W[J]}{\delta J(x) \delta J(y)} \tag{2.68}$$

it is fact that

$$\frac{\delta J_x}{\delta J_y} = \delta_{xy} = \int_z \frac{\delta J_x}{\delta \phi^c_z} \frac{\delta \phi^c_z}{\delta J_y} = \int_z \frac{\delta J_x}{\delta \phi^c_z} \frac{\delta^2 W}{\delta J_z \delta J_y} = -\int_z \frac{\delta^2 \Gamma}{\delta \phi^c_x \phi^c_z} \frac{\delta^2 W}{\delta J_z \delta J_y} \tag{2.69}$$

That is

$$\Gamma_2 = -G_2^{-1} \tag{2.70}$$

To lowest order, Γ_2 is just the quadratic piece of the classical action. This procedure can be extended easily. For example the full connected three-point function can be reconstructed out of the 1PI three-point function.

$$G_3(x_1, x_2, x_3) = -\int d^n y_1 \; \frac{\delta \phi_c(y_1)}{\partial J(x_3)} \; \frac{\delta}{\delta \phi^c(y_1)} \Gamma_2^{-1}(x_1, x_2) =$$
$$= \int d^n y_1 \; d^n y_2 \; d^n y_3 \; G_2(y_1, x_3) \; \Gamma_2^{-1}(x_1, y_2) \; \frac{\delta \Gamma_2(y_2, y_3)}{\delta \phi^c(y_1)} \; \Gamma_2^{-1}(x_2, y_2) =$$
$$= \int d^n y_1 \; d^n y_2 \; d^n y_3 \; G_2(y_1, x_3) \; G_2(x_1, y_1) \; \Gamma_3(y_2, y_3, y_1) \; G_2(x_2, y_2)$$

3

Gauge theories.

Gauge invariance is a formal invariance of the action with respect to transformations that depend of continuous parameters which are functions of the space-time point, say $g(x) \in G$. The group G can be abelian like $U(1)$, corresponding to electromagnetism, or non abelian, like $SU(n)$. Gauge transformations that approach the identity at infinity, G_0, are noy really symmetries, and the physical configuration space is A/G_0. These transformation are really redundancies of our description in terms of the variables A.

There are also in the non-abelian case *large gauge transformations* that do not tend to the identity at infinity, and relate different points in A/G_0 which are physically equivalent. Those are true symmetries.

3.0.1 Renormalization.

When computing all but the simplest diagrams, we find that they give rise to divergent integrals. For example the simplest scalar vacuum bubble reads

$$B \equiv \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} = \infty \tag{3.1}$$

Before even discussing what is the correct procedure to make sense out of this infinity we just encountered, we have to *regularize*, that is, modify our theory to a different one that yields finite answers and that reduces to the physical theory in some limit depending on continuous parameters. When this is done, we will have to study what happens when the continuous parameters go to the physical limit. This is what is called *renormalization.*

The simplest way to regularize is *momentum cutoff*, in which a factor of

$$\theta(\Lambda^2 - p^2) \tag{3.2}$$

is inserted into the momentum integrals. Our vacuum bubble regularized in such a way reads

$$B = C\Lambda^2 \tag{3.3}$$

The physical limit is then

$$\lim_{\Lambda\to\infty} \tag{3.4}$$

It can then be said that the bubble B is is quadratically divergent in the physical limit.

This procedure breaks Lorentz invariance as well as (soon to be discussed in detail) gauge symmetry. It is then preferable to work from the very beginning in a framework that respects as many symmetries as possible.

The *superficial degree of divergence* of a diagram is the difference between the number of momenta in the numerator of the integral and the number of momenta in the denominator. In spite of it being a very simple concept, its importance cannot be overestimated, because there is a famous theorem by Weinberg asserting that if a graph and all its subgraphs habe negative superficial degree of divergence„ then the full diagram is finite.

There is a quite simple way of computing it. The superficial degree of divergence in dimension n, just to get the general trend, is just

$$D \equiv nL - 2I_b - I_f \tag{3.5}$$

The fact that we can write the diagram in a paper means that the Euler characteristic must be one, so that

$$\chi = 1 = L - I + V \tag{3.6}$$

This means that

$$D = \sum n_i d_i + n(I - V + 1) - 2I_b - I_f = \sum n_i d_i + (n-2)I_b + (n-1)I_f - n(\sum n_i - 1) \tag{3.7}$$

The *law of conservation of boson ends* states that

$$E_b + 2I_b = \sum n_i b_i \tag{3.8}$$

where b_i is the number of boson lines stemming from a vertex labelled i, of which there are n_i in the given diagram. The *law of conservation of fermion ends* states that

$$E_f + 2I_f = \sum n_i f_i \tag{3.9}$$

Then the superficial degree of divergence reads

$$\begin{aligned} D &= \sum n_i d_i + (n-2)I_b + (n-1)I_f - n\sum n_i + n = \\ &= \sum n_i d_i + \frac{n-2}{2}\left(\sum n_i b_i - E_b\right) + \frac{n-1}{2}\left(\sum n_i f_i - E_f\right) - n\sum n_i + n = \\ &= -\frac{n-2}{2}E_b - \frac{n-1}{2}E_f + n + \sum n_i \delta_i \end{aligned} \tag{3.10}$$

where d_i is the number of derivatives attached to the vertex labelled i. The *index of divergence*

$$\delta_i \equiv \frac{n-2}{2}b_i + \frac{n-1}{2}f_i + d_i - n \tag{3.11}$$

In theories in which $\delta_i > 0$ the degree of divergence gets bigger and bigger as we add more vertices. Those theories are certainly non-renormalizable.

Assume there is no derivative vertex, Then $d_i = 0$. The condition for the index of divergence to be negative is

$$\frac{n-2}{2}b_i + \frac{n-1}{2}f_i \leq n \tag{3.12}$$

This means that in n=4 dimensions,

$$b_i + \frac{3}{2}f_i \leq 4 \tag{3.13}$$

vertices cannot have more than four bosonic legs, or two fermionic ones and one bosonic.

In n=2 dimensions,

$$\frac{1}{2}f_i \leq 2 \tag{3.14}$$

vertices can have an arbitrary number of bosonic legs, but only up to four fermionic ones.

Curiously enough, in n=6 dimensions the condition reads

$$2b_i + \frac{5}{2}f_i \leq 6 \tag{3.15}$$

so that the theory ϕ^3_6 looks renormalizable by power counting.

Renormalization is only the first step in order to be able to extract precise answers from QFT to compare eventually with experimental results. Even when we are guaranteed that all computations will yield non-divergent answers, we still have to compute them. This work is usually even harder that the one necessary to elimitate all infinities.

The only reason why a sample of these calculations is not included here is one of lack of space-time.

3.0.2 Dimensional regularization.

Analytic continuation in the spacetime dimension ('t Hooft and Veltman) regulates simultaneously IR and UV which is sometimes a nuisance, although usually convenient. The continuous parameter here is precisely the spacetime dimension, n and the physical divergences appear when the limit

$$\lim_{n\to 4} \tag{3.16}$$

is taken.

Wilson introduced an axiomatic formulation of this regularization. He was able to prove that the analytic function of the complex variable n

$$I(n; f) \equiv \int d^n p f(p) \tag{3.17}$$

is uniquely defined by

- Lineality

$$I(af + bg) = aI(f) + bI(g) \tag{3.18}$$

- Scaling.
  Define $(D_\lambda f)(p) \equiv f(\lambda p)$.

$$I(D_\lambda f) = \lambda^{-n} I(f) \tag{3.19}$$

- Translational invariance.
  Define $(T_q f)(p) \equiv f(p+q)$

$$I(T_q f) = I(f) \tag{3.20}$$

Recall the definition of Euler's beta function

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} = \int_0^\infty dx \frac{x^{z-1}}{(1+x)^{z+w}} \tag{3.21}$$

where Euler's Gamma function is given by

$$\Gamma(z) \equiv \int_0^\infty e^{-t}\, t^{z-1} \tag{3.22}$$

The euclidean integral

$$\int d^n p \frac{p^{2a}}{(p^2+m^2)^b} = \pi^{n/2} m^{n+2a-2b} \frac{\Gamma(a+n/2)\Gamma(b-a-n/2)}{\Gamma(n/2)\Gamma(b)} \tag{3.23}$$

implies in particular that

$$\int d^n p\; p^a = 0 \tag{3.24}$$

There is a quite useful (*proper time*) parameterization due to Schwinger

$$\frac{1}{(k^2+m^2)^a} = \frac{1}{\Gamma(a)} \int_0^\infty d\tau \tau^{a-1} e^{-\tau(k^2+m^2)} \tag{3.25}$$

Then several template integrals are easily done. First of all

$$\int d^n p \frac{1}{(p^2+2p.k+C)^a} = \pi^{n/2}(-k^2+C)^{n/2-a} \frac{\Gamma(a-n/2)}{\Gamma(a)} \tag{3.26}$$

thens

$$\int d^n p \frac{p_\mu}{(p^2+2p.k+C)^a} = -k_\mu \pi^{n/2}(-k^2+C)^{n/2-a} \frac{\Gamma(a-n/2)}{\Gamma(a)} \tag{3.27}$$

and finally

$$\int d^n p \frac{p_\mu p_\nu}{(p^2+2p.k+C)^a} = \frac{\pi^{n/2}}{\Gamma(a)}(-k^2+C)^{n/2-a} \times$$

$$\times \left(\Gamma(a-n/2) k_\mu k_\nu + \Gamma(a-1-n/2) \frac{-k^2+C}{2} \delta_{\mu\nu}\right) \tag{3.28}$$

In order to take the physical limit, we need to evaluate these analytic functions of the complex variable n in a neighborhood of the physical value $n = 4 + \epsilon$

$$\Gamma\left(\frac{4-n}{2}\right) = \Gamma\left(-\epsilon/2\right) = -\frac{1}{4-n} + O(1) = \frac{1}{n-4} + O(1) \tag{3.29}$$

This result stems from the well-known fact that

$$\Gamma\left(1+z\right) = z\Gamma\left(z\right) \tag{3.30}$$

When fermions are considered, we need to define also n-dimensionsl Dirac matrices. They obey

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu} = -2\delta_{\mu\nu} \tag{3.31}$$

Also

$$\mathrm{tr}\, 1 = 4 \tag{3.32}$$

which can be shown to be consistent, and implies

$$\begin{aligned}
&\gamma_\mu\gamma^\mu = n\\
&tr\,\gamma_\mu\gamma_\nu = 4g_{\mu\nu}\\
&\gamma_\mu \not{p} \gamma^\mu = (2-n)\not{p}\\
&\gamma_\mu \not{p}\not{q} \gamma^\mu = (n-4)\not{p}\not{q} + 4p.q\\
&\gamma_\mu \not{p}\not{q}\not{k} \gamma^\mu = -(n-4)\not{p}\not{q}\not{k} - 2\not{k}\not{q}\not{p}\not{p}
\end{aligned} \tag{3.33}$$

Canonical dimensions are now space-time dimension dependent

$$\begin{aligned}
&[A^0_\mu] = \frac{n-2}{2}\\
&[\psi^0] = \frac{n-1}{2}\\
&[e_0] = \frac{4-n}{2}
\end{aligned} \tag{3.34}$$

This fact will be relevant in a moment. This means that we have to distinguish (as we have done from the very beginning) between fields defined in n dimensions (which we shall dub *bare*) and decorate with a sub- or superindex 0, and physical fields with have the correct four-dimensional canonical dimension

$$\begin{aligned}
&A^0_\mu = \mu^{\frac{n-4}{2}} A_\mu\\
&\psi^0 = \mu^{\frac{n-4}{2}} \psi\\
&e_0 = \mu^{\frac{4-n}{2}}\, e
\end{aligned} \tag{3.35}$$

When we compute the divergences associated to different diagrams, we need to substract them somewhat. This is renormalization *sensu stricto*.

One way of doing that is just by substracting the divergent part, defined just as the residue at the pole. This is called *minimal substraction* (MS) scheme. It is possible to substract also some finite pieces (factors of 2π and the like) that simplify the algebra somewhat, as in the $\overline{MS}$ scheme, in which one substractas also a factor

$$-\gamma + \log(4\pi) \tag{3.36}$$

For example, given the result

$$\frac{1}{16\pi^2}\left(\frac{2}{\epsilon} - \gamma + \log\ 4\pi - \log\frac{A^2}{\mu^2}\right) \tag{3.37}$$

in MS we keep

$$\frac{1}{16\pi^2}\left(-\gamma + \log\ 4\pi - \log\frac{A^2}{\mu^2}\right) \tag{3.38}$$

whereas in $\bar{M}S$we would keep only

$$\frac{1}{16\pi^2}\log\frac{\mu^2}{A^2} \tag{3.39}$$

This is useful mainly to one loop order.

This has to be done order by order in perturbation theory. That is, we assume that it is consistent to neglect higher order (two loo and beyond) infinities when working at one loop, in spite of them being divergent, because they got a higher power of $\hbar$ in front. Actually the two-loop renormalization depends upon the one-loop results, that appear in subdivergences. The general structure of the renormalization constants in a theory with a coupling constant λ is of the type

$$\lambda^B = \mu^{4-n}\left(\lambda_R + a^1(\lambda_R)\frac{1}{4-n} + a^2(\lambda_R)\frac{1}{(4-n)^2} + \ldots\right) \tag{3.40}$$

with

$$a^1(\lambda_R) \equiv \sum_{n=2}^{\infty} a^1_n \lambda_R^n \tag{3.41}$$

and the coefficients of the expansion are finite for all values of the renormalized coupling constant λ_R.

All that renormalization physically means is that by changing (an infinite amount) what is meant by masses and other coupling constants, as well as by an (infinite) readjustment of the kinetic energy term, it is possible to extract finite predictions out of QFT.

Several comments are in order. First of all, this procedure, arbitrary as it seems at first sight, does not work for every QFT, but only for a small subset of the ones that are classically well defined; those are called *renormalizable QFT*. On second thought, this procedure is not as unnatural as it seems.

We cannot measure the bare couplings; only the renormalized ones appear in the physical expressions or S-matrix elements, and those are perfectly finite. Of course we need to check that physics is independent of all the arbitrariness we have introduced, such as different methods of regularization, different renormalization schemes, etc.

This can be argued to be the case at least for non-abelian gauge theories.

Gauge theories are theories in which there have been introduced redundant variables in order to make other symmetries (typically Lorentz invariance) manifest. This redundancy consists in invariances that depend on local (i.e. functions of the space-time point) parameters. This extra invariance does not imply extra Noether charges in addition to the ones stemming from the rigid (i.e., space-time independent) invariance. There are however some identities (Ward's) that are essential to the consistency of the quantum version of gauge theories.

All presently favored theories of the fundamental interactions are gauge theories. Even General Relativity is a gauge theory, although with some peculiarities.

3.1 Abelian $U(1)$ gauge theories

The simplest gauge theories are abelian ones, where the gauge group is just $U(1)$. The theory of the interactions of fermions with photons (quantum electyrodynamics or QED for short) is the most important example. Pure classical electromagnetism can be written in terms of gauge invariant quantities (namely electric and magnetic fields). Lorentz invariance is however obscure in terms of these variables . This is actually the reason why it took a certain time to discover the full invariance group of Maxwell's equations. The coupling of electromagnetism to fermions is even subtler and needs the vector potential.

Let us the stick to the Lorentz covariant notation. The Lagrangian written in terms of bare fields (we shall need to change the fields as time goes by) reads

$$L = -\frac{1}{4} F_{\mu\nu}(A_0)^2 - \frac{1}{2\alpha_0}(\partial_\mu A_0^\mu)^2 + \bar{\psi}_0 \left(i\not{D} + e_0 \not{A}_0 - m_0 \right) \psi_0 \tag{3.42}$$

Recall the Feynman rules for this theory

QED FEYNMAN RULES

1. Identify distinguishable connected diagrams.

2. Write down a vertex factor

$$-ig(2\pi)^4\delta^4(\sum k_j I_{ja})$$

$$i\lambda(2\pi)^4\delta^4(\sum k_j I_{ja})$$
s
$$-iq(\gamma^\mu)_{dc}(2\pi)^4\delta^4(\sum k_j I_{ja})$$

where c stands for incoming arrow and d outgoing arrow; μ corresponds to the photon
3. Associate $(2\pi)^4\delta^4(p_E - \sum k_j I_{jE})$ a los vértices externos.

4.Associate $\int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2}$ to every bosonic internal line.

5. Write the fermionic propagator $\int \frac{d^4p}{(2\pi)^4}\left(\frac{i}{\not p-m}\right)_{ab}$ to every internal fermion line

(arrow goes from a to b, and momentum in the direction of the arrow).
Spinor matrices are always multiplied together in the direction opposite to the charge arro
6. Write down $\int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2}\left(-\eta_{\mu\nu} + (1-\frac{1}{\alpha})\frac{p_\mu p_\nu}{p^2+i\epsilon}\right)$
for the photon propagator connecting μ and ν.

7. Write down the corresponding symmetry factor.

8. Write a minus sign for diagrams the differ in the swappig of external fermionic lines

9. Write down snother minus sign for any fermionic loop.

The renormalization procedure consist in appropiate rescalings of the bare fields and coupling constants in order to define the physical fields and coupling constants. Let us define

$$\begin{aligned} A_0^\mu &= Z_3^{1/2}A^\mu \\ \psi_0 &= Z_2^{1/2}\psi \\ e_0 &= Z_1 Z_2^{-1} Z_3^{-1/2}\mu^{\frac{4-n}{2}}e \\ \alpha_0 &= Z_3\alpha \\ m_0 &= Z_m m \end{aligned} \tag{3.43}$$

This is because the engeneering dimension of the bare fields and bare coupling constants is

$$
\begin{aligned}
&[A_0] = \frac{n-2}{2} = 1 + \frac{n-4}{2} \\
&[\psi_0] = \frac{n-1}{2} = \frac{3}{2} + \frac{n-4}{2} \\
&[e_0] = \frac{4-n}{2} \\
&[m_0] = [m] = 1
\end{aligned}
\tag{3.44}
$$

The renormalization constants turn out to diverge in the physical limit. The effect of all rescalings in the lagrangian can also be expressed in terms of the physical lagrangian with the addition of the so called *counterterms*

$$
L = -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2\alpha}(\partial_\mu A^\mu)^2 + \bar{\psi}\left(i\not{D} + \mu^{\frac{4-n}{2}}\ e\ \not{A} - m\right)\psi -
$$
$$
(Z_3-1)\frac{1}{4}F_{\mu\nu}^2 + (Z_2-1)\bar{\psi}i\not{D}\psi - (Z_1-1)\bar{\psi}e\mu^{\frac{4-n}{2}}\ \not{A}\psi - (Z_1 Z_m - 1)\ m\bar{\psi}\psi
$$

QED COUNTERTERMS FEYNMAN RULES

(Amputated diagrams; that is external lines stripped off.)

1. Fermion line

$$i\left((Z_2-1)\not{p} - ((Z_m-1)+(Z_2-1))\,m_R\right)$$

2. Photon line

$$-i(Z_3-1)\left(q^2\eta_{\mu\nu} - q_\mu q_\nu\right)$$

3.Vertex counterterm

$$-ie_R(Z_1-1)\gamma_\mu$$

4. Feynman Gauge fixing

$$-i(Z_3-1)q^2\eta^{\mu\nu}$$

It will be seen in due time that gauge invariance (through Ward's identities) implies that $Z_1 = Z_2$, that is, the charge renormalization is the same as the fermion wavefunction renormalization.

3.1.1 Electron self-energy.

Let us begin by considering the simplest one-particle irreducible (1PI) self-energy diagram. Let us amputate the external legs, which are not essential for our present discussion. Consider first the simpler $U(1)$ case. Let us also call the charge of the particle e.

$$\Sigma(p) = i\int \frac{d^n k}{(2\pi)^n}\left(-ie\mu^{\frac{4-n}{2}}\gamma_\mu\right)\frac{i(\not p - \not k + m)}{(p-k)^2 - m^2 + i\epsilon}$$
$$\left(-ie\mu^{\frac{4-n}{2}}\gamma_\nu\right)\frac{-i\eta^{\mu\nu}}{k^2 + i\epsilon} \tag{3.45}$$

Now it is a good moment to introduce Feynman parameters

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{(xA + (1-x)B)^2} = -\frac{1}{A-B}\,\frac{1}{x(A-B)+B}\Big|_0^1 = -\frac{1}{A-B}\left(\frac{1}{A} - \frac{1}{B}\right) \tag{3.46}$$

A useful alternative was provided by Schwinger, who championed the *proper time representation*

$$\frac{1}{(m^2 - p^2)^a} = \frac{1}{\Gamma(a)}\int d\tau\ \tau^{a-1}\ e^{-\tau\left(m^2 - k^2\right)} \tag{3.47}$$

The sort of integrals needed when using this representation are

$$\int d^n k\ e^{Ak^2 + 2bk} = i\left(\frac{\pi}{A}\right)^{\frac{n}{2}}\ e^{-\frac{b^2}{A}}$$
$$\int d^n k\ k^\mu\ e^{Ak^2 + 2bk} = i\left(\frac{\pi}{A}\right)^{\frac{n}{2}}\ \left(-\frac{b^\mu}{A}\right) e^{-\frac{b^2}{A}}$$
$$\int d^n k\ k^\mu k^\nu\ e^{Ak^2 + 2bk} = i\left(\frac{\pi}{A}\right)^{\frac{n}{2}}\ \left(-\frac{b^\mu b^\nu}{A^2} - \frac{1}{2A}\eta^{\mu\nu}\right) e^{-\frac{b^2}{A}}$$
$$\int d^n k\ k^\lambda k^\mu k^\nu\ e^{Ak^2 + 2bk} = i\left(\frac{\pi}{A}\right)^{\frac{n}{2}}\ \left(-\frac{b^\lambda b^\mu b^\nu}{A^3} + \frac{1}{2A^2}\left(b^\lambda \eta^{\mu\nu} + b^\mu \eta^{\lambda\nu} + b^\nu \eta^{\lambda\mu}\right)\right) e^{-\frac{b^2}{A}}$$

and so on and so forth.

Then, using

$$\gamma_\mu \not p \gamma^\mu = p_\rho\left(2\eta^{\rho\mu}\gamma_\mu - \gamma^\rho\gamma^\mu\gamma_\mu\right) = (2-n)\not p \tag{3.48}$$

we get

$$\Sigma(p) = -ie^2\mu^\epsilon \int_0^1 dx \int \frac{d^n k}{(2\pi)^n}\gamma^\mu \frac{(2-n)(\not p - \not k) + nm}{((1-x)k^2 + x((p-k)^2 - m^2) + i\epsilon)^2}\gamma_\mu =$$
$$= -ie^2\mu^\epsilon \int_0^1 dx \int \frac{d^n k}{(2\pi)^n}\frac{(2-n)(\not p - \not k) + nm}{(k^2 - 2xk.p + x(p^2 - m^2) + i\epsilon)^2} \tag{3.49}$$

The integral over momenta can be easily made

$$k \to xp$$
$$C \to x(p^2 - m^2) \tag{3.50}$$

It yields

$$\Sigma(p) = \frac{\alpha}{2\pi}\left(\frac{4\pi\mu^2}{p^2}\right)^{\epsilon/2}\Gamma(\epsilon/2)\int dx\,\Big((2-n)(1-x)\not{p}+nm\Big)\times$$
$$\times\Big(-x(1-x)+xm^2/p^2-i\epsilon\Big)^{-\epsilon/2} \quad (3.51)$$

Let us now take the physical limit $\epsilon\to 0$, and separate the pole

$$\Sigma(p) = -\frac{\alpha}{4\pi}\left(\left(\frac{2}{4-n}+log\,4\pi\mu^2/p^2-\gamma_E\right)\frac{\not{p}-4m}{2}-\frac{\not{p}-2m}{2}+\right.$$
$$\left.\int dx\,\Big(\not{p}(1-x)-2m\Big)\,log\,\Big(-x(1-x)+xm^2/p^2-i\epsilon\Big)\right) \quad (3.52)$$

The divergent part of the self-energy can be cancelled with a MS counterterm

$$Z_2-1=-\frac{\alpha}{4\pi}\frac{2}{4-n} \quad (3.53)$$

Mass renormalization is multiplicative. This means that the physical mass vanishes when the bare mass also vanishes.

$$Z_m-1=-\frac{3\alpha}{4\pi}\frac{2}{4-n} \quad (3.54)$$

In $SU(N)$ gauge theory the only difference is that there are extra factors T^a and T^b at each of the vertices. The propagator of the gauge bosons gets an extra δ_{ab}. The net result is then

$$\delta_{ab}\,T^a\,T^b\equiv C_2(F) \quad (3.55)$$

namely, an extra factor proportional to the second Casimir of the fermionic representatrion of $SU(N)$, For the fundamental representation, where for example in the $SU(3)$ case the generators are given ion terms of the eight Gell-Mann matrices as

$$T^a=\frac{1}{2}\lambda^a \quad (3.56)$$

$$C_2(F)=\frac{N^2-1}{N} \quad (3.57)$$

3.1.2 Vacuum polarization.

This is the traditional name for the photon self energy. The corresponding 1PI diagram is given by

$$
\begin{aligned}
&-i\Pi_{\alpha\beta}(q) = i\int \frac{d^n k}{(2\pi)^n} tr\left((-ie\gamma_\alpha)(\not k + m)(-ie\gamma_\beta)\left(-\not q + \not k + m\right)\right)\\
&\frac{i}{k^2 - m^2 + i\epsilon}\frac{i}{(q-k)^2 - m^2 + i\epsilon} =\\
&= ie^2\mu^{\epsilon/2}\int_0^1 dx \int \frac{d^n k}{(2\pi)^n} N_{\alpha\beta}(p.k)\frac{1}{\left((1-x)k^2 + x\left((q-k)^2 - m^2\right) + i\epsilon\right)^2} =\\
&= ie^2\mu^{\epsilon/2}\int \frac{d^n k}{(2\pi)^n} N_{\alpha\beta}(p.k)\frac{1}{\left(k^2 - 2xk.q + xq^2 - m^2 + i\epsilon\right)^2}
\end{aligned} \tag{3.58}
$$

The first thing is to perform the Dirac traces. This lead to

$$
N_{\alpha\beta} = 8k_\alpha k_\beta - 4\left(k_\alpha q_\beta + q_\beta k_\alpha\right) + 4g_{\alpha\beta}\left(k.q - k^2 + m^2\right) \tag{3.59}
$$

The divergent part of the full expression reads

$$
\begin{aligned}
-i\Pi_{\alpha\beta} &= -\left(q^2 g_{\alpha\beta} - q_\alpha q_\beta\right)\frac{2\alpha}{\pi}\left(4\pi\mu^2\right)^{\epsilon/2}\Gamma(\epsilon/2)\times\\
&\times\int dx \frac{x(1-x)}{\left(m^2 - x(1-x)q^2 - i\epsilon\right)^{\epsilon/2}} =\\
&-\left(q^2 g_{\alpha\beta} - q_\alpha q_\beta\right)\frac{\alpha}{3\pi}\left(\frac{2}{4-n} + \text{finite}\right)
\end{aligned} \tag{3.60}
$$

It is worth remarking that this counterterm is transverse, that is

$$
q^\lambda \Pi_{\lambda\alpha} = 0 \tag{3.61}
$$

The MS photon wavefunction renormalization is

$$
Z_3 - 1 = -\frac{\alpha}{3\pi}\frac{2}{4-n} \tag{3.62}
$$

In the non-abelian case, the modification is just the trace

$$
\text{tr} T^a T^b = I_R \delta^{ab} \tag{3.63}
$$

(For the fundamental,

$$
I_F = \frac{1}{2}) \tag{3.64}
$$

To sum up

$$
\Pi_{\alpha\beta} = i\left(q_\alpha q_\beta - q^2\eta_{\alpha\beta}\right)\ \pi(q^2) \tag{3.65}
$$

where

$$\pi(q^2) \equiv \frac{2\alpha}{\pi}\ \delta_{ab} \left\{ \log\ (4\pi\mu^2) - \int_0^1 dx\ x(1-x)\ \log\ \left[m^2 - x(1-x)q^2\right] \right\}$$

- The geometric series of the polarization contributions to the photon propagator can be summed yielding the result $G_{\mu\nu}$

$$G_{\mu\nu} = D_{\mu\nu} + D_{\mu\alpha}\Pi^{\alpha\beta}D_{\beta\nu} + D_{\mu\alpha}\Pi^{\alpha\beta}D_{\beta\gamma}\Pi^{\gamma\delta}D_{\delta\nu} + \ldots = D_{\mu\alpha}\left(\delta_{\alpha\beta} + \Pi_{\alpha\beta}G_{\beta\nu}\right) \tag{3.66}$$

It is fact that (using matrix notation)

$$D\Pi G = G - D \tag{3.67}$$

so that

$$(1 - D\Pi)\, G = D \quad \therefore \quad G^{-1}\,(1 - D\Pi)^{-1} = D^{-1} \tag{3.68}$$

that is

$$G^{-1} = D^{-1}\,(1 - D\Pi) = D^{-1} - \Pi \tag{3.69}$$

Putting back indices

$$G^{-1}_{\mu\nu} = i\left(k^2\eta_{\mu\nu} + \left(k^2\eta_{\mu\nu} - k_\mu k_\nu\right)\pi(k^2)\right) \tag{3.70}$$

that is

$$iG_{\mu\nu} = \frac{1}{k^4\,(1 + \pi(k^2))}\left(k^2 g_{\mu\nu} + \pi(k^2)k_\mu k_\nu\right) \tag{3.71}$$

The same procedure applied to the fermion self-energy leads to

$$\Gamma = \not{p} - m - \not{\Sigma} \tag{3.72}$$

It is worth remarking that owing to the regular behavior of the function $\pi(k^2)$ near $k^2 = 0$ the only solution of the equation

$$k^2\left(1 + \pi(k^2)\right) = 0 \tag{3.73}$$

is still

$$k^2 = 0 \tag{3.74}$$

that is, the photon mass does not renormalize. In one of the problem sheets you are invited to notice that this is not true anymore in n=2 dimensions where $k^2\pi(k^2)$ goes to a finite limit when $k = 0$ and the would-be photon develops a finite mass.

The logarithm gets an imaginary part for timelike q^2 when

$$m^2 - x(1-x)q^2 \leq 0 \tag{3.75}$$

Given that

$$x(1-x) \leq \frac{1}{4} \tag{3.76}$$

a branch cut appears when

$$q^2 = 4m^2 \tag{3.77}$$

The values of x for which this is possible are determined by the equation

$$x(1-x) \geq \frac{m^2}{q^2} \tag{3.78}$$

which only happen for

$$x_- \leq x \leq x_+ \tag{3.79}$$

where

$$x_\pm = \frac{1 \pm \sqrt{1 - \frac{4m^2}{q^2}}}{2} \tag{3.80}$$

Actually,

$$\text{Im}\,\pi\left(q^2 - i\epsilon\right)\frac{2\alpha}{\pi}\int_{x_-}^{x_+} dx x(1-x) = \frac{\alpha}{3}\left(1 + \frac{2m^2}{q^2}\right)\sqrt{1 - \frac{4m^2}{q^2}} \tag{3.81}$$

which is proportional to the cross section for production of a fermion-antifermion pair.

- This is a consequence of the *optical theorem*, which is in turn a cosequence of unitarity.

  Let us write

$$S = 1 + iT \tag{3.82}$$

  Then

$$SS^+ = 1 = 1 + i(T - T^+) + TT^+ \tag{3.83}$$

$$TT^+ = -i\left(T - T^+\right) \tag{3.84}$$

$$\sum_I \langle f|T|I\rangle\langle I|T^+|i\rangle = -i\left(\langle f|T|i\rangle - \langle f|T^+|i\rangle\right) = 2\ (2\pi)^4\delta^4(p_i - p_f)\ \text{Im}\ \mathcal{M}_{fi} =$$

$$= \sum_I (2\pi)^4\ \delta^4\left(p_f - p_I\right)\ (2\pi)^4\ \delta^4\left(p_i - p_I\right)\ M_{iI}\ M^*_{fI} \tag{3.85}$$

  This implies lots of relationships (*id est* the Cutkosky rules), the simplest one being the optical theorem proper, that asserts that the imaginary part of the forward scattering amplitude equals the total cross section for production of any possible final state.

$$\text{Im}\ M\left(k_1k_2 \to k_1k_2\right) = 2\ E\ p\ \sigma_T\ \left(k_1k_2 \to \text{all}\right) \tag{3.86}$$

- At a certain level, that is, in the static limit

$$k_0 = 0 \tag{3.87}$$

the whole effect of the vacuum polarization is the replacement of the electron charge e^2 by

$$\frac{e^2}{1+\pi(-\vec{k}^2)} \tag{3.88}$$

This is what happens in dielectric materials. The effective charge is defined in terms of the a *dielectric constant* $\epsilon(\vec{k})$ through

$$\frac{e^2}{\epsilon(\vec{k})} \tag{3.89}$$

In perturbation theory

$$\frac{e^2}{1+\pi(-\vec{k}^2)} \sim \frac{e^2}{\vec{k}^2}\left(1+\frac{\alpha}{15\pi}\frac{\vec{k}^2}{m^2}\right) \tag{3.90}$$

In position space this corresponds to the so-called Uehling potential

$$V = \frac{e^2}{4\pi r} + \frac{\alpha e^2}{15\pi m^2}\delta^{(3)}(\vec{x}) \tag{3.91}$$

3.1.3 Renormalized vertex.

Ths 1PI Vertex function reads

$$\Gamma_\mu = -i\int \frac{d^n k}{(2\pi)^n}\frac{1}{(p+k)^2-m^2+i\epsilon}\frac{1}{(q+k)^2-m^2+i\epsilon}\frac{1}{k^2+i\epsilon}\left(-ig^{\beta\alpha}\right)$$
$$\left(-ie\mu^{\epsilon/2}\gamma_\beta\right) i\left(\not{q}+\not{k}+m\right)\left(-ie\mu^{\epsilon/2}\gamma_\mu\right) i\left(\not{p}+\not{k}+m\right)\left(-ie\mu^{\epsilon/2}\gamma_\alpha\right) \tag{3.92}$$

Using Feynman's identity

$$\frac{1}{ABC} = 2\int_0^1 dx\ dy\ dz\ \delta(x+y+z-1)\frac{1}{(x\ A+y\ B+z\ C)^3} \tag{3.93}$$

we learn that

$$\Gamma_\mu^\infty = -4ie^3\int_0^1 dx\int_0^{1-x} dy\frac{d^k k}{(2\pi)^n}$$
$$\frac{\not{k}\gamma_\mu\not{k}}{(k^2+2k.(xp+yq)+xp^2+yq^2-(x+y)m^2+i\epsilon)^3} =$$
$$= -e\gamma_\mu\frac{\alpha}{4\pi}\frac{2}{4-n} \tag{3.94}$$

The renormalization is then given by

$$Z_1 - 1 = -\frac{\alpha}{4\pi}\frac{2}{4-n} \tag{3.95}$$

Indeed

$$Z_1 = Z_2 \tag{3.96}$$

which as we shall see in a moment, is a consequence of the gauge smmetry.

What changes in the non-abelian case? There are three matrices in the fermion representation; but the gauge boson propagator forces two of them to be equal. The extra factor is then

$$\begin{aligned} T_a T_b T_c \delta_{ac} &= T_a \left(T_a T_b + i f_{bad} T_d\right) = C_2(A) T_b + i f_{bad}\ T_a T_d = \\ &= C_2(F) T_b + i f_{bad}\ \frac{1}{2}\ [T_a, T_d] = C_2(A) T_b + i f_{bad}\ \frac{1}{2}\ i f_{adc} T_c = \\ &= C_2(F) T_b - \frac{1}{2} C_2(A) \delta_{bc} T_c = \left(C_2(F) - \frac{1}{2} C_2(A)\right) T_b \end{aligned} \tag{3.97}$$

For the group $SU(N)$ with fermions in the fundamental this is

$$\left(\frac{N^2-1}{2N} - \frac{N}{2}\right) T_b = -\frac{1}{2N} T_b \tag{3.98}$$

3.1.4 Mass dependent and mass independent renormalization.

MS is a *mass-independent* renormalization scheme. Traditionally, QED used to be renormalized in a different and in a certain sense more physical, *mass-dependent* scheme. The finite arbitrariness was fixed by the following requirements

- The physical mass of the electron is equal to m.

$$\Sigma(\not{p} = m) = 0 \tag{3.99}$$

- The residue of the pole of the fermionic propagator is set equal to 1.

$$\frac{d}{d\not{p}} \Sigma(\not{p}) \Big|_{\not{p}=m} = 0 \tag{3.100}$$

- The residue at the pole of the photon propagator is also set to 1.

$$\Pi(q^2 = 0) = 0 \tag{3.101}$$

- The electron charge is set to the physical value,e

$$-ie\Gamma_\mu(q - p = 0) = -ie\gamma_\mu \tag{3.102}$$

In fact QED is almost the only theory to which such an scheme can be made to work, because it is the only piece of the standard model which is *infrared free*, that is, weakly coupled in the infrared. *Quantum chromodynamics (QCD)* is the prototype of a theory that is strongly coupled in the infrared, where our ability to compute is limited to lattice simulations. Mass independent schemes (first employed in QCD) are now much more widely used, even for QED.

3.2 Nonabelian gauge theories

It is often the case that matter fields are invariant under a set of rigid (global) transformations, acting in some representation

$$h \in G \rightarrow D_R(h) \tag{3.103}$$

namely

$$\phi \rightarrow \phi' \equiv D_R(h)\phi \tag{3.104}$$

QED Dirac's lagrangian for the electron

$$L = \bar{\psi}(i\not{D} - m)\psi \tag{3.105}$$

enjoys abelian invariance $h \in G = U(1)$

$$\psi \rightarrow D(h)\psi \equiv e^{i\alpha}\psi \tag{3.106}$$

The parameters must be constant

$$(\partial_\mu \phi)' = \partial_\mu D(h)\phi + D(h)\partial_\mu \phi \tag{3.107}$$

for example

$$(\partial_\mu \psi)' = e^{i\alpha}(i\partial_\mu \alpha + \partial_\mu)\psi \tag{3.108}$$

Any rigid symmetry can be promoted to a local one by introducing a *covariant derivative* (that is, a *connection* in the precise mathematical sense),

$$\nabla_\mu \equiv \partial_\mu + iA_\mu \tag{3.109}$$

where A_μ is a gauge field. Its transformation is determined by the requirement that

$$(\nabla_\mu \phi)' = D(h)\nabla_\mu \phi \tag{3.110}$$

which leads to

$$A'_\mu \equiv A^{(g)}_\mu = D_h A_\mu D_h^{-1} + \frac{i}{g}\partial_\mu D_h D_h^{-1} \tag{3.111}$$

In a given representation R, of the algebra, the generators are given by T^a, $a = 1 \ldots d_G$

$$A_\mu \equiv A^a_\mu T^a \tag{3.112}$$

Structure constants are purely imaginary for hermitian generators

$$[T_a, T_b] = c^c_{ab} T_c \equiv i f_{abc}\, T_c \tag{3.113}$$

When the semisimple group is compact, it is always possible to choose a basis such that the structure constants are totally antisymmetric, Elements of the group are obtained by exponentiation

$$h = e^{i\omega^a T_a} \tag{3.114}$$

with real parameters, $\omega^a \in \mathbb{R}$. $G = SU(N)$, $T_a = T_a^+$ (physicist's convention). When generators are anti-hermitian (mathematicians preferred convention) then $h = e^{\omega^a T_a}$.

Close to the identity

$$h \sim 1 + i\omega^a T_a \tag{3.115}$$

the gauge transformation reads

$$\delta A^a_\mu = g f^a_{bc} A^b_\mu \omega^c - \partial_\mu \omega^a \tag{3.116}$$

The field strengh (the curvature of the connection) is given by

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu] \tag{3.117}$$

After a gauge transformation

$$F'_{\mu\nu} = D_h F_{\mu\nu} D_h^{-1} \tag{3.118}$$

In a given basis

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu \tag{3.119}$$

The (hermitian) adjoint representation is defined by

$$(T^a)_{bc} \equiv i f_{bac} \tag{3.120}$$

and acts on the algebra itself, of dimension d_G,which for $SU(N)$ is

$$d_{SU(N)} = N^2 - 1 \tag{3.121}$$

This is a representation by virtue of Jacobi's identity.

The *Dynkin index* of the representation is defined by

$$\mathrm{tr}_R\ T_a T_b \equiv I_R\ \delta_{ab} \tag{3.122}$$

The *Second Casimir invariant* of the group reads

$$C_2(R) \equiv \delta^{ab} T_a T_b \tag{3.123}$$

Casimir invariants take a constant value in each representation. This value depends on the representation considered and can actually be used to label different representations.

Tracing we learn that both objects are related through

$$C_2(R)\ d_R = I_R\ d_G \tag{3.124}$$

When considering the adjoint representation of the group, its dimension coincides with the one of the group, $d_R = d_G$, so that

$$C_2(G) \equiv I_{ad} \tag{3.125}$$

In the particular case of $G = SU(2)$, with generators in the fundamental representation

$$T_a(F) \equiv \frac{\sigma_a}{2}, \tag{3.126}$$

then $I_F = 1/2$, $C_2(F) = 3/2$ y $C_2(G) = 2$.

3.2.1 Yang-Mills action

Staring at the transformation (3.118) it is plain that an invariant action for the Yang-Mills gauge fields would be

$$S_{YM} \equiv -\frac{1}{4 I_R g^2} tr \int d^4x \; F_{\mu\nu}(A) F^{\mu\nu}(A) \tag{3.127}$$

so that, independently of the fermion representation

$$S_{YM} = -\frac{1}{4g^2} tr \int d^4x \; \delta^{ab} F_{a\mu\nu}(A) F_b^{\mu\nu}(A) \tag{3.128}$$

3.2.2 Ghosts.

The functional integration over all gauge fields is a redundant one because W and its gauge transform W^g represent the same physical state $\forall g(x) \in G$.

One possibility is to choose a gauge. This is nothing else than a representative for each gauge equivalence class, where

$$A \sim A' \quad \Leftrightarrow \quad \exists g \in G, A^g = A' \tag{3.129}$$

A good gauge condition

$$F(A) = 0 \tag{3.130}$$

is supposed to intersect every orbit once and only once.

Faddeev and Popov, proved that the integral of a gauge invariant functional $f(A) = f(A^g)$ is given by

$$\int \mathcal{D}A \; f(A) = \int \mathcal{D}g \; \int \mathcal{D}A \; \det \left| \frac{\delta F}{\delta A_\mu} D^\mu \right| \; \delta(F) \; f(A) \tag{3.131}$$

Here

$$\mathcal{D}g \equiv \prod_x d\mu(g(x)) \tag{3.132}$$

Let us prove this formally. Define $M(A)$ from

$$M(A) \int \mathcal{D}g \; \delta(F(A^g)) = 1 \tag{3.133}$$

The integration measure over a compact Lie group is well known to be given by a function of the group parameters, $\omega \equiv (\omega_1 \ldots \omega_n)$,

$$d\mu(g) \equiv e(\omega) \; d\omega_1 \wedge d\omega_2 \wedge \ldots \wedge d\omega_n. \tag{3.134}$$

We need now to generalize the well-known formula that states that

$$\delta\left(f(x)\right) = \sum_i \frac{1}{|f'(x_i)|} \; \delta(x - x_i) \tag{3.135}$$

where the sum runs over all zeros of the function

$$f(x_i) = 0 \tag{3.136}$$

Working to linear order, the gauge transformation means that

$$\begin{aligned} F(A_\mu^g(x)) &= F(A_\mu(x)) + \int d^n y \frac{\delta F\left(A_\mu(x)\right)}{\delta A^\lambda(y)}\ \delta A^\lambda(y) = \\ &= F\left(A_\mu(x)\right) + \int d^n y \frac{\delta F\left(A_\mu(x)\right)}{\delta A^\lambda(y)}\ D^\lambda \omega(y) \end{aligned} \tag{3.137}$$

which means that

$$\delta(F\left(A_\mu(x)\right)) = \frac{1}{\det\left|\frac{\delta F(A_\mu(x))}{\delta A_\mu} D^\mu\right|} \delta(\omega - \bar{\omega}) \tag{3.138}$$

where the parameters $\bar{\omega}(x)$ are defined as

$$F\left(A^{g(\bar{\omega}(x))}\right) = 0 \tag{3.139}$$

therefore the Faddev-Popov determinant is a functional one defined by

$$M(A) = \det\left|\frac{\delta F\left(A_\mu(x)\right)}{\delta A_\lambda(y)} D^\lambda\right|_{\omega=\bar{\omega}} \tag{3.140}$$

It is easy to show that this object is gauge invariant

$$M(A^g) = M(A) \tag{3.141}$$

The reason is that the measure of integration over a compact group is right as well as left invariant

$$\mathcal{D}g = \mathcal{D}\ (hg) = \mathcal{D}\ (gh) \tag{3.142}$$

We can then insert 1 in the desired integral

$$\begin{aligned} \int \mathcal{D}A\ f(A) &= \int \mathcal{D}A\ f(A)\ M(A) \int \mathcal{D}g\ \delta(F(A^g)) = \\ &= \int \mathcal{D}g \int \mathcal{D}A f(A) M(A) \delta(F(A^g)) = \\ &= \int \mathcal{D}g \int \mathcal{D}A\ \det\left|\frac{\delta F}{\delta A_\mu} D^\mu\right|\ \delta(F)\ f(A) \end{aligned} \tag{3.143}$$

where in the last line we have redefined

$$A^g \equiv B \tag{3.144}$$

and we have postulated that the functional measure for integration over gauge fields is gauge invariant

$$\mathcal{D}\ A = \mathcal{D}\ A^g \tag{3.145}$$

Now the integrand is independent of g, so the integration over the gauge group is just a divergent consnat, and we define the *physical* integration dividing by this constant as

$$\int \mathcal{D}A\ f(A)|_{\text{physical}} \equiv \int \mathcal{D}A\ \det \left| \frac{\delta F}{\delta A_\mu} D^\mu \right|\ \delta(F)\ f(A) \tag{3.146}$$

It has proven convenient to introduce some fermionic ghost fields, c^a and $\bar{c}^a$ to functionally represent the Faddeev-Popov determinant. Actually, this is the precise definition of what is meant by determinant in the present context. It could seem at first sight that this definition is a little bit circular, but those determinants can be defined in a quite satisfactory way using heat kernel techniques.

$$\det \left| \frac{\delta F}{\delta A_\mu} D^\mu \right| = \int \mathcal{D}c\mathcal{D}\bar{c} e^{-i\int d^4x\ \bar{c}^a\ \left(\frac{\delta F}{\delta A_\mu}\ D^\mu\right)_{ab}\ c^b} \tag{3.147}$$

The net outcome of this analysis has been the need to add to the gauge lagrangian a ghostly piece, given by

$$L_{gh} \equiv -i \int d^4x\ \bar{c}^a\ \left(\frac{\delta F}{\delta A_\mu}\ D^\mu\right)_{ab}\ c^b \tag{3.148}$$

Ghosts play an important part in gauge computations to one loop and beyond. They are also essential in ensuring unitarity and independence of the gauge fixing condition. Nevertheless, it is plain that there are no ghosts in external lines of the S-matrix; that is ghosts do not appear as asymptotic states.

We have still to take into account the gauge fixing condition. There is a trick to exponentiate the said gauge fixing condition as well. In order to do that, generalize the gauge condition to

$$F(A) = a(x) \tag{3.149}$$

In the functional integral now appears a term

$$\delta\left(F - a\right) \tag{3.150}$$

Given the fact that the result must be independent of the value of a, it is possible to integrate over a with a factor

$$e^{-\frac{i}{4\alpha} tr \int a(x)^2 dx} \tag{3.151}$$

this will chance the global normalization of the partition function, which is at any rate out of control and irrelevant for the computation of Green's functions. This implies a modification in the action

$$S_{gf} = -\frac{i}{4\alpha}\int d^4x F_a^2 \tag{3.152}$$

Let us now concentrate in the most popular of all gauge fixings, namely

$$F_a \equiv \partial_\mu A^\mu_a \tag{3.153}$$

The full gauge lagrangian reads

$$L = -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\nu + g f_{abc} A^b_\mu A^c_\nu)^2 - \frac{1}{4\alpha}(\partial_\mu A^\mu_a)^2 + \\ +\bar{c}^a\ \partial^\mu \left(\partial_\mu + f_{abc}\ A^b_\mu\right)\ c^c \tag{3.154}$$

Let us now recall the Feynman rules

- In order to determine the gauge boson propagator, consider the full quadratic part

$$L_{gauge} = \frac{1}{2} A^a_\mu \left(\Box\eta^{\mu\nu} - \left(1 - \frac{1}{2\alpha}\right)\partial^\mu\partial^\nu\right) A^a_\nu \tag{3.155}$$

  Then the propagator in momentum space is

$$-i\delta_{ab}\left[\frac{\eta_{\mu\nu}}{k^2 + i\epsilon} - \left(1 - \frac{1}{2\alpha}\right)\frac{k_\mu k_\nu}{(k^2 + i\epsilon)^2}\right] \tag{3.156}$$

- The ghost propagator

$$\frac{i\delta_{ab}}{k^2 + i\epsilon} \tag{3.157}$$

- The three gauge boson coupling is momentum-dependent and stems from

$$L_3 = -\frac{g}{2} f_{abc}\partial_\mu A^a_\nu A^\mu_b A^\nu_c \equiv \frac{1}{3!} V^{\mu\nu\rho}_{abc}(p, q, k)\ A^a_\mu A^\nu_b A^\rho_c \tag{3.158}$$

  It is necessary to antisymmetrize in color indices

$$L_3 = -\frac{g}{2} f_{abc}\frac{1}{6}\left(\partial_\mu A^a_\nu A^\mu_b A^\nu_c + \partial_\mu A^c_\nu A^\mu_a A^\nu_b + \partial_\mu A^b_\nu A^\mu_c A^\nu_a - \\ \partial_\mu A^a_\nu A^\mu_c A^\nu_b - \partial_\mu A^c_\nu A^\mu_b A^\nu_a - \partial_\mu A^b_\nu A^\mu_a W^\nu_c\right) \tag{3.159}$$

This yields the vertex

$$
\begin{aligned}
&-gf_{abc}\left(\eta_{\mu\nu}\left(q-p\right)_{\rho}+\eta_{\nu\rho}\left(k-q\right)_{\mu}+\eta_{\rho\mu}\left(p-k\right)_{\nu}\right)=\\
&-gf_{a_1a_2a_3}\left(\eta^{\mu_1\mu_2}\left(p_1-p_2\right)^{\mu_3}+\eta^{\mu_2\mu_3}\left(p_2-p_3\right)^{\mu_1}+\eta^{\mu_3\mu_1}\left(p_3-p_1\right)^{\mu_2}\right)
\end{aligned}
$$

Where all three momenta, p_1, p_2, p_3 are flowing into the vertex. The vertex written in this form is invariant under any relabelling of the vertices.

- The four gauge boson vertex stems from the term

$$L_4 = g^2 f_{abc}\ A^b_\mu\ A^c_\nu\ f_{auv}\ A^\mu_u\ A^\nu_v \tag{3.160}$$

It is compulsory to symmetrize in all pairs of indices (μa),(νb),(ρc) and (σd)

$$\frac{1}{4!}\ V^{\mu\nu\rho\sigma}_{abcd}\ A^a_\mu\ A^b_\nu\ A^c_\rho\ A^d_\sigma \tag{3.161}$$

Eventually the correct Feynman rule for the vertex is obtained

$$
\begin{aligned}
&-ig^2\Big\{f_{eab}f_{ecd}\left(\eta_{\mu\rho}\eta_{\nu\sigma}-\eta_{\mu\sigma}\eta_{\nu\rho}\right)+f_{eca}f_{edb}\left(\eta_{\mu\nu}\eta_{\rho\sigma}-\eta_{\mu\sigma}\eta_{\rho\nu}\right)+\\
&+f_{ead}f_{ecb}\left(\eta_{\mu\rho}\eta_{\sigma\nu}-\eta_{\mu\nu}\eta_{\rho\sigma}\right)\Big\}
\end{aligned}
\tag{3.162}
$$

- The gluon-ghost vertex is given by

$$L_{gff} = \bar{c}^a \partial^\mu\left(f_{abd}A^b_\mu c^d\right) \tag{3.163}$$

The Feynman rule reads

$$g f_{abc}\ p_\mu \tag{3.164}$$

(where p is the antighost momentum flowing into the vertex)

- The gauge boson-fermion-fermion coupling is

$$L_{int} = \bar{\psi}_A i\ A^a_\mu \gamma^\mu \left(T^a\right)_{AB}\ \psi_B \tag{3.165}$$

where $A, B\ldots$ are flavor indices and $(T^a)_{AB}$ yield the gauge algebra in some representation. The Feynman rule is

$$ig(\gamma)^\mu_{\alpha\beta}\ T^a_{AB} \tag{3.166}$$

3.3 One loop structure of gauge theories.

The loop order of a diagram stands for the number of independent integrations over momenta. Those diagrams in which all momenta are fixed by the delta functions, so that there are no momenta integrations left, are called *tree diagrams*. The loop expansion can also be understood as an expansion of the path integral in powers of $\hbar$. Were we to reinstate $\hbar \neq 0$ for a moment, then propagators get a factor of $\hbar$ whereas vertices get each a factor of $\hbar^{-1}$; altogether each amputated 1PI graph gets a factor of

$$\hbar^{I-V} \tag{3.167}$$

where I is the number of internal lines and V is the number of vertices. On the other hand

$$L = I - V + 1 \tag{3.168}$$

We learn that the power of each diagram is just

$$\hbar^{L-1} \tag{3.169}$$

The tree diagrams are of order $\frac{1}{\hbar}$; they are determined by the saddle point expansion and thus reproduce the perturbative classical physics.

Let us analyze the structure of the gauge theory to the lowest nontrivial order that is, one loop. The starting point will be

$$\begin{aligned} L_{eff} \equiv \bar{\psi}_R \left(i\not{D} - g_R \mu^\epsilon \not{A}_R . T_F - m_R \right) \psi_R - \frac{1}{4} F^2_{\mu\nu}(A_R, g_R \mu^\epsilon) - \frac{1}{2\alpha_R} \left(\partial A_R\right)^2 - \\ - \bar{c}^a_R \left(\partial_\mu \partial^\mu \delta_{ac} - g_R f_{abc} \partial_\mu A^\mu_{R\,c} \right) c^b_R \end{aligned} \tag{3.170}$$

3.3.1 Renormalized lagrangian

The full renormalized lagrangian reads

$$\begin{aligned} L_R(e_R, m_R, \lambda_R) = \bar{\psi}_R \left(i\not{D} - e_R \mu^\epsilon \not{A}_R - m_R \right) \psi_R - \frac{1}{4} F^R_{\mu\nu} F^{\mu\nu}_R \\ - \frac{1}{2\alpha_R} \left(\partial A_R\right)^2 + \bar{\psi}_R i\not{D} \psi_R (Z_\psi - 1) - \bar{\psi}_R e_R \mu^\epsilon \not{A}_R \psi_R (Z_1 - 1) \\ - m_R \bar{\psi}_R \psi_R (Z_\psi Z_m - 1) - \frac{1}{4} F^R_{\mu\nu} F^{\mu\nu}_R (Z_A - 1) \end{aligned} \tag{3.171}$$

Soon to be studied Ward identities guarantee that the renormalized lagrangian has the same gauge symmetry that the bare lagrangian.

Gluon self-energy

- This is the first diagram in which there is an important difference with the abelian case. There are three extra diagrams to begin with. The first is the one that includes two three-gluon couplings (with flowing momenta from the left, so that for the first vertex

$$\begin{aligned} p_1 &= q \\ p_2 &= -k \\ p_3 &= k - q \end{aligned} \tag{3.172}$$

whereas for the second one

$$\begin{aligned} p_1 &= -q \\ p_2 &= k \\ p_3 &= k - q \end{aligned} \tag{3.173}$$

The diagram reads

$$-i\Pi^1_{\alpha\beta,ab} = \frac{1}{2} i g^2 \mu^{2\epsilon} f_{acd} f_{dcb} \int \frac{d^n k}{(2\pi)^n} N_{\alpha\beta}(q,k) \frac{1}{k^2 + i\epsilon} \frac{1}{(q-k)^2 + i\epsilon} \tag{3.174}$$

The numerator is given by

$$N_{\alpha\beta}(q,k) = k_\alpha k_\beta (4n - 6) + q_\alpha q_\beta (n - 6) + (q_\alpha k_\beta + k_\alpha q_\beta)(-2n + 3) + (2k^2 - 2qk + 5q^2)\eta_{\alpha\beta}$$

The divergent part is computed by techniques by now standard

$$-i\Pi^{1,div}_{\alpha\beta,ab} = \frac{\alpha}{8\pi} \delta_{ab} N \left(\frac{19}{6} q^2 \eta_{\alpha\beta} - \frac{11}{3} q_\alpha q_\beta \right) \frac{2}{4-n} \tag{3.175}$$

Hum! It does not seem transverse!

- The second diagram corresponds to the four-gluon vertex (a tadpole of sorts)

$$-i\Pi^2_{\alpha\beta,ab} = \delta_{ab} V^4_{\alpha\beta} g^2 \mu^{2\epsilon} \int d^n k \frac{1}{k^2 + i\epsilon} = 0 \tag{3.176}$$

here V^4 is the complicated tensor that corresponds to the four-gluon vertex. The results holds because in dimensional regularization all integrals without a scale yield zero.

- Finally, there is the ghost loop

$$-i\Pi^3_{\alpha\beta,ab} = i g^2 N \delta_{ab} \int \frac{d^n k}{(2\pi)^n} \frac{1}{k^2 + i\epsilon} \frac{1}{(q-k)^2 + i\epsilon} (-k_\alpha q_\beta + k_\alpha k_\beta)$$

Its divergent part is

$$- i\Pi^{3,div}_{\alpha\beta,ab} = \frac{\alpha}{8\pi} N\delta_{ab}\left(\frac{1}{3}q_\alpha q_\beta + \frac{1}{6}q^2\eta_{\alpha\beta}\right)\frac{2}{4-n} \tag{3.177}$$

Doing now some numerics

$$\frac{19}{6} + \frac{1}{6} = \frac{10}{3} = -\frac{11}{3} + \frac{1}{3} \tag{3.178}$$

which is already transverse. To this result must be added the *abelian* diagram already computed, which was proportional to the Dynkin index T_F. The end result then reads

$$- i\Pi^{div}_{\alpha\beta,ab} = \frac{\alpha}{3\pi}\delta_{ab}\left(\frac{5}{4}N - T(F)n_f\right)\left(-q_\alpha q_\beta + q^2\eta_{\alpha\beta}\right)\frac{2}{4-n} \tag{3.179}$$

which is indeed transverse as it should.

The full counterterm then reads

$$-\frac{1}{4}\left(\partial_\alpha A^R_{\beta,a} - \partial_\beta A^R_{\alpha,a}\right)^2 (Z^1_A - 1) \tag{3.180}$$

with

$$Z^1_A = 1 + \frac{\alpha}{3\pi}\left(\frac{5}{4}N - T(F)n_f\right)\frac{2}{4-n} \tag{3.181}$$

It is worth remarking that the gauge fixing is not renormalized to one loop order

$$Z_\lambda Z_A = 1 \tag{3.182}$$

Fermion-fermion-gluon vertex

The non-abelian correction stems from a trilinear vertex and two Yukawa ones. The trilinear vertex has

$$\begin{aligned} p_1 &\equiv q_1 + q_2 \\ p_2 &= -q_1 - k \\ p_3 &\equiv k - q_2 \end{aligned} \tag{3.183}$$

The color factor is given by

$$f_{abc}\left(T^b T^c\right)_{ij} = \frac{1}{2} f_{abc}\left[T_b, T_c\right] = \frac{i}{2} f_{abc} f_{bcd} T_d = -\frac{i}{2} C(A)\delta_{ad} T_d = -\frac{i}{2} C(A) T_a \tag{3.184}$$

and the diagram itself reads

$$ig\Gamma^\mu(p^2) \equiv (ig)^2 g\int \frac{d^n l}{(2\pi)^n}\gamma^\rho \frac{i\not{k}}{k^2 + i0}\gamma^\nu \frac{-i}{(q_1+k)^2 + i0}\frac{-i}{(q_2-k)^2 + i0} N^{\mu\nu\rho}\left(k, q_1, q_2\right) \tag{3.185}$$

where the numerator is given by

$$N^{\mu\nu\rho}\left(k,q_1,q_2\right)\equiv\eta^{\mu\nu}\left(2q_1+q_2+k\right)^{\rho}+\eta^{\nu\rho}\left(-q_1+q_2-2k\right)^{\mu}+\eta^{\rho\mu}\left(k-2q_2-q_1\right)^{\nu} \tag{3.186}$$

In order to capture the pole it is enough to put all external momenta to zero. The dependence on those external momenta can then be restored by dimensional analysis.

$$\begin{aligned}\Gamma^{\mu}(0)&=g^2\int\frac{d^nk}{(2\pi)^n}\frac{\gamma^{\rho}\not{k}\gamma^{\nu}}{k^6}\left(\eta^{\mu\nu}k^{\rho}-2\eta^{\nu\rho}k^{\mu}+\eta^{\rho\mu}k^{\nu}\right)=\\&=g^2\int\frac{d^nk}{(2\pi)^n}\frac{1}{k^6}\left(\not{k}\not{k}\gamma^{\mu}-2\gamma_{\rho}\not{k}\gamma^{\rho}k^{\mu}-\gamma^{\mu}\not{k}\not{k}\right)=\\&=g^2\int\frac{d^nk}{(2\pi)^n}\frac{1}{k^6}\left(2k^2\gamma^{\mu}-2(2-n)\not{k}k^{\mu}\right)\end{aligned} \tag{3.187}$$

The integral is standard, and its divergent piece is:

$$\Gamma^{2,pole}_{\mu,a}=-gT^F_a\gamma_{\mu}\frac{3\alpha}{8\pi}N\frac{2}{4-n} \tag{3.188}$$

The total counterterm including both the abelian and non-abelian parts reads then

$$Z_1-1=-\frac{\alpha}{4\pi}\left(N+C_2(F)\right)\frac{2}{4-n} \tag{3.189}$$

In principle, the QCD renormalized lagrangian has many independent renprmalization constants, to wit

$$\begin{aligned}&L_R(g,m,\lambda)=L_{ef}(g,m,\lambda)+\bar{\psi}_R i\not{D}\psi(Z_{\psi}-1)-g\mu^{\epsilon}\bar{\psi}\not{A}_aT^F_a\psi(Z_1-1)-\\&m\bar{\psi}\psi(Z_mZ_{\psi}-1)-\frac{1}{4}(\partial_{\alpha}A_{\beta,a}-\partial_{\beta}A_{\alpha,a})^2(Z_A-1)+\\&+\frac{1}{2}g\mu^{\epsilon}f_{abc}A^{\alpha}_bA^{\beta}_c(\partial_{\alpha}A_{\beta,a}-\partial_{\beta}A_{\alpha,a})(Z_1'-1)\\&-\frac{1}{4}g^2\mu^{2\epsilon}f_{ebc}f_{evk}A^{\alpha}_bA^{\beta}_cA_{\alpha,v}A_{\beta,k}(Z_4-1)\\&-\bar{c}_a\partial^2c_a(Z_c-1)+g\mu^{\epsilon}f_{abc}\bar{c}_a\partial_{\mu}A^{\mu}_bc_c(Z_1''-1)\end{aligned}$$

Ward identities suggest a scheme such that

$$\begin{aligned}Z_1&=Z_{\psi}Z_A^{1/2}Z_g\\Z_1'&=Z_A^{3/2}Z_g\\Z_1''&=Z_cZ_A^{1/2}Z_g\\Z_4&=Z_A^2Z_g^2\end{aligned}$$

which is possible only if

$$Z_A^{\frac{1}{2}}Z_g=\frac{Z_1}{Z_{\psi}}=\frac{Z_1'}{Z_A}=\frac{Z''}{Z_c}=\left(\frac{Z_4}{Z_A}\right)^{\frac{1}{2}} \tag{3.190}$$

which is only possible thanks to gauge symmetry. Thse relations constitute in fact another way of writing down the Ward (Slavnov-Taylor) identities

Using all previous results we can derive the renormalization constant for the gauge coupling

$$Z_g = \frac{Z_1}{Z_\psi} Z_A^{-1/2} = 1 - \frac{\alpha}{4\pi} \left(\frac{11}{6} N - \frac{2}{3} T(F) n_f \right) \frac{2}{4-n} \tag{3.191}$$

It is not difficult to check it by computing other diagrams.

4

The renormalization group.

Let us consider a generic scalar field theory. (The spin does not play an important role, so that our results will be generic.)

$$S=\int d^4x\ \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-\frac{m^2}{2}\phi^2-\frac{g}{4!}\phi^4+g_1\phi^6+\ldots\right) \tag{4.1}$$

If we perform a scale transformation

$$x\equiv\lambda x' \tag{4.2}$$

The action transform as

$$S=\int d^4x'\ \lambda^4\ \left(\frac{1}{2}\lambda^{-2}\partial_{\mu'}\phi\partial^{\mu'}\phi-\frac{m^2}{2}\phi^2-\frac{g}{4!}\phi^4+g_1\phi^6+\ldots\right) \tag{4.3}$$

If we want this theory to have the same propagator as before, we have to rescale

$$\phi=\lambda^{-1}\phi' \tag{4.4}$$

In terms of the new fields

$$S=\int d^4x'\ \left(\frac{1}{2}\partial_{\mu'}\phi'\partial^{\mu'}\phi'-\frac{m^2\lambda^2}{2}\phi^2-\frac{g}{4!}\left(\phi'\right)^4+g_1\lambda^{-2}\left(\phi'\right)^6+\ldots\right) \tag{4.5}$$

In the vicinity of the gaussian fixed point, that is

$$S=\int d^4x'\ \left(\frac{1}{2}\partial_{\mu'}\phi'\partial^{\mu'}\phi'\right) \tag{4.6}$$

when we go towards the infrared ($\lambda\rightarrow\infty$) we see that

$$\begin{aligned} &m'\rightarrow\infty\\ &g_1\rightarrow 0\\ &g\rightarrow g \end{aligned} \tag{4.7}$$

Operators like $m^2\phi^2$ of dimension less than four are called *relevant.* Operators like $g_2\phi^6$ of scaling dimension bigger than four are called *irrelevant.* and operators like $\lambda\phi^4$ of scaling dimension exactly equal to four are called *marginal.*

Any physical observable should be independent of the scale μ, which has been introduced as an intermediate step in the regularization and is thus completely arbitrary. Observables then obey

$$0 = \frac{d}{d\mu} S\left[p_i, g_0, m_0\right]\Big|_{g_0,m_0} = \frac{d}{d\mu} S\left[p_i, g_R, m_R, \mu\right]\Big|_{g_0,m_0} =$$
$$= \left[\frac{\partial}{\partial \log \mu} + \beta(g_R, m_R)\ \frac{\partial}{\partial g_R}\Big|_{\mu,m_R} - \gamma_m(g_R, m_R)\frac{\partial}{\log m_R}\Big|_{g_R,\mu}\right] S\left[p_i, g_R, m_R, \mu\right]$$

This is dubbed the *renormalization group equation* (RGE). We have defined

$$\beta(g_R, m_R) \equiv \mu\frac{\partial}{\partial\mu} g_R(\mu)\Big|_{g_0,m_0}$$
$$\gamma_m(g_r, m_R) \equiv -\frac{\mu}{m_R}\frac{\partial}{\partial\mu}\ m_R(\mu)\Big|_{g_0,m_0} \tag{4.8}$$

Since the function $S\left[p_i, g_R, m_R, \mu\right]$ is analytic at $n = 4$, it is natural to expect that both functions β and γ_m are analytic as well. In order to compute these universal functions, and remembering that

$$m_0 = Z_m^{\frac{1}{2}}\ m_R \tag{4.9}$$

and

$$g_0 = Z_g\ g_R\ \mu^{\frac{4-n}{2}} \tag{4.10}$$

$$\beta(g_R, m_R) \equiv g_0\mu\frac{\partial}{\partial\mu}\frac{1}{Z_g\mu^{\frac{4-n}{2}}}\Big|_{g_0,m_0}$$
$$\gamma_m(g_r, m_R) \equiv -\frac{m_0}{m_R}\mu\frac{\partial}{\partial\mu}\ Z_m^{-\frac{1}{2}}\Big|_{g_0,m_0} \tag{4.11}$$

All this is much simpler in a mass independent renormalization scheme, where the renormalization constants are independent of m_R and μ. We have already discussed that MS (or $\overline{MS}$) are such a schemes.

It is plain that

$$\mu\ \frac{\partial}{\partial\mu}\ g_0 = 0 = \mu^{\frac{4-n}{2}}\left(\frac{4-n}{2} Z_g g_R + \mu\ \frac{\partial}{\partial\mu}\left(Z_g g_R\right)\right) \tag{4.12}$$

Now

$$g_R\ Z_g = g_R + \sum_{n=1}^{\infty} a_n(g_R)\left(\frac{2}{4-n}\right)^n \tag{4.13}$$

Let us make the Laurent-type ansatz (we shall see later that it is actually necessary)

$$\beta(g_R) \equiv \beta_1(g_R)\,(4-n) + \beta_0(g_R) + \ldots \tag{4.14}$$

We get

$$\frac{4-n}{2}\left(g_R + a_1(g_R)\frac{2}{4-n} + \ldots\right) + \beta_1(g_R)\,(4-n) + \beta_0(g_R) + \ldots +$$
$$+\frac{da_1(g_R)}{dg_R}\left\{\beta_1(g_R)\,(4-n) + \beta_0(g_R)\right\}\frac{2}{4-n} + \ldots = 0 \tag{4.15}$$

Terms of $O(n-4)$ (which are now seen to be neccessary) yield

$$\frac{g_R}{2} + \beta_1(g_R) = 0 \tag{4.16}$$

and terms of $O(1)$ imply

$$\beta(g_R, n-4) = -g_R\,\frac{4-n}{2} - a_1(g_R) + g_R\,\frac{da_1}{dg_R} \tag{4.17}$$

There are recursion relations worked out by 't Hooft to compute all $a_n, n > 1$ from the knowledge of a_1.

For the theory ϕ^4_4 the result is

$$\beta = 3\,\frac{\lambda^2}{(4\pi)^2} \tag{4.18}$$

In QED

$$\beta = \frac{e^3}{12\pi^2} \tag{4.19}$$

Whereas for the (also renormalizable) six-dimensional theory ϕ^3_6

$$\beta = -3\,\frac{\lambda^3}{4(4\pi)^3} \tag{4.20}$$

For a nonabelian $SU(N)$ gauge theory with n_f fermion flavors in the fundamental representation,

$$\beta = -\frac{g^3}{(4\pi)^2}\,\left(\frac{11}{3}N - \frac{4}{3}n_f T(F)\right) \tag{4.21}$$

Now, for a general beta function to assert

$$\beta \equiv b\lambda^3 \tag{4.22}$$

means that

$$\frac{d\lambda}{\lambda^3} = b\;d\,\log\,\mu \tag{4.23}$$

Integrating with the boundary conditions that

$$\lambda = \lambda_i \tag{4.24}$$

when

$$\mu = \mu_i \tag{4.25}$$

yields the dependence of the coupling constant on the RG scale, μ

$$\lambda^2 = \frac{\lambda_i^2}{1 - b\lambda_i^2 \log \frac{\mu}{\mu_i}} \tag{4.26}$$

When b is positive (like in ϕ_4^4 or QED), there is a *Landau pole* at

$$\mu = \Lambda \equiv \mu_i \ e^{\frac{1}{b\ \lambda_i^2}} \tag{4.27}$$

Trading μ_i by the scale Λ

$$\lambda^2 = \frac{\lambda_i^2}{b\lambda_i^2 \log \frac{\Lambda}{\mu}} \tag{4.28}$$

Those theories are *infrared safe*, but they do not enjoy an UV consistent limit.

When $b < 0$ (this what happens for ϕ_6^3 and also for ordinary gauge theories) there is a pole at

$$\mu = \Lambda \equiv \mu_i \ e^{-\frac{1}{|b|\ \lambda_i^2}} \tag{4.29}$$

The paradigm of these theories is QCD. They are *asymptotically free* but *infrared slave* The Landau pole is now located in the infrared region. Its scale is also denoted by Λ and the running coupling reads

$$g^2(\mu) = \frac{g_i^2}{2bg_i^2 \log \frac{\mu}{\Lambda}} \tag{4.30}$$

Λ is obviously renormalization group invariant and experimentally its value is

$$\Lambda \sim 217 MeV \tag{4.31}$$

and it signals the scale at which QCD starts being strongly coupled.

Green functions also obey some different renormalization group equations, because they are multiplicatively renormalized. The starting point is that

$$\mu \ \frac{\partial}{\partial\mu} \ \Gamma^0 = 0 \tag{4.32}$$

Then

$$\left\{\mu \ \frac{\partial}{\partial\mu}\bigg|_{g_R,m_R} + \beta(g_R)\frac{\partial}{\partial g_R}\bigg|_{\mu,m_R} - \gamma_m(g_R) \ \frac{\partial}{\partial m_R}\bigg|_{g_R\mu} - n\gamma_\phi(g_R)\right\} \Gamma_R\left(p_i, g_R, m_R, \mu\right) = 0$$

where we have defined the *anomalous dimension*

$$\gamma_\phi(g_R) \equiv \frac{1}{2}\ \mu\frac{\partial}{\partial\mu}\log\ Z_\phi = \frac{1}{2}\ \beta(g_R)\ \frac{\partial}{\partial g_R}\log\ Z_\phi \tag{4.33}$$

For the theory ϕ^3_6

$$\gamma_\phi(g) = \frac{1}{12}\frac{\lambda^3}{(4\pi)^3} + \frac{13}{432}\ \left(\frac{\lambda^2}{(4\pi)^3}\right)^2 \tag{4.34}$$

The RE equations for 1PI in gauge theories are best writing by first defining the operator

$$\mathcal{D} \equiv \mu\frac{\partial}{\partial\mu}+\beta(g_R)\frac{\partial}{\partial g_R}-\gamma_m(g_R)m_R\frac{\partial}{\partial m_R}+\delta(g_R,\lambda_R)\frac{\partial}{\partial\alpha_R}-n_A\gamma_A-n_f\gamma_\psi-n_c\gamma_c \tag{4.35}$$

The 1PI equation itself reads

$$\mathcal{D}\ \Gamma_{R,n}(g_R,m_R,\zeta_R) = 0 \tag{4.36}$$

The number of external gauge fields is (n_A), external fermions by(n_f) and ghosts (n_c), and their corresponding anomalous dimensions γ_A,γ_f,γ_c.

The generic definition of the *anomalous dimension* reads

$$\gamma \equiv \frac{1}{2}\mu\frac{\partial}{\partial\mu}log\,Z \tag{4.37}$$

These objects are in general gauge dependent.

Using the identity

$$Z_\lambda Z_A = 1 \tag{4.38}$$

$$\delta(g_R,\alpha_R) \equiv \mu\frac{\partial}{\partial\mu}\alpha_R = -\alpha_R\mu\frac{\partial}{\partial\mu}\log\ Z_A = -2\alpha_R\gamma_A \tag{4.39}$$

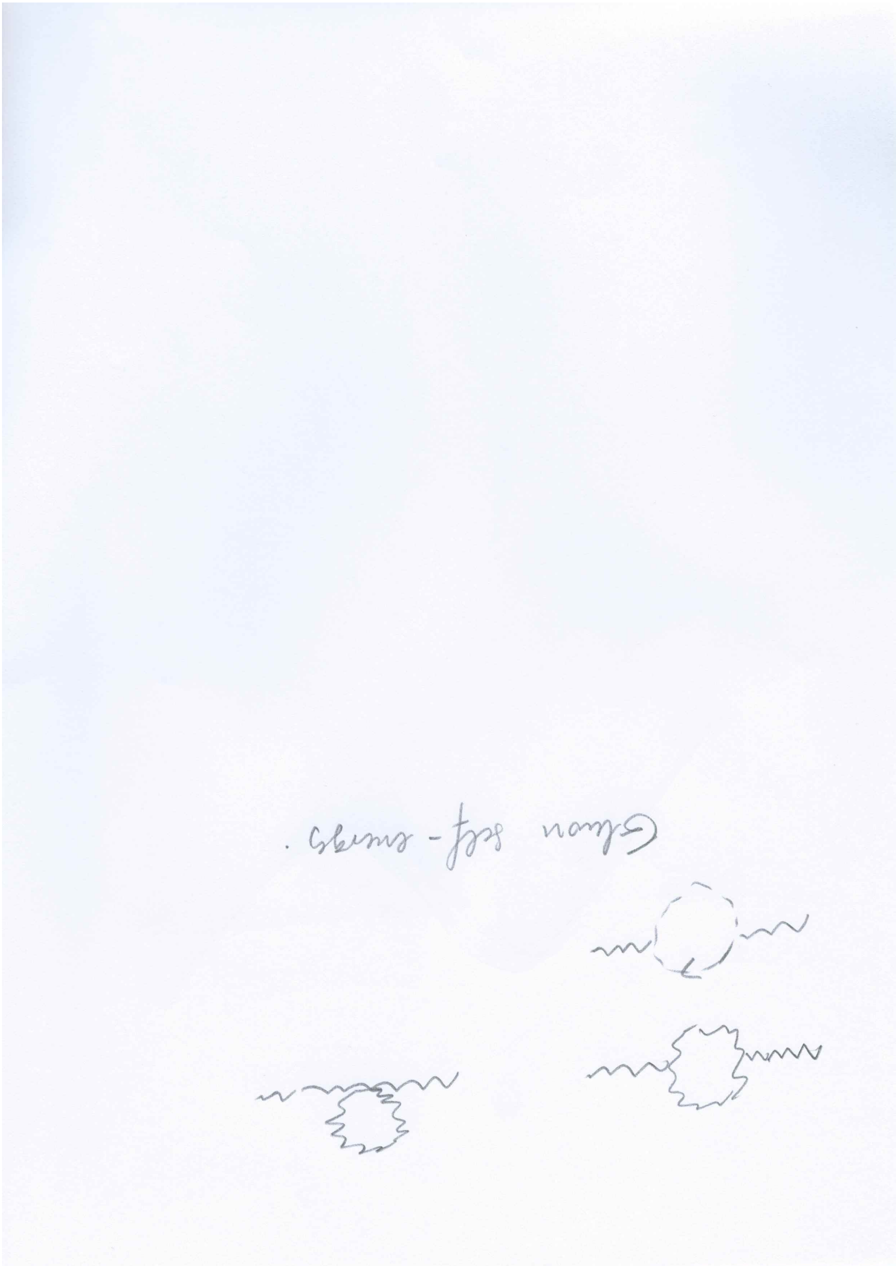

Figure 4.1: Gluon self-energy.

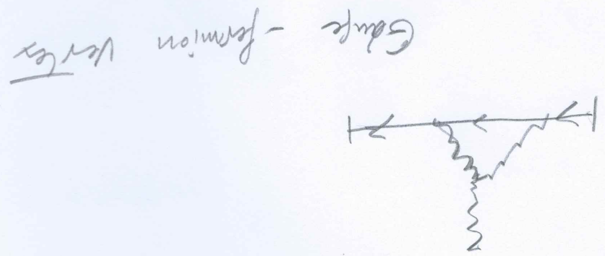

Figure 4.2: Gauge-fermion vertex.

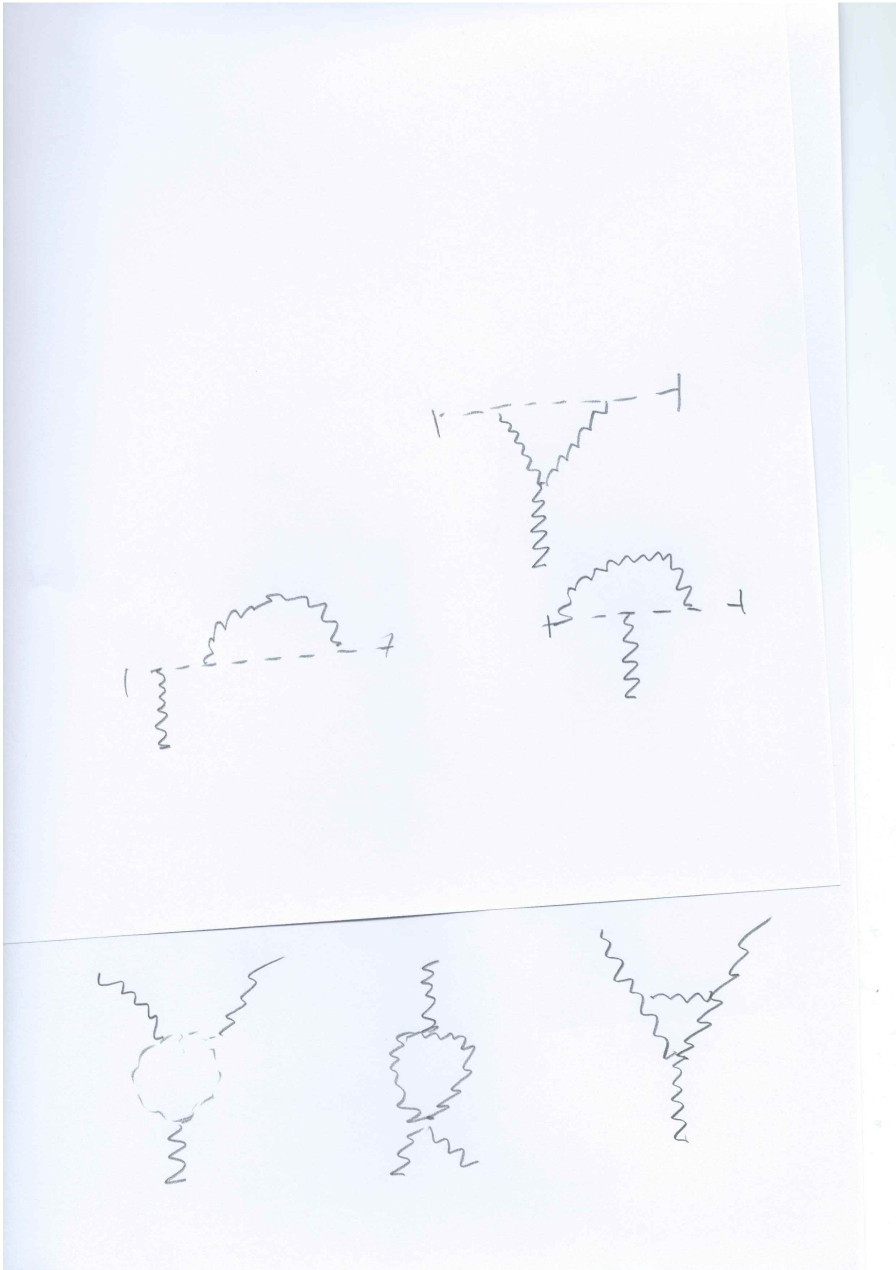

Figure 4.3: More diagrams.

5

Two loops in ϕ^3_6.

5.1 One loop

We have alrady seen that the six-dimensional theory ϕ^3_6 is renormalizable. The action reads

$$S = \int d^6x \left(\frac{1}{2}(\partial_\mu\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{g}{6}\phi^3\right) \tag{5.1}$$

In the figure we have drawn the divergent graphs to one loop order. The first diagram gives

$$\Gamma_a(p^2, m^2) = \frac{g^2}{2}(4\pi)^{-\frac{n}{2}}\Gamma(2-n/2)\int_0^1 dx \left[m^2 - x(1-x)p^2 - i\epsilon\right]^{n/2-2} \tag{5.2}$$

The pole at n=6 is given by

$$-\frac{g^2}{2}\frac{1}{(4\pi)^3}\left(\frac{2m^2}{6-n} - \frac{p^2}{3(6-n)}\right) \tag{5.3}$$

The counterterm lagrangian reads

$$\Delta L = \frac{1}{2}(\partial_\mu\phi)^2\,(Z_\phi - 1) - \frac{m^2}{2}\phi^2\,(Z_\phi Z_m - 1) - \frac{g}{6}\,\left(Z_\phi^{3/2} Z_g - 1\right)\phi^3 \tag{5.4}$$

Then in MS

$$Z_\phi - 1 = \frac{g^2}{(4\pi)^3}\left(-\frac{1}{12}\right)\,\frac{2}{6-n} \tag{5.5}$$

The mass counterterm is

$$\delta m^2 \equiv m^2\,(Z_\phi Z_m - 1) = \frac{g^2}{(4\pi)^3}\left(-\frac{1}{2}\right)\,\frac{2}{6-n} \tag{5.6}$$

Taking into account that the rules of the game are such that to one loop order

$$(Z_\phi Z_m - 1) = (Z_\phi - 1)\,(Z_m - 1) \tag{5.7}$$

we learn that

$$(Z_m - 1) = \frac{g^2}{(4\pi)^3}\left(-\frac{5}{12}\right)\frac{2}{6-n} \tag{5.8}$$

This means in particular, that the mass is multiplicatively renormalized

$$m_0 = Z_m^{1/2} m \tag{5.9}$$

This stands in contradistinction of the ϕ^3_4 theory, inwhich the same renormalizartion is additive

$$m_0^2 = m^2 + \delta m^2 \tag{5.10}$$

The triangle diagram yields

$$\Gamma_b(p_1,p_2,p_3) = -i\frac{(-ig)^3}{(2\pi)^n}\int d^n k \frac{i}{k^2 - m^2 + i\epsilon}\frac{i}{(p_1+k)^2 - m^2 + i\epsilon}\frac{i}{(p_2+k)^2 - m^2 + i\epsilon} \tag{5.11}$$

The divergent part reads

$$-\frac{g^3}{2(4\pi)^3}\left(\frac{2}{n-6} + \text{finite}\right) \tag{5.12}$$

Then

$$g\left(Z_\phi^{3/2}\, Z_g - 1\right) = -\frac{g^3}{2}\frac{1}{(4\pi)^3}\frac{2}{n-6} \tag{5.13}$$

which means

$$Z_g = -\frac{g^2}{2}\frac{1}{(4\pi)^3}\frac{3}{8}\frac{2}{n-6} \tag{5.14}$$

5.2 Two loops.

The contribution to Γ_2 from the first graph, reads

$$\begin{aligned} -i\Gamma_1 = & -\frac{i}{2}(-ig)^4 \int \frac{d^n k}{(2\pi)^n}\frac{i}{(p-k)^2 - m^2 + i\epsilon}\frac{i^2}{(k^2 - m^2 + i\epsilon)^2}\times \\ & \times \int \frac{d^n q}{(2\pi)^n}\frac{i}{q^2 - m^2 + i\epsilon}\frac{i}{(k-q)^2 - m^2 + i\epsilon} \end{aligned} \tag{5.15}$$

For $n < 4$ all integrals are convergent, and we may begin with whichever. Doing the integral over dq first

$$\begin{aligned} -i\Gamma_1 = & \frac{i}{2} g^4 \Gamma(2 - n/2)\int \frac{d^n k}{(2\pi)^n}\frac{1}{(p-k)^2 - m^2 + i\epsilon}\frac{1}{(k^2 - m^2 + i\epsilon)^2}\times \\ & \times \int_0^1 dx\, [x\,(1-x)]^{n/2-2}\left[\frac{m^2}{x(1-x)} - k^2 - i\epsilon\right]^{n/2-2} \end{aligned} \tag{5.16}$$

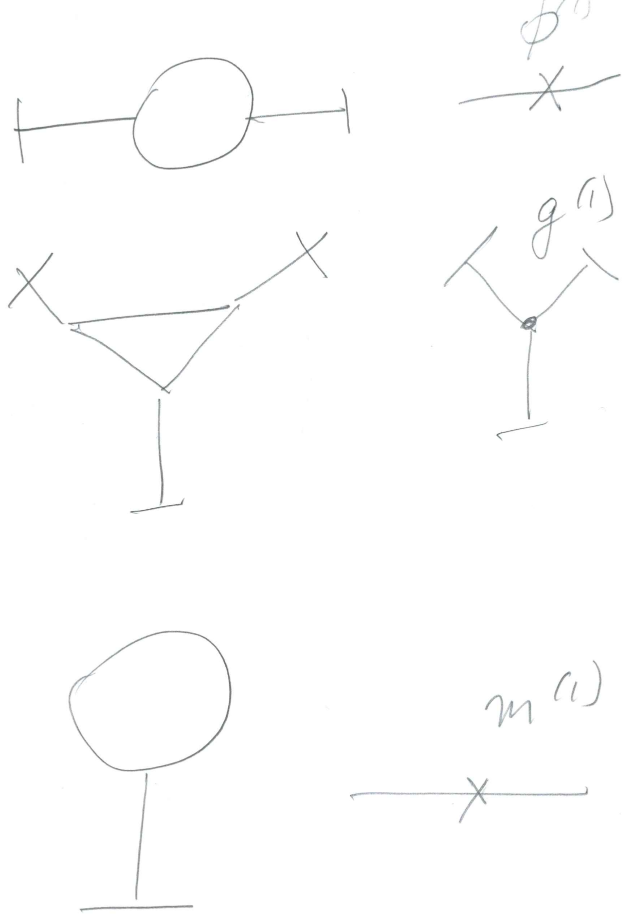

Figure 5.1: Divergences in ϕ_6^3.

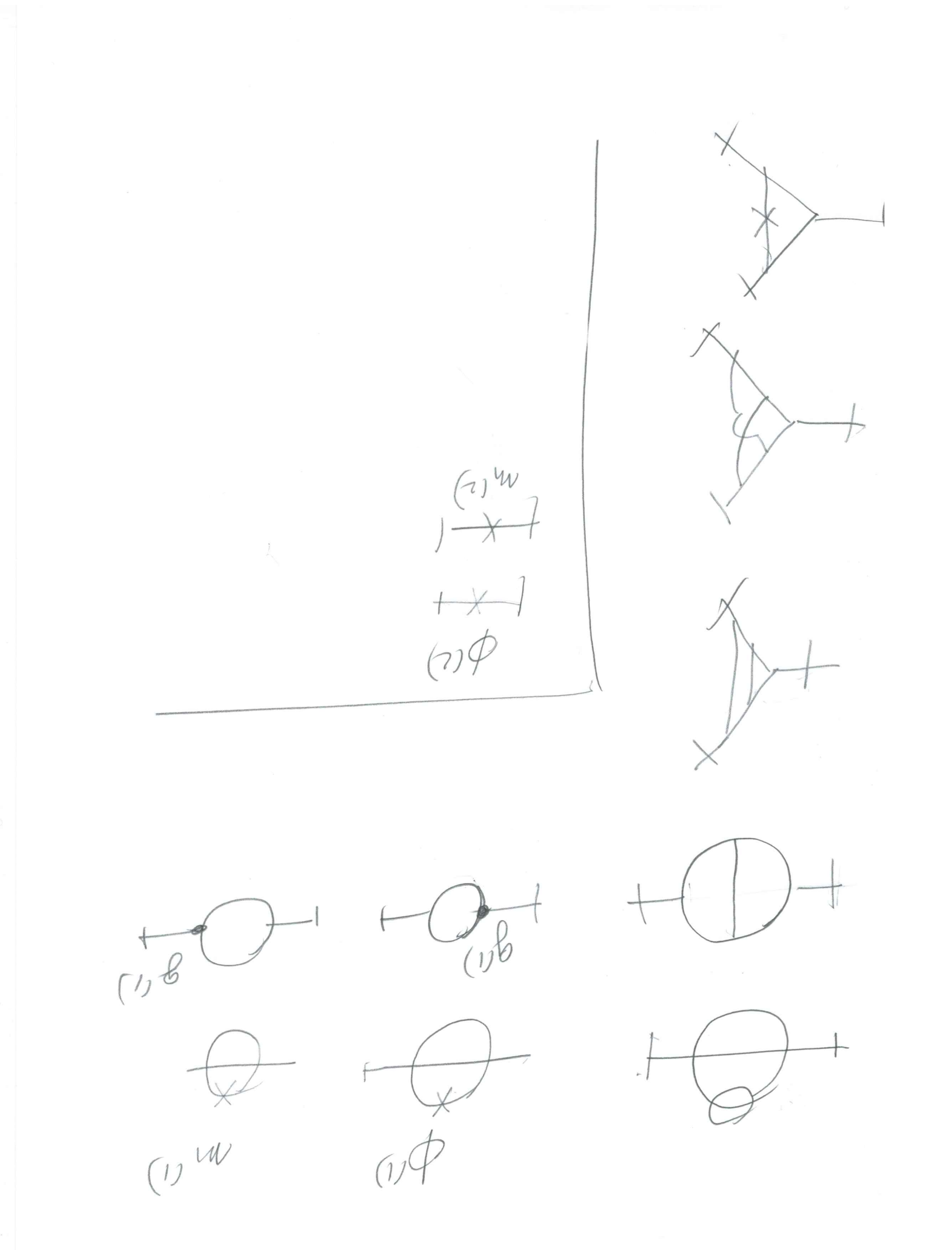

Figure 5.2: Two loop divergences in ϕ^3_6.

We need the general formula for the Feynman parameters

$$\prod_{i=1}^{N} \frac{1}{A^{\eta_i}} = \frac{1}{\prod_{i=1}^{N} \Gamma(\eta_i)} \times \Gamma\left(\sum_{i=1}^{N} \eta_i\right) \times$$
$$\times \int_0^1 dx_1\, x_1^{\eta_1 - 1} \ldots dx_N\, x_N^{\eta_N - 1} \times \frac{1}{\left(\sum_{i=1}^{N} x_i A_i\right)^{\sum_{i=1}^{N} \eta_i}} \times \delta\left(\sum_{i=1}^{N} x_i - 1\right)$$

This is valid for arbitrary values of the parameters η_i.

We then get

$$-i\Gamma_1 = g^4 \frac{(4\pi\mu^2)^{6-n}}{2(4\pi)^6}\, \Gamma(5-n) \int_0^1 dx\, [x(1-x)]^{n/2-2} \times$$
$$\int_0^1 dx_1 \int_0^{1-x_1} dx_2\, x_2^{1-n/2}\, (1-x_1-x_2) \left[m^2\left(1 - x_2 + \frac{x_2}{x(1-x)}\right) - p^2 x_1(1-x_1) - i\epsilon\right]^{n-5} \quad (5.17)$$

It is worth noticing that the factor $x_2^{1-n/2}$ is defined only for $n < 4$. We need to make an analytic continuation in the integral over dx_2. We do that by making two integration by parts. The result is

$$-i\Gamma_1 = g^4 \frac{(4\pi\mu^2)^{6-n}}{2(4\pi)^6}\, \Gamma(5-n) \int_0^1 dx\, [x(1-x)]^{n/2-2} \times$$
$$\int_0^1 dx_1 \frac{2}{6-n} \frac{2}{4-n} \left\{ \int_0^{1-x_1} dx_2\, x_2^{3-n/2}\, \frac{d^2}{dx_2^2}\left[(1-x_1-x_2)\, f(x,x_1,x_2)^{n-5}\right] + \right.$$
$$\left. + (1-x_1)^{\frac{6-n}{2}}\, f(x, x_1, (1-x_1))^{n-5} \right\} \quad (5.18)$$

where

$$f(x,x_1,x_2) \equiv -p^2 x_1(1-x_1) + m^2\left\{1 - x_2\left(1 - \frac{1}{x(1-x)}\right)\right\} \quad (5.19)$$

There are single and double poles at n=6

$$\text{pole part} = -\frac{1}{48}\, \frac{g^4}{(4\pi)^6}\, \frac{4}{(6-n)^2}\left(\frac{p^2}{3} + 3m^2\right) -$$
$$-\frac{1}{12}\, \frac{2}{6-n}\, m^2\, \frac{g^4}{(4\pi)^6} \int_0^1 dz\, \left\{g(z) - 5(1-z)\right\}\, \log\, \frac{m^2\, g(z)}{4\pi\mu^2} +$$
$$+\frac{1}{12}\, \frac{2}{6-n}\, \frac{g^4}{(4\pi)^6}\, p^2\, \left(-\frac{23}{72} + \frac{1}{6}\gamma_E\right) + \frac{1}{12}\, \frac{2}{6-n}\, m^2\, \frac{g^4}{(4\pi)^6}\left(\frac{9}{8} + \frac{3}{2}\gamma_E\right)$$

and

$$g(z) \equiv 1 - z(1-z)\, \frac{p^2}{m^2} \quad (5.20)$$

Please note that the pole terms with a log coefficient cannot be cancelled by a local counterterm, so that they have to cancel by themselves.

The second diagram in which we include the one loop counterterm as a vertex yields

$$-i\Gamma_2 = -i(-ig^2)\mu^{6-n} \int \frac{d^nk}{(2\pi)^n} \frac{i^2}{(k^2-m^2+i\epsilon)^2} \frac{i}{(p-k)^2-m^2+i\epsilon} \times$$
$$\times \frac{i}{2}\, g^2 \, \frac{1}{(4\pi)^3}\frac{2}{6-n} \left(m^2 - \frac{k^2}{6}\right) \tag{5.21}$$

Using the trick

$$m^2 - \frac{k^2}{6} = \frac{5}{6}m^2 - \frac{k^2-m^2}{6} \tag{5.22}$$

we get

$$-i\Gamma_2 = \frac{1}{2}\,\frac{2}{6-n}\,\frac{g^4}{(4\pi)^6}\, m^2 \left\{ \frac{5}{6}\,\Gamma\left(\frac{6-n}{2}\right) \int_0^1 dz\,(1-z) \left[\frac{m^2 g(z)}{4\pi\mu^2}\right]^{\frac{n-6}{2}} + \right.$$
$$\left. + \frac{1}{6}\,\Gamma\left(\frac{4-n}{2}\right) \int_0^1 dz\; g(z) \left[\frac{m^2 g(z)}{4\pi\mu^2}\right]^{\frac{n-6}{2}} \right\} \tag{5.23}$$

The pole terms are easily found to be

$$\text{pole terms} = \frac{1}{24}\,\frac{4}{(6-n)^2}\frac{g^4}{(4\pi)^6}\left(\frac{p^2}{3}+3m^2\right) +$$
$$+\frac{1}{12}\frac{2}{6-n}\frac{g^4}{(4\pi)^6}\, m^2 \int_0^1 dz\;\left(g(z)-5(1-z)\right)\,\log\frac{m^2 g(z)}{4\pi\mu^2} +$$
$$+\frac{1}{72}\frac{2}{6-n}\frac{g^4}{(4\pi)^6}p^2(1-\gamma_E) - \frac{1}{12}\frac{2}{6-n}\frac{g^4}{(4\pi)^6}m^2\left(1+\frac{3}{2}\gamma_E\right) \tag{5.24}$$

After taking care of the other diagrans, the full pole term reads

$$\Gamma_2^{\text{pole}} = \frac{g^4}{(4\pi)^6}p^2\left\{-\frac{5}{144}\frac{4}{(6-n)^2}+\frac{13}{864}\frac{2}{6-n}\right\}+\frac{g^4}{(4\pi)^6}m^2\left\{\frac{5}{16}\frac{4}{(6-n)^2}-\frac{23}{96}\frac{2}{6-n}\right\} \tag{5.25}$$

The two loop renormalization constants then read [9]

$$Z_\phi = \frac{g^4}{(4\pi)^6}\left\{\frac{5}{144}\frac{4}{(6-n)^2} - \frac{13}{864}\frac{2}{6-n}\right\}$$
$$(Z_\phi + Z_m) = \frac{g^4}{(4\pi)^6}\left\{\frac{5}{16}\frac{4}{(6-n)^2} - \frac{23}{96}\frac{2}{6-n}\right\}$$
$$\delta g = \frac{g^4}{(4\pi)^6}\; g\; \left\{\frac{5}{16}\frac{4}{(6-n)^2} - \frac{23}{96}\;\frac{2}{6-n}\right\} \tag{5.26}$$

6

Spontaneously broken symmetries.

6.1 Global (rigid) symmetries

Consider a charged scalar field transforming under $g \in U(1)$ as

$$\phi' \equiv g\phi = e^{i\alpha}\phi \tag{6.1}$$

with potential energy

$$V(\phi) = \frac{\lambda}{4!}(|\phi|^2 - v^2)^2 \tag{6.2}$$

Then the vacuum of the theory is not the Fock vacuum

$$a_k|0\rangle = 0 \tag{6.3}$$

because in Fock's vacuum

$$\langle 0|\phi|0\rangle = 0 \tag{6.4}$$

whereas the fundamental state of the potential (6.2)

$$\langle vac(\theta)|\phi|vac(\theta)\rangle = ve^{i\theta} \tag{6.5}$$

Under group transformations

$$\langle vac(\theta)|e^{i\alpha}\phi|vac(\theta)\rangle = ve^{i(\theta+\alpha)} = \langle vac(\theta+\alpha)|\phi|vac(\theta+\alpha)\rangle \tag{6.6}$$

that is

$$g|vac\rangle \neq |vac\rangle \tag{6.7}$$

which means that the symmetry is *spontaneously broken*. When there is an infinite number of degrees of freedom all thse vacua are orthononal to each other [**?**].

$$\langle vac_1|vac_2\rangle \equiv \langle v_1|v_2\rangle == 0 \tag{6.8}$$

Let us now change variables to the fields ρ and θ such defined

$$\phi = (\rho + v)e^{i\theta} \tag{6.9}$$

the lagrangian reads

$$L = \frac{1}{2}\partial_\mu\phi^*\partial^\mu\phi - \frac{\lambda}{4!}(|\phi|^2 - v^2)^2 =$$
$$\frac{1}{2}(\partial_\mu\rho)^2 + \frac{1}{2}(\rho+v)^2(\partial_\mu\theta)^2 - \frac{\lambda}{4!}((\rho+v)^2 - v^2)^2 \tag{6.10}$$

that is, the field ρ has got a mass

$$m^2 = \frac{\lambda}{3}v^2 \tag{6.11}$$

whereas the field θ remains massless.

This is the simplest instance of *Goldstone's theorem*.

Let us prove it in general. Consider a theory invariant under

$$\delta\phi_i = i\omega_a(T^a)_i^j\phi_j \tag{6.12}$$

The matrices T^a generate a representation R of a Lie algebra G.

In the preceding example $T = 1$. An invariant potential obeys

$$V(g\phi) = V(\phi) \tag{6.13}$$

with $g \equiv e^{i\omega_a T^a}$.

To be specific, the condition reads

$$\sum_{ij}\frac{\partial V}{\partial\phi_i}\,(T^a)_i^j\,\phi_j = 0 \tag{6.14}$$

Deriving once more

$$\sum_{ij}\frac{\partial^2 V}{\partial\phi_i\partial\phi_k}\,(T^a)_i^j\,\phi_j + \frac{\partial V}{\partial\phi_i}(T^a)_i^k = 0 \tag{6.15}$$

and evaluating it at the stationary points $v_j \equiv \langle\phi\rangle_j$ reads

$$M_{ik}\,(T^a)_i^j\,v_j = 0 \tag{6.16}$$

where the mass matrix is defined as

$$M_{ik} \equiv \left.\frac{\partial^2 V}{\partial\phi_i\partial\phi_k}\right|_{\phi=\langle\phi\rangle} \tag{6.17}$$

On the other hand, we know that $M_{ik} = \Delta_{ik}^{-1}(0)$.

The subgroup $H \subset G$ leaving the vacuum invariant is characterized by the condition

$$(H^a)_i^j \, v_j = 0 \tag{6.18}$$

Those generators correspond to the trivial zero eigenvalue of the matrix $M = \Delta^{-1}$. (In our example this subgroup was just the identity $H = 0$).

To avery generator of the group G which is not included in H, say K^a, it corresponds a nontrivial eigenvector with zero eingenvalue of the matrix

$$\Delta^{-1}_{kl}(0) \tag{6.19}$$

Then the theory contains $d_G - d_H$ massless fields, which are called *Goldstone bosons.*

In $n = 2$ dimensions there are no Goldstone bosons because the corresponding propagator violates positivity [**?**]. This fact was first proved by Coleman.

6.2 Spontaneously broken gauge symmetries.

Let is first consider the Higgs model, which is a simple extension of the model of the previous chapter. The lagrangian reads

$$L = \frac{1}{2}|D_\mu \phi|^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\lambda}{4!}(|\phi|^2 - v^2)^2 \tag{6.20}$$

where the covariant derivative is given by

$$D_\mu \phi \equiv \partial_\mu \phi + iqA_\mu \phi. \tag{6.21}$$

The gauge symmetry is

$$\delta\phi = i\epsilon q\phi. \tag{6.22}$$

Introducing again the polar variables

$$D_\mu \phi = (\partial_\mu \rho + i\,(\rho + v)\,\partial_\mu \theta + iq\,(\rho + v)\,A_\mu)\,e^{i\theta} \tag{6.23}$$

leads to

$$L = -\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} + \frac{1}{2}(\partial_\mu \rho)^2 + \frac{1}{2}(\rho + v)^2(\partial_\mu \theta + qA_\mu)^2 - V(\rho + v) \tag{6.24}$$

Redefining now the gauge field

$$W_\mu \equiv A_\mu + \frac{1}{q}\partial_\mu \theta \tag{6.25}$$

which is equivalent to working in the gauge

$$\theta = 0 \tag{6.26}$$

the full lagrangian reads

$$L = -\frac{1}{4}F_{\alpha\beta}(W)^2 + \frac{1}{2}(\partial_\mu \rho)^2 + \frac{q^2}{2}(\rho + a)^2 W_\mu^2 - V(\rho + v) \tag{6.27}$$

The Goldstone boson has disappeared and we have a massive vector boson instead

$$m^2(W) = q^2 v^2 \tag{6.28}$$

In the general case we can always make a change of variables (using a real field basis, which can always be done)

$$\tilde{\phi}_i \equiv g_i^j(x)\phi_j \tag{6.29}$$

where the transformations $g_i^j(x)$ are so chosen that

$$\tilde{\phi}_i \, (K^a)_i^j \, v_j = 0 \tag{6.30}$$

In our example, and using real fields

$$\phi \equiv \phi_1 + i\phi_2 \tag{6.31}$$

the only generator of the algebra $U(1) \sim SO(2)$ reads

$$K = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{6.32}$$

so that in the vacuum representative in which only the real part of the field is nonvanishing $\langle\phi_2\rangle = 0$, but $\langle\phi_1\rangle = v$,

$$\tilde{\phi}_2 = 0 \equiv \sin\ \theta \tag{6.33}$$

which is in our former notation precisely

$$\theta = 0 \tag{6.34}$$

This gauge completely eliminates Goldstone bosons. This is called the *unitarity gauge*

In the general case, consider the kinetic energy piece

$$\frac{1}{2}\left(D_\mu \tilde{\phi}_i\right)^2 \tag{6.35}$$

where the covariant derivative reads

$$D_\mu \tilde{\phi}_i \equiv \partial_\mu \tilde{\phi}_i - i A_\mu^a (T^a)_i^j \tilde{\phi}_j \tag{6.36}$$

Define now translated fields

$$\tilde{\phi}_i \equiv v_i + \sigma_i \tag{6.37}$$

The gauge condition reads now

$$(v_i + \sigma_i)\ K_{ij}\ v^j = \sigma_i K_{ij} v^j = 0 \tag{6.38}$$

(owing to antisymmetry). Expanding to quadratic order, the term mixing σ with A vanishes owing to our gauge condition (6.30),

$$\frac{1}{2}\left(\partial_\mu \sigma_i - i A^a (T^a)_i^j (v_j + \sigma_j)\right)^2 =$$
$$\frac{1}{2}\left((\partial_\mu \sigma_i)^2 - 2i\partial_\mu \sigma_i A^a_\mu (T^a)_i^j (v_j + \sigma_j) - A^a_\mu A^b_\mu (T^a)_i^j (v_j + \sigma_j) \left(T^b\right)_i^k (v_k + \sigma_k)\right)$$

Let us introduce the convenient notation

$$H_\mu \equiv A^a_\mu H^a$$
$$K_\mu \equiv A^a_\mu K^a \tag{6.39}$$

The trilinear gauge scalar scalar vertex reads, owing to our gauge condition

$$-2i\partial_\mu \sigma_i \left(H_\mu + K_\mu\right)_i^j (v_j + \sigma_j) = -2i\partial_\mu \sigma_i \left(A_\mu\right)_{ij} \sigma_j \tag{6.40}$$

id est, the quadratic piece mixing $A - \sigma$ has dissapeared.

The quartic coupling has the symbolic form

$$\left(H_\mu \sigma + K_\mu (v + \sigma)\right)_i \left(H_\mu \sigma + K_\mu (v + \sigma)\right)_i \tag{6.41}$$

Only survive

$$\left(H_\mu \sigma\right)\left(H_\mu \sigma\right) \tag{6.42}$$

and the mass term. The full quadratic piece then reads in this gauge

$$\frac{1}{2}\left(\partial_\mu \sigma_i\right)^2 - \frac{1}{2}\mu^2_{ab} A^a_\mu A^{b\mu} \tag{6.43}$$

and the gauge fields mass matrix reads

$$\mu^2_{ab} \equiv (K_a)_{ij} v^j (K_b)_{il} v^l \tag{6.44}$$

There are d_H massless gauge bosons and $d_G - d_H$ massive gauge bosons. This mechanism wears the name of *Higgs mechanism*

The gauge fields kinetic energy reads

$$M^{ab}_{\mu\nu} = -\delta^{ab}\left(\eta_{\mu\nu}\Box - \partial_\mu \partial_\nu\right) - (\mu^2)^{ab}\eta_{\mu\nu} \tag{6.45}$$

In order to find the propagator of the gauge fields we have to solve

$$\left(\delta_{ab}\left(k^2 \eta_{\mu\nu} - k_\mu k_\nu\right) - \left(\mu^2\right)_{ab}\eta_{\mu\nu}\right)\left(\Delta^{bc}\right)^{\mu\nu}(k) = \delta^c_a \delta^\lambda_\mu \tag{6.46}$$

Let us make the ansatz

$$\Delta \equiv A_{ab}\eta_{\mu\nu} + B_{ab} k_\mu k_\nu \tag{6.47}$$

so that

$$(k^2 - \mu^2)\ A = 1$$
$$(k^2 - \mu^2)\ B - A - B\ k^2 = 0 \tag{6.48}$$

Then

$$\Delta^{\mu\nu}_{bc}(k) = \left(k^2 - \mu^2\right)^{-1}_{bd} \left(\eta_{\nu\lambda}\delta_{dc} - (\mu^{-2})_{dc} k_\nu k_\lambda\right) \tag{6.49}$$

whose UV limit ($|k| \rightarrow \infty$) is a constant, which spoils the power counting necessary for renormalizability.

In order to renormalize the theory is better to work in another gauge, discovered by 't Hooft, where however the physical particle content is obscure. The great advantage is that in this new gauge propagator decreases at infinity as $\frac{1}{k^2}$.

This has been reformulated much more clearly by Fujikawa, Lee y Sanda (FLS) who introduced the so called *ξ-renormalizable gauge* which somewhat interpolates between both gauges, 't Hooft and unitarity.

The gauge fixing term reads as usual

$$L_{gf} = -\frac{1}{2\xi} F_a F^a \tag{6.50}$$

and is chosen in such a way that

$$F_a = \partial_\mu A^\mu_a - i\xi (T_a)^i_j \sigma_i v_j \tag{6.51}$$

Again, it cancels the gauge-scalar mixing

When $\xi \rightarrow \infty$ the unitarity gauge is recovered, whereas when $\xi = 0$ the FLS gauge reduces to the one of Landau,

$$\partial_\mu A^\mu_a = 0 \tag{6.52}$$

Under a gauge transformation the gauge fixing gives

$$\delta F_a = \Box \epsilon_a - i f_{abc} \partial_\mu (\epsilon_b A^\mu_c) + \xi (T_a v)_i \epsilon_b (T_b \phi)_i \tag{6.53}$$

This means that the ghost lagrangian is given by

$$L(c, b) = b_a (\Box c_a - i f_{abc} \partial_\mu (c_b A^\mu_c) + \xi (T_a v)_i c_b (T_b \phi)_i) \tag{6.54}$$

Putting together all the pieces, the full quadratic lagrangian then reads

$$L^{(2)} = -\frac{1}{4} \sum_a (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 - \frac{1}{2} \sum_{ab} \mu^2_{ab} A^a_\mu A^\mu_b - \frac{1}{2\xi} \sum_a (\partial_\mu A^\mu_a)^2$$
$$\frac{1}{2} \sum_i (\partial_\mu \sigma_i)^2 - \frac{1}{2} M^2_{ij} \sigma_i \sigma_j - \partial b_a \partial^\mu c_a - \xi \sum_{ab} \mu^2_{ab} b_a c_b$$

where

$$\mu_{ij}^2 = \sum_a (T_a v)_i (T_a v)_j$$
$$M_{ij}^2 = V_{ij} + \frac{\xi}{2}\mu_{ij}^2 \tag{6.55}$$

and

$$V_{ij} \equiv \left.\frac{\partial^2 V}{\partial\phi_i \partial\phi_j}\right|_v \tag{6.56}$$

and γ_i the Yukawa coupling matrix.

That is, ghosts get gauge dependent masses

$$m_{\text{gh}} = \sqrt{\xi} m_{\text{gauge}} \tag{6.57}$$

In the unitarity gauge $\xi \rightarrow \infty$ what happens is that the Goldtone bosons get so heavy that they decouple wheas the other bosonic mases remanin finite.

Gauge propagators (with the same ansatz as before) have to obey

$$A = (k^2 - \mu^2)^{-1}$$
$$\left(k^2 - (1 - \frac{1}{\xi})k^2 - \mu^2\right) \; B = (1 - \frac{1}{\xi}) \; A \tag{6.58}$$

namely

$$\Delta_{\mu\nu}^{ab} = \left[\frac{1}{k^2 - \mu^2}\left(\eta_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2 - \xi\mu^2}\right)\right]_{ab} \tag{6.59}$$

which has the good UV behavior except when $\xi \rightarrow \infty$. In practice almost always one works in Feynman gauge $\xi = 1$. The scalar propagator is to be found from the ansatz

$$\Delta = A \; \delta_{ij} + B_{ij} \tag{6.60}$$

and reads

$$\Delta_{ij} = \left[1 - \frac{\xi\mu^2}{k^2 - m^2 - \frac{1}{2}\xi\mu^2}\right]\frac{1}{k^2 - m^2} \tag{6.61}$$

It is possible to show that the non-physical ξ-dependent poles cancel between the gauge and scalar pieces.

The ghost propagator, on the other hand, reads

$$\Delta_{ab} = \left[\frac{1}{k^2 + \xi\mu^2}\right]_{ab} \tag{6.62}$$

7

BRST

7.1 The adjoint representation

Let us work out in detail the consequences of the Jacobi identity for the Lie algebra of a simple group. This states that for any three operators

$$[[T_a, T_b], T_c] + [[T_b, T_c], T_a] + [[T_c, T_a], T_b] = 0 \tag{7.1}$$

This implies

$$f_{abd} f_{dce} + f_{bcd} f_{dae} + f_{cad} f_{dbe} = 0 \tag{7.2}$$

Let us define now the antisymetric hermitian $d_G \times d_G$ matrices

$$(T^a)_{bc} = -(T^a)_{cb} \equiv i f_{bac} = (T^c)_{ab} = \left(T^b\right)_{ca} \tag{7.3}$$

Jacobi's identity then can be read as

$$-\left(T^b T^c\right)_{ae} + i f_{bcd}\left(T^d\right)_{ae} + \left(T^c T^b\right)_{ae} = 0 \tag{7.4}$$

that is

$$[T_c, T_b] = i f_{cbd}\, T_d \tag{7.5}$$

This is the d_G-dimensional *adjoint representation* of the Lie algebra. Many identities amongst structure constants are best understood in terms of the adjoint repesentation. A more intrinsic definition is as follows. Given two elements of the Lie algebra $X, Y \in \mathrm{Lie(G)}$, we define an endomorphism of L as

$$\mathrm{ad}\ X(Y) \equiv [X, Y] \in L \tag{7.6}$$

Given a normalized basis

$$X \equiv X^a T_a \tag{7.7}$$

then one can define

$$\mathrm{ad}\ X(Y) \equiv A_{ab}\, Y^b\, T^a \tag{7.8}$$

where

$$A_{cd} = i f_{cad} X^a \tag{7.9}$$

7.2 Symmetries of the gauge fixed action

The gauge fixed action, in spite of not being gauge invariant, still enjoys a symmetry, first uncovered by Becchi, Rouet y Stora, and independently, Tyutin.

Let us begin by parametrizing gauge fixing by auxiliary fields

$$L_{gf} \equiv B^a \partial_\mu A^\mu_a + \frac{\alpha}{2} B_a B^a \tag{7.10}$$

EM

$$\alpha B_a + \partial_\mu A^\mu_a = 0 \tag{7.11}$$

so that

$$L_{gf} = \left(-\frac{1}{\alpha} + \frac{\alpha}{2}\frac{1}{\alpha^2}\right)(\partial_\mu A^\mu_a)^2 = -\frac{1}{2\alpha}\left(\partial_\mu A^\mu_a\right)^2 \tag{7.12}$$

($\alpha = 0$ corresponds to the so-called *Landau gauge*). The ghost lagrangian reads

$$L_{gh} \equiv -i\partial_\mu \; b_a (D^\mu c)^a \equiv b^a M_{ab} c^b \tag{7.13}$$

The fields $c^a = (c^a)^+$ and $b^a = (b^a)^+$ (this is the field formely denoted by $\bar{c}_a$; we changed its name here for clarity) are independent hermitian fields, so that hey are not related by charge conjugation. The BRST symmetry reads

$$\begin{aligned} & sA^a_\mu = (D_\mu c)^a \\ & sB^a = 0 \\ & sc^a = -\frac{g}{2} f_{abc}\; c^b c^c \equiv -i\;\frac{g}{2}\; T^a_{bc}\; c^b c^c \\ & sb^a = -iB^a \end{aligned} \tag{7.14}$$

The gauge action is invariant, because on physical fields this is just a gauge transformation. On the gauge-fixing piece

$$sL_{gf} = B_a \partial^\mu (D_\mu c)^a \tag{7.15}$$

The ghost variation

$$sL_{gh} = \partial_\mu B^a (D^\mu c)_a + \partial_\mu b_a s[(D^\mu c)^a] \tag{7.16}$$

The BRST variation of the ghost covariant derivative vanishes

$$\begin{aligned} & s[(D_\mu c)^a] = s\big(\partial_\mu c^a + g f_{auv} A^u_\mu c^v\big) = s\big(\partial_\mu c^a + igT^a_{uv}\; A^u_\mu c^v\big) = \\ & -i\frac{g}{2}\partial_\mu (T^a_{uv}\; c^u c^v) + igT^a_{bc}\; (D_\mu c)^b\, c^c + igT^a_{bc}\; A^b_\mu \left(-\frac{g}{2}\right) iT^c_{uv}\; c^u c^v = \\ & = -i\frac{g}{2}\partial_\mu (T^a_{uv}\; c^u c^v) + igT^a_{bc}\; \partial_\mu\; c^b c^c - \\ & -g^2 T^a_{bc} T^b_{uv} A^u_\mu c^v c^c + \frac{g^2}{2} T^a_{bc}\; A^b_\mu T^c_{uv}\; c^u c^v \end{aligned} \tag{7.17}$$

We have represented here the hermitian matrices in the adjoint representation by

$$f_{abc} \equiv iT^a_{bc} = iT^b_{ca} = iT^c_{ab} \tag{7.18}$$

Lert us first study the terms containing derivatives

$$-\frac{ig}{2}T^a_{bc}\left(\partial_\mu(c^b c^c) - 2\partial_\mu c^b c^c\right) = -\frac{ig}{2}T^a_{bc}\left(\partial_\mu(c^b c^c) - \partial_\mu c^b c^c + \partial_\mu c^c c^b\right) = 0$$

The terms without derivatives read

$$\frac{g^2}{2}A^b_\mu c^u c^v \left(T^a_{bc}T^c_{uv} - T^a_{cv}T^c_{bu} + T^a_{cu}T^c_{bv}\right) \tag{7.19}$$

Now the term into parenthesis can be written as

$$-\left(T^a T^u\right)_{bv} + \left(T^u T^a\right)_{bv} - if_{uac}T^c_{bv} = 0 \tag{7.20}$$

Then the variation of the ghost lagrangian plus the variation of the gauge fixing term yields a total derivative, letting the action invariant.

This is a nilpotent symmetry

$$s^2_{BRST} = 0 \tag{7.21}$$

The only non-obvious piece is

$$s^2 c^a = s\left(-\frac{ig}{2}\;T^a_{bc}c^b c^c\right) = -\frac{g^2}{4}\;T^a_{bc}\left(T^b_{uv}c^u c^v c^c + c^b T^c_{uv}c^u c^v\right) =$$
$$= -\frac{g^2}{4}\;c^u c^v c^c\left(T^a_{bc}T^b_{uv} + T^a_{cb}T^b_{uv}\right) = 0 \tag{7.22}$$

owing to antisymmetry of the structure constants.

Besides the whole combination of ghosts plus gauge fixing is BRST exact

$$L_{gf} + L_{gh} = s\Psi \equiv -s\left[\bar{c_a}\partial_\mu A^\mu_a + \frac{\alpha}{2}b_a B_a\right] \tag{7.23}$$

7.3 The physical subspace.

Consider the standard Noether BRST current

$$J^{BRST}_\mu = \frac{\partial L}{\partial(\partial_\mu A^a_\rho)}\;sA^a_\rho + \frac{\partial L}{\partial(\partial_\mu B)}\;sB + \frac{\partial L}{\partial(\partial_\mu c)}\;sc + \frac{\partial L}{\partial(\partial_\mu b)}\;sb =$$
$$= F^a_{\mu\rho}D^\rho c^a + i\partial_\mu b^a\left(-\frac{g}{2}\right)f_{abc}c^b c^c - (D_\mu c)^a(-B^a) \tag{7.24}$$

and its corresponding charge

$$Q_{BRST} = \int d^3x\left[F^a_{0\rho}D^\rho c^a - i\frac{g}{2}\partial_0 b^a\;f_{abc}\;c^b c^c - (D_0 c)^a(-B^a)\right] =$$
$$= \int d^3x\Big[B_a(D_0 c)^a - \dot{B}^a c^a + \frac{ig}{2}f_{abc}\dot{b}^a c^b c^c\Big] \tag{7.25}$$

where the EM have been used

$$\begin{aligned}
&(D^\mu F_{\mu\nu})^a = \partial_\nu B^a - igf_{abc}\partial_\nu b_b c_c \\
&\partial^\mu A^a_\mu + \alpha B^a = 0 \\
&\partial^\mu (D_\mu c)^a = 0 \\
&(D^\mu \partial_\mu b)^a = 0
\end{aligned} \tag{7.26}$$

Let us now discuss in gory detail the reality of the ghost field. With

$$L_{gh} = -i\partial^\mu b_a D_\mu c^a \equiv -i\partial^\mu b^a \left(\partial_\mu c^a + f_{abc} A^b_\mu c^c\right) \tag{7.27}$$

If we assume

$$b_a = c_a^+ \tag{7.28}$$

then the first term (without the i in front) would be hermitian. But there is no way in which the second term (namely $-i\partial^\mu b^a\ f_{abc}A^b_\mu c^c$) can be self adjoint inless we assume that both ghost are already hermitian to begin with.

The Noether charge associated to the invariance of L_{fp} under ghost rescalings

$$\begin{aligned}
&c^a \rightarrow e^\lambda c^a \\
&b^a \rightarrow e^{-\lambda} b^a
\end{aligned} \tag{7.29}$$

is aptly named *ghost charge*

$$j^\mu_{gh} \equiv \sum \frac{\partial L}{\partial(\partial_\mu \phi)}\ \delta\phi \equiv -iD_\mu c^a(-\lambda b^a) + i\partial^\mu b \lambda c^a \tag{7.30}$$

$$Q_{gh} \equiv i\int d^3x (b^a (D_0 c)^a - \dot{b}^a c^a) \tag{7.31}$$

It is important to notice that this charge is hermiytian

$$Q^+_{gh} = -i\int d^3x \left(b_a D_0 c_a - \dot{b}_a\ c_a\right) = Q_{gh} \tag{7.32}$$

The ghost charge can be expressed in terms of canonical momenta

$$\begin{aligned}
&\pi^a \equiv \frac{\partial_L L}{\dot{c}_a} = i\dot{b}^a \\
&\text{and} \\
&\bar{\pi}^a \equiv \frac{\partial_L L}{\partial \dot{b} a} = -i(D_0 c)^a)
\end{aligned} \tag{7.33}$$

,

$$Q_{gh} = -\int d^3x \left[b^a \bar{\pi}_a + \pi_a c^a\right] \tag{7.34}$$

$$\{\pi^a(t,\vec{x}), c_b(t,\vec{y})\} = -i\delta_{ab}\delta^{(3)}(\vec{x}-\vec{y}) \tag{7.35}$$

and

$$\{\bar{\pi}^a(t,\vec{x}), b_b(t,\vec{y})\} = -i\delta_{ab}\delta^{(3)}(\vec{x}-\vec{y}), \tag{7.36}$$

so that eventually

$$\begin{aligned}[] [iQ_{gh}, c^a(x)] &= c^a(x) \\ [iQ_{gh}, b^a(x)] &= -b^a(x) \end{aligned} \tag{7.37}$$

The eigenvalues of Q_{gh} are imaginary ($\in i\mathbb{Z}$) in spite of the fact that the charge itself is hermitian. This fact was clarified by Kugo and Ojima and stems from the fact that our ghosts are hermitian fields.

Let us denotes eigenstates of the ghost charge by

$$G_{gh}|in\rangle \equiv in|in\rangle \tag{7.38}$$

Then owing to the fact that

$$Q_{gh}^{+} = Q_{gh} \tag{7.39}$$

$$\langle in|im\rangle = \langle in\left|\frac{Q_{gh}}{im}\right|im\rangle = \langle in\left|\frac{Q_{gh}}{-in}\right|im\rangle = \delta_{n+m} \tag{7.40}$$

The last equality is just a normalization. There are then null eigenvectors

$$\langle in|in\rangle = 0 \tag{7.41}$$

This conveys the fact that the scalar product in Fock space is not positive semidefinite. Defining

$$|\lambda\rangle \equiv |in\rangle + \lambda| - in\rangle \tag{7.42}$$

then

$$\langle\lambda|\lambda\rangle = 2Re\,\lambda \tag{7.43}$$

which does not have a definite sign, as advertised.

The commutation rules of the charge read

$$\begin{aligned} \{Q_{BRS}, Q_{BRS}\} &= 0 \\ [iQ_{fant}, Q_{BRS}] &= Q_{BRS} \end{aligned} \tag{7.44}$$

The BRST charge is exact

$$Q_{BRST} = -isQ_{gh} \tag{7.45}$$

which in turn means that

$$2Q_{BRST}^2 = \{Q_{BRST}, Q_{BRST}\} = sQ_{BRST} = s(-isQ_{gh}) = 0 \tag{7.46}$$

Let us now define the *physical subspace* as the set of all vectors in Fock's space which are annihilated by the BRST charge (this interpretation was first proposed again by Kugo and Ojima). This is a clever generalization of the old-fashioned Gupta-Bleuler condition in QED.

$$\mathcal{H}_{fis} \equiv \{|\Phi\rangle, \quad Q_{BRS}|\Phi\rangle = 0\} \tag{7.47}$$

We postulate that the vacuum is a physical state.

$$Q_{BRS}|0\rangle = 0 \tag{7.48}$$

Actually this is necessary in order to be able to prove that the ensuing theory is independent of the gauge fixing, or what is the same, of the odd quantity Ψ. Let us study the variation of an arbitrary overlapping under a variation of the gauge fixing condition.

$$\delta\langle u|v\rangle = \langle u|\delta S|v\rangle = \langle u|s\ \delta\Psi|v\rangle \equiv \langle u|[Q_{BRS}, \delta\Psi]|v\rangle \tag{7.49}$$

In order for this matrix element to vanish, it is necessary that

$$Q_{BRS}|u\rangle = Q_{BRS}|v\rangle = 0 \tag{7.50}$$

7.4 BRST for QED.

Let us study BRST in the simplest QED instance. Previous attempts to define a physical subspace by imposing constraings have difficulties. For example

$$\partial_\mu A^\mu \mid \Psi\rangle = 0 \tag{7.51}$$

This contradicts

$$[\partial_\mu A^\mu(x), A_\nu(y)] = i\partial_\mu D^{\mu\nu}(x-y) \neq 0 \tag{7.52}$$

This is the reason why Gupta and Bleuler proposed to impose only half of the constraint

$$(\partial_\mu A^\mu)^+ \mid \Psi\rangle = 0 \tag{7.53}$$

which implies

$$\langle\psi|\,(\partial_\mu A^\mu)^- = 0 \tag{7.54}$$

so that expectation values of the contraints vanish

$$\langle\Psi|\,\partial_\mu A^\mu \mid\Psi\rangle \equiv \langle\Psi \mid \left\{ (\partial_\mu A^\mu)^- + (\partial_\mu A^\mu)^- \right\} \mid \Psi\rangle = 0 \tag{7.55}$$

But even that has got problems with the coupling of the electromagnetic field to other fields. Let us now derive the elegant way in which this problem is solved in the BRST approach.

$$
\begin{aligned}
sA_\mu &= \partial_\mu c \\
sb &= \partial_\mu A^\mu \\
sc &= 0
\end{aligned}
\tag{7.56}
$$

Representing fields in Fock space

$$
\begin{aligned}
A_\mu(x) &= \int \frac{d^3p}{\sqrt{(2\pi)^3 2\omega_p}} \left[a_\mu(p) e^{-ipx} + a^+_\mu(p) e^{ipx} \right] \\
b(x) &= \int \frac{d^3p}{\sqrt{(2\pi)^3 2\omega_p}} \left[b(p) e^{-ipx} + b^+(p) e^{ipx} \right] \\
c(x) &= \int \frac{d^3p}{\sqrt{(2\pi)^3 2\omega_p}} \left[c(p) e^{-ipx} + c^+(p) e^{ipx} \right]
\end{aligned}
\tag{7.57}
$$

Let us state some facts.

- Given an arbitrary physical state

$$
Q_{BRS}|\chi\rangle = 0 \tag{7.58}
$$

the state with a transverse photon added

$$
|\epsilon\,\chi\rangle \equiv \epsilon_\mu a^{\mu +}|\chi\rangle \tag{7.59}
$$

is a physicsl state as well, provided

$$
\epsilon p = 0 \tag{7.60}
$$

because

$$
[Q_{BRST}, a^\mu(p)] = -p^\mu c(p) \tag{7.61}
$$

and so

$$
Q_{BRST}|\epsilon\,\chi\rangle Q_{BRST}\epsilon_\mu a^{\mu +}|\chi\rangle = \epsilon_\mu \left[Q_{BRST}, a^{\mu +}(p) \right] = (\epsilon.p)\;\; c|\chi\rangle = 0 \tag{7.62}
$$

- On the other hand,

$$
\{Q_{BRST}, b^+(p)\} = p^\mu a^+_\mu(p) \tag{7.63}
$$

which implies:

$$
Q_{BRST} b^+|\chi\rangle = p^\mu a^+_\mu|\chi\rangle \tag{7.64}
$$

This just means that the polarization ϵ_μ is physically equivalent to the polarization $\epsilon_\mu + \lambda p_\mu$,because the difference is BRST-exact.

- Antighosts are not physical

$$Q_{BRST}b^{+}|\chi\rangle = p^{\mu}a_{\mu}^{+}(p)|\chi\rangle \neq 0 \tag{7.65}$$

- Ghosts are BRST exact, owing to the fact that

$$\left[Q, a_{\mu}^{+}(p)\right] = p_{\mu}c^{+}(p) \tag{7.66}$$

Then

$$c^{+}(p)|\chi\rangle = \frac{1}{\epsilon.p}\, Q_{BRST}\ \epsilon^{\mu}a_{\mu}^{+}|\chi\rangle \tag{7.67}$$

7.5 Positiveness

Three problems which are always difficult to solve in any covariant formalism of gauge theories are as follows First of all, show that the hamiltonian in the full Fock space is self-adjoint

$$H = H^{+} \tag{7.68}$$

Second, show that there is an invariant subpsace of the Fock space, the *physical subspace*, invariant under the evolution operator

$$H\mathcal{H}_{\text{phys}} \subset \mathcal{H}_{\text{phys}} \tag{7.69}$$

Third, prove that the scalar product in the physical subspace is positive semidefinite

$$\{|\psi\rangle \in \mathcal{H}_{\text{phys}}\} \Rightarrow \langle\psi|\psi\rangle\ \geq 0 \tag{7.70}$$

In practice, all that we shall be able to prove is that there are indeed states that violate positivity, but thosed atates appear only in null-norm combinations.

Owing to BRST nilpotency, all *BRS-exact states* ,

$$|\psi\rangle = Q_{BRS}|\phi\rangle \tag{7.71}$$

are physical states according to our definition. Nevertheless, all those are orthogonal to any other physical state, since

$$\langle phys|\psi\rangle = \langle phys|Q_{BRS}|\phi\rangle = 0 \tag{7.72}$$

and besides, they have zero norm

$$\langle\psi|\psi\rangle = \langle\psi|Q_{BRS}|\phi\rangle = 0 \tag{7.73}$$

Treating those states as trivial states is akin to definig an equivalence relation: two states are physically equivalent provided their difference is BRST exact.

The BRST cohomology is defined as

$$H(Q) \equiv ker\, Q_{BRS}/Im\, Q_{BRS} \tag{7.74}$$

The kernel of the BRST operator

$$ker\, Q_{BRS} \equiv \{|\psi\rangle, Q_{BRS}|\psi\rangle = 0\}, \tag{7.75}$$

The image of the BRST operator is the set of vectors such that

$$\mathrm{Im}\, Q_{BRS} \equiv \{|\chi\rangle = Q_{BRS}|\kappa\rangle\}. \tag{7.76}$$

- The set of elements in $H(Q)$ will be dubbed *singlet states* . We shall denote the sunset of ghpst number n as $H^n(Q)$.

- Singlet states can have vanishing ghost number $|\chi_0\rangle \in H^0(Q)$ (those are the ones that correspond to physical particles *sensu stricto*) or else a non-vanishing one, $|\chi_n\rangle \in H^n(Q)$.

- In the latter case they can be a *unpaired singlet* $|\chi_n\rangle$. This means that there is no physical states among the ghost number complements of the type $|\sigma_{-n}\rangle$, and such that $\langle\chi_n|\sigma_{-n}\rangle = 1$. Then it is not possible either that a descendent state

$$Q|\xi_{n-1}\rangle \tag{7.77}$$

has got nonvanishing scalar product with our unpaired singlet, because if this were true, then

$$\langle\xi_{-n-1}|Q|\chi_n\rangle = 0 \tag{7.78}$$

owing to the fact that $|\chi_n \in H(Q)$.

This in turn means that no such complement state is in $H(Q)$, so that the restriction of the scalar product to $H(Q)$ is degenerate in the sense that this unpaired singlet is orthogonal to the whole $H(Q)$. This is considered not to be a harmful situation.

- There can slso be a *singlet pair*, with equal and opposite ghost number. There are then physical states of the form

$$|\xi\rangle \equiv |\chi_n\rangle + \xi|\sigma_{-n}\rangle \tag{7.79}$$

which have negative norm.

$$\langle\chi_n + \xi\sigma_{-n} \mid \chi_n + \xi\sigma_{-n}\rangle = \xi + \bar{\xi} \tag{7.80}$$

Then a consistent reduction of the Fock space is not possible.

This situation is however not realized in ordinary gauge theories.

The reason stems from a small theorem asserting that in any cohomology class in $H(Q)$ there is a representative with vanishing ghost number. Indeed, any physical state with ghost number $-k$ can be represented as

$$|-k\rangle \equiv b_{a_1}\ldots b_{a_k} t^{[a_1\ldots a_k]}|g\rangle + \ldots \tag{7.81}$$

where the state $|g\rangle$ does not involve antighosts, so that it has vanishing ghost number. We are then concentrating in the component with the minimum possible number of antighosts. Other components have more antighosts (and also more ghosts to keep even the balance).

Now, we know that this whole state is annihilated by the BRST charge. The component with $N_{gh} = -(k-1)$ must then annihilate by itself. Recall that

$$sb_a = [iQ_{BRS}, b_a] = iB_a \tag{7.82}$$

then

$$0 = Q_{BRS} b_{a_1} b_{a_2}\ldots b_{a_k} t^{[a_1\ldots a_k]}|g\rangle \sim B_{a_1} b_{a_2}\ldots b_{a_k} t^{[a_1\ldots a_k]}|g\rangle \tag{7.83}$$

This is possible in general only if

$$b_{a_2}\ldots b_{a_k} t^{[a_1\ldots a_k]} = B_{a_0} b_{a_2}\ldots b_{a_k} u^{[a_0 a_1\ldots a_k]} \tag{7.84}$$

so that the whole thing cancels by antisymmetry Then

$$|-k\rangle = b_{a_1} B_{a_0} b_{a_2}\ldots b_{a_k} u^{[a_0 a_1\ldots a_k]}|g\rangle = Q_{BRS} b_{a_1} b_{a_0} b_{a_2}\ldots b_{a_k} u^{[a_0 a_1\ldots a_k]}]|g\rangle \tag{7.85}$$

This shows that the component with minimum possible number of antighosts is exact. Induction proceeds until we reach a representative without any antighost.

- All non-physical states $|\lambda_n\rangle$ give rise to a BRST doublet

$$\{|\lambda_n\rangle, |\delta_{n+1}\rangle \equiv Q_{BRS}|\lambda_n\rangle\}, \tag{7.86}$$

where the other element of the doublet is BRST exact (and then nilpotent). It the scalar product is to be non-degenerate there must exist at least one state with non-vanishing scalar product with it. This state has to have opposite ghost number

$$\langle\delta_{n+1}|\delta_{-1-n}\rangle = 1 \tag{7.87}$$

Finally, $|\lambda_{-n}\rangle \equiv Q_{BRS}|\delta_{-1-n}\rangle$ is the last member of this *quartet*. These do appear in ordinary gauge theories

It can be the case that all physical states can be described in terms of asymptotic fields, that is *asymptotic completeness* (this does not happen in unbroken non-abelian gauge theories owing to confinement, but this fact is unfortunately beyond BRST techniques).

The operators that create these states out of the vacuum

$$\begin{aligned}
&|\lambda_n\rangle \equiv a_n^+|0\rangle \\
&|\delta_{n+1}\rangle \equiv Q_{BRS}|\lambda_n\rangle = -ic^+_{n+1}|0\rangle \\
&|\delta_{-(n+1)}\rangle \equiv -b^+_{-n-1}|0\rangle \\
&|\lambda_{-n}\rangle \equiv Q_{BRS}|\delta_{-(n+1)}\rangle = -b^+_{-n}|0\rangle
\end{aligned} \tag{7.88}$$

We can assume with no loss of generality that n is even, $n \in 2\mathbb{Z}$.

A quartet that always appear in ordinary gauge theories is generated by the pure gauge fields

$$\begin{aligned}
&|\lambda_{n=0}\rangle \sim A^a_\mu(x) \sim \partial_\mu a^a(x) + \ldots \\
&|\delta_1\rangle \sim (D_\mu c)^a \sim \partial_\mu c^a(x) + \ldots \\
&|\delta_{-1}\rangle \sim b^a(x) \sim b^a(x) + \ldots \\
&|\delta_{(-n=0)}\rangle \sim B^a(x) \sim b^a(x) + \ldots
\end{aligned} \tag{7.89}$$

Let us now show that quartets appear only as zero norm combinations. They are then undetectable.

- It is a fact that asymptotic states corresponding to physical particles are always orthogonal to quartets.

  The only dangerous matrix element is the overlap of a physical state with one of those animals

  $$\langle phys|\lambda_n\rangle \tag{7.90}$$

  and this is possible when $n = 0$ only.

  But in this case we can choose another representative of the cohomology class such that

  $$|PHYS\rangle \equiv |phys\rangle - \sum_m |\lambda_{-m}\rangle\langle\lambda_m|phys\rangle = |phys\rangle - Q_{BRS}\sum_m |\delta_{-m-1}\rangle\langle\lambda_m|phys\rangle \tag{7.91}$$

  clearly

  $$\langle\lambda_n|PHYS\rangle = \langle\lambda_n|phys\rangle - \sum_m \langle\lambda_n|\lambda_{-m}\rangle\langle\lambda_m|phys\rangle = 0. \tag{7.92}$$

  because

  $$\langle\lambda_n|\lambda_{-m}\rangle = \delta_{nm} \tag{7.93}$$

- Let us now define projectors (details can be found in Kugo-Ojima's [?]) in the subspace of $\mathcal{H}_{phys}$ with N quartets.

$$\begin{aligned} &\mathcal{P}_N^2 = \mathcal{P}_N^+ = \mathcal{P}_N \\ &\mathcal{P}_N\mathcal{P}_M = \mathcal{P}_M\mathcal{P}_N = \delta_{NM}\mathcal{P}_N \\ &\sum_N \mathcal{P}_N = 1 \end{aligned} \tag{7.94}$$

Those projectors are BRST invariant

$$[Q_{BRS}, \mathcal{P}_N] = 0 \tag{7.95}$$

and there is a resolution of the identity in the form

$$1 - \mathcal{P}_0 = \{iQ_{BRS}, \mathcal{R}\} \tag{7.96}$$

Ths subspace of genuine physical particles is the projection into the subspace without any quartet from the whole physical space

$$H^0_{phys} \equiv \mathcal{P}_0\mathcal{H}_{phys} \tag{7.97}$$

- It is then a fact that any BRST closed state

$$|f\rangle \in \mathcal{P}_N H \tag{7.98}$$

with $N \neq 0$ is necessarily exact. This is indeed plain because

$$|f\rangle = \mathcal{P}_N|f\rangle \Rightarrow (1-\mathcal{P}_0)|f\rangle = |f\rangle = \{iQ_{BRS}, \mathcal{R}\}|f\rangle \tag{7.99}$$

that is

$$|f\rangle \sim \ \mathcal{R}Q_{BRST}|f\rangle \equiv |0\rangle \tag{7.100}$$

8

Ward identities

8.1 The equations of motion.

The postulate that the functional integral of a functional derivative vanishes implies the equations of motion (Schwinger Dyson, plus contact terms.

$$\int \mathcal{D}\phi \, \frac{\delta}{\delta\phi(x)} \left\{ e^{i\left[S[\phi]+\int d(vol)\; J(x)\phi(x)\right]} \right\} = 0 \tag{8.1}$$

namely

$$\int \mathcal{D}\phi \, \left(\frac{\delta S}{\delta\phi(x)} + J(x) \right) \; e^{i\left[S[\phi]+\int d(vol)\; J(x)\phi(x)\right]} = 0 \tag{8.2}$$

Which implies for example that

$$\int \mathcal{D}\phi \, \frac{\delta}{\delta J(y)} \left\{ \left(\frac{\delta S}{\delta\phi(x)} + J(x) \right) \; e^{i\left[S[\phi]+\int d(vol)\; J(x)\phi(x)\right]} \right\} = 0 \tag{8.3}$$

That is

$$\langle 0|T\frac{\delta S}{\delta\phi(x)} \; \phi(y)|0\rangle + \delta^n(x-y) + \langle 0|J(x)\phi(y)|0\rangle = 0 \tag{8.4}$$

8.2 Ward

When the lagrangian enjoys a symmetry, some manipulations on the fuctional integral imply relationships between Green's functions. Those identities are generically known as Ward identities and are the quantum mechanical version of Noether's theorem

Assume the action to be invariant under rigid ($\partial_\mu \epsilon = 0$) transformations of the form

$$\phi_i'(x) = \phi_i(x) + \epsilon\delta\phi_i(x) = \phi_i(x) + \epsilon t_i{}^j \phi_j(x) \tag{8.5}$$

This means that under local ($\partial_\mu \epsilon \neq 0$) transformations

$$S[\phi'] = S[\phi] + \int d^n x \; J^\mu_{Noether}(x) \; \partial_\mu \epsilon. \tag{8.6}$$

Assuming that the functional measure is also invariant the partiction function reads

$$Z' = \int \mathcal{D}\phi' \; e^{iS[\phi'] + i\int J^i \phi'_i} = Z = \int \mathcal{D}\phi \; e^{iS[\phi] + i\int d^n x \; J^i \phi_i} \times$$
$$\times \left[1 + i \int d^n x \; \epsilon(x) \; \partial_\mu J^\mu_N(x) + i \int d^4 x \; \epsilon(x) \; J^i t_i^j \phi_j \right] \tag{8.7}$$

Functionaly differentiating with respect to $J(x_1) \ldots J(x_n)$ and evaluating the result at the point when all sources vanish $J = 0$),

$$\langle 0_+ | T\phi_{i_1}(x_1) \ldots \phi_{i_n}(x_n) \int d^4 x \; \epsilon(x) \; \partial_\mu J^\mu_{N.}(x) | 0_- \rangle =$$
$$= \sum_{k=1}^{k=n} \epsilon(x_k) \langle 0_+ | T\phi_{i_1}(x_1) \ldots \delta\phi_{i_k}(x_k) \ldots \phi_{i_n}(x_n) | 0_- \rangle \tag{8.8}$$

which can be rewritten as

$$\partial^\mu_x \langle 0_+ | T\phi_{i_1}(x_1) \ldots \phi_{i_n}(x_n) \partial_\mu J^\mu_N(x) | 0 \rangle = \sum_{k=1}^{k=n} \delta(x - x_k) \langle 0 | T\phi_{i_1}(x_1) \ldots \delta\phi_{i_k}(x_k) \ldots \phi_{i_n}(x_n) | 0_- \rangle \tag{8.9}$$

Please note that this procedure yields nothing new (over the rigid case) for gauge theories.

8.3 Charge conservation

Let us work the simplest example in detail. When dealing with complex scalar fields

$$\phi \equiv \frac{1}{\sqrt{2}} (\phi_2 + i\phi_2) \tag{8.10}$$

charge conservation stems from the phase symmetry

$$\delta\phi = i\alpha(x)\phi \tag{8.11}$$

The Noether current reads

$$J_\mu = i \left(\partial_\mu \phi \phi^* - \phi \partial_\mu \phi^* \right) \tag{8.12}$$

The Ward hierarchy of identities stems from

$$\partial^\mu \langle 0_+ | T\phi_{i_1}(x_1) \ldots \phi_{i_n}(x_n) i \left(\partial_\mu \phi(x) \phi^*(x) - \phi(x) \partial_\mu \phi^*(x) \right) \phi^*(y_1) \ldots \phi^*(y_m) | 0_- \rangle_J =$$
$$= \sum_{k=1}^{k=n} \delta(x - x_k) \langle 0_+ | T\phi_{i_1}(x_1) \ldots \phi_{i_k}(x_k) \ldots \phi_{i_n}(x_n) \phi^*(y_1) \ldots \phi^*(y_m) | 0_- \rangle_J +$$
$$- \sum_{l=1}^{l=m} \delta(x - y_k) \langle 0_+ | T\phi_{i_1}(x_1) \ldots \phi_{i_k}(x_k) \ldots \phi_{i_n}(x_n) \phi^*(y_1) \ldots \phi^*(y_l) \phi^*(y_m) | 0_- \rangle_J \tag{8.13}$$

8.4 QED

- Let us consider the particularly important case of QED abelian gauge invariance. The reason why this gives a nontrivial result is that there is a associated rigid symmetry acting on the charged matter, namely

$$\begin{aligned}\delta\psi &= ie\psi\\ \delta\bar{\psi} &= -ie\bar{\psi}\end{aligned} \tag{8.14}$$

Noether's current reads

$$j_N^\mu(x) = i\bar{\psi}(x)\gamma^\mu\psi(x) \tag{8.15}$$

and Ward's identity in the simplest nontrivial instance reads

$$\begin{aligned}&\langle 0|T\partial_\mu j_N^\mu(x)\psi(y)\bar{\psi}(z)|0\rangle = \delta^{(4)}(x-y)\langle 0|Tie\psi(y)\bar{\psi}(z)|0\rangle +\\ &\delta^{(4)}(x-z)\langle 0|T\psi(y)(-ie)\bar{\psi}(z)|0\rangle\end{aligned} \tag{8.16}$$

Fourier transforming $x_1 = x - z$ y $x_2 = y - z$ and defining

$$S_F(p) \equiv \int d^4x e^{ipx}\langle 0|T\psi(x)\bar{\psi}(0)|0\rangle \tag{8.17}$$

we easily derive:

$$\begin{aligned}&p_1^\mu G_\mu(p_1,p_2) \equiv \int d^4x_1 d^4x_2 e^{i(p_1x_1+p_2x_2)}\langle 0|T\partial_\mu j_N^\mu(x_1)\psi(x_2)\bar{\psi}(0)|0\rangle =\\ &= iS_F(p_1+p_2) - iS_F(p_2)\end{aligned} \tag{8.18}$$

and defining the amputated function

$$\Gamma_\mu(p_1,p_2) \equiv S_F^{-1}(p_1+p_2)G_\mu(p_1,p_2)S_F^{-1}(p_2) \tag{8.19}$$

The QED Ward's identity reduces to

$$p_1^\mu\Gamma_\mu(p_1,p_2) = iS_F^{-1}(p_2) - iS_F^{-1}(p_1+p_2) \tag{8.20}$$

This is the old-fashioned form of the identity found for example, in Bjorken and Drell's book . It can be written in a more symmetric form as The QED Ward's identity reduces to

$$(p^\mu - q^\mu)\,\Gamma_\mu(p_1,p_2) = iS_F^{-1}(q) - iS_F^{-1}(p) \tag{8.21}$$

In the limit when $p \to q$ this is equivalent to

$$\Gamma_\mu(q) = i\frac{\partial}{\partial q^\mu}\, S_F^{-1}(q) \tag{8.22}$$

This implies immediatly that

$$Z_1 = Z_2 \tag{8.23}$$

- A natural question to ask is whether the gauge symmetry does not have any more direct consequances for the photon sector. The short answer is yes. Transversality. Consider the photon four-momentum

$$k^\mu = E\,(1,0,0,1) \tag{8.24}$$

In this frame the two physical polarizations of the photon are just

$$\begin{aligned}\epsilon_1 &\equiv (0,1,0,0)\\ e_2 &\equiv (0,0,1,0)\end{aligned} \tag{8.25}$$

Now what happens is thet the little group does not leave invariant the subspace (ϵ_1, ϵ_2). Consider for example the Lorentz transformation

$$L^\mu{}_\nu \equiv \begin{pmatrix} \frac{3}{2} & 1 & 0 & -\frac{1}{2} \\ 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ \frac{1}{2} & 1 & 0 & \frac{1}{2} \end{pmatrix} \tag{8.26}$$

First of all, $L \in SO(1,3)$ because

$$L^T \eta L = \begin{pmatrix} \frac{3}{2} & 1 & 0 & \frac{1}{2} \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{2} & -1 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{3}{2} & 1 & 0 & -\frac{1}{2} \\ 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ \frac{1}{2} & 1 & 0 & \frac{1}{2} \end{pmatrix} =$$
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \equiv \eta \tag{8.27}$$

It also belongs to the little group, because

$$L^\mu{}_\nu k^\nu = k^\mu \tag{8.28}$$

But

$$L^\mu{}_\nu \epsilon_1^\nu = \epsilon_1^\mu + E^{-1} k^\mu \tag{8.29}$$

(It leaves invariant the subspace generated by ϵ_2 though). Physical amplitudes are of the form

$$\mathcal{M} \equiv \epsilon^\mu \mathcal{M}_\mu \tag{8.30}$$

where

$$\epsilon^\mu = C_1 \epsilon_1^\mu + C_2 \epsilon_2^\mu \tag{8.31}$$

Under a Lorentz transformation

$$\mathcal{M}' = \left(C_1 \left(\epsilon_1^\mu + E^{-1} k^\mu\right) + C_2 \epsilon_2^\mu\right) \mathcal{M}'_\mu \tag{8.32}$$

But there are no physical states with longitudinal polarizations. The only way out is that

$$k^\mu \mathcal{M}_\mu = 0 \tag{8.33}$$

That is physical amplitudes involving physical photons must be transverse.

- In real life, one has to take into account gauge fixing terms and ghosts. To be specific, with the gauge fixing term

$$L_{gf} \equiv \frac{1}{2\alpha}(\partial_\mu A^\mu)^2 \tag{8.34}$$

ghosts do not couple to the physical fields and can be ignored.

Under a gauge transformation the only non-invariance comes from sources as well as gauge fixing. This implies a Ward identity as usual

$$\left(-\frac{1}{2\alpha}\Box\partial_\mu\frac{\delta}{\delta J_\mu(x)} + \bar{\eta}\frac{\delta}{\delta\bar{\eta}(x)} - \eta\frac{\delta}{\delta\eta(x)}\right)W[J] + \partial_\mu J^\mu = 0 \tag{8.35}$$

Legendre transforming we arrive to Ward's identity in terms of the 1PI effective action

$$\left(\partial_\mu\frac{\delta}{\delta A_\mu(x)} + \psi\frac{\delta}{\delta\psi(x)} - \bar{\psi}\frac{\delta}{\delta\bar{\psi}(x)}\right)\Gamma[A] + \Box\partial_\mu A^\mu = 0 \tag{8.36}$$

We have denoted classical fields as if they were ordinary fields in order not to clutter the notation unnecessarily.

8.5 Non-abelian Ward (Slavnov-Taylor) identities.

Let us now include sources for some composite operators, namely

$$L \to L(A,b,c) + \int d^4x\left[J^a_\mu A^\mu_a + \bar{\xi}_a c_a + b_a\xi_a + \chi_a B^a + K^{a\mu}(D_\mu c)_a - \frac{1}{2}L_a f_{abc}c_b c_c\right] \tag{8.37}$$

Under al BRST transformation, only the trems including sources are noninvariant

$$\int d^4x\langle 0|T J_{a\mu}(D_\mu c)_a + \bar{\xi}(-\frac{1}{2}f_{abc}c_b c_c) - B_a\xi_a|0\rangle_J = 0 \tag{8.38}$$

which can be rewritten

$$\int d^4x\left[J_{a\mu}\frac{\delta}{\delta K_{a\mu}} + \bar{\xi}\frac{\delta}{\delta L_a} - \xi_a\frac{\delta}{\delta\chi_a}\right]Z(J) = 0 \tag{8.39}$$

Functionally deriving with respect to sources, a whole hierarchy of equations relating different Green functions is easily derived. These were first obtained through a much more involved argument by Slavnov and Taylor.

The antighost EM is easily obtained by performing a translation in the functional integral

$$b \to b + \Delta b \tag{8.40}$$

or else through the fact that the functional integral of a total functional derivative vanishes

$$\int \mathcal{D}A\mathcal{D}b\mathcal{D}c \, \frac{\delta}{\delta b(x)} \, e^{iS} = 0 \tag{8.41}$$

$$\langle 0 \, | T M_{ab} c_b + \xi_a | \, 0 \rangle_J = 0 \tag{8.42}$$

In terms of the partition function

$$\left[M_{ab} \left(\frac{\delta}{i \delta J} \right) c_b + \xi_a \right] \; Z(J) = 0 \tag{8.43}$$

and taking into account that

$$M_{ab} c_b = \partial_\mu (Dc)^a \tag{8.44}$$

(in the standard gauge $F_a = \partial_\mu A^\mu_a$), reads

$$\left[\frac{1}{i} \partial_\mu \frac{\delta}{\delta K_{a\mu}}(x) + \xi_a(x) \right] Z(J) = 0 \tag{8.45}$$

8.6 Renormalizaction of non-abelian gauge theories

Let us study the simplest case in which the gauge fixing condition is linear in the gauge fields. Other more complicated cases are treated similarly.

Let us recall Ward's identity (8.45) for the free energy $W \equiv i \log Z$ in the covariant gauge

$$F_a \equiv \partial_\mu A^\mu_a \tag{8.46}$$

to wit

$$\partial_\mu \frac{\delta W}{\delta K_{a\mu}(x)} = \xi_a(x) \tag{8.47}$$

Start by defining as usual *classical fields*

$$\begin{aligned}
A^{(cl)}_{a\mu}(J) &= \frac{\delta W}{\delta J_{a\mu}} \\
c^{(cl)}_a(J) &= \frac{\delta^L W}{\delta \bar{\xi}_a} \\
b^{(cl)}_a(J) &= -\frac{\delta^L W}{\delta \xi_a}
\end{aligned} \tag{8.48}$$

Classical fields are nothing else than the vacuum expectation value of the field operator in the presence of external sources. Legendre transforming leads to the effective action (the generator of 1PI Green functions)

$$\Gamma(A, c, b, K, L) \equiv W(J, \xi, \bar{\xi}, K, L) - \int d^4x (JA + \bar{\xi}c + b\xi) \tag{8.49}$$

Classical fields are represented by the same symbol as ordinary fields; this should not lead to any confusion, because in this section only classical fields will be discussed

$$\begin{aligned}
A_{a\mu} &= \frac{\delta W}{\delta J_{a\mu}} \qquad J_{a\mu} = -\frac{\delta \Gamma}{\delta A_{a\mu}} \\
c_a &= \frac{\delta W}{\delta \bar{\xi}_a} \qquad \bar{\xi}_a = \frac{\delta \Gamma}{\delta c_a} \\
b_a &= -\frac{\delta W}{\delta \xi_a} \qquad \xi_a = \frac{\delta \Gamma}{\delta b_a}
\end{aligned} \tag{8.50}$$

Given the fact that

$$\frac{\delta\Gamma}{\delta A(x)} \equiv \int d^4y \frac{\delta W}{\delta J(y)}\frac{\delta J(y)}{\delta A(x)} - \int d^4y \frac{\delta J(y)}{\delta A(x)} A(y) - J(x)$$
$$\frac{\delta\Gamma}{\delta c_x} \equiv \int d^4y \frac{\delta\bar{\xi}(y)}{\delta c(x)}\frac{\delta W}{\delta\bar{\xi}(y)} - \int d^4y \frac{\delta\bar{\xi}(y)}{\delta c(x)} c(y) + \bar{\xi}(x)$$
$$\frac{\delta\Gamma}{\delta b_x} \equiv \int d^4y \frac{\delta\xi(y)}{\delta b(x)}\frac{\delta W}{\delta\xi(y)} - \int d^4y b(y)\frac{\delta\xi(y)}{\delta b(x)} - \xi(x) \tag{8.51}$$

From the very definition it is plain that derivatives with respect to any parameter not involved in Lagendre's transform are the same for the effective action as for the free energy

$$\frac{\delta W}{\delta P(x)} = \frac{\delta\Gamma}{\delta P(x)} \tag{8.52}$$

This means that Ward's identity

$$\int d^4x \left[J_{a\mu}\frac{\delta}{\delta K_{a\mu}(x)} + \bar{\xi}\frac{\delta}{\delta L_a(x)} + \xi_a \frac{\delta}{\delta \chi_a(x)}\right] Z(J) = 0$$

can be written in terms of the effective action

$$\int d^4x \left(-\frac{\delta\Gamma}{\delta A_{a\mu}}(x)\frac{\delta\Gamma}{\delta K_{a\mu}(x)} + \frac{\delta\Gamma}{\delta c_a(x)}\frac{\delta\Gamma}{\delta L_a(x)} - \frac{\delta\Gamma}{\delta b_a(x)}\partial_\mu A^\mu_a\right) = 0$$
$$\frac{\delta\Gamma}{\delta b_a(x)} + \partial_\mu \frac{\delta\Gamma}{\delta K_{a\mu}(x)} = 0 \tag{8.53}$$

We have identified B_a with $-F_a = -\partial_\mu A^\mu_a$, which stems from the EM. It is possible to simplify the preceding identities by a slight modification in the effective action :

$$\tilde{\Gamma} \equiv \Gamma - \frac{1}{2}\int d^4x \, (\partial_\alpha A^\alpha_a)^2 \tag{8.54}$$

by defining the star product (Zinn-Justin)

$$\tilde{\Gamma} * \tilde{\Gamma} \equiv \int d^4x \left(-\frac{\delta\tilde{\Gamma}}{\delta A_{a\mu}(x)}\frac{\delta\tilde{\Gamma}}{\delta K_{a\mu}(x)} + \frac{\delta\tilde{\Gamma}}{\delta c_a(x)}\frac{\delta\tilde{\Gamma}}{\delta L_a(x)}\right) = 0$$
$$\partial_\mu \frac{\delta\tilde{\Gamma}}{\delta K_{a\mu}(x)} + \frac{\delta\tilde{\Gamma}}{\delta b_a(x)} = 0 \tag{8.55}$$

When working woth the modified action, $\tilde{\Gamma}$, the only thing that changes is

$$\frac{\delta\tilde{\Gamma}}{\delta A^a_\mu(x)} = \frac{\delta\Gamma}{\delta A^a_\mu(x)} - \Box A^\mu_a \tag{8.56}$$

The second equation of the set above remains the same, whereas the first one wins an extra term

$$\partial_\alpha A^\alpha_a \ \partial_\mu \frac{\delta\Gamma}{\delta K_{a\mu}} = -\partial_\mu \frac{\delta\Gamma}{\delta b_a(x)} \tag{8.57}$$

(where the second equation gas been used). This trick then leaves the Zinn-Justin equation without any linear terms.

Let us now sketch an inductive argument in order to show that the gauge lagrangian

$$L = L_{gauge} + L_{gf} + L_{gh} \tag{8.58}$$

is stable under renormalization.

Divergences in the $\hbar^{n+1}$ loop order Γ_{n+1} can be cancelled just by adding to $\tilde{\Gamma}$ a counterterm such that the renormalized action $\tilde{\Gamma}$ to the desired order $\hbar^{n+1}$ obeys Ward's identities. In that way, Ward's identities can be shown to be stable under renormalization.

It then will follow that the renormalized action $\tilde{\Gamma}_{(ren)}$ is the most general dimension four local polynomial obeyng Ward's identities.

Remembering the dimensions of the different fields involved

$$\begin{aligned} & d(A) = d(c) = d(\xi) = 1 \\ & d(K) = d(L) = d(B) = d(\chi) = 2 \\ & d(J) = 3 \end{aligned} \tag{8.59}$$

Also the ghost number asignments read $gh(K) = -1$ and $gh(L) = -2$.

Dimensional analysis and ghost number conservation then imply that

$$\tilde{\Gamma}_{(ren)} = \int d^4x \left(K_{a\mu} (D^{(ren)}_\mu (A) c)_a - \frac{1}{2} f^{(ren)}_{abc} L_a c_b c_c + L^{(ren)}(A, c, b) \right) \tag{8.60}$$

where the renormalized covariant derivative is defined as

$$(D^{ren}_\mu c)_k = \partial_\mu c_k + f^{ren}_{kbj} A^b_\mu c^j \tag{8.61}$$

and is given in terms of some renormalized constants f^{ren}_{abc}.

The second identity (8.55) then implies

$$\partial^\mu (D^{(ren)}_\mu c)_a(x) + \int d^4y \frac{\delta L^{(ren)}(x)}{\delta b_a(y)} = 0 \tag{8.62}$$

so that the antighost dependence is fixed. The dependence on ghosts then follows by ghost number conservation.

$$\begin{aligned} \tilde{\Gamma}_{(ren)} = \int d^4x \ \Big(& K_{a\mu} (D^{(ren)}_\mu (A) c)_a - \frac{1}{2} f^{(ren)}_{abc} L_a c_b c_c - \\ & - b_a \partial_\mu (D^{(ren)}_\mu c)^a + L^{(ren)}(A) \Big) \end{aligned} \tag{8.63}$$

The first Ward identity of the set (8.55) leads to

$$\int d^4x \left[-\left(K_{a\mu} \frac{\delta D^{(ren)}_{ab\mu}}{\delta A_{k\lambda}} c_b + \frac{\delta L^{(ren)}}{\delta A_{k\lambda}} - b_a \partial_\mu \frac{\delta D^{(ren)}_{ab\mu}}{\delta A_{k\lambda}} c^b \right) D^{(ren)}_{\lambda kj} c_j \right.$$
$$\left. -\frac{1}{2} f^{(ren)}_{kij} c_i c_j \Big(K_{a\mu} D^{(ren)}_{ak\mu} - f^{(ren)}_{akc} L_a c_c + b_a \partial_\mu D^{(ren)}_{ak\mu} \Big) \right] = 0 \qquad (8.64)$$

All this has got to be true for arbitrary sources. Let us analyse its consequences in some detail.

- First of all, demanding the vanishing of the coefficient of the fermionic source $K_{a\mu}(x)$ we learn that

$$\int d^4x \left(\frac{\delta D^{(ren)}_{\mu ai}}{\delta A_{k\lambda}} c_i D^{(ren)}_{kj\lambda} c_j + \frac{1}{2} D^{(ren)}_{ak\mu} f^{(ren)}_{kij} c_i c_j \right) = 0 \qquad (8.65)$$

  To be specific, the integrand reads

$$\mathcal{K} \equiv f^{ren}_{aki} c^i \left(\partial_\mu c_k + f^{ren}_{kbj} A^b_\mu c^j \right) + \frac{1}{2} \left(f^{ren}_{aij} \partial_\mu (c^i c^j) + f^{ren}_{abk} A^b_\mu f^{ren}_{kij} c^i c^j \right) = 0 \qquad (8.66)$$

  Consider first the set of terms that do not involve gauge fields

$$f^{ren}_{aij} c^j \partial_\mu c^i = -\frac{1}{2} f^{ren}_{aij} (\partial_\mu c^i c^j + c^i \partial_\mu c^j) \qquad (8.67)$$

  which vanishes as long as the renormalized constants are antisymmetric

$$f^{ren}_{a(ij)} = 0 \qquad (8.68)$$

  On the other hand, the terms contaning gauge fields read

$$f^{ren}_{aki} f^{ren}_{kbj} c^i c^j + \frac{1}{2} f^{ren}_{kij} c^i c^j f^{ren}_{abk} \qquad (8.69)$$

  which is equivalent to

$$f^{ren}_{aki} f^{ren}_{kbj} - f^{ren}_{akj} f^{ren}_{kbi} + f^{ren}_{kij} f^{ren}_{abk} \qquad (8.70)$$

  That is, this terms also vanishes provided the constants

$$f^{ren}_{ijk} \qquad (8.71)$$

  obey Jacobi's identity.

- Vanishing of the coefficient of the sources $L(x)$ implies

$$\int d^4x \big(- f^{(ren)}_{lij} c_i f^{(ren)}_{alc} c_j c_c \big) = 0 \qquad (8.72)$$

  namely, Jacobi's identity again.

- Consider finally, the term without sources

$$-\frac{\delta L^{(ren)}}{\delta A_{l\lambda}} D^{(ren)}_{\lambda l b} c_b + \partial_\mu \left[\left(f_{akb} c^b (D^{ren}_\mu c)_k + \frac{1}{2} f_{kij} c^i c^j D^{ren}_{ak\mu} \right) \, b_a \right]$$
$$= -\frac{\delta L^{(ren)}}{\delta A_{l\lambda}} D^{(ren)}_{\lambda l b} c_b + \partial_\mu \left[\mathcal{K} b_a \right] = 0 \tag{8.73}$$

  Using the fact that $\mathcal{K} = 0$ it reads

$$\int \frac{\delta L^{(ren)}}{\delta A_{l\lambda}} D^{(ren)}_{\lambda l b} c_b = 0 \tag{8.74}$$

  This implies that $D^{(ren)}$ obey the same equations as D, so that this is just a fancy way of writing down gauge invariance

  Besides, the constants $f^{(ren)}_{abc}$ obey the (Jacobi) identity neccessary for them to qualify as structure constants of a group G.

- Finally, the last equation tells us that the lagrangian $L^{(ren)}(A)$ is again gauge invariant.

  When the group G is simple, it is plain by continuity arguments that both the symmetry group and the representations involved must be the same as in the bare lagrangian.

The semisimple case can be treated using much the same techniques.

Also more general gauge conditions can be considered.

Nevertheless, in order to renormalize spontaneously broken gauge theories some extra work has to be done.

9

Effective field theories

For phenomenological reasons it is often advisable not to commit ourselves on what the theory should be at arbitrarily high energies, and behave as if all we knew is an *effective theory* valid for energies lower than a certain cutoff scale

$$E \leq \Lambda \tag{9.1}$$

Quite often there are universal predictions, depending only on the symmetries of the problem and on the available degrees of freedom.

9.1 Composite operators.

Sometimes, namely in order to define the energy-momentum tensor, or else the equations of motion (EM), nor to speak on the operator product expansion (OPE) we need to compute correlators including fields evaluated at the same space-time point. Those have divergences in addition to the ones already renormalized, and new counterterms are necessary. We work in momentum space except for the operator, which remains in position space.

Consider, for example [3] the operator ϕ^2 in ϕ^3_6. We are interested in

$$G \equiv \langle 0|T\phi_p\phi_q\phi^2_z|0\rangle \equiv \int d^6x\ d^6y\ e^{i(p_1x+p_2y)}\ \langle 0|T\phi_x\phi_y\phi^2_z|0\rangle \tag{9.2}$$

The lowest order graph yields

$$G_{lo} = \frac{i}{p_1^2 - m^2 + i\epsilon}\ \frac{i}{p_2^2 - m^2 + i\epsilon} \tag{9.3}$$

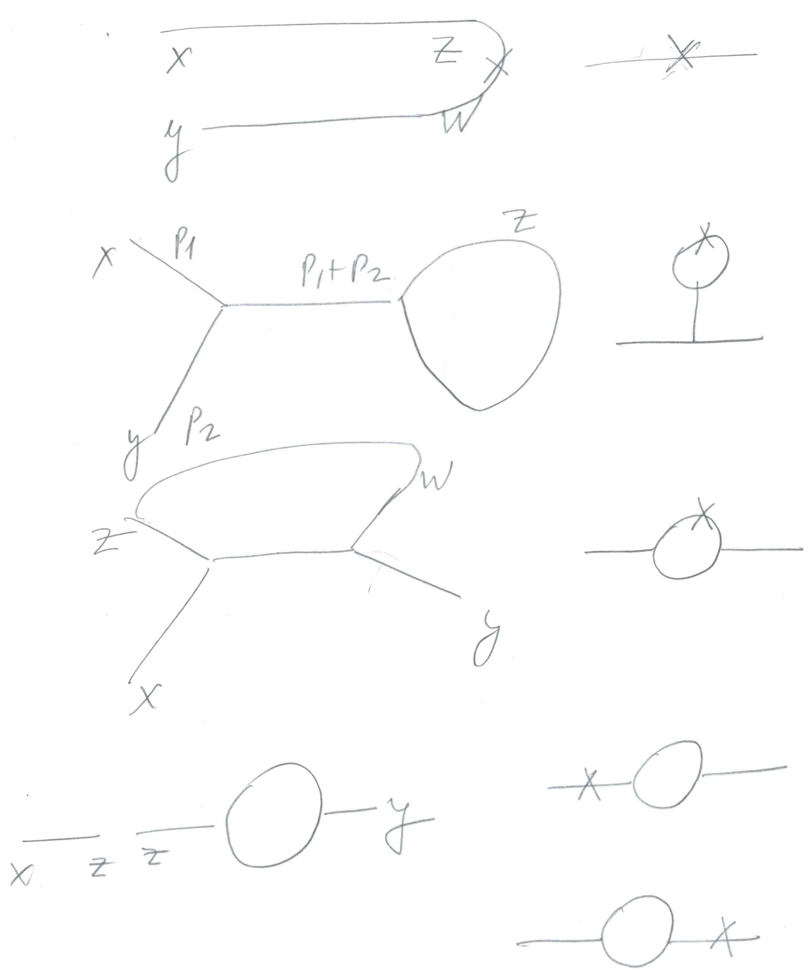

Figure 9.1: Simplest diagrams for ϕ^2.

The first one-loop graph yields

$$G_1 = \frac{i}{p_1^2 - m^2 + i\epsilon} \frac{i}{p_2^2 - m^2 + i\epsilon} \frac{ig^2\mu^{6-n}}{(2\pi)^n} \int d^n k \frac{1}{(k^2 - m^2 + i\epsilon)((k-p_1)^2 - m^2 + i\epsilon)((k+p_2)^2 - m^2 + i\epsilon)}$$
$$= \frac{i}{p_1^2 - m^2 + i\epsilon} \frac{i}{p_2^2 - m^2 + i\epsilon} \frac{g^2}{64\pi^3} \Gamma(3 - \frac{n}{2}) \int_0^1 dx \int_0^{1-x} dy$$
$$\left[\frac{m^2 - p_1^2 y(1-x-y) - p_2^2 x(1-x-y) - (p_1+p_2)^2 xy}{4\pi\mu^2}\right]^{\frac{n}{2}-3}$$

The corresponding counterterm reads

$$-\frac{1}{(p_1^2 - m^2)(p_2^2 - m^2)} \frac{g^2}{64\pi^3} \frac{1}{n-6} \tag{9.5}$$

In such a way that the renormalized graph reads

$$-\frac{1}{(p_1^2 - m^2)(p_2^2 - m^2)} \frac{g^2}{64\pi^3} \left\{ -\frac{\gamma}{2} - \int_0^1 dx \int_0^{1-x} dy \right.$$
$$\left. \log \left[\frac{m^2 - (p_1^2 y + p_2^2 x)(1-x-y) - (p_1+p_2)^2 xy}{4\pi\mu^2}\right]\right\} \tag{9.6}$$

The second one-loop graph reads

$$G_2 = \frac{-g\mu^{3-\frac{n}{2}}}{(p_1^2 - m^2)(p_2^2 - m^2)((p_1+p_2)^2 - m^2)} \frac{ig\mu^{3-n/2}}{2(2\pi)^n} \int d^n k \frac{1}{(k^2 - m^2)((p_1+p_2+k)^2 - m^2)} =$$
$$= \frac{-g\mu^{3-\frac{n}{2}}}{(p_1^2 - m^2)(p_2^2 - m^2)((p_1+p_2)^2 - m^2)} \frac{-g\mu^{n/2-3}}{128\pi^3} \Gamma(2 - n/2) \int_0^1 dx \frac{\left[m^2 - (p_1+p_2)^2 x(1-x)\right]^{n/2-2}}{(4\pi\mu^2)^{n/2-3}}$$

with counterterm

$$\frac{-g\mu^{3-n/2}}{(p_1^2 - m^2)(p_2^2 - m^2)((p_1+p_2)^2 - m^2)} \frac{g\mu^{n/2-3}}{64\pi^3} \left(m^2 - \frac{(p_1+p_2)^2}{6}\right) \frac{1}{n-6} \tag{9.7}$$

In such a way that the renormalized graph reads

$$-\frac{g}{(p_1^2 - m^2)(p_2^2 - m^2)((p_1+p_2)^2 - m^2)} \frac{-g}{128\pi^3} \left\{(\gamma - 1)\left(m^2 - \frac{(p_1+p_2)^2}{6}\right) + \int_0^1 dx \right.$$
$$\left. \left(m^2 - (p_1+p_2)^2 x(1-x)\right) \log \left[\frac{m^2 - (p_1+p_2)^2 x(1-x)}{4\pi\mu^2}\right]\right\} \tag{9.8}$$

The counterterm are just vertices for ϕ^2 and for $\left(\frac{1}{6}\Box + m^2\right)\phi$, in such a way that the total renormalized G reads

$$G = \left\{1 + \frac{g^2}{64\pi^3} \frac{1}{n-6}\right\} \langle 0|T\phi_{p_1}\phi_{p_2} \frac{1}{2}\phi^2(0)|0\rangle + \left\{\frac{g\mu^{n/2-3}}{64\pi^3} \frac{1}{n-6}\right\} \langle 0|T\phi_{p_1}\phi_{p_2} \left(m^2 + \frac{1}{6}\Box\right) \phi(0)|0\rangle + O(g^4)$$

This means that the (renormalized) composite operator $[\phi^2]$ is defined by

$$\frac{1}{2}\left[\phi^2\right] \equiv \left\{1+\frac{g^2}{64\pi^3}\frac{1}{n-6}\right\}\frac{\phi^2}{2}+\left\{\frac{g\mu^{n/2-3}}{64\pi^3}\frac{1}{n-6}\left(m^2+\frac{1}{6}\Box\right)\right\}\phi+O(g^4) \quad (9.9)$$

In fact, a renormalized operator can always be defined by a formula of the type (please remember that the canonical dimension of the field ϕ in six dimension is 2)

$$\frac{1}{2}\left[\phi^2\right] \equiv Z_a\frac{\phi^2}{2}+\mu^{n/2-3}Z_b m^2\phi+\mu^{n/2-3}Z_c\Box\phi \quad (9.10)$$

where the Z depend only on (g, n). The operators that appear as conterterms have dimension lees than or equal to that of the original operator. A closed set of operators under renormalization is then

$$\begin{pmatrix}\frac{1}{2}[\phi^2]\\ \phi\\ \Box\phi\end{pmatrix} = \begin{pmatrix}Z_a\ \mu^{n/2-3} & Z_b\ m^2 & \mu^{n/2-3}\ Z_c\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\frac{1}{2}\phi^2\\ \phi\\ \Box\phi\end{pmatrix} \quad (9.11)$$

9.2 Warmup on proper time.

Remember that

$$\frac{i}{A+i\epsilon} = \int_0^\infty ds\ e^{is(A+i\epsilon)} \quad (9.12)$$

Also

$$\frac{1}{AB} = -\int_0^\infty ds_1\int_0^\infty ds_2 e^{is_1 A+s_2 B} \quad (9.13)$$

Replace

$$\begin{aligned} s_1 &\equiv x\tau\\ s_2 &\equiv (1-x)\tau \end{aligned} \quad (9.14)$$

Then

$$\frac{1}{AB} = -\int_0^1 dx\int_0^\infty \tau\ d\tau\ e^{i\tau(xA+(1-x)B)} = \int_0^1\frac{dx}{\left(ax+B(1-x)\right)^2} \quad (9.15)$$

This yields a nice interpretation of Feynman parameter $x = \frac{s_1}{s_1+s_2}$ as how much one particle is lagging the other obe running in the loop.

Then we can represent Feynman's propagator as

$$D_F(x,y) = \int\frac{d^4p}{(2\pi)^4}\ e^{ip(x-y)}\frac{i}{p^2-m^2+i\epsilon} = \int\frac{d^4p}{(2\pi)^4}\ e^{ip(x-y)}\int_0^\infty ds\ e^{is\left(p^2-m^2+i\epsilon\right)} =$$
$$= -\frac{i}{16\pi^2}\int_0^\infty dss^2\ e^{-i\left(\frac{(x-y)^2}{4s}+sm^2-i\epsilon s\right)} \quad (9.16)$$

Let us introduce, following Schwinger, one-particle Hilbert space spanned by $|x\rangle$ and sich that

$$\langle p|x\rangle = e^{ipx} \tag{9.17}$$

defining

$$\begin{aligned}&\hat{p}^{\mu}|p\rangle = p^{\mu}|p\rangle\\ &\hat{H} \equiv -\hat{p}^2\end{aligned} \tag{9.18}$$

(a non-relativistic hamiltonian; this the priging of the fact that the dimsnions $[s] = -2$) in such a way that

$$e^{isp^2}\langle p|x\rangle \equiv \langle p|e^{-is\hat{H}}|x\rangle \tag{9.19}$$

we get

$$D_F(x,y) = \int_0^{\infty} dse^{-s\epsilon}\ e^{-ism^2}\ \langle y|e^{-is\hat{H}}|x\rangle \equiv \int_0^{\infty} dse^{-s\epsilon}\ e^{-ism^2}\ \langle y;0|x;s\rangle$$

yielding a nice interpretation of the propagator as the amplitude to propagate from the point x to the point y in proper time s and integrated over all proper time. More is true.

$$\begin{aligned}D_F(x,y) &= \int \frac{d^4p}{(2\pi)^4}\ e^{ip(x-y)}\frac{i}{p^2-m^2+i\epsilon} = \int \frac{d^4p}{(2\pi)^4}\ \langle y|p\rangle\langle p|\frac{i}{\hat{p}^2-m^2+i\epsilon}|x\rangle \equiv \langle y|\hat{G}|x\rangle =\\ &= \int_0^{\infty} ds\ e^{-s\epsilon}\ e^{-ism^2}\ \langle y|e^{-i\hat{H}s}|x\rangle\end{aligned} \tag{9.20}$$

9.3 Simplest examples.

Let us work out some simple examples to begin with. Consider, for example, a Yukawa interaction at low energies.

$$S = \int d^4x\left\{\bar{\psi}i\partial\!\!\!/\psi - \frac{1}{2}\phi\left(\Box + m^2\right)\phi + \lambda\bar{\psi}\psi\right\} \tag{9.21}$$

If we neglect scalar loops, we can write

$$\phi \equiv \lambda\left(\Box + m^2\right)^{-1}\bar{\psi}\psi \tag{9.22}$$

and the action reads

$$S = \int d^4x\left\{\bar{\psi}i\partial\!\!\!/\psi + \frac{\lambda^2}{2}\bar{\psi}\psi\left(\Box + m^2\right)^{-1}\bar{\psi}\psi\right\} \tag{9.23}$$

When we are interested in energies much smaller than the mass of the Yukawa particle,

$$E << m \tag{9.24}$$

this in turn can be expanded in a series of local operators

$$S = \int d^4x \left\{ \bar{\psi} i\partial\!\!\!/ \psi + \frac{\lambda^2}{2m^2} \bar{\psi}\psi\bar{\psi}\psi + \frac{\lambda^2}{2m^4} \bar{\psi}\psi \Box \bar{\psi}\psi + \ldots \right\} \tag{9.25}$$

Let us now do a slightly more complicated exercise, namely, the effective lagrangian for the case aof a constant field strength $F_{\mu\nu}$. This was first derived by Euler and Heisenberg in 1936. Let us start by studying the fermion propagator in an external electromagnetic background.

Once the fermions are integrated out the effective action reads

$$L_e = -\frac{1}{4} F_{\mu\nu}^2 - i\text{Tr}\ \log\ (iD\!\!\!\!/ - m) \tag{9.26}$$

taking one derivative over the mass

$$\frac{d}{dm^2} L_e = \frac{i}{2m} \text{Tr}\ \frac{iD\!\!\!\!/ + m}{-D\!\!\!\!/^2 - m^2} = \frac{i}{2} \text{Tr}\ \frac{1}{-D\!\!\!\!/^2 - m^2} = \frac{1}{2} \int_0^\infty ds e^{-ism^2} \text{Tr}\ e^{-iD\!\!\!\!/^2 s} \tag{9.27}$$

Integrating now over dm^2 we learn

$$L_e = -\frac{1}{4} F_{\mu\nu}^2 + \frac{i}{2} \int_0^\infty \frac{ds}{s} e^{-ism^2} \text{Tr}\ e^{-iD\!\!\!\!/^2 s} + C \tag{9.28}$$

Now acting on some arbitrary function

$$\begin{aligned} D\!\!\!\!/^2 f(x) &= (\partial\!\!\!/ + ieA\!\!\!/)^2 \psi = \left(\partial_\mu \partial_\nu + ie\partial_\mu A_\nu + ieA_\nu \partial_\mu + ieA_\mu \partial_\nu - e^2 A_\mu A_\nu\right) \gamma^\mu \gamma^\nu f(x) = \\ &= \left(\Box + ie\partial_\mu A^\mu + ie\frac{1}{2} F_{\mu\nu} \gamma^{\mu\nu} + 2ieA_\lambda \partial^\lambda - e^2 A_\mu A^\mu\right) f(x) = \\ &= \left(D_\mu D^\mu + \frac{1}{2} eF_{\mu\nu} \sigma^{\mu\nu}\right) f(x) \end{aligned} \tag{9.29}$$

where

$$\sigma_{\mu\nu} = i\gamma_{\mu\nu} \tag{9.30}$$

Let us dub

$$H \equiv D_\mu D^\mu + \frac{1}{2} eF_{\mu\nu} \sigma^{\mu\nu} \tag{9.31}$$

Then

$$L_e = -\frac{1}{4} F_{\mu\nu}^2 + \frac{i}{2} \int_0^\infty \frac{ds}{s}\ e^{-ism^2}\ \text{Tr}\ \langle x | e^{-iHs} | x \rangle \tag{9.32}$$

We are using here covariant first quantized formalism, where

$$[\hat{x}^\mu, \hat{p}^\nu] = -i\eta^{\mu\nu} \tag{9.33}$$

In particular, we recover

$$\left[\hat{x}^i, \hat{p}^l\right] = i\delta^{ij} \tag{9.34}$$

The states $|x\rangle$ obey

$$\hat{x}^\mu|x\rangle = x^\mu|x\rangle \tag{9.35}$$

In the Heisenberg picture

$$\hat{x}^\mu_H \equiv e^{i\hat{H}s}\hat{x}^\mu e^{-i\hat{H}s} \tag{9.36}$$

The Schroedinger state is by definition

$$|x;s\rangle \equiv e^{-i\hat{H}s}|x\rangle \tag{9.37}$$

It is then a fact that

$$i\partial_s\langle y;0|x;s\rangle = \langle y|e^{-i\hat{H}s}\hat{H}|x\rangle \tag{9.38}$$

as well as

$$\langle y|e^{-i\hat{H}s}\hat{x}^\mu(s) = \langle y|\hat{x}^\mu e^{-i\hat{H}s} = y^\mu\langle y|e^{-i\hat{H}s} \tag{9.39}$$

Let us introduce the operator

$$\hat{\pi}^\mu \equiv p^\mu - eA^\mu(\hat{x}) \tag{9.40}$$

We have

$$[\hat{x}^\mu(s), \hat{\pi}^\nu(s)] = -i\eta^{\mu\nu} \tag{9.41}$$

and for constant field strength

$$[\hat{\pi}^\mu(s), \hat{\pi}^\nu(s)] = -ieF^{\mu\nu} \tag{9.42}$$

It is clear that

$$\hat{H}(s) = -\hat{\not{\pi}}^2 = -\hat{\pi}_\mu(s)\hat{\pi}^\mu(s) + \frac{e}{2}F_{\mu\nu}\sigma^{\mu\nu} \tag{9.43}$$

The Heisenberg EM implies that, whenever F is constant

$$\frac{d\pi^\mu}{ds} = i\left[\hat{H}, \pi^\mu(s)\right] = 2eF^\mu{}_\nu\pi^\nu \tag{9.44}$$

Then

$$\pi^\mu(s) = \left(e^{2esF^\mu{}_\nu}\right)\ \pi^\nu(0) \tag{9.45}$$

It is also a fact that

$$\frac{dx^\mu}{ds} = i\left[\hat{H}, x^\mu\right] = 2\pi^\mu \tag{9.46}$$

This is easily checked to be solved by

$$x^\mu(s) = x^\mu_0 + (eF^\mu{}_\nu)^{-1}(\pi^\nu(s) - \pi^\nu(0)) = x^\mu(0) + \left(\frac{e^{2sF}-1}{eF}\right)^\mu{}_\nu\ \pi^\nu(0) =$$

$$= x(0) + 2\ s\ e^{esF}\ \left(\frac{\sinh(esF)}{esF}\right)\ \pi(0) \tag{9.47}$$

This implies that

$$\pi(0) = e^{-esF} \left(\frac{eF}{2\ \sinh\ (esF)}\right) (x(s) - x(0)) \tag{9.48}$$

and consequently

$$\pi(s) = \left(e^{2esF}\right)\ \pi(0) = e^{esF} \left(\frac{eF}{2 \sinh (esF)}\right) (x(s) - x(0)) \tag{9.49}$$

We can now rewrite the hamiltonian as

$$\hat{H} = -\hat{\pi}_\mu(s)\hat{\pi}^\mu(s) + \frac{e}{2}\sigma_{\mu\nu}\ F^{\mu\nu} =$$
$$\left(e^{esF}\right)_{\mu\lambda} \left(\frac{eF}{2\sinh(esF)}\right)^\lambda{}_\sigma (x(s)-x(0))^\sigma \left(e^{esF}\right)^\mu_\delta \left(\frac{eF}{2\sinh(esF)}\right)^\delta_\epsilon (x(s)-x(0))^\epsilon -$$
$$-\frac{e}{2}\text{tr}\ \ (\sigma F) = -\left(x(s)-x(0)\right)\ K\ \left(x(s)-x(0)\right) - \frac{e}{2}\text{tr}\ \ (\sigma F) \tag{9.50}$$

with

$$K \equiv \frac{e^2 F^2}{4\ \sinh^2\ (esF)} \tag{9.51}$$

owing to the fact that

$$\left(e^{esF}\right)_{\mu\lambda} \left(e^{esF}\right)^\mu_\delta = \left(e^{es\left(F+F^T\right)}\right) = 1 \tag{9.52}$$

Let us now move all the $x(s)$ to the left and the $x(0)$ to the right

$$\pi(s)\pi(s) = x(s)Kx(s) - x(s)Kx(0) + x(0)Kx(0) - x(s)Kx(0) - K\left[x(0), x(s)\right] \tag{9.53}$$

(Please note that K is a constant matrix).

Now it is a fact that

$$K_{\mu\nu}\left[x^\mu(0), x^\nu(s)\right] = \ K_{\mu\nu}\left[x^\mu(0), x^\nu(0) + 2e^{esF}\ \frac{\sinh\ esF}{eF}\pi^\nu(0)\right] =$$
$$= -\frac{i}{2}\ \text{Tr}\ \left\{eF + eF\text{coth}\ esF\right\} = -\frac{i}{2}\ \text{Tr}\ \left\{eF\text{coth}\ esF\right\} \tag{9.54}$$

owing to the fact that

$$2Ke^{esF}\ \frac{\sinh\ esF}{eF} = \frac{eFe^{esF}}{2\sinh\ esF} = \frac{eF}{2}\ \left(1 + \coth\ esF\right) \tag{9.55}$$

We conclude that

$$\hat{H} = -x(s)Kx(s) + 2x(s)Kx(0) - x(0)Kx(0) - \frac{i}{2}\text{Tr}\ \left\{eF\text{coth}\ esF\right\} - \frac{e}{2}\text{Tr}\ \sigma F \tag{9.56}$$

Then we can work out the expression

$$i\frac{\partial}{\partial s}\langle y;0|x;s\rangle = \langle y|e^{-iHs}H|x\rangle = \langle y|e^{-iHs}\Big\{ -x(s)Kx(s)+2x(s)Kx - xKx\Big\}|x\rangle + \ldots =$$
$$= \langle y|\Big\{ -yKy+2yKx-xKx\Big\}e^{-iHs}+\ldots|x\rangle = \langle y|\Big\{ -yKy+2yKx-xKx\Big\}+\ldots|x;s\rangle =$$
$$-\langle y|\,(y-x)\,K\,(y-x)+\ldots|x;s\rangle \tag{9.57}$$

In this way Schrodinger's equation now is just an ODE and reads

$$-i\partial_s\langle y:0|x;s\rangle = -\Big[(y-x)\frac{e^2F^2}{4\sinh^2\ (esF)}(y-x)+\frac{i}{2}\text{ Tr }\{eF\coth(esF)\}+\frac{e}{2}\text{Tr }\sigma F\Big]\langle y;0|x;s\rangle \tag{9.58}$$

The general solution of the ODE is readily found and reads

$$\langle y;0|x;s\rangle = C(x,y)e^{-i(y-x)\frac{eF}{4}\coth(esF)(y-x)+\frac{1}{2}\text{Tr }\ \log\ \frac{\sinh(esF)}{eF}-i\frac{e}{2}\text{Tr }\ (\sigma F)s} \tag{9.59}$$

This holds for any value of the function $C(x,y)$. In order to determine it, there is some additional information we may use, namely,

$$\left(-i\frac{\partial}{\partial x}-eA\right)\langle y;0|x;s\rangle = \langle y;0|\,\pi(0)|x;s\rangle = e^{-esF}\frac{eF}{2\sinh\ (esF)}(y-x)\langle y;0|x;s\rangle \tag{9.60}$$

as well as

$$\left(i\frac{\partial}{\partial y}-eA\right)\langle y;0|x;s\rangle = e^{esF}\frac{eF}{2\sinh\ (esF)}(y-x)\langle y;0|x;s\rangle \tag{9.61}$$

Owing to the fact that

$$\frac{eF}{2}\frac{e^{esF}+e^{esF}-2e^{-esF}}{e^{esF}-e^{esF}} = \frac{eF}{2} \tag{9.62}$$

Those equations imply

$$\left(-i\frac{\partial}{\partial x}-eA-\frac{e}{2}F(x-y)\right)C(x,y) = \left(-i\frac{\partial}{\partial y}-eA-\frac{e}{2}F(x-y)\right)C(x,y) = 0 \tag{9.63}$$

The solution of those reads

$$C(x,y) = C\ e^{ie\int_x^y dz^\mu\left(A_\mu(z)+\frac{1}{2}\ F_{\mu\nu}(z^\nu-y^\nu)\right)} \tag{9.64}$$

which is independent on the path $z^\mu(\lambda)$ between the two points x and Y because the differential form to be integrated is closed. The remaining consntant, C, is fixed by the demand that we recover the correct result when $A=0$. This determines

$$C = -\frac{i}{16\pi^2 s^2} \tag{9.65}$$

Let us now come back to our effective lagrangian. We have

$$L_e = -\frac{1}{4}F_{\mu\nu}^2 + \frac{i}{2}\int_0^\infty \frac{ds}{s} e^{-ism^2}\mathrm{Tr}\,\{\langle x|e^{-i\hat{H}s}|x\rangle = -\frac{1}{4}F_{\mu\nu}^2 + \\ -\frac{1}{32\pi^2}\mathrm{Tr}\int_0^\infty \frac{ds}{s^3}\ e^{-ism^2 - i\frac{es}{2}\mathrm{Tr}\,(\sigma F) + \frac{1}{2}\mathrm{Tr}\ \ \log\ \frac{\sinh\ (esF)}{seF}} \tag{9.66}$$

Now

$$\mathrm{Tr}\ (\sigma F)^2 = \mathrm{Tr}\ \left(2F_{\mu\nu}^2 + i\gamma_5\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}\right) \equiv 8\,(\mathcal{F} - i\gamma_5\mathcal{G}) \tag{9.67}$$

where

$$\mathcal{F} \equiv \frac{1}{2}\left(B^2 - E^2\right) \\ \mathcal{G} \equiv \vec{E}\vec{B} \tag{9.68}$$

The eigenvalues of Tr (σF) are

$$\lambda = \pm\sqrt{8\,(\mathcal{F} \pm i\mathcal{G})} \tag{9.69}$$

Then

$$\mathrm{Tr}\ e^{i\frac{es}{2}\mathrm{Tr}\,(\sigma F)} = e^{\frac{ies}{2}\sqrt{8(\mathcal{F}+i\mathcal{G})}} + e^{-\frac{ies}{2}\sqrt{8(\mathcal{F}+i\mathcal{G})}} + e^{\frac{ies}{2}\sqrt{8(\mathcal{F}-i\mathcal{G})}} + e^{-\frac{ies}{2}\sqrt{8(\mathcal{F}-i\mathcal{G})}} = \\ = 2\cos\left(es\sqrt{2\,(\mathcal{F} + i\mathcal{G})}\right) + 2\cos\left(es\sqrt{2\,(\mathcal{F} - i\mathcal{G})}\right) = 4\mathrm{Re}\ \cos\ (esX) \tag{9.70}$$

with

$$X \equiv \sqrt{\frac{1}{2}F_{\mu\nu}^2 + \frac{i}{2}F_{\mu\nu}\tilde{F}^{\mu\nu}} = \sqrt{2\,(\mathcal{F} + i\mathcal{G})} = \sqrt{\left(\vec{B} + i\vec{E}\right)^2} \tag{9.71}$$

Next

$$\frac{1}{2}\mathrm{Tr}\ \log\ \frac{\sinh\,(eFs)}{esF} = \log\sqrt{\lambda_1\lambda_2\lambda_3\lambda_4} \tag{9.72}$$

The eigenvalues are easily determined form the corresponding eigenvalues of F, which read

$$\Lambda_i = \pm\frac{i}{\sqrt{2}}\left(\sqrt{\mathcal{F} + i\mathcal{G}} \pm \sqrt{\mathcal{F} - i\mathcal{G}}\right) \tag{9.73}$$

which leads to

$$\frac{1}{2}\mathrm{Tr}\ \log\ \frac{\sinh\,(eFs)}{esF} = \frac{(es)^2\mathcal{G}}{\mathrm{Im}\ \cos\ (esX)} \tag{9.74}$$

The final result for the Euler-Heisenberg effective legrangian is

$$L_e = -\frac{1}{4}F_{\mu\nu}^2 - \frac{e^2}{32\pi^2}\int_0^\infty ds\frac{1}{s}e^{-im^2 s}\frac{\mathrm{Re}\ \cos\ (esX)}{\mathrm{Im}\cos\,(esX)}F_{\mu\nu}\tilde{F}^{\mu\nu} \tag{9.75}$$

Expanding perturbatively in e, we find

$$\frac{\text{Re}\ \cos\ (esX)}{\text{Im}\cos(esX)} F_{\mu\nu}\tilde{F}^{\mu\nu} = \frac{4}{e^2 s^2} + \frac{2}{3}F_{\mu\nu}^2 - \frac{e^2 s^2}{45}\left(F_{\mu\nu}^4 + \frac{7}{4}\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right)^2\right) + \ldots \tag{9.76}$$

The first two terms in this expansion are UV divergent. The very first is just the vacuum energy density. With a proper time cutoff

$$L_e = -\frac{1}{4}F_{\mu\nu}^2\left(1 - \frac{e^2}{12\pi^2}\log\frac{m^2}{\Lambda^2} + \ldots\right) \tag{9.77}$$

This is an effect of the vacuum polarization corresponding to

$$\beta = \frac{e^3}{12\pi^2} \tag{9.78}$$

Consider now the case when the electric and magnetic fields are parallel. Then

$$L_e = \frac{1}{2}\left(E^2 - B^2\right) - \frac{e^2}{8\pi^2}\int_0^\infty \frac{ds}{s}\ e^{i\epsilon\sigma}\ e^{-sm^2}\left(EB\text{cot}\ (esE)\coth\ (esB) - -\frac{1}{e^2 s^2} - \frac{B^2 - E^2}{3}\right) \tag{9.79}$$

Given the fact that the singularities are associated with the electric field, we can consider the limit $B \sim 0$, where

$$L_e \sim \frac{1}{2}E^2 - \frac{1}{8\pi^2}\int_0^\infty \frac{ds}{s^3}e^{i\epsilon s}e^{-sm^2}\left(eEs\text{cot}\ (eEs) - 1 + \frac{1}{3}(esE)^2\right) \tag{9.80}$$

which has poles for real E whenever

$$s \in \frac{\pi}{eE}\mathbb{N} \tag{9.81}$$

This physically means that strong electric fields can create electron-positron pairs, by pair production (Schwinger). The probability per unit time and volume that any number of pairs are created is

$$2\text{Im}\ L_e = \frac{1}{4\pi}\sum_{n=1}^\infty \frac{1}{s_n^2}e^{-m^2 s_n} = \frac{\alpha E^2}{2\pi^2}\sum_{n=1}^\infty \frac{1}{n^2}e^{\frac{-n\pi m^2}{eE}} \tag{9.82}$$

This is negligible until

$$E \sim E_c = \frac{m^2}{e} \sim 10^{18}\ volts/m \tag{9.83}$$

This fact is sometimes invokes to explain why the periodic table has less than $\alpha^{-1} = 137$ elements, the reason being that further elements would not be stable owing to this mechanism.

10

Advanced topics.

10.1 The background field technique.

Let us expand around an arbitrary background comfiguration

$$W_\mu = \bar{A}_\mu + A_\mu \tag{10.1}$$

The background is sometimes denoted the *classical field* while the fluctuation is refered to as the *quantum field*. In the functional integral we integrate over quantum fields only.

$$Z[\bar{A} \equiv \int \mathcal{D}A_\mu \; e^{-S[\bar{A}+A]} \tag{10.2}$$

The background is assumed to obey the classical equations of motion

$$\left.\frac{\delta S}{\delta A_\mu}\right|_{\bar{A}_\mu} = 0 \tag{10.3}$$

The full quantum gauge tramsformations, under which the background field remains inert read

$$\begin{aligned} \bar{A}'_\mu &= \bar{A}_\mu \\ A'_\mu &= g\left(\bar{A}_\mu + A_\mu + \partial_\mu\right) g^{-1} - \bar{A}_\mu \end{aligned} \tag{10.4}$$

This is the gauge transformation that we have got to gauge fix. The thing is that there is another, *background gauge* transformation, which can be kept even when gauge fixing [10.4]. Namely

$$\begin{aligned} \bar{A}'_\mu &= g\left(\bar{A}_\mu + \partial_\mu\right) g^{-1} \\ A'_\mu &= g\; A_\mu \; g^{-1} \end{aligned} \tag{10.5}$$

under which the quantum fields rotate in the adjoint. For example, we could use the gauge fixing

$$F \equiv \bar{D}_\mu A^\mu \tag{10.6}$$

where $\bar{D}$ stands for the covariant derivative with respect to the background gauge field. When doing so all counterterms must respect background gauge covariance which greatly restricts them.

The Yang-Mills action reads now

$$\begin{aligned} S[\bar{A}+A] = -\frac{1}{4}\left(\bar{F}^a_{\mu\nu} + \bar{D}_\mu A_\nu - \bar{D}_\nu A_\mu + f_{abc} A^b_\mu A^c_\nu\right)^2 + \\ -\frac{1}{2\alpha}\left(\bar{D}_\mu A^\mu\right)^2 - \bar{D}_\mu \bar{c}_a \left(\bar{D}^\mu c_a - f_{abc} c^b A^\mu_c\right) \end{aligned} \tag{10.7}$$

Terms linear in the quantum fields vanish when the background field lies on shell. The quadratic piece gives then rise to two determinants

$$\begin{aligned} W[\bar{A}] = -\frac{1}{2}\text{Tr}\ \log\ \left(\left(\bar{D}_\mu \bar{D}^\mu \eta_{\alpha\beta} - \left(1-\frac{1}{\alpha}\right)\bar{D}_\alpha \bar{D}_\beta\right)\delta_{bc} - 2\bar{F}^a_{\alpha\beta} f_{abc}\right)\Big|_{\text{gauge}} + \\ +\text{Tr}\ \log\ \left(\bar{D}_\mu \bar{D}^\mu\right)\delta_{bc}\Big|_{\text{ghost}} \end{aligned} \tag{10.8}$$

Renormalizarion constants are defined as usual

$$\begin{aligned} g_0 &= Z_g g \\ \bar{A}_0 &= Z_A^{\frac{1}{2}} \bar{A} \end{aligned} \tag{10.9}$$

Then the field strength renormalizes as

$$\bar{F}_0 = Z_A^{\frac{1}{2}}\left(\partial\bar{A} - \partial\bar{A} - g Z_g Z_A^{\frac{1}{2}} f \bar{A}\bar{A}\right) \tag{10.10}$$

which is background gauge invariant only when

$$Z_g Z_A^{\frac{1}{2}} = 1 \tag{10.11}$$

The coupling constant renormalizations is then related to the wave function renormalization.

Let us finally remark for future use that when $\alpha = 1$ all relevant operators are of the form

$$D_\mu D^\mu + Y \tag{10.12}$$

with

$$D_\mu \equiv \partial_\mu + X_\mu \tag{10.13}$$

Namely, for the gauge operator

$$\begin{aligned} X_\mu &= A_\mu \\ Y &= -2F_{\mu\nu} \end{aligned} \tag{10.14}$$

whereas for the ghost determinant

$$\begin{aligned} X_\mu &= A_\mu \\ Y &= 0 \end{aligned} \tag{10.15}$$

10.2 Functional determinants

It is a fact that to one loop order, only the terms quadratic in the quantum fields contribute. This because the topological identity

$$L = I - V + 1 \tag{10.16}$$

collapses in this case to

$$I = V \tag{10.17}$$

which means that every one loop diagram is just a circle with classical fields attached in trilinear vertices with one classical field and two quantum fields that propagate in the circle.

10.2.1 Feynman diagrams

As a matter of fact

$$\begin{aligned}
&log\, det\,(-\Box + m^2 + \bar{M}^2) = tr\, log\,(-\Box + m^2)\left(1 + (-\Box + m^2)^{-1}\bar{M}^2\right) = \\
&tr\, log\,(-\Box + m^2) + tr\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}\left((-\Box + m^2)^{-1}\bar{M}^2\right)^n = \\
&C + \sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}\int_{x_1\ldots x_m}(-\Box + m^2)^{-1}_{x_1x_2}\bar{M}^2_{x_2}(-\Box + m^2)^{-1}_{x_2x_3}\bar{M}^2_{x_3}\ldots(-\Box + m^2)^{-1}_{x_mx_1}\bar{M}^2_{x_1} = \\
&C + \sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}\int_{x_1\ldots x_m}\int_{p_1\ldots p_{2m}} e^{ip_1(x_1-x_2)}e^{ip_2x_2}\frac{\bar{M}^2_{p_2}}{p_1^2 + m^2}\ldots e^{ip_{2m-1}(x_m-x_1)}e^{ip_{2m}x_1}\frac{\bar{M}^2_{p_{2m}}}{p^2_{2m-1} + m^2} = \\
&C + \sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}\int_{p_1\ldots p_{2m}}\delta(p_1 + p_{2m-1} - p_{2m})\delta(-p_1 + p_2 + p_3)\ldots\frac{\bar{M}^2_{p_2}}{p_1^2 + m^2}\ldots\frac{\bar{M}^2_{p_{2m}}}{p^2_{2m-1} + m^2} = \\
&C + \sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}\int_{p_2\ldots p_{2m}}\delta(p_2 + p_4 + \ldots + p_{2m})\bar{M}_{p_2}\ldots\bar{M}_{p_{2m}}\mathcal{D}^{(m)}(p_2\ldots p_{2m})
\end{aligned} \tag{10.18}$$

where

$$\begin{aligned}
&(-\Box + m^2)^{-1}_{xy} \equiv \int\frac{d^np}{(2\pi)^n}e^{ip(x-y)}\frac{1}{p^2 + m^2} \\
&\bar{M}^2_x \equiv \int\frac{d^np}{(2\pi)^n}e^{ipx}\bar{M}^2_p
\end{aligned} \tag{10.19}$$

The nontrivial piece of the determinant is

$$\mathcal{D}^{(m)}(p_2\ldots p_{2m}) \equiv \int_{p_1\ldots p_{2m-1}}\delta(p_1 + p_{2m-1} - p_{2m})\delta(-p_1 + p_2 + p_3)\ldots\frac{1}{p_1^2 + m^2}\ldots\frac{1}{p^2_{2m-1} + m^2} \tag{10.20}$$

There are m Dirac deltas, of which $m-1$ are efficient in killing a momentum integration. Given tha fact that there werpreviously m of those, there is one momentum integrtation left, that is, all those diagrams are one-loop ones.

The final expression for $\mathcal{D}^{(m)}$ is

$$\mathcal{D}^{(m)}(p_2 \ldots p_{2m}) \equiv \int \frac{d^n p}{(2\pi)^n} \frac{1}{p^2+m^2} \frac{1}{(p-p_2)^2+m^2} \cdots \frac{1}{(p-p_2-p_4-\ldots-p_{2m-2})^2+m^2} \tag{10.21}$$

In $d = 4$ dimensions, the first two terms are divergent (although the term $m = 1$ is taken to be zero in dimensional regularization), and the rest are given by finite integrals.

The *effective potential* corresponds to the coefficient to the zero mode,i.e.

$$\bar{M}_p^2 = (2\pi)^n \, \delta^{(n)}(p) \, \bar{M}^2 \left(\bar{\phi}\right) \tag{10.22}$$

We have

$$V_{eff} = C + \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \left((2\pi)^n \bar{M}\right)^m \int \frac{d^n p}{(2\pi)^n} \left(\frac{1}{p^2+m^2}\right)^m =$$
$$C + \int \frac{d^n p}{(2\pi)^n} \, log \left(1 + (2\pi)^n \bar{M}^2 \frac{1}{p^2+m^2}\right) \tag{10.23}$$

This is similar to the formula by Iliopoulos et al.

At any rate, it is much easier to use the zeta-function approach (to be explained in a moment) to get, in four dimensions:

$$V_{eff} = \frac{1}{2} m^2 \bar{\phi}^2 + V\left(\bar{\phi}\right) + \frac{\left(m^2 + \bar{M}(\bar{\phi})^2\right)^2}{64\pi^2} \left(log \, \frac{m^2 + \bar{M}(\bar{\phi})^2}{\mu^2} - 3/2\right) \tag{10.24}$$

If we follow Coleman and Weinberg and define the coupling constant in the massless ϕ_4^4 theory as

$$\lambda \equiv \left. \frac{d^4 V_{eff}(\bar{\phi})}{d\bar{\phi}^4} \right|_{\bar{\phi}=M} \tag{10.25}$$

we get [**?**]

$$V_{eff} = \frac{1}{2} m^2 \bar{\phi}^2 + \lambda \frac{\bar{\phi}^4}{24} + \frac{\lambda^2 \bar{\phi}^2}{256\pi^2} \left(log \, \frac{\bar{\phi}^2}{M^2} - 25/6\right) \tag{10.26}$$

10.3 Heat kernel

Let us now follow a slightly different route which is however intimately related. We begin, following Schwinger, by considering the divergent integral which naively is independent of λ

$$I(\lambda) \equiv \int_0^{\infty} \frac{dx}{x} \, e^{-x\lambda} \tag{10.27}$$

The integral is actually divergent, so before speaking about is it has to be regularized. It can be defined through

$$I(\lambda) \equiv \lim_{\epsilon\to 0} I(\epsilon,\lambda) \equiv \lim_{\epsilon\to 0}\int_\epsilon^\infty \frac{dx}{x} e^{-x\lambda} \tag{10.28}$$

such that

$$\lim_{\epsilon\to 0}\frac{\partial I(\epsilon,\lambda)}{\partial\lambda} = -\frac{1}{\lambda} \tag{10.29}$$

It follows

$$\therefore \quad I(\lambda) = -\log\lambda + C \tag{10.30}$$

It is natural to define (for trace class [1]) operators

$$\log\det\Delta = \mathrm{tr}\log\Delta \equiv \sum_n \log\lambda_n \tag{10.31}$$

Now given an operator (with purely discrete, positive spectrum) we could generalize the above idea (Schwinger)

$$\log\det\Delta \equiv -\int_0^\infty \frac{d\tau}{\tau}\mathrm{tr}\; e^{-\tau\Delta} \tag{10.32}$$

The trace here encompasses not only discrete indices, but also includes an space-time integral. Let is define now the *heat kernel* associated to that operator as the operator

$$K(\tau) \equiv e^{-\tau\Delta} \tag{10.33}$$

Formally the inverse operator is given through

$$\Delta^{-1} \equiv \int_0^\infty d\tau\; K(\tau) \tag{10.34}$$

where the kernel obeys the heat equation

$$\left(\frac{\partial}{\partial\tau} + \Delta\right) K(\tau) = 0 \tag{10.35}$$

In all case that will interest us, the operator Δ will be a differential operator. Then the heat equation is a parabolic equation

$$\left(\frac{\partial}{\partial\tau} + \Delta\right) K(\tau; x, y) = 0 \tag{10.36}$$

which need to be solved with the boundary condition

$$K(x, y, 0) = \delta^{(n)}(x - y) \tag{10.37}$$

[1] In the physical Lorentzian signatura, all quantities will be computed from analytic continuations from riemannian configurations where they are better defined. This procesire is not always unambiguous when gravity is present.

The mathematicians have studied operators which are deformations of the laplacian of the type

$$\Delta \equiv D^\mu D_\mu + Y \tag{10.38}$$

where D_μ is a gauge covariant derivative

$$D_\mu \equiv \nabla_\mu + X_\mu \tag{10.39}$$

and ∇_μ is the usual covariant space-time derivative.

In the simplest case $X = Y = 0$ and $\nabla_\mu = \partial_\mu$, the flat space solution corresponding to (minus) the euclidean laplacian is given by

$$K_0(x, y; \tau) = \frac{1}{(4\pi\tau)^{n/2}} e^{-\frac{\sigma(x,y)}{2\tau}} \tag{10.40}$$

where the world function in flat space is simply

$$\sigma(x, y) \equiv \frac{1}{2}(x - y)^2 \tag{10.41}$$

To be precise, the heat equation in the flat case reads

$$\left(\frac{\partial}{\partial\tau} - \Box\right) K\left(\tau; x, y\right) = 0 \tag{10.42}$$

where

$$\Box \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu \tag{10.43}$$

is a flat metric (either minkowkian or eudlidean) and

$$\sigma(x, y) \equiv \frac{1}{2}\eta_{\mu\nu}\left(x - y\right)^\mu \left(x - y\right)^\nu \tag{10.44}$$

In the massive case the heat kernel is modified to

$$K_0^m(x, y; \tau) = K_0\left(\tau; x, y\right)\; e^{-\frac{\mu^2}{\mu^2}\tau} \tag{10.45}$$

This can be easily checked by direct computation.

It is unfortunately quite difficult to get explicit solutions of the heat equation except in very simple cases. This limits the applicability of the method for computing finite determinants. These determinants are however divergent in all cases of interest in QFT, , and their divergence is due to the lower limit of the proper time integral. It we were able to know the solution close to the lower limit, we could get at least some information on the structure of the divergences. This is exactly how far it is possible to go.

The small proper time expansion of Schwinger and DeWitt is given by a Taylor expansion

$$K\left(\tau; x, y\right) = K_0\left(\tau; x, y\right)\; \sum_{p=0}^{\infty}\, a_p\left(x, y\right)\tau^p \tag{10.46}$$

with

$$a_0(x,x) = 1 \tag{10.47}$$

The coefficients $a_p(x,y)$ characterize the operator whose determinant is to be computed. Actually, for the purpose at hand, only their diagonal part, $a_n(x,y)$ is relevant.

The integrated diagonal coefficients will be denoted by capital letters

$$A_n \equiv \int \sqrt{|g|}\; d^n x\; a_n(x,x) \tag{10.48}$$

in such a way that

$$A_0 = vol \equiv \int_M \sqrt{|g|}\; d^n x \tag{10.49}$$

10.3.1 Propagators

The famous integral

$$\int_0^\infty dx\; x^{\nu-1}\; e^{-\frac{\beta}{x}-\gamma x} = \left(\frac{\beta}{\gamma}\right)^{\frac{\nu}{2}}\; K_\nu\left(2\sqrt{\beta\gamma}\right) \tag{10.50}$$

defines the mother of all propagators

$$\left(\Delta + m^2\right)^{-1} \equiv \int_0^\infty d\tau\; K_0^m(\tau; x, t) = \frac{1}{2\pi}\left(\frac{m}{2\pi\,|x-y|}\right)^{\frac{n}{2}-1}\; K_{\frac{n}{2}-1}\left(m\,|x-y|\right) \tag{10.51}$$

where

$$\Delta \equiv -\sum_{i=1}^n \partial_i^2$$

$$|x|^2 \equiv \sum_{i=1}^n x_i^2 \tag{10.52}$$

and $K_n(x)$ is Bessel's function with imaginary argument.

It is quite interesting to apply the Schwinger-deWitt expansion to the heat kernel definition of the propagator

$$\Delta^{-1}(x,y) \equiv \int_0^\infty d\tau\;\; K(x,y;\tau) \tag{10.53}$$

The result is

$$\Delta^{-1}(x,y) = \sum_{p=0}^\infty \frac{a_p(x,y)}{(4\pi)^{\frac{n}{2}}}\left(\frac{\sigma(x,y)}{2}\right)^{p-\frac{n}{2}+1}\Gamma\left(\frac{n}{2}-p-1\right) \tag{10.54}$$

In $n=4$ dimensions this yields

$$\Delta^{-1}(x,y) = \frac{a_0(x,y)}{(4\pi)^2}\frac{2}{\sigma(x,y)} + \frac{a_1(x,y)}{(4\pi)^{\frac{n}{2}}}\left(\frac{\sigma(x,y)}{2}\right)^{\frac{4-n}{2}}\Gamma\left(\frac{n-4}{2}\right) + \ldots \tag{10.55}$$

Terms with

$$p \geq 2 \tag{10.56}$$

are proportional to positive power of σ and they vanish in the coincidence limit $\sigma(x,y) \to 0$. Expanding everything around $n = 4$ this yields (keeping only terms that are singular when $x \to y$)

$$\Delta^{-1}(x,y) = \frac{a_0(x,y)}{(4\pi)^2}\frac{2}{\sigma(x,y)} + \frac{a_1(x,y)}{(4\pi)^2}\left(1 + \frac{4-n}{2}\log\ 4\pi\right) \times \times \left(1 + \frac{4-n}{2}\log\ \frac{\sigma(x,y)}{2}\right)\left(\frac{2}{n-4} - \gamma\right) = \frac{a_0(x,y)}{(4\pi)^2}\frac{2}{\sigma(x,y)} + + \frac{a_1(x,y)}{(4\pi)^2}\frac{2}{n-4} - \frac{a_1(x,y)}{(4\pi)^2}\left(\gamma + \log\ \left(2\pi\sigma(x,y)\right)\right) \tag{10.57}$$

The behavior of the propagator in the coincidence limit is said to be of the Hadamard type [6]. Actually this behavior was derived for hyperbolic equations; we see here that it is a consequence of the assumed Schwinger-DeWitt expansion.

A different way to proceed is to put explicit IR (μ) and UV (Λ) proper time cutoffs, such that $\frac{\Lambda}{\mu} >> 1$. It should be emphasized that these cutoffs are not cutoffs in momentum space; they respect in particular all gauge symmetries the theory may enjoy. The propagator reads then in four dimensions

$$\Delta^{-1} \equiv \int_{\Lambda^{-2}}^{\mu^{-2}} d\tau\ K(x,y;\tau) = \int_{\Lambda^{-2}}^{\mu^{-2}} d\tau\ \frac{1}{(4\pi\tau)^2}\sum_p a_p(x,y)\ \tau^p\ e^{\frac{-\sigma(x,y)}{2\tau}} = = \frac{1}{16\pi^2}\ \sum_p a_p(x,y)\ \left(\frac{\sigma(x,y)}{2}\right)^{p-1}\left(\Gamma(1-p,\Lambda^{-2}) - \Gamma(1-p,\mu^{-2},\right) = = \frac{1}{16\pi^2}\left\{\left(\Gamma(1,\Lambda^{-2}) - \Gamma(1,\mu^{-2})\right)\ a_0(x,y)\frac{2}{\sigma(x,y)} + + \left(\Gamma(0,\Lambda^{-2}) - \Gamma(0,\mu^{-2})\right)\ a_1(x,y) + \ldots\right\}$$

where $\Gamma(z,w)$ is the incomplete gamma function.

Let us now consider the minkowskian signature. It is well known that the position space propagator computed with Feynman's boundary conditions is

$$\frac{1}{\sigma(x,y) - i\epsilon} \tag{10.58}$$

This $i\epsilon$ factor is responsible for the logarithmic piece of the singular behavior of the propagator in the coincidence limit.

How can we get this imaginary part from the heat equation? The simplest guess would read

$$K_0(x,y;\tau) = \frac{1}{(4\pi\tau)^{n/2}}e^{-\frac{\sigma(x,y)-i\epsilon}{2\tau}} \tag{10.59}$$

which obeys the modified heat equation

$$\left(\frac{\partial}{\partial\tau} - \left(\Box - i\frac{\epsilon}{2\tau^2}\right)\right) K\left(\tau; x, y\right) = 0 \tag{10.60}$$

which corresponds to $p^2 \to p^2 + i\epsilon$.

10.3.2 Determinants

The determinant of the operator is then given by an still divergent integral. Let us recall that the trace operation involves in particular taking $x = y$ and integrating over the whole spacetime. The short time expansion does not arrange anything in that respect. This integral has to be regularized by some procedure. One of the possibilities is to keep $x \neq y$ in the exponent, so that

$$\log\det\Delta \equiv -\int_0^\infty \frac{d\tau}{\tau}\mathrm{tr}\,K(\tau) \equiv -\lim_{\sigma\to 0}\int_0^\infty \frac{d\tau}{\tau}\frac{1}{(4\pi\tau)^{n/2}}\sum_{p=0}^\infty \tau^p \mathrm{tr}\,a_p(x,y)\,e^{-\frac{\sigma}{4\tau}} \tag{10.61}$$

We have regularized the determinant by point-splitting. For consistency, also the off-diagonal part of the short-time coefficient ought to be kept.

All ultraviolet divergences are given by the behavior in the $\tau \sim 0$ end-point. Changing the order of integration, and performing first the proper time integral, the Schwinger-de Witt expansion leads to

$$\log\det\Delta = -\int d(vol)\lim_{\sigma\to 0}\sum_{p=0}^\infty \frac{\sigma\left(x,y\right)^{p-n/2}}{4^p\pi^{n/2}}\,\Gamma\left(\frac{n}{2}-p\right)\,\mathrm{tr}\,a_p\left(x,y\right) \tag{10.62}$$

Here it has not been not included the possible σ dependence of

$$\lim_{\sigma\to 0} a_n\left(x,y\right) \tag{10.63}$$

In flat space this corresponds to

$$(x-y)^2 = 2\sigma \to 0 \tag{10.64}$$

Assuming this dependence is analytic, this could only yield higher powers of σ, as will become plain in a moment.

The term $p = 0$ diverges in four dimensions when $\sigma \to 0$ as

$$\frac{1}{\sigma^2} \tag{10.65}$$

but this divergence is common to all operators and can be absorbed by a counterterm proportional to the total volume of the space-time manifold. This renormalizes the the cosmological constant.

The next term corresponds to $p = 2$, and is independent on σ.In order to pinpoint the divergences, When $n = 4 - \epsilon$ it is given by

$$\log\det\,\Delta|_{n=4} \equiv \frac{1}{156\pi^2\,(4-n)}\;A_2 \tag{10.66}$$

From this term on, the limit $\sigma \to 0$ kills everything.

Using again proper time cutoffs,

$$\log\det\Delta \equiv -\int \frac{d\tau}{\tau}\text{tr}\;K(\tau) \equiv -\int_{\frac{1}{\Lambda^2}}^{\frac{1}{\mu^2}} \frac{d\tau}{\tau}\frac{1}{(4\pi\tau)^{n/2}}\sum_{p=0}\tau^p\text{tr}\;A_p\,[\Delta] \tag{10.67}$$

This yields, for example in $n = 4$ dimensions

$$\log\det\Delta = \frac{1}{(4\pi)^2}\left(\frac{1}{2}\Lambda^4\;Vol + A_1\,[\Delta]\;\;\Lambda^2 + A_2\,[\Delta]\;\;\log\;\frac{\Lambda^2}{\mu^2}\right) \tag{10.68}$$

10.4 Zeta function

Let us recall the definition of Riemann's ζ-function

$$\zeta_R(s) \equiv \sum_{n=1}^{\infty} n^{-s} \tag{10.69}$$

This series converge only when

$$\text{Re}\;s \geq 1 \tag{10.70}$$

The function can be analytically continued to the whole complex plane, in such a way that

$$\begin{aligned}\zeta_R(0) &= -\frac{1}{2}\\ \left.\frac{d\zeta_R}{ds}\right|_{s=0} &= -\frac{1}{2}log\,2\pi\end{aligned} \tag{10.71}$$

This analytic continuation yields a sum for the divergent series

$$\sum_{n=1}^{\infty} 1 = -\frac{1}{2} \tag{10.72}$$

Given an operator M such that

$$M\phi_n = \lambda_n\phi_n \tag{10.73}$$

we define by analogy the ζ-function associated with the operator M, namely,

$$\zeta(s) \equiv \sum_{n=0}^{\infty} \lambda_n^{-s} \tag{10.74}$$

so that

$$\log\det M \equiv -\left.\frac{d\zeta}{ds}\right|_{s=0} \tag{10.75}$$

It is a fact of life that the zeta function can easily be reconstructed out of the heat kernel, because

$$\begin{aligned}&\frac{1}{\Gamma(s)}\int_0^\infty d\tau\,\tau^{s-1}\,tr\,K(\tau) = \sum_{n=0}^\infty \lambda_n^{-s} \equiv \zeta(s) = \\ &= \frac{1}{\Gamma(s)}\int_0^\infty d\tau\,\tau^{s-1}\,tr\int\sqrt{|g|}d^nx\,K(x,x;\tau)\end{aligned} \tag{10.76}$$

Let us compute the zeta function using the short time Schwinger-DeWitt expansion for the heat kernel

$$\begin{aligned}\zeta(s) &= \frac{1}{\Gamma(s)}\,\frac{1}{(4\pi)^{\frac{n}{2}}}\,\sum_p\int_0^\infty d\tau\tau^{s-1+p-\frac{n}{2}}a_p\,(x,y)\;e^{\frac{-\sigma(x,y)}{2\tau}} = \\ &= \frac{1}{\Gamma(s)}\frac{1}{(4\pi)^{\frac{n}{2}}}\,\sum_p 4^{\frac{n}{2}-p-s}\;a_p\,(x,y))\;\sigma\,(x,y)^{s+p-\frac{n}{2}}\;\Gamma\left(\frac{n}{2}-p-s\right)\end{aligned}$$

Now the properties of Euler's gamma function

$$\begin{aligned}&\Gamma(1+z) = z\,\Gamma(z)\\ &\Gamma(z)\;\Gamma(1-z) = \frac{\pi}{\sin\;\pi z}\\ &\left.\frac{1}{\Gamma(z)}\right|_{z\sim 0} = z+\gamma z^2+\frac{6\gamma^2-\pi^2}{12}\;z^3+O(z^4)\\ &\Gamma(z-n)|_{z\sim 0} = \frac{(-1)^n}{n!}\left(\frac{1}{z}-\gamma\right)+O(z)\end{aligned} \tag{10.77}$$

determine the zeta function around $s=0$

$$\begin{aligned}\zeta(s) &= \frac{1}{\frac{1}{s}-\gamma}\frac{1}{(4\pi)^{\frac{n}{2}}}\,\sum_p 4^{\frac{n}{2}-p}\,(1-s\log\;4)\,a_p\,(x,y))\;\sigma\,(x,y)^{p-\frac{n}{2}}\,\times\\ &\times(1+s\log\sigma(x,y))\;\Gamma\left(\frac{n}{2}-p\right)\left(1-s\;\psi_0\left(\frac{n}{2}-p\right)\right)\end{aligned} \tag{10.78}$$

where $\psi_0(z)$ is the polygamma function

$$\psi_0(z) \equiv \frac{\Gamma'(z)}{\Gamma(z)} \tag{10.79}$$

For example, close to four dimensions

$$\begin{aligned}\zeta(s) &= \frac{1}{\frac{1}{s}-\gamma}\frac{1}{(4\pi)^{\frac{n}{2}}}\,\sum_p 4^{\frac{n}{2}-p}\,(1-s\log\;4)\,a_p\,(x,y))\;\sigma\,(x,y)^{p-\frac{n}{2}}\,\times\\ &\times(1+s\log\sigma(x,y))\;\Gamma\left(\frac{n-4}{2}+2-p\right)\left(1-s\;\psi_0\left(\frac{n-4}{2}+2-p\right)\right)\end{aligned}$$

That is

$$\zeta(s) = s\Big\{\frac{1}{\pi^2}\ \frac{1}{\sigma(x,y)^2}\ a_1(x,y) + \frac{1}{16\pi^2}\ \left(1 + \frac{4-n}{2}\log\ 4\pi\right) \times$$
$$\times\ a_2\,(x,y)\,\frac{2}{n-4}\ \left(1 + \frac{4-n}{2}\log\ \sigma(x,y))\right)\Big\} + O(s^2) \tag{10.80}$$

so that

$$\zeta'(s=0) = \frac{1}{\pi^2}\ \frac{1}{\sigma(x,y)^2}\ a_1(x,y) + \frac{1}{16\pi^2}\ a_2\,(x,y)\left(\frac{2}{n-4} - \log\ (4\pi\sigma\,(x,y))\right) \tag{10.81}$$

The rest of the terms carry positive powers of $\sigma(x,y)$.

It is also possible to perform a purely four dimensional calculation, by introducing again proper time cutoffs.

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_{\Lambda^{-2}}^{\mu^{-2}} d\tau\ \tau^{s-1}\ \frac{1}{(4\pi\tau)^2}\ \sum_{p=0}^{\infty}\ a_p(x,y)\ \tau^p = \frac{1}{16\pi^2\Gamma(s)} \times$$
$$\times\sum_{p=0}^{\infty} a_p(x,y)\ \frac{1}{2-s-p}\ \left(\Lambda^{2(s+p-2)} - \mu^{2(s+p-2)}\right) =$$
$$= \frac{1}{16\pi^2\Gamma(s)}\Big\{a_0(x,y)\ \frac{1}{2-s}\ \left(\Lambda^{2(s-2)} - \mu^{2(s-2)}\right) + a_1(x,y)\ \frac{1}{1-s}\ \left(\Lambda^{2(s-1)} - \mu^{2(s-1)}\right) +$$
$$-a_2(x,y)\ \frac{1}{s}\ \left(\Lambda^{2s} - \mu^{2s}\right) + \text{analytic}\Big\} \tag{10.82}$$

In the neighborhood of $s=0$ this yields

$$\zeta(s) = -\frac{s}{16\pi^2}\ a_2(x,y)\ \log\ \frac{\Lambda^2}{\mu^2} \tag{10.83}$$

conveying the fact that

$$\zeta'(s=0) = -\frac{1}{16\pi^2}\ a_2(x,y)\ \log\ \frac{\Lambda^2}{\mu^2} \tag{10.84}$$

It is also frequent the use of the regularization corresponding to

$$\begin{aligned}\Lambda^{-2} &= 0\\ \mu^{-2} &= 1\end{aligned} \tag{10.85}$$

in which case in the same neighborhood

$$\zeta(s) = -\frac{1}{16\pi^2}\,(1+\gamma s)\ a_2(x,y) \tag{10.86}$$

where we have used

$$\frac{1}{\Gamma(s)} = s\left(1 + \gamma s + O(s^2)\right) \tag{10.87}$$

This yields

$$\zeta(0) = -\frac{1}{16\pi^2}\ a_2(x,y)$$
$$\zeta'(0) = -\frac{\gamma}{16\pi^2}\ a_2(x,y) \tag{10.88}$$

10.5 The Coleman-Weinberg effective potential

For the Laplace operator in flat space, which is the starting point in all perturbative calculations,

$$\mu^2 M = -\sum_{i=1}^{n}\left(\frac{\partial}{\partial x^i}\right)^2 + m^2 \tag{10.89}$$

We have introduced an arbitrary mass parameter, μ, to make the eigenvalues dimensionless. One finds

$$K(x,y;\tau) = \mu^n\,(4\pi\tau)^{-n/2}\,e^{-\frac{\mu^2(x-y)^2}{4\tau}-\frac{m^2}{\mu^2}\tau} \tag{10.90}$$

This leads inmediatly to

$$\zeta(s) = \mu^n V\left(\frac{m^2}{4\pi\mu^2}\right)^{n/2-s}\frac{\Gamma(s-n/2)}{\Gamma(s)} = \mu^n V\left(\frac{m^2}{4\pi\mu^2}\right)^{n/2-s}\frac{1}{(s-1)(s-2)\ldots(s-n/2)} \tag{10.91}$$

where

$$V \equiv \int d^n x \tag{10.92}$$

and we have assumed that $n \in 2\mathbb{Z}$. The corresponding derivative is then

$$\frac{d\zeta(s)}{ds} = (4\pi)^{-n/2}\frac{V m^n}{(s-1)(s-2)\ldots(s-n/2)}\left(-log\,\frac{m^2}{\mu^2} - \right.$$
$$\left. -\frac{1}{s-n/2} - \frac{1}{s-(n/2-1)} - \ldots - \frac{1}{s-1}\right) \tag{10.93}$$

This means that for any even dimension,

$$\frac{1}{2} log\, det\, M = -\frac{1}{2}\left.\frac{d\zeta(s)}{ds}\right|_{s=0} = (4\pi)^{-n/2}\frac{V m^n}{(n/2)!}\left(log\,\frac{m^2}{\mu^2} - \left(1 + \frac{1}{2} + \ldots + \frac{1}{n/2}\right)\right) \tag{10.94}$$

In $n = 4$ dimensions, in particular, this yields

$$\frac{1}{2} log\, det\, M = \frac{V m^4}{32\pi^2}\left(log\,\frac{m^2}{\mu^2} - 3/2\right) \tag{10.95}$$

When computing in background field in a massless ϕ^4_4 theory

$$m^2 \equiv \frac{\lambda}{2}\bar{\phi}^2 \tag{10.96}$$

in such a way that

$$V_e(\bar{\phi}) = \frac{\lambda}{4!}\bar{\phi}^4 + \frac{\lambda^2\bar{\phi}^4}{256\pi^2}\left(\log\frac{\lambda\bar{\phi}^2}{2\mu^2} - \frac{3}{2}\right) \tag{10.97}$$

In order to compare with Coleman-Weinberg's result it is necessary to remember that they define a coupling constant such that

$$\lambda_M \equiv \left.\frac{d^4 V_e(\bar{\phi})}{d\bar{\phi}^4}\right|_{\bar{\phi}=M} \tag{10.98}$$

This determines a running coupling constant

$$\lambda_M = \lambda + \frac{3\lambda^2}{32\pi^2}\left(\log\frac{\lambda M^2}{2\mu^2} + \frac{8}{3}\right) \tag{10.99}$$

and the effective potential now reads

$$V_e(\bar{\phi}) = \frac{\lambda}{4!}\bar{\phi}^4\left(\lambda_M - \frac{3\lambda_M^2}{32\pi^2}\left(\log\frac{\lambda_M M^2}{2\mu^2} + \frac{8}{3}\right)\right) + \\ + \frac{\lambda_M^2\bar{\phi}^4}{256\pi^2}\left(\log\frac{\lambda\bar{\phi}^2}{2\mu^2} - \frac{3}{2}\right) \tag{10.100}$$

The dependence in μ cancels as it should, and we are left with

$$V_e(\bar{\phi}) = \lambda_M\frac{\bar{\phi}^4}{24} + \frac{\lambda_M^2\bar{\phi}^4}{256\pi^2}\left(\log\frac{\bar{\phi}^2}{M^2} - \frac{25}{6}\right) \tag{10.101}$$

10.6 The conformal anomaly

In the case of the standard scalar laplacian,

$$\Delta \equiv \nabla^2 \equiv \frac{1}{\sqrt{g}}\partial_\mu\left(g^{\mu\nu}\sqrt{g}\partial_\nu\right) \tag{10.102}$$

the conformal weight coincides with its mass dimension, $\lambda = 2$.

The new zeta function after a global Weyl transformation is given in general by

$$\widetilde{\zeta}(s) = \Omega^{-\lambda s}\ \zeta(s) \tag{10.103}$$

so that the determinant defined through the ζ-function scales as

$$\det\ \widetilde{\Delta} = \Omega^{-\lambda\zeta(0)}\ \det\ \Delta \tag{10.104}$$

and this modifies correspondingly the effective action

$$\widetilde{W} = W + \ \lambda\ \omega\ \zeta(0). \tag{10.105}$$

where we have defined

$$\Omega \equiv 1 + \omega \tag{10.106}$$

The energy-momentum tensor is *defined* in such a way that under a general variation of the metric the variation of the effective action reads

$$\delta W \equiv \frac{1}{2}\int d(vol)\ T_{\mu\nu}\ \delta g^{\mu\nu} \tag{10.107}$$

which in the particular case that this variation is proportional to the metric tensor itself (like in a conformal transformation at the lineal level),

$$\delta g^{\mu\nu} = -2\ \omega\ g^{\mu\nu} \tag{10.108}$$

yields the integrated trace of the energy-momentum tensor

$$\delta W = -\int d(vol)\ \omega T. \tag{10.109}$$

Conformal invariance in the above sense then means that the energy-momentum tensor must be traceless. When quantum corrections are taken into account, it follows that

$$-\int d(vol) T = \lambda\ \zeta(0). \tag{10.110}$$

We have already seen that

$$\zeta(0) \equiv \lim_{s\to 0} s\int_0^\infty\ d\tau\ \tau^{s-1} K(\tau) = \lim_{s\to 0} s\int_0^1\ d\tau\ \tau^{s-1} K(\tau) = a_{\frac{d}{2}} \tag{10.111}$$

where d is the specific value of the spacetime dimension. The conformal anomaly is usually then written as

$$-\int d(vol) T = \lambda a_{\frac{d}{2}}. \tag{10.112}$$

The Schwinger-de Witt coefficient corresponding to the physical dimension, $n = \frac{d}{2}$ precisely coincides with the divergent part of the effective action when computed in dimensional regularization as indicated above. This means that in order to compute the one loop conformal anomaly in many cases it is enough to compute the corresponding counterterm.

This argument shows clearly that when the conformal weight of the operator of interest vanishes, $\lambda = 0$ all eigenvalues remain invariant and there is no conformal anomaly for determinants defined through the zeta function.

10.7 Flat space determinants

Let us see in detail how the heat equation can be iterated to get the coefficients of the short time expansion for operators pertaining to flat space gauge theories.

The small proper time expansion of the heat kernel should be substituted into the heat equation for the gauge operator as above, It follows

$$\frac{\partial}{\partial\tau} K\left(\tau; x, y\right) = \frac{1}{(4\pi)^{\frac{n}{2}}}\, e^{-\frac{(x-y)^2}{4\tau}} \sum_{p=0} \left(a_p \frac{(x-y)^2}{4} + \left(p - \frac{n}{2} - 1\right) a_{p-1} \right) \tau^{p-2-\frac{n}{2}} \tag{10.113}$$

Now derive with respect to the coordinates

$$\partial_\mu K = \frac{1}{(4\pi\tau)^{\frac{n}{2}}} e^{-\frac{(x-y)^2}{4\tau}} \sum_p \left(-\frac{\sigma_\mu}{2\tau} a_p + \partial_\mu a_p \right) \tau^p$$

$$D_\mu K\left(\tau; x, y\right) = \frac{1}{(4\pi\tau)^{\frac{n}{2}}} e^{-\frac{(x-y)^2}{4\tau}} \sum_p \left(-\frac{\sigma_\mu}{2\tau} a_p + D_\mu a_p \right) a_p \tag{10.114}$$

Finally,

$$D_\mu D^\mu K\left(\tau; x, y\right) =$$

$$= \frac{1}{(4\pi)^{\frac{n}{2}}}\, e^{-\frac{(x-y)^2}{4\tau}} \sum_p \left(-\frac{n}{2\tau} a_p + \frac{(x-y)^2}{4\tau^2} a_p - \sum_\mu \frac{\sigma^\mu}{\tau} D_\mu a_p + D^2 a_p \right) \tau^{p-\frac{n}{2}} =$$

$$= \frac{1}{(4\pi)^{\frac{n}{2}}}\, e^{-\frac{(x-y)^2}{4\tau}} \sum_p \left(-\frac{n}{2}\, a_{p-1} + \frac{(x-y)^2}{4} a_p - \sum_\mu \sigma^\mu\, D_\mu a_{p-1} + D^2 a_{p-2} \right) \tau^{p-2-\frac{n}{2}}$$

Now in order to be an elliptic operator,in flat space we have got to define when $X = Y = 0$

$$\Delta \equiv -\partial_\mu \partial^\mu \tag{10.115}$$

This can be easily checked with the form of the fundamental flat space solution. Let us then agree that

$$\Delta \equiv -D_\mu D^\mu + Y \tag{10.116}$$

It follows

$$-\Delta K\left(\tau; x, y\right) = \left(D_\mu^2 - Y\right) K =$$

$$= \frac{1}{(4\pi)^{\frac{n}{2}}}\, e^{-\frac{(x-y)^2}{4\tau}} \sum_p \left(-\frac{n}{2}\, a_{p-1} + \frac{(x-y)^2}{4} a_p - \sum_\mu \sigma^\mu\, D_\mu a_{p-1} - \Delta a_{p-2} \right) \tau^{p-2-\frac{n}{2}}$$

(where $\sigma_\mu \equiv (x_\mu - y_\mu)$).

The more divergent terms are those in $\tau^{-2-\frac{n}{2}}$, but they do not give anything new

$$\frac{a_0}{4}(x-y)^2 = \frac{a_0}{4}(x-y)^2 \tag{10.117}$$

The next divergent term (only even p contribute to the expansion without boundaries) is $\tau^{1-\frac{n}{2}}$

$$-\frac{n}{2}a_0 = -\frac{n}{2}a_0 - \sigma.Da_0 \tag{10.118}$$

ao that we learn that

$$\sigma.Da_0 = 0 \tag{10.119}$$

Generically,

$$\left(p - \frac{n}{2} - 1\right) a_{p-1} = -\frac{n}{2}a_{p-1} - \sigma.Da_{p-1} - \Delta a_{p-2} \tag{10.120}$$

which is equivalent to

$$(p+1)a_{p+1} = -\sigma.Da_{p+1} - \Delta a_p \tag{10.121}$$

Taking the covariant derivative of the first equation,

$$D_\lambda(\sigma^\mu D_\mu a_0) = 0 = D_\lambda a_0 + \sigma^\mu D_\lambda D_\mu a_0 \tag{10.122}$$

the first *coincidence limit* follows

$$[D_\mu a_0] = 0 \tag{10.123}$$

(please note that $[a_0] = 1$ which we knew already, does not imply the result.) Taking a further derivative, we get

$$[(D_\mu D_\nu + D_\nu D_\mu)a_0] = 0 \tag{10.124}$$

whose trace reads

$$\left[D^2 a_0\right] = 0 \tag{10.125}$$

The usual definition

$$W_{\mu\nu} \equiv [D_\mu, D_\nu] \tag{10.126}$$

implies

$$[D_\mu D_\nu \; a_0] = \frac{1}{2}\left[(\left[D_\mu, D_\nu\right]_- + \{D_\mu, D_\nu\})a_0\right] = \frac{1}{2}W_{\mu\nu} \tag{10.127}$$

where the fact has been used that

$$[a_0] = 1 \tag{10.128}$$

Taking $p = 0$ in (10.121)

$$-a_1 = \Delta a_0 + \sigma.Da_1 \tag{10.129}$$

so that

$$[a_1] = -[\Delta a_0] = \left[D^2 - Y\right] a_0 = -Y \tag{10.130}$$

(since $\Delta = -D^2 + Y$). When $p = 1$ in (10.121)

$$-2a_2 = \Delta a_1 + \sigma.Da_2 \tag{10.131}$$

so that

$$[a_2] = -\frac{1}{2}[\Delta a_1] \tag{10.132}$$

Let us derive again the $p = 0$ expression before the coincidence limit, namely We get

$$-D_\mu a_1 = D_\mu \Delta a_0 + D_\mu(\sigma.Da_1) = D_\mu \Delta a_0 + D_\mu a_1 + \sigma^\lambda D_\mu D_\lambda a_1 \tag{10.133}$$

Then

$$-2D_\mu a_1 = D_\mu \Delta a_0 + \sigma^\lambda D_\mu D_\lambda a_1 \tag{10.134}$$

Deriving once more, this implies at the coincidence limit

$$-2\left[D^2 a_1\right] = \left[D^2 \Delta a_0\right] + \left[D^2 a_1\right] \tag{10.135}$$

Then

$$\left[D^2 a_1\right] = -\frac{1}{3}\left[D^2 \Delta a_0\right] \tag{10.136}$$

in such a way that

$$-[\Delta a_1] \equiv \left[D^2 a_1\right] - [Y a_1] = \frac{1}{3}\left[D^2 D^2 a_0\right] - \frac{1}{3}D^2 Y + Y^2 \tag{10.137}$$

Now deriving three times the equation (10.119)

$$\begin{aligned}
&\left(D_{\mu_2}D_{\mu_1} + D_{\mu_1}D_{\mu_2} + \sigma^\lambda D_{\mu_2}D_{\mu_1}D_\lambda\right) a_0 = 0\\
&\left(D_{\mu_3}D_{\mu_2}D_{\mu_1} + D_{\mu_3}D_{\mu_1}D_{\mu_2} + D_{\mu_2}D_{\mu_1}D_{\mu_3} + \sigma^\lambda D_{\mu_3}D_{\mu_2}D_{\mu_1}D_\lambda\right) a_0 = 0\\
&(D_{\mu_4}D_{\mu_3}D_{\mu_2}D_{\mu_1} + D_{\mu_4}D_{\mu_3}D_{\mu_1}D_{\mu_2} + D_{\mu_4}D_{\mu_2}D_{\mu_1}D_{\mu_3} + D_{\mu_3}D_{\mu_2}D_{\mu_1}D_{\mu_4} +\\
&+\sigma^\lambda D_{\mu_4}D_{\mu_3}D_{\mu_2}D_{\mu_1}D_\lambda\big) a_0 = 0
\end{aligned} \tag{10.138}$$

$$(D_\delta D_\sigma D_\rho D_\mu + D_\delta D_\sigma D_\mu D_\rho + D_\delta D_\rho D_\mu D_\sigma + D_\sigma D_\rho D_\mu D_\delta + \sigma^\lambda D_\delta D_\sigma D_\rho D_\mu D_\lambda) a_0 = 0 \tag{10.139}$$

Contracting with $\eta^{\mu_4\mu_3}\eta^{\mu_1\mu_2}$

$$2\left[(D^2D^2 + D^\mu D^2 D_\mu)a_0\right] = 0 \tag{10.140}$$

and contracting instead with $\eta^{\mu_4\mu_2}\eta^{\mu_3\mu_1}$

$$2\left[(D^\mu D^\nu D_\mu D_\nu)a_0\right] + \left[D^\mu D^2 D_\mu a_0\right] + \left[D^2 D^2 a_0\right] = 0 \tag{10.141}$$

so that also

$$\left[(D^\mu D^\nu D_\mu D_\nu)a_0\right] = 0 \tag{10.142}$$

Now

$$[(D^\sigma D^\mu D_\mu D_\sigma)a_0] = [(D^\mu D^\sigma D_\mu D_\sigma a_0 + W^{\sigma\mu} D_\mu D_\sigma a_0] \tag{10.143}$$

It follows that

$$\left[D^\alpha D^2 D_\alpha a_0\right] = 0 + W^{\sigma\mu}\left[D_\mu D_\sigma a_0\right] = -\frac{1}{2}W^2 \tag{10.144}$$

so that

$$\left[D^2 D^2 a_0\right] = \frac{1}{2}W^2 \tag{10.145}$$

and finally

$$[a_2] = -\frac{1}{2}\left[\Delta a_1\right] = \frac{1}{6}\left[D^2 D^2 a_0\right] + \frac{1}{2}Y^2 + \frac{1}{6}D^2 Y = \frac{1}{12}W^2 + \frac{1}{2}Y^2 + \frac{1}{6}D^2 Y \tag{10.146}$$

The final expression for the divergent piece of the determinant of the flat space gauge operator reads

$$\log\ \det \Delta = -\frac{2}{(4-n)}\ \frac{i}{(4\pi)^2}\int d^n x\ \mathrm{tr}\ \left(\frac{1}{12}W^{\mu\nu}W_{\mu\nu} + \frac{1}{2}Y^2\right) \tag{10.147}$$

(the term in $D^2 Y$ vanishes as a surface term).

10.8 The Casimir effect

Let us work out the simplified case of scalar Dirichlet boundary conditions in

$$x = 0, a \tag{10.148}$$

The frequencies are now quantized

$$\lambda_n = \frac{\pi}{a}\mathbb{N} \tag{10.149}$$

Define as usual

$$\zeta(s) \equiv \sum_n \lambda_n^{-s} = \sum_n \left(\frac{\pi n}{a}\right)^{-s} \tag{10.150}$$

This is closely related to the original Riermann's zeta function

$$\zeta_R(s) \equiv \sum_{n=1}^{\infty} n^{-s} \tag{10.151}$$

and actually the vacuum energy can be defined to be given by

$$E(a) \equiv \frac{1}{2}\sum \lambda_n \equiv \frac{1}{2}\zeta_R(-1) = -\frac{1}{24a} \tag{10.152}$$

In three dimensions the resulting force is

$$F(a) \equiv -\frac{dE(a)}{da} = -\frac{\pi^2 \hbar c}{240 a^4} A \tag{10.153}$$

where A is the area of the walls.

Let us now examine to what extent this result is regularization independent. Casimir himself proposed the following general regularization

$$E(a) \equiv \frac{\pi}{2}\sum_n \frac{n}{a} f\left(\frac{n}{a\Lambda}\right) \tag{10.154}$$

where $f(x)$ is a function of which more in a moment. Consider a finite space with length $0 \leq x \leq L$ with the plates inside separated by a distance $0 \leq x \leq a$. The total energy of the $L - a$ side of the plates will be

$$E\,(L-a) = \frac{\pi}{2}\,(L-a)\,\Lambda^2 \sum_n \frac{n}{(L-a)^2\Lambda^2}\; f\left(\frac{n}{(L-a)\Lambda}\right) \tag{10.155}$$

In the continuum limit $(L \rightarrow \infty)$

$$E\,(L-a)\,0 = \frac{\pi}{2}\; L\;\Lambda^2 \int dx\; x\; f(x) - \frac{\pi}{2}\; a\;\Lambda^2 \int dx\; x\; f(x) \tag{10.156}$$

The first term is independent of a. Now, doing the change of variables $x \equiv \frac{n}{a\Lambda}$ in the second, we have

$$E \equiv E(a) + E(L-a) = C_L + \frac{\pi}{2a}\left(\sum_n n f(\frac{n}{a\Lambda}) - \int n dn f\left(\frac{n}{a\Lambda}\right)\right) \tag{10.157}$$

This is given by the Euler-MacLaurin formula

$$\sum_{n=1}^{N} F(n) - \int_0^N F(n)dn = \frac{F(0)+F(N)}{2} + \frac{F'(N)-F'(0)}{12} + \ldots + B_j \frac{F^{(j-1)}(N)-F^{(j-1)}(0)}{j!} + \ldots \tag{10.158}$$

where B_j are Bernoulli's numbers. For us

$$F(n) \equiv nf\left(\frac{n}{a\Lambda}\right) \tag{10.159}$$

We conclude that

$$E = C_L - \frac{\pi f(0)}{24a} - \frac{B_4}{4!}\frac{3\pi}{2a^3\Lambda^2}\ f''(0) + \ldots \tag{10.160}$$

Then the class of regulator with

$$\begin{aligned} &f(0) = 1 \\ &\lim_{x\to\infty}\ x\ f^{(j)}(0) = 0 \end{aligned} \tag{10.161}$$

all yield the same value for the Casimir energy.

10.9 First quantized QFT and one dimensional quantum gravity.

Let us consider the lagrangian for a free massive relativistic particle

$$S = \frac{1}{2}\int e(\tau)d\tau\left(\frac{1}{e^2}\dot{x}^2 + m^2\right) \tag{10.162}$$

Under a reparameterization the *einbein* behaves as

$$ed\tau = e'd\tau' \tag{10.163}$$

Then the canonical momenta

$$p_\mu = m\frac{\dot{x}_\mu}{e} \tag{10.164}$$

ao that the hamiltonian reads

$$H = p_\mu \dot{x}^\mu - L = \frac{ep^2}{m} - \frac{ep^2}{2m} - \frac{em^2}{2} = e\frac{p^2 - m^2}{2} \tag{10.165}$$

Let us now consider the probability amplitude from propagation from a point x to another pont y. The only invariant of the path from the two pints is the proper length, T.

$$K(x,y) \equiv \int_0^\infty dT\ \langle y|e^{-TH}|x\rangle = \int_0^\infty dT \int \frac{d^np}{(2\pi)^n}e^{ip(x-y)}e^{-\frac{T}{2}(p^2-m^2)} = \int \frac{d^np}{(2\pi)^n}e^{ip(x-y)}\ \frac{2}{p^2-m^2} \tag{10.166}$$

In order to take into account interatcions, consider a one-dimensional graph connecting a number of external points $x_1 \ldots x_n$

$$\Gamma\left(x_1 \ldots x_n\right) \tag{10.167}$$

through some vertices located at points $y_1 \ldots y_m$.

We have to integrate over all proper lengths of the internal lines connecting the different points $y_1 \ldots y_m$. This yields a propagator for each internal line.

The integration over the positions of the vertices $y_1 \ldots y_m$ themselves then yield momentum conservation at each vertex.

Eventually all Feynman rules are reconstructed in the first quantized formalism.

This shows an intimate relationship between QFT and one dimensional quantum gravity, not unlike the relationship between string theory and CFT_2. Let us now be more specific and compute the scalar effective action in an electromagnetic background. We start from

$$L = \phi^+ D^2 \phi \tag{10.168}$$

where

$$D_\mu \equiv \partial_\mu - igA_\mu \tag{10.169}$$

We have

$$\Gamma[A] = -\log\det\left(-D^2\right) = \int_0^\infty \frac{d\tau}{\tau}\int \frac{d^4p}{(2\pi)^4}\langle p|e^{-\frac{1}{2}e\tau(p-gA)^2}|p\rangle =$$
$$= \int_0^\infty \frac{d\tau}{\tau} N\int \mathcal{D}x\; e^{-\int_0^\tau ds\frac{1}{2e}\dot{x}^2+igA[x(s)].\dot{x}} \tag{10.170}$$

With minkowskian signature

$$\Gamma[A] = \int_0^\infty \frac{d\tau}{\tau} N\int \mathcal{D}x\; e^{-\int_0^\tau ds\frac{1}{2e}\dot{x}^2}\;\; e^{ig\oint A[x(s)]} \tag{10.171}$$

which is just the expectation value of a Wilson loop. We shall assume this formula to remail true also in the non-abelian case, with the trivial modifications.

Let us now expand this action to order g^N.

$$\Gamma_N[A] = \frac{(ig)^N}{N!}\int_0^\infty \frac{d\tau}{\tau} N\int \mathcal{D}x\; e^{-\int_0^\tau ds\frac{1}{2e}\dot{x}^2}\;\mathrm{Tr}\;\prod_{i=1}^N\int_0^\tau ds_i A\left[x(s_i)\right].\dot{x}(s_i) \tag{10.172}$$

Insert now for A_μ a sum of states with outgoing momentum k_i, polarization ϵ_i and gauge charge T_a

$$A^\mu(x) \equiv \sum_{i=1}^N T^{a_i}\epsilon_i^\mu e^{ik_i x} \tag{10.173}$$

where T^{a_i} belongs to the appropiate representation for the scalar field. Keeping onpy the terms in which each mode appears precisely once,

$$\Gamma_N\left(k_1\ldots k_N\right) = (ig)^N\int_0^\infty \frac{d\tau}{\tau} N\int \mathcal{D}x\; e^{-\int_0^\tau ds\frac{1}{2e}\dot{x}^2}$$
$$\mathrm{Tr}\;\;\left(T^{a_N}\ldots T^{a_1}\right)\prod_{i=1}^N\int_0^{s_{i+1}} ds_i\;\epsilon_i.\dot{x}(s_i)\;e^{ik_i x(s_i)} \tag{10.174}$$

Let us first neglect the polarizations. The total source reads

$$J^\mu(s) = \sum_{j=1}^N ik_j^\mu\delta\left(\sigma-\sigma_j\right) \tag{10.175}$$

Then

$$\Gamma_N\left(k_1\ldots k_N\right)=\frac{4(ig)^N}{(4\pi)^2e^2}\ \mathrm{Tr}\ \left(T^{a_N}\ldots T^{a_1}\right)\int_0^\infty\frac{d\tau}{\tau^3}\prod_{i=1}^N ds_i\ \times$$
$$\times e^{-\frac{1}{2}\ \int_0^\tau ds\int_0^\tau ds' J^\mu(s)G_B(s,s')\eta_{\mu\nu}J^\nu(s')}=$$
$$=\Gamma_N\left(k_1\ldots k_N\right)=\frac{4(ig)^N}{(4\pi)^2e^2}\ \mathrm{Tr}\ \left(T^{a_N}\ldots T^{a_1}\right)\int_0^\infty\frac{d\tau}{\tau^3}\prod_{i=1}^N ds_i\ \times$$
$$\times e^{\frac{1}{2}\sum_{i,j=1}^N\ k_i.k_jG_B(s_i,s_j)} \tag{10.176}$$

Here G_B is a one-dimensional Green's function, to be discussed in a moment. Before doing that, let us take care of the polarizations. The standard methos is to exponentiate them, keeping only linear terms. This changes the sources

$$J^\mu(s)=\sum_{i=1}^N\delta(s-s_i)\left(\epsilon_i^\mu\partial_{s_i}+ik_i^\mu\right) \tag{10.177}$$

Now we get

$$\Gamma_N\left(k_1\ldots k_N\right)=\Gamma_N\left(k_1\ldots k_N\right)=\frac{4(ig)^N}{(4\pi)^2e^2}\ \mathrm{Tr}\ \left(T^{a_N}\ldots T^{a_1}\right)\int_0^\infty\frac{d\tau}{\tau^3}\prod_{i=1}^N ds_i\ \times$$
$$\times e^{\frac{1}{2}\sum_{i,j=1}^N k_i.k_jG_B(s_j-s_i)-2ik_i.\epsilon_j\partial_{s_j}G_B(s_j-s_i)-\epsilon_i.\epsilon_j\partial^2_{s_js_i}G_B(s_j-s_i)}\Big|_{\text{linear in each }\epsilon_i} \tag{10.178}$$

The bosonic Green function obeys

$$e^{-1}\frac{\partial^2}{\partial s^2}\ G_B\left(s,s'\right)=\delta\left(s-s'\right) \tag{10.179}$$

On the real line the solution reads

$$G_B\left(s,s'\right)=\frac{e}{2}\log\ |s-s'|+C_1+C_2s \tag{10.180}$$

On a circle of circumference τ we have to modify the equation

$$e^{-1}\frac{\partial^2}{\partial s^2}G_B\left(s,s'\right)=\delta\left(s-s'\right)-\frac{1}{\tau} \tag{10.181}$$

and the periodic solution is

$$G_B\left(s,s'\right)=\frac{e}{2}\left(\log\ |s-s'|-\frac{\left(s-s'\right)^2}{\tau}\right) \tag{10.182}$$

Now do a few things

- Owing to the fact that $G_B(s,s)=0$ as well as $\partial G_B(s,s)=0$, the terms in $\epsilon_i.k_i$ as well as the terms with k_i^2 can be removed without the use of the EM.

- Replace
$$s_i \to u_i \tau \tag{10.183}$$
This factors out N powers of τ.
- The integral over ds_N is trivial. Also fix the origin of proper time by choosing
$$s_N \equiv \tau \tag{10.184}$$
- Choose the gauge
$$e = 2 \tag{10.185}$$
- Perform the momentum integrals in
$$n = 4 - \epsilon \tag{10.186}$$
dimensions.

Then we get

$$\Gamma_N \left(k_1 \dots k_N\right) = \frac{\left(ig\mu^{\frac{\epsilon}{2}}\right)^N}{(4\pi)^{2-\frac{\epsilon}{2}}} \text{ Tr } \left(T^{a_N} \dots T^{a_1}\right)$$
$$\int_0^\infty \frac{d\tau}{\tau^{3-N\frac{\epsilon}{2}}} \int_0^1 du_{N-1} \int_0^{u_{N-1}} du_{N-2} \dots \int_0^{u_2} du_1 \; e^{\sum_{i<j=1}^N k_i.k_j G_B^{ji}}$$
$$e^{\sum_{i<j=1}^N (-ik_i.\epsilon_j - k_j.\epsilon_i)\dot{G}_B^{ji} + \epsilon_i.\epsilon_j \ddot{G}_B^{ji}}\Big|_{\text{linear in each } \epsilon} \tag{10.187}$$

The Green functions read

$$\begin{aligned} G_B &= \tau\left(|u-u'| - (u-u')^2\right) \\ \dot{G}_B &= \text{sign}\left(u-u'\right) - 2\left(u-u'\right) \\ \ddot{G}_B &= \frac{2}{\tau}\left(\delta\left(u-u'\right) - 1\right) \end{aligned} \tag{10.188}$$

The term

$$e^{\sum_{i<j=1}^N k_i.k_j G_B^{ji}} = e^{\tau \sum_{i<j=1}^N k_i.k_j \left(|u-u'| - (u-u')^2\right)} \tag{10.189}$$

after integrating over $d\tau$ yields the usual Feynman parametrized denominator corresponding to a scalar loop integral.

The integral

$$\int_0^\infty d\tau \tau^a e^{\sum_{i<j=1}^N k_i.k_j G_B^{ji}} = \frac{\Gamma\left(1+a\right)}{\left(-\sum_{i<j=1}^N k_i.k_j\left(|u-u'| - (u-u')^2\right)\right)^{1+a}} \tag{10.190}$$

and the remaining term

$$e^{\sum_{i<j=1}^N (-ik_i.\epsilon_j - k_j.\epsilon_i)\dot{G}_B^{ji} + \epsilon_i.\epsilon_j \ddot{G}_B^{ji}}\Big|_{\text{linear in each } \epsilon} \tag{10.191}$$

provides the numerator of the Feynman parameter integral. More details can be found in Strassler's paper [14] and references therein.

11

The singlet chiral anomaly.

Consider a set of left and right fermions in an external gauge field

$$L = \bar{\psi} i \not{D}(A) \psi = \bar{\psi}_L i \not{D}(A) \psi_L + \bar{\psi}_R i \not{D}(A) \psi_R \tag{11.1}$$

When necessary, we shall use Weyl's representation of Dirac's gamma matrices

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{11.2}$$

and

$$\gamma^i = \begin{pmatrix} 0 & -\vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \tag{11.3}$$

In this form, the operator $i\not{\partial}$ reads

$$\not{\partial} = \begin{pmatrix} 0 & i\partial_0 + i\vec{\sigma}\vec{\nabla} \\ i\partial_0 - i\vec{\sigma}\vec{\nabla} & 0 \end{pmatrix} \tag{11.4}$$

Finally

$$\gamma_5 \equiv i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{11.5}$$

is such that

$$\{\gamma_5, \gamma_\mu\} = 0 \tag{11.6}$$

In that way the left and right projectors

$$P_L \equiv \frac{1}{2}(1 + \gamma_5) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{11.7}$$

as well as

$$P_R \equiv 1 - P_L = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{11.8}$$

. To be specific

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \tag{11.9}$$

It is plain that

$$\bar{\psi}_L \equiv (P_L\psi)^+\gamma_0 = \psi^+ P_L \gamma_0 = \bar{\psi} P_R \tag{11.10}$$

Kinetic energy terms do not mix chiralities

$$L = \bar{\psi} i \not{D} \psi = \bar{\psi}_L i \not{D} \psi_L + \bar{\psi}_R i \not{D} \psi_R \tag{11.11}$$

which is not the case with either masses or Yulawa couplings

$$L_m \equiv \bar{\psi} m \psi = \bar{\psi}_L m \psi_R + \bar{\psi}_R m \psi_L \tag{11.12}$$

Charge conjugates are defined by

$$\psi^c = -\gamma_2 \psi^* = \begin{pmatrix} \sigma_2 \psi_R^* \\ -\sigma_2 \psi_L^* \end{pmatrix} \tag{11.13}$$

Let us also recall the well-known fact that the whole action cal be expressed in terms of left-handed fields

$$(\bar{\psi}_c)_L = (0, \psi_R^T \sigma_2) \tag{11.14}$$

Also

$$(\bar{\psi}_c)_R = (-\psi_L^T \sigma_2, 0) \tag{11.15}$$

In fact

$$\begin{aligned} &(\bar{\psi}_c)_L i \not{D} (\psi_c)_L = \psi_R \sigma_2 (i\partial_0 - i\vec{\sigma}\vec{\nabla})\sigma_2 \psi_R^* = \\ &\psi_R^T (i\partial_0 + i\vec{\sigma}^*\vec{\nabla})\psi_R^* = -i\partial_0 \psi_R^+ \psi_R - i\vec{\nabla}\psi_R^+ \vec{\sigma} \psi_R \end{aligned} \tag{11.16}$$

(we have taken into account that $\sigma_i^* = -\sigma_2 \sigma_i \sigma_2$). Integrating by parts this yields

$$i\psi_R^+ \partial_0 \psi_R + i\psi_R^+ \vec{\sigma} \vec{\nabla} \psi_R \tag{11.17}$$

which is precisely

$$\bar{\psi}_R i \not{D} \psi_R \tag{11.18}$$

All this holds independently of the structure of any non-spinorial indices the fermions may have

For example, if we have a Dirac spinor with a flavor index $i = 1 \ldots N$, we can always define a $2N$ left component spinor

$$\Psi \equiv \begin{pmatrix} \psi_L \\ \psi_L^c \end{pmatrix} \tag{11.19}$$

The kinetic energy piece reads

$$L = \bar{\Psi} i \not{D} \Psi \tag{11.20}$$

and is $U(2N)$ invariant under

$$\delta \Psi = i U \Psi \tag{11.21}$$

Majorana spinors ara self-conjugate $\psi = \psi^c$. Then

$$\psi^M = \begin{pmatrix} \psi_L \\ -\sigma_2 \psi_L^* \end{pmatrix} \tag{11.22}$$

Both Weyl and Majorana spinors have only two complex independent components, which is half those of a Dirac spinor.

Majorana masses are couplings of the form

$$M \bar{\psi}_M \psi_M = -M \left(\psi_L^T \sigma_2 \psi_L + \psi_L^+ \sigma_2 \psi_L^* \right) \tag{11.23}$$

and they violate fermion number conservation.

This lagrangian is invariant under two different global transformations. This first is the *vector* one

$$\delta \psi = i \epsilon \psi \tag{11.24}$$

that is

$$\begin{aligned} \delta \psi_L &= i \epsilon \psi_L \\ \delta \psi_R &= i \epsilon \psi_R \end{aligned} \tag{11.25}$$

The corresponding Noether current is fermion number conservation

$$j_\mu = \bar{\psi} \gamma_\mu \psi \tag{11.26}$$

The second symmetry is the *axial* or *chiral*

$$\delta \psi = i \epsilon \gamma_5 \psi \tag{11.27}$$

that is

$$\delta \psi_L = i \epsilon \psi_L \tag{11.28}$$

$$\delta \psi_R = -i \epsilon \psi_R \tag{11.29}$$

The corresponding Noether current reads

$$j_5^\mu \equiv \bar{\psi} \gamma_5 \gamma^\mu \psi \tag{11.30}$$

It is plain that

$$\begin{aligned} \bar{\psi} \gamma^\mu \psi &= \psi_L \gamma^\mu \psi_L + \bar{\psi}_R \gamma^\mu \psi_R \\ \bar{\psi} \gamma_5 \gamma^\mu \psi &= \psi_L \gamma^\mu \psi_L - \bar{\psi}_R \gamma^\mu \psi_R \end{aligned} \tag{11.31}$$

What happens is that in quantum mechanics there is an anomaly in the latter current. Besides, thie anomaly is a total derivative.

$$\partial_\mu j_5^\mu \equiv \mathcal{A} = \frac{g^2}{16\pi^2}\text{tr } \epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu} = \frac{g^2}{4\pi^2}\epsilon^{\alpha\beta\gamma\delta}\text{Tr}\,\partial_\alpha\left(A_\beta\partial_\gamma A_\delta + \frac{2}{3}A_\beta A_\gamma A_\delta\right) \tag{11.32}$$

The abelian anomaly does not signal any inconsistency, but it has some important phenomenological consequences.

The fact that we keep conservation of the vector current implies that the left anomaly is equal and opposite in sign from the right anomaly.

$$\partial_\mu j_L^\mu \equiv \partial_\mu(\bar{\psi_L}\gamma^\mu\psi_L) = -\partial_\mu j_R^\mu \equiv -\partial_\mu(\bar{\psi_R}\gamma^\mu\psi_R) = \frac{1}{2}\mathcal{A} \tag{11.33}$$

This anomaly is called the *abelian anomaly*, although it is present also in nonabelian theories.

There is also a *non abelian anomaly* in the non abelian current

$$j_\mu^a \equiv \bar{\psi}\gamma_\mu T^a\ P_+\ \psi \tag{11.34}$$

namely

$$D_\mu j_a^\mu = \frac{1}{24\pi^2}\text{Tr }\ T_a\ \epsilon^{\alpha\beta\gamma\delta}\partial_\alpha\left(A_\beta\partial_\gamma A_\delta + \frac{1}{2}A_\beta A_\gamma A_\delta\right) \tag{11.35}$$

It is striking the similarity of this expression with the abelian anomaly [11.32]. It is no less remarkable the difference betewen the factor $\frac{2}{3}$ and the factor $\frac{1}{2}$ in the trilinear terms.

Anomalous theories are believed to be inconsistent insofar as the anomalous current is coupled to gauge fields. In the standard model they cancel generation by generation between left and right contributions. Let us first concentrate in the abelian anomaly.

11.0.1 The Adler-Bell-Jackiw computation.

It is still worth looking in detail to the old fashioned perturbative calculation in massless QED with external vector and axial sources. Define

$$\Delta^5_{\lambda\mu\nu}(k_1,k_2) \equiv \mathcal{F}\langle 0|T J^5_\lambda(0) J_\mu(x_1) J_\nu(x_2)|0\rangle \tag{11.36}$$

The diagrams give

$$\Delta^5_{\lambda\mu\nu}(k_1,k_2) = i\int\frac{d^4p}{(2\pi)^4}\text{Tr }\gamma_\lambda\gamma_5\frac{1}{\not p - \not k_1 - \not k_2}\gamma_\nu\frac{1}{\not p - \not k_1}\gamma_\mu\frac{1}{\not p} + \\ +\gamma_\lambda\gamma_5\frac{1}{\not p - \not k_1 - \not k_2}\gamma_\mu\frac{1}{\not p - \not k_2}\gamma_\nu\frac{1}{\not p} \tag{11.37}$$

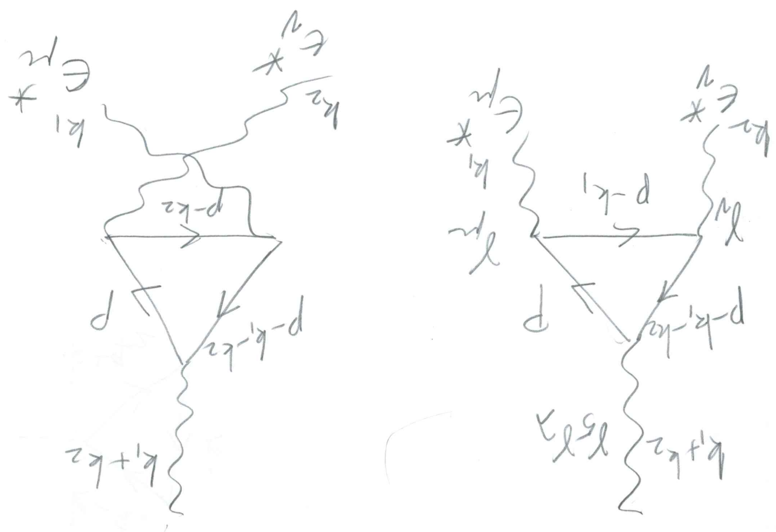

Figure 11.1: Triangle diagram

and

Ward's identity for the vector current (current conservation) implies

$$k_1^\mu \Delta^5_{\lambda\mu\nu} = k_2^\nu \Delta^5_{\lambda\mu\nu} = 0 \tag{11.38}$$

and the axial Ward identity

$$(k_1 + k_2)^\lambda \Delta^5_{\lambda\mu\nu} = 0 \tag{11.39}$$

Let us impose the vector current conservation, that is

$$k_1^\mu \Delta^5_{\lambda\mu\nu}(k_1, k_2) = i \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr} \left\{ \gamma_\lambda \gamma_5 \frac{1}{\not p - \not k_1 - \not k_2} \gamma_\nu \frac{1}{\not p - \not k_1} \not k_1 \frac{1}{\not p} + \right.$$
$$\left. \gamma_\lambda \gamma_5 \frac{1}{\not p - \not k_1 - \not k_2} \not k_1 \frac{1}{\not p - \not k_2} \gamma_\nu \frac{1}{\not p} \right\} \tag{11.40}$$

The integrand can also be written as

$$k_1^\mu \Delta^5_{\lambda\mu\nu}(k_1, k_2) = i \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr} \left\{ \gamma_\lambda \gamma_5 \frac{1}{\not p - \not q} \gamma_\nu \left(\frac{1}{\not p - \not k_1} - \frac{1}{\not p} \right) + \right.$$
$$\left. + \gamma_\lambda \ \gamma_5 \ \left(\frac{1}{\not p - \not q} - \frac{1}{\not p - \not k_2} \right) \ \gamma_\nu \ \frac{1}{\not p} \ \right\} =$$
$$i \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr} \left(\gamma_\lambda \gamma_5 \left\{ \frac{1}{\not p - \not q} \gamma_\nu \frac{1}{\not p - \not k_1} - \frac{1}{\not p - \not k_2} \gamma_\nu \frac{1}{\not p} \right\} \right) \tag{11.41}$$

This naively vanishes; it is just necessary to make the change of integration variable

$$p - k_1 \to p \tag{11.42}$$

But this is not kosher, because the integral does not converge, and we better be careful.

Let us examine what are the consequences of an arbitrary shift in the integration variable

$$p \to p + a \tag{11.43}$$

$$k_1^\mu \Delta^5_{\lambda\mu\nu}(a, k_1, k_2) = i \int \frac{d^4p}{(2\pi)^4} tr \, \gamma_\lambda \gamma_5 \left\{ \frac{1}{\not p + \not a - \not q} \gamma_\nu \frac{1}{\not p + \not a - \not k_1} - \right.$$
$$\left. - \ \frac{1}{\not p + \not a - \not k_2} \ \gamma_\nu \ \frac{1}{\not p + \not a} \right\}$$

We shall compute the difference

$$\left[k_1 \Delta^5 \right]_{\lambda\nu} \equiv k_1^\mu \Big(\Delta^5_{\lambda\mu\nu}(a, k_1, k_2) - \Delta^5_{\lambda\mu\nu}(a = 0, k_1, k_2) \Big) \tag{11.44}$$

Using Stokes'theorem ,

$$\int d^n p \left(f(p+a) - f(p) \right) = \int d^n p \; a^\mu \frac{\partial}{\partial p_\mu} \; f(p) + \ldots$$
$$= a^\mu n_\mu \; f_\infty \; S_{n-1} \tag{11.45}$$

(where

$$n_\mu \equiv \lim_{p\to\infty} \frac{p^\mu}{p} \in S_{n-1}$$
$$\lim_{p\to\infty} \frac{p^\mu p^\nu}{p^2} = \eta^{\mu\nu} \tag{11.46}$$

that is, assuming that the asymptotic value of the function does not depend on the polar angles

$$\lim_{p\to\infty} \; f(p) = \; f_\infty \tag{11.47}$$

Our function is given by

$$f(p) \equiv \text{Tr} \left(\gamma_\lambda \gamma_5 \frac{1}{\not p - \not k_2} \gamma_\nu \frac{1}{\not p} \right) = \epsilon_{\lambda\mu\nu\rho} \; \frac{k_2^\mu p^\rho}{(p-k_2)^2 p^2} \tag{11.48}$$

What we actually would really like to compute is the difference between doing that with $a_1 = a$ and doing it with $a_2 = a - k_1$. This is

$$k_\mu^1 \lim_{\Lambda\to\infty} \frac{\Lambda^\mu}{\Lambda} (2\pi^2 k^3) \frac{\gamma_\lambda \gamma_5 (\not\Lambda - \not k_2) \gamma_\nu \not\Lambda}{(\Lambda - k_2)^2 \Lambda^2} = -2\pi^2 i k_\mu^1 \lim_{\Lambda\to\infty} \frac{\Lambda^\mu}{\Lambda} \Lambda^3 \frac{1}{\Lambda^4} \epsilon_{\lambda\rho\nu\sigma} k_2^\rho \Lambda^\sigma =$$
$$= -i 2\pi^2 k_\mu^1 \epsilon_{\lambda\rho\nu\sigma} k_2^\rho \eta^{\mu\sigma} \tag{11.49}$$

This yields a nonvanishing, independent on the value of the shift a, and finite value for the vector Ward identity.

$$k_1^\mu \Delta^5_{\lambda\mu\nu} = 2 i \pi^2 \frac{1}{(2\pi)^4} k_1^\mu k_2^\rho \epsilon_{\lambda\rho\nu\mu} \tag{11.50}$$

It looks that there is no possible way to keep vector symmetry in the quantum theory. This would mean that QED does imply quantum violation of electric charge conservation.

In order to clarify the issue, let us go back to basics and define yet another shift-dependent correlator, before imposing any Ward identity. Our purpose in life is to examine what are the possibilities of imposing current conservation.

$$\Delta^5_{\lambda\mu\nu}(a, k_1, k_2) = i \int \frac{d^4 p}{(2\pi)^4} \text{Tr} \left\{ \gamma_\lambda \gamma_5 \frac{1}{\not p + \not a - \not k_1 - \not k_2} \gamma_\nu \frac{1}{\not p + \not a - \not k_1} \gamma_\mu \frac{1}{\not p + \not a} + \right.$$
$$\left. \gamma_\lambda \gamma_5 \frac{1}{\not p + \not a - \not k_1 - \not k_2} \gamma_\mu \frac{1}{\not p + \not a - \not k_2} \gamma_\nu \frac{1}{\not p + \not a} \right\} \tag{11.51}$$

Let us compute again the object $[\Delta]$. The function we have to analyze is now

$$f(p) \equiv \text{Tr}\ \gamma_\lambda \gamma_5 \frac{1}{\not p - \not q} \gamma_\nu \frac{1}{\not p - \not k_1} \gamma_\mu \frac{1}{\not p} = \text{Tr}\ \frac{\gamma_\lambda \gamma_5 \left(\not p - \not q\right) \gamma_\nu \left(\not p - \not k_1\right) \gamma_\mu \not p}{(p-q)^2 \left(p-k_1\right)^2 p^2} \tag{11.52}$$

Traces are easily computed

$$\text{Tr}\ \left(\gamma_\lambda \gamma_5 \not p \gamma_\nu \not p \gamma_\mu \not p\right) = \text{Tr}\ \gamma_\lambda \gamma_5 (2\eta_{\nu\mu_1} - \gamma_\nu \gamma_{\mu_1}) \gamma_{\mu_2} (2\eta_{\mu_3 \nu} - \gamma_{\mu_3} \gamma_\nu) p^{\mu_1} p^{\mu_2} p^{\mu_3} =$$
$$= \text{Tr}\ \gamma_\lambda \gamma_5 \gamma_\nu p^2 \gamma_{\mu_3} \gamma_\mu p^{\mu_3} \tag{11.53}$$

This leads to the expression

$$\left[\Delta_5^{\lambda\mu\nu}\right] = \frac{4i}{8\pi^2} \lim \frac{k_\alpha k_\beta}{k^2} \epsilon^{\beta\nu\mu\lambda} + (\mu, k_1 \rightarrow \nu k_2) = \frac{i}{8\pi^2} a_\alpha \epsilon^{\alpha\nu\mu\lambda} \tag{11.54}$$

Let us now consider the most general shift we can imagine, namely a linear combination of the momenta k_1 and k_2

$$a \equiv x(k_1 + k_2) + y(k_1 - k_2) \tag{11.55}$$

leading to

$$\left[\Delta^5_{\lambda\mu\nu}\right] \equiv \Delta^5_{\lambda\mu\nu}(a) - \Delta^5_{\lambda\mu\nu}(a=0) = \frac{iy}{4\pi^2} \epsilon_{\lambda\mu\nu\sigma} (k_1 - k_2)^\sigma \tag{11.56}$$

because the piece linear in $q \equiv k_1 + k_2$ disappears when symmetrizing on $(\mu, k_1) \leftrightarrow (\nu k_2)$.

There is an ambiguity in the momentum over which we integrate. The most general choice would be

$$k \rightarrow k + \lambda_1 k_1 + \lambda_2 k_2 \tag{11.57}$$

Doing that we get for the vector current

$$k_1^\mu M^5_{\lambda\mu\nu} = \frac{1}{4\pi^2} \left(1 - \lambda_1 + \lambda_2\right)\ \epsilon^{\lambda\nu\rho\sigma}\ k^1_\rho k^2_\sigma \tag{11.58}$$

as well as

$$q^\lambda M^5_{\lambda\mu\nu} = \frac{1}{4\pi^2} \left(\lambda_1 - \lambda_2\right)\ \epsilon^{\mu\nu\rho\sigma}\ k^1_\rho k^2_\sigma \tag{11.59}$$

Then we can choose

$$\lambda_2 - \lambda_1 = 1 \tag{11.60}$$

in order to implement vector current conservation; but then

$$q^\lambda M^5_{\lambda\mu\nu} = \frac{1}{4\pi^2}\ \epsilon^{\mu\nu\rho\sigma}\ k^1_\rho k^2_\sigma \tag{11.61}$$

11.0.2 Pauli-Villars regularization.

The Pauli-Villars regularization is a gauge invariant way of introducing a cutoff. The main idea stems from the fact that the difference of two propagators behaves much better at infinity than each one separately.

$$\frac{1}{p^2-m^2}-\frac{1}{p^2-M^2}=\frac{m^2-M^2}{(p^2-m^2)(p^2-M^2)} \tag{11.62}$$

Of course the minus sign in from of the propagator is not physical, and indicates that the corresponding particle is a ghost. One must make sure that all unwanted ghostly efects are gone when $M\rightarrow\infty$. This regularization works best with fermion loops (like the one appearing in the abelian vacuum polarization diagram), which can be understood as the determinant of Dirac's operator

$$\det\, i\not{D}_m \tag{11.63}$$

where

$$i\not{D}_m \equiv i\not{\partial} - e\not{A} - m \tag{11.64}$$

Then we substitute instead of the determinant the quantity

$$\det\, i\not{D}_m \prod_{i=i}^{i=n} \left(\det\, i\not{D}_{M_i}\right)^{c_i} \tag{11.65}$$

or what is the same,

$$\text{Tr}\,\log\; i\not{D}_m + \sum_{i=i}^{i=n} c_i \text{Tr}\,\log\,(\; i\not{D}_{M_i}) \tag{11.66}$$

The coefficients c_i cannot be all positive, because they have to obey

$$\begin{aligned}&\sum c_i + 1 = 0\\&\sum_i c_i M_i^2 + m^2 = 0\end{aligned} \tag{11.67}$$

This means that in general the regulators will violate the spin-statistics theorem,*id est* , they are ghosts.

In order to compute the j-th determinant we write

$$\begin{aligned}&\text{Tr}\,\log\; i\not{D}_{M_j} = \text{Tr}\,\log\;(i\not{\partial} - M_j)\left(1 - e(i\not{\partial}-M_j)^{-1}\not{A}\right) =\\&= \text{Tr}\,\log\;(i\not{\partial} - M_j) + \text{Tr}\,\log\left(1 - e(i\not{\partial}-M_j)^{-1}\not{A}\right) =\\&N + \sum_{n_1}^{\infty}\frac{(-e)^n}{n}\text{Tr}\int d^4x_1 d^4x_2\ldots d^4x_n\;\not{A}(x_1)S_j(x_1-x_2)\not{A}(x_2)\ldots\not{A}(x_n)S_j(x_n-x_1)\end{aligned}$$

where N is a divergent constant and

$$(i\not{\partial} - M_j)^{-1} \equiv S_j(x-y) \tag{11.68}$$

(we can include as well the physical mass as $M_0 \equiv m$). The Pauli-Villars'regulator loop in momentum space is proportional to

$$\int d^4k_1 \ldots d^4k_n \int d^4p \frac{\mathrm{Tr}\left(\gamma_{\mu_1}(\not p + M_j)\gamma_{\mu_2}(\not p + \not k_1 + M_j)\ldots\gamma_{\mu_n}(\not p + \not k_{n-1} + M_j)\right)}{(p^2 - M_j^2)((p+k_1)^2 - M_j^2)\ldots((p+k_{n-1})^2 - M_j^2)} \times$$
$$\times A^{\mu_1}(k_1)\ldots A^{\mu_n}(k_n)\ \delta^{(4)}(k_1 + k_2 + \ldots k_n) \tag{11.69}$$

Given the fact that the numerator of the integrand has mass dimension n whereas the denominator has mass dinension $2n$, the superficial degree of divergence of this diagram is

$$D = 4 - n \tag{11.70}$$

This means that all terms with $n \leq 4$ will be divergent. We can represent the integrand as a power series in the masses ($P_\lambda(p)$ represents a polynomial in p of degree λ).

$$\frac{P_n(p) + M_j^2 P_{n-2}(p) + \ldots + M_j^n}{P_{2n}(p) + M_j^2 P_{2n-2}(p) + \ldots + M_j^{2n}} = \frac{P_n(p)\left(1 + M_j^2 \frac{P_{n-2}(p)}{P_n(p)} + \ldots + M_j^n \frac{1}{P_n(p)}\right)}{P_{2n}(p)\left(1 + M_j^2 \frac{P_{2n-2}(p)}{P_{2n}(p)} + \ldots + M_j^{2n} \frac{1}{P_{2n}(p)}\right)} =$$
$$= \frac{P_n(p)}{P_{2n}(p)}\left(M_j^2\left(\frac{P_{n-2}(p)}{P_n(p)} - \frac{P_{2n-2}(p)}{P_{2n}(p)}\right) + \ldots\right) \tag{11.71}$$

The net contribution of the regulators is the sum of all this terms weighted with c_j. The coefficient of M_j^λ behaves at large momenta as $p^{-n-\lambda}$. If the weights are chosen to obey the conditions as above, this cancels the terms in M_j^0 (behaving as Λ^{4-n}) and M_j^2 (behaving as Λ^{2-n}). This is enough in or case. In other situations, we maight have to impose extra conditions to the coefficients c_j.

For our purposes, it is enough to consider a single regulator of mass M. The physical limit will be $m \to 0$ and $M \to \infty$. In the regularized theory, with finite M, we can safely perform changes of variables in the finite integrals

$$\Delta^{PV}_{\lambda\mu\nu}(k_1, k_2) \equiv \Delta_{\lambda\mu\nu}(m) - \Delta_{\lambda\mu\nu}(M) \tag{11.72}$$

The axial Ward identity reads

$$q^\lambda \Delta_{\lambda\mu\nu} \equiv \lim_{M\to\infty}\left[2m\Delta_{\mu\nu}(m) - 2M\Delta_{\mu\nu}(M)\right] \tag{11.73}$$

Let us compute the diagram corresponding to the regulator

$$\Delta_{\mu\nu}(M) = -i\int \frac{d^4p}{(2\pi)^4} tr\left(\frac{i}{\not p - M + i\epsilon}\gamma_5 \frac{i}{\not p - \not q - M + i\epsilon}\gamma_\nu \frac{i}{\not p - \not k_1 - M + i\epsilon}\gamma_\mu - \right.$$
$$\left.\frac{i}{\not p - M + i\epsilon}\gamma_5 \frac{i}{\not p - \not q - M + i\epsilon}\gamma_\mu \frac{i}{\not p - \not k_2 - M + i\epsilon}\gamma_\nu\right) \tag{11.74}$$

Introducing Feynman parameters,

$$\Delta_{\mu\nu}(M) = -\int \frac{d^4p}{(2\pi)^4} 2\int_0^1 dx_1 \int_0^{1-x_1} dx_2$$
$$\frac{\text{Tr } (\not{p}+M)\gamma_5(\not{p}-\not{q}+M)\gamma_\nu(\not{p}-\not{k}_1+M)\gamma_\mu}{[(p^2-M^2)x_2+((p-q)^2-M^2)(1-x_1-x_2)+((p-k_1)^2-M^2)x_1]^3} -$$
$$(k_1 \leftrightarrow k_2)(\mu \leftrightarrow \nu) \tag{11.75}$$

The only way we could possibly get a nonvanishing trace is with four Dirac matrices besides the γ_5. The full set of terms in the numerator reads

$$\begin{aligned} &\not{p}\gamma_5\not{q}\gamma_\nu M\gamma_\mu + \not{p}\gamma_5 M\gamma_\nu(\not{p}-\not{k}_1)\gamma_\mu + \\ &+M\gamma_5(\not{p}-\not{q})\gamma_\nu(\not{p}-\not{k}_1)\gamma_\mu - M\gamma_5\not{p}\gamma_\nu\not{k}_1\gamma_\mu \end{aligned} \tag{11.76}$$

All those terms cancel but one.

$$M\text{Tr } \gamma_5\not{q}\gamma_\nu\not{k}_1\gamma_\mu = M4i\epsilon_{\beta\nu\alpha\mu}k_2^\mu k_1^\alpha + (k_1 \leftrightarrow k_2)(\mu \leftrightarrow \nu) \tag{11.77}$$

ending up with

$$\Delta_{\mu\nu} = \int \frac{d^4p}{(2\pi)^4} 2\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{2M4i\epsilon_{\mu\nu\alpha\beta}k_1^\alpha k_2^\beta}{[p^2-2pk-N^2]^3} \tag{11.78}$$

where

$$k \equiv q(1-x_1-x_2)+k_1x_1 \tag{11.79}$$

and

$$N^2 \equiv M^2 - q^2(1-x_1-x_2) - k_1^2 x_1 \tag{11.80}$$

The momentum integral is a particular instance of

$$\int \frac{d^np}{(p^2-2pk-N^2)^a} = i^{1-2a}\pi^{n/2}\frac{\Gamma(a-n/2)}{\Gamma(a)}\frac{1}{(k^2+N^2)^{a-n/2}} \tag{11.81}$$

The final result is then

$$\lim_{M\to\infty} 2M\Delta_{\mu\nu}(M) = \lim_{M\to\infty} \frac{1}{(2\pi)^4}\frac{\pi^2}{2i}\frac{1}{M^2}2M2M4i\epsilon_{\mu\nu\alpha\beta}k_1^\alpha k_2^\beta 2\int_0^1 dx_1 \int_0^{1-x_1} dx_2 =$$
$$\frac{1}{2\pi^2}\epsilon_{\mu\nu\alpha\beta}k_1^\alpha k_2^\beta \tag{11.82}$$

From this Pauli-Villars viewpoint, all the anomaly comes from the regulator.

11.0.3 Dimensional Regularization.

The problem of dimensionally regularizing chiral fermions is a nototious one. The reason is that the number of components of a fermion grows as 2^n, as do the dimensions of gamma-matrices. Also, the definition of γ_5 (which is defined at any rate in even dimensions only) is dimension-dependent.

After some trial and error it is nowadays clear that the best definition of γ_5 in dimensional regularization is the one initially proposed by 't Hooft y Veltman [3]:

$$\begin{aligned} &\{\gamma_5, \gamma_\mu\} = 0 \quad (\mu = 0 \ldots 3) \\ &[\gamma_5, \gamma_\mu] = 0 \quad (\mu = 4 \ldots n-1) \end{aligned} \tag{11.83}$$

The diagram we have to consider is

$$\begin{aligned} &\delta_{\lambda\mu\nu} = -\int \frac{d^4 p}{(2\pi)^4} \frac{d^{n-4} P}{(2\pi)^{n-4}} tr \, \frac{1}{\not{p} + \not{P}} \gamma_\lambda \gamma_5 \frac{1}{\not{p} + \not{P} - \not{q}} \gamma_\nu \frac{1}{\not{p} + \not{P} - \not{k}_1} \gamma_\mu \\ &-(k_1 \leftrightarrow k_2)(\mu \leftrightarrow \nu) \end{aligned} \tag{11.84}$$

where we have been careful in distinguishing

$$\not{p} \equiv \sum_0^3 \gamma_\mu p^\mu \tag{11.85}$$

from the extra components

$$\not{P} \equiv \sum_4^{n-1} P^\mu \gamma_\mu \tag{11.86}$$

Again, once the theory is regularized, we can translate the integration variables

$$p \to p + k_1 \tag{11.87}$$

The axial Ward identity reads

$$\begin{aligned} &q^\lambda \Delta_{\lambda\mu\nu} = -\int \frac{d^4 p}{(2\pi)^4} \frac{d^{n-4} P}{(2\pi)^{n-4}} \frac{tr \, (\not{p} + \not{P} + \not{k}_1)\not{q}\gamma_5(\not{p} + \not{P} - \not{k}_2)\gamma_\nu(\not{p} + \not{P})\gamma_\mu}{[(p+k_1)^2 - P^2][(p-k_2)^2 - P^2][p^2 - P^2]} \\ &+(k_1 \leftrightarrow k_2)(\mu \leftrightarrow \nu) \end{aligned} \tag{11.88}$$

The rules of the game mean that

$$\begin{aligned} &\not{p}\not{p} = p^2 \\ &\not{P}\not{P} = -P^2 \\ &\not{p}\not{P} + \not{P}\not{p} = 0 \\ &(\not{p} + \not{P})(\not{\phi} + \not{P}) = p^2 - P^2 \end{aligned} \tag{11.89}$$

There are 18 different terms in the numerator. The computation simplifies using

$$\mathrm{Tr}\, \gamma_5 \gamma_\alpha \gamma_\beta \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma = \frac{i}{2} \Big(\eta_{\rho\sigma} \epsilon_{\alpha\beta\mu\nu} - \eta_{\nu\sigma} \epsilon_{\alpha\beta\mu\rho} + \eta_{\sigma\mu} \epsilon_{\alpha\beta\nu\rho} - \eta_{\beta\sigma} \epsilon_{\alpha\mu\nu\rho} + \eta_{\sigma\alpha} \epsilon_{\beta\mu\nu\rho} \Big) \tag{11.90}$$

The only surviving terms after taking the trace are the ones proportional to $\not{P}\not{P}$:

$$4\ \mathrm{Tr}\ \gamma_5\gamma_\mu\gamma_\nu\not{k}_1\not{p}\int\frac{d^4p}{(2\pi)^4}\frac{d^{n-4}P}{(2\pi)^{n-4}}\frac{P^2}{[p^2-P^2][(p+k_1)^2-P^2][(p-k_2)^2-P^2]} \tag{11.91}$$

Introducing Feynman parameters and performing the momentum integral we get

$$16i\epsilon_{\mu\nu\alpha\beta}k_1^\alpha k_2^\beta\frac{1}{(2\pi)^4}\frac{\pi^2}{2i}2\int_0^1 dx_1\int_0^{1-x_1}dx_2\frac{d^{n-4}P}{(2\pi)^{n-4}}\frac{P^2}{P^2+f(x_1,x_2)} \tag{11.92}$$

The last integral is a particular instance of

$$\int\frac{d^nP}{(2\pi)^n}\frac{(P^2)^a}{(P^2+f)^b}=\frac{f^{a+b+n/2}}{(2\sqrt{\pi})^n}\frac{\Gamma(a+n/2)\Gamma(b-a-n/2)}{\Gamma(n/2)\Gamma(b)} \tag{11.93}$$

so that the physical four-dimensional limit

$$\lim_{n\to 4}\int\frac{d^{n-4}P}{(2\pi)^{n-4}}\frac{P^2}{P^2+f(x_1,x_2)}=-1 \tag{11.94}$$

where the finite value is the result of a cancellation

$$0\times\infty \tag{11.95}$$

due to the product

$$\frac{\Gamma(-\epsilon)}{\Gamma(\epsilon)} \tag{11.96}$$

These operators are often dubbed *evanescent operators.*

Finally we recover the result

$$q^\lambda\Delta_{\lambda\mu\nu}=-\frac{1}{2\pi^2}\epsilon_{\mu\nu\alpha\beta}k_1^\alpha k_2^\beta \tag{11.97}$$

11.0.4 Anomalies as due to non-invariante of the functional integral measure.

This way of looking to the anomaly is due to Fujikawa [**?**]. The starting point is the formal definition of Berezin's functional integral measure

$$\prod_x\mathcal{D}\bar{\psi}(x)\mathcal{D}\psi(x) \tag{11.98}$$

Giving the fact that

$$\int d\psi\psi=1, \tag{11.99}$$

then

$$\int d(\lambda\psi)\lambda\psi=1 \tag{11.100}$$

which implies

$$d(\lambda\psi) = \frac{1}{\lambda} d\psi. \tag{11.101}$$

The infinitesimal version of the jacobian of the transformation (11.28)

$$\begin{aligned}
&\psi'(x) = e^{i\epsilon\gamma_5}\psi(x)\\
&\bar{\psi}' = \bar{\psi}e^{i\epsilon\gamma_5}\\
&\mathcal{D}\psi'\mathcal{D}\bar{\psi}' = e^{-2i\epsilon\gamma_5}\mathcal{D}\psi\mathcal{D}\bar{\psi}
\end{aligned} \tag{11.102}$$

will then be

$$J \equiv det\,(1 - 2i\epsilon(x)\gamma_5) \tag{11.103}$$

id est

$$\log\, J = -2\, i\, \mathrm{tr}\, \epsilon(x)\gamma_5\delta(x-y) \tag{11.104}$$

The only thing that remains is to give some precise sense to the above expression. In order to perform the trace, we shall use a complete set of eigenfunctions of Dirac's operator

$$\not{D}\phi_n(x) \equiv (\not{\partial} - ig\not{A})\phi_n(x) = \lambda_n\phi_n(x). \tag{11.105}$$

Let us regularize as follows

$$\begin{aligned}
&\frac{i}{2}\log\, J = \sum_n \int d^4x\; d^4y\; \phi_n(x)^+\; \epsilon(x)\; \gamma_5\; \delta_{xy}\phi_n(y) \equiv\\
&= \lim_{\Lambda\to\infty} \int d^4x\; \epsilon(x)\; \sum_n \phi_n^+(x)\; \gamma_5\; e^{-\frac{\lambda_n^2}{\Lambda^2}}\; \phi_n(x) =\\
&= \lim_{\Lambda\to\infty} \int d^4x\;\; \epsilon(x)\; \sum_n \phi_n^+(x)\; \gamma_5\; e^{-\frac{\not{D}^2}{\Lambda^2}}\; \phi_n(x) =\\
&= \lim_{\Lambda\to\infty} \int d^4x\; \epsilon(x)\; \mathrm{tr} \int \frac{d^4k}{(2\pi)^4}\; e^{-ikx}\; \gamma_5\; e^{-\frac{\not{D}^2}{\Lambda^2}} e^{ikx}
\end{aligned} \tag{11.106}$$

where in the last line we have changed to a plane wave basis.

It is not difficult to check the following facts

$$\frac{\not{D}^2}{\Lambda^2} = \frac{1}{\Lambda^2}\left(D^\mu D_\mu + \frac{1}{4}[\gamma^\mu, \gamma^\nu][D_\mu, D_\nu]\right) \tag{11.107}$$

$$D_\mu e^{ikx} = (\partial_\mu - igA_\mu)e^{ikx} = (ik_\mu - igA_\mu)e^{ikx} \tag{11.108}$$

$$D^\mu D_\mu e^{ikx} = (-k^2 - ig\partial.A + 2gk.A - g^2 A_\alpha A^\alpha)e^{ikx} \tag{11.109}$$

$$[D_\mu, D_\nu]e^{ikx} = igF_{\mu\nu}e^{ikx} \tag{11.110}$$

What is left to compute is precisely

$$\lim_{\Lambda\to\infty} \text{Tr} \int \frac{d^4k}{(2\pi)^4}\ e^{-ikx}\ \gamma_5\ e^{\frac{1}{\Lambda^2}[-k^2-ig\partial.A+2gk.A-g^2A_\alpha A^\alpha+\frac{i}{4}g\gamma^{\mu\nu}F_{\mu\nu}]}e^{ikx}. \tag{11.111}$$

where

$$\gamma^{\mu\nu} \equiv [\gamma^\mu, \gamma^\nu]. \tag{11.112}$$

Rescaling $k = p\Lambda$ and keeping the exponential of momenta in the integral, the only surviving term after tracing and regulating is

$$\int \frac{d^4p}{(2\pi)^4} e^{-p^2} \frac{1}{2!}\text{Tr}\ \frac{-g^2}{16}\gamma_5 (F_{\mu\nu}\gamma^{\mu\nu})^2 = \frac{ig^2}{32\pi^2}\text{Tr}\ \epsilon^{\mu\nu\rho\sigma}\ F_{\mu\nu}\ F_{\rho\sigma}$$

given that the volume of the unit three-sphere is $V(S_3) = 2\pi^2$, as well as

$$\int_0^\infty p^3 dp e^{-p^2} = \frac{1}{2}$$
$$\text{Tr}\ \gamma_5\gamma_{\mu\nu}\gamma_{\rho\sigma} = -16i\epsilon_{\mu\nu\rho\sigma}. \tag{11.113}$$

All this means that taking into account the jacobian, the axial current is not conserved anymore, but rather

$$\partial_\mu\langle\bar{\psi}\gamma^\mu\gamma_5\psi\rangle = \frac{g^2}{8\pi^2}\text{Tr}\ \int d^4x * F^{\mu\nu}\ F_{\mu\nu} \tag{11.114}$$

where the dual field strength has been defined

$$* F_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} \tag{11.115}$$

The preceding analysis is related to the index theorem (cf. [**?**]). What we are evaluating is actually

$$\sum \phi_n^+\gamma_5\phi_n = n_{(+)} - n_{(-)} \tag{11.116}$$

namely the difference between the number of positive and negative chirality eigenmodes of Dirac's operator. Neverteless it is the case that only zero modes can be chiral, because given some eigenfunction

$$\not{D}\phi_n^{(+)} = \lambda_n\phi_n^{(+)} \tag{11.117}$$

the action on γ_5 on it

$$\phi^{(-)} \equiv \gamma_5\phi^{(+)} \tag{11.118}$$

obeys

$$\not{D}\ \phi_n^{(-)} = -\lambda_n\phi_n^{(-)} \tag{11.119}$$

This means that nonvanishing eigenvalues just come in pairs with opposite sign, and the only mismatch can only stem from the zero modes, for which our arguments do not apply. The quantity (11.116) is exactly what mathematicians call the *index* of the Dirac operator, and what Fujikawa just proved with physicist's techniques, is that

$$\mathrm{ind} \not{D} = -\frac{1}{16\pi^2} \mathrm{Tr} \int d^4x * F^{\mu\nu} \; F_{\mu\nu} \qquad (11.120)$$

11.0.5 The Heat kernel approach.

- It is clear that the singlet anomaly is given by

$$A(x) \equiv \lim_{\tau\to 0} \text{Tr}\ \gamma_5 K(\tau, x, x) \tag{11.121}$$

It is approppiate to substitute here the short proper time expansion for the diagnonal piece of the heat kernel

$$K(\tau; x, x) = \frac{1}{(4\pi\tau)^{\frac{n}{2}}} \sum_p a_p(x)\ \tau^p \tag{11.122}$$

It so happens that the trace of the first two terms vanish, and the only nonvanishing contribution comes from

$$\text{Tr}\ \ (\gamma_5\ a_2) \tag{11.123}$$

The calculation of this coefficient is essentially the same as the one done by Fujikawa, which has been extended somewhat by Nepomechie.

- It is also possible to define the anomaly through the zeta function

$$A(x) \equiv \lim_{s\to 0} \text{Tr}\ \ (\gamma_5 \zeta(s)) \tag{11.124}$$

Both regularizations are related through

$$\zeta(s) \equiv \frac{1}{\Gamma(s)} \int_0^\infty dz z^{s-1} K(z; x, x) \tag{11.125}$$

Here we can divide the integral into two pieces, $\int_0^\epsilon$ and $\int_\epsilon^\infty$. The second piece is some analytical function of s, say $f(s)$.

$$A(x) \equiv \lim_{s\to 0} \left\{ \frac{1}{\Gamma(s)} \int_0^\epsilon dz\ \frac{1}{(4\pi z)^{\frac{n}{2}}} \sum_p a_p(x)\ z^p + \frac{f(s)}{\Gamma(s)} \right\} \tag{11.126}$$

Defining the integral for Re s big enough, this yields

$$A(x) \equiv \lim_{s\to 0} \left\{ \frac{s}{\Gamma(1+s)} \sum_p a_p(x)\ \frac{\epsilon^{s-\frac{n}{2}+p}}{s-\frac{n}{2}+p} + \frac{f(s)}{\Gamma(s)} \right\} = \text{Tr}\ \left(\gamma_5 a_{\frac{n}{2}}\right) \tag{11.127}$$

The operator whose heat kernel we are dealing with is the *square root* of

$$\not{D}^2 = D^\mu D_\mu + ig\frac{1}{4}\gamma^{\mu\nu}\ F_{\mu\nu} = \Box - ig\partial_\mu A^\mu - 2igA^\mu\partial_\mu - g^2 A_\alpha A^\alpha + ig\frac{1}{4}\gamma^{\mu\nu}\ F_{\mu\nu} \tag{11.128}$$

This operator is already of the canonical form.

Let us regularize the determinant through point splitting plus dimensional regularization; id est

$$\log\det\ \Delta \equiv -\int d(vol)\ \lim_{\sigma\to 0}\sum_{p=0}^{\infty}\frac{\sigma^{p-\frac{n}{2}}}{4^p\ \pi^{\frac{n}{2}}}\ \Gamma\left(\frac{n}{2}-p\right)\ \text{Tr}\ \ a_p(x,y) =$$

$$-\int d(vol)\lim_{\sigma\to 0}\text{Tr}\ \left(\frac{1}{(\pi\sigma)^{\frac{n}{2}}}\Gamma\left(\frac{n}{2}\right)a_0+\frac{1}{4\pi^{\frac{n}{2}}}\sigma^{1-\frac{n}{2}}\Gamma\left(\frac{n-2}{2}\right)+\right.$$

$$\left.+\frac{\sigma^{\frac{4-n}{2}}}{16\pi^{\frac{n}{2}}}\Gamma\left(\frac{n-4}{2}\right)a_2+\ldots\right)$$

where the coincidence limit of the world function

$$\sigma\equiv\frac{(x-y)^2}{2} \tag{11.129}$$

kills all the terms with

$$p>\frac{n}{2} \tag{11.130}$$

Now the square of our operator [11.128] has got

$$\begin{aligned} X_\mu &= -igA_\mu \\ Y &= \frac{ig}{4}\gamma^{\mu\nu}F_{\mu\nu} \end{aligned} \tag{11.131}$$

Defining

$$W_{\mu\nu}\equiv\partial_\mu X_\nu-\partial_\nu X_\mu+[X_\mu,X_\nu] \tag{11.132}$$

the heat kernel coefficient reads

$$a_2=\frac{1}{12}\ W_{\mu\nu}^2+\frac{1}{2}Y^2+\frac{1}{6}\ D^2Y \tag{11.133}$$

The γ_5 factor in the trace kills everything except for

$$\text{Tr}\,\gamma_5\ a_2=-\frac{g^2}{16}\ \text{Tr}\ \ \gamma_5\ \gamma_{\mu\nu}\ \gamma_{\rho\sigma}F^{\mu\nu}F^{\rho\sigma}=-\frac{g^2}{16}\ \epsilon_{\mu\nu\rho\sigma}\ F^{\mu\nu}F^{\rho\sigma} \tag{11.134}$$

- Let us now turn to the non-abelian anomaly. The current of interest is now

$$J^\mu_a\equiv\bar\psi P_- T_a\gamma^\mu\psi \tag{11.135}$$

The operator of interest is

$$D\equiv\gamma^\mu P_- D_\mu \tag{11.136}$$

where

$$D_\mu\equiv\partial_\mu-iA^a_\mu T^a \tag{11.137}$$

Let s define

$$D^{-1}(x,y) = S_F(x,y) \equiv \langle \psi(x)\bar{\psi}(y)\rangle \qquad (11.138)$$

It is such that

$$\begin{aligned} D_x D^{-1}(x,y) &= \delta(x-y) \\ D^{-1}(x,y)\overleftarrow{D}^y &= -\delta(x-y) \end{aligned} \qquad (11.139)$$

where

$$\overleftarrow{D}^y \equiv -\overleftarrow{\partial}^y - ieA \qquad (11.140)$$

Then for example

$$\begin{aligned} &\mathrm{Tr}\,\partial_\mu\left(\bar{\psi}\gamma^\mu T^a\psi\right) = \mathrm{Tr}\,\partial_\mu\bar{\psi}\gamma^\mu T^a\psi + \bar{\psi}\gamma^\mu T^a\partial_\mu\psi = -\mathrm{Tr}\,\left(\psi\partial_\mu\bar{\psi}\gamma^\mu T^a + \partial_\mu\psi\bar{\psi}\gamma^\mu T^a\right) = \\ &\mathrm{Tr}\,\lim_{y\to x}\left(\partial_\mu S_F(x,y)\gamma^\mu T^a - S_F(x,y)\overleftarrow{\partial}^y_\mu\gamma^\mu T^a\right) \end{aligned} \qquad (11.141)$$

Analogously,

$$\left(D_\mu j^\mu\right)_a = \partial_\mu j^\mu_a + f_{abc}A^b_\mu j^\mu_c = \mathrm{Tr}\,\left(T_a D S_F - T_a S_F\overleftarrow{D}\right) \qquad (11.142)$$

The Dirac operator D is not of the canonical form whose heat kernel small proper time expansion we can control. Let us define another operator

$$\bar{D} \equiv P_+\not{D} \qquad (11.143)$$

Then

$$\begin{aligned} D\bar{D} &= P_-\Delta \\ \bar{D}D &= P_+\Delta \end{aligned} \qquad (11.144)$$

where

$$\Delta \equiv \not{D}^2 = D^2 + \frac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu} \qquad (11.145)$$

Then we define the regularized propagator as

$$S(x,x) \equiv \bar{D}\int_0^\infty K_{D\bar{D}}(\tau) \qquad (11.146)$$

It is natural to define here

$$j^\mu_a \equiv \int_0^\infty d\tau\mathrm{Tr}\,\left(T_a P_-\bar{D}K_{D\bar{D}}(\tau)\right) \qquad (11.147)$$

Then

$$\begin{aligned} \left(D_\mu j^\mu\right)_a &= \int_0^\infty d\tau\mathrm{Tr}\,\left(T_a D\bar{D}K_{D\bar{D}}(\tau) - T_a\bar{D}K_{D\bar{D}}(\tau)\,\overleftarrow{D}\right) = \\ &= \lim_{\tau\to 0}\,\mathrm{Tr}\,\left(T_a K_{D\bar{D}}(\tau) - T_a K_{\bar{D}D}(\tau)\right) \end{aligned} \qquad (11.148)$$

owing to the fact that

$$K_{D\bar{D}}\overleftarrow{D} = DK_{\bar{D}D} \tag{11.149}$$

This leads to an expression in terms of the small proper time heat kernel expansion

$$(D_\mu j^\mu)_a = \frac{1}{16\pi^2}\ \text{Tr}\ \left(T_a a_2^{D\bar{D}} - T_a a_2^{\bar{D}D}\right) \tag{11.150}$$

This in turn leads to a covariant expression for the anomaly

$$(D_\mu j^\mu)_a = \frac{1}{16\pi^2}\text{Tr}\ \left(T_a\gamma_5 a_2^\Delta\right) = \frac{1}{32\pi^2}\frac{1}{4}\text{Tr}\ \left(T_a\gamma_5\left(\gamma^\mu\gamma^\nu F_{\mu\nu}\right)^2\right) =$$
$$= -\frac{i}{32\pi^2}\epsilon^{\alpha\beta\gamma\delta}\text{Tr}\ \left(T_a F_{\alpha\beta}F_{\gamma\delta}\right) \tag{11.151}$$

It is also possible to derive an expression for the consistent anomaly by using similar techniques.

11.1 Consistency conditions

Consider the free energy as a function of the background gauge field

$$e^{-W[A]} \equiv \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\ e^{-\int d^n x\ \bar{\psi} i\not{D}\psi} \tag{11.152}$$

under an infinitesimal gauge transformation

$$\delta_v A = dv + [A, v] \equiv Dv \tag{11.153}$$

the variation of the free energy reads

$$\delta_v W[A] = \int d^n x\ \frac{\delta W[A]}{\delta A_\mu^a}\ (D_\mu v)^a = -\int d^n x\ v^a \left(D_\mu \frac{\delta W[A]}{\delta A_\mu^a}\right)^a =$$
$$= -\int d^n x\ v^a D^\mu J_\mu^a \equiv -\int d^n x\ v^a \mathcal{A}^a(x, A) \tag{11.154}$$

That ism the generator of gauge transformations acting on functionals of the gauge field is

$$v^a D_\mu\ \frac{\delta}{\delta A_\mu^a(x)} \equiv -v^a \mathcal{J}^a(x) \tag{11.155}$$

Let us now compute the commutator $\left[\mathcal{J}^a(x), \mathcal{J}^b(y)\right]$. It is useful to represent

$$W_a^\mu(x) \equiv\ \frac{\delta W}{\delta A_\mu^a(x)} \tag{11.156}$$

as well as

$$W^{\mu\nu}_{ab}(x,y) \equiv \frac{\delta^2 W}{\delta A^a_\mu(x)\delta A^b_\nu(y)} \tag{11.157}$$

It is a fact of life that

$$W^{\mu\nu}_{ab}(x,y) = W^{\nu\mu}_{ba}(y,x) \tag{11.158}$$

$$\begin{aligned}
&\mathcal{J}^a(x)\mathcal{J}^b(y)\; W = D^x_\mu \; \frac{\delta}{\delta A^a_\mu(x)} D^y_\nu \; \frac{\delta W}{\delta A^b_\nu(x)} = D_\mu \; \frac{\delta}{\delta A^a_\mu(x)} \\
&\Big(\partial^y_\nu \; W^\nu_b(y) + f_{bb_1b_2} A^{b_1}_\nu(y)\; W^\nu_{b_2}(y)\Big) = D^x_\mu \Big\{ \partial^y_\nu \; W^{\mu\nu}_{ab}(x,y) + \\
&+ f_{bab_2}\; \eta_{\mu\nu}\; \delta_{xy}\; W^\nu_{b_2}(y) \Big\} = \partial^x_\mu \partial^y_\nu \; W^{\mu\nu}_{ab}(x,y) + f_{bab_2}\; \eta_{\mu\nu}\; \partial^x_\mu \delta_{xy}\; W^\nu_{b_2}(y) + \\
&+ f_{aa_1a_2} A^{a_1}_\mu(x) \Big\{ \partial_\nu W^{\mu\nu}_{a_2b}(x,y) + f_{ba_2b_2} W^\mu_{b_2}(x) \Big\}
\end{aligned} \tag{11.159}$$

Jacobi tells us that

$$f_{aa_1a_2}\; f_{a_2b_2b} = -f_{b_2aa_2}\; f_{a_2a_1b} - f_{a_1b_2a_2}\; f_{a_2ab} \tag{11.160}$$

Then

$$\begin{aligned}
&= \partial^x_\mu \partial^y_\nu \; W^{\mu\nu}_{ab}(x,y) - f_{bab_2}\; \eta_{\mu\nu}\; \delta_{xy}\; \partial^x_\mu W^\nu_{b_2}(y) + \\
&+ f_{aa_1a_2} A^{a_1}_\mu(x) \partial_\nu W^{\mu\nu}_{a_2b}(x,y) - (f_{b_2aa_2}\; f_{a_2a_1b} + f_{a_1b_2a_2}\; f_{a_2ab})\; A^{a_1}_\mu(x) W^\mu_{b_2}(x) = \\
&= \partial^x_\mu \partial^y_\nu \; W^{\mu\nu}_{ab}(x,y) + f_{aa_1a_2} A^{a_1}_\mu(x) \partial^y_\nu W^{\mu\nu}_{a_2b}(x,y) - f_{b_2aa_2}\; f_{a_2a_1b} A^{a_1}_\mu(x) W^\mu_{b_2}(x) + \\
&+ f_{abb_2}\; \eta_{\mu\nu}\; \delta_{xy}\; \partial^y_\mu W^\nu_{b_2}(y) - f_{a_1a_2b_2}\; f_{a_2ab} A^{a_1}_\mu(x) W^\mu_{b_2}(x)
\end{aligned} \tag{11.161}$$

The next to the last line is symmetric versus the change (ab). This stems from the fact that

$$\partial^y_\nu D^x_\mu W^{\mu\nu}_{ab}(x,y) = \partial^x_\mu D^y_\nu W^{\nu\mu}_{ba}(y,x) \tag{11.162}$$

Finally, taking into account that

$$f_{abc}\; \delta_{xy}\; \mathcal{J}_c W = f_{abc}\; \delta_{xy}\; D^\lambda W^c_\lambda = f_{abc}\; \delta_{xy} \left(\partial^\lambda W^\lambda_c(y) + f_{cc_1c_2}\; A^{c_1}_\lambda(y) W^\lambda_{c_2}(y)\right) \tag{11.163}$$

we get the commutation relations

$$\Big[\mathcal{J}^a(x), \mathcal{J}^b(y)\Big] = i f_{abc}\; \delta^4(x-y)\; \mathcal{J}^c(x) \tag{11.164}$$

This immediatly implies a consistency relation for the anomaly, namely

$$\mathcal{J}^a(x)\mathcal{A}^b(y,A) - \mathcal{J}^b(y)\mathcal{A}^a(x,A) = i f_{abc}\; \delta^4(x-y)\; \mathcal{A}^c(y,A) \tag{11.165}$$

Those are the Wess-Zumino consistency conditions. Remember now that BRST transformations act as

$$
\begin{aligned}
sA^a_\mu &= \partial_\mu c^a + f^{abc}\ A^b_\mu\ c^c \\
sc^a &= -\frac{1}{2}\ f^{abc}\ c^b c^c
\end{aligned}
\tag{11.166}
$$

It is quite useful to consider gauge fields as G-valued one-forms on the cotangent space

$$
A \equiv A^a_\mu(x) T^a dx^\mu \tag{11.167}
$$

and besides, assume that

$$
\{s, d\} = 0 \tag{11.168}
$$

Now define the anticommuting integrated anomaly as

$$
\mathcal{A}(c, A) \equiv \int d^4x\ c^a(x)\ \mathcal{A}^a(x, A) \tag{11.169}
$$

It is a fact that the Wess-Zumino consistency relations are equivalent to the demand that the object $\mathcal{A}(c, A)$ be BRST closed.

$$
\begin{aligned}
& s\ \mathcal{A}(c, A) = \int d^4x\ \ (s\ c^a(x))\ \ \mathcal{A}^a(x, A) - c^a(x)\ \ (s\ \mathcal{A}^a(x, A)) = \\
& = \int d^4x \Big\{ -\frac{1}{2} f_{abc} c^b c^c \mathcal{A}^a(x, A) - c^a\ \int d^4y \left(D_\mu c^b(y)\ \frac{\delta \mathcal{A}^a(x, A)}{\delta A^b_\mu(y)} \right) \Big\} = \\
& \int d^4x \Big\{ -\frac{1}{2} f_{abc}\ c^b(x)\ c^c(x)\ \mathcal{A}^a(x, A) - c^a(x) \int d^4y\ c^b(y)\ \mathcal{J}^b(y)\ \mathcal{A}^a(x) \Big\}
\end{aligned}
\tag{11.170}
$$

This is possible only when

$$
\mathcal{J}^a(x)\mathcal{A}^b(y) - \mathcal{J}^b(y)\mathcal{A}^a(x) = f_{abc}\ \mathcal{J}\mathcal{A}^c d_{xy}(y = \tag{11.171}
$$

It is natural top identify anomalies with the BRST cohomology, computed in the space of local functionals of ghost number one.

12

Conformal invariance

12.1 Scale invariance.

In flat space, a *scale transformation* is defined as

$$x'_\mu = \lambda x_\mu \tag{12.1}$$

Scale transformations belong to the *conformal group*, $SO(2,n)$, which includes besides the *special conformal transformations*

$$x'_\mu = \frac{x^\mu - a^\mu x^2}{1 - 2a.x + a^2 x^2} \tag{12.2}$$

as well as the whole Poincaré group. The special conformal transformations C are a combination of a translation

$$T_a : \; x^\mu \rightarrow x^\mu + a^\mu \tag{12.3}$$

and an inversion

$$I : x^\mu \rightarrow -\frac{x^\mu}{x^2} \tag{12.4}$$

namely

$$C = T_a \circ I \circ T_a \tag{12.5}$$

Altogether, there are 15 parameters in the conformal group. The infinitesimal generators can be chosen [**?**] as

$$\begin{aligned} M_{\mu\nu} &= M_{\mu\nu} \quad M_{65} = D \\ M_{5\mu} &= \frac{1}{2}\left(P_\mu - K_\mu\right) \quad J_{6\mu} = \frac{1}{2}\left(P_\mu + K_\mu\right) \end{aligned} \tag{12.6}$$

where D is the generator of dilatations, K_μ generate the special conformal transformations and P_μ are the ordinary translations. $M_{AB}\, A = 1 \ldots 6$ are the generators of $SO(2,4)$.

There is a remarkable 11-dimensional subgroup incorporating only the dilatations, besides the Lorentz group. This is precisely the little group of the origin in Minkowski space, $SO_0(2,4)$. It is remarkable that the quotient space

$$SO(2,n)/SO_0(2,n) \tag{12.7}$$

has dimension n always.

Working in a local inertial frame in a ganeral spacetime it is sometimes possible to trade scale transformations for drigid (global) Weyl transformations

$$\left(e^a_\mu\right)' \equiv \lambda e^a_\mu \tag{12.8}$$

In order to be able to do that, it is needed that the scaling of the term

$$\partial_\mu \phi \partial^\mu \phi \tag{12.9}$$

is the same as the Weyl scaling of the term

$$\sqrt{g}\ g^{\mu\nu}\ \nabla_\mu \phi \nabla_\nu \phi \tag{12.10}$$

which is $n-2$. This means that this idea works only in $n=4$ dimensions.

If we are willing to give up diff invariance and remain with the smaller invariance under area preserving (transverse) diffs only, then this is true in any dimension for the kinetic energy term

$$g^{\frac{2}{n}}\ g^{\mu\nu}\ \nabla_\mu \phi \nabla_\nu \phi \tag{12.11}$$

Of course we can always change the rescaling of the metric to

$$g'_{\alpha\beta} = \Omega^{\frac{4}{n-2}} g_{\alpha\beta} \tag{12.12}$$

and this reproduces the scaling of the laplacian for any dimension.

When the spacetime is not flat, gauge (local) Weyl transformations are defined by

$$g'_{\mu\nu} \equiv \Omega^2(x) g_{\mu\nu}(x) \tag{12.13}$$

This is the more general sense in which conformal transformations might be considered.

Let us come back to the linearized approximation in flat space, when $\lambda = 1+\epsilon$,

$$\delta x^\mu = \epsilon x^\mu$$

and fields transform as

$$\phi'(x) = \lambda^D \phi(\lambda x) \tag{12.14}$$

$$\delta\phi = \epsilon D \phi + \epsilon x^\mu \partial_\mu \phi$$

The corresponding Noether current is

$$j_\mu \equiv x^\lambda T_{\lambda\mu} \tag{12.15}$$

provided an adequate definition of the energy-momentum tensor is used. Let us quickly review how this comes about.

$$S \equiv \int d^n x L(\phi_a, \partial_\mu \phi_a) \tag{12.16}$$

Assume in general that there is a symmetry under

$$\begin{aligned} &(x')^\mu - x^\mu \equiv \delta x^\mu \equiv \xi^\mu(x) \\ &\delta\phi_a \equiv \phi'_a - \phi_a(x) \end{aligned} \tag{12.17}$$

This means that

$$\int d^n x' L(\phi'_a, \partial'_\mu \phi'_a) - \int d^n x L(\phi_a, \partial_\mu \phi_a) = 0 \tag{12.18}$$

First of all,

$$d^n x' = d^n x \, (1 + \partial_\mu \xi^\mu) \tag{12.19}$$

We define internal variations as

$$\delta\phi_a \equiv \phi'_a(x) - \phi_a(x) \tag{12.20}$$

so that for example an scalar field obeys

$$\phi'(x') = \phi(x) \tag{12.21}$$

$$\delta\phi = -\xi^\alpha \partial_\alpha \phi \tag{12.22}$$

It is convenient to represent the ξ-independent piece of the variation by another symbol:

$$\delta\phi \equiv -\xi^\alpha \partial_\alpha \phi + \bar{\delta}\phi \tag{12.23}$$

(in the case of an scalar field,

$$\bar{\delta}\phi = 0 \tag{12.24}$$

but for a multiplet such

$$\bar{\delta}\phi_a = \omega_a{}^b \phi_b \tag{12.25}$$

This yields

$$\begin{aligned} &\int d^n x \sum_a \left(\frac{\partial L}{\partial \phi_a} \left(-\xi^\rho \partial_\rho \phi_a + \bar{\delta}\phi_a \right) + \frac{\partial L}{\partial(\partial_\mu \phi_a)} \left(-\partial_\mu \left(\xi^\rho \partial_\rho \phi_a \right) + \right.\right. \\ &\left.\left. + \partial_\mu \bar{\delta}\phi_a \right) + L \partial_\mu \xi^\mu \right) = 0 \end{aligned} \tag{12.26}$$

Integrating by parts we easily arrive to

$$\int d^n x \; \partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \phi^a)} \bar{\delta}\phi_a \right) + \frac{\delta S}{\partial \phi_a} \bar{\delta}\phi_a - \partial_\mu \left(\xi_\nu T^{\mu\nu} \right) + \xi^\rho \partial_\rho \phi_a \frac{\delta S}{\delta \phi_a} - \xi^\lambda \partial_\lambda L = 0 \tag{12.27}$$

The Euler Lagrange equations as well as conservation of the canonical energy-momentum tensor, id est,

$$\frac{\delta S}{\partial \phi_a} \equiv \frac{\partial L}{\partial \phi_a} - \partial_\mu \frac{\partial L}{\partial(\partial_\mu \phi_a)} = 0$$
$$\partial_\rho T^\rho_\mu \equiv \partial_\rho \left(\frac{\partial L}{\partial(\partial_\rho \phi_a)} \partial_\mu \phi_a - \partial^\mu L \eta^\rho_\mu \right) = 0 \tag{12.28}$$

then imply the conservation of the Noether current

$$\partial_\mu J^\mu_N \equiv \partial_\mu \left(\sum_a \frac{\partial L}{\partial(\partial_\mu \phi_a)} \delta\phi_a - \xi^\rho T^\mu{}_\rho \right) = 0 \tag{12.29}$$

The conservation of the energy momentum itself corresponds to the particular case of the symmetry under spacetime translations:

$$\xi^\mu = a^\mu \tag{12.30}$$

In this conditions

$$\partial_\lambda L = 0 \tag{12.31}$$

The internal variation of the fields vanishes in this case

$$\bar{\delta}\phi_a = 0 \tag{12.32}$$

and it follows that

$$J^\mu_N = -a^\rho T^\mu_\rho \tag{12.33}$$

so that

$$0 = \int d^n x \sum_a \frac{\delta S}{\partial \phi_a} \delta\phi_a + \partial_\mu T^{\mu\nu} a^\sigma \partial_\sigma \phi_a - a^\delta \partial_\mu T^\mu_\delta \tag{12.34}$$

For diagonal dilatations,

$$\xi^\mu = \lambda x^\mu$$
$$\delta\phi_a = \lambda d_a \phi_a \tag{12.35}$$

This yields for an scalar field with canonical kinetic energy term the so called *virial current*, J^μ_V,

$$J^\mu_N = J^\mu_V - x^\rho T^\mu_\rho \tag{12.36}$$

It is a fact that

$$J^\mu_V = \partial_\nu J^{\nu\mu} \tag{12.37}$$

with

$$J^\mu \equiv \frac{1}{2} \partial^\mu \left(\phi^2\right) \tag{12.38}$$

$$J^{\mu\nu} \equiv \frac{1}{2} \phi^2 \eta^{\mu\nu} \tag{12.39}$$

If we define for an scalar field the improved energy-momentum tensor (which is best understood in terms of a nonminimal coupling to gravitation, dubbed Ricci gauging in [**?**])

$$T_{\mu\nu}^{imp} \equiv T_{\mu\nu}^{can} - \frac{1}{6}\left(\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box\right)\phi^2 \tag{12.40}$$

then we can change the dilatation current by adding a divergenceless piece

$$J^\mu_{new} \equiv J^\mu_{can} + \frac{1}{6}\partial_\sigma\left(x^\mu\partial^\sigma - x^\sigma\partial^\mu\right)\phi^2 = T^{\mu\nu}_{imp}x_\nu \tag{12.41}$$

This means that $\partial_\alpha j^\alpha = 0$ is equivalent to

$$\partial_\mu j^\mu = T^\mu{}_\mu = 0 \tag{12.42}$$

12.2 The conformal group $C(1,3) \sim SO(2,4) \sim SU(2,2)$

Let us define a mapping between n-dimensional Minkowski space, $M_{1,n-1}$, and the lightcone at the origin of The (n+2)-dimensional Minkowski space , with two times, $M_{2,n}$. A point in $M(2,n)$ will be represented by ξ^A and its two-times metric by η_{AB}. This lightcone is defined as

$$N_0 \equiv \{\xi_{0'}^2 + \xi_0^2 - \sum_{i=1}^{i=n} \xi_i^2 = 0\} \tag{12.43}$$

Its coordinates are defined up to a multiplicative factor, that is

$$\xi^A \to \lambda \xi^A \tag{12.44}$$

We introduce lightcone coordinates

$$\xi_\pm \equiv \xi_{0'} \pm \xi_n \tag{12.45}$$

The lighcone N_0 now reads

$$\xi^2 \equiv \xi_0^2 - \sum_{i=1}^{i=n-1} \xi_i^2 = -\xi_+ \xi_- \tag{12.46}$$

Now define a mapping

$$N_0 \to M_n \tag{12.47}$$

For $\mu = 0 \ldots n-1$,

$$\xi \to x^\mu \equiv \frac{\xi^\mu}{\xi_+} \tag{12.48}$$

Then

$$x^2 \xi_+^2 = \xi^2 \tag{12.49}$$

The inverse map

$$M_n \to N_0 \tag{12.50}$$

is given by

$$\begin{aligned} \xi^\mu &= \xi_+ x^\mu \\ \xi_- &= -\xi_+ x^2 = -\frac{\xi^2}{\xi_+} \end{aligned} \tag{12.51}$$

It is so that the defining equations [12.46] are satisfied.

It is clear that all transformations of $SO(2,n)$

$$\xi^A \to M^A{}_B \xi^B \tag{12.52}$$

with

$$M^A{}_B M^C{}_D\, \eta_{AC} = \eta_{BD} \tag{12.53}$$

induce conformal transformations of M_n,because

$$(x')^2 = \Omega(x)^2 x^2 \tag{12.54}$$

means that

$$\frac{(\xi')^2}{(\xi'_+)^2} = \Omega^2 \frac{\xi^2}{\xi_+^2} \tag{12.55}$$

which is always true when N_0 is preserved, because then this is simply

$$(\xi'_-)^2 = \Omega^2 \xi_-^2 \tag{12.56}$$

which tields the value of Ω.

Consider the most nontrivial one, the one that swaps

$$\xi_\pm \rightarrow \xi_\mp \tag{12.57}$$

Then

$$x'^\mu = \frac{\xi^\mu}{\xi_-} = x^\mu \frac{\xi_+}{\xi_-} = -\frac{x^\mu}{x^2} \tag{12.58}$$

Penrose's compactification of the four-dimensional Minkoski space proceeds as follows [**?**][**?**]. There is an embedding of $\mathbb{R}_{(1,3)}$ into the light cone of the origin in $\mathbb{R}_{(2,4)}$.

$$x \in \mathbb{R}_{(1,3)} \rightarrow \xi \equiv \sigma(x) \left(x^\mu, \frac{1}{\sqrt{2}}\left(1 + \frac{x^2}{2}\right), \frac{1}{\sqrt{2}}\left(1 - \frac{x^2}{2}\right)\right) \tag{12.59}$$

Where the signature of the six-dimensionsl flat space is $(+, -^4 .+)$, so that

$$\xi_A = \sigma\left(x_\mu, -\frac{1}{\sqrt{2}}\left(1 + \frac{x^2}{2}\right), \frac{1}{\sqrt{2}}\left(1 - \frac{x^2}{2}\right)\right) \tag{12.60}$$

and

$$\xi^2 \equiv \xi_A \xi^A = 0 \tag{12.61}$$

The six dimensional Minkowski metric is then mapped into a metric conformal to the four-dimensional Minkowski metric

$$\eta_{AB} d\xi^A d\xi^B = \sigma^2 \eta_{\mu\nu} dx^\mu dx^\nu \tag{12.62}$$

Any six-dimensional Poincare transformation

$$\xi^A \rightarrow L^A{}_B \xi^B \tag{12.63}$$

then induces a conformal transformation of the four-dimensional space.

For example, inversions are obtained through a first translation

$$a_1^A = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} \end{pmatrix} \tag{12.64}$$

then a Lorentz transformation

$$L^A{}_B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{12.65}$$

Write everything in terms of

$$y^\mu \equiv \frac{x^\mu}{x^2} \tag{12.66}$$

with

$$\sigma \equiv \frac{1}{x^2} = y^2 \tag{12.67}$$

and finally a second translation

$$a_2^A = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} \end{pmatrix} \tag{12.68}$$

It is plain that there are points in the six-dimensional cone that do not belong to the four dimensional minkowski space, for example

$$N \equiv (a^\mu, -b, b) \tag{12.69}$$

(with $a^2 = 0$). Penrose then adds to M_4 points at infinity to get useful correspondence. Namely one for every null direction, plus another one corresponding to $a = 0$.

Twistors transform under $SU(2,2)$ as

$$Z^\alpha \rightarrow T^\alpha{}_\beta Z^\beta \tag{12.70}$$

with

$$T^\alpha{}_\beta \overline{T}^\gamma{}_\delta \Lambda_{\alpha\gamma} = \Lambda_{\beta\delta} \tag{12.71}$$

where the matrix of signature $(+^2, -^2)$

$$\Lambda = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{12.72}$$

There is a mapping between antisymmetric $SU(2,2)$ twistors and $SO(2,4)$ vectors [**?**] given by

$$\xi^A \equiv \Sigma^A_{\alpha\beta} Z^{\alpha\beta} \tag{12.73}$$

12.3 The Callan-Symanzik equations

It is more or less obvious that scale invariance is not maintained in general in the quantum theory. There are two reasons for it. First of all, the beta function

$$\beta(g) = \frac{\partial g}{\partial \log\ \mu} \quad = -\frac{\partial g}{\partial \log\ x} \tag{12.74}$$

This means that the Lagrangian transforms as

$$\delta L \equiv T^{\mu}_{\mu} = \beta(g)\frac{\partial}{\partial g}\ L \tag{12.75}$$

For example, writing the QED lagrangian as

$$L = -\frac{1}{4e^2}\ F_{\mu\nu}F^{\mu\nu} \tag{12.76}$$

we learn that

$$T^{\mu}_{\mu} = \frac{\beta(e)}{2e^3}\quad F_{\mu\nu}F^{\mu\nu} \tag{12.77}$$

The second reason is that if there is a non-constant gravitational field (even as a background), there is a length scale associated to it.

The Ward identities associated with the anomalous behavior under dilatations can be obtained by the following procedure. The renormalized 1PI functions are defined as:

$$\Gamma^{(n)}(p_1 \ldots p_n) = Z_3^{n/2}\Gamma^{(n)}_0(p_1 \ldots p_n) \tag{12.78}$$

Green function with $m_0^2\phi^2$ insertion

$$i\Gamma^{(n)}{}_{(0)\delta}(0; p_1 \ldots p_n) = m_0\frac{\partial}{\partial m_0}\Gamma^{(n)}_{(0)}(p_1 \ldots p_n) \tag{12.79}$$

The corresponding renormalized quantity needs a new constant $Z(\Lambda)$:

$$\Gamma^{(n)}{}_{\delta}(0; p_1 \ldots p_n) = ZZ_3^{n/2}\Gamma^{(n)}{}_{(0)\delta}(0; p_1 \ldots p_n) \tag{12.80}$$

it follows

$$i\Gamma^{(n)}_{\delta}\ (0; p_1 \ldots p_n) = Zm_0\frac{\partial}{\partial m_0}\Gamma^{(n)}\ (p_1 \ldots p_n) - \frac{n}{2}Zm_0\frac{\partial \log Z_3}{\partial m_0}\Gamma^{(n)}\ (p_1 \ldots p_n) \tag{12.81}$$

and using the chain rule

$$i\Gamma^{(n)}_{\delta}\ (0; p_1 \ldots p_n) = \left(\left(Zm_0\frac{\partial m}{\partial m_0}\right)\frac{\partial}{\partial m} + \left(Zm_0\frac{\partial \lambda}{\partial m_0}\right)\frac{\partial}{\partial \lambda} - \frac{n}{2}\left(Zm_0\frac{\partial \log Z_3}{\partial m_0}\right)\right)\Gamma^{(n)}\ (p_1 \ldots p_n) \tag{12.82}$$

One can choose constants in such a way that

$$Zm_0 \frac{\partial m}{\partial m_0} = m \tag{12.83}$$

we define

$$\beta(\lambda) \equiv Zm_0 \frac{\partial \lambda}{\partial m_0} \tag{12.84}$$

as well as

$$\gamma(\lambda) \equiv \frac{1}{2} Zm_0 \frac{\partial \log Z_3}{\partial m_0} \tag{12.85}$$

This is the famous Callan-Symanzik equation

$$\left(m \frac{\partial}{\partial m} + \beta(\lambda) \frac{\partial}{\partial \lambda} - n\gamma(\lambda) \right) \Gamma^{(n)}(p_1 \ldots p_n) = i\Gamma_\delta^{(n)}(0; p_1 \ldots p_n) \tag{12.86}$$

There are two main differences with the naive result

$$\left(m \frac{\partial}{\partial m} \right) \Gamma^{(n)}(p_1 \ldots p_n) = i\Gamma_\delta^{(n)}(0; p_1 \ldots p_n) \tag{12.87}$$

the *anomalous dimension* of the field

$$D = 1 + \gamma(\lambda) \tag{12.88}$$

and the anomaly proporcional to $\beta(\lambda)$.

12.4 Dimensional regularization

We define Weyl transformations in a general space-time as

$$\tilde{g}_{\alpha\beta} \equiv \Omega^2(x) g_{\alpha\beta}$$

such that when

$$\Omega^2 \equiv 1 + w(x)$$
$$\delta g_{\mu\nu} = w(x) g_{\mu\nu} \tag{12.89}$$

We define the (expectation value of the) energy momentum tensor from the quantum effective action

$$Z[g] \equiv \int \mathcal{D}\phi \; e^{iS(\phi,g)} \equiv e^{iW[g]}$$
$$T_{\mu\nu} \equiv \frac{2}{\sqrt{g}} \frac{\delta W}{\delta g^{\mu\nu}} \tag{12.90}$$

obeys

$$\delta W = 0 = \frac{1}{2} w \int d^n x \sqrt{g} T_{\alpha\beta} g^{\alpha\beta} \tag{12.91}$$

The massless scalar field

$$S = \frac{1}{2} \int d^n x \sqrt{g} \partial_\mu \phi g^{\mu\nu} \partial_\nu \phi \tag{12.92}$$

is Weyl invariant in $n = 2$ dimensions. In four dimensions $n = 4$ Weyl invariance can be reached through a non minimal coupling (cf. Callan, Coleman y Jackiw) .

$$S_\xi = \frac{1}{2} \int d^n x \sqrt{g} \, \left(\partial_\mu \phi g^{\mu\nu} \partial_\nu \phi - \xi R \phi^2 \right) \equiv \frac{1}{2} \int d^n x \sqrt{g} \; L_\xi \tag{12.93}$$

Under a Weyl transformation,

$$\delta g_{\mu\nu} \equiv \omega g_{\mu\nu} \tag{12.94}$$

so that $\delta\sqrt{g} = \frac{n}{2}\omega\sqrt{g}$, ([**?**]):

$$\delta R = -\omega R - (n-1) \nabla^2 \omega \tag{12.95}$$

The variation of the non minimal action reads

$$\delta S_\xi = \frac{1}{2} \int d^n x \, \sqrt{|g|} \; \frac{n}{2} \omega L_\xi - \omega (\nabla\phi)^2 - \xi \phi^2 \left(-\omega R - (n-1) \nabla^2 \omega \right) =$$
$$\frac{1}{2} \int d^n x \; \omega \; \left[\left(\frac{n}{2} - 1 + 2\,(n-1)\,\xi \right) (\nabla\phi)^2 - (\frac{n}{2} - 1) \xi R \phi^2 \right] +$$
$$(n-1) \xi \nabla_\alpha \left(\phi^2 \nabla^\alpha \; \omega - 2 \phi \nabla^\alpha \phi \; \omega \right) + 2(n-1) \xi \; \phi \nabla^2 \phi \omega \tag{12.96}$$

when the boundary terms do not contribute, this vanishes provided the parameter ξ takes the *conformal* value

$$\xi_c = -\frac{n-2}{4\,(n-1)}$$

and furthermore

$$(\frac{n}{2}-1)R\phi^2 - 2(n-1)\phi\nabla^2\phi = 0 \tag{12.97}$$

The equations of motion read in general

$$\nabla^2\phi + \xi R\phi = 0$$

and reduce for the conformal choice of $\xi = \xi_c$ to [12.97] The energy-momentum tensor reads

$$T^{\alpha\beta} = \frac{1}{2}g^{\alpha\beta}[(\nabla\phi)^2 - \xi R\phi^2] - \nabla^\alpha\phi\nabla^\beta\phi - \xi\phi^2(R^{\alpha\beta} - \frac{1}{2}Rg^{\alpha\beta}) - \xi(g^{\alpha\beta}\nabla^2\phi^2 - \nabla^\alpha\nabla^\beta\phi^2) \tag{12.98}$$

so that its trace

$$g^{\alpha\beta}T_{\alpha\beta} = \left(\frac{n}{2} - 1 - 2\xi\,(n-1)\right)(\nabla\phi)^2 - \xi R\phi^2 - 2\xi\,(n-1)\,\phi\nabla^2\phi$$

vanishes on shell for conformal coupling ξ_c.

Forgetting for the time being non minimal coupling, the Weyl variation of the scalar lagrangian is

$$\delta L = \frac{n-2}{2}\delta w(x) L \tag{12.99}$$

If the theory is scale invariant in d dimensions, then

$$\delta L = \frac{\epsilon}{2} L \tag{12.100}$$

with $\epsilon \equiv n - d$.

For example, for the Maxwell lagrangian

$$L = -\frac{1}{4}F_{\mu\nu}F_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma} \tag{12.101}$$

$$\delta L = -2\delta w(x) L \tag{12.102}$$

which leads to

$$\delta\left(\sqrt{|g|}L\right) = (n/2-2)L \tag{12.103}$$

This also holds for gravitational counter terms. For example in $d = 2$ the only candidate with the correct dimension is

$$L_{ct} = \alpha R \tag{12.104}$$

$$\delta R = -\delta w(x) R - \nabla^2 \delta w \tag{12.105}$$

so that we reproduce (12.100) under *global* Weyl ($\nabla_\mu \delta w = 0$).

$$\delta[\sqrt{g}R] = \frac{\epsilon}{2}[\sqrt{g}R] \tag{12.106}$$

In arbitrary dimension

$$\delta R_{\alpha\beta\gamma\delta} = \delta w(x) R_{\alpha\beta\gamma\delta} + g_{\delta[\alpha}\nabla_{\beta]}\nabla_\gamma \delta w - g_{\gamma[\alpha}\nabla_\beta \nabla_\delta \delta w \tag{12.107}$$

Ricci:

$$\delta R_{\alpha\beta} = -\frac{n-2}{2}\nabla_\alpha \nabla_\beta \delta w - \frac{1}{2} g_{\alpha\beta}\nabla^2 \delta w \tag{12.108}$$

The variation of the quadratic invariants reads

$$\delta R^2 = -2\delta w R^2 \tag{12.109}$$

$$\delta R_{\alpha\beta}R^{\alpha\beta} = -2\delta w R_{\alpha\beta}R^{\alpha\beta} \tag{12.110}$$

and

$$\delta R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = -2\delta w R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} \tag{12.111}$$

so that including the variation of the volume element $\delta\sqrt{|g|} = \frac{nw}{2}\sqrt{|g|}$ we recover (12.100) in $d = 4$ dimensions.

If we consider now the effect of the counterterms

$$L = L_{class} + \frac{1}{\epsilon} L_{count} \tag{12.112}$$

is plain that

$$T^\mu_\mu = L_{count} \tag{12.113}$$

The gravitational contributions to the generic one loop counterterm(cf. [**?**]) reads:

$$\delta L = \frac{1}{8\pi^2 \epsilon}\sqrt{g}[\frac{1}{2}(\frac{1}{6}R - X)^2 + \frac{1}{12}Y_{\mu\nu}Y^{\mu\nu} + \frac{1}{60}H + \frac{1}{180}G] \tag{12.114}$$

where H and G are determined by the Euler density

$$E_4 \equiv R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2 \tag{12.115}$$

as well as the Weyl tensor squared,

$$W^2 = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 2R_{\alpha\beta}R^{\alpha\beta} + \frac{1}{3}R^2 \tag{12.116}$$

$$\begin{aligned} G &= E_4 \\ H &= \frac{1}{2}(W^2 - E_4) \end{aligned} \tag{12.117}$$

12.5 Interacting theories

There is a simple argument

$$\int \sqrt{|g|}\ d^n x\ T^\alpha_\alpha = 2\int d^n x\ g^{\alpha\beta}\ \frac{\delta S}{\delta g^{\alpha\beta}} = 2\frac{\delta W}{\delta w} \tag{12.118}$$

Now, a rigid Weyl variation

$$\delta g_{\alpha\beta} = \delta w g_{\alpha\beta} \tag{12.119}$$

is equivalent to a scale transformation

$$x^\mu \rightarrow (1+\delta w)^{1/2} x^\mu \tag{12.120}$$

that is

$$\Lambda \rightarrow (1+\delta w)^{-1/2}\Lambda \tag{12.121}$$

In dimensional regularization, with $\epsilon \equiv 4-n$,

$$\begin{aligned} \phi_0(\epsilon) &= Z^{1/2}(\epsilon)\phi \\ m_0(\epsilon) &= Z_m^{1/2}(\epsilon) m_R \\ g_0(\epsilon) &= Z_g(\epsilon) g_R \mu^{\frac{\epsilon}{2}} \end{aligned} \tag{12.122}$$

Wilsonian β function,

$$\beta(g_0) \equiv \frac{\partial g_0}{\partial log\,\Lambda} \tag{12.123}$$

$$\frac{\delta log\,\Lambda}{\delta w} = -\frac{1}{2} \tag{12.124}$$

$$\frac{\delta W}{\delta w} = \frac{\partial W}{\partial g_0}\beta(g_0)(-\frac{1}{2}) \tag{12.125}$$

If a lattice regularization is used, we can be more specific. We do not want the renormalized coupling to depend on the lattice spacing, so that

$$a\frac{d}{da} g_R = 0 = \left(a\frac{\partial}{\partial a} - \frac{\partial g_0}{\partial log\ a}\frac{\partial}{\partial g_0}\right) g_R \equiv \left(a\frac{\partial}{\partial a} - \beta_{Latt.}(g_0)\frac{\partial}{\partial g_0}\right) g_R \tag{12.126}$$

On the other hand,

$$\frac{\partial}{\partial log\ a} g_R(g_0, m_R a) = \frac{\partial}{\partial log\ m_R} g_R(g_0, m_R a) \equiv \beta(g_R) \tag{12.127}$$

This means that

$$\beta(g_R) = \beta_{Latt}(g_0)\frac{\partial}{\partial \log\ m_R}\ g_R(g_0, m_R a) \tag{12.128}$$

Indeed

$$
\begin{aligned}
&\beta(g_R) = \beta_1 g_R^2 + \beta_2 g_R^3 + \dots \\
&\beta_{Latt}(g_0) = \hat{\beta}_1 g_0^2 + \hat{\beta}_2 g_0^3 + \dots \\
&\beta_1 = \hat{\beta}_1 \\
&\beta_2 = \hat{\beta}_2 \\
&\beta_2 \neq \hat{\beta}_3
\end{aligned}
\tag{12.129}
$$

$$
T^\alpha_\alpha = -\beta(g_0)\frac{\partial W}{\partial g_0} \tag{12.130}
$$

We could also use

$$
\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g}\right) W = 0 \tag{12.131}
$$

Drummond y Hathrell have computed the corrections to the gravitational QED trace anomaly, Theor result reads

$$
T^\alpha_\alpha = \frac{2}{3}\frac{\alpha_e}{4\pi}\langle N\left(F^{\mu\nu}F_{\mu\nu}\right)\rangle - \frac{1}{20(4\pi)^2}\left(3 + \frac{35\alpha_e}{72\pi}\right)\left(W^2 + \frac{2}{3}\Box R\right) + \frac{1}{(4\pi)^2}\left(\frac{73}{360} + \frac{\alpha_e}{6\pi}\right)E_4
$$

13

Problems

13.1 AQFT. Problem sheet 1

- 1.- Find the combinatoric structure of the two point function in the ϕ^3 scalar theory.
$$V(\phi) \equiv \frac{g}{3!}\phi^3 \tag{13.1}$$
using the Schwinger-Dyson approach.

- 2.- Idem in the ϕ^4 theory.
$$V(\phi) = \frac{\lambda}{4!}\phi^4 \tag{13.2}$$
This time, draw the answer from the path integral.

- 3.- Write the expression for the tadpoles (one point functions) in the theory with potential
$$V(\phi) = \frac{\lambda}{4!}(\phi^2 - v^2)^2 \tag{13.3}$$

- 3.-Compute the order α correction to the fermion propagator in QED in the presence of fermionic sources
$$S_{sources} \equiv \int d^4x \left(\bar{\eta}\psi + \bar{\psi}\eta\right) \tag{13.4}$$

13.2 AQFT. Problem sheet 2

- 1.- Check the gauge tranformation of the non-abelian field strength
$$\tilde{F}_{\mu\nu} = U F_{\mu\nu} U^{+} \tag{13.5}$$

- 2. Compute
$$[D_\mu, D_\mu] \tag{13.6}$$
acting on Diract spinors.

- 3.- What is the dimension of the gaume coupling constant in $n = 3$ spacetime dimensions? And in $n = 6$ spacetime dimensions?

- 4.- Compute the Dynkin index and the second Casimir of SU(3) in the representation provided by Gell-Mann matrices.

$$\lambda_1 \equiv \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 \equiv \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\lambda_4 \equiv \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 \equiv \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$
$$\lambda_7 \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 \equiv \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{13.7}$$

- 5.- Prove the Fierz-like identity
$$\sum_{A=1}^{A=8} (\lambda_A)^a_b \, (\lambda_A)^c_d = 2\delta^a_b \delta^c_d - \frac{2}{3}\delta^a_b \delta^c_d \tag{13.8}$$

13.3 Examen TCA. March 2016. Schwinger's model

Remember the fact that in QED_4 the photon mass remains zero after renormalization, owing to gauge invariance. Consider now the same theory in two dimensions (Schwinger's model), QED_2 with massless fermions. Compute the photon mass renormalization in this theory.

Here there are a few properties that perhaps you might find useful.

- First, than in two dimensions any two-form can be written as
$$\omega_2 = \phi d(vol) \equiv \phi \; dx^0 \wedge dx^1 \tag{13.9}$$

- It is also a fact that, calling $\bar{\gamma}$ the two-dimensional analogue of the four dimensional γ_5 matrix, then
$$\gamma^\mu \bar{\gamma} = -\epsilon^{\mu\nu}\gamma_\nu \tag{13.10}$$

- The chiral current, $j_5^\mu \equiv \bar{\psi}\bar{\gamma}\gamma^\mu\psi$ is not conserved, but rather
$$\partial_\mu j_5^\mu = \frac{e}{2\pi}\,\epsilon^{\mu\nu}F_{\mu\nu} \tag{13.11}$$

How is your result compatible with gauge invariance?

13.3.1 Solution

Let us choose

$$\gamma_0 = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
$$\gamma_1 = -i\sigma_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \tag{13.12}$$

Then

$$\bar{\gamma} \equiv \gamma^0\gamma^1 = \sigma_3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{13.13}$$

A brute force calculation in dimensional regularization around $n = 2$ leads to

$$\pi^{\mu\nu} = \frac{2e^2}{k^2\pi}\left(k^2\eta^{\mu\nu} - k^\mu k^\nu\right) \tag{13.14}$$

It so happens that $\pi(k^2)$ has a pole at $k = 0$, which is precisely what does not happen in four dimensions. This is the reason why the photon remains massless in four dimensiona, but fails to do so in two dimensions.

The EM of the theory are

$$\partial_\nu F^{\mu\nu} = \partial_\nu\phi\ \epsilon^{\mu\nu} = ej^\mu = e\epsilon^{\mu\nu}\ j^5_\nu \tag{13.15}$$

that is

$$\Box\phi = \frac{e^2}{2\pi}\ \epsilon^{\mu\nu}\ \epsilon_{\mu\nu}\phi = -\frac{e^2}{\pi}\phi \tag{13.16}$$

The photon has converted into a scalar and got a (dimensionless) mass

$$m_\gamma^2 = \frac{e^2}{\pi} \tag{13.17}$$

This is the first example in the literature of (dynamical) spontaneous symmetry breaking.

13.4 Problem sheet 5

- The Higgs sector of the standard model reads

$$L = -\frac{1}{4}(W^a_{\mu\nu})^2 - \frac{1}{4}B^2_{\mu\nu} + (D_\mu H)^+ D_\mu H + m^2 H^+ H - \lambda(H^+H)^2 \tag{13.18}$$

  where B_μ is the hypercharge gauge vector boson, and W^a_μ are the weak $SU(2)$ gauge fields.

$$D_\mu H = \partial_\mu H - igW^a_\mu\tau^a H - \frac{ig'}{2}B_\mu H \tag{13.19}$$

Expend the Higgs field as

$$H = e^{2i\frac{\pi^a \tau^a}{v}} \begin{pmatrix} 0 \\ \frac{v+h}{\sqrt{2}} \end{pmatrix} \tag{13.20}$$

with $v \equiv \frac{m}{\sqrt{\lambda}}$. The physical fields are

$$\begin{aligned} Z_\mu &\equiv \cos\ \theta\ W^3_\mu - \sin\ \theta\ B_\mu \\ A_\mu &\equiv \sin\ \theta\ W^3_\mu + \cos\ \theta\ B_\mu \end{aligned} \tag{13.21}$$

where

$$\tan\ \theta \equiv \frac{g'}{g} \tag{13.22}$$

Compute the gauge boson propagators in the unitary gauge ($\pi^a = 0$)-

13.5 Problem sheet 6

- Consider a theory of two-index Maxwell field

$$L = -\frac{1}{4}\partial_{[\mu} A_{\nu\rho]}\partial^{[\mu} A^{\nu\rho]} \tag{13.23}$$

Determine its gauge invariance. Compute the ghost sector. How do you deal with the fact that the ghost lagrangian has itself a gauge invariance?

- Consider the following lagrangians for spin 2 in momentum space. The first is the Fierz-Pauli one.

$$S_{FP} = \int d^n x\ h_{\mu\nu} \Big\{ -\Box\left(\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} - 2\eta^{\mu\nu}\eta^{\rho\sigma}\right)$$
$$+\left(\partial^\mu\partial^\rho\eta^{\nu\sigma} + \partial^\nu\partial^\sigma\eta^{\mu\rho} + \partial^\mu\partial^\sigma\eta^{\nu\rho} + \partial^\nu\partial^\rho\eta^{\mu\sigma} - 2\partial^\mu\partial^\nu\eta^{\rho\sigma} - 2\partial^\rho\partial^\sigma\eta^{\mu\nu}\right)\Big\} h_{\rho\sigma}$$

Determine its gauge symmetry and ghost lagrangian.

The second is the unimodular one.

$$S^U = \int d^n x\; h_{\mu\nu}\Big\{ -\frac{1}{8}\Box\left(\eta_{\mu\rho}\eta_{\nu\sigma}+\eta_{\mu\sigma}\eta_{\nu\rho}\right)+\frac{1}{8}\left(\partial_\nu\partial_\sigma\eta_{\mu\rho}+\eta_{\mu\sigma}\partial_\nu\partial_\rho+\partial_\nu\partial_\rho\eta_{\mu\sigma}+\partial_\nu\partial_\sigma\eta_{\mu\rho}\right)- \\ -\frac{1}{2n}\left(\eta_{\mu\nu}\partial_\rho\partial_\sigma+\eta_{\rho\sigma}\partial_\mu\partial_\nu\right)+\frac{n+2}{4n^2}\Box\eta_{\mu\nu}\eta_{\rho\sigma}\Big\}h_{\rho\sigma} \tag{13.24}$$

Determine also the gauge symmetry and the ghost content.

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