The Gravitational Field

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Introduction

This text grew out of lectures from an introductory graduate course on gravitational physics. The emphasis is in field theory, and in the analogies of the gravitational interaction with all other fundamental interactions. The necessary background in differential geometry is included only in a terse manner, and should be supplemented with personal study of the references.

In general the viewpoint of this lectures is closer to the classic texts of Landau and Schrödinger than to many modern texts. Only a few basic things are included, but these in some detail. In some sense, the aim of this book is to put these admirable books in a modern context, relating the gravitational interaction with the other three fundamental interactions. It is believed, with strong experimental support that the other interactions, namely the electroweak and strong interactions are described by *gauge theories*. General relativity is also a gauge theory in some sense. One of our aims is precisely to nuance this statement, putting it in context.

There is an immediate difference between these four interactions which is obvious, and this is that out of the four only the electromagnetic and gravitational ones are long range, which means that they have classical manifestations. The classical equations descibing the behavior of those two fields are Maxwell's and Einstein's equations respectively. In spite of the fact that both fields have classical (that is, non quantum) manifestations, both set of equations are quite different. Maxwell's are *linear* partial differential equations (PDE), whereas Einstein's are nonlinear PDE, and in that sense, similar to the sort of non-abelian classical equations corresponding to the Yang-Mills theories describing the other two interactions, namely the weak and strong ones. In these cases we are interested mainly in the quantum behavior of those fields, owing to the short range of their interactions, which make their analysis even more difficult. Actually the existence of a *mass gap* for Yang-Mills theory is one of the Millenium Prize Problems of the Clay Mathematical Foundation. For the time being, the only evidence we have that glueballs are massive are the lattice simulations of the corresponding path integral. While convincing for the majority of the physics community, apparently they are not good enough for the Clay foundation. One of the main (I would even dare to say *the main*) open problems in theoretical physics is to understand the *strong coupling* regime, in which perturbative analysis is not applicable. This we would like to do in *quantum field theory* (QFT), but the sad reality is that we do not understand this regime even in nonlinear classical field theory. Actually other of the Clay Millenium Prizes is offered for the proof of the Navier-Stokes existence and smoothness of solutions. Now the Navier-Stokes system of PDE is a baby version of Einstein's (or even Yang-Mills) equations. Many surprises are in store for when we understand these equations fully. There are even indications that string theory may represent some aspects of this nonperturbative sector, at least in some cases.

To summarize, our understanding of the gravitational interaction is somewhat perplexing. On the one hand, we have a classical theory which works extremely well, and there is no reason to doubt its validity so far. It is true that until recently, the theory was checked in the quasi-linear approximation. The so called pametrized post-newtonian (PPN) approach was devised precisely with the purpose of doing that in a systematic way. Since the discovery of the first binary pulsar by Hulse and Taylor there is astrophysical evidence of variation of its orbital period with time, interpreted as due to loos of energy by gravitational radiation. The cumulative agreement of the data with the Einstein formula is impressive.

There are in cosmolgy however indications that there is something missing in our understanding, which bears the *dark* names. *Dark matter* means unknown matter that weighs as normal matter does. *Dark energy* is the name given to a mechanism that produces a cosmological constant, which weighs in a weird way and should explain the observed cosmological acceleration. Many people has tried (and keeps trying) to look whether some modification of Einstein's gravity could help explain some of these phenomena.

The corners of parameter space in which it can be modified while keeping agreement with observations are growing thinner and thinner as time goes by and experiments achieve better precision.

On the other hand, if gravity is a fundamental interaction, it should be compatible with quantum mechanics. This has not yet been achieved, and many of the most interesting open problems in theoretical physics are related to this fact. The point of view that gravitation is not a fundamental theory, but rather *emergent* in some sense at a macroscopic level is not without problems of its own. We have tried to give a flavor of these problems as well.

It is perhaps not superfluous to insist that the only way of learning physics is through personal work and experience.

The most important part of the learning process is however the next one, when one asks and tries to solve her own questions. It is far more important to learn to ask important questions than to be able to learn how to solve somebody else's.

Throughout the lectures we sometimes use the acronyms FIDO (fiducial observer) to mean an observer at rest in a given gravitational field and FREFO (free falling) to mean an observer in a free falling frame.

I would like to thank here all my students, collaborators and teachers from whom I learned everything I know.

2

Energy-momentum tensors.

Our first task will be to find a variational principle for the Maxwell equations. Given a general action depending on a set of fields and their derivatives

$$S = \int d^4x L(\phi_i, \partial_\mu \phi_i, \partial_\mu \partial_\nu \phi_i \ldots) \tag{2.1}$$

the action principle tells us that the action must be stationary under those variations that vanish at the boundary of the spacetime integration region

$$\delta S = \int d^4x \left(\frac{\partial L}{\partial \phi_i} \delta\phi_i + \frac{\partial L}{\partial(\partial_\mu \phi_i)} \delta \partial_\mu \phi_i + \frac{\partial L}{\partial(\partial_\mu \partial_\nu \phi_i)} \delta \partial_\mu \partial_\nu \phi_i + \ldots \right) \tag{2.2}$$

integrating by parts

$$\delta S \equiv \int d^4x \ \frac{\delta S}{\delta \phi_i} \ \delta\phi_i = \int d^4x \ \delta\phi_i$$
$$\left(\frac{\partial L}{\partial \phi_i} - \partial_\mu \frac{\partial L}{\partial(\partial_\mu \phi_i)} + \partial_\mu \partial_\nu \frac{\partial L}{\partial(\partial_\mu \partial_\nu \phi_i)} + \ldots \right) \tag{2.3}$$

the equations of motion read

$$\frac{\delta S}{\delta \phi_i} = 0 \tag{2.4}$$

In Maxwell's case, trial and error leads to

$$S = \int \left[-\frac{1}{c^2} A_\alpha j^\alpha - \frac{1}{4c} F_{\alpha\beta} F^{\alpha\beta} \right] d^4x \tag{2.5}$$

The source current $j^\mu(x)$ should be conserved (i.e., $\partial_\rho j^\rho = 0$) in order for the action to be gauge invariant . For example, when the source of the field is a point particle of charge q, the current is give by [1]

$$j^\alpha(x) \equiv q \int u^\alpha(\lambda) \delta^4(x^\rho - x^\rho(\lambda)) d\lambda \tag{2.6}$$

[1] It is plain that $\partial_\alpha j^\alpha = \int u^\alpha(\lambda) \partial_\alpha \delta^4(x^\rho - x^\rho(\lambda)) = -\int \frac{d}{d\lambda} \delta^4(x^\rho - x^\rho(\lambda)) d\lambda = \delta^4(x^\rho - x^\rho(\lambda_0)) - \delta^4(x^\rho - x^\rho(\lambda_1))$, which vanishes almost everywhere.

The variation of the action reads

$$\delta S = \int d^4x \left[-\frac{1}{c^2}\delta A_\alpha j^\alpha - \frac{1}{2c} F_{\alpha\beta}\delta F^{\alpha\beta} \right] \tag{2.7}$$

so that an integration by parts yields

$$\partial_\alpha F^{\alpha\beta} = \frac{1}{c} j^\beta \tag{2.8}$$

From the conceptual point of view the scalar spin zero field is the simplest field: just a scalar function over space-time $\phi(x)$. When performing an arbitary space-time transformation (not only for Lorentz transformations)

$$\phi'(x') \equiv \phi(x) \tag{2.9}$$

The recently discovered Higgs particle corresponds to a short range *massive* scalar field $m_H \sim 125 \, \frac{GeV}{c^2}$. They are spinless (s=0). We shall see in a moment why.

We shall show in a moment that its static potential (the analogous of the Coulomb potential) is of the Yukawa form

$$V_y \sim \frac{e^{-\frac{mcr}{\hbar}}}{r} \tag{2.10}$$

and its range is given by the Compton wavelength of the corresponding particle

$$l \sim \frac{\hbar}{mc} \tag{2.11}$$

2.1 Conserved currents from Poincaré invariance.

Let us consider a lagrangian depending on a set of fields,ϕ_i (they do not have to be spinless) and their first derivatives, $L(\phi_i, \partial_\mu \phi_j)$.

The transformations to be considered are

$$\begin{aligned} &\delta x^\mu \equiv \xi^\mu(x) \\ &\delta\phi_i \equiv \phi'_i(x) - \phi_i(x) = -\xi^\alpha \partial_\alpha \phi_i + \bar{\delta}\phi_i \\ &\bar{\delta}\phi_i \equiv D_i^j \phi_j + d_i^{j\mu} \partial_\mu \phi_j + t_i \end{aligned} \tag{2.12}$$

When $\xi = 0$, we speak of *internal transformations*. The matrices D_i^j and d_i^j depend on continuous parameters, say $\epsilon_1 \ldots \epsilon_N$.

The set (2.12) define a symmetry when $\delta S = 0$ *off shell* that is, whithout employing the EM. For example the lagrangian for N scalar fields ϕ_i,

$$S = \int d^4x \frac{1}{2} \delta_{ij} \partial_\alpha \phi^i \partial^\alpha \phi^j \tag{2.13}$$

is invariant inder the $N(N-1)/2$ transformations

$$\begin{aligned} \xi^\mu &= 0 \\ d_i^{j\mu} &= 0 \\ D_i^j(\xi) &= \omega_i^j \end{aligned} \tag{2.14}$$

provided $\omega_{(ij)} = 0$. These generate the N-dimensional rotation group, $SO(N)$), which is the internal symmetry of this action principle. The $N(N-1)/2$ parameters are the $\epsilon \sim \omega_{ij}$.

2.2 Internal transformations

Let is first discuss the simplest example of a pure internal transformation. Let us perform a variation letting the parametrs ϵ_a conatined in the matrix D_i^j depend on the spacetime point. The variation of the action will not vanish in general, but it will be proportional to $\partial_\mu \epsilon_a$ (because it has to vanish when $\partial_\mu \epsilon_a = \partial_\mu D_i^j = 0$).

$$\delta S = \int d^4x \left\{ \frac{\partial L}{\partial \phi_i} D_i^j \phi_j + \frac{\partial L}{\partial(\partial_\mu \phi_j)} \partial_\mu (D_j^k \phi_k) \right\} = \int d^4x \left\{ \frac{\delta S}{\partial \phi_i} D_i^j \phi_j + \partial_\mu (J_N^\mu)_i^j D_i^j \right\} \tag{2.15}$$

The variation of the lagrangian will in general be a total derivative (usually represented as partial derivatives!)

$$\delta S = \int d^4x \; \partial_\mu \Lambda^\mu \tag{2.16}$$

The current is given by

$$(J_N^\mu)_i^j \equiv \frac{\partial L}{\partial(\partial_\mu \phi_i)} \phi_j - \Lambda^\mu \tag{2.17}$$

We can now choose the parameters ϵ_a is such a way that

$$D_i^j \phi_j = 0 \tag{2.18}$$

at the boundary. This variations correspond to the action principle, so that

$$\delta S = 0 \tag{2.19}$$

This means that *on shell*

$$\partial_\mu J_N^\mu = 0 \tag{2.20}$$

The Noether current is conserved.

2.3 The general case

Let us work out the transformation of the derivatives of the fields.

$$\partial_{\mu'}\phi'\left(x+\xi\right)=\partial_{\mu'}x^{\rho}\ \partial_{\rho}\phi(x) \tag{2.21}$$

This means that

$$\delta\partial_{\mu}\phi\equiv\partial_{\mu'}\phi'(x)-\partial_{\mu}\phi(x)=-\xi^{\lambda}\partial_{\lambda}\partial_{\mu}\phi-\partial_{\mu}\xi^{\lambda}.\partial_{\lambda}\phi=-\partial_{\mu}\delta\phi \tag{2.22}$$

On the other hand the measure also varies

$$d(vol)'\equiv d^{n}x'=\left|\det\frac{\partial x'}{\partial x}\right|d^{n}x=\left(1+\partial_{\mu}\xi^{\mu}\right)d^{n}x \tag{2.23}$$

That is,

$$\delta d(vol)=\partial_{\mu}\xi^{\mu}d(vol) \tag{2.24}$$

The full variation of the lagrangian is then

$$\delta L=\sum_{a}\left(\frac{\partial L}{\partial\phi_{a}}\left(-\xi^{\rho}\partial_{\rho}\phi_{a}+\bar{\delta}\phi_{a}\right)+\frac{\partial L}{\partial(\partial_{\mu}\phi_{a})}\left(-\partial_{\mu}\left(\xi^{\rho}\partial_{\rho}\phi_{a}\right)+\partial_{\mu}\bar{\delta}\phi_{a}\right)+\xi^{\lambda}\partial_{\lambda}L+L\partial_{\mu}\xi^{\mu}\right)=0 \tag{2.25}$$

where we have included the variation of the Lagrangian itself as a scalar,

$$\delta^{*}L\equiv L'(x')-L(x)=\xi^{\lambda}\partial_{\lambda}L \tag{2.26}$$

Leibnitz rule implies

$$\int d^{n}x\,\partial_{\mu}\left(\frac{\partial L}{\partial(\partial_{\mu}\phi^{a})}\bar{\delta}\phi_{a}\right)+\frac{\delta S}{\delta\phi_{a}}\bar{\delta}\phi_{a}-\partial_{\mu}\left(\xi_{\nu}T^{\mu\nu}\right)-\xi^{\rho}\partial_{\rho}\phi_{a}\frac{\delta S}{\delta\phi_{a}}+\partial_{\mu}\left(L\xi^{\mu}\right)-\partial_{\mu}\left(\xi^{\mu}L\right) \tag{2.27}$$

Here we have defined the *canonical energy-momentum tensor*,

$$T_{\mu}^{\rho}\equiv\frac{\partial L}{\partial(\partial_{\rho}\phi_{a})}\partial_{\mu}\phi_{a}-L\eta_{\mu}^{\rho} \tag{2.28}$$

Now the hypothesis that the action is invariant, that is

$$\delta S=0 \tag{2.29}$$

then implies the conservation of the Noether current

$$\partial_{\mu}J_{N}^{\mu}\equiv\partial_{\mu}\left(\sum_{a}\frac{\partial L}{\partial(\partial_{\mu}\phi_{a})}\delta\phi_{a}-\xi^{\rho}T^{\mu}{}_{\rho}\right)=0 \tag{2.30}$$

To begin with, let us apply this to pure (constant) translations. Then

$$\begin{aligned}\xi^{\alpha}&=a^{\alpha}\\ \bar{\delta}\phi_{a}&=0\end{aligned} \tag{2.31}$$

In this case the Noether current coincides with the energy-momentum tensor itself. It follows

$$\partial_{\mu}T^{\mu\nu}=0 \tag{2.32}$$

2.4 Scale transformations

An important example of spacetime dependent symmetry is *dilatation symmetry*. The free scalar action

$$S \equiv \int d^n x \ \frac{1}{2}\partial_\mu \phi \partial^\mu \phi \tag{2.33}$$

is invariant under a rescaling of the spacetime coordinates

$$x^{\mu'} = (1+\lambda) x^\mu \tag{2.34}$$

provided that the field is correspondingly rescaled

$$\phi'(x) = \frac{\phi(x)}{1+\lambda} \tag{2.35}$$

This means that

$$\begin{aligned} \xi^\mu &= \lambda x^\mu \\ \delta\phi &= -\lambda\phi \end{aligned} \tag{2.36}$$

This yields the so called *virial current*, J_V^μ (which is defined as a quantiuty such that

$$T \equiv \partial_\mu J_V^\mu), \tag{2.37}$$

as well as the *dilatation current*

$$\begin{aligned} J_N^\mu &= J_V^\mu - x^\rho T_\rho^\mu \\ J_V^\mu &\equiv -\frac{1}{2}\partial^\mu \left(\phi^2\right) = \partial_\nu J^{\nu\mu} \end{aligned} \tag{2.38}$$

with

$$L^{\mu\nu} \equiv -\frac{1}{2}\phi^2 \eta^{\mu\nu} \tag{2.39}$$

Acually, the existence of this current means thatmthe theory is *conformal invariant*, and not just scala invariant. The *conformal current* is given by

$$K_{\mu\nu} \equiv \left(2x_\rho x_\nu - x^2 \eta_{\rho\nu}\right) T^\rho{}_\mu - 2x_\nu J_\mu^V + 2L_{\mu\nu} \tag{2.40}$$

It is easy to check that indeed

$$\partial_\mu K^{\mu\nu} = 0 \tag{2.41}$$

If we redefine for an scalar field the improved energy-momentum tensor (which is best understood in terms of a nonminimal coupling to gravitation, dubbed as Ricci gauging)

$$T_{\mu\nu}^{imp} \equiv T_{\mu\nu}^{can} + I_{\mu\nu} \equiv T_{\mu\nu}^{can} - \frac{1}{6}\left(\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box\right)\phi^2 \tag{2.42}$$

The added piece is conserved and does not contribute to the total energy. It is not traceless, bu rather

$$I^{\mu}_{\mu} \equiv \frac{1}{2}\Box\phi^2 = (\partial\phi)^2 + \phi\Box\phi \tag{2.43}$$

Given the fact that

$$x^{\nu} I_{\mu\nu} \equiv -\frac{1}{6} x^{\nu}\left(\partial_{\mu}\partial_{\nu} - \eta_{\mu\nu}\Box\right)\phi^2 = -\frac{1}{6}\left(x^{\nu}\partial_{\mu}\partial_{\nu} - x_{\mu}\Box\right)\phi^2 \tag{2.44}$$

as well as

$$\begin{aligned} &\partial_{\mu} J^{\mu}_{V} \equiv \partial_{\mu}\frac{1}{6}\partial_{\sigma}\left(x^{\mu}\partial^{\sigma} - x^{\sigma}\partial^{\mu}\right)\phi^2 \equiv 0 \\ &\frac{1}{6}\partial_{\sigma}\left(x^{\mu}\partial^{\sigma} - x^{\sigma}\partial^{\mu}\right)\phi^2 = \frac{1}{6}\left(\partial^{\mu} + x^{\mu}\Box - 4\partial^{\mu} - x^{\sigma}\partial_{\sigma}\partial^{\mu}\right)\phi^2 = -\frac{1}{2}\partial^{\mu}\left(\phi^2\right) + x^{\nu} I^{\mu}_{\nu} \end{aligned} \tag{2.45}$$

we can change the dilatation current by adding a divergenceless piece

$$J^{\mu}_{new} \equiv J^{\mu}_{can} + \frac{1}{6}\partial_{\sigma}\left(x^{\mu}\partial^{\sigma} - x^{\sigma}\partial^{\mu}\right)\phi^2 = T^{\mu\nu}_{imp} x_{\nu} \tag{2.46}$$

This means that $\partial_{\alpha} J^{\alpha}_{new} = 0$ is equivalent to

$$\partial_{\mu} J^{\mu}_{new} = T^{\mu}_{imp\,\mu} = 0 \tag{2.47}$$

The improved energy-momentum tensor is traceless. The canonical energy-momentum tensor for a massless scalar field is given by

$$T_{\mu\nu} = \partial_{\mu}\phi\partial_{\nu}\phi - \frac{1}{2}\partial_{\alpha}\phi\partial^{\alpha}\phi\eta_{\mu\nu} \tag{2.48}$$

which fails to be traceless in four dimensions (although it is so in two dimensions). It is easy to check that the corresponding improved energy-momentum tensor is indeed traceless on shell.

2.5 Conserved charges out of conserved currents.

Given a conserved current, $\partial_{\alpha} j^{\alpha} = 0$, it is possible to define a *conserved charge* which is time independent,

$$Q \equiv \int d^3x j^0(t, \vec{x}) \tag{2.49}$$

$\frac{dQ}{dt} = \int d^3x \partial_0 j^0 = \int d^3x \nabla\vec{j} = 0$

The charge corresponding to the energy-momentum tensor is given by

$$P^{\mu} \equiv \int d^3x T^{\mu 0} \equiv \left(P^0 = E, \vec{P}\right) \tag{2.50}$$

2.6 Invariance under Lorentz transformations.

A Lorentz transformation has

$$\xi^\mu = \omega^\mu_\nu x^\nu \tag{2.51}$$

The transformation of a scalar field

$$\phi'(x^\mu + \omega^\mu_\nu x^\nu) = \phi(x) \tag{2.52}$$

so that

$$\phi'(x^\mu) = \phi(x^\mu - \omega^\mu_\nu x^\nu) = \phi(x) - \omega^\mu_\nu x^\nu \partial_\mu \phi = \phi(x) + \omega^{\mu\nu} \frac{1}{2} \left(x^\mu \partial^\nu - x^\nu \partial^\mu\right) \phi \tag{2.53}$$

This means that

$$\delta\phi = \omega.D\phi \equiv \omega.D\phi \tag{2.54}$$

with

$$D_{\alpha\beta} \equiv \frac{1}{2} \left(x_\alpha \partial_\beta - x_\beta \partial_\alpha\right) \tag{2.55}$$

Fields with spin have extra indices, say ϕ_a; the transformation of the field is then given by

$$D_{\alpha\beta} \to D_{\alpha\beta} + \Sigma_{\alpha\beta} \tag{2.56}$$

where $\Sigma_{\alpha\beta}$ acts on the extra indices and is called the *spin matrix*. For example, for a vector field (please check)

$$\left(\Sigma_{\alpha\beta}\right)^\rho_\mu = \frac{1}{2} \left(\delta^\rho_\beta \eta_{\alpha\mu} - \delta^\rho_\alpha \eta_{\beta\mu}\right) \tag{2.57}$$

It is possible to verify that this corresponds with the representation $1 \equiv (1/2, 0) \oplus (0, 1/2)$. In our case it is true that

$$\delta\partial_\rho \phi = \partial_\rho \left(\omega.D\phi\right) \tag{2.58}$$

The equation 2.30 asserting the conservation of Noether's current then tells us that

$$0 = \partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \phi)} \omega.\Sigma\phi - \xi_\rho T^{\mu\rho} \right) \tag{2.59}$$

Taking into account that

$$\partial_\mu \xi^\rho = \omega^\rho{}_\mu \tag{2.60}$$

we get the interesting equation

$$\partial^\rho \left(\frac{\partial L}{\partial(\partial_\rho \phi)} \Sigma_{\mu\nu}\phi \right) = T_{[\mu\nu]} \tag{2.61}$$

This means that we can define a new symmetric energy-momentum tensor (called the *Belinfante tensor*)

$$T^{bel}_{\mu\nu} \equiv T_{\mu\nu} - \partial^\rho \left[\frac{\partial L}{\partial(\partial^\rho \phi_i)} \Sigma_{\mu\nu\, i}{}^j \phi_j + \frac{\partial L}{\partial(\partial^\nu \phi_i)} \Sigma_{\rho\mu\, i}{}^j \phi_j + \frac{\partial L}{\partial(\partial^\mu \phi_i)} \Sigma_{\nu\rho\, i}{}^j \phi_j \right] \tag{2.62}$$

It is natural to define an *angular momentum tensor*

$$M^{\lambda\mu\nu} \equiv x^\mu T^{\lambda\nu}_{bel} - x^\nu T^{\lambda\mu}_{bel} \tag{2.63}$$

which is conserved because the Belinfante tensor is symmetric

To give an explicit example, the canonical Maxwell energy-momentum tensor reads

$$T_{\mu\nu} = \frac{1}{2}\left(\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \eta_{\mu\nu} - F_{\rho\mu} \partial_\nu A^\rho\right) \tag{2.64}$$

which is neither symmetric nor gauge invariant

The Belinfante tensor reads

$$T^{bel}_{\mu\nu} = \frac{1}{2}\left(\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \eta_{\mu\nu} - F_{\rho\mu} F^\rho_\nu\right) \tag{2.65}$$

It is easy to show that the energy density of the field is given by

$$T_{00} = \frac{1}{2}\left(\vec{E}^2 + \vec{B}^2\right) \tag{2.66}$$

as well as the corresponding momentum dendity (Poynting vector):

$$T^{0i} = -\frac{1}{2}\left(\left(\vec{E} \times \vec{B}\right)^i\right) \tag{2.67}$$

Finally the *Rosenfeld tensor* is defined by first coupling the theory to a external metric

$$\eta_{\mu\nu} \to g_{\mu\nu} \tag{2.68}$$

and then definint

$$T^{\rm Ros}_{\mu\nu} \equiv 2\, \frac{\delta S}{\delta g^{mn}}\bigg|_{g_{\mu\nu} = \eta_{\mu\nu}} \tag{2.69}$$

3

Gauge invariance.

3.1 Propagators.

Let us first derive the different Green's functions corresponding to the operator

$$K \equiv \frac{d^2}{dt^2} + \omega^2 \tag{3.1}$$

To begin, let us analyze the equation for a delta-source

$$KG(t) = \delta(t) \tag{3.2}$$

as well as the homogeneous one

$$KG_H(t) = 0 \tag{3.3}$$

In momentum space

$$G(t) = \int \frac{dk}{2\pi} e^{ikt} G(k) \tag{3.4}$$

so that

$$\left(-k^2 + \omega^2\right) G(k) = 1 \tag{3.5}$$

as well as

$$\left(-k^2 + \omega^2\right) G_H(k) = 0 \tag{3.6}$$

Two particular solutions of the homogeneous equation are

$$G_{\pm}^H \equiv -\frac{i}{2\omega} e^{\pm i\omega t} = \int \frac{dk}{2\pi} e^{ikt} \frac{2\pi}{i} \theta(\pm k) \delta(k^2 - \omega^2) \tag{3.7}$$

They are as such hat is, two generators of the full space of solutions, G_H and represent on shell "particles". This means that

$$G_H(t) = \int \frac{dk}{2\pi} e^{ikt} g(k) \delta\left(-k^2 + \omega^2\right) = a\, cos\, \omega\, t + b \sin\, \omega\, t \tag{3.8}$$

It is a fact that

$$
\begin{aligned}
&G^H_+(t) = G^H_-(-t)\\
&G^{H*}_+(t) = -G^H_-(t)\\
&G^{H*}_+(t) = -G^H_+(-t)\\
&\text{Re } G^H_\pm = \pm\frac{1}{2\omega}\sin\,\omega t\\
&\text{Im } G^H_\pm = \frac{1}{2\omega}\cos\,\omega t
\end{aligned}
\tag{3.9}
$$

where $g(k)$ is an arbitrary function, giving rise upon integration to the two arbitrary constants.

On the other hand, turning back to the inhomogeneous equation

$$
G(t) = \int \frac{dk}{2\pi} e^{ikt} \frac{1}{-k^2+\omega^2}
\tag{3.10}
$$

There are two poles on the integration circuit. There are several possibilities. At any rate we can always write

$$
G(t) = -i\theta(t) Res_+ \frac{e^{ikt}}{(k-\omega)(k+\omega)} + i\theta(-t) Res_- \frac{e^{ikt}}{(k-\omega)(k+\omega)}
\tag{3.11}
$$

closing the contour on the upper (positive) half-plane for positive t, and on the lower (negative) half-plane for negative t.

The four different possibilities are drawn in the figures.

- The Dyson boundary conditions The prescription

$$
\omega^2 + i\epsilon
\tag{3.12}
$$

Or equivalently,

$$
k \to k e^{-i\epsilon}
\tag{3.13}
$$

places the positive pole on the upper half of the complex plane, whereas the negative pole stays on the lower half. This leads to the so-called Dyson propagator,

$$
G_D(t) = -\frac{i}{2\omega}\left(\theta(t) e^{i\omega t} + \theta(-t) e^{-i\omega t}\right) = -\frac{i}{2\omega} e^{i\omega|t|} \equiv \theta(t) G^H_+ + \theta(-t) G^H_-
\tag{3.14}
$$

where

$$
G_\pm\,(t) \equiv -\frac{i}{2\omega} e^{\pm i\omega t}
\tag{3.15}
$$

To summarize,

$$
G_D = -\frac{i}{2\omega} e^{i\omega|t|}
\tag{3.16}
$$

- Retarded The prescription

$$\omega^2 + i\epsilon\, sign\,(k) \tag{3.17}$$

or else

$$k \to k - i\epsilon \tag{3.18}$$

places the two poles in the upper half of the complex plane. This leads to the retarded propagator:

$$G_R(t) = -\frac{i}{2\omega}\theta(t)\left(e^{i\omega t} - e^{-i\omega t}\right) = \theta(t)\left(G_+^H - G_-^H\right) = \frac{1}{2}\,\theta(t)\frac{1}{\omega}sin\,\omega t \tag{3.19}$$

$$G_R(t) = \theta(t)\frac{sin\,\omega t}{\omega} = \pm G_\pm^H(t) \tag{3.20}$$

- Advanced Were both poles to be placed in the lower half of the complex plane, id est

$$\omega^2 - i\epsilon\, sign\,(k) \tag{3.21}$$

that is

$$k \to k + i\epsilon \tag{3.22}$$

we would have found the advanced propagator

$$G_A(t) = \frac{i}{2\omega}\theta(-t)\left(e^{i\omega t} - e^{-i\omega t}\right) = \theta(-t)\left(G_-^H - G_+^H\right) = -\theta(-t)\frac{1}{\omega}sin\,\omega t \tag{3.23}$$

$$G_A(t) = -\theta(-t)\frac{sin\,\omega t}{\omega} = -\frac{1}{2}\theta(-t)\text{Re }G_+ \tag{3.24}$$

It is plain that

$$G_A(-t) = G_R(t) \tag{3.25}$$

- Feynman Finally,

$$\omega^2 - i\epsilon \tag{3.26}$$

id est

$$k \to ke^{i\epsilon} \tag{3.27}$$

leads to the Feynman propagator, namely,

$$\begin{aligned} G_F\,(t) = \frac{i}{2\omega}\left(\theta(t)e^{-i\omega t} + \theta(-t)e^{i\omega t}\right) = \\ -\theta(t)G_-^H - \theta(-t)G_+^H = -\theta(t)G_- - \theta(-t)G_+ \end{aligned} \tag{3.28}$$

that is,

$$G_F = \frac{i}{2\omega}e^{-i\omega|t|} \tag{3.29}$$

It propagates positive frequencies towards the future and negative frequencies towards the past. It is an even function:

$$G_F(t) = G_F(-t). \tag{3.30}$$

and

$$G_D^* = G_F \tag{3.31}$$

It is a fact of life that

$$\begin{aligned} G_F + G_D &= 2ReG_F = G_R + G_A = \frac{sin\,\omega|t|}{\omega} \\ G_F - G_D &= 2iImG_F = i\frac{cos\,\omega t}{\omega} = i\frac{\sqrt{1-\omega^2(G_A+G_R)^2}}{\omega} \end{aligned} \tag{3.32}$$

as well as

$$\begin{aligned} 2G_+^H &= G_R - G_A + G_F^* - G_F \\ 2G_-^H &= G_A - G_R + G_F^* - G_F \\ G_F - G_R &= -G_+^H \\ G_F - G_A &= -G_-^H \end{aligned} \tag{3.33}$$

The real part of the Feynman propagator is just the half sum of the advanced and the retarded propagators, and the imaginary part is the time-symmetric homogeneous solution

$$\begin{aligned} G_F(t) = \frac{1}{2}\left(G_R(t) + G_A(t)\right) + \frac{i}{2\omega}\cos\,\omega t = \\ \frac{G_R(t)+G_A(t)}{2} - \frac{G_+(t) - G_-(t)}{2} \end{aligned} \tag{3.34}$$

In momentum space

$$\begin{aligned} G\pm &= \frac{2\pi}{i}\theta(\pm k)\delta(k^2-\omega^2) \equiv -2\pi i\delta^{\pm}(k) \\ G_F &= \frac{-1}{k^2-\omega^2+i\epsilon} \\ G_{R,A} &= \frac{-1}{k^2-\omega^2\mp i\epsilon k} \end{aligned} \tag{3.35}$$

A fact that will turn out to be useful in due time is:

$$\begin{aligned} 2\omega G_R(k) &= \frac{1}{k+\omega-i\epsilon} - \frac{1}{k-\omega-i\epsilon} = P\frac{1}{k+\omega} - P\frac{1}{k-\omega} - i\pi\delta(k-\omega) + i\pi\delta(k+\omega) \\ 2\omega G_A(k) &= \frac{1}{k+\omega+i\epsilon} - \frac{1}{k-\omega+i\epsilon} = P\frac{1}{k+\omega} - P\frac{1}{k-\omega} + i\pi\delta(k-\omega) - i\pi\delta(k+\omega) \\ 2\omega G_F(k) &= \frac{1}{k+\omega-i\epsilon} - \frac{1}{k-\omega+i\epsilon} = P\frac{1}{k+\omega} - P\frac{1}{k-\omega} + i\pi\delta(k-\omega) + i\pi\delta(k+\omega) \\ 2\omega G_D(k) &= \frac{1}{k+\omega+i\epsilon} - \frac{1}{k-\omega-i\epsilon} = P\frac{1}{k+\omega} - P\frac{1}{k-\omega} - i\pi\delta(k-\omega) - i\pi\delta(k+\omega) \end{aligned}$$

In conclusion

$$G_A(k) = G_R^*(k) \tag{3.36}$$

$$G_F(k) = \theta(k)G_A(k) + \theta(-k)G_R(k) \tag{3.37}$$

as well as many other relationships easily unveiled.

- The euclidean propagator The euclidean theory is defined uniquely by the requirement that

$$e^{iS} \tag{3.38}$$

goes to

$$e^{-S_E} \tag{3.39}$$

with S_E positive semidefinite. This fixes

$$t = -i\tau \tag{3.40}$$

The euclidean Green function is defined as

$$\left(\frac{d^2}{d\tau^2} - \omega^2\right) G_E(\tau) = -\delta(\tau), \tag{3.41}$$

$$G_E(\tau) \equiv \int \frac{d\kappa}{2\pi} e^{i\kappa\tau} G_E(\kappa), \tag{3.42}$$

$$G_E(\tau) = \int \frac{d\kappa}{2\pi} e^{i\kappa\tau} \frac{1}{\kappa^2 + \omega^2} = \frac{1}{2\omega} \left(\theta(\tau)e^{-\omega\tau} + \theta(-\tau)e^{\omega\tau}\right) =$$
$$\theta(\tau)G_E^-(\tau) + \theta(-\tau)G_E^+(\tau) \tag{3.43}$$

where the two independent solutions of the homogeneous equation are chosen as:

$$G_E^\pm \equiv \frac{1}{2\omega} e^{\pm\omega\tau}. \tag{3.44}$$

The full euclidean propagator is then

$$G_E = \frac{1}{2} e^{-\omega|\tau|} \tag{3.45}$$

The euclidean propagator is related to Feynman's propagator by the analytic continuation $\tau = it$ such that the piece which goes with $\theta(\tau)$ in the euclidean function goes into $\sqrt{-1}$ times the piece that is proportional to $\theta(t)$ in Minkowskian time and the piece which goes with $\theta(-\tau)$ in the euclidean function goes into $\sqrt{-1}$ times the piece that is proportional to $\theta(-t)$ in Minkowskian time .

3.2 Current-current interaction.

Let us consider an scalar field with an external current

$$S = \int d^n x - \frac{1}{2}\phi\left(\Box + m^2\right)\phi + J(x)\phi(x) \tag{3.46}$$

There is simple way to extract thefull depende of the action on the external sources. This will determine the strength of the interaction between those sources. We start with the identity

$$-\frac{1}{2}\phi\mathcal{O}\phi + J\phi = -\frac{1}{2}\Phi^T\mathcal{O}^{-1}\Phi + \frac{1}{2}\ J\ \mathcal{O}^{-1}\ J \tag{3.47}$$

where the fields

$$\Phi(x) \equiv \mathcal{O}\phi - J \tag{3.48}$$

$$\Phi^T \equiv \phi\mathcal{O} - J \tag{3.49}$$

represent a *point transformation* of the canonical variables. The inverse of the operator is formally defined as

$$\int d^n z\ \mathcal{O}\left(x - z\right)\mathcal{O}^{-1}\left(z - y\right) \equiv \delta^{(n)}\left(x - y\right) \tag{3.50}$$

The moral of all this is that the interaction between the sources is given by the Green's function of the quadratic operator $\mathcal{O}$. In the static approximation it is enough to consider the euclidean one

$$G_E(r) \equiv \int d^{n-1}p\ e^{i\sum_{i=1}^n p_i x^i}\ \frac{1}{\sum_{i=1}^n p_i p^i + m^2} \sim e^{-mr}\ \frac{1}{r^{n-3}} \tag{3.51}$$

This gives the Yukawa potential in 3+1 dimensions (and a logarithmic potential in 2+1 dimensions.

$$V_{\text{yuk}} \sim \frac{e^{-mr}}{r} \tag{3.52}$$

It is a fact of life that in the physically most interesting case of *gauge theories* it is not possible to invert the oparator untl after fixing the gauge. The reason is that the operator has a zero mode, precisely in the gauyge invariant direction. For example, in the EM case,

$$\mathcal{O}_{\alpha\beta} \equiv \frac{1}{2}\ \left(\Box\eta_{\alpha\beta} - \partial_\alpha\partial_\beta\right) \tag{3.53}$$

obeys

$$\mathcal{O}_{\alpha\beta}\partial^\beta\Lambda = 0 \tag{3.54}$$

This means that this operator does not have an inverse. In the Lorenz gauge however the operator reduces to

$$\mathcal{O}_{\alpha\beta} = \frac{1}{2}\ \Box\eta_{\alpha\beta} \tag{3.55}$$

so that the current-current interaction is given by the Lienard-Wiechert potentials

$$J^{\mu}\ (\Box)^{-1}\ \eta_{\mu\nu}\ J^{\nu} \tag{3.56}$$

There is another caveat. In order for the current piece to be gauge invariant, this current must be conserved.

In both the scalar and vector cases the *sign* of the interaction is arbitary, depending of the sign of the currents. In the case of the gravitational field, which corresponds to spin 2, the current is the energy-momentum tensor, and from the fact that the energy is positive, the sign of the gravitational interaction between external sources is fixed to be attractive.

3.3 From rigid to gauge.

Consider the lagrangian corresponding to a charged scalar field ($\equiv$ two real scalar fields)

$$S = \int d^4x \left[\partial_\alpha\phi^*\partial^\alpha\phi - \frac{1}{2}m^2\phi\phi^*\right] \tag{3.57}$$

$$S = \int d^4x \sum_{j=1,2}\left[\frac{1}{2}\partial_\alpha\phi_j\partial^\alpha\phi_j - \frac{1}{2}m^2\phi_j\phi_j\right] \tag{3.58}$$

The EM are again given by the Klein-Gordon equation

$$\frac{\delta S}{d\phi} = -(\Box + m^2)\phi = 0 \tag{3.59}$$

The lagrangian is invariant under constant (*rigid,global*) changes of phase

$$\phi' = \phi\, e^{iq\theta} \tag{3.60}$$

($\partial_\mu\theta = 0,\ q \in \mathbb{R}$). $e^{iq\theta} \in U(1) \sim \mathbb{R}^2 \sim SO(2)$. The real form is

$$\phi'_i \equiv M_{ij}\phi_j \tag{3.61}$$

with

$$\delta^{ij}M_{il}M_{jk} = \delta_{lk} \tag{3.62}$$

The Noether current reads

$$j^\mu \equiv \frac{\partial L}{\partial(\partial_\mu\phi)}\delta\phi = iq(\partial^\mu\phi^*\phi - \partial^\mu\phi\phi^*) \tag{3.63}$$

$$(j^\mu)^+ = j^\mu \tag{3.64}$$

and the charge

$$Q \equiv iq\int d^3x(\dot{\phi}^*\phi - \dot{\phi}\phi^*) \tag{3.65}$$

We would like this invariance to hold true when the parametrer is not constant (*gauge,local*),

$$\partial_\mu \theta \neq 0 \tag{3.66}$$

In order to get that we need to couple the scalar field to the electromagnetic field. The *minimal coupling* is defined by replacing all derivatives by the *covariant derivatives*

$$D_\mu \equiv \partial_\mu - iqA_m \tag{3.67}$$

It is a fact that it transforms as

$$(D_\mu \phi)' = e^{iq\theta} D_\mu \phi \tag{3.68}$$

The full lagrangian reads

$$S = \int d^4x \left[\frac{1}{2} D_\alpha \phi^* D^\alpha \phi - \frac{1}{2} m^2 \phi \phi^* - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right] \tag{3.69}$$

and is gauge invariant under $U(1)$ *abelian* transformations.

If the starting point is a set of Dirac fermions

$$S = \sum_{i=1}^{N} \int d^4x \bar{\psi}_i \gamma^\mu \partial_\mu \psi_i - m \bar{\psi}_i \psi_i \tag{3.70}$$

The action is invariant under

$$\delta \psi_i = i\epsilon \sum_j T_{ij} \psi_j \tag{3.71}$$

as long as the matrix T is hermitian

$$T^*_{ij} = T_{ji} \tag{3.72}$$

This means thet the finite transformation is unitary

$$U = e^{i\epsilon T} \in U(N) \tag{3.73}$$

Now we can ask under what circumstances can we promote this summetry to a local one, id est, $\partial_\mu \epsilon \neq 0$? The problem is clearly with the derivative, because (using matrix notation)

$$\partial_\mu U \psi \neq U \partial_\mu \psi \tag{3.74}$$

In order to fix this, we need to introduce what mathematicians call a *connection* and physicists a *gauge field*. This animal has an index for the derivative, and then the same set of indices as the matrix U. The fixed derivative is called the *covariant derivative*

$$D_\mu \psi \equiv \partial_\mu \psi - A_\mu \psi \tag{3.75}$$

What we want is that

$$(D_\mu\psi)' \equiv \partial_\mu (U\psi) - A'_\mu U\psi = U(\partial_\mu\psi - A_\mu\psi) \tag{3.76}$$

which would be true provided

$$UA_\mu = \partial_\mu U + A'_\mu U \tag{3.77}$$

that is

$$A'_\mu = UA_\mu U^+ - \partial_\mu U U^+ \tag{3.78}$$

Gauge fields (both abelian and non-abelian) pervade our current understanding of the fundamental interactions. We shall see that the gravitational field can also be understood using a similar set of ideas. It is a simple exercise to show that

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + A_\mu A_\nu - A_\nu A_\mu \tag{3.79}$$

transforms as

$$F'_{\mu\nu} \equiv UF_{\mu\nu}U^+ \tag{3.80}$$

This means that

$$L \equiv -\frac{1}{4}\text{tr } F_{\mu\nu}F^{\mu\nu} \tag{3.81}$$

is gauge invariant and the natural candidate for the kinetic energy term of a gauge theory. Indeed

$$\begin{aligned}
&\partial_\mu A'_\nu - \partial_\nu A'_\mu = \partial_\mu \left(UA_\nu U^+ - \partial_\nu U U^+\right) - \partial_\nu \left(UA_\mu U^+ - \partial_\mu U U^+\right) \\
&A'_\mu A'_\nu - A'_\nu A'_\mu \equiv \left(UA_\mu U^+ - \partial_\mu U U^+\right)\left(UA_\nu U^+ - \partial_\nu U U^+\right) - \\
&\left(UA_\nu U^+ - \partial_\nu U U^+\right)\left(UA_\mu U^+ - \partial_\mu U U^+\right)
\end{aligned} \tag{3.82}$$

and the result follows using

$$\begin{aligned}
\partial_\mu U U^+ &= -U\partial_\mu U^+ \\
\partial_\mu U^+ U &= -U^+\partial_\mu U
\end{aligned} \tag{3.83}$$

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Feynman kernel

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Abstract:

Figure 3.1: The Feynman contour.

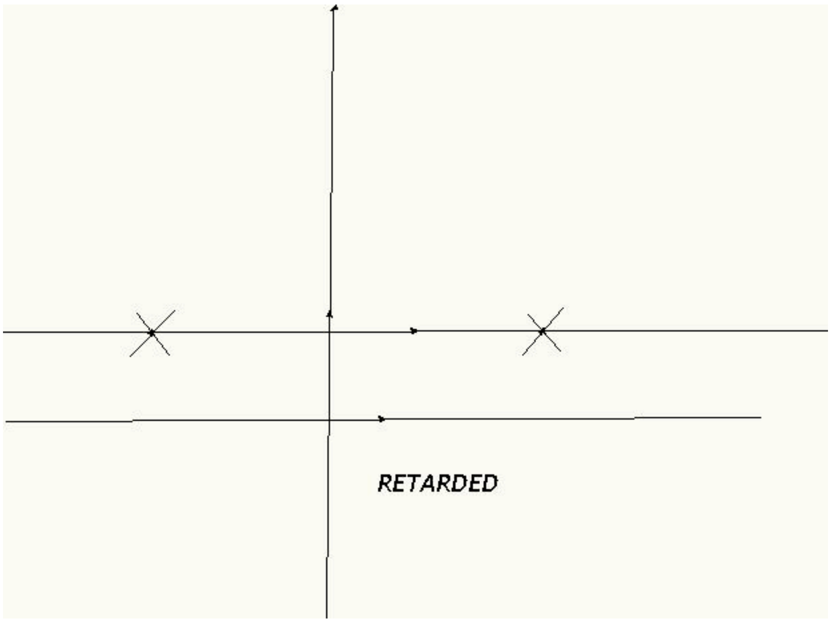

Figure 3.2: The retarded contour.

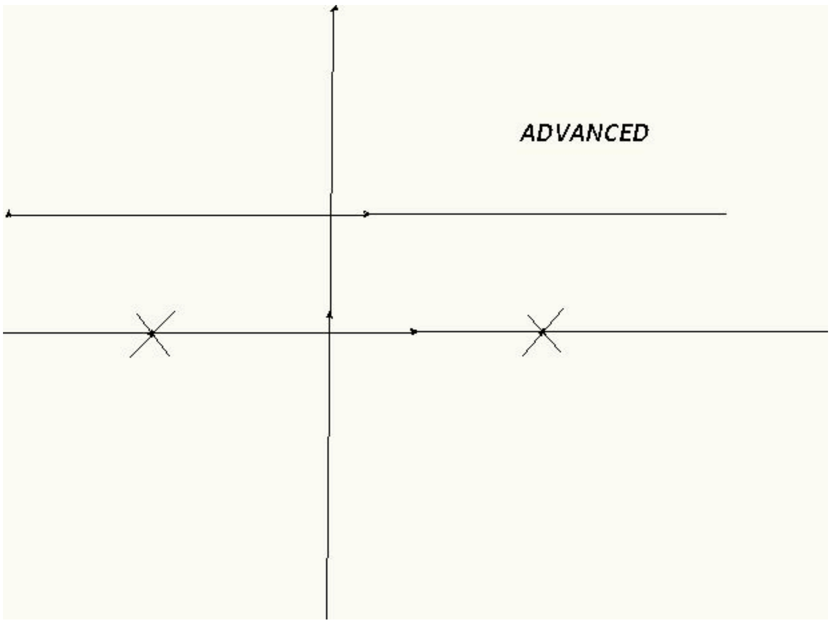

Figure 3.3: The advanced contour.

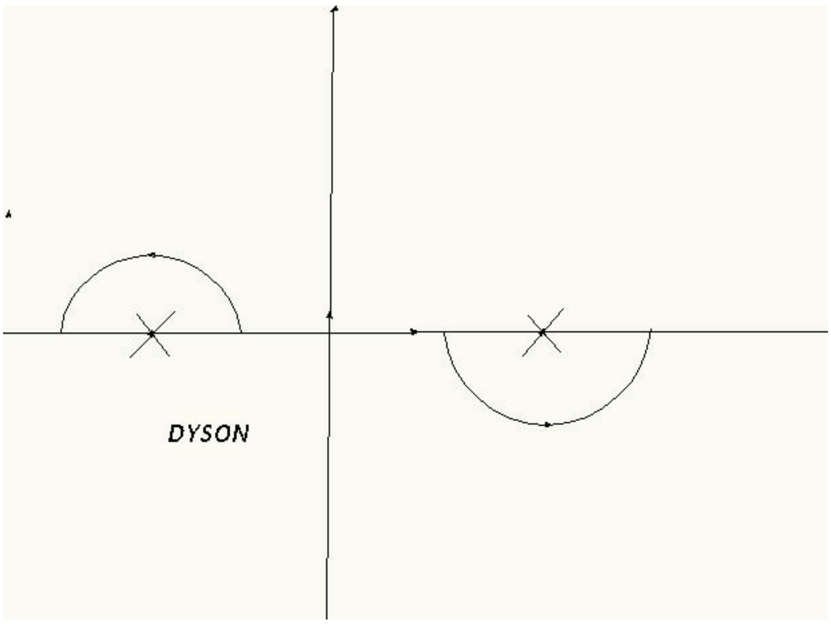

Figure 3.4: The Dyson Contour.

4

The Fierz-Pauli spin 2 theory.

Let us review one viewpoint on the origin of gauge invariance which is found in Tini Veltman's Les Houches lectures. Let us begin wity the simplest case of spin 1. When the vector particle is massive, it has three polarizations $(2s+1)$. Choosing the frame in such a way that the momentum is

$$k = (m, 0, 0, 0) \tag{4.1}$$

The three polarizations can then be chosen as

$$\begin{aligned} \epsilon_1 &\equiv e_1 \equiv (0,1,0,0) \\ \epsilon_2 &\equiv e_2 \equiv (0,0,1,0) \\ \epsilon_3 &\equiv e_3 \equiv (0,0,0,1) \end{aligned} \tag{4.2}$$

Now let us imagine that we want to take the massless linit, $m \to 0$. The momentum is now null, so the best we can do is

$$k = (1, 0, 0, 1) \tag{4.3}$$

The polarizations are still three, namely

$$\begin{aligned} \epsilon_1 &= e_1 \\ \epsilon_2 &= e_2 \\ \epsilon_3 &= k \end{aligned} \tag{4.4}$$

It is actually possible to get propagators from unitarity. The amplitude for creating a photon with polarization $\epsilon^a_\mu(x)$ at a given spacetime point, x, is proportional to the polarization itself. The amplitude for this same photon to be absorbed at the point y is proportional to $\epsilon^a_\nu(y)$. In order to get the full propagator, we have to sum over all three polarizations.

$$D_{\mu\nu} \equiv K^{-1}_{\mu\nu} = \sum_{a=1}^{3} \epsilon^a_\mu \epsilon^a_\nu = \theta_{\mu\nu} \equiv \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \tag{4.5}$$

The transverse metric is really a projector, which does not have an inverse. On shell it can be sustituted by

$$\theta\mu\nu \to \theta^{TOS}_{\mu\nu} \equiv \eta_{\mu\nu} - \frac{k_\mu k_\nu}{m^2} \tag{4.6}$$

Then the ensuing lagrangian

$$L \sim A^\mu \left(\theta^{-1}_{TOS}\right)_{\mu\nu} A^\nu \sim A^\mu \left(\left(k^2 - m^2\right)\eta_{\mu\nu} + k_\mu k_\nu\right) A^\nu \tag{4.7}$$

Normalizing properly, this yields the lagrabgian for a massive photon

$$L = -\frac{1}{4}F^2_{\mu\nu} - \frac{1}{2}m^2 A^2_\mu \tag{4.8}$$

The only known way to stay with only two polarizations (massless fields) is to make the identification

$$\epsilon \sim \epsilon + k \tag{4.9}$$

that is, longitudinal polarizations are pure gauge. In position space this reads

$$\epsilon_\mu \sim \epsilon_\mu + \partial_\mu \lambda \tag{4.10}$$

Let us now turn to spin 2. There are now five polarizations in the massive case.

$$e_i \otimes e_j + e_j \otimes e_i - \frac{2}{3}\left(\sum_k e_k \otimes e_k\right)\delta_{ij} \tag{4.11}$$

In the massless limit we have

$$\epsilon_3 \equiv k \otimes e_2 + e_2 \otimes k = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\epsilon_4 \equiv k \otimes e_1 + e_1 \otimes k = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\epsilon_5 \equiv k \otimes k = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

$$\epsilon_1 \equiv e_1 \otimes e_2 + e_2 \otimes e_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\epsilon_2 \equiv \epsilon_1 \otimes e_1 - e_2 \otimes e_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.12}$$

The last two rotate among themselves under the little group. The smallest gauge invariance we need to stay with only two polarizations is

$$\epsilon_{\alpha\beta} \sim \epsilon_{\alpha\beta} + \partial_\alpha \xi_\beta + \partial_\beta \xi_\alpha \tag{4.13}$$

(Actually it is enough to as these conditions for transverse veciors,

$$\partial_\rho \xi^\rho = 0 \tag{4.14}$$

but we shall not pursue this here.) The most general form of the propagator is

$$D_{\mu\nu\lambda\sigma} \equiv \sum_A \epsilon^A_{\mu\nu}\epsilon^A_{\lambda\sigma} = c_1 \eta_{\mu\nu}\eta_{\lambda\sigma} + c_2 \left(\eta_{\mu\nu} k_\lambda k_\sigma + k_\mu k_\nu \eta_{\lambda\sigma}\right) + c_3 \left(\eta_{\mu\lambda}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\lambda}\right) +$$
$$c_4 \left(k_\mu k_\sigma \eta_{\nu\lambda} + k_\mu k_\lambda \eta_{\nu\sigma} + k_\nu k_\sigma \eta_{\mu\lambda} + k_\nu k_\lambda \eta_{\mu\sigma}\right) + c_5 k_\mu k_\nu k_\lambda k_\sigma \tag{4.15}$$

Demanding transversality and tracelessness is enough to fix it up to a constant. Transversality and tracelessness can also be argued for from the fact that an on shell graviton cannot decay neither in an scalar particle nor in a vector one.

$$D_{\mu\nu\lambda\sigma} = c_1 \left(\theta_{\mu\nu}\theta_{\lambda\sigma} - \frac{3}{2}\left(\theta_{\mu\lambda}\theta_{\nu\sigma} + \theta_{\mu\sigma}\theta_{\nu\lambda}\right)\right) \tag{4.16}$$

In order to find the lagrangian we proceed as in the U(1) case and substitute

$$\theta_{\mu\nu} \to \theta^{TOS}_{\mu\nu} \tag{4.17}$$

The ensuing propagator reads

$$K^{-1}_{\mu\nu\rho\sigma} \equiv D_{\mu\nu\lambda\sigma} = c_1 \left(\theta^{TOS}_{\mu\nu}\theta^{TOS}_{\lambda\sigma} - \frac{3}{2}\left(\theta^{TOS}_{\mu\rho}\theta^{TOS}_{\nu\sigma} + \theta^{TOS}_{\mu\sigma}\theta^{TOS}_{\nu\sigma}\right)\right) \tag{4.18}$$

Computing the inverse and normalizing properly, we get the massive Fierz-Pauli theory, describing a massive spin 2 particle in Minkowski spacetime. This theory was first considered by Fierz and Pauli in 1939, 24 years after Einstein wrote down General Relativity in 1915. It is nevertheless interesting to understand why and in what sense does not work.

It simplifies the computation to use projectors. We start with the longitudinal and transverse projectors

$$\theta_{\alpha\beta} \equiv \eta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}$$
$$\omega_{\alpha\beta} \equiv \frac{k_\alpha k_\beta}{k^2} \tag{4.19}$$

They obey

$$\theta + \omega \equiv \theta^\nu_\mu + \omega^\nu_\mu = \delta^\nu_\mu \equiv 1$$
$$\theta^2 \equiv \theta^\beta_\alpha \theta^\gamma_\beta = \theta^\gamma_\alpha \equiv \theta$$
$$\omega^2 \equiv \omega^\beta_\alpha \omega^\gamma_\beta = \omega^\gamma_\alpha \equiv \omega \tag{4.20}$$

as well as

$$
\begin{aligned}
&tr\ \theta = n-1\\
&tr\ \omega = 1
\end{aligned}
\tag{4.21}
$$

The four-indices projectors are

$$
\begin{aligned}
&P_2 \equiv \frac{1}{2}\left(\theta_{\mu\rho}\theta_{\nu\sigma}+\theta_{\mu\sigma}\theta_{\nu\rho}\right)-\frac{1}{n-1}\theta_{\mu\nu}\theta_{\rho\sigma}\\
&P_1 \equiv \frac{1}{2}\left(\theta_{\mu\rho}\omega_{\nu\sigma}+\theta_{\mu\sigma}\omega_{\nu\rho}+\theta_{\nu\rho}\omega_{\mu\sigma}+\theta_{\nu\sigma}\omega_{\mu\rho}\right)\\
&P_0^s \equiv \frac{1}{n-1}\theta_{\mu\nu}\theta_{\rho\sigma}\\
&P_0^w \equiv \omega_{\mu\nu}\omega_{\rho\sigma}\\
&P_0^{sw} \equiv \frac{1}{\sqrt{n-1}}\theta_{\mu\nu}\omega_{\rho\sigma}\\
&P_0^{ws} \equiv \frac{1}{\sqrt{n-1}}\omega_{\mu\nu}\theta_{\rho\sigma}
\end{aligned}
\tag{4.22}
$$

Their physical meaning can be seen as follows.

The projectors obey

$$
\begin{aligned}
&P_i^a P_j^b = \delta_{ij}\delta^{ab}P_i^b\\
&P_i^a P_j^{bc} = \delta_{ij}\delta^{ab}P_j^{ac}\\
&P_i^{ab} P_j^{c} = \delta_{ij}\delta^{bc}P_j^{ac}\\
&P_i^{ab} P_j^{cd} = \delta_{ij}\delta^{bc}\delta^{ad}P_j^{a}
\end{aligned}
\tag{4.23}
$$

as well as

$$
\begin{aligned}
&tr\ P_2 \equiv \eta^{\mu\nu}(P_2)_{\mu\nu\rho\sigma} = 0\\
&tr\ P_0^s = \theta_{\rho\sigma}\\
&tr\ P_0^w = \omega_{\rho\sigma}\\
&tr\ P_1 = 0\\
&tr\ P_0^{sw} = \sqrt{n-1}\ \omega_{\rho\sigma}\\
&tr\ P_0^{ws} = \frac{1}{\sqrt{n-1}}\ \theta_{\rho\sigma}\\
&P_2+P_1+P_0^w+P_0^s = \frac{1}{2}\left(\delta_\mu^\nu\delta_\rho^\sigma+\delta_\mu^\sigma\delta_\rho^\nu\right)
\end{aligned}
\tag{4.24}
$$

Any symmetric operator can be written as

$$
K = a_2P_2 + a_1P_1 + a_wP_0^w + a_sP_0^s + a_\times P_0^\times \tag{4.25}
$$

(where $P_0^\times \equiv P_0^{ws} + P_0^{sw}$). Then

$$
K^{-1} = \frac{1}{a_2}P_2 + \frac{1}{a_1}P_1 + \frac{a_s}{a_sa_w - a_\times^2}P_0^w + \frac{a_w}{a_sa_w - a_\times^2}P_0^s - \frac{a_\times}{a_sa_w - a_\times^2}P_0^\times \tag{4.26}
$$

It is a fact that

$$
\begin{aligned}
&(P_2)_{\mu\nu}{}^{\rho\sigma}\,(P_{tr})_{\rho\sigma}{}^{\lambda\delta} = P_2\\
&P_0^s P_{tr} = P_0^s - \frac{n-1}{n}P_0^s - \frac{\sqrt{n-1}}{n}P_0^{sw}\\
&P_0^w P_{tr} = P_0^w - \frac{\sqrt{n-1}}{n}P_0^{ws} - \frac{1}{n}P_0^w\\
&P_1 P_{tr} = P_1\\
&P_0^{sw} P_{tr} = P_0^{sw} - \frac{\sqrt{n-1}}{n}P_0^{ws} - \frac{1}{n}P_0^w\\
&P_0^{ws} P_{tr} = P_0^{ws} - \frac{\sqrt{n-1}}{n}P_0^{sw} - \frac{n-1}{n}P_0^s
\end{aligned} \tag{4.27}
$$

The end result of the Fierz-Pauli lagrangian is

$$
S = \int d^4x \left\{ \frac{1}{4}(\partial_\mu h_{\rho\sigma}\partial_\mu h^{\rho\sigma} - \frac{1}{2}\partial_\mu h^{\mu\nu}\partial^\rho h_{\nu\rho} + \frac{1}{2}\partial^\mu h \partial_\rho h^{\mu\rho} - \frac{1}{4}\partial_\mu h \partial^\mu h - \frac{m^2}{4}\left(h_{\alpha\beta}h^{\alpha\beta} - h^2\right) \right\} \tag{4.28}
$$

It can be easily checked that in the massless case, $m = 0$, this is the *unique* lagrangian which is invariant under the gauge symmetry

$$
\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \tag{4.29}
$$

It is interesting to study a little bit the FP EM.

$$
\begin{aligned}
K_{\mu\nu\rho\sigma} \equiv\ & \frac{1}{8}\left(k^2 - m^2\right)(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}) - \frac{1}{8}(k_\mu k_\rho \eta_{\nu\sigma} + k_\mu k_\sigma \eta_{\nu\rho} + k_\nu k_\rho \eta_{\mu\sigma} + k_\nu k_\sigma \eta_{\mu\rho}) +\\
& \frac{1}{4}(\eta_{\mu\nu}k_\rho k_\sigma + k_\mu k_\nu \eta_{\rho\sigma}) - \frac{1}{4}\left(k^2 - m^2\right))\eta_{\mu\nu}\eta_{\rho\sigma}
\end{aligned} \tag{4.30}
$$

The EM in momentum space read

$$
4K_{\mu\nu\rho\sigma}h^{\rho\sigma} = \left(k^2 - m^2\right)h_{\mu\nu} - k_\mu k^\rho h_{\rho\nu} - k_\nu k^\rho h_{\mu\rho} + \eta_{\mu\nu}k^\rho k^\sigma h_{\rho\sigma} + k_\mu k_\nu h - \left(k^2 - m^2\right)\eta_{\mu\nu}h = 0 \tag{4.31}
$$

The divergence of the EM yields

$$
k^\nu K_{\mu\nu\rho\sigma}h^{\rho\sigma} = -2m^2\left(k^\rho h_{\rho\mu} - k_\mu h\right) = 0 \tag{4.32}
$$

It follows that

$$
k^2 h = k_\rho k_\sigma h^{\rho\sigma} \tag{4.33}
$$

On the other hand, the trace of the EM is

$$
\eta^{\mu\nu}K_{\mu\nu\rho\sigma}h^{\rho\sigma} = -2(1-n)m^2 h \tag{4.34}
$$

This tells us that in fact,

$$
\begin{aligned}
&h = k_\mu k_\nu h^{\mu\nu} = 0\\
&k^\mu h_{\mu\nu} = 0
\end{aligned} \tag{4.35}
$$

and the EMK imply the Klein-Gordon equation

$$\left(\Box + m^2\right) h_{\mu\nu} = 0 \tag{4.36}$$

We have seen that were we interested in the massless limit, we better impose the abelian gauge invarian

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \tag{4.37}$$

where the gauge parameter is now a Minkowski vector, ξ_μ. In the quadratic approximetion the most general Lorentz invariant is given by

$$L = \frac{1}{4}\partial_\mu h^{\nu\rho}\partial^\mu h_{\nu\rho} - \beta\frac{1}{2}\partial_\mu h^{\mu\rho}\partial_\nu h^\nu_\rho + a\frac{1}{2}\partial^\mu h \partial^\rho h_{\mu\rho} - b\frac{1}{4}\partial_\mu h \partial^\mu h \tag{4.38}$$

It is quite easy to check that gauge invariance implies uniquely that

$$\beta = a = b = 1 \tag{4.39}$$

That is, demanding that a quadratic Lorentz invariant lagrangian have exactly this gauge symmetry fixes the lagrangian to the massless Fierz-Pauli form: (the choice

$$\begin{aligned} \beta &= 1 \\ a &= \frac{2}{n} \\ b &= \frac{n+2}{n^2} \end{aligned} \tag{4.40}$$

corresponds to the linear limit of *unimodular gravity*, an interesting variant theory of which we can not say more here.

$$S = \int d^4x \left\{ \frac{1}{4}(\partial_\mu h_{\rho\sigma}\partial_\mu h^{\rho\sigma} - \frac{1}{2}\partial_\mu h^{\mu\nu}\partial^\rho h_{\nu\rho} + \frac{1}{2}\partial^\mu h \partial_\rho h^{\mu\rho} - \frac{1}{4}\partial_\mu h \partial^\mu h \right\} \tag{4.41}$$

The quadratic operator

$$S \equiv \int d^4x\ h_{\mu\nu}\ \mathcal{O}^{\mu\nu\rho\sigma}\ h_{\rho\sigma} \tag{4.42}$$

is given by

$$\begin{aligned} \mathcal{O}^{\mu\nu\rho\sigma} &= -\frac{1}{8}\left(\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\right)\Box + \frac{1}{8}\left(\partial^\mu\partial^\rho\eta^{\nu\sigma} + \partial^\mu\partial^\sigma\eta^{\nu\rho} + \partial^\nu\partial^\rho\eta^{\mu\sigma} + \partial^\nu\partial^\sigma\eta^{\mu\rho}\right) + \\ &-\frac{1}{2}\eta^{\mu\nu}\partial^\rho\partial^\sigma + \frac{1}{4}\ \eta^{\mu\nu}\eta^{\rho\sigma}\Box = -\frac{k^2}{4}\left(P_2 - 2P_0^s\right) \end{aligned} \tag{4.43}$$

A very convenient gauge is the de Donder or harmonic gauge,

$$L_{gf}^H \equiv \frac{1}{2}\left(\partial_\lambda h^{\lambda\mu} - \frac{1}{2}\partial^\mu h\right)^2 = -\frac{k^2}{4}\left(P_1 + \frac{n-1}{2}P_0^s + \frac{1}{2}P_0^w - \frac{\sqrt{n-1}}{2}P^\times\right) \tag{4.44}$$

In this gauge the lagrangian simplifies enormously

$$\begin{aligned}\mathcal{O}_H^{\mu\nu\rho\sigma} &= -\tfrac{1}{8}\left(\eta^{\mu\rho}\eta^{\nu\sigma}+\eta^{\mu\sigma}\eta^{\nu\rho}\right)\Box+\tfrac{1}{8}\,\eta^{\mu\nu}\eta^{\rho\sigma}\Box = \\ &= -\tfrac{k^2}{4}\left\{P_2+P_1-\tfrac{n-3}{2}P_0^s+\tfrac{1}{2}P_0^w-\tfrac{\sqrt{n-1}}{2}P^\times\right\}^{\mu\nu\rho\sigma}\end{aligned} \tag{4.45}$$

The interaction between two external sources will then be

$$W \equiv \int d^4z\; T^1_{\mu\nu}(x)\;\left(\mathcal{O}^{-1}\right)^{\mu\nu\rho\sigma}(x-y)\;T^2_{\rho\sigma}(y) \tag{4.46}$$

In n=4 dimensions

$$\left(\mathcal{O}^{-1}\right)^{\mu\nu\rho\sigma}(x-y)=\frac{1}{4}\;K^{\mu\nu\rho\sigma}\Box^{-1}(x-z) \tag{4.47}$$

$$\frac{1}{4}\int d^4z\;K^{\mu\nu\rho\sigma}(x-z)K_{\rho\sigma\alpha\beta}(z-y)=\left(\frac{1}{2}\left(\delta^\mu_\alpha\delta^\nu_\beta+\delta^\mu_\beta\delta^\nu_\alpha\right)+(n-4)\,\eta^{\mu\nu}\eta_{\alpha\beta}\right)\delta(x-y) \tag{4.48}$$

so that the interaction energy is proportional to

$$W=\frac{1}{k^2}\;\left(T^{(1)}_{\mu\nu}T^{(2)\mu\nu}-\frac{1}{2}T^{(1)}T^{(2)}\right) \tag{4.49}$$

The same calculation in the massive case (no need for gauge fixing there) yields the value

$$W=\frac{1}{k^2}\;\left(3\;T^{(1)}_{\mu\nu}T^{(2)\mu\nu}-T^{(1)}T^{(2)}\right) \tag{4.50}$$

It is then plain that the massless Fierz-Pauli propagator (in the harmonic gauge) it is not the limit of the massive propagator when $m \to 0$. This phenomenon is dubbed the *van Dam and Veltman discontinuity*, in honor of its discoverers.

In terms of projectors

$$\Delta=-\frac{4}{k^2}\left\{P_2+P_1-\frac{1}{n-2}P_0^s+\frac{n-3}{n-2}P_0^w-\frac{\sqrt{n-1}}{n-2}P^\times\right\} \tag{4.51}$$

The residue of P_0^s is negative. Consider the ADM decomposition of the graviton

$$h_{\mu\nu}\equiv h^{TT}_{\mu\nu}+\partial_\mu A_\nu+\partial_\nu A_\mu+\frac{1}{n}\eta_{\mu\nu}\phi+\left(\partial_\mu\partial_\nu-\frac{1}{n}\Box\eta_{\mu\nu}\right)a \tag{4.52}$$

with

$$\begin{aligned}h^{TT}_{\mu\nu}\eta^{\mu\nu}&=0\\ \partial^\lambda h^{TT}_{\lambda\mu}&=0\\ \partial_\mu A^\mu&=0\end{aligned} \tag{4.53}$$

in such a way that

$$
\begin{aligned}
h^{\mu\nu}h_{\mu\nu} &= \phi \\
\partial^{\nu}h_{\mu\nu} &= \Box A_{\mu} + \tfrac{1}{n}\partial_{\mu}\phi + \tfrac{n-1}{n}\Box\partial_{\mu}a \\
\partial^{\mu}\partial^{\nu}h_{\mu\nu} &= \tfrac{1}{n}\Box\phi + \tfrac{n-1}{n}\Box^{2}a
\end{aligned} \tag{4.54}
$$

Then

$$
\begin{aligned}
(P_2 h)_{\mu\nu} &= h^{TT}_{\mu\nu} \\
(P_1 h)_{\mu\nu} &= \partial_{\mu}A_{\nu} + \partial_{\nu}A_{\mu} \\
\theta^{\mu\nu}h_{\mu\nu} &= \tfrac{n-1}{n}\left(\phi - k^2\, a\right) \\
\omega^{\rho\sigma}h_{\rho\sigma} &= \tfrac{1}{n}\left(\phi + (n-1)k^2\, a\right)
\end{aligned} \tag{4.55}
$$

$$
\begin{aligned}
(P^s_0 h)_{\mu\nu} &= \tfrac{n-1}{n}\left(\phi - k^2 a\right)\theta_{\mu\nu} \\
h P^s_0 h &= \tfrac{n-1}{n^2}\left(\phi - k^2 a\right)^2
\end{aligned} \tag{4.56}
$$

$$
\begin{aligned}
(P^w_0 h)_{\mu\nu} &= \omega_{\mu\nu}\tfrac{1}{n}\left(\phi + (n-1)k^2 a\right) \\
h P^w_0 h &= \tfrac{1}{n^2}\left(\phi + (n-1)k^2 a\right)^2
\end{aligned} \tag{4.57}
$$

$$
\begin{aligned}
(P^{\times} h)_{\mu\nu} &= \tfrac{1}{n\sqrt{n-1}}\left(\phi + (n-1)k^2 a\right)\theta_{\mu\nu} + \tfrac{n-1}{n}\left(\phi - k^2 a\right)\omega_{\mu\nu} \\
h P^{\times} h &= 2\tfrac{1}{(n-1)n^2}\left(\phi + (n-1)k^2 a\right)(\phi - k^2 a)
\end{aligned} \tag{4.58}
$$

Besides, under a FP gauge transformation

$$
\begin{aligned}
\xi_{\mu} &\equiv \xi^{T}_{\mu} + \partial_{\mu}\xi \\
\partial^{\mu}\xi_{\mu} &= \Box\xi \\
\partial^{\mu}\xi^{T}_{\mu} &= 0
\end{aligned} \tag{4.59}
$$

this fields transform as

$$
\begin{aligned}
\delta h^{TT}_{\mu\nu} &= 0 \\
\delta A_{\mu} &= \xi^{T}_{\mu} \\
\delta a &= 2\xi \\
\delta\phi &= 2\Box\xi
\end{aligned} \tag{4.60}
$$

in such a way that the field

$$
\Phi \equiv \phi - \Box a \tag{4.61}
$$

is gauge invariant.

The FP lagrangian is expressed in terms of this fields as

$$L = -\tfrac{1}{4}\Big\{ h^{TT}_{\mu\nu}\Box h^{\mu\nu}_{TT} + \tfrac{3}{8}\partial_\mu\Phi\partial^\mu\Phi \Big\} \tag{4.62}$$

It is a fact of life that this is invariant under linearized TDiffs

$$\delta h^{TT}_{\mu\nu} = \partial_\mu\lambda_\nu + \partial_\nu\lambda_\mu \tag{4.63}$$

with

$$\partial_\mu\lambda^\mu = 0 \tag{4.64}$$

and under CKV

$$\partial\Phi = \partial_\mu\eta^\mu \tag{4.65}$$

Actually

$$\delta L \sim \partial_\mu\partial_\lambda\eta^\lambda\partial^\mu\Phi \sim \Box\partial_\lambda\eta^\lambda\Phi = 0 \tag{4.66}$$

because the CKV equation

$$\partial_\mu\eta_\nu + \partial_\nu\eta_\mu = \frac{2}{n}\partial_\lambda\eta^\lambda\eta_{\mu\nu} \tag{4.67}$$

do imply

$$\Box\partial_\lambda\eta^\lambda = 0 \tag{4.68}$$

for any dimension $n \neq 2$.

TDiff invariance reduces the number of DOF from 5 to 2, and CKV seems to kill the massless scalar field (this argument works for *any* massless scalar field).

4.1 The coupling to matter

The main argument in favor of considering spin 2 as a candidate theory of gravitation is the positivity of the classical energy density. It is then natural to try a coupling for example to a scalar field of the type

$$L_{\text{int}} \equiv \kappa h^{\mu\nu}T_{\mu\nu}[\phi] = \kappa h^{\mu\nu}\left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\left(\partial_\rho\phi\partial^\rho\phi - \frac{m^2}{2}\phi^2\right)\eta_{\mu\nu}\right) \tag{4.69}$$

Here κ is a coupling constant with mass dimension -1.

The EM would now read

$$K_{\mu\nu\rho\sigma}h^{\rho\sigma} = \kappa T_{\mu\nu} \tag{4.70}$$

The first member is divergence-free

$$\partial^\mu K_{\mu\nu\rho\sigma} = 0 \tag{4.71}$$

It follows that the second member must also enjoy a vanishing divergence if the EM are to be consistent. But this is not trie anynore because the scalar EM have been modified due to the interaction term. The new conserved canonical energy-momentum tensor is

$$T^{\text{can}}_{\mu\nu} = T_{\mu\nu} + \kappa\left(\left(h^{\rho}_{\mu}\partial_{\rho}\phi - h\partial_{\mu}\phi\right)\partial_{\nu}\phi - h^{\alpha\beta}T_{\alpha\beta}\eta_{\mu\nu}\right) \tag{4.72}$$

This means that we should modify the couping accordingly. But in doing so, we modify again the conserved energy-momentum tensor. There is clearly an infinite series of terms that we have somewhat to add. There is an exceedingly clever way of doing it in a fell swoop due to Stanley Deser. First of all, let us rewrite the Fierz-Pauli lagrangian in first order form as

$$L \equiv -\kappa\bar{h}^{\mu\nu}\left[\partial_{\mu}\Gamma^{\rho}_{\nu\rho} - \partial_{\rho}\Gamma^{\rho}_{\mu\nu}\right] + \eta^{\mu\nu}\left[\Gamma^{\lambda}_{\mu\rho}\Gamma^{\rho}_{\nu\lambda} - \Gamma^{\lambda}_{\mu\nu}\Gamma^{\rho}_{\lambda\rho}\right] \tag{4.73}$$

where

$$\begin{aligned} &\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\, h\ \eta_{\mu\nu} \\ &\Gamma^{\alpha}_{[\beta\gamma]} = 0 \end{aligned} \tag{4.74}$$

The EM are given by

$$\begin{aligned} &\frac{\delta S}{\delta \bar{h}^{\mu\nu}} = -\partial_{\rho}\Gamma^{\rho}_{\mu\nu} + \frac{1}{2}\left(\partial_{\mu}\Gamma_{\nu} + \partial_{\nu}\Gamma_{\mu}\right) \\ &\frac{\delta S}{\delta \Gamma^{\alpha}_{\mu\nu}} = \partial_{\alpha}\bar{h}^{\mu\nu} - \frac{1}{2}\left(\delta^{\nu}_{\alpha}\partial_{\sigma}\bar{h}^{\mu\sigma} + \delta^{\mu}_{\alpha}\partial_{\sigma}\bar{h}^{\nu\sigma}\right) - \eta^{\mu\nu}\Gamma_{\alpha} - \\ &-\frac{1}{2}\left(\Gamma^{\mu}\delta^{\nu}_{\alpha} + \Gamma^{\nu}\delta^{\mu}_{\alpha}\right) + \Gamma^{\nu}_{\alpha\sigma}\eta^{\mu\sigma} + \Gamma^{\mu}_{\alpha\beta}\eta^{\nu\beta} \end{aligned} \tag{4.75}$$

where

$$\begin{aligned} &\Gamma_{\mu} \equiv \Gamma^{\lambda}_{\lambda\mu} \\ &\Gamma^{\mu} \equiv \eta^{\rho\sigma}\Gamma^{\mu}_{\rho\sigma} \end{aligned} \tag{4.76}$$

The trace of the h-equation tells us that

$$\partial^{\lambda}\Gamma_{\lambda} = \partial_{\mu}\Gamma^{\mu} \tag{4.77}$$

whereas the trace δ^{α}_{μ} of the Gamma-equation yields

$$\Gamma^{\nu} = \partial_{\sigma}\bar{h}^{\nu\sigma} \tag{4.78}$$

and taking the trace $\eta^{\mu\nu}$

$$\partial_{\alpha}\bar{h} - \partial_{\lambda}\bar{h}^{\lambda}_{\alpha} - n\Gamma_{\alpha} - \Gamma^{\lambda}\eta_{\lambda\alpha} + \Gamma_{\alpha} + \Gamma_{\alpha} = 0 \tag{4.79}$$

so that

$$\Gamma_\alpha = -\frac{1}{n-2}\partial_\alpha \bar{h} \tag{4.80}$$

and then the full Gamma equation tells us that

$$-\partial_\alpha \bar{h}_{\mu\nu} + \frac{1}{n-2}\partial_\alpha \bar{h}\ \eta_{\mu\nu} + \Gamma_{\nu,\alpha\mu} + \Gamma_{\mu,\alpha\nu} = 0 \tag{4.81}$$

where

$$\Gamma_{\mu,\alpha\beta} \equiv \eta_{\lambda\mu}\ \Gamma^\lambda_{\alpha\beta} \tag{4.82}$$

In terms of the variables $h_{\mu\nu}$ this reads

$$\Gamma_{\nu;\alpha\mu} + \Gamma_{\mu;\nu\alpha} = +\partial_\alpha h_{\mu\nu} \tag{4.83}$$

Cycic permutations are

$$\begin{aligned} \Gamma_{\mu;\nu\alpha} + \Gamma_{\alpha;\mu\nu} &= +\partial_\nu h_{\alpha\mu} \\ \Gamma_{\alpha;\mu\nu} + \Gamma_{\nu;\alpha\mu} &= +\partial_\mu h_{\nu\alpha} \end{aligned} \tag{4.84}$$

Summing 1+2-3 yields

$$\Gamma_{\mu;\nu\alpha} = \frac{1}{2}\left(\partial_\nu h_{\alpha\mu} + \partial_\alpha h_{\mu\ } - \partial_\mu h_{\nu\alpha}\right) \tag{4.85}$$

which yields the linear piece of Christoffel's symbols

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}\eta^{\alpha\beta}\left(\partial_\mu h_{\beta\nu} + \partial_\nu h_{\beta\mu} - \partial_\beta h_{\mu\nu}\right) \tag{4.86}$$

The Deser action enjoys *on shell* the gauge invariance

$$\begin{aligned} \delta h_{\mu\nu} &= \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \\ \delta \Gamma^\alpha_{\beta\gamma} &= \partial_\beta \partial_\gamma \xi^\alpha \end{aligned} \tag{4.87}$$

(That is the variation of the action is proportional to the EM). The EM are equivalent to

$$R^L_{\mu\nu}[h] = 0 \tag{4.88}$$

and the linearized Bianchi identities tell us that

$$\partial_\mu R_L^{\mu\nu} = \frac{1}{2}\partial^\nu R_L \tag{4.89}$$

Now we change again the action. The extra term reads

$$\Delta S \equiv -\kappa \int d^4x\ \bar{h}^{\mu\nu}\left[\Gamma^\lambda_{\mu\rho}\Gamma^\rho_{\nu\lambda} - \Gamma^\lambda_{\mu\nu}\Gamma^\rho_{\lambda\rho}\right] \tag{4.90}$$

The full lagrangian now reads

$$L \equiv -\kappa \bar{h}^{\mu\nu}\left[\partial_\mu \Gamma^\rho_{\nu\rho} - \partial_\rho \Gamma^\rho_{\mu\nu}\right] + \left(\eta^{\mu\nu} - \kappa \bar{h}^{\mu\nu}\right)\left[\Gamma^\lambda_{\mu\rho}\Gamma^\rho_{\nu\lambda} - \Gamma^\lambda_{\mu\nu}\Gamma^\rho_{\lambda\rho}\right] \tag{4.91}$$

Let is call

$$\eta^{\mu\nu} - \kappa\bar{h}^{\mu\nu} \equiv f^{\mu\nu} \tag{4.92}$$

and its inverse

$$f_{\alpha\lambda} f^{\lambda\beta} = \delta^\beta_\alpha \tag{4.93}$$

so that the lagrangian reads

$$L \equiv (f^{\mu\nu} - \eta^{\mu\nu}) \left[\partial_\mu \Gamma^\rho_{\nu\rho} - \partial_\rho \Gamma^\rho_{\mu\nu}\right] + f^{\mu\nu} \left[\Gamma^\lambda_{\mu\rho}\Gamma^\rho_{\nu\lambda} - \Gamma^\lambda_{\mu\nu}\Gamma^\rho_{\lambda\rho}\right] \tag{4.94}$$

The new EM read

$$\begin{aligned}
&\frac{\delta S}{\delta f^{\mu\nu}} = R_{\mu\nu}[\Gamma] \\
&\frac{\delta S}{\delta \Gamma^\alpha_{\mu\nu}} = \partial_\alpha f^{\mu\nu} - \frac{1}{2}\left(\delta^\nu_\alpha \partial_\sigma f^{\mu\sigma} + \delta^\mu_\alpha \partial_\sigma f^{\nu\sigma}\right) - f^{\mu\nu}\Gamma_\alpha - \\
&-\frac{1}{2}\left(\Gamma^\mu f^\nu_\alpha + \Gamma^\nu f^\mu_\alpha\right) + \Gamma^\nu_{\alpha\sigma} f^{\mu\sigma} + \Gamma^\mu_{\alpha\beta} f^{\nu\beta}
\end{aligned} \tag{4.95}$$

This equations are similar to the ones we solved previously. One gets

$$\Gamma^\mu_{\alpha\beta} f^{\alpha\beta} = -\partial_\sigma f^{\sigma\mu} \tag{4.96}$$

as well as

$$\Gamma_\lambda = \frac{1}{n-2} f_{\alpha\beta}\partial_\lambda f^{\alpha\beta} = -\frac{1}{n-2} F^{-1}\partial_\alpha F \tag{4.97}$$

where

$$F \equiv \det\ f_{\mu\nu} \tag{4.98}$$

The full Gamma equation now reads

$$\partial_\alpha f^{\mu\nu} + \frac{1}{n-2} f^{\mu\nu} F^{-1}\partial_\alpha F + \Gamma^\mu_{\sigma\alpha} f^{\nu\sigma} + \Gamma^\nu_{\alpha\sigma} f^{\sigma\mu} = 0 \tag{4.99}$$

which can be written in the form

$$\partial_\alpha \left(f^{\mu\nu}\ F^{\frac{1}{n-2}}\right) + \Gamma^\mu_{\sigma\alpha} f^{\nu\sigma}\ F^{\frac{1}{n-2}} + \Gamma^\nu_{\alpha\sigma} f^{\sigma\mu}\ F^{\frac{1}{n-2}} = 0 \tag{4.100}$$

It is natural to define

$$g^{\mu\nu} \equiv F^{\frac{1}{n-2}}\ f^{\mu\nu} \tag{4.101}$$

so that

$$f^{\mu\nu} = \sqrt{g} g^{\mu\nu} \tag{4.102}$$

Miraculously, the full lagrangian is diff invariant with the measure

$$d(vol) \equiv \sqrt{g}\ d^n x \tag{4.103}$$

The equation can again be solved by cyclic permutations, resulting in the connection taking the Christoffel value

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} \left(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}\right) \tag{4.104}$$

so that

$$R_{\mu\nu} \equiv R_{\mu\nu}[g] \tag{4.105}$$

the full nonlinear Einstein equation.

5

The principle of equivalence and the space-time manifold.

There is a mistery in Newton's law of gravitation. The force that a given body of *mass* m_1 exerts on another body of mass m_2 is given by

$$\vec{F} = G m_1 m_2 \frac{\vec{r_1} - \vec{r}_2}{r_{12}^3} \tag{5.1}$$

where

$$r_{12} \equiv |\vec{r}_1 - \vec{r}_2| \tag{5.2}$$

First of all, if we compare it with the Coulomb force between two electrically charged objects, with charges q_1 and q_2, which reads in Gauss units

$$\vec{F} = q_1 q_2 \frac{\vec{r_1} - \vec{r}_2}{r_{12}^3} \tag{5.3}$$

we see immediatly that the masses play the rôle of the charges. The first difference is that the *gravitational charges*, that is the *active gravitational masses* as we will dub them from now on, are always positive. This a mystery in Newtonian physics. But if we now want to compute the acceleration that this force impinges on the particle number two, we should use Newton's second law

$$m_2^i \ddot{\vec{r}}_2 = G m_1^g m_2^g \frac{\vec{r_1} - \vec{r}_2}{r_{12}^3} \tag{5.4}$$

Here we have put m^i to indicate that what apppears in Newton's second law is the *inertial mass* whose physical meaning lies in the effectiveness of a given external force to produce acceleration on that body, which in principle has nothing to do with the ability of that body to create a gravitational field, which is proportional to its gravitational charge, *id est* the *active gravitational mas*, which we have denoted by m^g. The experimental fact that these teo masses are equal

$$m^i = m^g \tag{5.5}$$

means that we can simplify the above equation

$$\ddot{\vec{r}}_2 = G m_1^g \frac{\vec{r_1} - \vec{r}_2}{r_{12}^3} \tag{5.6}$$

which now implies that *all bodies are subjected to the same acceleration*, independently of their mass. This is one of the more important experiments in the history of physics. The present experimental limit, due to the Eöt-Wash group at the university of Washington in Seattle by using a torsion balance is (in the particular case of berilium and titanium)

$$\eta\,(\mathrm{Be},\mathrm{Ti}) \equiv 2\frac{\left(\frac{m_g}{m_i}\right)^{\mathrm{Be}} - \left(\frac{m_g}{m_i}\right)^{\mathrm{Ti}}}{\left(\frac{m_g}{m_i}\right)^{\mathrm{Be}} + \left(\frac{m_g}{m_i}\right)^{\mathrm{Ti}}} \;\leq\; 10^{-13} \tag{5.7}$$

This parameter η is usually called the Eötvös parameter, and it should vanish if the equivalence principle is correct. More information can be easily found in the home page of the Washington group: http://www.npl.washington.edu/eotwash/

Einstein postulated on this physical basis the *strong equivalence principle*, asserting that *all* physics in a free falling frame is equivalent to physics in absence of gravitation. This means that at any point in spacetime there is a reference system in which the laws of special relativity are exactly valid. This reference system varies from point to point (even in Newtonian gravity).

The great leap forward, one of the biggest intellectual achievements in the history of mankind taken by Einstein in 1916 is to atribute this to tha fact that four dimensional spacetime wa not flat as is Minkowski spacetime, but curved instead, and that the precise amount of curvature was dictated by some precise equations, the are now denoted as Einstein equations. This means that spacetime is a curved four dimensional space, what mathematicians call a *differential manifold*, which looks locally like a flat space. The prototypical example is a sphere. Locally, as we can assert from everyday experience, a two-sphere looks like a two-dimensional plane. A point in spacetime is again represented by four coordinates

$$x^\mu \equiv \left(x^0 \equiv ct, x^1, x^2, x^2\right) \tag{5.8}$$

We shall represent the three spatial coordinates by

$$x^i \quad i = 1, 2, 3. \tag{5.9}$$

The metric of the spacetime will then be given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{5.10}$$

where the 10 components of the *metric tensor* $g_{\mu\nu}(x^0, x^1, x^2, x^3)$ sre to be determined by the Einstein equations, a set of nonlinear PDE. Since GR was discovered, gravitation is expressed in the language of differential geometry. To speak gravitation, it is necessary to know the language. How do we characterize the free falling frames? We can call them *locally inertial frames* LIF, because in them the standard laws of SR do hold. Now, Lorentz transformations are *linear*. This means that coordinates and vectors transform the same way. This is not true anymore when transformations are nonlinear, which is the case that will concern us when studying the gravitational field in a general setting. This means that LIF can be characterized by a basis of vectors in the tangent space of the spacetime manifold. When studying vectors and tensors in general it is useful to consider specific basis, which in this context, will be called *frames*. The are many possible *frames* in Minkowski space. The canonical one will consist in four unit vector, one timelike and the other three spacelike.

$$\begin{aligned}
&u^2 = 1\\
&e_1^2 = e_2^2 = e_3^2 = -1\\
&u.e_i = 0\\
&e_i.e_j = -\delta_{ij}
\end{aligned} \tag{5.11}$$

It is convenient to call $u \equiv e_0$. If we represent in a vary explicit way the indices, this is

$$e_a^\mu \tag{5.12}$$

with $a = 0, 1, 2, 3$ labels the four vectors of the frame, and the contravariant index indicates that each object is indedeed a four-vector. The orthogpnality properties then tell us that

$$\eta_{\mu\nu} e_a^\mu e_b^\mu = \eta_{ab} \tag{5.13}$$

Any vector in the Minkowski space can be written as

$$v^\mu = \sum v^\alpha e_a^\mu \tag{5.14}$$

where

$$v_a \equiv v_\mu e_a^\mu \tag{5.15}$$

The position of the indices is material.

The mathematical consequence of the equivalence principle is that at each point $x \in V$ of spacetime the is such a frame (that will depend on the point) $e_a^\mu(x)$. This defines at each point a 4×4 matrix, which must not be singular. This is an essential hypothesis of the whole approach. Let us now consider the inverse matrix, which we shall call E_μ^b

$$e_a^\mu . E_\mu^b = \delta_a^b \tag{5.16}$$

Now let us consider

$$\left(\eta^{ab} e_a^\mu e_b^\nu\right) E_\nu^c \eta_{cd} = e_d^\mu \tag{5.17}$$

$$E^a_\mu = g_{\mu\nu} e^\nu_b \eta^{ab} \equiv e^a_\mu \tag{5.18}$$

From now on, no distinction will be made between the matrices e and E, because the position of the indices is enough to identify them unambiguously. It is important to distiguish between *latin indices*, $a, b, c, \ldots$, which we will call following Zumino *Lorentz indices* (sometimes called flat indices, that are lowered or raised with the flat Minkowski metric η_{ab}, and the greek indices $\mu, \nu, \rho, \ldots$, which we will call *Einstein indices*, or curved indices, lowered or raised with the spacetime metric, which is precisely the position-dependent matrix

$$\eta^{ab} e^\mu_a e^\nu_b \equiv g^{\mu\nu}(x) \tag{5.19}$$

More on this later.

From the moving frame e^μ_a we define quantities in the frame by projecting

$$V_a \equiv e^\mu_a V_\mu \tag{5.20}$$

Leibnitz rule implies that

$$\partial_\mu V_a = \partial_\mu e^\lambda_a . V_\lambda + e^\lambda_a \partial_\mu V_\lambda \tag{5.21}$$

Given a frame at T_x, e_a, any Lorentz-rotated frame with an (point-dependent) transformation is physically equivalent to it, and in particular gives rise to exactly the same space-time metric

$$(e')^\mu_a \equiv L(x)^b_a e^\mu_b \tag{5.22}$$

This stems from the fact that

$$\eta^{\alpha\beta} (e')^\mu_a (e')^\nu_b = \eta^{\alpha\beta} L^c_a L^d_b e^\mu_c e^\nu_d = \eta^{cd} e^\mu_c e^\nu_d \equiv g^{\mu\nu}(x) \tag{5.23}$$

On the other hand, each of the four vectors in a frame transforms as a vector under spacetime diffeomorphisms.

5.1 Differential forms.

We shall identify tangent vectors $\vec{v} \in T_x$ with directional derivatives of functions defined at a given point of the manifold

$$\vec{v}(f) \equiv v^\mu \partial_\mu f \tag{5.24}$$

A particular basis is given by the vectors

$$\partial_\mu \tag{5.25}$$

Given an arbitrary function, its differential is defined as $df \in T^*_x$

$$df\,(\vec{v}) \equiv \vec{v}\,(f) \tag{5.26}$$

Differential forms are antisymmetric linear maps

$$\omega_1 \; : v \in \mathbb{R}^n \rightarrow \omega(v) \in \mathbb{R} \tag{5.27}$$

A local basis is given by

$$dx^a(\partial_b) = \delta^a_b \tag{5.28}$$

$$\omega_2 \,(v, w) \in \mathbb{R}^n \times \mathbb{R}^n \rightarrow \omega_2(v, w) \in \mathbb{R} \tag{5.29}$$

We shall write in local coordinates

$$\alpha \equiv \frac{1}{p!} \alpha_{\mu_1 \ldots \mu_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p} \tag{5.30}$$

It is exceedingly useful to introduce the Kronecker symbol

$$\begin{aligned}
&\epsilon^{\lambda_1 \ldots \lambda_p}_{\mu_1 \ldots \mu_p} \equiv p! \; \delta^{\lambda_1}_{[\mu_1} \ldots \delta^{\lambda_p}_{\mu_p]} \\
&\epsilon^{\rho_1 \ldots \rho_p}_{\mu_1 \ldots \mu_p} \alpha_{\rho_1 \ldots \mu_p} = p! \; \alpha_{\mu_1 \ldots \mu_p} \\
&\epsilon^{\lambda_1 \ldots \lambda_q}_{\mu_1 \ldots \mu_q} \epsilon^{\mu_1 \ldots \mu_q \sigma_1 \ldots \sigma_p}_{\nu_1 \ldots \nu_{p+q}} = q! \; \epsilon^{\lambda_1 \ldots \lambda_q \sigma_1 \ldots \sigma_p}_{\nu_1 \ldots \nu_{p+q}} \\
&\epsilon^{\lambda_1 \ldots \lambda_q \rho_1 \ldots \rho_p}_{\mu_1 \ldots \mu_q \rho_1 \ldots \rho_p} = p! \; \epsilon^{\lambda_1 \ldots \lambda_q}_{\mu_1 \ldots \mu_q} \\
&\eta_{\mu_1 \ldots \mu_n} \equiv \sqrt{|g|} \; \epsilon^{1 \ldots n}_{\mu_1 \ldots \mu_n} \\
&\eta^{\mu_1 \ldots \mu_n} = \frac{1}{\sqrt{|g|}} \; \epsilon^{\mu_1 \ldots \mu_n}_{1 \ldots n} \\
&\eta_{\lambda_1 \ldots \lambda_p \lambda_{p+1} \ldots \lambda_n} \eta^{\lambda_1 \ldots \lambda_p \mu_{p+1} \ldots \mu_n} = p! \; \epsilon^{\mu_{p+1} \ldots \mu_n}_{\lambda_{p+1} \ldots \lambda_n} \\
&\nabla_\rho \eta_{\mu_1 \ldots \mu_n} = 0 \\
&d(vol) \equiv \eta_{\mu_1 \ldots \mu_n} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_n} = \sqrt{|g|} dx^1 \wedge \ldots \wedge dx^n \\
&dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_n} = \eta^{\mu_1 \ldots \mu_n} d(vol)
\end{aligned} \tag{5.31}$$

To verify these formulas is excellent gymnastics.

- Exterior product. The exterior product of two one-forms yields a two-form

  $$(\omega_1 \wedge \alpha_1)(v_1, v_2) \equiv \det \begin{pmatrix} \omega_1(v_1) & \alpha_1(v_1) \\ \omega_1(v_2) & \alpha_1(v_2) \end{pmatrix} \tag{5.32}$$

  In the general case, the product of a p-form and a q-form is a (p+q)-form

  $$(\omega_k \wedge \omega_l)\,(v_1 \ldots v_{k+l}) \equiv \sum \pm \omega_k(v_{i_1} \ldots v_{i_k}) \omega_l(v_{i_{k+1}} \ldots v_{i_{k+l}}) \tag{5.33}$$

  A general formula is given by

  $$\alpha \wedge \beta = \frac{1}{p!} \frac{1}{q!} \; \alpha_{\lambda_1 \ldots \lambda_p} \beta_{\mu_1 \ldots \mu_q} dx^{\lambda_1} \wedge \ldots \wedge dx^{\lambda_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_q} \tag{5.34}$$

The basic identity reads

$$\omega_p \wedge \omega_q = (-1)^{pq} \ \omega_q \wedge \omega_p \tag{5.35}$$

Sometimes we shall write

$$dx^{\mu_1 \ldots \mu_p} \equiv dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p} \tag{5.36}$$

This means that for every odd degree form

$$\omega_{2p+1} \wedge \omega_{2p+1} = 0 \tag{5.37}$$

- Coordinate basis In the basis of the tangent space associated to a local chart, (x^α),

$$\omega_k \equiv \sum_{\iota_1 < \ldots < \iota_k} \omega_{\iota_1 \ldots \iota_k} dx^{\iota_1} \wedge \ldots \wedge dx^{\iota_k} \tag{5.38}$$

- Exterior differential The differential of a function is given by a one-form

$$df \equiv \sum \partial_a f dx^a \tag{5.39}$$

In the general case, the differential of a p-form is a (p+1)-form

$$d\omega \equiv \sum_{\iota_1 < \ldots < \iota_k} d\omega_{\iota_1 \ldots \iota_k} \wedge dx^{\iota_1} \wedge \ldots \wedge dx^{\iota_k} \tag{5.40}$$

A general formula can also be given

$$(d\alpha)_{\mu_0 \mu_1 \ldots \mu_p} \equiv \frac{1}{(p+1)!} \ \epsilon^{\lambda_0 \lambda_1 \ldots \lambda_p}_{\mu_0 \mu_1 \ldots \mu_p} \partial_{\lambda_0} \alpha_{\lambda_1 \ldots \lambda_p} \tag{5.41}$$

The uselfuness of exterior calculus stems essentially from the basic fact that

$$d^2 = 0 \tag{5.42}$$

It is also a fact that the graded Leibnitz rule holds, id est,

$$d\left(\alpha_p \wedge \beta_q\right) = d\alpha_p \wedge \beta_q + (-1)^p \alpha_p \wedge d\beta_q \tag{5.43}$$

- Hodge The Hodge operator maps p-forms into (n-p)-forms. It is defined by

$$(*\alpha)_{\mu_{p+1} \ldots \mu_n} \equiv \frac{1}{p!} \ \eta_{\mu_1 \ldots \mu_n} \alpha^{\mu_1 \ldots \mu_p} \tag{5.44}$$

Its square depends on the dimension of spacetime as well as on the degree of the form

$$*^2 = (-1)^{p(n-p)} \tag{5.45}$$

In four dimensions (actually, in any even dimension)

$$*^2 = (-1)^p \tag{5.46}$$

In three-dimensions it is always +1

$$*^2 = +1. \tag{5.47}$$

The *exterior codifferential* is the adjoint of the exterior differential

$$(\alpha, \delta\beta) \equiv (d\alpha, \beta) \tag{5.48}$$

It is given by

$$\delta \equiv (-1)^p *^{-1} d* \tag{5.49}$$

It is possible to give a simple formula

$$(\delta\alpha)_{\rho_1 \ldots \rho_{p-1}} = -\frac{1}{p!}\, \epsilon^{\mu_1 \ldots \mu_p}_{\nu\rho_1 \ldots \rho_{p-1}} \nabla^\nu \alpha_{\mu_1 \ldots \mu_p} \tag{5.50}$$

The *interior product* of a p-form and a vector, X, is the (p-1)-form given by

$$(i(X)\omega)\,(v_1 \ldots v_{p-1}) \equiv \omega_p\,(X, v_1 \ldots v_{p-1}) \tag{5.51}$$

- Stokes' theorem We start from the properties of the volume defined by an elementary cell of $\mathbb{R}^3$
  - It vanishes if the vectors are linearly dependent.
  - It stays the same when we add to a given vector a linear combination of the other vectors.
  - Depends in a linear way on all vectors.

Al these properties are enjoyed by the elementary formula

$$V = \sum \epsilon_{ijk} v_1^i v_2^j v_3^k = \eta\,(\vec{v}_1, \vec{v}_2, \vec{v}_3) \tag{5.52}$$

where the *volume element* is defined by

$$\eta \equiv dx^1 \wedge dx^2 \wedge dx^3 \tag{5.53}$$

This leads in a natural way to define volumes through intagration

$$\int_{\partial V} \omega = \int_V d\omega \tag{5.54}$$

The classical theorems of Gauss, Stokes and the divergence are but particular instamces of this. For example

$$\int_{S_2} dA_1 = \int_{C_1 \equiv \partial S_2} A_1 \tag{5.55}$$

If A_1 is a 1-form of $\mathbb{R}^3$

$$A_1 \equiv A_i dx^i \tag{5.56}$$

then

$$dA_2 = \frac{1}{2} \left(\partial_i A_j - \partial_j A_i\right) dx^i \wedge dx^j \tag{5.57}$$

It is customary to define the *rotational* or *curl* as

$$(\mathrm{rot} A)_i \equiv \epsilon_{ijk} \partial_j A_k \tag{5.58}$$

The surface integral

$$\int_S dA_2 = \int_S \frac{1}{2} \left(\partial_i A_j - \partial_j A_i\right) dx^i \wedge dx^j = \tag{5.59}$$

It is customary to define

$$n^i dS \equiv \frac{1}{2} \epsilon_{ijk} dx^j \wedge dx^k \tag{5.60}$$

so that

$$\sum_i (\mathrm{rot} A)_i n_i dS = \sum_{jk} \left(\partial_j A_k - \partial_k A_j\right) dx^j \wedge dx^k \tag{5.61}$$

and we recover Stokes'original theorem

$$\int_S \vec{\mathrm{rot}} \vec{A}\ \vec{n}\ dS = \int_{\partial S} \vec{A}\ d\vec{x} \tag{5.62}$$

Let us now apply it to

$$\int_{V_3} d\omega_2 = \int_{\partial V_3} \omega_2 \tag{5.63}$$

Write

$$\omega_2 \equiv \frac{1}{2} \omega_{ij} dx^i \wedge dx^j \tag{5.64}$$

so that

$$d\omega_2 \equiv \frac{1}{2} \partial_k \omega_{ij} dx^k \wedge dx^i \wedge dx^j = \frac{1}{2} \partial_k \omega_{ij} \epsilon^{kij} dV \tag{5.65}$$

Now we define the dual one-form

$$\Omega_i dx^i \equiv (*\omega_2)_1 \equiv \frac{1}{2} \epsilon_{ijk} \omega_{jk} \tag{5.66}$$

then

$$d\omega_2 = \partial_k \Omega_k \equiv \mathrm{div} \vec{\Omega} \tag{5.67}$$

and we recover Gauss'divergence teorem

$$\int_V \mathrm{div}\ \vec{\Omega}\ dV = \int_{\partial V} \vec{\Omega}\ \vec{n}\ dS \tag{5.68}$$

Many other examples can be found for example, in Flanders'book

- Lie derivative The Lie derivative of a function is defined as the directional derivative
$$\vec{v}(f) = \pounds(\vec{v})f \tag{5.69}$$
The Lie derivative of a one-form is defined in a natural way.
$$\pounds(\vec{v})df \equiv d\vec{v}(f) \tag{5.70}$$
This definition extends to a general case simply by postulating that Leibnitz' rule holds true
$$\pounds(\vec{v})a_a d\xi^a = (\pounds(\vec{v})a_a)d\xi^a + \alpha_a \pounds(\vec{v})d\xi^a \tag{5.71}$$
In the case of vectors we use the dual application
$$\pounds(\vec{v})\langle\alpha, \vec{X}\rangle = \langle\pounds(\vec{v})\alpha, \vec{X}\rangle + \langle\alpha, \pounds(\vec{v})\vec{X}\rangle \tag{5.72}$$
It is a fact that
$$\begin{aligned} &\pounds(\vec{X})\vec{Y} = [\vec{X}, \vec{Y}] \\ &\pounds(\vec{X}) = i(\vec{X})d + di(\vec{X}) \end{aligned} \tag{5.73}$$

- Diffeomorfisms An active diffeomorphism
$$\xi : x \in M \rightarrow y = \xi(x) \in M \tag{5.74}$$
Acting on vectors, given $g : y \rightarrow \mathbb{R}$, then $g \circ \xi : x \rightarrow \mathbb{R}$ and $v \in T_x$, we define a different vector $\xi_* v \in T_y$ through
$$\xi_*(v)(g) \equiv v(g \circ \xi) \tag{5.75}$$
In a local coordinate basis
$$(\xi_* v)^\mu(y) = v^\rho \partial_\rho \xi^\mu(x) \tag{5.76}$$
Given a one-form $\omega \in T^*_y$ we define another form $\xi^*\omega \in T_x$ through
$$\xi^*\omega(v) \equiv \omega(\xi_* v) \tag{5.77}$$
In a local coordinate basis
$$(\xi^*\omega)_\alpha(x) = \omega_\mu(y)\partial_\alpha \xi^\mu(x) \tag{5.78}$$
If it were a 2-form
$$(\xi^*\omega)(v, w) = \omega(v, w) \tag{5.79}$$
that is
$$(\xi^*\omega)_{\alpha\beta}(x) = \omega_{\mu\nu}(y)\partial_\alpha \xi^\mu \partial_\beta \xi^\nu \tag{5.80}$$

5.2 Particle motion in external gravitational fields.

It is natural to postulate that the action for an otherwise free particle moving in an external (background) gravitational field will be given by

$$S \equiv -mc\int_\gamma ds = -mc\int_\gamma \sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} d\lambda \tag{5.81}$$

where the ds is the pseudo-riemannian element of length as given by the metric. The integral is extended over the parameterized curve γ

$$x^\mu = x^\mu(\lambda) \tag{5.82}$$

and we have denoted by

$$\dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda} \tag{5.83}$$

In the timelike case we can normalize the tangent vector

$$u^\mu \equiv \frac{\dot{x}^\mu}{\sqrt{\dot{x}^2}} \tag{5.84}$$

The extrema of the action are by definition the *geodesics* of the manifold. We get

$$\delta S = -mc\int d\lambda \left\{\partial_\rho g_{\mu\nu}\delta x^\rho \dot{x}^\mu \dot{x}^\nu + \ 2g_{\mu\nu}\dot{x}^\mu \delta\dot{x}^\nu\right\} = -mc\int d\lambda\ \delta x^\rho \Big\{$$
$$\partial_\rho g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu - \left(\partial_\lambda g_{\mu\rho} + \partial_\mu g_{\lambda\rho}\right)\dot{x}^\lambda \dot{x}^\mu - 2g_{\mu\rho}\ddot{x}^\mu\Big\} \tag{5.85}$$

Expressed in the form of four ordinary differential equations for the four functions of one variable $x^\mu(s)$ they read

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0 \tag{5.86}$$

Here the Christoffel symbols are given by

$$\Gamma_{\lambda,\mu\nu} \equiv g_{\lambda\rho}\Gamma^\rho_{\mu\nu} \equiv g_{\lambda\rho}\ \frac{1}{2}g^{\rho\sigma}\left(-\partial_\sigma g_{\mu\nu} + \partial_\mu g_{\nu\sigma} + \partial_\mu g_{\nu\sigma}\right) \tag{5.87}$$

The Christoffel symbols are in a sense the gauge field associated to diffeomorphisms, that is, given a vector

$$V^{\mu'}(x') \equiv \frac{\partial x^{\mu'}}{\partial x^\rho}\ V^\rho\ (x) \tag{5.88}$$

It is obvious that

$$\partial_\alpha V^\mu \tag{5.89}$$

does not transform as a tensor unless the diffeomorphism is a linear one. Then the gauge fields are defined in such a way that

$$\nabla_\rho V^\mu \equiv \partial_\rho V^\mu + \Gamma^\mu_{\rho\sigma}\ V^\sigma \tag{5.90}$$

does transform as a tensor, that is

$$\nabla_{\rho'} V^{\mu'}(x') = \frac{\partial x^\sigma}{\partial x^{\rho'}} \frac{\partial x^{\mu'}}{\partial x^\lambda} \nabla_\sigma V^\lambda \tag{5.91}$$

This yields immediatly the transformation properties of the connection, that is

$$\Gamma^{\mu'}_{\rho'\lambda'} \frac{\partial x^{\lambda'}}{\partial x^\lambda} + \frac{\partial^2 x^{\mu'}}{\partial x^\lambda \partial x^\sigma} \frac{\partial x^\sigma}{\partial x^{\rho'}} = \frac{\partial x^\sigma}{\partial x^{\rho'}} \frac{\partial x^{\mu'}}{\partial x^\delta} \Gamma^\delta_{\sigma\lambda} \tag{5.92}$$

Because of the inhomogeneous term the Christoffel symbols are NOT tensors. Thay are connections, that is, gauge fields. It is useful exercise to check at least the the Christoffel symbols are a solution of these equations. Actually they are the *unique* solution involving the metric tensor alone.

It is also useful to check that for covariant tensors.

$$\nabla_\mu \omega_\nu \equiv \partial_\mu \omega_\nu - \Gamma^\lambda_{\mu\nu} \omega_\lambda \tag{5.93}$$

Using this formula, it is plain to check that the metric is covariantly constant

$$\nabla_\alpha g_{\beta\gamma} = \partial_\alpha g_{\beta\gamma} - \Gamma^\lambda_{\alpha\beta} g_{\lambda\gamma} - \Gamma^\lambda_{\alpha\gamma} g_{\lambda\beta} = 0 \tag{5.94}$$

The analogous to the field strength tensor for gauge theories is the *Riemann-Christoffel tensor*

$$R^\mu{}_{\nu\alpha\beta} \equiv \partial_\alpha \Gamma^\mu_{\alpha\beta} - \partial_\beta \Gamma^\mu_{\nu\alpha} + \Gamma^\mu_{\sigma\alpha}\Gamma^\sigma_{\nu\beta} - \Gamma^\mu_{\sigma\beta}\Gamma^\sigma_{\nu\alpha} \tag{5.95}$$

The *Ricci tensor* is defined by contracting indices

$$R_{\mu\nu} \equiv R^\lambda{}_{\mu\lambda\nu} \tag{5.96}$$

The Riemann tensor is, by construction, antisymmetric in the first two indices, and also in the last two indices; and is symmetric to the interchange of the first couple of indices by the second couple of indices. It obeys also the identity

$$R^\mu{}_{\alpha\beta\gamma} + R^\mu{}_{\gamma\alpha\beta} + R^\mu{}_{\beta\gamma\alpha} = 0 \tag{5.97}$$

This leaves

$$\frac{n^2(n^2-1)}{12} \tag{5.98}$$

independent components (20 in n=4 dimensions).

There are also some differential identities, the *Bianchi identities*

$$\nabla_\alpha R^\mu{}_{\nu\beta\gamma} + \nabla_\gamma R^\mu{}_{\nu\alpha\beta} + \nabla_\beta R^\mu{}_{\nu\gamma\alpha} = 0 \tag{5.99}$$

Contracting δ^β_μ
$$\nabla_\alpha R_{\nu\gamma} - \nabla_\gamma R_{\nu\alpha} + \nabla_\mu R^\mu{}_{\nu\gamma\alpha} = 0 \tag{5.100}$$
Contracting again $g^{\nu\alpha}$
$$\nabla^\alpha R_{\alpha\gamma} - \nabla_\gamma R + \nabla_\mu R^\mu_\gamma = 0 \tag{5.101}$$
We shall derive most of these equations in a short while. Many useful formulas of tensor calculus are to be found in Eisenhart's book, still indispensable.

Also very useful are the *Ricci identities* that state that
$$[\nabla_\alpha, \nabla_\beta]\, \omega_\gamma = \; R_{\alpha\beta\gamma\delta}\; \omega^\delta \tag{5.102}$$
This can actually be taken as the definition of the Riemann tensor, as is done in many books.

All this means that the geodesic equations can be written as
$$u^\mu \nabla_\mu u^\alpha = 0 \tag{5.103}$$
where the four-velocity of the massive particle is given by
$$u^\alpha \equiv \frac{dx^\alpha}{ds} \tag{5.104}$$
This allows us to identify the nonrelativistic limit, in which the action must reduce to
$$S = -mc \int \left(c - \frac{v^2}{2c} + \frac{V_g}{c} \right) dt \tag{5.105}$$
Where V_g is the gravitational potential. This means that we have to identify in this limit
$$ds^2 = \left(c^2 + 2V_g \right) dt^2 - dl^2 \tag{5.106}$$
where
$$v \equiv \frac{dl}{dt} \tag{5.107}$$
This leads to the identification of the newtonian potential in the limit
$$g_{00} = 1 + \frac{2V_g}{c^2} \tag{5.108}$$
Proper times at different places obey
$$d\tau = \frac{1}{c}\sqrt{g_{\mu\nu} dx^\mu dx^\nu} \tag{5.109}$$
Frequencies (corresponding to atomic transitions) obey the inverse law. If the emitting object is a rest
$$\frac{\omega_1}{\omega_2} = \sqrt{\frac{g^2_{00}}{g^1_{00}}} \sim 1 + \frac{V_g^2 - V_g^1}{c^2} \tag{5.110}$$

If the atom is raised over a height h over the Earth surface

$$V_g \sim -\frac{GM}{h+R_\oplus} \tag{5.111}$$

$$\frac{\omega(h)}{\omega(0)} \sim 1 - \frac{GM}{c^2 R_\oplus} + \frac{GM}{c^2\left(R_\oplus + h\right)} \sim 1 - \frac{GM}{c^2 R_\oplus} + \frac{GM}{c^2 R_\oplus}\left(1 - \frac{h}{R_\oplus}\right) =$$
$$= 1 - \frac{GM}{c^2 R_\oplus^2}\, h \tag{5.112}$$

This makes sense from the photon viewpoint. It costs energy for the photon to get out of the gravitational potential; the atome raised at the height h has less gravitational energy to fight upon. This is a generic prediction of GR, independent of the equations of motion, and was as such realized by Einstein. It was experimentally verified by Pound and Rebka in 1960 in a laboratory experiment. Please google it for recent experiments.

5.3 The Space-Time Manifold. Moving Frames.

As a matter of principle, let us discuss the different types of connections that can exist in a manifold, following the excellent reference [4]. First of all, there are *general connections on vector bundles*, what physicists understamd as gauge fields. In the particular case when the vector bundle is the tangent bundle of a manifold, those are the *affine connections.* Closely related is the *frame bundle*, the set of all frames in the tangent space at each point of the manifold. Namely, in a local chart

$$\vec{e}_a \equiv e_a^\mu \partial_\mu \tag{5.113}$$

The determinant of the matrix $e_a^\mu \neq 0$.

The corresponding coframe is defined by

$$\underline{e}^a \equiv \left(e^{-1}\right)^a_{\ \mu} dx^\mu \tag{5.114}$$

where

$$\left(e^{-1}\right)^a_{\ \mu} e_b^\mu \equiv \delta_b^a \tag{5.115}$$

When there is a metric in the manifold, it makes sense to choose *orthonormalized frames*, and to study whether the connection is compatible with orthonormalization. This last case is the one of interest in general relativity and its natural extensions. Namely

$$\vec{e}_a.\vec{e}_b = \eta_{ab} \tag{5.116}$$

We claim that

$$\left(e^{-1}\right)^a_{\ \mu} = \eta^{ab} g_{\mu\nu} e_b^\nu \tag{5.117}$$

Indeed

$$\left(\eta^{ab} g_{\mu\nu} e^{\nu}_{b}\right) e^{\sigma}_{a} \equiv e^{a}_{\mu} e^{\sigma}_{a} \equiv J^{\sigma}_{\mu} \tag{5.118}$$

$$J^{\sigma}_{\mu} J^{\mu}_{\lambda} = J^{\sigma}_{\lambda} = \delta^{\sigma}_{\lambda} \tag{5.119}$$

It follows that

$$e^{a}_{\mu} e_{a\nu} = g_{\mu\nu} \tag{5.120}$$

The gravitational field is then represented in GR as the fact that the spacetime metric

$$da^2 = g_{\mu\nu}(x)\; dx^{\mu}\; dx^{\nu} \tag{5.121}$$

is not flat; to the extent that it differs from the flat metric, it indicates the presence of a gravitational field. At each point there are tensors (or spinors) that represent physical observables. For example, the energy momentum tensor

$$T_{\mu\nu}(x) \tag{5.122}$$

This tensor *live* in the tangent space; the set of all tangent spaces of the manifold is the tangent bundle. A *frame* is a basis of the tangent vector space at a given point of the space-time manifold. This four vectors are represented by

$$\vec{e} \equiv e^{\mu}_{a} \partial_{\mu} \tag{5.123}$$

where the index $a = 0, 1, 2, 3$ labels the four different vectors. The simplest possibility is to choose one of them timelike (this is the one labeled $\vec{e}_0$) , and the other three spacelike. Furthermore, they can be normalized in such a way that

$$g_{\mu\nu} e^{\mu}_{a} e^{\nu}_{b} = \eta_{ab} \tag{5.124}$$

This is the reason why latin indices are dubbed *Lorentz* indices, whereas the ordinary spacetime indices are called Einstein indices. Such a frame is precisely a LIF (where FREFOS live) and the physical observables measured in the LIF are simply

$$T_{ab} \equiv T_{\mu\nu}\, e^{\mu}_{a} e^{\nu}_{b} \tag{5.125}$$

The determinant of e considered as a matrix cannot vanish. We can then *define* the *coframe* made out of the dual one-forms

$$\underline{e}^{a} \left(\vec{e}_b\right) \equiv e^{a}_{\mu} e^{\mu}_{b} = \delta^{a}_{\beta} \tag{5.126}$$

When indices are put in place, this is equivalent to computing the inverse matrix

$$\begin{aligned} e^{a}_{\mu} e^{\mu}_{b} &\equiv \delta^{a}_{b} \\ e^{a}_{\mu} e^{\nu}_{a} &= \delta^{\nu}_{\mu} \end{aligned} \tag{5.127}$$

From the normalization condition

$$g_{\mu\nu} E^{\mu}_{a} E^{\nu}_{b} = \eta_{ab}$$

and multiplying both members by the dual form e^a_σ

$$\Rightarrow e^a_\mu = g_{\mu\nu}\eta^{ab}e^\nu_b$$

This means that the dual form is simply the frame with the Einstein indices lowered with the spacetime metric, and the Lorentz indices raised with the Lorentz metric. Following most physicists we shall represent both the frame and the coframe with the same letter, although when neccessary we will indicate explicitly its nature, as in

$$\vec{e}_a \equiv e^\mu_a \partial_\mu$$
$$\underline{e}_a \equiv e_{a\mu}dx^\mu \tag{5.128}$$

The parallel propagator is defined once frames at different points are selected by some mechanism

$$g^\alpha{}_{\beta'}(x,x') \equiv e^\alpha_a(x)e^a_{\alpha'}(x') \tag{5.129}$$

Then physical quantities at different points are related through

$$A^\alpha(x) \equiv g^\alpha{}_{\beta'}(x,x')a^{\alpha'}(x') \tag{5.130}$$

For n-dimensional spheres in stereographic coordinates

$$ds^2 = \Omega^2\ \delta_{\mu\nu}dx^\mu dx^\nu \tag{5.131}$$

where

$$\Omega \equiv \frac{1}{1+\frac{x^2}{4L^2}} \tag{5.132}$$

and the coframe is defined by

$$e^a_\mu = \Omega\delta^a_\mu \tag{5.133}$$

in such a way that the frame itself is given by

$$e^\mu_a = \frac{1}{\Omega}\delta^\mu_a \tag{5.134}$$

The S_n Christoffels read

$$\Gamma^\alpha_{\beta\gamma} = \frac{\Omega_\beta}{\Omega}\ \delta^\alpha_\gamma + \frac{\Omega_\gamma}{\Omega}\ \delta^\alpha_\beta - \frac{\Omega^\alpha}{\Omega}\ \delta_{\beta\gamma} \tag{5.135}$$

Under a local Lorentz transformation

$$\vec{e}_{a'} = L_{a'}{}^b(x)\,\vec{e}_b \tag{5.136}$$

$e^\mu_a(x)$ is a nonsingular square $n\times n$ matrix. The commutators are given by

$$[\vec{e}_a,\vec{e}_b] = C^c_{ab}\vec{e}_c$$

It is a fact that

$$d\underline{e}^a = \partial_{[\mu} e^a_{\rho]} dx^\rho \wedge dx^\mu = \frac{1}{2}\left(\partial_\mu e^a_\nu - \partial_\nu e^a_\mu\right) dx^\mu \wedge dx^\nu = \frac{1}{2}\left(\partial_\mu e^a_\nu - \partial_\nu e^a_\mu\right) e^\mu_c e^\nu_d e^c \wedge e^d =$$
$$= \frac{1}{2}\left(e_c(e^a_\nu) e^\nu_d - e_d(e^a_\mu) e^\mu_c\right) e^c \wedge e^d = \frac{1}{2}\left(e_d(e^\mu_c) e^a_\mu - e^a_\nu e_c(e^\nu_d)\right) e^c \wedge e^d =$$
$$= \frac{1}{2}\left[\vec{e}_d, \vec{e}_c\right]^\mu e^a_\mu \, \underline{e}^c \wedge \underline{e}^d = -\frac{1}{2} C^a_{cd} \, \underline{e}^c \wedge \underline{e}^d \tag{5.137}$$

To be specific, the structure constants read

$$C^c_{ab} = e^c_\mu \left(e^\lambda_a \partial_\lambda e^\mu_b - e^\lambda_b \partial_\lambda e^\mu_a\right) = e^c_\mu \left(e^\lambda_a \nabla_\lambda e^\mu_b - e^\lambda_b \nabla_\lambda e^\mu_a\right) \tag{5.138}$$

(The Christoffels cancel when taking the antisymmetric part). In our S_n example,

$$C^c_{ab} = \frac{\Omega_b}{\Omega^2}\,\delta^c_a - \frac{\Omega_a}{\Omega^2}\,\delta^c_b \tag{5.139}$$

Under a local Lorentz transformation the vierbein transforms as

$$\underline{e}^{a'} = L^a{}_b(x)\underline{e}^b \tag{5.140}$$

This is not true of the derivatives of the vierbein, de^a, owing to the term in $dL^a{}_b$. We would like to introduce a gauge field (connection) in the LIF, the so called *spin connection*, such that the two-form

$$De^a \equiv de^a + \omega^a{}_b \wedge e^b \tag{5.141}$$

transforms as

$$(De^a)' = L^a{}_b De^b \tag{5.142}$$

For this to be true we need

$$d\left(L^a{}_b e^b\right) + (\omega')^a{}_b \wedge \left(L^b{}_c e^c\right) = L^a{}_b\left(de^b + \omega^b{}_c \wedge e^c\right) \tag{5.143}$$

This is equivalent to

$$dL^a{}_b \wedge e^b + (\omega')^a{}_b \wedge L^b{}_c e^c = L^a{}_b \omega^b{}_c \wedge e^c \tag{5.144}$$

which is kosher provided

$$dL^a{}_c + (\omega')^a{}_b L^b{}_c = L^a{}_b \omega^b{}_c \tag{5.145}$$

Lorentz transformations are such that

$$L^{ac} L_{ad} = \delta^c_d = L^{ca} L_{da} \tag{5.146}$$

Finally we get the transformation law for the gauge field

$$(\omega')^a{}_d = L^a{}_b\, \omega^b{}_c\, L_d{}^c - dL^a{}_c L_d{}^c \tag{5.147}$$

At the linear level,

$$L_{ab} \equiv \eta_{ab} + \lambda_{ab} \tag{5.148}$$

$$\delta\omega^a{}_{b\mu} = -\partial_\mu \lambda^a{}_b + [\lambda, \omega_\mu]^a{}_b \tag{5.149}$$

- Assume now a field transforming with a given representation of the tangent group, say

$$\psi \to D(L)\psi \equiv (1 + d(\lambda))\,\psi \tag{5.150}$$

$$\delta\phi^i = \frac{1}{2}\lambda^{ab}\left(\Sigma_{ab}\right)^i{}_j\phi^j$$

where the Σ represent the algebra of $\mathfrak{SO}(n)$

$$[\Sigma_{ab}, \Sigma_{cd}] \equiv C_{ab,cd}{}^{ef}\Sigma_{ef} = \Sigma_{ad}\eta_{bc} + \Sigma_{bc}\eta_{ad} - \Sigma_{ac}\eta_{bd} - \Sigma_{bc}\eta_{ac}$$

For a Dirac spinor, for example,

$$\Sigma_{ab} \equiv \frac{1}{2}\gamma_{ab} \equiv \frac{1}{4}[\gamma_a, \gamma_b]$$

The representation that we were implicitly using until now is the fundamental of $\mathfrak{SO}(n)$

$$d^F\left(\omega_\mu\right)^a_b \equiv \omega^c_{d\mu}\left(T_c{}^d\right)^a{}_b \tag{5.151}$$

where

$$\left(T_c{}^d\right)^a{}_b \equiv \frac{1}{2}\left(\delta^a_c\delta^d_b - \delta_{cb}\delta^{da}\right) \tag{5.152}$$

as well as

$$d^F(\lambda) = \lambda_{ab} \tag{5.153}$$

In ths new language

$$\delta d^F\left(\omega_\mu\right)^a{}_b = -\partial_\mu d^F\left(\lambda\right)^a{}_b + \left[d^F\left(\lambda\right), d^F\left(\omega_\mu\right)\right]^a{}_b \tag{5.154}$$

and this relationship clearly goes through in any representation. Please note that the adjoint operators also obey

$$[\Sigma^+_{ab}, \Sigma^+_{cd}] \equiv -C_{ab,cd}{}^{ef}\Sigma^+_{ef} = -\Sigma^+_{ad}\eta_{bc} - \Sigma^+_{bc}\eta_{ad} + \Sigma^+_{ac}\eta_{bd} + \Sigma^+_{bc}\eta_{ac}$$

Define the covariant derivative as

$$\nabla^T_\mu\psi \equiv \partial_\mu\psi + \Omega_\mu\psi \tag{5.155}$$

where Ω is a matrix in the representation considered. We demand that

$$\begin{aligned}\left(\nabla^T_\mu\psi\right)' &= \partial_\mu\left(D(L)\psi\right) + \Omega'_\mu D(L)\psi \\ &= D(L)\nabla^T_\mu\psi = D(L)\left(\partial_\mu\psi + \Omega_\mu\psi\right)\end{aligned} \tag{5.156}$$

The condition for that to be true reads

$$\Omega'_\mu = D(L)\Omega_\mu D^{-1}(L) - \partial_\mu D(L)D^{-1}(L) \tag{5.157}$$

At the linearized level

$$\delta\Omega_\mu = [d(\lambda), \Omega_\mu] - \partial_\mu d(\lambda) \tag{5.158}$$

This is obeyed by

$$\Omega_\mu \equiv d\left(\omega_\mu\right) \tag{5.159}$$

- Knowing that the quantity $\bar{\psi}\psi$ is a Lorentz scalar, it has got to be true that

$$D_\mu\overline{\psi}.\psi = \partial_\mu\left(\overline{\psi}.\psi\right) \tag{5.160}$$

This fact determines uniquely

$$D_\mu\overline{\psi} \equiv \partial_\mu\overline{\psi} - \omega_\mu^{ab}\overline{\psi}\Sigma_{ab} \tag{5.161}$$

in contradistinction to

$$D_\mu\psi \equiv \partial_\mu\overline{\psi} + \omega_\mu^{ab}\Sigma_{ab}\psi \tag{5.162}$$

In fact, from

$$\delta\psi = \frac{1}{2}\left(\omega.\Sigma\right)\psi \tag{5.163}$$

we deduce that

$$\delta\psi^+ = \frac{1}{2}\psi^+\left(\omega.\Sigma^+\right) \tag{5.164}$$

and then

$$\delta\overline{\psi} = \frac{1}{2}\overline{\psi}\left(\gamma_0\omega.\Sigma^+\gamma_0\right) = -\frac{1}{2}\overline{\psi}\left(\omega.\Sigma\right) \tag{5.165}$$

because

$$\Sigma_{ab}^+ = -\gamma_0\Sigma_{ab}\gamma_0 \tag{5.166}$$

- We have just seen this to be valid for any field living in the LIF that transforms with a representation of the Lorentz group. But any field can be so represented. For example, a vector field, V^μ is projected on the LIF by a FREFO as $V^a \equiv e^a_\mu V^\mu$. We want that its Lorentz covariant derivative is also the projection of Einstein's covariant derivative, that is

$$\nabla^L_\mu\left(V^a\right) \equiv \partial_\mu V^a + \omega^a{}_{b\mu}V^b = e^a_\rho\left(\nabla^E_\mu V\right)^\rho \equiv e^a_\rho\left(\partial_\mu V^\rho + \Gamma^\rho_{\mu\sigma}V^\sigma\right) \tag{5.167}$$

This physical requirement determines the relationship between Lorentz and Einstein connections; otherwise, those two connections could be completely independent.

$$\omega^a{}_{b\sigma} = e^a_\lambda\Gamma^\lambda_{\mu\sigma}e^\mu_b - e^\rho_b\partial_\sigma e^a_\rho \tag{5.168}$$

Using the known formula for the Christoffel's symbols, this leads after a few manipulations, to

$$\omega_{ab|c} \equiv \omega_{ab\mu} e^{\mu}_{c} = \frac{1}{2}\Big\{ e_{b\mu}\left(\partial_a e^{\mu}_{c} - \partial_c e^{\mu}_{a}\right) + e_{c\sigma}\left(\partial_a e^{\sigma}_{b} - \partial_b e^{\sigma}_{a}\right) + e_{a\sigma}\left(\partial_c e^{\sigma}_{b} - \partial_b e^{\sigma}_{c}\right) \Big\} =$$

$$= \frac{1}{2}\Big\{ e_{b\mu} C_{ac}{}^{d} e^{\mu}_{d} + e_{c\sigma} C_{ab}{}^{d} e^{\sigma}_{d} + e_{a\sigma} C_{cb}{}^{d} e^{\sigma}_{d} \Big\} = \frac{1}{2}\Big\{ C_{ac|b} + C_{ab|c} + C_{cb|a} \Big\} \tag{5.169}$$

Please note that the structure constants are not totally antisymmetric in general. Nevertheless

$$\omega_{ab|c} = -\omega_{ba|c} \tag{5.170}$$

- It is a fact (confer [25]) that the *torsion* can be defined through the *connection* ω^{a}_{b} by

$$de^{a} + \omega^{a}_{b} \wedge e^{b} \equiv T^{a} \equiv \frac{1}{2} T^{a}_{bc} e^{b} \wedge e^{c}$$

In geberal we can define the *non-metricity* through

$$\nabla_a \eta_{bc} \equiv -Q_{bca} \tag{5.171}$$

Demanding that the tangent metric is covariantly constant we learn that

$$\nabla_a \eta_{bc} \equiv e^{\mu}_{a} \nabla^{L} \eta_{bc} = 0 = e^{\mu}_{a}\left(-\omega^{d}_{\mu b}\eta_{dc} - \omega^{d}_{\mu c}\eta_{db} \equiv -\omega_{\mu|bc} - \omega_{\mu|cb}\right) \tag{5.172}$$

When the torsion vanishes, and in tensor form

$$\partial_\rho e^{a}_{\sigma} - \partial_\sigma e^{a}_{\rho} + \omega^{a}{}_{b\rho} e^{b}_{\sigma} - \omega^{a}{}_{b\sigma} e^{b}_{\rho} = 0 \tag{5.173}$$

it follows that

$$\omega_{ad|c} - \omega_{ac|d} = \left(\partial_\rho e_{a\sigma} - \partial_\sigma e_{a\rho}\right) e^{\sigma}_{d} e^{\rho}_{c} \equiv \underline{e}_a \left(\partial_d \vec{e}_c - \partial_c \vec{e}_d\right) =$$
$$= \underline{e}_a\, C_{dc}{}^{u} \vec{e}_u = C_{dc|a} \tag{5.174}$$

where we have used the fact that

$$\vec{e}_b \partial_c \underline{e}_a = -\underline{e}_a \partial_c \vec{e}_b \tag{5.175}$$

This means that the torsion-free condition completely determines the antisymmetric part of the connection. One often is interested in the case when the connection lies in the Lie algebra of a simple group. For example, if $\omega_\mu \in \mathfrak{SO}(n))$

$$\omega_{\mu|ab} = -\omega_{\mu|ba} \tag{5.176}$$

For spheres we have

$$\omega_{a|bc} = \frac{1}{2}\left(\frac{\Omega_c}{\Omega^2}\ \delta_{ab} - \frac{\Omega_b}{\Omega^2}\ \delta_{ac}\right) \tag{5.177}$$

$$2\omega_{\mu|ab} = \frac{\Omega_b}{\Omega}\delta_{\mu a} - \frac{\Omega_a}{\Omega}\ \delta_{\mu b} = \left(\frac{\Omega_b}{\Omega}\ \delta_{a\mu} + \frac{\Omega_\mu}{\Omega}\ \delta_{ab} - \frac{\Omega_a}{\Omega}\ \delta_{b\mu}\right) - \frac{\Omega_\mu}{\Omega}\ \delta_{ab} \tag{5.178}$$

We see that this is equivalent to our physical postulate of FREFOs and FIDOS. The curvature of the connection is defined through

$$d\omega^a_b + \omega^a_c \wedge \omega^c_b \equiv R^a{}_b \equiv \frac{1}{2}R^a_{bcd}e^c \wedge e^d$$

It is asy to check that this a true Lorentz tensor; that is, under a local Lorentz transformation

$$R^a_b \rightarrow L^a{}_c R^c{}_d L_b{}^d \tag{5.179}$$

This leads immediately to Bianchi identities

$$\begin{aligned}
&dT^a = d\omega^a_b \wedge e^b - \omega^a_b \wedge de^b = (R^a_b - \omega^a_c \wedge \omega^c_b) \wedge e^b - \omega^a_b \wedge (T^b - \omega^b_c \wedge e^c) = \\
&= R^a_b \wedge e^b - \omega^a_b \wedge T^b \\
&dR^a_b = d\omega^a_c \wedge \omega^c_b - \omega^a_c \wedge d\omega^c_b = \\
&(R^a_c - \omega^a_d \wedge \omega^d_c) \wedge \omega^c_b - \omega^a_c \wedge (R^c_b - \omega^c_d \wedge \omega^d_b) = R^a_c \wedge \omega^c_b - \omega^a_c \wedge R^c_b
\end{aligned} \tag{5.180}$$

For a Levi-Civita connection the algebraic Bianchi identity in a natural basis reads

$$R^a_b \wedge e^b = 0 = \frac{1}{2}R^a_{b\mu\nu}e^b_\lambda dx^{\mu\nu\lambda} \tag{5.181}$$

In gory detail

$$R^\alpha{}_{[\lambda\mu\nu]} = 0 = R^\alpha{}_{\lambda\mu\nu} + R^\alpha{}_{\mu\nu\lambda} + R^\alpha{}_{\nu\lambda\mu} \tag{5.182}$$

Clever use of this identity allows to prove that

$$R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta} \tag{5.183}$$

Let us see it. We start with

$$\begin{aligned}
&R_{\alpha\lambda\mu\nu} + R_{\alpha\mu\nu\lambda} + R_{\alpha\nu\lambda\mu} = 0 \\
&R_{\lambda\alpha\mu\nu} + R_{\lambda\nu\alpha\mu} + R_{\lambda\mu\nu\alpha} = 0
\end{aligned} \tag{5.184}$$

Substracting

$$2R_{\alpha\lambda\mu\nu} + R_{\alpha\mu\nu\lambda} + R_{\alpha\nu\lambda\mu} - R_{\lambda\nu\alpha\mu} - R_{\lambda\mu\nu\alpha} = 0 \tag{5.185}$$

The same equation with the indices interchanged

$$(\alpha\lambda) \rightarrow (\mu\nu) \tag{5.186}$$

$$2R_{\mu\nu\alpha\lambda} + R_{\mu\alpha\lambda\nu} + R_{\mu\lambda\nu\alpha} - R_{\nu\lambda\mu\alpha} - R_{\nu\alpha\lambda\mu} = 0 \tag{5.187}$$

conveys the fact that

$$R_{\mu\nu\alpha\lambda} = R_{\alpha\lambda\mu\nu} \tag{5.188}$$

We have then a symmetric tensor R_{IJ} where each index is in the antisymmetric $[\alpha\beta]$ (that is, $D \equiv \frac{n(n-1)}{2}$ values). This yields

$$\frac{D(D+1)}{2} - \binom{n}{4} = \frac{n^2(n^2-1)}{12} \tag{5.189}$$

(we withdraw $\binom{n}{4}$ because of the algebraic Bianchi identity) independent components. Id est, 20 in n=4 dimensions. The differential identity in a natural basis reads

$$\nabla_{[\alpha} R^{\mu}{}_{\overline{\beta}\gamma\delta]} \equiv \nabla_{\alpha} R^{\mu}{}_{\beta\gamma\delta} + \nabla_{\gamma} R^{\mu}{}_{\beta\delta\alpha} + \nabla_{\delta} R^{\mu}{}_{\beta\alpha\gamma} = 0 \tag{5.190}$$

where the overline on an index means that this particular index is absent from the antisymmetrization. Now

$$\begin{aligned}\nabla_{[\alpha} R^{\mu}{}_{\overline{\beta}\gamma\delta]} &\equiv \partial_{[\alpha} R^{\mu}{}_{\overline{\beta}\gamma\delta]} + \Gamma^{\mu}_{[\alpha\overline{\sigma}} R^{\sigma}{}_{\overline{\beta}\gamma\delta]} - \Gamma^{\sigma}_{[\alpha\overline{\beta}} R^{\mu}{}_{\overline{\sigma}\gamma\delta]} - \Gamma^{\sigma}_{[\alpha\gamma} R^{\mu}{}_{\overline{\beta}\overline{s}\sigma\delta]} - \Gamma^{\sigma}_{\alpha\delta} R^{\mu}{}_{\overline{\beta}\gamma\overline{\sigma}]} = \\ &= \partial_{[\alpha} R^{\mu}{}_{\overline{\beta}\gamma\delta]} + \Gamma^{\mu}_{[\alpha\overline{\sigma}} R^{\sigma}{}_{\overline{\beta}\gamma\delta]} - \Gamma^{\sigma}_{[\alpha\overline{\beta}} R^{\mu}{}_{\overline{\sigma}\gamma\delta]}\end{aligned} \tag{5.191}$$

Using the relationship between $\omega^{a}_{b\mu}$ and $\Gamma^{\alpha}_{\beta\mu}$ derived above we are done. On the other hand

$$\partial_{\alpha} R^{\mu}{}_{\beta\gamma\delta} = \partial_{\alpha}\left(e^{\mu}_{a} e^{b}_{\beta} R^{a}{}_{b\gamma\delta}\right) = \left(\partial_{\alpha} e^{\mu}_{a}\right) e^{b}_{\beta} R^{a}{}_{b\gamma\delta} + e^{\mu}_{a}\left(\partial_{\alpha} e^{b}_{\beta}\right) R^{a}{}_{b\gamma\delta} + e^{\mu}_{a} e^{b}_{\beta} \partial_{\alpha} R^{a}{}_{b\gamma\delta} \tag{5.192}$$

It is a fact of life that

$$\begin{aligned}&\nabla_{e_a}(e_b) \equiv \Gamma^{c}_{ab} e_c \\ &T^{a}_{bc} = \Gamma^{a}_{bc} - \Gamma^{a}_{cb} - C^{a}_{bc} \\ &R^{a}_{b,cd} = E_c \Gamma^{a}_{db} - E_d \Gamma^{a}_{cb} + \Gamma^{e}_{db} \Gamma^{a}_{ce} - \Gamma^{e}_{cb} \Gamma^{a}_{de} - C^{e}_{cd} \Gamma^{a}_{eb}\end{aligned} \tag{5.193}$$

5.4 Commuting Spinors.

It is sometimes useful to take advantage of the fact that the Lorentz group and the group of unit-determinant complex two-dimensional matrices are simply related

$$SO(1,3) \sim SL(2,\mathbb{C})/\mathbb{Z}_2 \tag{5.194}$$

It works at follows. To any vector $v^a \partial_a \in T(M)$ we map the two-dimensional hermitian matrix $\tilde{v} = \tilde{v}^+$

$$v \rightarrow \tilde{v} \equiv \begin{pmatrix} v^{0\dot{0}} & v^{1\dot{0}} \\ v^{0\dot{1}} & v^{1\dot{1}} \end{pmatrix} \equiv \begin{pmatrix} v^0 + v^3 & v^1 - iv^2 \\ v^1 + iv^2 & v^0 - v^3 \end{pmatrix} \tag{5.195}$$

Clearly

$$\det\, \tilde{v} = v^2 \equiv v_0^2 - \sum v_i^2 \tag{5.196}$$

This means that the transformations that preserve the determinant of the two dimensional matrix, as well as its hermitian character are equivalent to Lorentz transformations. Those are precisely the $SL(2, \mathbb{C}$ transformations

$$M \rightarrow M\tilde{v}M^+ \tag{5.197}$$

We now introduce spinors as elements of a two dimensional complex space S which transform under the $(1/2, 0)$ representation of the group $SL(2, \mathbb{C})$ as

$$\psi^A \rightarrow M^A{}_B \psi^B \tag{5.198}$$

Elements of the dual space S^* are denoted by ξ_A so that

$$\xi(\psi) \equiv \xi_A \psi^A \tag{5.199}$$

The symplectic structure is denoted by

$$[\psi, \eta] \equiv \epsilon_{AB} \psi^A \eta^B \tag{5.200}$$

where $\epsilon_{AB} = -\epsilon_{BA}$. Spinors in the $(0, 1/2)$ are dubbed *dotted spinors* and transform with the comjugate matric

$$\bar{\chi}^{A'} \rightarrow (M^*)^{A'}{}_{B'}\, \bar{\chi}^{B'} \tag{5.201}$$

In GR it is computationally convenient to define spinors as commuting objects. This means that

$$\epsilon_{AB} \psi^A \psi^B = 0 \tag{5.202}$$

This is inconsistent with the classical $(\hbar \rightarrow 0)$ limit of quantum field theory, which yields anticommuting spinors. These spinors are best looked instead at as a tool to perform computations which are essentially bosonic in character rather than as fundamental fields in the theory.

The Levi-Civita tensor defines in formal terms a symplectic structure which in turn defines the determinant of the two-dimensional matrix through

$$\epsilon_{AB} M^A{}_C M^B{}_D = \det\, M \epsilon_{CD} \tag{5.203}$$

It also defines a natural (NW/SE) isomorphism between S and S^*

$$\xi_A = \xi^B \epsilon_{BA} \tag{5.204}$$

Owing to the fact that spin space is two-dimensional, the Jacobi identity follows

$$\epsilon_{A[B}\epsilon_{CD]} = 0 \tag{5.205}$$

It is convenient to define the dual object with a minus sign in it

$$\epsilon^{AB} = -\left(\epsilon^{-1}\right)^{AB} \tag{5.206}$$

This is so because consistency of

$$\xi^C = \epsilon^{CB}\xi_B = \epsilon^{CB}\xi^D\epsilon_{DB} \tag{5.207}$$

demands

$$\epsilon^{CB}\epsilon_{AB} = \delta^C{}_A \tag{5.208}$$

It is worth remarking that

$$\epsilon_C{}^A = \epsilon^{AB}\epsilon_{CB} = -\epsilon^{AB}\epsilon_{BC} = -\epsilon^A{}_C \tag{5.209}$$

The equivalent of a vierbein is now a *spin basis* that consists of two spinors,

$$[\circ\iota] \equiv \epsilon_{AB}\circ^A\iota^B = 1 \tag{5.210}$$

Then it is easy to check that

$$\epsilon^{AB} = \circ^A\iota^B - \iota^A\circ^B \tag{5.211}$$

Given a spin basis $(\circ,\iota)$ there is a *associated Newman-Penrose (NP) null tetrad*

$$\begin{aligned} l^\alpha &\equiv \circ^A\bar{\circ}^{A'} \\ n^\alpha &\equiv \iota^A\bar{\iota}^{A'} \\ m^a &\equiv \circ^A\bar{\iota}^{A'} \\ \bar{m}^a &\equiv \iota^A\bar{\circ}^{A'} \end{aligned} \tag{5.212}$$

It is plain that

$$l_a n^a = -m^a\bar{m}_a = 1 \tag{5.213}$$

A useful fact is that *only symmetric spinors matter*, so that for example

$$\tau_{AB} = \tau_{(AB)} + \frac{1}{2}\epsilon_{AB}\tau_C{}^C \tag{5.214}$$

The tangent matric is given by

$$g_{ABA'B'} = \epsilon_{AB}\epsilon_{A'B'} = l_\alpha n_\beta + l_\beta n_\alpha - m_\alpha\bar{m}_\beta - m_\beta\bar{m}_\alpha \tag{5.215}$$

It is a fact of life that

$$\epsilon_{AB}\epsilon_{A'B'} = \eta_{ab}\sigma^a_{AA'}\sigma^b_{BB'} \tag{5.216}$$

where the Infeld-van der Waerden symbols are given by the Pauli matrices

$$\sigma_a^{AA'} \equiv \frac{1}{\sqrt{2}}\sigma_a \tag{5.217}$$

where

$$\sigma_a = (1, \sigma^i) \tag{5.218}$$

Then

$$\sigma_{aA'A} \equiv \frac{1}{\sqrt{2}}(1, -\sigma^i) \tag{5.219}$$

because

$$\sigma_{aA'A} = \epsilon_{B'A'}\epsilon_{BA}\sigma_a^{B'B} = \left(i\sigma_2\sigma_a\left(-i\sigma_2\right)\right)^T = \sigma_2\sigma_a^T\sigma_2 = (1, -\vec{\sigma}) \tag{5.220}$$

We have

$$v_a = \sigma_{aAA'}v^{AA'} \tag{5.221}$$

In particular

$$\epsilon_{AB}\epsilon_{A'B'}\sigma_a^{AA'}\sigma_b^{BB'} = \frac{i}{2}\text{ tr }\sigma_2^T\sigma_a i\sigma_2\sigma_b^T = \frac{1}{2}\text{tr }\sigma_2\sigma_a\sigma_2\sigma_b^T = \frac{1}{2}\text{tr }\sigma_a^T\sigma_b^T = \eta_{ab} \tag{5.222}$$

using the magic of Pauli matrices

$$\sigma_i^T = -\sigma_2\sigma_i\sigma_2 \tag{5.223}$$

as well as

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij} \tag{5.224}$$

which implies

$$\text{tr }\sigma_i\sigma_j = 2\delta_{ij} \tag{5.225}$$

The Riemann tensor written in spinor language reads

$$R_{ABCDA'B'C'D'} = \epsilon_{A'B'}\epsilon_{C'D'}\left(\Psi_{ABCD} - 2\Lambda\epsilon_{A(C}\epsilon_{D)B}\right) + \epsilon_{A'B'}\epsilon_{CD}\Phi_{ABC'D'} + \epsilon_{AB}\epsilon_{CD}\left(\overline{\Psi}_{A'B'C'D'} - 2\bar{\Lambda}\epsilon_{A'(C'}\epsilon_{D')B'}\right) + \epsilon_{AB}\epsilon_{C'D'}\overline{\Phi}_{A'B'CD} \tag{5.226}$$

where the Bianchi identity implies that

$$\overline{\Lambda} = \Lambda \tag{5.227}$$

and Φ is hermitian. Besides, the traceless piece of Riemann's tensor, dubbed the Weyl tensor (to be precisely defined in a moment) reads

$$W_{\mu\nu\rho\sigma} = \Psi_{ABCD}\epsilon_{A'B'} + \overline{\Psi}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD} \tag{5.228}$$

and the traceful piece, the Ricci tensor

$$R_{\mu\nu} = R_{ABA'B'} = -2\Phi_{ABA'B'} + 6\Lambda\epsilon_{AB}\epsilon_{A'B'} = -2\Phi_{\mu\nu} + 6\Lambda g_{\mu\nu} \tag{5.229}$$

where the traceless Ricci is given by

$$-2\Phi_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu} \tag{5.230}$$

Finally

$$R = 24\Lambda \tag{5.231}$$

The tracefree part of the Ricci tensor can be decomposed as

$$\begin{aligned}
\Phi_{00} &\equiv \Phi_{ABA'B'}\, o^A\, o^B \bar{o}^{A'} \bar{o}^{B'}\\
\Phi_{01} &\equiv \Phi_{ABA'B'}\, o^A\, o^B \bar{o}^{A'} \bar{\iota}^{B'}\\
\Phi_{02} &\equiv \Phi_{ABA'B'}\, o^A\, o^B \bar{\iota}^{A'} \bar{\iota}^{B'}\\
\Phi_{10} &\equiv \Phi_{ABA'B'}\, o^A\, \iota^B \bar{o}^{A'} \bar{o}^{B'}\\
\Phi_{11} &\equiv \Phi_{ABA'B'}\, o^A\, \iota^B \bar{o}^{A'} \bar{\iota}^{B'}\\
\Phi_{12} &\equiv \Phi_{ABA'B'}\, o^A\, \iota^B \bar{\iota}^{A'} \bar{\iota}^{B'}\\
\Phi_{20} &\equiv \Phi_{ABA'B'} \iota^A \iota^B \bar{o}^{A'} \bar{o}^{B'}\\
\Phi_{21} &\equiv \Phi_{ABA'B'} \iota^A \iota^B \bar{o}^{A'} \bar{\iota}^{B'}\\
\Phi_{22} &\equiv \Phi_{ABA'B'} \iota^A \iota^B \bar{\iota}^{A'} \bar{\iota}^{B'}
\end{aligned} \tag{5.232}$$

There are 9 complex quantities such that

$$\Phi_{ab} = \overline{\Phi}_{ba} \tag{5.233}$$

which yield

$$9 = 10 - 1 \tag{5.234}$$

real quantities. Also, the ten components of the Weyl tensor can be packed into five complex scalars defined as

$$\begin{aligned}
\Psi_0 &\equiv \Psi_{ABCD}\, o^A\, o^B\, o^C\, o^D\\
\Psi_1 &\equiv \Psi_{ABCD}\, o^A\, o^B\, o^C\, \iota^D\\
\Psi_2 &\equiv \Psi_{ABCD}\, o^A\, o^B \iota^C \iota^D\\
\Psi_3 &\equiv \Psi_{ABCD}\, o^A\, \iota^B \iota^C \iota^D\\
\Psi_4 &\equiv \Psi_{ABCD} \iota^A \iota^B \iota^C \iota^D
\end{aligned} \tag{5.235}$$

5.5 Conformal invariance.

Under a Weyl rescaling

$$\tilde{g}_{\alpha\beta} \equiv e^{2\sigma} g_{\alpha\beta} \tag{5.236}$$

where $\sigma(x)$ is a function of the space-time point, the Riemann tensor transforms as

$$e^{-2\sigma}\ \tilde{R}_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} + g_{\alpha\delta}\sigma_{\beta\gamma} + g_{\beta\gamma}\sigma_{\alpha\delta} - g_{\alpha\gamma}\sigma_{\beta\delta} - \\ -g_{\beta\delta}\sigma_{\alpha\gamma} + (\nabla\sigma)^2 \left(g_{\alpha\delta}g_{\beta\gamma} - g_{\alpha\gamma}g_{\beta\delta}\right) \tag{5.237}$$

where

$$\sigma_{\alpha\beta} \equiv \nabla_\beta\nabla_\alpha\sigma - \nabla_\alpha\sigma\nabla_\beta\sigma \tag{5.238}$$

The Ricci tensor transforms as

$$\tilde{R}_{\beta\gamma} = R_{\beta\gamma} - (n-2)\sigma_{\beta\gamma} - \left(\Delta\sigma + (n-2)(\nabla\sigma)^2\right) g_{\beta\gamma} \tag{5.239}$$

The curvature scalar

$$e^{2\sigma}\tilde{R} = R - 2(n-1)\Delta\sigma - (n-2)(n-1)(\nabla\sigma)^2 \tag{5.240}$$

By eliminating $\Delta\sigma$ from the last couple of equations, and thel plugging the resulting value of $\sigma_{\alpha\beta}$ in the equation for the Riemann tensor we learn that the so called Weyl tensor

$$W_{\alpha\beta\gamma\delta} \equiv R_{\alpha\beta\gamma\delta} - \frac{1}{n-2}\left(g_{\alpha\gamma}R_{\beta\delta} - g_{\alpha\delta}R_{\beta\gamma} - g_{\beta\gamma}R_{\delta\alpha} + g_{\beta\delta}R_{\gamma\alpha}\right) + \frac{1}{(n-1)(n-2)} R\left(g_{\alpha\gamma}g_{\delta\beta} - g_{\alpha\delta}g \right. \tag{5.241}$$

is such that

$$\tilde{W}^{\alpha}{}_{\beta\gamma\delta} = W^{\alpha}{}_{\beta\gamma\delta} \tag{5.242}$$

The Weyl tensor has exactly the same symmetries of the Riemann tensor. The only possible trace would be

$$W_{\beta\gamma} \equiv W^{\lambda}{}_{\beta\lambda\gamma} = R_{\beta\gamma} - \frac{1}{n-2}\left((n-2)R_{\beta\gamma} + Rg_{\beta\gamma}\right) + \frac{1}{(n-1)(n-2)} R\left(ng_{\beta\gamma} - g_{\beta\gamma}\right) = 0 \tag{5.243}$$

and it vanishes identically. This means that the necessary (also sufficient) condition for a manifold (of dimension bigger than three) to be conformally flat is that its Weyl tensor vanishes.

In three dimensions

$$R_{\alpha\beta\gamma\delta} = g_{\alpha\gamma}R_{\beta\delta} - g_{\alpha\delta}R_{\beta\gamma} - g_{\beta\gamma}R_{\delta\alpha} + g_{\beta\delta}R_{\gamma\alpha} - \frac{R}{2} R\left(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}\right) \tag{5.244}$$

so that Weyl's tensor vanishes identically and yields no information.

It is useful to consider the Schouten tensor defined as

$$K_{\mu\nu} \equiv \frac{1}{n-2}\left(R_{\mu\nu} - \frac{1}{2(n-1)} Rg_{\mu\nu}\right) \tag{5.245}$$

which is such that

$$K = \frac{1}{2(n-1)}\ R \tag{5.246}$$

as well as

$$\nabla_\mu K^{\mu\nu} = \frac{1}{2}\nabla^\nu R \tag{5.247}$$

The Weyl tensor is simpler when expressed in terms of Schounten's

$$W_{\alpha\beta\gamma\delta} \equiv R_{\alpha\beta\gamma\delta} - (g_{\alpha\gamma}K_{\beta\delta} - g_{\alpha\delta}K_{\beta\gamma} - g_{\beta\gamma}K_{\delta\alpha} + g_{\beta\delta}K_{\gamma\alpha}) \tag{5.248}$$

The differential Bianchi identity tells us that

$$\nabla_\epsilon W_{\alpha\beta\gamma\delta}+\nabla_\delta W_{\alpha\beta\epsilon\gamma}+\nabla_\gamma W_{\alpha\beta\delta\epsilon} = g_{\alpha\gamma}C_{\beta\epsilon\delta}+g_{\alpha\delta}C_{\beta\gamma\epsilon}+g_{\beta\gamma}C_{\alpha\delta\epsilon}+g_{\beta\delta}C_{\alpha\gamma\epsilon}+g_{\beta\epsilon}C_{\alpha\gamma\delta}+g_{\alpha\epsilon}C_{\beta\delta\gamma} \tag{5.249}$$

The antisymmetrized covariant derivative of the Schounten tensor is the Cotton tensor

$$C_{\alpha\rho\sigma} \equiv 2\nabla_{[\sigma}K_{\rho]\alpha} \tag{5.250}$$

It is a fact that

$$C^\alpha{}_{\alpha\sigma} = 0 \tag{5.251}$$

as well as

$$\nabla^\lambda W_{\lambda\beta\gamma\delta} = (3-n)\, C_{\beta\gamma\delta} \tag{5.252}$$

Under a Weyl rescaling,

$$\tilde{C}_{\alpha\beta\gamma} = C_{\alpha\beta\gamma} + (n-2)\partial_\lambda \delta W^\lambda{}_{\beta\gamma\alpha} \tag{5.253}$$

Remembering that in three dimensions the Weyl tensor vanishes, we learn that the vanishing of the Cotton tensor is the necessary and sufficient condition for a three dimensional manifold to be conformally flat.

In two dimensions all manifolds are conformally flat, baceuse there

$$R_{\mu\nu\rho\sigma} = \frac{R}{n(n-1)}\,(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \tag{5.254}$$

The Bach tensor is defined out of the Weyl and Cotton tensors as

$$B_{\alpha\beta} \equiv \nabla^\lambda C_{\lambda\alpha\beta} + K^{\lambda\rho}W_{\lambda\alpha\beta\rho} \tag{5.255}$$

It is traceless (because both Cotton and Weyl are so) and conformally invariant as well in n=4.

It is then a fact of life that the scalar

$$\sqrt{|g|}W_{\alpha\beta\gamma\delta}W^{\alpha\beta\gamma\delta} = \sqrt{|g|}R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - \frac{4}{n-2}R_{\alpha\beta}R^{\alpha\beta} + \frac{2}{(n-1)(n-2)}R^2 \equiv \sqrt{|g|}W_4 \tag{5.256}$$

is conformally invariant in four and only in four dimensions. The variational derivative of the action constructed out of this is proportional to Bach's tensor

$$\frac{\delta}{\delta g^{\alpha\beta}}\int W_4 \sim B_{\alpha\beta} \tag{5.257}$$

There is a generalization of this to arbitrary dimension, namely Branson's Q-curvature

$$\int d(vol)Q \tag{5.258}$$

The Q-curvature itself is not pointwise conformal invariant, but its integral over a compact manifold is conformal invariant.

On the other hand, the Euler characteristic in four dimensions is given by

$$\chi(M) = \frac{1}{32\pi^2}\int d^4x\sqrt{|g|}\ E_4 \tag{5.259}$$

where the Euler density reads

$$E_4 \equiv R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2 \tag{5.260}$$

This means that whenever a term Riemann squared appears in the lagrangian, it can be traded out for Ricci squared and curvature squared, as follows

$$\int d(vol)R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = 32\pi^2\chi(M) + \int d(vol)\left(4R_{\mu\nu}R^{\mu\nu} - R^2\right) \tag{5.261}$$

In four dimensions, the Q-curvature is

$$6Q = -\Delta R + R^2 - 3R_{\mu\nu}R^{\mu\nu} \tag{5.262}$$

In general, there is a formula by Graham and Juhl, that states that in even dimension n,

$$2nC_{\frac{n}{2}}Q = nv^{(n)} + \sum_{k=1}^{\frac{n}{2}-1}(n-2k)p^*_{2k}v^{(n-2k)} \tag{5.263}$$

where

$$C_{\frac{n}{2}} \equiv \frac{(-1)^{\frac{n}{2}}}{2^n\left(\frac{n}{2}\right)!\left(\frac{n}{2}-1\right)!} \tag{5.264}$$

The constructs $v^{(2j)}$ are the coefficients appearing in the asymptotic expansion pf the volume form of a Poincaré metric for the metric g. For example

$$\begin{aligned}
v^{(2)} &= -\frac{1}{2}P\\
v^{(4)} &= \frac{1}{8}\left(P^2 - |P|^2\right)\\
v^{(6)} &= \frac{1}{48}\left(-\frac{2}{n-4}P^{\mu\nu}P_{\mu\nu} + 3P|P|^2 - P^3 - 2P^{\mu\nu}P^\lambda_\mu P_{\lambda\nu}\right)
\end{aligned} \tag{5.265}$$

The differential operators p_{2k} are those that appear in the xpansion of a harmonic function for the Poincaré metric; for example

$$\begin{aligned}
-2(n-2)p_2 &= \Delta\\
8(n-2)(n-4)p_4 &= \Delta^2 + 2P\Delta + 2(n-2)P^{\mu\nu}\nabla_\mu\nabla_\nu + (n-2)\nabla^\mu P\nabla_\mu
\end{aligned} \tag{5.266}$$

Details can be found in the Graham-Juhl paper just cited.

Its integral is a linear combination of the integral of W_4 and E_4.

There is a generalization of Bach's tensor for arbitrary dimension, namely the Fefferman-Graham obstruction tensor $\mathcal{O}_{\mu\nu}$ is tracefree and symmetric, conformally invariant and vanishes for conformally Einstein metrics. The EM derived from this action are

$$\frac{\delta}{\delta g^{\mu\nu}}\int Qd(vol) = (-1)^{\frac{n}{2}}\frac{n-2}{2}\mathcal{O}_{\mu\nu} \tag{5.267}$$

Let us finally mention that spacetimes can be classified (Petrov) by the eigenvectors of the Weyl tensor. The eigenvectors define some null vectors, the *principal null directions* (PND). This is best treated using commuting spinors [22]. Actually

$$C_{\mu\nu\rho\sigma} = \Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \bar{\psi}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD} \tag{5.268}$$

where

$$\Psi_{ABCD} = \alpha_{(A}\beta_B\gamma_C\delta_{D)} \tag{5.269}$$

Grosso modo, the Petrov types are as follows

- **Type I** Four simple PND, $(1,1,1,1)$ THis is the algebraically general case.
- **Type II** One double PND and two simple PND, $(2,1,1)$.
- **Type D** Two double PND, $(2,2)$.
- **Type III**One triple PND and one simple PND, $(3,1)$.
- **Type N** One cuadruple PND,(4).
- **Type 0** Weyl=0. These spaces are conformally flat

5.6 Timelike congruences.

A *timelike congruence* is a field of (normalized) timelike vectors. In fancy language, a chapter of the tangent bundle. It can represent a family of observers defined in every point of spacetime.

$$u^2 \equiv g_{\mu\nu}(x)u^\mu(x)u^\nu(x) = 1 \tag{5.270}$$

This implies

$$u^\mu\nabla_\rho u_\mu = 0 \tag{5.271}$$

The projection on the space orthogonal is

$$h^\mu_\nu \equiv \delta^\mu_\nu - u^\mu u_\nu \tag{5.272}$$

By construction

$$h^m_\nu u^\nu = 0 \tag{5.273}$$

It is plain that it is indeed a projector in the mathematical sense

$$h^\mu_\nu h^\nu_\rho = h^\mu_\rho \tag{5.274}$$

A conguence is geodesic when

$$\dot{u}_\mu \equiv u^\lambda \nabla_\lambda u_\mu = 0 \tag{5.275}$$

We can assume that each geodesic is characterized by a label, say λ, and s represents the arc length on each geodesic. That is

$$u^\mu(s,\lambda) \equiv \frac{\partial x^\mu(s,\lambda)}{\partial s} \tag{5.276}$$

It is natural to define a *deviation vector* ξ^α between neighboring geodesics through

$$\xi^\mu \equiv \frac{\partial x^\mu}{\partial \lambda} \tag{5.277}$$

By definition

$$\frac{\partial u^\alpha}{\partial \lambda} = \xi^\lambda \nabla_\lambda u^\alpha = \frac{\partial \xi^\alpha}{\partial s} = u^\rho \nabla_\rho \xi^\alpha \equiv \dot{\xi}^\alpha \tag{5.278}$$

Another useful equation can be easyly proved: The scalar product $\xi.u$ is a constant of motion:

$$u^\mu \nabla_\mu (u.\xi) = u^\mu \nabla_\mu \xi^\rho . u_\rho + \xi^\rho u^\mu \nabla_\mu u_\rho = \xi^\mu \nabla_\mu u^\rho . u_\rho = \frac{1}{2} \xi^\mu \nabla_\mu \left(u^2\right) = 0 \tag{5.279}$$

Let us compute the quantity

$$\ddot{\xi}^\alpha \equiv u^\lambda \nabla_\lambda \left(u^\rho \nabla_\rho \xi^\alpha\right) = u^\lambda \nabla_\lambda \left(\xi^\rho \nabla_\rho u^\alpha\right) = u^\lambda \nabla_\lambda \xi^\rho \left(\nabla_\rho u^\alpha\right) + \left(u^\lambda \nabla_\lambda \nabla_\rho u^\alpha\right) \xi^\rho =$$
$$\xi^\mu \nabla_\mu u^\rho \nabla_\rho u^\alpha + \xi^\rho u^\lambda \left(\nabla_\rho \nabla_\lambda u^\alpha + R_{\lambda\rho}{}^\alpha{}_\beta u^\beta\right) = \xi^\mu \nabla_\mu u^\rho \nabla_\rho u^\alpha + \xi^\rho u^\lambda R_{\lambda\rho}{}^\alpha{}_\beta u^\beta + \xi^\rho \nabla_\rho \left(u^\lambda \nabla\right.$$
$$-\xi^\rho \nabla_\rho u^\lambda \nabla_\lambda u^\alpha = \xi^\rho u^\lambda R_{\lambda\rho}{}^\alpha{}_\beta u^\beta$$

The resulting Jacobi equation

$$\ddot{\xi}^\alpha = R^\alpha{}_{\beta\gamma\delta} u^\beta u^\gamma \xi^\delta \tag{5.281}$$

is known in the physics literature as the *geodesic deviation equation*, and is of fundamental importance. Its solutions are called *Jacobi fields*.

For example, this gives the difference between an homogeneous gravitational field and a central one (like the one of the Earth).

Let us now consider the tensor

$$\nabla_\beta u_\alpha \tag{5.282}$$

First of all, this tensor is purely transverse. It is also a fact that

$$\nabla_\beta u_\alpha \xi^\beta = u^\rho \nabla_\rho \xi_\alpha \tag{5.283}$$

so that physically this tensor measures the extent to which ξ^α fails to be parallel transported by the congruence u. A canonical decomposition of this tensor reads

$$\nabla_\beta u_\alpha = \omega_{\alpha\beta} + \sigma_{\alpha\beta} + \frac{1}{n-1}\,\theta h_{\alpha\beta} \tag{5.284}$$

The scalar

$$\theta \equiv \nabla_\alpha u^\alpha \tag{5.285}$$

is called the *expansion* of the congruence. The congruence is expanding if $\theta > 0$, and it is converging otherwise. The symmetric tracefree part,

$$\sigma_{\alpha\beta} \equiv \nabla_{(\beta} u_{\alpha)} - \frac{1}{n-1}\theta\; h_{\alpha\beta} \tag{5.286}$$

is called the *shear*; and the antisymmetric piece

$$\omega_{\alpha\beta} \equiv \nabla_{[\beta} u_{\alpha]} \tag{5.287}$$

the *rotation* of the congruence. It is possible to show that when the rotation vanishes, then the congruence is hypersurface orthogonal, that is, there are a couple of scalars $\psi(x)$ and $\phi(x)$ such that

$$u_\mu = \psi(x)\nabla_\mu \phi(x) \tag{5.288}$$

Actually,

$$\omega_{\alpha\beta} = \nabla_{[\alpha}\psi\nabla_{\beta]}\phi \tag{5.289}$$

But on the other hand,

$$\omega_{\alpha\beta}u^\beta = 0 = \nabla_{[\alpha}\psi\nabla_{\beta]}\phi\psi\nabla^\beta\phi \tag{5.290}$$

implies

$$\nabla_\alpha\psi \sim \nabla_\alpha\phi \tag{5.291}$$

Let us now derive Raychaudhuri's equation

$$u^\lambda\nabla_\lambda\nabla_\beta u_\alpha = u^\lambda\nabla_\beta\nabla_\lambda u_\alpha + u^\lambda R_{\lambda\beta\alpha\rho}u^\rho = \nabla_\beta\left(u^\lambda\nabla_\lambda u_\alpha\right) - \left(\nabla_\beta u^\lambda\right)\left(\nabla_\lambda u^\alpha\right) +$$
$$+u^\lambda R_{\lambda\beta\alpha\rho}u^\rho = -\nabla_\beta u_\lambda\nabla^\lambda u_\alpha + u^\lambda R_{\lambda\beta\alpha\rho}u^\rho \tag{5.292}$$

Taking the trace,

$$\dot\theta = -\frac{1}{n-1}\theta^2 - \sigma_{\alpha\beta}\sigma^{\alpha\beta} + \omega_{\alpha\beta}\omega^{\alpha\beta} - R_{\alpha\beta}u^\alpha u^\beta \tag{5.293}$$

This clearly implies the *focusing theorem* in the hypersurface orthogonal case, when the rotation of the congruence vanishes. Then

$$\dot\theta \leq 0 \tag{5.294}$$

in agreement with the atractive character of the gravitational interaction.

5.7 Normal coordinates

We shall show that there is a system of coordinates where at a givan point $P \in M$,

$$g_{\alpha\beta}|_P = \eta_{\alpha\beta}$$
$$\partial_\sigma g_{\alpha\beta}|_P = 0 \tag{5.295}$$

Geodesics are defined by the ODE

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0 \tag{5.296}$$

It is plain that

$$\frac{d^3x^\mu}{ds^3} = -\partial_\rho\Gamma^\mu_{\alpha\beta}\frac{dx^\rho}{ds}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} + \Gamma^\mu_{\alpha\beta}\Gamma^\alpha_{\delta\epsilon}\frac{dx^\delta}{ds}\frac{dx^\epsilon}{ds}\frac{dx^\beta}{ds} + \Gamma^\mu_{\alpha\beta}\Gamma^\beta_{\delta\epsilon}\frac{dx^\alpha}{ds}\frac{dx^\delta}{ds}\frac{dx^\epsilon}{ds} \equiv$$
$$\equiv \Gamma^\mu_{\rho\alpha\beta}\frac{dx^\rho}{ds}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} \tag{5.297}$$

and so on. Power series solution through the point x^μ_0 at$s = 0$ reads

$$x^\mu(s) = x^\mu_0 + \dot{x}^\mu_0 s - \frac{1}{2}\left.\Gamma^\mu_{\alpha\beta}\right|_0 \dot{x}^\alpha_0\dot{x}^\beta_0 s^2 + \ldots \tag{5.298}$$

We want now to define new coordinates $x^{\mu'}$ such that in the new system of coordinates (*normal coordinates*) the solutions are just linear functions of the arc length

$$y^\mu = \dot{x}^\mu_0 s \tag{5.299}$$

withought any higher order term. The origin of arc length $s = 0$ is taken at the point P

$$y^\mu|_{s=0} = \dot{x}^\mu_0 \tag{5.300}$$

To be specific,

$$x^\mu(s) = x^\mu_0 + y^\mu - \frac{1}{2}\left.\Gamma^\mu_{\alpha\beta}\right|_0 y^\alpha y^\beta + \ldots \tag{5.301}$$

This means that

$$\left.\Gamma^{\alpha'}_{\beta'\gamma'}\right|_0 = 0$$
$$\left.\Gamma^{\alpha'}_{(\beta'\gamma'\delta')}\right|_0 = 0$$
$$\ldots \tag{5.302}$$

Now, it is a fact of life that

$$\partial_\alpha g_{\beta\gamma} = \{\beta\alpha;\gamma\} + \{\gamma\alpha;\beta\} \tag{5.303}$$

so that by a constant linear transformation we can get

$$\gamma_{\alpha\beta}|_0 = \eta_{\alpha\beta}$$
$$\partial_\alpha g_{\beta\gamma}|_0 = 0 \tag{5.304}$$

From the general expression of Riemann's tensor in terms of derivatives of the metric

$$\begin{aligned}
&2R^\mu{}_{\nu\alpha\beta} \equiv \partial_\alpha \Gamma^\mu_{\nu\beta} - \partial_\beta \Gamma^\mu_{\nu\alpha} + \Gamma^\mu_{\sigma\alpha}\Gamma^\sigma_{\nu\beta} - \Gamma^\mu_{\sigma\beta}\Gamma^\sigma_{\nu\alpha} = \\
&\partial_\alpha g^{\mu\sigma}\left(-\partial_\sigma g_{\nu\beta} + \partial_\nu g_{\beta\sigma} + \partial_\beta g_{\nu\sigma}\right) + g^{\mu\sigma}\left(-\partial_\alpha\partial_\sigma g_{\nu\beta} + \partial_\alpha\partial_\nu g_{\beta\sigma} + \partial_\alpha\partial_\beta g_{\nu\sigma}\right) - \\
&-\partial_\beta g^{\mu\sigma}\left(-\partial_\sigma g_{\nu\alpha} + \partial_\alpha g_{\sigma\nu} + \partial_\nu g_{\sigma\alpha}\right) - g^{\mu\sigma}\left(-\partial_\beta\partial_\sigma g_{\nu\alpha} + \partial_\beta\partial_\alpha g_{\sigma\nu} + \partial_\beta\partial_\nu g_{\sigma\alpha}\right) + \\
&+g^{\mu\lambda}\left(-\partial_\lambda g_{\sigma\alpha} + \partial_\sigma g_{\lambda\alpha} + \partial_\alpha g_{\lambda\sigma}\right) g^{\sigma\delta}\left(-\partial_\delta g_{\nu\beta} + \partial_\nu g_{\delta\beta} + \partial_\beta g_{\nu\delta}\right) - \\
&g^{\mu\lambda}\left(-\partial_\lambda g_{\sigma\beta} + \partial_\sigma g_{\beta\lambda} + \partial_\beta g_{\lambda\sigma}\right) g^{\sigma\delta}\left(-\partial_\delta g_{\nu\alpha} + \partial_\nu g_{\delta\alpha} + \partial_\alpha g_{\delta\nu}\right)
\end{aligned} \tag{5.305}$$

we learn that in normal coordinates

$$2R_{\mu\alpha\nu\beta} = -\partial_\nu\partial_\mu g_{\alpha\beta} + \partial_\alpha\partial_\nu g_{\beta\mu} + \partial_\beta\partial_\mu g_{\nu\alpha} - \partial_\beta\partial_\alpha g_{\mu\nu} \tag{5.306}$$

Besides from the condition

$$\partial_{(\mu}\Gamma^\nu_{\alpha\beta)} = 0 \tag{5.307}$$

we learn that

$$-\partial_\lambda\partial_\mu g_{\alpha\beta} - \partial_\alpha\partial_\lambda g_{\beta\mu} - \partial_\beta\partial_\lambda g_{\mu\alpha} + 2\partial_\mu\partial_\alpha g_{\lambda\beta} + 2\partial_\mu\partial_\beta g_{\lambda\alpha} + 2\partial_\alpha\partial_\beta g_{\lambda\mu} = 0 \tag{5.308}$$

which can be put in the form

$$\partial_\mu\partial_\nu g_{\alpha\beta} + \partial_\alpha\partial_\nu g_{\beta\mu} + \partial_\beta\partial_\nu g_{\alpha\mu} = 2\left(\partial_\mu\partial_\alpha g_{\nu\beta} + \partial_\mu\partial_\beta g_{\nu\alpha} + \partial_\alpha\partial_\beta g_{\mu\nu}\right) \tag{5.309}$$

Swapping $[\mu\nu]$ yields

$$\partial_\mu\partial_\nu g_{\alpha\beta} + \partial_\alpha\partial_\mu g_{\beta\nu} + \partial_\beta\partial_\mu g_{\alpha\nu} = 2\left(\partial_\nu\partial_\alpha g_{\mu\beta} + \partial_\nu\partial_\beta g_{\mu\alpha} + \partial_\alpha\partial_\beta g_{\mu\nu}\right) \tag{5.310}$$

and adding the two

$$4\partial_\alpha\partial_\beta g_{\mu\nu} + \partial_\mu\partial_\alpha g_{\nu\beta} + \partial_\nu\partial_\alpha g_{\mu\beta} + \partial_\mu\partial_\beta g_{\alpha\nu} + \partial_\nu\partial_\beta g_{\alpha\mu} = 2\partial_\mu\partial_\nu g_{\alpha\beta} \tag{5.311}$$

so that

$$\left(4\partial_\alpha\partial_\beta g_{\mu\nu} + 2\partial_\mu\partial_\alpha g_{\nu\beta} + 2\partial_\nu\partial_\alpha g_{\mu\beta}\right) x^\alpha x^\beta = \left(2\partial_\mu\partial_\nu g_{\alpha\beta}\right) x^\alpha x^\beta \tag{5.312}$$

This means that

$$R_{\mu\alpha\nu\beta} x^\alpha x^\beta = -\frac{3}{2}\, \partial_\alpha\partial_\beta g_{\mu\nu} x^\alpha x^\beta \tag{5.313}$$

and the expression of the spacetime metric in normal coordinates to second order in a Taylor expansion around the origin reads

$$g_{\mu\nu} = \eta_{\mu\nu} - \frac{1}{3}\, R_{\mu\alpha\nu\beta} x^\alpha x^\beta + O(x^3) \tag{5.314}$$

Consider the geodesics through an arbitrary point, P

$$\frac{d^2x^\mu}{ds^2}+\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}=0 \tag{5.315}$$

From it, deriving once

$$\frac{d^3x^\mu}{ds^3}+\partial_\lambda\Gamma^\mu_{\alpha\beta}\frac{dx^\lambda}{ds}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}+\Gamma^\mu_{\alpha\beta}\left(-\Gamma^\alpha_{\rho\sigma}\frac{dx^\rho}{ds}\frac{dx^\sigma}{ds}\frac{dx^\beta}{ds}-\Gamma^\beta_{\rho\sigma}\frac{dx^\alpha}{ds}\frac{dx^\rho}{ds}\frac{dx^\sigma}{ds}\right)=0 \tag{5.316}$$

The equation of the geodesic through P can be written as

$$x^\mu=x^\mu_0+\xi^\mu s-\frac{1}{2}\left(\Gamma^\mu_{\alpha\beta}\right)_P\xi^\alpha\xi^\beta s^2-\frac{1}{3!}\left(\Gamma^\mu_{\alpha\beta\rho}\right)_P\xi^\alpha\xi^\beta\xi^\rho s^3+\ldots \tag{5.317}$$

Riemann normal coordinates (RNC) are defined such that the geodesic equation is the equation of a straight line

$$y^\mu=\xi^\mu s \tag{5.318}$$

They are given in an implicit way by

$$x^\mu=x^\mu_0+y^\mu-\frac{1}{2}\left(\Gamma^\mu_{\alpha\beta}\right)_P y^\alpha y^\beta-\frac{1}{3!}\left(\Gamma^\mu_{\alpha\beta\rho}\right)_P y^\alpha y^\beta y^\rho+\ldots \tag{5.319}$$

It is clear that in those coordinates

$$\begin{aligned}&\left(\frac{\partial x^\alpha}{\partial y^\beta}\right)_P=\delta^\alpha_\beta\\&\left(\frac{\partial^2x^\mu}{\partial y^\alpha y^\beta}\right)_P=\left(\Gamma^\mu_{\alpha\beta}\right)_P=0\\&\left(\frac{\partial^3x^\mu}{y^\gamma\partial y^\alpha\partial y^\beta}\right)_P=\left(\partial_{(\gamma}\Gamma^\mu_{\alpha\beta)}\right)_P=0\\&\ldots\end{aligned} \tag{5.320}$$

This implies many useful relationships. In any system

$$\begin{aligned}&\nabla_\mu g_{\alpha\beta}=0\ \Rightarrow\ \partial_\mu g_{\alpha\beta}=\Gamma^\rho_{\mu\alpha}g_{\rho\beta}+\Gamma^\rho_{\mu\beta}g_{\rho\alpha}\\&\partial_\nu\partial_\mu g_{\alpha\beta}=\partial_\nu\Gamma^\rho_{\mu\alpha}g_{\rho\beta}+\Gamma^\rho_{\mu\alpha}\left(\Gamma^\sigma_{\nu\rho}g_{\sigma\beta}+\Gamma^\sigma_{\nu\beta}g_{\sigma\rho}\right)+\\&\partial_\nu\Gamma^\rho_{\mu\beta}g_{\rho\alpha}+\Gamma^\rho_{\mu\beta}\left(\Gamma^\sigma_{\nu\beta}g_{\sigma\alpha}+\Gamma^\sigma_{\nu\alpha}g_{\rho\sigma}\right)\end{aligned}$$

So that in RNC

$$\partial_\nu\partial_\mu g_{\alpha\beta}=\partial_\nu\Gamma^\rho_{\mu\alpha}g_{\rho\beta}+\partial_\nu\Gamma^\rho_{\mu\beta}g_{\rho\alpha} \tag{5.321}$$

Using now

$$\partial_\rho\Gamma^\mu_{\nu\sigma}+\partial_\nu\Gamma^\mu_{\sigma\rho}+\partial_\sigma\Gamma^\mu_{\nu\rho}=0 \tag{5.322}$$

$$
\begin{aligned}
R^{\mu}_{\nu\rho\sigma} &= \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} = 2\partial_\rho \Gamma^\mu_{\nu\sigma} + \partial_\nu \Gamma^\mu_{\rho\sigma} \\
R^{\mu}_{\rho\nu\sigma} &= \partial_\nu \Gamma^\mu_{\rho\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} = 2\partial_\nu \Gamma^\mu_{\rho\sigma} + \partial_\rho \Gamma^\mu_{\nu\sigma} \\
R^{\mu}_{(\nu\rho)\sigma} &= 3\partial_{(\rho} \Gamma^\mu_{\nu)\sigma}
\end{aligned}
\tag{5.323}
$$

Indeed

$$
g_{\alpha\beta} = \eta_{\alpha\beta} + \frac{1}{2}\left(\partial_\mu \partial_\nu g_{\alpha\beta}\right)_P y^\mu y^\nu + \ldots = \eta_{\alpha\beta} + \frac{1}{2}\left(\frac{1}{3}R^{\beta}_{(\mu\nu)\alpha} + \frac{1}{3}R^{\alpha}_{(\mu\nu)\beta}\right) y^\mu y^\nu + \ldots
\tag{5.324}
$$

One can continue in this way

$$
\begin{aligned}
g_{\alpha\beta} &= \eta_{\alpha\beta} - \frac{1}{3}R_{\alpha\mu\beta\lambda} y^\mu y^\lambda - \frac{1}{3!}\partial_\mu R_{\alpha\gamma\beta\delta} y^\mu y^\gamma y^\delta + \\
&+ \frac{1}{5!}\left(-6\nabla_\mu \nabla_\lambda R_{\alpha\delta\beta\gamma} + \frac{16}{3} R_{\lambda\beta\mu}{}^{\rho} R_{\gamma\alpha\delta\rho}\right) y^\lambda y^\mu y^\gamma y^\delta + \ldots
\end{aligned}
\tag{5.325}
$$

5.8 Fluid form of the energy-momentum tensor

Given a family of observers with unit tangent vector u, it is poossible to write an arbitrary energy-moemntum tensor in the following way

$$
T_{\alpha\beta} \equiv (\rho + p)\, u_\alpha u_\beta - p g_{\alpha\beta} + q_\alpha u_\beta + q_\beta u_\alpha + \pi_{\alpha\beta}
\tag{5.326}
$$

where the *heat flow vector* obeys

$$
q - u = 0
\tag{5.327}
$$

and the *shear tensor* $\pi_{\alpha\beta}$

$$
\begin{aligned}
\pi_{\alpha\beta} u^\alpha &= 0 \\
\pi^\alpha_\alpha &= 0
\end{aligned}
\tag{5.328}
$$

A *perfect fluid* does not have heat conduction not shear. There is also a coserved particle numner (such as baryon number, for example), which is represented by a four current

$$
j^\mu \equiv n u^\mu
\tag{5.329}
$$

The fact that it is conserved implies

$$
\nabla_\mu j^\mu = \dot{n} + n\theta = 0
\tag{5.330}
$$

where given any functiom we represent by an overdot the derivative in the direction of the observer, that is

$$
\dot{f} \equiv u^\alpha \nabla_\alpha f
\tag{5.331}
$$

and the so called *expansion* of the congruence u is defined as

$$\theta \equiv \nabla_\alpha u^\alpha \tag{5.332}$$

The covariant conservation of the energy-momentum tensor

$$\nabla_\mu T^{\mu\nu} = (\rho + p)\,(\theta u^\nu + \dot{u}^\nu) + (\dot{\rho} + \dot{p})\,u^\nu - \nabla^\nu p = 0 \tag{5.333}$$

The tangent projection yields

$$u_\nu \nabla_\mu T^{\mu\nu} = \dot{\rho} + (\rho + p)\,\theta = 0 \tag{5.334}$$

and the normal component (remember that $u.\dot{u} = 0$)

$$(\rho + p)\,\dot{u}_\mu - h^\rho_\mu \nabla_\rho p = 0 \tag{5.335}$$

$$h_{\mu\nu} \equiv g_{\mu\nu} - u_\mu u_\nu \tag{5.336}$$

5.9 Null congruences.

The Newman-Penrose (NP) null tetrad consists in four null vectors, of which two real, l and n, and one complex, m.

$$\begin{aligned} &l^2 = n^2 = m^2 = \bar{m}^2 = 0 \\ &l.m = l.\bar{m} = n.m = n.\bar{m} = 0 \\ &l.n = m\bar{m} = 1 \end{aligned} \tag{5.337}$$

It has proved itself useful in many circumstances. For this, and many other, reasons, it is interesting to study also the null case, where the tangent vector $l^2 = 0$. We shall also define a deviation vector ξ in an analogous way as in the timelike situation, that is

$$l.\xi = [\xi, l] = 0 \tag{5.338}$$

To define what *transverse* means now, we need another null vector

$$n^2 = 0 \tag{5.339}$$

normalized in such a way that

$$ln = 1 \tag{5.340}$$

Then we define the transverse projector

$$h_{\mu\nu} \equiv g_{\mu\nu} - l_\mu n_\nu - \nu_\mu l_\nu \tag{5.341}$$

which obeys

$$h^{\lambda}_{\mu}h^{\nu}_{\lambda} = h^{\nu}_{\mu}$$
$$h_{\mu\nu}n^{\nu} = h_{\mu\nu}l^{\nu} = 0$$
$$h^{\mu}_{\mu} = n-2 \tag{5.342}$$

We can write as in the timelike situation the transverse part of the derivative of the tangent vector

$$h_{\alpha\beta}\nabla_{\mu}l^{\beta} = \nabla_{\mu}l^{\beta} - l_{\alpha}n_{\beta}\nabla_{\mu}l^{\beta} \tag{5.343}$$

The deviation vector is not neccessarily transverse; actually,

$$\xi^{\mu}_{T} \equiv h^{\mu}_{\lambda}\xi^{\lambda} = \xi^{\mu} - \xi.nl^{\mu} \tag{5.344}$$

Let us compute its variation in the direction of the flow

$$\left(l^{\lambda}\nabla_{\lambda}\xi^{\mu}_{T}\right)_{T} = h^{\alpha}_{\lambda}l^{\mu}\nabla_{\mu}\left(h^{\lambda}_{\sigma}\xi^{\sigma}\right) = h^{\alpha}_{\lambda}h^{\lambda}_{\sigma}l^{\mu}\nabla_{\mu}\xi^{\sigma} + h^{\alpha}_{\lambda}l^{\mu}\xi^{\sigma}\nabla_{\mu}h^{\lambda}_{\sigma} =$$
$$h^{\alpha}_{\sigma}\xi^{\mu}\nabla_{\mu}l^{\sigma} - h^{\alpha}_{\lambda}l^{\mu}\xi^{\sigma}n_{\sigma}\nabla_{\mu}l^{\lambda} = h^{\alpha}_{\sigma}\left(\xi^{\mu} - l^{\mu}\left(\xi - n\right)\right)\nabla_{\mu}l^{\lambda} =$$
$$h^{\alpha}_{\sigma}\xi^{\mu}_{T}\nabla_{\mu}l^{\sigma} = \xi^{\mu}_{T}h^{\alpha}_{\sigma}h^{\rho}_{\mu}\nabla_{\rho}l^{\sigma} = \xi^{\mu}_{T}\left(\nabla_{\mu}l^{\alpha}\right)_{T} \tag{5.345}$$

Let us write

$$(\nabla_{\beta}l_{\alpha}) \equiv \sigma_{\alpha\beta} + \omega_{\alpha\beta} + \frac{1}{n-2}\theta h_{\alpha\beta} \tag{5.346}$$

It is easy to check that

$$\theta = \nabla_{\lambda}l^{\lambda} \tag{5.347}$$

Frobenius' theorem is still valid for null congruences. There is a small subtlety though. When the rotation vanishes, then there is a family of surfaces

$$f(x) = C \tag{5.348}$$

such that

$$l_{\alpha} \sim \partial_{\alpha}f \tag{5.349}$$

Then the vector l is at the same time parallel and orthogonal to the surface; so that the geodesics lie in fact on the surface, and are called the *null generators* of it. On the other hand Raychaudhiri's equation is still valid, with the only change of

$$-\frac{1}{n-1}\theta^{2} \rightarrow -\frac{1}{n-2}\theta^{2} \tag{5.350}$$

The NP tetrad is a natural by-product of a spin basis, $(\circ, \iota)$, in the sense that given a spin basis, the NP tratrad is naturally defined, namely

$$l^{\mu} \equiv \circ^{A}\bar{\circ}^{A'}$$
$$n^{\mu} \equiv \iota^{A}\bar{\iota}^{A'}$$
$$m^{\mu} \equiv \circ^{A}\bar{\iota}^{A'}$$
$$\bar{m}^{\mu} \equiv \iota^{A}\bar{\circ}^{A'} \tag{5.351}$$

The directional derivatives along the NP vectors are conventionally denoted by

$$
\begin{aligned}
D &\equiv l^{\mu}\nabla_{\mu} \\
\Delta &\equiv n^{\mu}\nabla_{\mu} \\
\delta &\equiv m^{\mu}\nabla_{\mu} \\
\bar{\delta} &\equiv \overline{m}^{\mu}\nabla_{\mu}
\end{aligned}
\tag{5.352}
$$

There is a sum rule

$$
\nabla_{\mu} = n_{\mu}D + l_{\mu}\Delta - \overline{m}_{\mu}\delta - m_{\mu}\bar{\delta} \tag{5.353}
$$

The connection is given in terms of the 12 complex rotation coefficients

$$
\begin{aligned}
&\kappa \equiv \circ^{A} D \circ_{A} = m^{\alpha} D l_{\alpha} \\
&\epsilon \equiv \circ^{A} D \iota_{A} = \frac{1}{2}\left(n^{\lambda} D l_{\lambda} - \overline{m}^{\lambda} D m_{\lambda}\right) \\
&\pi \equiv \iota^{A} D \iota_{A} = -\overline{m}^{\lambda} D n_{\lambda} \\
&\tau \equiv \circ^{A} \Delta \circ_{A} \\
&\gamma \equiv \circ^{A} \Delta \iota_{A} \\
&\nu \equiv \iota^{A} \Delta \iota_{A} \\
&\sigma \equiv \circ^{A} \delta \circ_{A} \\
&\beta \equiv \circ^{A} \delta \iota_{A} \\
&\mu \equiv \iota^{A} \delta \iota_{A} \\
&\rho \equiv \circ^{A} \bar{\delta} \circ_{A} \\
&\alpha \equiv \circ^{A} \bar{\delta} \iota_{A} \\
&\lambda \equiv \iota^{A} \bar{\delta} \iota_{A}
\end{aligned}
\tag{5.354}
$$

Einstein's equations simplify enormously in a NP tetrad is many cases.

6

The gravitational action principle.

Under a diffeomorphism connected withg the identity, $\xi \in \text{Diff}_0(M)$, which in local coordinate chart reads

$$x^\mu \rightarrow x^{\alpha'} = x^\alpha - \xi^\alpha(x) \tag{6.1}$$

the variation of the metric is

$$\delta_\xi g_{\alpha\beta} = \nabla_\alpha \xi_\beta + \nabla_\beta \xi_a = \pounds(\xi) g_{\alpha\beta} = \xi^\lambda \partial_\lambda g_{\alpha\beta} + \partial_\alpha \xi^\lambda g_{\lambda\beta} + \partial_\beta \xi^\lambda g_{\alpha\lambda} \tag{6.2}$$

the variation of the determinant is

$$g^{-1}\delta_\xi g = g^{\alpha\beta}\delta_\xi g_{\alpha\beta} = 2\nabla_\alpha \xi^\alpha = 2\frac{1}{\sqrt{|g|}}\partial_\mu \left(\sqrt{|g|}\xi^\mu\right) \tag{6.3}$$

so that

$$\delta_\xi \sqrt{|g|} = \partial_\mu \left(\sqrt{|g|}\xi^\mu\right) \tag{6.4}$$

Now any scalar function, no matter how complicated (it will include all fields and derivatives) transforms as

$$\delta_\xi \Phi = \xi^\alpha \partial_\alpha \Phi \tag{6.5}$$

And magically

$$\delta_\xi \left(\sqrt{|g|}\Phi\right) = \partial_\mu \left(\sqrt{|g|}\xi^\mu\right)\Phi + \sqrt{|g|}\xi^\alpha \partial_\alpha \Phi = \partial_\mu \left(\sqrt{|g|}\xi^\mu \Phi\right) \tag{6.6}$$

so that the product

$$\sqrt{|g|}\Phi \tag{6.7}$$

transforms under a diffeomorphism into a total derivative, so that its integral is invariant provided appropriate boundary conditions are imposed.

It is also the case that covariant derivatives can be integrated by parts.

$$\int \sqrt{|g|}\ d^n x\ \Lambda \nabla_\mu \Sigma^\mu = \int \sqrt{|g|}\ d^n x\ \Lambda \frac{1}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|} \Sigma^\mu \right) =$$
$$\int d^n x\ \left(\partial_\mu \left(\Lambda \sqrt{|g|}\ \Sigma^\mu \right) - \sqrt{|g|} \partial_\mu \Lambda\ \Sigma^\mu \right) = -\int \sqrt{|g|}\ d^n x\ \nabla_\mu \Lambda\ \Sigma^\mu \quad (6.8)$$

provided Λ and Σ^μ decay fast enough at infinity.

6.1 The Einstein-Hilbert lagrangian.

$$S = -\frac{c^3}{16\pi G} \int_V d^n x \sqrt{|g|}\ (R + 2\lambda) + S_{\text{matter}} + S_{\partial V} \quad (6.9)$$

It is customary to write

$$\kappa^2 \equiv 8\pi G \quad (6.10)$$

The negative sign in front of the Einstein-Hilbert term is determined by the sign of the matter action. To compute the variations it pays to be careful

$$\delta R^\mu_{\nu\alpha\beta} = \delta \left(\partial_\alpha \Gamma^\mu_{\nu\beta} - \partial_\beta \Gamma^\mu_{\nu\alpha} + \Gamma^\mu_{\sigma\alpha} \Gamma^\sigma_{\nu\beta} - \Gamma^\mu_{\sigma\beta} \Gamma^\sigma_{\nu\alpha} \right) = \partial_\alpha \delta \Gamma^\mu_{\nu\beta} - \partial_\beta \delta \Gamma^\mu_{\nu\alpha} \quad (6.11)$$

Even the variations of the connections are well-defined tensors (which the connection itself is not)

$$\delta \Gamma^\rho_{\nu\sigma} = \frac{1}{2} g^{\rho\sigma} \left(-\nabla_\lambda \delta g_{\nu\sigma} + \nabla_\nu \delta g_{\lambda\sigma} + \nabla_\sigma \delta g_{\lambda\nu} \right) \quad (6.12)$$

We also need the variation of the determinant

$$\delta g = g g^{\alpha\beta} \delta g_{\alpha\beta} = -g g_{\alpha\beta} \delta g^{\alpha\beta} \quad (6.13)$$

so that

$$\delta \sqrt{|g|} = -\frac{1}{2} \sqrt{|g|} g_{\alpha\beta} \delta g^{\alpha\beta} \quad (6.14)$$

This leads easily to

$$\delta R_{\mu\nu} = \frac{1}{2} \left(g_{\beta\mu} g_{\nu\alpha} \nabla^2 - g_{\beta\mu} \nabla_\alpha \nabla_\nu - g_{\beta\nu} \nabla_\alpha \nabla_\mu + g_{\alpha\beta} \nabla_\mu \nabla_\nu \right) \delta g^{\alpha\beta} \quad (6.15)$$

as well as

$$\delta R = \delta g^{\nu\sigma} R_{\nu\sigma} + \nabla_\rho t^\rho = -\delta g_{\nu\sigma} R^{\nu\sigma} + \nabla_\rho t^\rho \quad (6.16)$$

The total covariant derivative behaves indeed a total derivative when integrated with the diff invariant measure because

$$\int \sqrt{|g|}\ d^n x\ \nabla_\alpha t^\alpha = \int \sqrt{|g|}\ d^n x\ \frac{1}{\sqrt{|g|}} \partial_\alpha \left(\sqrt{|g|}\ t^\alpha \right) = \int\ d^n x\ \partial_\alpha \left(\sqrt{|g|}\ t^\alpha \right) \quad (6.17)$$

Altogether we have got (assuming for the time being that the integration domain is boundaryless),

$$\delta S_{EH} = -\frac{c^3}{16\pi G}\int \sqrt{|g|}\, d^n x \left(R_{\alpha\beta} - \frac{1}{2}\ (R\ + 2\lambda)\ g_{\alpha\beta}\right)\delta g^{\alpha\beta} + \int d^n x\, \frac{\delta S_{\text{matt}}}{\delta g^{\alpha\beta}}\delta g^{\alpha\beta} \tag{6.18}$$

The definition of the *metric* energy-momentum tensor is

$$T_{\alpha\beta} \equiv \frac{2c}{\sqrt{|g|}}\frac{\delta S_{\text{matt}}}{\delta g^{\alpha\beta}} \tag{6.19}$$

This leads to Einstein's equations (derived by him without using the action principle)

$$S_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2}(R + 2\lambda)\ g_{\alpha\beta} = \frac{8\pi G}{c^4}T_{\alpha\beta} \equiv \frac{\kappa^2}{c^4}T_{\alpha\beta} \tag{6.20}$$

The first member is called Einstein's tensor and it contracted covariant derivative vanishes identically by virtue of Bianchi's identity

$$\nabla_\mu S^{\mu\nu} = 0 \tag{6.21}$$

The integrability condition of Einstein's equation is then precisely the covariant conservation of the energy-momentum tensor of the matter

$$\nabla_\mu T^{\mu\nu} = 0 \tag{6.22}$$

In the above derivation wa have assumed that $\partial V = 0$. This is not the case in mosr circumstances. For example, one can integrate on the slice of spacetime defined by

$$t_i \leq t \leq t_f \tag{6.23}$$

where t is some *cosmic time*. Then the boundary includes the hypersurfaces Σ_i and Σ_f, where

$$\Sigma \equiv t = \text{constant} \tag{6.24}$$

Let us then repeat the analysis keeping the boundary terms. This is hard work.

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} + \left(g_{\mu\nu}\nabla^2 - \nabla_\mu\nabla_\nu\right)\delta g^{\mu\nu} \tag{6.25}$$

The boundary term in the Einstein-Hilbert variation then reads

$$S_{\partial V} = -\frac{c^3}{16\pi G}\int_{\partial V} d^{n-1}y\sqrt{|h|}\, n_\rho \left(g_{\mu\nu}\nabla^\rho \delta g^{\mu\nu} - \nabla_\nu \delta g^{\rho\nu}\right) \equiv -\frac{c^3}{16\pi G}\int_{\partial V} d^{n-1}y\sqrt{|h|}\, n_\rho J^\rho \tag{6.26}$$

Taking into account that

$$\delta g_{\alpha\beta}|_{\partial V} = 0 \tag{6.27}$$

as well as

$$\left(g_{\mu\nu}\Box - \nabla_\mu \nabla_\nu\right) \delta g^{\mu\nu} = \nabla_\mu \left(\nabla_\lambda \delta g^{\mu\lambda} - \nabla^\mu g_{\alpha\beta} \delta g^{\alpha\beta}\right) \tag{6.28}$$

(remember that $g_{\alpha\beta} = n_\alpha n_\beta + h_{\alpha\beta}$), it is possible to write

$$\begin{aligned} &n^\rho J_\rho = n^\rho \left(g_{\mu\nu} \nabla_\rho \delta g^{\mu\nu} - g_{\rho\sigma} \nabla_\nu \delta g^{\sigma\nu}\right) = \\ &n_\rho g_{\alpha\beta} \left(\nabla^\alpha \delta g^{\rho\beta} - \nabla^\rho \delta g^{\alpha\beta}\right) = n_\rho \left(n_\alpha n_\nu + h_{\alpha\beta}\right) \left(\nabla^\alpha \delta g^{\rho\beta} - \nabla^\rho \delta g^{\alpha\beta}\right) = \\ &= -n^\rho h^{\mu\nu} \partial_\rho \delta g_{\mu\nu} \end{aligned} \tag{6.29}$$

The product of three normals are symmetric in $(\rho\alpha)$, whereas the factor in the variations is antisymmetric with respect to thsese same indices $[\rho\alpha]$. The product then vanishes.

Besides, tangential derivatives of the variation of the metric must vanish as well: $h_{\mu\nu}\partial^\nu \delta g^{\rho\sigma} = 0$. We are then left with the stated term only.

This surface variation can be cancelled with the boundary action

$$S_{\partial V} \equiv \frac{c^3}{8\pi G} \int_{\partial V} \sqrt{|h|} d^{n-1}y \; K \tag{6.30}$$

where

$$K \equiv \nabla_\alpha n^\alpha = \left(n^\alpha n^\beta + h^{\alpha\beta}\right) \nabla_\beta n_\alpha = h^{\alpha\beta} \nabla_\beta n_\alpha \tag{6.31}$$

In fact (on the boundary $\delta h_{\alpha\beta} = 0$), and

$$\delta \Gamma^\rho_{\alpha\beta} = \frac{1}{2} g^{\rho\lambda} \left(-\nabla_\lambda \delta g_{\alpha\beta} + \nabla_\alpha \delta g_{\lambda\beta} + \nabla_\beta \delta g_{\alpha\lambda}\right) \tag{6.32}$$

$$\delta K = -h^{\alpha\beta} \delta \Gamma^\rho_{\alpha\beta} n_\rho = \frac{1}{2} h^{\alpha\beta} n^\lambda \partial_\lambda \delta g_{\alpha\beta} \tag{6.33}$$

which precisely cancels the boundary term in the variation of the bulk piece of the Einstein-Hilbert action.

A perfect fluid as the one that is usually taken to represent the coarse grained material content in cosmology has

$$T_{\alpha\beta} \equiv (\rho + p)\, u_\alpha u_\beta - p g_{\alpha\beta} \tag{6.34}$$

So that a cosmological constant corresponds to

$$\begin{aligned} \rho &= -p \\ p &= \frac{1}{\kappa^2} \lambda \end{aligned} \tag{6.35}$$

The Einstein vacuum equations reduce to the Ricci flatness condition for the corresponding space

$$R_{\mu\nu} = 0 \tag{6.36}$$

6.1.1 Schrödinger's $\Gamma - \Gamma$ noninvariant lagrangian-

The EH lagrangian has got clearly two pieces in it. Let us dub

$$S_{\alpha\beta} \equiv \Gamma^{\sigma}_{\beta\rho}\Gamma^{\rho}_{\sigma\alpha} - \Gamma^{\sigma}_{\sigma\rho}\Gamma^{\rho}_{\alpha\beta} \tag{6.37}$$

and

$$L_S \equiv \sqrt{g}\; g^{\alpha\beta} S_{\alpha\beta} \tag{6.38}$$

It is a fact that

$$L_{EH} = \sqrt{|g|}\left(g^{\alpha\beta}\partial_\beta\Gamma^{\sigma}_{\alpha\sigma} - \partial_\rho\Gamma^{\rho}_{\alpha\beta}\right) + L_S = \partial_\beta\left(\sqrt{|g|}g^{\alpha\beta}\Gamma^{\sigma}_{\alpha\sigma}\right) - \partial_\rho\left(\sqrt{|g|}\; g^{\alpha\beta}\Gamma^{\rho}_{\alpha\beta}\right) -$$
$$-\partial_\beta\left(\sqrt{|g|}g^{\alpha\beta}\right)\Gamma^{\sigma}_{\alpha\sigma} + \partial_\rho\left(\sqrt{|g|}g^{\alpha\beta}\right)\Gamma^{\rho}_{\alpha\beta} + L_S = \partial_\beta\left(\sqrt{|g|}g^{\alpha\beta}\Gamma^{\sigma}_{\alpha\sigma}\right) - \partial_\rho\left(\sqrt{|g|}\; g^{\alpha\beta}\Gamma^{\rho}_{\alpha\beta}\right) -$$
$$-L_S =$$

where we have used

$$\partial_\rho\left(\sqrt{|g|}g^{\alpha\beta}\right) = \sqrt{|g|}\left(\frac{\partial_\rho\sqrt{|g|}}{\sqrt{|g|}}g^{\alpha\beta} - \Gamma^{\alpha}_{\rho\mu}g^{\mu\beta} - \Gamma^{\beta}_{\rho\mu}g^{\alpha\mu}\right) = \sqrt{|g|}\left(\Gamma^{\sigma}_{\sigma\rho}g^{\alpha\beta} - \Gamma^{\alpha}_{\rho\mu}g^{\mu\beta} - \Gamma^{\beta}_{\rho\mu}g^{\alpha\mu}\right)$$

so that

$$-\partial_\beta\left(\sqrt{|g|}g^{\alpha\beta}\right)\Gamma^{\sigma}_{\alpha\sigma} + \partial_\rho\left(\sqrt{|g|}g^{\alpha\beta}\right)\Gamma^{\rho}_{\alpha\beta} = -2L_S \tag{6.39}$$

This shows that the Einstein-Hilbert lagrangian and Schrödinger's $\Gamma - \Gamma$ lagrangian differ by a total derivative, and thus yield the same equations of motion.

This is remarkable, because L_S is *not* diffeomorphism invariant. On the other hand, it depends only on first derivatives of the metric; this then gives a new insight insight explaining why Einstein's equations are second order.

There are many lessons to be drawn fro this fascinating lagrangian; for example, that the symmetries of the equations of motion do not have to coincide with the ones of the lagrangian; they can be enhanced, as is the case here.

6.2 The first order formalism.

$$L = -\frac{1}{2\kappa^2}R^a_b \wedge \Sigma_{ab} \tag{6.40}$$

It is very easy to derive a first order action principle provided one is willing to postulate that the torion of the connection vanishes.

$$S = \int d^n x\sqrt{|g|}g^{\mu\nu}\; R_{\mu\nu}(\Gamma) \tag{6.41}$$

Here the connection is itself a variable, so that we cannot integrate by partis covariant derivatives.

The so called *Palatini identity* tells us that

$$\delta R^{\mu}{}_{\nu\alpha\beta} = \nabla_\alpha \delta\Gamma^{\mu}_{\nu\beta} - \nabla_\beta \delta\Gamma^{\mu}_{\nu\alpha} \tag{6.42}$$

and

$$\delta R_{\mu\nu} = \nabla_\lambda \delta\Gamma^{\lambda}_{\mu\nu} - \nabla_\nu \delta\Gamma^{\lambda}_{\mu\lambda} \tag{6.43}$$

We can then write

$$\delta S_p = \int \sqrt{|g|}\; d^n x \Big\{ \Big(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \Big) \delta g^{\mu\nu} + \\ + g^{\mu\nu} \Big(\nabla_\lambda \delta\Gamma^{\lambda}_{\mu\nu} - \nabla_\nu \delta\Gamma^{\lambda}_{\mu\lambda} \Big) \Big\} \tag{6.44}$$

It is useful to use again the variable

$$\sqrt{|g|} g^{\mu} \;\equiv \mathfrak{g}^{\mu\nu} \tag{6.45}$$

$$\delta S_p = \int d^n x \Big\{ \Big(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \Big) \delta \mathfrak{g}^{\mu\nu} + \mathfrak{g}^{\mu\nu} \Big(\nabla_\lambda \delta\Gamma^{\lambda}_{\mu\nu} - \nabla_\nu \delta\Gamma^{\lambda}_{\mu\lambda} \Big) \Big\} = \\ \int d^n x \Big\{ \Big(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \Big) \delta \mathfrak{g}^{\mu\nu} + \nabla_\lambda \Big(\mathfrak{g}^{\mu\nu} \delta\Gamma^{\lambda}_{\mu\nu} - \mathfrak{g}^{\lambda\mu} \delta\Gamma^{\sigma}_{\mu\sigma} \Big) + \nabla_\lambda \mathfrak{g}^{\mu\nu} \delta\Gamma^{\lambda}_{\mu\nu} + \nabla_\nu \mathfrak{g}^{\mu\nu} \delta\Gamma^{\sigma}_{\mu\sigma} \Big\} = \\ \int d^n x \Big\{ \Big(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \Big) \delta \mathfrak{g}^{\mu\nu} - \nabla_\lambda \mathfrak{g}^{\mu\nu} \delta\Gamma^{\lambda}_{\mu\nu} + \nabla_\nu \mathfrak{g}^{\mu\nu} \delta\Gamma^{\sigma}_{\mu\sigma} \Big\} \tag{6.46}$$

because for a tensor density

$$\nabla_\lambda \mathfrak{t}^\lambda = \partial_\lambda \mathfrak{t}^\lambda \tag{6.47}$$

This means that

$$\nabla_\lambda \mathfrak{g}^{\mu\nu} = \delta^\lambda_\nu \nabla_\rho \mathfrak{g}^{\mu\rho} \tag{6.48}$$

which is easily seen to imply

$$\nabla_\lambda \mathfrak{g}^{\mu\nu} = 0 \tag{6.49}$$

so that the metric is covariantly constant and the connection is the Levi-Civita one.

6.3 The physical meaning of diffeomorphism invariance.

GR is in some sense a gauge theory. There are however some subtle differences.

GAUGE THEORY		GENERAL RELATIVITY
Gauge field A_μ	$\leftrightarrow$	Levi Civita connection $\Gamma^\mu_{\nu\rho}$
nothing	$\leftrightarrow$	spacetime metric
Field strength $F_{\mu\nu}$	$\leftrightarrow$	Riemann Christoffel tensor $R^\mu{}_{\nu\rho\sigma}$
nothing	$\leftrightarrow$	Action linear in curvature $\sqrt{\lvert g\rvert}\ R$
Action quadratic in curvature $\int \mathrm{tr}\ F_{\mu\nu}F^{\mu\nu}$	$\leftrightarrow$	Action quadratic in curvature $\int \sqrt{\lvert g\rvert}\left(aR^2 + bR_{\mu\nu}R^{\mu\nu}\right)$

A general diffeomorphism is characterized by a vector field

$$x' = x + \xi \tag{6.50}$$

It is a fact that the variation of a geometric emtity under a diffeomorphism is another geometric entity

$$\delta V^\alpha = [V, \xi]^\alpha \equiv \pounds(\xi) V^\alpha \tag{6.51}$$

$$\delta\omega_\alpha = \pounds(\xi)\omega_\alpha \tag{6.52}$$

as well as

$$\delta g_{\mu\beta} = \pounds(\xi) g_{\alpha\beta} = \nabla_\alpha \xi_b + \nabla_\beta \xi_a = -\xi^\rho \partial_\rho g_{\alpha\beta} + \partial_\alpha \xi^\rho g_{\rho\beta} + \partial_\beta \xi^\rho g_{\rho\alpha} \tag{6.53}$$

There are particular diffeomorphisms that leave invariant the metric. Those are dubbed *isometries*. The generator of an isometry is called a Killing field. They are specific to a given metric, and obey

$$\pounds(\xi) g_{\alpha\beta} = \nabla_\alpha \xi_b + \nabla_\beta \xi_a = -\xi^\rho \partial_\rho g_{\alpha\beta} + \partial_\alpha \xi^\rho g_{\rho\beta} + \partial_\beta \xi^\rho g_{\rho\alpha} = 0 \tag{6.54}$$

Let us denote

$$H_{\mu\nu} \equiv \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0 \tag{6.55}$$

Consider

$$\begin{aligned} 0 &= \nabla_\mu H_{\nu\rho} + \nabla_\rho H_{\nu\mu} - \nabla_\nu H_{\rho\mu} = \left[\nabla_\mu, \nabla_\nu\right]\xi_\rho + \left[\nabla_\rho, \nabla_\nu\right]\xi_\mu + \left\{\nabla_\mu, \nabla_\rho\right\}\xi_\nu = \\ &= R_{\rho\nu\mu\lambda}\xi^\lambda + R_{\mu\nu\rho\lambda}\xi^\lambda + 2\nabla_\mu\nabla_\rho\xi_\nu + R_{\rho\mu\nu\lambda}\xi^\lambda \end{aligned} \tag{6.56}$$

And using the algebraic Bianchi identity,

$$\nabla_\mu \nabla_\rho \xi_\nu = R_{\rho\nu\mu\lambda}\xi^\lambda \tag{6.57}$$

The set of all isometries of a given space-time form a group. Let us show that the commutator of two isometries is indeed another isometry. We shall do that by a brute force very useful calculation.

$$\begin{aligned}
&\nabla_\alpha \left[\xi_1,\xi_2\right]_\beta + \nabla_\beta \left[\xi_1,\xi_2\right]_\alpha = \nabla_\alpha \left(\xi_1^\mu \nabla_\mu \xi_\beta^2 - \xi_2^\mu \nabla_\mu \xi_\beta^1\right) + \nabla_\beta \left(\xi_1^\lambda \nabla_\lambda \xi_\alpha^2 - \xi_2^\lambda \nabla_\lambda \xi_\alpha^1\right) = \\
&\xi_1^\mu \nabla_\alpha \nabla_\mu \xi_\beta^2 - \xi_2^\mu \nabla_\alpha \nabla_\mu \xi_\beta^1 + \xi_\mu^1 \nabla_\beta \nabla_\mu \xi_\beta^2 - \xi_2^\lambda \nabla_\beta \nabla_\lambda \xi_\alpha^1 = \\
&-\xi_1^\mu \nabla_\alpha \nabla_\beta \xi_\mu^2 + \xi_2^\lambda \nabla_\alpha \nabla_\beta \xi_\lambda^1 - \xi_1^\mu \nabla_\beta \nabla_\alpha \xi_\mu^2 + \xi_2^\lambda \nabla_\beta \nabla_\alpha \xi_\lambda^1 = \\
&-\xi_1^\mu \xi_2^\lambda R_{\alpha\lambda\beta\mu} + \xi_2^\lambda \xi_1^\mu R_{\alpha\mu\beta\lambda} - \xi_1^\mu \xi_2^\lambda R_{\beta\lambda\alpha\mu} + \xi_2^\lambda \xi_1^\mu R_{\beta\mu\alpha\lambda} = \\
&\xi_1^\mu \xi_2^\lambda \left(-R_{\alpha\lambda\beta\mu} + R_{\alpha\mu\beta\lambda} - R_{\alpha\mu\beta\lambda} + R_{\alpha\lambda\beta\mu}\right) = 0
\end{aligned} \tag{6.58}$$

This in turn implies that the said commutator is a linear combination of other Killings with constant coefficients, the structure constants of the isometry group. In general if $\xi_a, a = 1\ldots r$ represents a basis of the linear space of Killing fields of a manifold, the following is true

$$[\xi_a, \xi_b] = C^c_{ab}\xi_c \tag{6.59}$$

This isometry group is *simply transitive* if the Killing vectors are linearly independent. Otherwise the group is multiply transitive. The orbits of the group are homogeneus spaces, which have the same dimension as the group in the simply transitive case. When a given space enjoys a timelike isometry the space is *stationary*. It is useful to consider *coordinates adapted to the isometry*, in which the isometry reads

$$\xi = (1,0,0,0) \tag{6.60}$$

Naturally, then

$$\xi_0 = g_{00} \tag{6.61}$$

There is no need for $g_{0i} = 0$. When they do, we say that the spacetime is *static*. Ay any rate the equation for the isometry guarantees that time is a cyclic coordinate

$$\partial_0 g_{\alpha\beta} = 0 \tag{6.62}$$

When there is an isometry, there is a conserved current

$$J(\xi)^\mu \equiv \xi_\alpha T^{\alpha\mu} \tag{6.63}$$

It is conserved in the sense that

$$\nabla_\mu J(\xi)^\mu = 2\nabla_{(\mu}\xi_{\alpha)} T^{\alpha\mu} + \xi_\alpha \nabla_\mu T^{\alpha\mu} = 0 \tag{6.64}$$

But it is a fact of life that for vectors (and *only for vectors*)

$$\nabla_\alpha J^\alpha = \frac{1}{\sqrt{|g|}}\partial_\alpha\left(\sqrt{|g|}J^\alpha\right) \tag{6.65}$$

This means that there is a conserved charge in the usual sense by integrating the current over a codimension one hypersurface (such as Σ_t in case there is a globally defined time coordinate)

$$Q(\Sigma) \equiv \int J^\alpha d\Sigma_\alpha \tag{6.66}$$

Indeed, integrating over the cylinder capped by Σ_1 and Σ_2

$$0 = \int \nabla_\alpha J^\alpha = \int_{\Sigma_2} J^\alpha d\Sigma_\alpha - \int_{\Sigma_1} J^\alpha d\Sigma_\alpha = Q(\Sigma_2) - Q(\Sigma_1) \tag{6.67}$$

Diffeomorphism invariance does not imply any conserved charge in general. Noether's theorem however allows us to derive Bianchi identities in another way. The covariant conservation of the energy-momentum tensor is a particular instance of the above.

$$\delta S \equiv \delta \int \sqrt{|g|} d^4x \; L\left(\phi_i, g_{\mu\nu}\right) = \int \sqrt{|g|} d^4x \; \frac{\delta S}{\delta g_{\alpha\beta}} \; \delta g^{\alpha\beta} = \int \sqrt{|g|} d^4x \; \frac{\delta S}{\delta g_{\alpha\beta}} \; \left(\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha\right) =$$
$$2 \int \sqrt{|g|} d^4x \left\{ \nabla_\alpha \left(\frac{\delta S}{\delta g_{\alpha\beta}} \; \xi_\beta \right) - \xi_\alpha \nabla_\beta \frac{\delta S}{\delta g_{\alpha\beta}} \right\} \tag{6.68}$$

This means that to the extent that there are diffeomorphisms that vanish on the boundary of spacetime

$$\xi|_{\partial M} = 0 \tag{6.69}$$

we get the identity

$$\nabla_\beta \frac{\delta S}{\delta g_{\alpha\beta}} \equiv 0 \tag{6.70}$$

As advertised, this includes the contracted Bianchi identities when considering the pure gravitational piece of the action, as well as the covariant conservation of the energy-momentum tensor when considering the matter piece.

6.4 The weak field limit.

The weak field expansion corresponds to small curvatures

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \tag{6.71}$$

(so that

$$g^{\mu\nu} = \eta^{\mu\nu} - \kappa h^{\mu\nu}) \tag{6.72}$$

A simple calculation tells us that

$$R_{\mu\nu} = \kappa \left(-\frac{1}{2} \Box h_{\mu} \; + \frac{1}{2} \partial_\mu \partial_\lambda h^\lambda_\mu + \frac{1}{2} \partial_\nu \partial_\lambda h^\lambda_\mu - \frac{1}{2} \partial_\mu \partial_\nu h \right) + O(\kappa^2) \tag{6.73}$$

The Ricci flatness condition coincides with the Fierz-Pauli massless equation.

It is interesting also to consider small deviations from an arbitraruy background, $\bar{g}_{\mu\nu}$

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \kappa h_{\mu\nu} \tag{6.74}$$

Then

$$R^{\mu}{}_{\nu\alpha\beta} = \bar{R}^{\mu}{}_{\nu\alpha\beta} + \frac{1}{2}\kappa \left(\bar{\nabla}_{\alpha}\left(-\bar{\nabla}^{\mu}h_{\nu\beta} + \bar{\nabla}_{\nu}h^{\mu}_{\beta} + \bar{\nabla}_{\beta}h^{\mu}_{\nu}\right) + \bar{\nabla}_{\beta}\left(\bar{\nabla}^{\mu}h_{\nu\alpha} - \bar{\nabla}_{\nu}h^{\mu}_{\alpha} - \bar{\nabla}_{\alpha}h^{\mu}_{\nu}\right)\right) \tag{6.75}$$

The Ricci tensor reads

$$R_{\nu\beta} = \bar{R}_{\nu\beta} + \frac{1}{2}\kappa\left(-\bar{\nabla}^2 h_{\nu\beta} + \bar{\nabla}_{\mu}\bar{\nabla}_{\nu}h^{\mu}_{\beta} + \bar{\nabla}_{\mu}\bar{\nabla}_{\beta}h^{\mu}_{\nu} - \bar{\nabla}_{\beta}\bar{\nabla}_{\nu}h\right) \tag{6.76}$$

and the curvature scalar

$$R = \bar{R} + \kappa\left(\bar{\nabla}_{\mu}\bar{\nabla}_{\nu}h^{\mu\nu} - \bar{\nabla}^2 h\right) \tag{6.77}$$

The Weyl tensor reads

$$\begin{aligned}
&\frac{2}{\kappa}\left(W^{\mu}{}_{\nu\alpha\beta} - \bar{W}^{\mu}{}_{\nu\alpha\beta}\right) = \bar{\nabla}_{\alpha}\left(-\bar{\nabla}^{\mu}h_{\nu\beta} + \bar{\nabla}_{\nu}h^{\mu}_{\beta} + \bar{\nabla}_{\beta}h^{\mu}_{\nu}\right) + \bar{\nabla}_{\beta}\left(\bar{\nabla}^{\mu}h_{\nu\alpha} - \bar{\nabla}_{\nu}h^{\mu}_{\alpha} - \bar{\nabla}_{\alpha}h^{\mu}_{\nu}\right) + \\
&\frac{1}{n-2}\left(\bar{R}^{\mu}_{\alpha}h_{\nu\beta} + \bar{R}_{\nu\beta}h^{\mu}_{\alpha} - \bar{R}^{\mu}_{\beta}h_{\nu\alpha} - \bar{R}_{\nu\alpha}h^{\mu}_{\beta}\right) - \frac{1}{(n-1)(n-2)}\bar{R}\left(\bar{g}^{\mu}_{\alpha}h_{\nu\beta} - \bar{g}^{\mu}_{\beta}h_{\nu\alpha} + h^{\mu}_{\alpha}\bar{g}_{\nu\beta} - h\right. \\
&\frac{1}{n-2}\left(\bar{g}_{\nu\beta}R^{\mu(1)}_{\alpha} + R^{(1)}_{\nu\beta}\bar{g}^{\mu}_{\alpha} - R^{(1)\mu}_{\beta}\bar{g}_{\nu\alpha} - R^{(1)}_{\nu\alpha}h^{\mu}_{\beta}\right) - \\
&-\frac{1}{(n-1)(n-2)}\left(\bar{\nabla}_{\lambda}\bar{\nabla}_{\sigma}h^{\lambda\sigma} - \bar{\nabla}^2 h\right)\left(\bar{g}^{\mu}_{\alpha}\bar{g}_{\nu\beta} - \bar{g}^{\mu}_{\beta}\bar{g}_{\nu\alpha}\right)
\end{aligned}$$

The *harmonic gauge* also dubbed *de Donder* gauge corresponds to

$$\partial_{\lambda}h^{\lambda}_{\mu} = \frac{1}{2}\partial_{\mu}h \tag{6.79}$$

This means that the Ricci flat condition is simply the wave equation

$$\Box h_{\mu\nu} = 0 \tag{6.80}$$

In the static case, Einstein equations should reduce in this regime to the well-known Poisson equation

$$\Delta V_g = 4\pi G\rho \tag{6.81}$$

Actually, Einstein's equation can be written as

$$R_{\alpha\beta} = \kappa^2\left(T_{\alpha\beta} - \frac{1}{2}T g_{\alpha\beta}\right) \tag{6.82}$$

Assuming a perfect fluid form for the energy-momentum tensor, in the non-relativistic limit the pressure is negligible, so that the only nonvanishing component of the second member is coming from the energy density:

$$R^0_0 = \frac{\kappa^2}{2}\rho \tag{6.83}$$

A simple calculation reveals that in this limit

$$R^0_0 \sim \frac{1}{c^2}\Delta\Phi \tag{6.84}$$

so that Einstein's equation reduce to

$$\Delta\Phi = \frac{c^2\kappa^2}{2}\rho \tag{6.85}$$

This leads to the identification

$$\kappa^2 \equiv \frac{8\pi G}{c^2} \tag{6.86}$$

6.5 Gravitational Waves.

Dimensional arguments tell us that in order to be able to carry away energy to infinity, the radiation field should decay far from the source as

$$4\pi r^2 A^2 < \infty \tag{6.87}$$

This means

$$A \sim \frac{1}{r} \tag{6.88}$$

It should depend also on the time derivative of some moment of the charge density. In electromagnetism the first moment is the charge, which is conserved, so that its time derivatice vanishes. This means that there is no monopolar radiation. The electric or magnetic dipole are not conserved, so dipolar EM radiation is allowed.

The analogous reasoning in gravitation tells us that there should not be monopolar gravitational radiation (GR) because te energy is conserved; nor dipolar radiation, owing to conservation of linear momentum. The first nontrivial momentum must be the quadrupole.

Assume the dimensionless perturbation produced bt GR sue to a source of linear extent R to be

$$h \sim \frac{G}{c^4}\frac{1}{r}\frac{\partial^2}{\partial t^2}\left(MR^2\right) \tag{6.89}$$

Assume some sort of binary such that

$$M \equiv m_1 + m_2 \tag{6.90}$$

and the reduced mass is

$$\mu \equiv \frac{m_1 m_2}{M} \tag{6.91}$$

The orbital frequency is

$$\Omega^2 a^3 \sim M \tag{6.92}$$

Assume

$$\frac{\partial}{\partial t^2} \sim \Omega^2 \tag{6.93}$$

This means that the dimensionless variable

$$h \sim \frac{G}{c^4}\frac{1}{r}\frac{M^2}{a} \sim 10^{-22}\left(\frac{M}{2.8 M_\odot}\right)\left(\frac{0.01\text{sec}}{T}\right)^{\frac{2}{3}}\left(\frac{100\text{ Mpc}}{r}\right) \tag{6.94}$$

This is quite small owing to the prefactor. Nevertheless the flux of energy is huge:

$$F \sim h^2\omega^2 \sim \frac{c^3}{G}h^2\omega^2 \tag{6.95}$$

This yields tipically 100erg/cm^{2}sec. This contrasts with Sirius' flux which is is $\sim 10^{-4}$erg/cm^{2}sec. Unfortunately it is difficult to detect such a huge amount of energy due to the small value of Newton's constant which governs the coupling of this energy to the measuring apparatus.

We have already seen that in the linear approximation and in the harmonic gauge (HG) the gravitational perturbations in vacuum obey the wave equation, because

$$R_{\mu\nu} = -\frac{1}{2}\Box h_{\mu\nu} \tag{6.96}$$

When matter is present the second member does not vanish

$$-\frac{1}{2}\Box\left(h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu}\right) \equiv -\frac{1}{2}\Box\bar{h}_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{6.97}$$

Once in the HG,

$$\partial_\mu h^{\mu\nu} = \frac{1}{2}\partial^\nu h \tag{6.98}$$

The HG condition does not fix completely the gauge. Actually, it is easy to see that

$$\delta HG_\mu = \Box\xi_\mu \tag{6.99}$$

It is possible to use this residual gauge freedom, namely those diffeomorphisms such that

$$\Box\xi^\mu = 0 \tag{6.100}$$

to reach in a source-free region the radiation gauge (this is not really a *gauge sensu stricto* in that we have to use the EM to reach it)

$$h = h_{0\mu} = 0 \tag{6.101}$$

Let us see in detail how this comes about. We write at the initial time t=0 the conditions

$$2\partial_\mu \xi^\mu(0,\vec{x}) + h(0,\vec{x}) = 0$$
$$\partial_0 \xi_i(0,\vec{x}) + \partial_i \xi_0(0,\vec{x}) + h_{0i}(0,\vec{x}) = 0 \tag{6.102}$$

as well as their time derivative (please note that $\ddot{\xi}^\mu = \Delta \xi^\mu$ because the gauge parameters are also solutions of the wave equation)

$$2\left(\Delta\xi_0(0,\vec{x}) - \vec{\nabla}\dot{\vec{\xi}}(0,\vec{x})\right) + \dot{h}(0,\vec{x}) = 0$$
$$\Delta\xi_i(0,\vec{x}) + \partial_i \dot{\xi}_0(0,\vec{x}) + \dot{h}_{0i}(0,\vec{x}) = 0 \tag{6.103}$$

These are eight PDE in $\mathbb{R}^3$ for the eight quantities $\xi^\mu(0,\vec{x})$, $\dot{\xi}^\mu(0,\vec{x})$. This picks a gauge such that

$$h(0,\vec{x}) = h_{0i}(0,\vec{x}) = 0 \tag{6.104}$$

Since both h and h_{0i} obey the wave equation, this remains true for all time. On the other hand, the harmonic gauge

$$\partial_0 h^{00} + \partial_i h^{i0} = \frac{1}{2}\partial^0 h \tag{6.105}$$

implies now that

$$\dot{h}_{00} = 0 \tag{6.106}$$

so that the wave equation reduces to

$$\Delta h_{00} = 0 \tag{6.107}$$

so that in vacuum,

$$h_{00} = 0 \tag{6.108}$$

as well. This is then a close analogous of the *radiation gauge* in electromagnetism.

The solutions can then be written as linear combinations of

$$h_{\mu\nu} = \epsilon_{\mu\nu}\, e^{ikx} \tag{6.109}$$

where there are a priori 10 independent polarizations. The gauge conditions, however tell us that

$$k^\mu \epsilon_{\mu i} = 0 \quad (3 \quad \text{conditions})$$
$$\epsilon_{0\mu} = 0 \quad (4 \quad \text{conditions})$$
$$\epsilon^\mu_\mu = 0 \quad (1 \quad \text{condition}) \tag{6.110}$$

This means that there are only $2 = 10 - 8$ polarizations left, which is the correct amount for a massless particle according to Wigner's general analysis.

It is customary to denote the two polarizations by the names of *plus* and *cross*.

$$\epsilon_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{6.111}$$

Right and left handed circular polarizations are defined by

$$\begin{aligned} h_R &\equiv \frac{1}{\sqrt{2}}\left(h_+ + ih_\times\right) \\ h_L &\equiv \frac{1}{\sqrt{2}}\left(h_+ - ih_\times\right) \end{aligned} \tag{6.112}$$

The geodesic deviation equation teaches us that the tidal force in an adapted RS is

$$\ddot{X}^i = R^i{}_{00\lambda} X^\lambda = \frac{1}{2}\ddot{h}_{ij} X^j \tag{6.113}$$

valid in the radiation gauge. The expected magnitude of the perturbation in a GW is of the order

$$|h| \sim 10^{-17} \tag{6.114}$$

which is quite small. This is the same order of the fractional relative displacement

$$\frac{\Delta l}{l} \tag{6.115}$$

Over a distance of 1 Km this means that we need a precision of

$$10^5 \times 10^{-17}\ cm \sim 10 \text{ fermi} \tag{6.116}$$

The amazing thing is that it is claimed that this is reachable in ongoing experiments like LIGO.

The metric disturbance can in principle be computed by the retarded propagator (when matter is present we have to go back to the de Donder gauge where the EM reads

$$\Box\bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}) \tag{6.117}$$

$$\overline{h}_{\mu\nu}(x) = 4\int_{N_x^-} T_{\mu\nu}\frac{1}{|x - x'|} dS' \tag{6.118}$$

where the integral is extended to the past light cone of the point $x \in M$, denoted here N_x^-.

We are using here the fundamental retarded solution

$$G_{\text{ret}} \equiv \frac{\delta(t-r)}{4\pi\,|t-r|} \tag{6.119}$$

which obeys

$$\Box G_{ret}(x) = -\delta^{(4)}(x) \tag{6.120}$$

Let us concentrate our efforts in the purely spatial components (the other components can be obtained from the gauge condition). Let us Fourier transform in the time coordinate only

$$\bar{h}_{ij}(\omega,\vec{x}) = 4G \int T_{ij}(\omega,\vec{x}')\; e^{i\omega|\vec{x}-\vec{x}'|}\;\frac{1}{|x-x'|} dS' \tag{6.121}$$

Far away from the source, we can pull out the exponential term out of the integral and replace it by

$$\frac{e^{i\omega r}}{r} \tag{6.122}$$

The remaining integral is

$$\int T^{ij} d^3x = \int \partial_k \left(T^{kj} x^i\right) d^3x - \int \partial_k T^{kj} x^i d^3x = i\omega \int T^{0j} x^i d^3x =$$
$$\frac{i\omega}{2} \int \left(T^{0j} x^i + T^{0i} x^j\right) d^3x = \frac{i\omega}{2} \int \partial_l \left(T^{0l} x^i x^j\right) d^3x - \int \partial_l T^{0l} x^i x^j d^3x =$$
$$\frac{\omega^2}{2} \int T^{00} x^i x^j d^3x \tag{6.123}$$

where we have used

$$\partial_k T^{kj} = -\partial_0 T^{0j}$$
$$\partial_l T^{0l} = -\partial_0 T^{00} \tag{6.124}$$

This means that in this approximation

$$\bar{h}_{ij} = -\frac{2G}{3}\omega^2 \frac{e^{i\omega r}}{r}\; D_{ij}(\omega) \tag{6.125}$$

where the *quadrupole moment* is defined as

$$D_{ij} \equiv \int T^{00} \left(3x^i x^j - \delta_{ij} r^2\right) d^3x \tag{6.126}$$

We have implicitly assumed that $i \neq j$. In position space

$$\bar{h}_{ij}(x) = \frac{2G}{3r}\; \frac{d^2 D_{ij}}{dt^2}\bigg|_{\text{ret}} \tag{6.127}$$

Let us consider a simple plane wave in the x-direction. This means that all components are assumed to be functions of the variable $x - ct$. The gauge condition reads

$$\partial_\nu \bar{h}^{\mu\nu} \equiv \partial_\nu \left(h^{\mu\nu} - \frac{1}{2} h \eta^{\mu\nu} \right) = 0 \tag{6.128}$$

Using that for all these functions $\partial_1 = -\frac{1}{c}\partial_0$ we deduce

$$\partial_0 \bar{h}^0_\mu + \partial_1 \bar{h}^1_\mu = 0 = \partial_0 \left(\bar{h}^0_\mu - \frac{1}{c}\, \bar{h}^1_\mu \right) \tag{6.129}$$

In gory detail

$$\begin{aligned} &\bar{h}^{00} = \bar{h}^0_0 = \bar{h}^1_0 = \bar{h}^{10} \\ &\bar{h}^0_1 = -\bar{h}^{01} = -\bar{h}^1_1 = \bar{h}^{11} \\ &\bar{h}^{21} = \bar{h}^{20} \\ &\bar{h}^{30} = \bar{h}^{31} \end{aligned} \tag{6.130}$$

Using the residual gauge transformations $\xi_\mu(ct - x)$ one can put to zero

$$\bar{h}^{01} = \bar{h}^{02} = \bar{h}^{03} = \bar{h}^{22} + \bar{h}^{33} = 0 \tag{6.131}$$

We only have to take into account that

$$\bar{h}^{22} + \bar{h}^{33} = h_{00} - h_{11} \tag{6.132}$$

and that we have four arbitary functions, namely the four components of $\xi_\mu(x - ct)$. The only non vanishing fluctuations are

$$\begin{aligned} &\bar{h}^{23} \neq 0 \\ &\bar{h}^{22} - \bar{h}^{33} \neq 0 \end{aligned} \tag{6.133}$$

Let us now turn to estimate the energy such waves carry. The pseudo tensor energy-momentum (to be defined later in the main text) is easily shown to be

$$t^{01} = \frac{c^2}{16\pi G} \left(\dot{h}^2_{23} + \frac{1}{4} \left(\dot{h}_{22} - \dot{h}_{33} \right)^2 \right) = \frac{G}{36\pi c^6 r^2} \left(\left(\frac{\dddot{D}_{22} - \dddot{D}_{33}}{2} \right)^2 + \dddot{D}^2_{23} \right) \tag{6.134}$$

At any rate, the order of magnitude is clear, just by analogy with the corresponding formula in electromagnetism, where the energy dentity is proportional to $E^2 + B^2$. This is all we need for an order of magnitude estimate. The intensity of the radiation is then given in general by

$$\frac{dI}{d\Omega} = \frac{G}{36\pi c^5} \left(\frac{1}{4} \left(\dddot{D}_{ij} n^i n^j \right)^2 + \frac{1}{2} \dddot{D}^2_{ij} - \dddot{D}_{ij} \dddot{D}_{ik} n^j n^k \right) \tag{6.135}$$

The total rate of energy loss is then given by averaging

$$-\frac{dE}{dt} = \frac{G}{45c^5}\dddot{D}_{ij}^2 \tag{6.136}$$

The total energy flux emitted is of the order of

$$F = 3\left(\frac{f}{1\ kHz}\right)\left(\frac{h}{10^{-22}}\right)\frac{ergs}{cm^2 sec} \tag{6.137}$$

For comparison, the solar neutrino flux at the earth is of the order of

$$F_\nu \sim 10^4\ \frac{ergs}{cm^2 sec} \tag{6.138}$$

It is possible to compute (and measure) the rate of variation of the radius of the orbits of two bodies gravitationally bound owing to the loss of energy due to emission of gravitational radiation. Starting from the formula

$$-\frac{dE}{dt} = \frac{G}{45c^5}\dddot{D}_{ij}^2 \tag{6.139}$$

Assuming circular orbits, Landau and Lifshitz derive (or rather propose as an exercise)

$$-\frac{dE}{dt} = \frac{32G}{5c^5}\left(\frac{M_1 M_2}{M_1+M_2}\right)^2 R^4\Omega^6 \tag{6.140}$$

where the frequency is assumed to be

$$\Omega^2 R^3 = G\left(M_1+M_2\right) \tag{6.141}$$

Given the fact that

$$E = -G\frac{M_1 M_2}{R} \tag{6.142}$$

we easily get

$$\dot{R} = -\frac{64G^3}{5c^5 R^3}M_1 M_2\left(M_1+M_2\right) \tag{6.143}$$

and the variation in the orbital period

$$\frac{\dot{T}}{T} = -\frac{96G^3}{5c^5 R^2}M_1 M_2\left(M_1+M_2\right) \tag{6.144}$$

This has been verified experimentally through the observation by Hulse and Taylor [29] of the binary pulsar PSR B1913+16. The decrease in the orbital period owing to energy loss by gravitational radiation agrees with this formula to an amazing factor

$$0.997 \pm 0.002 \tag{6.145}$$

This result is often interpreted as indirect evidence for gravitational radiation.

6.6 Exact plane waves.

Bondi, Pirani and Robinson discovered in 1959 an exact solution with many analogies with electromagnetic plane waves. Let us write these in the form proposed by Synge.

$$F_{\mu\nu} = B(u)\left((k_\mu l_\nu - k_\nu l_\mu)\cos\theta(u) + (k_\mu m_\nu - k_\nu m_\mu)\sin\theta(u)\right) \tag{6.146}$$

where k is the null propagation vector $k^2 = 0$; l and m are constant spacelike vectors orthogonal to k

$$\begin{aligned} k^2 &= l^2 = -1 \\ k.l &= k.m = l.m = 0 \end{aligned} \tag{6.147}$$

The amplitude $B(u)$ and the polarization $\theta(u)$ are functions of the variable $u \equiv k.x$. Waves propagating in the positive x-direction are described by the explicit choice

$$\begin{aligned} k &= (1,1,0,0) \\ l &= (0,0,1,0) \\ m &= (0,0,0,1) \end{aligned} \tag{6.148}$$

This means that

$$u = t - x \tag{6.149}$$

The field strength is invariant under a 5 parameter subgroup of $ISO(1,3)$ that leaves invariant u to wit

$$\begin{pmatrix} y^0 \\ y^1 \\ y^2 \\ y^3 \end{pmatrix} = \begin{pmatrix} 1-\frac{b^2}{2} & -\frac{b^2}{2} & b_2 & b_3 \\ \frac{b^2}{2} & 1-\frac{b^2}{2} & b_2 & b_3 \\ b_1 & -b_1 & 1 & 0 \\ b_2 & -b_2 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} + \begin{pmatrix} a_0 + \frac{1}{2}ab \\ a_1 + \frac{1}{2}ab \\ a_2 \\ a_3 \end{pmatrix} \tag{6.150}$$

where $a^0 = a^1$ and $b_k \quad k = 3,4$ are the five independent parameters. $b^2 \equiv sum_k b_k b_k$. This is equivalent to $y^\mu \equiv L^\mu{}_\nu(a,b)x^\nu$, where

$$\begin{aligned} y^0 &= x^0 + a^0 + \sum_k b_k\left(x_k + \frac{1}{2}a_k + \frac{1}{2}b_k u\right) \\ y^1 &= x^1 + a^1 + \sum_k b_k\left(x_k + \frac{1}{2}a_k + \frac{1}{2}b_k u\right) \\ y^k &= x^k + a^k + b^k u \end{aligned} \tag{6.151}$$

where $a^0 = a^1$. First, we observe that this transformation leaves u invariant

$$u' = u \tag{6.152}$$

According to Wigner, it can be written as the product of a boost, with velocity

$$v \equiv \left(\frac{b^2}{2}, b_2, b_3\right) \tag{6.153}$$

and a spatial rotation,

$$R = \frac{1}{4+b^2}\begin{pmatrix} 4-b^2 & 4b_2 & 4b_3 \\ -4b_2 & 4+b_3^2-b_2^2 & -2b_2b_3 \\ -4b_3 & -2b_2b_3 & 4+b_2^2-b_3^2 \end{pmatrix} \tag{6.154}$$

The world-line of the origin,

$$\begin{aligned} X^0 &= \tau \\ x^1 &= x^2 = x^3 = 0 \end{aligned} \tag{6.155}$$

gets transformed under a null rotation into

$$\begin{aligned} y^0 &= \left(1+\frac{b^2}{2}\right)\tau \\ y^1 &= \frac{b^2}{2}\tau \\ y^k &= b^k\tau \end{aligned} \tag{6.156}$$

Let us denote the infinitesimal generators of this transformations by $T(a,b)$. Then

$$\begin{aligned} T^\mu(a^0, b=0) &= (1,1,0,0) \\ T^\mu(a^2, b=0) &= (0,0,1,0) \\ T^\mu(a^3, b=0) &= (0,0,0,1) \end{aligned} \tag{6.157}$$

and

$$T(a=0,b_2) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{6.158}$$

$$T(a=0,b_3) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix} \tag{6.159}$$

The non-vanishing commutators are

$$\begin{aligned} [T_{a_2}, T_{b_2}] &= T_{a^0} \\ [T_{a_3}, T_{b_3}] &= T_{a^0} \end{aligned} \tag{6.160}$$

We want to define plane gravitational waves in as similar a way as possible. The corresponding spacetime metric reads

$$ds^2 = e^{2\phi(u)}\left(d\tau^2 - d\xi^2\right) - u^2\left(\cosh\,(2\beta(u))\left(d\eta^2 + d\zeta^2\right) + \right.$$
$$\left. \sinh\,(2\beta(u))\,\cos\,(2\theta(u))\left(d\eta^2 - d\zeta^2\right) - 2\sinh\,(2\beta(u))\,\sin\,(2\theta(u))\,d\eta d\zeta\right)$$

where $u \equiv \tau - \xi$. The function $\beta(u)$ defines the *amplitude* of the wave and the function $\theta(u)$ the *polarization.* Let us hereinafter restrict to waves with a fixed plane of polarization, $\theta = 0$. In addition, Ricci flatness demands

$$2\phi'(u) = u(\beta'(u))^2 \tag{6.161}$$

In the vierbein

$$\begin{aligned} e_0 &= e^{-\phi}\,\frac{\partial}{\partial\tau} \\ e_1 &\equiv e^{-\phi}\,\frac{\partial}{\partial\xi} \\ e_2 &\equiv \frac{1}{u}\,e^{-\beta}\,\frac{\partial}{\partial\eta} \\ e_3 &\equiv \frac{1}{u}\,e^{\beta}\,\frac{\partial}{\partial\zeta} \end{aligned} \tag{6.162}$$

The nonvanishing components of the Riemann tensor reads

$$R_{3130} = -R_{3131} = R_{1212} = R_{1220} = \sigma \tag{6.163}$$

where

$$\sigma \equiv \frac{1}{u^2}\,e^{-2\phi(u)}\,\left(\beta'' + 2\frac{\beta'}{u} - u(\beta')^3\right) \tag{6.164}$$

Let us consider *sandwich waves* with amplitude non-vanishing for a finite range of $u_i \leq u \leq u_f$ only (not including $u = 0$). Elsewere, spacetime is flat. To be specific, in the overlapping regions where both the plane wave ansatz as well as Minkowski coordinates are allowed, we can choose, for example the flat form

$$\phi = \phi_0 \tag{6.165}$$

$$\beta = \beta_0 \tag{6.166}$$

In the filling the wave is a smooth deformation of this. Minkowski coordinates are defined by

$$\begin{aligned} \tau - \xi &= t - x \equiv u \\ \tau + \xi &= e^{-2\phi_0}\,\left(t + x - \frac{y^2 + z^2}{t - x}\right) \\ \eta &= e^{-\beta_0}\,\frac{y}{u} \\ \zeta &= e^{\beta_0}\,\frac{z}{u} \end{aligned} \tag{6.167}$$

The mathing with the flat metric when $u \to u_i^-$ involves

$$(\beta^-, \phi^-) \equiv lim_{u\to u_i^-}(\beta, \phi) \tag{6.168}$$

whereas the one at $u \to u^+$ involves

$$(\beta^+, \phi^+) \equiv lim_{u\to u_i^+}(\beta, \phi) \tag{6.169}$$

The matching can be done in a smooth manner in a finite overlapping region. The BPR solutions can be generalized [26] giving an interesting family of vacuum exact solutions (Petrov type N) is that of *plane-fronted parallel waves with parallel rays* (PP waves). Define null coordinates through

$$\begin{aligned} u &\equiv t - x_n \\ v &\equiv t + x_n \end{aligned} \tag{6.170}$$

These spaces are characterized by the existence of a parallel null vector, that is

$$\nabla_\mu Z_\nu = 0 \tag{6.171}$$

This means that Z is a Killing vector field, and also that it is a gradient. If it does not vanish anywhere, we can define a null coordinate such that

$$Z \equiv \frac{\partial}{\partial v} \tag{6.172}$$

The Killing equation can be written as

$$\pounds(Z) g_{\mu\nu} = 0 = Z^\lambda \partial_\lambda g_{\mu\nu} - \partial_\mu Z^\lambda g_{\lambda\nu} - \partial_\nu Z^\lambda g_{\lambda\mu} = \partial_v g_{\mu\nu} \tag{6.173}$$

On the other hand, the fact that

$$\nabla_{[\mu} Z_{\nu]} = 0 \tag{6.174}$$

conveys the fact that locally

$$Z_\mu \equiv g_{\mu v} = \partial_\mu u(x) \tag{6.175}$$

The metric has then the form

$$ds^2 = dudv + K(u, x_T) du^2 + 2A_a(u, x_T) dx^a du + g_{ab}(u, x_T) dx^a dx^b \tag{6.176}$$

where the transverse coordinates

$$x_T \equiv (x^a) \quad (a = 1 \ldots n-2) \tag{6.177}$$

In fact we are going to be specially interested in the case where

$$g_{ab} = \delta_{ab} \tag{6.178}$$

which are the real *plane-fronted waves with parallel rays* (often shortened to pp waves). Note that the wave fronts

$$u = C \tag{6.179}$$

are flat, then planar. The other part of the name (parallel rays) refers to the existence of a parallel null vector. Shifts of the coordinate v

$$\delta v = \Lambda(u, x_T) \tag{6.180}$$

belong to the residual gauge symmetry. Then

$$\begin{aligned} \delta K &= \frac{1}{2}\partial_a \Lambda \\ \delta A_a &= \partial_a \Lambda \end{aligned} \tag{6.181}$$

Plane waves are a particular case where $A_a = 0$. Then

$$ds^2 = dudv - A_{ab}(u) x^a x^b du^2 - \delta_{ab} dx^a dx^b \tag{6.182}$$

where

We shall see in a moment that a plane wave is flat iff $A_{ab} = 0$.

It is easy to check that the only nonvanishing component of the Riemann tensor is

$$R_{uaub} = -A_{ab} \tag{6.183}$$

The Ricci tensor, in turn, has as the only nonvanishing component

$$R_{uu} = -\delta^{ab} A_{ab} \tag{6.184}$$

in such a way that the scalar curvature vanishes

$$R = 0 \tag{6.185}$$

The manifold is Ricci flat whenever the transverse matrix A is traceless.

The nonvanishing components of the Weyl tensor are given in by

$$W_{uaub} = -\left(A_{ab} - \frac{1}{n-2}\,\delta_{ab}\ \text{tr}\ A\right) \tag{6.186}$$

which vanishes iff A_{ab} is a pure trace. As a consequence, all curvature invariants of a plane wave vanish.

The geodesic deviation equation (to be derived later) tells us that

$$\frac{d^2}{du^2}\delta x^a = A_{ab}\delta x^b \tag{6.187}$$

which is an harmonic oscillator equation. This shows in particular that the tidal forces become infinite whenever A_{ab} diverges.

One interesting aspect of PP waves is that they are generic in the sense that every space time admits a PP wave as some limit, the Pentose limit of it. The said Penrose limit is associated to a particular null geodesic congruence, $\gamma(\lambda)$, whose tangent vector is represented in adapted coordinates as

$$\dot{\gamma} = \partial_u \tag{6.188}$$

so that the geodesics themselves are parametrized by constant values of v and the traneverse coordinates x_T^k. In Penrose's coordinates

$$g_{uu} = g_{ui} = 0 \tag{6.189}$$

so that the full metric reads

$$ds^2 = -2dudv - a(u,v,x_T)dV^2 - 2b(u,v,x_T)dY^i dV - g_{ij}(u,v,x_T)dx_T^i dx_T^j \tag{6.190}$$

We can interpret the coordinates (v, x_T^i) labelling different geodesics in the congruence, whereas u is an affine parameter along each geodesic.

Penrose first instructs us to first make a boost

$$(u,v,x_T) \to (u/\lambda, \lambda u, x_T) \tag{6.191}$$

At the same time, we rescale all coordinates by

$$(u,v,x_T) \to (\lambda u, \lambda v, \lambda x_T) \tag{6.192}$$

so that altogether we have

$$(u,v,x_T) \to \left(u, \lambda^2 v, \lambda x_T\right) \tag{6.193}$$

Next we perform an overall rescaling of the metric

$$ds^2 \to \frac{1}{\lambda^2} ds^2 \tag{6.194}$$

The end result is

$$ds^2 = dudv - \lambda^2 a(u, \lambda^2 v, \lambda x_T^k)dv^2 - 2\lambda b(u, \lambda^2 v, \lambda x_T^k)dx_T^i dv - g_{ij}(u, \lambda^2 v, \lambda x_T^k)dx_T^i dx_T^j \tag{6.195}$$

Taking now the limit when $\lambda \to 0$ yields

$$ds_\gamma^2 = dudv - g_{ij}(u)dx_T^i dx_T^j \tag{6.196}$$

It is a fact that if two geodesics are related by an isometry, the Penrose limits are themselves isometric. The limit preserves Ricci flatness, conformal flatness and local symmetry.

Let us work out an example in detail. Consider the manifold with metric

$$ds^2 = R_1^2\left(dt^2 - \sin^2 t\ d\Omega_{n_1-1}^2\right) - R_2^2\left(d\theta^2 - \sin^2 \theta d\Omega_{n_2-2}^2\right) \tag{6.197}$$

We shall see later on that this represents the direct product of a n_1-dimensional constant curvature anti de Sitter space and an ordinary positive curvature n_2-dimensional sphere. This manifold is interesting because it can be an exact background of string theory and as such is one of the components of the celebrated AdS/CFT Maldacena duality. Let us introduce null coordinates

$$\begin{aligned} u &\equiv R_1\ t - R_2\theta \\ v &\equiv R_2\theta + R_1\ t \end{aligned} \tag{6.198}$$

so that

$$ds^2 = dudv - R_1^2 \sin^2 \frac{u-v}{2R_1}\ d\Omega^2_{n_1-1} - R_2^2 \sin^2 \frac{u+v}{R_1}\ d\Omega^2_{n_2-1} \tag{6.199}$$

The Penrose limit is now easily obtained in Rosen coordinates

$$ds_P^2 = dudv - \sin^2 \frac{u}{2R_1} dy_T^2 - \sin^2 \frac{u}{2R_2} dz_T^2 \tag{6.200}$$

6.7 Stationary spacetimes

We have already emphasized that the GR generalization of stationary gravitational field consist in demanding the there is a timelike Killing vector, which can then be taken along the time direction, *id est*,

$$\pounds(\xi) g_{\alpha\beta} = \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0 \tag{6.201}$$

Then in the coordinate system adapted to the Killing vector,

$$\xi = \frac{\partial}{\partial t} \tag{6.202}$$

this condition is simply

$$\frac{\partial}{\partial t} g_{\alpha\beta} = 0 \tag{6.203}$$

In general,

$$g_{0i} \neq 0 \tag{6.204}$$

Spacetime is said to be static if the Killing is hypersurface orthogonal, which is the case when

$$\xi \wedge d\xi = 0 \tag{6.205}$$

(by Frobenius' theorem).

It is fact of life that the tangent vector to a geodesic, say k, then obeys

$$k^\alpha \nabla_\alpha \left(k.\xi\right) = k^\alpha \xi^\beta \nabla_\alpha k_\beta + k^\beta k^\alpha \nabla_\alpha \xi_\beta = 0 + 0 = 0 \tag{6.206}$$

The general form of the metric of a stationary space-time is

$$ds^2 = A^2(x^2,x^3) dt^2 - B^2(x^2,x^3) \left(d\phi - \omega dt\right)^2 - C^2(x^2,x^3) \left(dx_2^2 + dx_3^2\right) \tag{6.207}$$

where the two Killing vectors are given by

$$\xi \equiv \frac{\partial}{\partial t}$$
$$\chi \equiv \frac{\partial}{\partial \phi} \tag{6.208}$$

A tetrad defining a LIF is

$$e_0 = Adt$$
$$e_1 \equiv \omega B dt - B d\phi$$
$$e_2 \equiv C dx^2$$
$$e_3 \equiv C dx^3 \tag{6.209}$$

A given observer will have a four velocity

$$u^\alpha \equiv \frac{dx^\alpha}{ds} = \gamma\,(1, \Omega, \dot{x_2}, \dot{x_3}) \tag{6.210}$$

where

$$\Omega \equiv \frac{d\phi}{dt} \equiv \dot{\phi}$$
$$\gamma \equiv \frac{dt}{ds} = \frac{1}{\sqrt{A^2 - B^2\,(\Omega - \omega)^2 - C^2 v_T^2}} \tag{6.211}$$

where

$$v_T^2 \equiv \dot{x_2}^2 + \dot{x_3}^2 \tag{6.212}$$

In the LIF defined by our tetrad the four velocity is measured as

$$u^a \equiv e^a_\mu \frac{dx^\mu}{dt} = (A, B\,(\omega - \Omega)\,, C\dot{x}_2, C\dot{x}_3) \tag{6.213}$$

An observer at rest in a LIF (which means that $\Omega = \omega$) will have angular velocity ω in a coordinate frame. An observer at rest in a coordinate frame (FIDO) which means $\Omega = 0$ will have angular velocity $-\omega$ in a LIF. This *dragging of inertial frames* is due to the angular momentum of the source (proportional to ω) and has been studied by Lense and Thirring.

6.8 Noether charges and superpotentials

There are two cases to consider essentially different.

- **Non-covariant appproach without boundary terms.** Let us review here the non-covariant approach leadinbg to pseudotensors, first pioneered by Einstein [19] himself.

The Einstein-Hilbert lagrangian reads (we suppress the overall factr $-\frac{1}{2\kappa^2} = -\frac{c^3}{16\pi G}$).

$$L = \sqrt{|g|}\Big\{ -\tfrac{1}{2} g^{\nu\beta} g^{\lambda\delta} g^{\mu\epsilon} \partial_\mu g_{\epsilon\delta} \left(-\partial_\lambda g_{\nu\beta} + \partial_\beta g_{\nu\lambda} + \partial_\nu g_{\beta\lambda}\right) +$$
$$+\tfrac{1}{2} g^{\nu\beta} g^{\mu\lambda} \left(-\partial_\mu\partial_\lambda g_{\nu\beta} + \partial_\mu\partial_\beta g_{\nu\lambda} + \partial_\mu\partial_\nu g_{\beta\lambda}\right) + \tfrac{1}{2} g^{\mu\epsilon} g^{\lambda\delta} \partial_\beta g_{\epsilon\delta} \left(-\partial_\lambda g_{\mu\nu} + \partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu}\right) +$$
$$-\tfrac{1}{2} g^{\mu\lambda} \left(-\partial_\beta\partial_\lambda g_{\mu\nu} + \partial_\beta\partial_\mu g_{\nu\lambda} + \partial_\beta\partial_\nu g_{\mu\lambda}\right) +$$
$$+\tfrac{1}{4} g^{\beta\nu} g^{\mu\lambda} \left(-\partial_\lambda g_{\mu\sigma} + \partial_\mu g_{\lambda\sigma} + \partial_\sigma g_{\mu\lambda}\right) g^{\sigma\delta} \left(-\partial_\delta g_{\beta\nu} + \partial_\nu g_{\beta\delta} + \partial_\beta g_{\nu\delta}\right) -$$
$$-\tfrac{1}{4} g^{\beta\nu} g^{\mu\lambda} \left(-\partial_\lambda g_{\beta\sigma} + \partial_\beta g_{\lambda\sigma} + \partial_\sigma g_{\beta\lambda}\right) g^{\sigma\delta} \left(-\partial_\delta g_{\mu\nu} + \partial_\nu g_{\mu\delta} + \partial_\mu g_{\nu\delta}\right)\Big\} \quad (6.2$$

It follows that

$$\frac{\partial L_{EH}}{\partial g_{\alpha\beta}} = \sqrt{|g|}\left(\frac{1}{2} g^{\alpha\beta} R - R^{\alpha\beta}\right) \tag{6.215}$$

and

$$\frac{\partial L_{EH}}{\partial\left(\partial_\mu g_{\alpha\beta}\right)} = \sqrt{|g|}\left(-\Gamma^\beta_{\rho\sigma} g^{\rho\sigma} g^{\alpha\mu} + \Gamma^\mu_{\rho\sigma} g^{\rho\alpha} g^{\beta\sigma} + \Gamma^\alpha_{\rho\sigma} g^{\rho\sigma} g^{\beta\mu} + \Gamma^\mu_{\rho\sigma} g^{\rho\sigma} g^{\alpha\beta}\right) \tag{6.216}$$

as well as

$$\frac{\partial L_{EH}}{\partial\left(\partial_\mu\partial_\nu g_{\alpha\beta}\right)} = \frac{1}{2}\sqrt{|g|}\left\{\left(g^{\mu\alpha} g^{\beta\nu} + g^{\nu\alpha} g^{\mu\beta}\right) - g^{\mu\nu} g^{\alpha\beta}\right\} \tag{6.217}$$

A general variation of the lagrangian can be written as

$$\delta L_{EH} \equiv \frac{\partial L_{EH}}{\partial g_{\alpha\beta}} \delta g_{\alpha\beta} + \frac{\partial L_{EH}}{\partial\left(\partial_\mu g_{\alpha\beta}\right)} \delta\partial_\mu g_{\alpha\beta} + \frac{\partial L_{EH}}{\partial\left(\partial_\mu\partial_\nu g_{\alpha\beta}\right)} \delta\partial_\mu\partial_\nu g_{\alpha\beta} =$$
$$= \frac{\partial L_{EH}}{\partial g_{\alpha\beta}} \delta g_{\alpha\beta} + \partial_\mu\left(\frac{\partial L_{EH}}{\partial\left(\partial_\mu g_{\alpha\beta}\right)} \delta g_{\alpha\beta}\right) - \left(\partial_\mu \frac{\partial L_{EH}}{\partial\left(\partial_\mu g_{\alpha\beta}\right)}\right) \delta g_{\alpha\beta} +$$
$$+\partial_\mu\left(\frac{\partial L_{EH}}{\partial\left(\partial_\mu\partial_\nu g_{\alpha\beta}\right)} \delta\partial_\nu \delta g_{\alpha\beta}\right) + \left(\partial_\mu\partial_\nu \frac{\partial L_{EH}}{\partial\left(\partial_\mu\partial_\nu g_{\alpha\beta}\right)}\right) \delta g_{\alpha\beta} - \partial_\nu\left(\partial_\mu \frac{\partial L_{EH}}{\partial\left(\partial_\mu\partial_\nu g_{\alpha\beta}\right)} \delta g_{\alpha\beta}\right) =$$
$$= \frac{\delta S_{EH}}{\delta g_{\alpha\beta}} \delta g_{\alpha\beta} + \partial_\mu\left(\frac{\partial L_{EH}}{\partial\left(\partial_\mu g_{\alpha\beta}\right)} \delta g_{\alpha\beta} + \frac{\partial L_{EH}}{\partial\left(\partial_\mu\partial_\nu g_{\alpha\beta}\right)} \delta\partial_\nu \delta g_{\alpha\beta} - \partial_\nu \frac{\partial L_{EH}}{\partial\left(\partial_\mu\partial_\nu g_{\alpha\beta}\right)} \delta g_{\alpha\beta}\right) \equiv$$
$$= \frac{\delta S_{EH}}{\delta g_{\alpha\beta}} \delta g_{\alpha\beta} + \partial_\mu j^\mu \tag{6.218}$$

where the canonical EM are given by

$$\frac{\delta S_{EH}}{\delta g_{\alpha\beta}} \equiv \frac{\partial L_{EH}}{\partial g_{\alpha\beta}} - \partial_\mu \frac{\partial L_{EH}}{\partial\left(\partial_\mu g_{\alpha\beta}\right)} + \partial_\mu\partial_\nu \frac{\partial L_{EH}}{\partial\left(\partial_\mu\partial_\nu g_{\alpha\beta}\right)} \tag{6.219}$$

It so happens that

$$\frac{\delta S_{EH}}{\delta g_{\alpha\beta}} = \tfrac{1}{2} g^{\alpha\beta} R - R^{\alpha\beta} - \partial_\mu\left(\sqrt{|g|}\left(-\Gamma^\beta_{\rho\sigma} g^{\rho\sigma} g^{\alpha\mu} + \Gamma^\mu_{\rho\sigma} g^{\rho\alpha} g^{\beta\sigma} + \Gamma^\alpha_{\rho\sigma} g^{\rho\sigma} g^{\beta\mu} +\right.\right.$$
$$\left.\left.+\Gamma^\mu_{\rho\sigma} g^{\rho\sigma} g^{\alpha\beta}\right)\right) + \partial_\mu\partial_\nu\left(\tfrac{1}{2}\sqrt{|g|}\left(\left(g^{\mu\alpha} g^{\beta\nu} + g^{\nu\alpha} g^{\mu\beta}\right) - g^{\mu\nu} g^{\alpha\beta}\right)\right) \tag{6.220}$$

Under a diffeomorphism

$$\begin{aligned}\delta g_{\alpha\beta} &= \xi^\mu \partial_\mu g_{a\beta} + \partial_\alpha \xi^\lambda g_{\lambda\beta} + \partial_\beta \xi^\lambda \gamma_{\alpha\lambda}\\ \delta\partial_\nu g_{\alpha\beta} &= \partial_\nu \xi^\mu \partial_\mu g_{a\beta} + \xi^\mu \partial_\nu \partial_\mu g_{a\beta} + \partial_\nu \partial_\alpha \xi^\lambda g_{\lambda\beta} + \partial_\alpha \xi^\lambda \partial_\nu g_{\lambda\beta} + \\ &\quad + \partial_\nu \partial_\beta \xi^\lambda g_{\alpha\lambda} + \partial_\beta \xi^\lambda \partial_\nu g_{\alpha\lambda} \end{aligned} \tag{6.221}$$

Now

$$\delta L = \partial_\mu \left(\xi^\mu L\right) \tag{6.222}$$

Then we learn that on shell

$$\partial_\mu \left(-L\xi^\mu + j^\mu\right) \equiv \partial_\mu \left(\xi^\lambda \, T^\mu_\lambda + \partial_\lambda \xi^\sigma \, U^{\mu\lambda}_\sigma + \partial_\lambda \partial_\sigma \xi^\delta \, V^{\mu(\lambda\sigma)}_\delta\right) = 0 \tag{6.223}$$

Taking into account that ξ^μ is arbitary, we get the cascade equations of Julia and Silva [14]

$$\begin{aligned}\xi^\lambda \left(\partial_\mu T^\mu_\lambda\right) &= 0\\ \partial_\mu \xi^\lambda \left(T^\mu_\lambda + \partial_\rho U^{\rho\mu}_\lambda\right) &= 0\\ \partial_\lambda \partial_\sigma \xi^\delta \left(U^{\lambda\sigma}_\delta + \partial_\mu V^{\mu\lambda\sigma}_\delta\right) &= 0\\ \partial_\mu \partial_\lambda \partial_\sigma \xi^\delta \left(V^{(\mu(\lambda\sigma))}_\delta\right) &= 0\end{aligned} \tag{6.224}$$

The energy-momentum pseudotensor is given by

$$\begin{aligned}T^\mu_\lambda &= -L\delta^\mu_\lambda + \left(\tfrac{\partial L}{\partial\partial_\mu g_{\alpha\beta}} - \partial_\nu \tfrac{\partial L}{\partial\partial_\mu \partial_\nu g_{\alpha\beta}}\right)\partial_\lambda g_{\alpha\beta} + \tfrac{\partial L}{\partial_\mu \partial_\nu g_{\alpha\beta}} \partial_\nu \partial_\lambda g_{\alpha\beta} = \\ &= -R\delta^\mu_\lambda + \sqrt{|g|}\left(\left(-\Gamma^\beta_{\rho\sigma} g^{\rho\sigma} g^{\alpha\mu} + \Gamma^\mu_{\rho\sigma} g^{\rho\alpha} g^{\beta\sigma} + \Gamma^\alpha_{\rho\sigma} g^{\rho\sigma} g^{\beta\mu} + \Gamma^\mu_{\rho\sigma} g^{\rho\sigma} g^{\alpha\beta}\right)\partial_\lambda g_{\alpha\beta} + \right.\\ &\quad \left. + \tfrac{1}{2}\left(g^{\mu\alpha} g^{\beta\nu} + g^{\nu\alpha} g^{\mu\beta} - g^{\mu\nu} g^{\alpha\beta}\right)\partial_\nu \partial_\lambda g_{\alpha\beta}\right) = \end{aligned} \tag{6.225}$$

- It is also possible to proceed as follows. Starting from

$$\delta R^\mu{}_{\nu\alpha\beta} = \nabla_\alpha \delta\Gamma^\mu_{\nu\beta} - \nabla_\beta \delta\Gamma^\mu_{\nu\alpha} \tag{6.226}$$

where

$$\delta\Gamma^\mu_{\alpha\beta} \equiv \frac{1}{2} g^{\mu\lambda}\left(-\nabla_\lambda \delta g_{\alpha\beta} + \nabla_\alpha \delta g_{\lambda\beta} + \nabla_\beta \delta g_{\lambda\alpha}\right) \tag{6.227}$$

The variation of the curvature scalar

$$\begin{aligned}\delta R &\equiv g^{\nu\beta}\delta R_{\nu\beta} = g^{\nu\beta}\nabla_\mu \delta\Gamma^\mu_{\nu\beta} - \nabla_\beta \delta\Gamma^\mu_{\nu\mu} = \tfrac{1}{2} g^{\nu\beta}\Big\{ g^{\mu\lambda} \times \\ &\times \nabla_\mu \left(-\nabla_\lambda \delta g_{\nu\beta} + \nabla_\beta \delta g_{\lambda\nu} + \nabla_\nu \delta g_{\lambda\beta}\right) - g^{\mu\lambda}\nabla_\beta \times \\ &\times \left(-\nabla_\lambda \delta g_{\nu\alpha} + \nabla_\nu \delta g_{\lambda\alpha} + \nabla_\alpha \delta g_{\nu\lambda}\right)\Big\}\end{aligned} \tag{6.228}$$

and

$$\delta\sqrt{|g|} = \frac{1}{2}\sqrt{|g|}g^{\alpha\beta}\delta g_{\alpha\beta} \tag{6.229}$$

Then

$$\delta S_E = \int d(vol)\Big\{ \left(-R^{\alpha\beta} + \tfrac{1}{2}\, R\, g^{\alpha\beta}\right) \delta g_{\alpha\beta} + \nabla_\mu j^\mu \left[g_{\alpha\beta}, \delta g_{\alpha\beta}\right] \Big\}$$

where the precise expression for the current j^μ will be given momentarily.

- We have just seen that the full variation of the Einstein-Hilbert lagrangian can be expressed as

$$\delta S_{EH} = -\frac{1}{2\kappa^2}\int \Big\{\frac{\delta S}{\delta g_{\mu\nu}}\delta g_{\mu\nu} d(vol) + \nabla_\mu j^\mu\Big\} d(vol) \tag{6.230}$$

with

$$j^\mu \equiv \nabla^\mu g_{\rho\sigma}\delta g^{\rho\sigma} - \nabla_\sigma \delta g^{\mu\sigma} = g^{\mu\lambda}g_{\rho\sigma}\nabla_\lambda \delta g^{\rho\sigma} - \nabla_\sigma\delta g^{\mu\sigma} =$$
$$= g^{\mu\lambda}g_{\rho\sigma}\left(\partial_\lambda \delta g^{\rho\sigma} + \Gamma^\rho_{\lambda\delta}\delta g^{\delta\sigma} + \Gamma^\sigma_{\lambda\delta}\delta g^{\rho\delta}\right) - \partial_\lambda\delta g^{\mu\lambda} - \Gamma^\mu_{\lambda\delta}\delta g^{\gamma\lambda} - \Gamma^\lambda_{\lambda\delta}\delta g^{\mu\delta}$$

Now, under a diffeomorphism

$$\delta g^{\mu\lambda} = -g^{\mu\sigma}g^{\lambda\rho}\delta g_{\sigma\rho} = \xi^\delta\partial_\delta g^{\mu\lambda} - \partial_\delta\xi^\mu g^{\delta\lambda} - \partial_\delta \xi^\lambda g^{\mu\delta} \tag{6.231}$$

Then

$$\begin{aligned} j^\mu = g^{\mu\lambda}g_{\rho\sigma}\Big\{&\partial_\lambda\left(\xi^\delta\partial_\delta g^{\rho\sigma} - \partial_\delta\xi^\rho g^{\delta\sigma} - \partial_\delta\xi^\sigma g^{\rho\delta}\right) + \\ &+\Gamma^\rho_{\lambda\delta}\left(\xi^\alpha\partial_\alpha g^{\delta\sigma} - \partial_\alpha\xi^\delta g^{\alpha\sigma} - \partial_\alpha\xi^\sigma g^{\delta\alpha}\right) + \\ &+\Gamma^\sigma_{\lambda\delta}\left(\xi^\alpha\partial_\alpha g^{\rho\delta} - \partial_\alpha\xi^\rho g^{\alpha\delta} - \partial_\alpha\xi^\delta g^{\alpha\rho}\right)\Big\} - \\ &-\partial_\lambda\left(\xi^\alpha\partial_\alpha g^{\mu\lambda} - \partial_\alpha\xi^\mu g^{\alpha\lambda} - \partial_\alpha\xi^\lambda g^{\mu\alpha}\right) - \\ &-\Gamma^\mu_{\lambda\delta}\left(\xi^\alpha\partial_\alpha g^{\delta\lambda} - \partial_\alpha\xi^\delta g^{\alpha\lambda} - \partial_\alpha\xi^\lambda g^{\mu\alpha}\right) - \\ &-\Gamma^\lambda_{\lambda\delta}\left(\xi^\alpha\partial_\alpha g^{\mu\delta} - \partial_\alpha\xi^\mu g^{\alpha\delta} - \partial_\alpha\xi^\delta g^{\mu\alpha}\right) \end{aligned} \tag{6.232}$$

Grouping terms with the same number of derivatives of the generator ξ^α

$$\sqrt{|g|}\, j^\mu \equiv \xi^\rho\, T^\mu_\rho + \partial_\nu\xi^\rho\, U^{\mu\nu}_\rho + \partial_\delta\partial_\nu\xi^\rho\, V^{\mu\nu\delta}_\rho \tag{6.233}$$

Now for a diffeomorphism

$$\delta S_{EH} = 0 \tag{6.234}$$

Then *on shell* (that is when $\frac{\delta S_{EH}}{\delta g_{\mu\nu}}=0$),

$$\nabla_\mu j^\mu = \frac{1}{\sqrt{|g|}}\,\partial_\mu\left(\sqrt{|g|}j^\mu\right)=0 \tag{6.235}$$

Now different order of derivatives of the arbitrary parameters ξ^ρ are totally independent, so that their coefficients have got to vanish.

It follows that on shell

$$\partial_\mu T^\mu_\rho = 0 \tag{6.236}$$

as well as

$$\partial_\mu\xi^\rho\, T^\mu_\rho + \partial_\mu\xi^\rho\,\partial_\lambda\, U^{\lambda\mu}_\rho = 0 \tag{6.237}$$

Then

$$T^\mu_\rho = -\partial_\lambda\, U^{\lambda\mu}_\rho \tag{6.238}$$

The objects $U^{\lambda\mu}_\rho$ are traditionally called *superpotentials* in the literature.

It is also the case that

$$U^{\mu\nu}_\rho = -\partial_\lambda V^{\lambda(\nu\mu)}_\rho \tag{6.239}$$

where $V^{\lambda(\nu\mu)}_\rho$ constitute another set of superpotentials, which are constrained by

$$V^{(\mu\nu\delta)}_\rho = 0 \tag{6.240}$$

Let us proceed to the computation of the superpotentials. For that purpose it is better to rewrite the Noether current corresponding to Diff as

$$\begin{aligned} j^\mu = \nabla^\mu g_{\rho\sigma}\delta g^{\rho\sigma} - \nabla_\sigma\delta g^{\mu\sigma} &= -2\nabla^\mu\nabla_\lambda\xi^\lambda + \nabla_\sigma\left(\nabla^\mu\xi^\sigma + \nabla^\sigma\xi^\mu\right) = \\ &= \left(g^{\alpha\beta}\delta^\mu_\lambda - g^{\alpha\mu}\delta^\beta_\lambda\right)\nabla_\alpha\nabla_\beta\xi^\lambda + R^\mu_\lambda\xi^\lambda \end{aligned} \tag{6.241}$$

owing to Ricci identity's

$$\left[\nabla_\mu,\nabla_\lambda\right]\xi^\rho = R_{\mu\lambda\rho\sigma}\xi^\sigma \tag{6.242}$$

Now

$$\nabla_\mu j^\mu = \left(g^{\alpha\beta}\delta^\mu_\lambda - g^{\alpha\mu}\delta^\beta_\lambda\right)\nabla_\mu\nabla_\alpha\nabla_\beta\xi^\lambda + \nabla_\mu R^\mu_\lambda\,\xi^\lambda + R^\mu_\lambda\,\nabla_\mu\xi^\lambda$$

Let us work out the triple covariant derivative

- **Let us consider now boundary terms.** In this case there is an extra contribution to the variation, namely

$$\int_{\partial M} \sqrt{|h|} d^3y\, j^\mu n_\mu \tag{6.243}$$

where n is the unit normal to the boundary, and h_{ij} is the induced metric on it. The full spacetime metric is related to the induced metric through

$$g_{\alpha\beta} = h_{\alpha\beta} \pm n_\alpha n_\beta \tag{6.244}$$

in case $n^2 = \pm 1$. The null case is more complicated and should be dealt with separately.

In the above derivation wa have assumed that $\partial V = 0$. This is not the case in mosr circumstances. For example, one can integrate on the slice of spacetime defined by

$$t_i \leq t \leq t_f \tag{6.245}$$

where t is some *cosmic time*. Then the boundary includes the hypersurfaces Σ_i and Σ_f, where

$$\Sigma \equiv t = \text{constant} \tag{6.246}$$

Let us then repeat the analysis keeping the boundary terms. This is hard work.

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} + \left(g_{\mu\nu}\nabla^2 - \nabla_\mu\nabla_\nu\right)\delta g^{\mu\nu} \tag{6.247}$$

The boundary term in the Einstein-Hilbert variation then reads

$$S_{\partial V} = -\frac{c^3}{16\pi G}\int_{\partial V} d^{n-1}y\sqrt{|h|}\, n_\rho \left(g_{\mu\nu}\nabla^\rho \delta g^{\mu\nu} - \nabla_\nu \delta g^{\rho\nu}\right) \equiv -\frac{c^3}{16\pi G}\int_{\partial V} d^{n-1}y\sqrt{|h|}\, n_\rho J^\rho \tag{6.248}$$

Taking into account that

$$\delta g_{\alpha\beta}|_{\partial V} = 0 \tag{6.249}$$

as well as

$$\left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right)\delta g^{\mu\nu} = \nabla_\mu\left(\nabla_\lambda \delta g^{\mu\lambda} - \nabla^\mu g_{\alpha\beta}\delta g^{\alpha\beta}\right) \tag{6.250}$$

(remember that $g_{\alpha\beta} = n_\alpha n_\beta + h_{\alpha\beta}$), it is possible to write

$$\begin{aligned} & n^\rho J_\rho = n^\rho \left(g_{\mu\nu}\nabla_\rho \delta g^{\mu\nu} - g_{\rho\sigma}\nabla_\nu \delta g^{\sigma\nu}\right) = \\ & n_\rho g_{\alpha\beta}\left(\nabla^\alpha \delta g^{\rho\beta} - \nabla^\rho \delta g^{\alpha\beta}\right) = n_\rho \left(n_\alpha n_\nu + h_{\alpha\beta}\right)\left(\nabla^\alpha \delta g^{\rho\beta} - \nabla^\rho \delta g^{\alpha\beta}\right) = \\ & = -n^\rho h^{\mu\nu}\partial_\rho \delta g_{\mu\nu} \end{aligned} \tag{6.251}$$

The product of three normals are symmetric in $(\rho\alpha)$, whereas the factor in the variations is antisymmetric with respect to thsese same indices $[\rho\alpha]$. The product then vanishes.

Besides, tangential derivatives of the variation of the metric must vanish as well: $h_{\mu\nu}\partial^\nu \delta g^{\rho\sigma} = 0$. We are then left with the stated term only.

This surface variation can be cancelled with the boundary action

$$S_{\partial V} \equiv \frac{c^3}{8\pi G}\int_{\partial V}\sqrt{|h|}d^{n-1}y\ K \tag{6.252}$$

where

$$K \equiv \nabla_\alpha n^\alpha = \left(n^\alpha n^\beta + h^{\alpha\beta}\right)\nabla_\beta n_\alpha = h^{\alpha\beta}\nabla_\beta n_\alpha \tag{6.253}$$

In fact (on the boundary $\delta h_{\alpha\beta} = 0$), and

$$\delta\Gamma^\rho_{\alpha\beta} = \frac{1}{2}g^{\rho\lambda}\left(-\nabla_\lambda\delta g_{\alpha\beta} + \nabla_\alpha\delta g_{\lambda\beta} + \nabla_\beta\delta g_{\alpha\lambda}\right) \tag{6.254}$$

$$\delta K = -h^{\alpha\beta}\delta\Gamma^\rho_{\alpha\beta}n_\rho = \frac{1}{2}h^{\alpha\beta}n^\lambda\partial_\lambda\delta g_{\alpha\beta} \tag{6.255}$$

which precisely cancels the boundary term in the variation of the bulk piece of the Einstein-Hilbert action.

A perfect fluid as the one that is usually taken to represent the coarse grained material content in cosmology has

$$T_{\alpha\beta} \equiv (\rho + p)\,u_\alpha u_\beta - p g_{\alpha\beta} \tag{6.256}$$

So that a cosmological constant corresponds to

$$\begin{aligned}\rho &= -p\\ p &= \frac{1}{\kappa^2}\lambda\end{aligned} \tag{6.257}$$

The Einstein vacuum equations reduce to the Ricci flatness condition for the corresponding space

$$R_{\mu\nu} = 0 \tag{6.258}$$

6.8.1 The $\Gamma - \Gamma$ noninvariant lagrangian-

The EH lagrangian has got clearly two pieces in it. Let us dub

$$S_{\alpha\beta} \equiv \Gamma^\sigma_{\beta\rho}\Gamma^\rho_{\sigma\alpha} - \Gamma^\sigma_{\sigma\rho}\Gamma^\rho_{\alpha\beta} \tag{6.259}$$

and

$$L_S \equiv \sqrt{g}\; g^{\alpha\beta} S_{\alpha\beta} \tag{6.260}$$

It is a fact that

$$L_{EH} = \sqrt{|g|}\left(g^{\alpha\beta}\partial_\beta \Gamma^\sigma_{\alpha\sigma} - \partial_\rho \Gamma^\rho_{\alpha\beta}\right) + L_S = \partial_\beta\left(\sqrt{|g|} g^{\alpha\beta}\Gamma^\sigma_{\alpha\sigma}\right) - \partial_\rho\left(\sqrt{|g|}\; g^{\alpha\beta}\Gamma^\rho_{\alpha\beta}\right) -$$
$$-\partial_\beta\left(\sqrt{|g|} g^{\alpha\beta}\right)\Gamma^\sigma_{\alpha\sigma} + \partial_\rho\left(\sqrt{|g|} g^{\alpha\beta}\right)\Gamma^\rho_{\alpha\beta} + L_S = \partial_\beta\left(\sqrt{|g|} g^{\alpha\beta}\Gamma^\sigma_{\alpha\sigma}\right) - \partial_\rho\left(\sqrt{|g|}\; g^{\alpha\beta}\Gamma^\rho_{\alpha\beta}\right.$$
$$-L_S =$$

where we have used

$$\partial_\rho\left(\sqrt{|g|} g^{\alpha\beta}\right) = \sqrt{|g|}\left(\frac{\partial_\rho\sqrt{|g|}}{\sqrt{|g|}} g^{\alpha\beta} - \Gamma^\alpha_{\rho\mu} g^{\mu\beta} - \Gamma^\beta_{\rho\mu} g^{\alpha\mu}\right) = \sqrt{|g|}\left(\Gamma^\sigma_{\sigma\rho} g^{\alpha\beta} - \Gamma^\alpha_{\rho\mu} g^{\mu\beta} - \Gamma^\beta_{\rho\mu} g^{\alpha\mu}\right.$$

so that

$$-\partial_\beta\left(\sqrt{|g|} g^{\alpha\beta}\right)\Gamma^\sigma_{\alpha\sigma} + \partial_\rho\left(\sqrt{|g|} g^{\alpha\beta}\right)\Gamma^\rho_{\alpha\beta} = -2L_S \tag{6.261}$$

This shows that the Einstein-Hilbert lagrangian and Schrödinger's $\Gamma - \Gamma$ lagrangian differ by a total derivative, and thus yield the same equations of motion.

This is remarkable, because L_S is *not* diffeomorphism invariant. On the other hand, it depends only on first derivatives of the metric; this then gives a new insight insight explaining why Einstein's equations are second order.

There are many lessons to be drawn fro this fascinating lagrangian; for example, that the symmetries of the equations of motion do not have to coincide with the ones of the lagrangian; they can be enhanced, as is the case here.

6.8.2 The first order formalism.

Assume there is a frame $e^a(x)$ (a section of the cotangent bundle)

$$L = -\frac{1}{2\kappa^2} R^a{}_b \wedge \frac{1}{2}\epsilon_{abcd} e^c \wedge e^d \tag{6.262}$$

Define a spin (Lorentz) connection such that under a Lorentz transformation (such that $L^{-1}_{ab} = L_{ba}$)

$$\omega^a{}_b \rightarrow L^a{}_u \omega^u{}_v (L^{-1})^v{}_b - (L^{-1})^c{}_b\, dL^a{}_c \tag{6.263}$$

That is, when $\omega^a{}_b \equiv \delta^a_b + \lambda^a_b$

$$\delta\omega^a{}_b = \lambda^a{}_u\, \omega^u{}_b - \omega^a{}_v\, \lambda^v{}_b - d\lambda^a{}_b \tag{6.264}$$

We define the curvature two-form

$$R^a{}_b \equiv d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \tag{6.265}$$

the torsion two-form

$$T^a \equiv De^a \equiv de^a + \omega^a{}_b \wedge e^b \tag{6.266}$$

and the non-metricity one-form

$$\Omega^{ab} \equiv D\eta^{ab} = d\eta^{ab} + \omega^{ab} + \omega^{ba} \tag{6.267}$$

From the definitions some identities (Bianchi) can be derived

$$dR^a{}_b = d\omega^a{}_c \wedge \omega^c{}_b - \omega^a{}_c \wedge d\omega^c{}_b = \left(R^a{}_c - \omega^a{}_d \wedge \omega^d{}_c\right) \wedge \omega^c{}_b - \\ -\omega^a{}_c \wedge \left(R^c{}_b - \omega^c{}_d \wedge \omega^d{}_b\right) = R^a{}_c \wedge \omega^c{}_b - \omega^a{}_c \wedge R^c{}_b \tag{6.268}$$

which can also be written as

$$DR^a{}_b = 0 \tag{6.269}$$

It is also a fact that

$$dT^a = d\omega^a{}_b \wedge e^b - \omega^a{}_b \wedge de^b = (R^a{}_b - \omega^a{}_c \wedge \omega^c{}_b) \wedge e^b - \\ -\omega^a{}_b \wedge \left(T^b - \omega^c{}_d \wedge e^d\right) = R^a{}_b \wedge e^b - \omega^a{}_b \wedge T^b \tag{6.270}$$

that is

$$DT^a = R^a{}_b \wedge e^b \tag{6.271}$$

Finally

$$d\Omega^{ab} = d\omega^{ab} + d\omega^{ba} \tag{6.272}$$

that is

$$D\omega^{ab} \equiv d\Omega^{ab} + \omega^a{}_c \wedge \left(\omega^{cb} + \omega^{bc}\right) + \omega^b{}_c \left(\omega^{ac} + \omega^{ca}\right) = \\ = d\omega^{ab} + d\omega^{ba} + \omega^a{}_c \wedge \left(\omega^{cb} + \omega^{bc}\right) + \omega^b{}_c \left(\omega^{ac} + \omega^{ca}\right) = R^{ab} + R^{ba}$$

The variation yields

$$\delta S = -\frac{1}{2\kappa^2} \int d^4x \left(D\delta\omega^a{}_b \wedge \frac{1}{2}\epsilon_{abcd} e^c \wedge e^d + R^a{}_b \wedge \frac{1}{2}\epsilon_{abcd} \left(\delta e^c \wedge e^d + e^c \wedge \delta e^d\right)\right) \tag{6.273}$$

The EM then read

$$\begin{aligned} \epsilon_{abcd}\, D\left(e^c\wedge e^d\right) = 0 \\ \epsilon_{abcd}\, R^a{}_b\wedge e^c = 0 \end{aligned} \tag{6.274}$$

Under local Lorentz transformations the Noether currrent reads

$$\begin{aligned} j_L \equiv \delta_L \omega^a{}_b \wedge \frac{\partial L}{\partial d\omega^a{}_b} = \frac{1}{2}\left(\lambda^a{}_u\,\omega^u{}_b - \omega^a{}_v\,\lambda^v{}_b - d\lambda^a{}_b\right)\wedge\epsilon_{abcd}e^c\wedge e^d \equiv \\ \equiv \lambda^u{}_v j^v_u + d\lambda^u{}_v \wedge U^v{}_u \end{aligned} \tag{6.275}$$

where

$$\begin{aligned} j^v{}_u = \left(\omega^v{}_b\epsilon_{ubcd} - \omega^a{}_u\epsilon_{avcd}\right)\wedge e^c\wedge e^d \\ U^v{}_u = -\epsilon_{vucd}e^c\wedge e^d \end{aligned} \tag{6.276}$$

Now the condition

$$dj_L = 0 \tag{6.277}$$

yields a Julia-Silva cascade

$$\begin{aligned} j^v_u + dU^v{}_u = 0 \\ dj^v{}_u = 0 \end{aligned} \tag{6.278}$$

Under Diff

$$\delta x^\mu = \xi^\mu(x) \tag{6.279}$$

the variation of the different forms is

$$\begin{aligned} \delta\omega^a{}_b = \left(di_\xi + i_\xi d\right)\omega^a{}_b \\ \delta e^a = \left(di_\xi + i_\xi d\right)e^a \end{aligned} \tag{6.280}$$

$$\delta L = \pounds(\xi)L = \left(di_\xi + i_\xi d\right)L = di_\xi L \tag{6.281}$$

Noether tells us that

$$dj_\xi \equiv d\left(i_\xi L - \pounds(\xi)\omega^a{}_b\wedge\epsilon_{abcd}e^c\wedge e^d\right) = 0 \tag{6.282}$$

Now using the fact that

$$\pounds(fX)\alpha = f\pounds(X)\alpha + df\wedge i_X\alpha \tag{6.283}$$

when $fX = \xi^\rho\partial_\rho$,

$$\pounds(\xi)\omega = \xi^\rho\pounds(\partial_\rho)\omega + d\xi^\rho i_{\partial_\rho}\omega \tag{6.284}$$

it follows that

$$j_\xi = \xi^\rho\tau_\rho + d\xi^\rho\wedge\sigma_\rho \tag{6.285}$$

with

$$\tau_\rho = i(\partial_\rho)L - \pounds(\partial_\rho)\omega^a{}_b \wedge \epsilon_{abcd}e^c \wedge e^d$$
$$\sigma_\rho = i(\partial_\rho)\omega^a{}_b \wedge \epsilon_{abcd}e^c \wedge e^d \tag{6.286}$$

The cascade leads to the superpotential

$$\tau_\rho = d\sigma_\rho \tag{6.287}$$

The theory is also Weyl invariant in some sense under

$$\delta\omega^a{}_b = \omega(x)\delta^a_b \tag{6.288}$$

This is quite trivial, because when the non-metricty vanishes, the connection is antisymmetric

$$\omega_{ab} = -\omega_{ba} \tag{6.289}$$

In case there is a symmetric piece

$$\omega = \omega^S + \omega^A \tag{6.290}$$

they decpuple in the sense that

$$R = R^S + R^A \tag{6.291}$$

and R^S does not contribute to the action.

More

It is vary easy to derive a first order action principle provided one is willing to postulate that the torion of the connection vanishes.

$$S = \int d^n x \sqrt{|g|} g^{\mu\nu}\ R_{\mu\nu}(\Gamma) \tag{6.292}$$

Here the connection is itself a variable, so that we cannot integrate by partis covariant derivatives.

The so called *Palatini identity* tells us that

$$\delta R^\mu{}_{\nu\alpha\beta} = \nabla_\alpha \delta\Gamma^\mu_{\nu\beta} - \nabla_\beta \delta\Gamma^\mu_{\nu\alpha} \tag{6.293}$$

and

$$\delta R_{\mu\nu} = \nabla_\lambda \delta\Gamma^\lambda_{\mu\nu} - \nabla_\nu \delta\Gamma^\lambda_{\mu\lambda} \tag{6.294}$$

We can then write

$$\delta S_p = \int \sqrt{|g|}\ d^n x \Big\{ \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}\right) \delta g^{\mu\nu} + \\ + g^{\mu\nu} \left(\nabla_\lambda \delta\Gamma^\lambda_{\mu\nu} - \nabla_\nu \delta\Gamma^\lambda_{\mu\lambda}\right) \Big\} \tag{6.295}$$

It is useful to use again the variable

$$\sqrt{|g|}g^{\mu} \equiv \mathfrak{g}^{\mu\nu} \tag{6.296}$$

$$\delta S_p = \int d^n x \Big\{ \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) \delta \mathfrak{g}^{\mu\nu} + \mathfrak{g}^{\mu\nu} \left(\nabla_\lambda \delta\Gamma^\lambda_{\mu\nu} - \nabla_\nu \delta\Gamma^\lambda_{\mu\lambda} \right) \Big\} =$$
$$\int d^n x \Big\{ \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) \delta \mathfrak{g}^{\mu\nu} + \nabla_\lambda \left(\mathfrak{g}^{\mu\nu} \delta\Gamma^\lambda_{\mu\nu} - \mathfrak{g}^{\lambda\mu} \delta\Gamma^\sigma_{\mu\sigma} \right) + \nabla_\lambda \mathfrak{g}^{\mu\nu} \delta\Gamma^\lambda_{\mu\nu} + \nabla_\nu \mathfrak{g}^{\mu\nu} \delta\Gamma^\sigma_{\mu\sigma}$$
$$\int d^n x \Big\{ \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) \delta \mathfrak{g}^{\mu\nu} - \nabla_\lambda \mathfrak{g}^{\mu\nu} \delta\Gamma^\lambda_{\mu\nu} + \nabla_\nu \mathfrak{g}^{\mu\nu} \delta\Gamma^\sigma_{\mu\sigma} \Big\} \quad (6$$

because for a tensor density

$$\nabla_\lambda \mathfrak{t}^\lambda = \partial_\lambda \mathfrak{t}^\lambda \tag{6.298}$$

This means that

$$\nabla_\lambda \mathfrak{g}^{\mu\nu} = \delta^\lambda_\nu \nabla_\rho \mathfrak{g}^{\mu\rho} \tag{6.299}$$

which is easily seen to imply

$$\nabla_\lambda \mathfrak{g}^{\mu\nu} = 0 \tag{6.300}$$

so that the metric is covariantly constant and the connection is the Levi-Civita one.

7

The Schwarzschild vacuum solution.

In any static space-time, the frequency of any radiation observed by a FIDO

$$u \equiv \frac{\xi}{\sqrt{\xi^2}} \tag{7.1}$$

is given by

$$\omega \equiv \frac{k.u}{\sqrt{\xi^2}} \tag{7.2}$$

A possible definition of a spherically symmetric space-time is

$$M_2 \times S_2 \tag{7.3}$$

where M_2 is a two-dimensional lorentzian manifold. The metric reads

$$ds^2 = B^2(t,r)dt^2 - A^2(t,r)dr^2 - r^2 d\Omega_2^2 \tag{7.4}$$

There is a natural tetrad, namely

$$\begin{aligned} e_0 &\equiv B\ dt \\ e_1 &\equiv A\ dr \\ e_2 &\equiv r\ d\theta \\ e_3 &\equiv r\sin\,\theta\ d\phi \end{aligned} \tag{7.5}$$

In the corresponding LIF

$$R_{02} = 2\frac{\partial_0 A}{rBA^2} \tag{7.6}$$

so that

$$A = A(r) \tag{7.7}$$

only. On the other hand, still in the LIF

$$R_{22} - R_{00} = \frac{2}{r}\frac{1}{A^2}\partial_r \log\ (AB) \tag{7.8}$$

We learn that

$$AB = f(t) \tag{7.9}$$

It is plain that we can always redefine the time coordinate in such a way that $f(t) = 0$. This is actually *Birkhoff's theorem*: the space-time external to an spherical mass is necessarily static. The other equation reads

$$\{R_{33} = 0\} \quad \Leftrightarrow \quad \left\{2\frac{\partial_r A}{A}\frac{1}{A^2 r} = \frac{1}{r^2}\left(1 - \frac{1}{A^2}\right)\right\} \tag{7.10}$$

The solution to this equation is

$$\frac{1}{A^2} = 1 - \frac{r_S}{r} \tag{7.11}$$

The Schwarzschild metric (Petrov Type D) is given by

$$ds^2 = \left(1 - \frac{r_S}{r}\right) dt^2 - \frac{dr^2}{1 - \frac{r_S}{r}} dr^2 - r^2 d\Omega_2^2 \tag{7.12}$$

where the Schwarzschild radius is give by

$$r_S \equiv \frac{2GM}{r} \tag{7.13}$$

The Killing vector is given by

$$\xi = \frac{\partial}{\partial t} \tag{7.14}$$

so that its square is

$$\xi^2 = g_{00} = 1 - \frac{r_S}{r} \tag{7.15}$$

For the Sun, $r_S \sim 3\ km$. The formula above then tells us that

$$\frac{\omega_1}{\omega_2} = \sqrt{\frac{k_2^2}{k_1^2}} = \sqrt{\frac{1 - \frac{r_S}{r_2}}{1 - \frac{r_S}{r_1}}} \tag{7.16}$$

This is nothing but a fancier derivation of the gravitational redshift formula. A FIDO at $r_0 = \infty$ will measure a frequency

$$\omega_0 = \omega_r \sqrt{1 - \frac{r_S}{r}} \tag{7.17}$$

which is *smaller* than ω_r (that is, the waves are *redshifted*) and actually vanishes when $r \sim r_S$; there is an infinite redshift for all waves emitted at the horizon. In order to observe a finite frequency at infinity, the frequency close to the horizon (that is, at a distance $r = r_S + \epsilon$) must be

$$\omega_\epsilon = 2\omega_0 \frac{r_S}{\epsilon} \tag{7.18}$$

which is transplanckian for

$$\epsilon \leq 2\frac{\omega_0}{M_P}\, r_S \tag{7.19}$$

7.1 Timelike geodesics.

We shall follow the thorough analysis in [3]. The equivalent lagrangian reads

$$L = \frac{1}{2}\left\{ \left(1 - \frac{r_S}{r}\right)\dot{t}^2 - \frac{\dot{r}^2}{1 - \frac{r_S}{r}} - r^2\,\dot{\theta}^2 - r^2 \sin^2\theta\;\dot{\phi}^2 \right\} \tag{7.20}$$

Cyclic coordinates:

$$\begin{aligned} p_t &\equiv \left(1 - \frac{r_S}{r}\right)\dot{t} = E = \text{const} \\ p_\phi &\equiv r^2 \sin^2\theta\;\dot{\phi} = \text{const} \end{aligned} \tag{7.21}$$

On the other hand,

$$\frac{d}{d\tau}\left(r^2\dot{\theta}\right) = r^2 \sin\theta\,\cos\theta\;\dot{\phi}^2 \tag{7.22}$$

so that if we assign $\theta = \frac{\pi}{2}$ when $\dot{\theta} = 0$, then $\ddot{\theta} = 0$, and $\theta = \frac{\pi}{2}$ forever. We shall therefore restrict ourselves to the plane $\theta = \frac{\pi}{2}$. Then

$$r^2\dot{\phi} \equiv L \tag{7.23}$$

The normalization of the four velocity now takes the form

$$\frac{E^2}{1 - \frac{r_S}{r}} - \frac{\dot{r}^2}{1 - \frac{r_S}{r}} - \frac{L^2}{r^2} = (1, 0) \tag{7.24}$$

for timelike or null geodesics, respectively. Let us restrict to the timelike case for the time being. Then

$$\begin{aligned} &\left(\frac{dr}{d\tau}\right)^2 + \left(1 - \frac{r_S}{r}\right)\left(1 + \frac{L^2}{r^2}\right) = E^2 \\ &\frac{d\phi}{d\tau} = \frac{L}{r^2} \end{aligned} \tag{7.25}$$

$$\frac{dr}{d\phi} = \sqrt{(E^2 - 1)\frac{r^4}{L^2} + \frac{r_S}{L^2}r^3 - r^2 + r_S r} \tag{7.26}$$

and in terms of the convenient variable $u \equiv r^{-1}$

$$\frac{du}{d\phi} = \sqrt{r_S u^3 - u^2 + \frac{r_S u}{L^2} - \frac{1 - E^2}{L^2}} \tag{7.27}$$

Once this equation is solved the problem is completely determined through

$$\frac{d\tau}{d\phi} = \frac{1}{L u^2} \tag{7.28}$$

$$\frac{dt}{d\phi} = \frac{E}{L u^2 (1 - r_S u)} \tag{7.29}$$

- Let us begin by considering the simplest instance, namely the radial geodesics. Those have zero angular momentum The equations are

$$\begin{aligned} \frac{dr}{d\tau} &= \sqrt{\frac{r_S}{r} - (1 - E^2)} \\ \frac{dt}{d\tau} &= \frac{E}{1 - \frac{r_S}{r}} \end{aligned} \tag{7.30}$$

  We shall consider boundary conditions such that they start with

$$\begin{aligned} r &= r_i \\ \dot{r} &= 0 \end{aligned} \tag{7.31}$$

  Then

$$r_i = \frac{r_S}{1 - E^2} \tag{7.32}$$

  We define an auxiliary variable η such that

$$r = \frac{r_S}{1 - E^2} \frac{1 + \cos\eta}{2} = \frac{r_S}{1 - E^2} \cos^2 \frac{\eta}{2} = r_i \cos^2 \frac{\eta}{2} \tag{7.33}$$

  It is plain that

$$\eta = 0 \tag{7.34}$$

  when

$$r = r_i \tag{7.35}$$

  The horizon crossing ($r = r_S$) is located at

$$\eta_H \equiv 2 \sin^{-1} E \tag{7.36}$$

  and the singularity at $r = 0$ is reached when

$$\eta = \pi \tag{7.37}$$

The equations to be integrated are

$$\frac{dr}{d\tau} = -\sqrt{1-E^2}\text{tg}\ \frac{\eta}{2} = -\sqrt{\frac{r_S}{r_i}}\text{tg}\ \frac{\eta}{2}$$
$$\frac{dr}{d\eta} = -r_i\text{sin}\ \frac{\eta}{2}\ \cos\ \frac{\eta}{2} \tag{7.38}$$

Now

$$\frac{dt}{d\eta} \equiv \frac{dt}{d\eta}\frac{d\tau}{dr}\frac{dr}{d\eta} = \frac{E\cos^2\frac{\eta}{2}}{\cos^2\frac{\eta}{2} - \cos^2\frac{\eta_H}{2}} \tag{7.39}$$

It follows

$$\frac{d\tau}{d\eta} = \sqrt{\frac{r_i^3}{r_S}}\cos^2\ \frac{\eta}{2} = \sqrt{\frac{r_i^3}{r_S}}\ (1+\cos\ \eta) \tag{7.40}$$

so that normalizing such that $\tau = 0$ at $\eta = 0$

$$\tau = \sqrt{\frac{r_i^3}{r_S}}\ (\eta + \sin\ \eta) \tag{7.41}$$

This means that the particle crosses the horizon at a finite proper time

$$\tau_H = \sqrt{\frac{r_i^3}{r_S}}\ (\eta_H + \sin\ \eta_H) \tag{7.42}$$

and reaches the singularity in a finite proper time as well

$$\tau_0 = \pi\ \sqrt{\frac{r_i^3}{4r_S}} \tag{7.43}$$

This is what would have measured a FREFO.

To obtain the corresponding coordinate time (as measured by a FIDO) we have to integrate

$$\frac{dt}{d\eta} = E\sqrt{\frac{r_i^3}{r_S}}\frac{\cos^4\frac{\eta}{2}}{\cos^2\frac{\eta}{2} - \cos^2\frac{\eta_H}{2}} \tag{7.44}$$

The result of the quadrature is

$$t = E\sqrt{\frac{r_i^3}{r_S}}\left(\frac{\eta + \sin\ \eta}{2} + (1-E^2)\eta\right) + r_S\ \log\ \frac{\text{tg}\ \frac{\eta_H}{2} + \text{tg}\ \frac{\eta}{2}}{\text{tg}\ \frac{\eta_H}{2} - \text{tg}\ \frac{\eta}{2}} \tag{7.45}$$

It is plain that the FIDO time diverges logartithmically when $\eta \to \eta_H$.

- Let us now consider in detail the bound orbits (they correspond to $E^2 < 1$)

They are defined through

$$\frac{du}{d\phi} = \sqrt{f(u)} \equiv \sqrt{r_S u^3 - u^2 + \frac{r_S}{L^2}u - \frac{1-E^2}{L^2}} \tag{7.46}$$

The roots of

$$f(u) = 0 \tag{7.47}$$

it being a cubic polynomial, fall into two classes: either there are three real roots, or else there is one real root and two complex conjugate ones. At any rate

$$\begin{aligned} &\lim_{u\rightarrow\pm\infty} = \pm\infty \\ &f(u=0) = -\frac{1-E^2}{2} < 0 \\ &u_1 u_2 u_3 = \frac{1-E^2}{r_S L^2} \\ &u_1 + u_2 + u_3 = \frac{1}{r_S} \end{aligned} \tag{7.48}$$

In fact there are five possibilities for the real roots. Let us call the roots u_i

- i. $0 < u_1 < u_2 < u_3$.
  Then there are two distict orbits confined to the intervals in which $f(u) \geq 0$, that is, either

  $$u_1 \leq u \leq u_2 \tag{7.49}$$

  (this is an orbit that oscillates between two extreme values of r, namely, $\frac{1}{u_1}$ and $\frac{1}{u_2}$) or else

  $$u \geq u_3 \tag{7.50}$$

  (which starting at certain aphelion distance (namely $\frac{1}{u_3}$ finishes at the singularity. These two classes of orbits are dubbed *first and second kind*
- ii. $0 < u_1 = u_2 < u_3$.
  Then the orbit of the first kind is a stable circular orbit of zero eccentricity.
- iii. $0 < u_1 < u_2 = u_3$
  Then the orbit of the first kind starts at aphelion $\frac{1}{u_1}$ and spirals towards the circle of radius $\frac{1}{u_3}$. The orbit of the second kind spirals towards the singularity.
- iv. $0 < u_1 = u_2 = u_3$
  There is an ustable circular orbit of radius $\frac{1}{u_1}$

– v. $0 < u_1 \quad u_2 = u_3^*$.
In this case all orbits finish at the singularity

Let us now concentrate on the **orbits of the first kind**. We define the *latus rectum*, l and the *eccentricity*, $0 \leq e < 1$ through

$$\begin{aligned} u_1 &\equiv \frac{1-e}{l} \\ u_2 &\equiv \frac{1+e}{l} \\ u_3 &\equiv \frac{1}{r_S} - \frac{2}{l} \end{aligned} \tag{7.51}$$

The ordering $u_2 < u_3$ that we have assumed implies that

$$l \geq r_S(3+e) \tag{7.52}$$

Let us define

$$\mu \equiv \frac{r_S}{2l} \tag{7.53}$$

so that

$$\mu \leq \frac{1}{2(3+e)} \tag{7.54}$$

or what is the same thing,

$$1 - 6\mu - 2\mu e \geq 0 \tag{7.55}$$

Now

$$f(u) = r_s \left(u - \frac{1-e}{l}\right)\left(u - \frac{1+e}{l}\right)\left(u - \frac{1}{r_S} + \frac{2}{l}\right) \tag{7.56}$$

and consistency implies

$$\begin{aligned} \frac{r_S}{2L^2} &= \frac{2l - r_S(3+e^2)}{2l^2} \\ \frac{1-E^2}{L^2} &= \frac{(1-2r_S)(1-e^2)}{l^3} \end{aligned} \tag{7.57}$$

that is

$$\begin{aligned} \frac{1}{L^2} &= 2\frac{1-\mu(3+e^2)}{lr_S} \\ \frac{1-E^2}{L^2} &= \frac{(1-4\mu)(1-e^2)}{l^2} \end{aligned} \tag{7.58}$$

from which it easily follows that

$$\mu < \frac{1}{3+e^2} \tag{7.59}$$

as well as

$$\mu < \frac{1}{4} \tag{7.60}$$

It is a fact that

$$\frac{E^2}{L^2} = 2\frac{(2\mu - 1)^2 - 4\mu^2 e^2}{l r_S} \tag{7.61}$$

Let us now make the substitution

$$u = \frac{1 + e \cos \chi}{l} \tag{7.62}$$

where χ is the *relativistic anomaly*, which is such that $\chi = \pi$ at aphelion, and $\chi = 0$ at perihelion. Then

$$\left(\frac{d\chi}{d\rho}\right)^2 = 1 - 2\mu\,(3 + e\cos\,\chi) = (1 - 6\mu + 2\mu e) - 4\mu\; e\; \cos^2\,\frac{\chi}{2} \tag{7.63}$$

so that

$$\frac{d\chi}{d\rho} = \sqrt{1 - 6\mu + 2\mu e}\sqrt{1 - k^2\cos^2\frac{\chi}{2}} \tag{7.64}$$

with

$$k^2 \equiv \frac{4\mu e}{1 - 6\mu + 2\mu e} \tag{7.65}$$

Our previous inequalities mean that

$$k^2 \leq 1 \quad 1 - 6\mu + 2\mu e > 0 \tag{7.66}$$

Thie means that

$$\phi = \frac{2}{\sqrt{1 - 6\mu + 2\mu e}} F\left(\frac{\pi - \chi}{2}, k\right) \tag{7.67}$$

where the *Jacobian elliptic integral* is defined as

$$F(\psi, k) \equiv \int_0^\psi \frac{dx}{\sqrt{1 - k^2\,\sin^2\,x}} \tag{7.68}$$

and the origin of ϕ has been chosen at the aphelion $\chi = \pi$. Perihelion occurs at $\chi = 0$ ($\psi = \frac{\pi}{2}$). The solution includes

$$\tau = \frac{1}{L}\int\frac{d\phi}{u^2} = \frac{1}{L}\int\frac{d\phi}{\chi}\frac{d\chi}{u^2} =$$
$$\sqrt{\frac{2l^3\,(1 - \mu(3 + e^2))}{r_S}}\int_\chi^\pi d\chi\,\frac{1}{(1 + e\,\cos\,\chi)^2\,\sqrt{1 - 2\mu\,(3 + e\,\cos\,\chi)}} =$$
$$\frac{T_N}{2\pi}\sqrt{(1 - e^2)^3\,(1 - \mu(3 + e^2))} \tag{7.69}$$

as well as

$$t = \frac{E}{L}\int \frac{d\phi}{d\chi}\frac{d\chi}{u^2\,(1-ur_S)} = \sqrt{\frac{2l^3\,((2\mu-1)^2-4\mu^2e^2)}{r_S}}\times$$
$$\int_\chi^\pi d\chi\,\frac{1}{(1-2\mu(1+e\,\cos\,\chi)\,(1+e\,\cos\,\chi)^2\,\sqrt{1-2\mu\,(1-2\mu(3+e\,\cos\,\chi)}} =$$
$$\frac{T_N}{2\pi}\sqrt{(1-e^2)^3\,((2\mu-1)^2-4\mu^2e^2)} \tag{7.70}$$

where the Newtonian period of the orbit with the same eccentricity and latus rectum

$$T_N \equiv \sqrt{\frac{8\pi^2 l^3}{(1-e^2)^3 r_S}} \tag{7.71}$$

Let us now consider the case ii) e=0 when the two roots $u_1 = u_2$ coincide and the case iii) $2\mu(3+e) = 1$ when the roots $u_2 = u_3$.

- The case $e = 0$. In this case the orbit is a circle w ith

$$r_c = l$$
$$\mu = \frac{r_S}{2r_c} \tag{7.72}$$

Using

$$\frac{1}{L^2} = \frac{2-\frac{3r_S}{r_c}}{r_c r_S}$$
$$\frac{E^2}{L^2} = 2\frac{\left(\frac{r_S}{r_c}-1\right)^2}{r_c r_S} \tag{7.73}$$

The first equation can be rewritten as

$$r_c^2 - 2\frac{L^2}{r_S}r_c + 3L^2 = 0 \tag{7.74}$$

so that

$$r_c = \frac{L^2}{r_S}\left(1 \pm \sqrt{1-3\frac{r_S^2}{L^2}}\right) \tag{7.75}$$

This means that no circular orbit is possible when

$$\frac{L}{M} < 2\sqrt{3} < \tag{7.76}$$

and for the minimum value of this ratio

$$r_c = 3r_S$$
$$E^2 = \frac{8}{9} \tag{7.77}$$

On the other hand, the largest root of the quadratic equation corresponds to a minimum of the potential energy, while the smaller root corresponds to a maximum. The respective allowed ranges read

$$\begin{aligned} &3r_S < r_c < \infty \quad \text{(stable)} \\ &\frac{3}{2}r_S \leq r_c \leq 3r_S \quad \text{(unstable)} \end{aligned} \tag{7.78}$$

The periods are given by

$$\begin{aligned} \tau_P &= T_N \sqrt{\frac{1-3\mu}{1-6\mu}} \\ t_P &= T_N \frac{1}{\sqrt{1-6\mu}} \end{aligned} \tag{7.79}$$

When $\mu = \frac{1}{6}$ and $r_c = 3r_S$ then $t_P = \infty$.

- **The case** $2\mu\,(3+e) = 1$**.** The perihelion and aphelion distances are given by

$$\begin{aligned} r_P &\equiv \frac{1}{1+e} = r_S\frac{3+e}{1+e} \\ r_A &\equiv r_S\frac{e+3}{1-e} \end{aligned} \tag{7.80}$$

They are restricted to the interval

$$2r_S \leq r_P < 3r_S \tag{7.81}$$

Besides

$$\begin{aligned} \frac{L^2}{r_S^2} &= \frac{(3+e)^2}{(3-e)(1+e)} \\ 1-E^2 &= \frac{1-e^2}{9-e^2} \end{aligned} \tag{7.82}$$

The modulus $k^2 = 1$ and besides

$$\left(\frac{d\chi}{d\phi}\right)^2 = 4\mu e \sin^2\frac{\chi}{2} \tag{7.83}$$

which means that

$$\phi = -\frac{1}{\sqrt{\mu e}}\log\,\mathrm{tg}\,\frac{\chi}{4} \tag{7.84}$$

so that $\phi = 0$ whenever $\chi = \pi$, and $\phi = \infty$ when $\chi = 0$ at the perihelion approach. The orbit approaches the circle at r_P asymptotically, spiralling around it an infinite number of times. Actually, this orbot continues into the interior of the corcle as an orbit of the seconfd kind to plunge eventually into the singularity.

- **The post-newtonial approximation.**

The quantity
$$\mu \equiv \frac{r_S}{2l} \sim 10^{-6} \tag{7.85}$$
is usually quite small. Expanding to first order in μ we learn that
$$-\,d\phi = d\chi\,(1 + 3\mu + \mu e\,\cos\,\chi) \tag{7.86}$$
namely
$$-\,\phi = (1 + 3\mu)\,\chi + \mu e\,\sin\,\chi + C \tag{7.87}$$
The change in ϕ after one complete revolution during which $\Delta\chi = 2\pi$ is given by $2(1 + 3\mu)\pi$. The *advance of the perihelion* (Einstein) per revolution is then given by
$$\Delta\phi \equiv 3\pi\frac{r_s}{l} = 3\pi\frac{r_S}{a(1-e^2)} \tag{7.88}$$
where a denotes the semi-major axis of the Keplerian ellipse.

- **Orbits of the second Kind**

These have their aphelions at $\frac{1}{u_3}$ and eventually plunge into the singilarity at $r = 0$. Given the fact that
$$u_1 + u_2 + u_3 = \frac{1}{r_S} \tag{7.89}$$
and
$$u_1 + u_2 > 0 \tag{7.90}$$
as well as
$$u_3 < \frac{1}{r_S} \tag{7.91}$$
all these orbits stay outside the horizon. Let us make the substitution
$$u \equiv \left(\frac{1}{r_S} - \frac{2}{l}\right) + \left(\frac{1}{r_S} - \frac{3+e}{l}\right)\mathrm{tg}^2\,\frac{\xi}{2} \tag{7.92}$$
Then when $\xi = 0$,
$$u = u_3 = \frac{1}{r_S} - \frac{2}{l} \tag{7.93}$$
and when $\xi \to \infty$
$$u \to \infty \tag{7.94}$$
and the ODE reduces to
$$\left(\frac{\delta\xi}{d\phi}\right)^2 = (1 - 6\mu + 2\mu e)\left(1 - k^2\,\sin^2\,\frac{\xi}{2}\right) \tag{7.95}$$

so that

$$\phi = \frac{2}{\sqrt{1-6\mu+2\mu e}}\ F\left(\frac{\xi}{2}, k\right) \tag{7.96}$$

At aphelion, when $\xi = \phi = 0$; and at the sigunlarity, $\xi \to \pi$ and

$$\phi = \phi_0 \equiv \phi = \frac{2}{\sqrt{1-6\mu+2\mu e}}\ K\left(k\right) \tag{7.97}$$

where $K(k)$ denotes the complete elliptic integral

$$K(k) \equiv \int_0^{\frac{\pi}{2}} \frac{dz}{\sqrt{1-k^2\,\sin^2\,z}} \tag{7.98}$$

Proper and coordinate time can be found through

$$\begin{aligned} \frac{d\tau}{d\xi} &= \frac{1}{Lu^2}\,\frac{d\phi}{d\xi} \\ \frac{dt}{d\xi} &= \frac{E}{Lu^2\left(1-ur_S\right)}\frac{d\phi}{d\xi} \end{aligned} \tag{7.99}$$

This means that the parts of the orbits with $r < r_S$ are inaccesible to a FIDO outside the horizon. Let us now consider two special cases:

- **e=0** Then $k^2 = 0$ and we can write

$$\xi = \sqrt{1-6\mu}\ \left(\phi - \phi_0\right) \tag{7.100}$$

as well as

$$u = \frac{1}{l} + \left(\frac{1}{r_S} - \frac{3}{l}\right)\ \sec^2\ \left(\frac{\sqrt{1-6\mu}}{2}\left(\phi-\phi_0\right)\right) \tag{7.101}$$

In spite of having zero eccenticity, this orbit is not a circle. Starting at an aphelion distance $\frac{1}{u_3}$ (so that

$$\frac{3}{2}r_S \leq \frac{1}{u_3} \leq 3r_S) \tag{7.102}$$

when $\phi = \phi_0$, it reaches the singularity when

$$\phi - \phi_0 = \frac{\pi}{\sqrt{1-6\mu}} \tag{7.103}$$

after circling one or more times. The circle at $\frac{1}{u_3}$ is the envelope of tehse solutions. The case $e = 0$ and $6\mu = 1$ must be treated separately. Then all three roots of the equation $f(u) = 0$ coincide and

$$u_1 = u_2 = \frac{1}{3r_S} \tag{7.104}$$

Then

$$\left(\frac{du}{d\phi}\right)^2 = r_S\left(u - \frac{1}{3r_S}\right)^3 \tag{7.105}$$

which means that

$$u = \frac{1}{3r_S} + \frac{4}{r_S\,(\phi-\phi_0)^2} \tag{7.106}$$

The orbit approaches the circle at $r = 3r_S$ asimptotically.

- **The case** $2\mu(3+e) = 1$ Then the roots of $f(u) = 0$ read

$$\begin{aligned} u_1 &= \frac{1-e}{l} \\ u_2 = u_3 &= \frac{1}{2r_S} - \frac{1-e}{2l} = \frac{1+e}{l} \end{aligned} \tag{7.107}$$

and the obvious substitution is

$$u = \frac{1+e+2e\ \mathrm{tg}^2\ \frac{\xi}{2}}{l} \tag{7.108}$$

Then, when $\xi = 0$,

$$u_1 = u_2 = u_3 = \frac{1+e}{l} \tag{7.109}$$

whereas when $\xi = \pi$,

$$u \to \infty \tag{7.110}$$

Besides,

$$\left(\frac{d\xi}{d\phi}\right)^2 = 4\mu e\ \sin^2\frac{\xi}{2} \tag{7.111}$$

so that

$$\phi = -\frac{1}{\sqrt{\mu e}}\ \log\ \mathrm{tg}\ \frac{\xi}{2} \tag{7.112}$$

This means that when $\xi = \pi$, then $\phi = 0$, wheras when $\xi \to 0$, then $\phi \to \infty$, and the preihelion is approached at

$$r \equiv r_P = \frac{l}{1+e} \tag{7.113}$$

The orbit approaches the circle at $r = r_P$ by spiralling around it an infinite number of times.

- **The orbits with imaginary eccentricities** In this case the equation

$$f(u) = 0 \tag{7.114}$$

allows one real root (positive for the bound orbits) and a pair of complex-conjugate complex roots. They can start at some finite aphelion distance, but they eventually fall into the siongularity, though they may circle the origin one or more times before doing so. We shall use the imaginary eccentricity ie (with $e > 0$) so that

$$u_1 = \frac{1}{r_S} - \frac{2}{l}$$
$$u_2 = \frac{1 + ie}{l}$$
$$u_3 = \frac{1 - ie}{l} \tag{7.115}$$

It is then plain that

$$\frac{1}{L^2} = 2\frac{1 - \mu(3 - e^2)}{lr_S} \tag{7.116}$$

as well as

$$\frac{1 - E^2}{L^2} = \frac{(1 - 4\mu)(1 + e^2)}{l^2} \tag{7.117}$$

$$\frac{E^2}{L^2} = 2\frac{(2\mu - 1)^2 + 4\mu^2 e^2}{lr_S} \tag{7.118}$$

(this implies that $l > 0$). As we are considering bound orbits, $E^2 < 1$, so that $4\mu < 1$ and

$$1 - 2\mu + \mu e^2 > 0 \tag{7.119}$$

(There is no upper limit to the value of e^2. The ODE to be solved is

$$\left(\frac{du}{d\phi}\right)^2 = r_S\left(u - \frac{1}{r_S} + \frac{2}{l}\right)\left((u - \frac{1}{l})^2 + \frac{e^2}{l^2}\right) \tag{7.120}$$

Under the substitution

$$u \equiv \frac{1 + e \ \mathrm{tg}\ \frac{\xi}{2}}{l} \tag{7.121}$$

Rememberging the range of u

$$\frac{1}{r_S} - \frac{2}{l} \leq u < \infty \tag{7.122}$$

the range of ξ is determined to be

$$\xi_0 \leq \xi < \pi \tag{7.123}$$

where

$$\sin\frac{\xi_0}{2} = -\frac{6\mu - 1}{\Delta}$$
$$\cos\frac{\xi_0}{2} = \frac{2\mu e}{\Delta} \tag{7.124}$$

with

$$\Delta \equiv \sqrt{(6\mu - 1)^2 + 4\mu^2 e^2} \tag{7.125}$$

The ODE reduces to

$$\frac{\delta \xi}{d\phi} = \pm\sqrt{2\left((6\mu - 1) + 2\mu e \, \sin \, \xi + (6\mu - 1) \, \cos \, \xi\right)} \tag{7.126}$$

and the solution is again given in terms of a Jacobian elliptic integral

$$\pm \, \phi = \frac{1}{\sqrt{\Delta}} \int^{\psi} \frac{dy}{\sqrt{1 - k^2 \, \sin^2 \, y}} \tag{7.127}$$

where

$$k^2 \equiv \frac{\Delta + 6\mu + 1}{2\Delta} \tag{7.128}$$

and

$$\sin^2 \, \psi = \frac{\Delta - 2\mu e \, \sin \, \xi - (6\mu - 1) \, \cos \, \xi}{\Delta + 6\mu - 1} =$$
$$\frac{\Delta + 6\mu - 1 - 2\left(2\mu e \, \sin \, \frac{\xi}{2} + (6\mu - 1) \, \cos \, \frac{\xi}{2}\right) \cos \, \frac{\xi}{2}}{\Delta + 6\mu - 1} \tag{7.129}$$

We have that

$$\sin^2 \, \psi = 1 \tag{7.130}$$

both when $\xi = \xi_0$ (aphelion) as well as when $\xi = \pi$ (the singularity). Besides

$$\sin^2 \, \psi = 0 \tag{7.131}$$

whenever

$$\xi = \mathrm{tg}^{-1} \, \frac{2\mu e}{6\mu - 1} \tag{7.132}$$

This means that ψ assumes the value $\psi = 0$ withinn the range of ξ. Therefore

$$-\frac{\pi}{2} \leq \psi \leq \frac{\pi}{2} \tag{7.133}$$

so that, assuming that $\phi = 0$ at the singularity, where $\xi = \pi$ and $\psi = \frac{\pi}{2}$

$$\phi = \frac{K(k) - F(\psi, k)}{\sqrt{\Delta}} \tag{7.134}$$

- **The unbound orbits ($E^2 > 1$)** In this case the equation $f(u) = 0$ must allow for a negative root. It is not possible that the three roots are negative. We can distinguish orbits of two kinds when two roots are positive. Orbits of the first kind restricted to the interval

$$0 \leq u \leq u_2 \tag{7.135}$$

which are the analogues of the hyperbolic orbits in Newtonian theory. Then there are the orbits of the second kind, with

$$u \geq u_3 \tag{7.136}$$

which are in essence no different from bound orbits of the second kind- When $u_2 = u_3$ the two orbits coalesce. Whe the equation $f(u) = 0$ allows for a pair of complex conjugate roots (besides a negative real root) the orbits have imaginary eccentricity.

- **Orbits of the first and second kind** The eccentricity $e \geq 1$. Then

$$\begin{aligned} u_1 &= -\frac{e-1}{l} \\ u_2 &= \frac{e+1}{l} \\ u_3 &= \frac{1}{r_S} - \frac{2}{l} \end{aligned} \tag{7.137}$$

Now we still have

$$1 - 6\mu - 2\mu e \geq 0 \tag{7.138}$$

(this dependes only on the fact that

$$u_1 \leq u_2 \leq u_3 \tag{7.139}$$

Besides

$$\frac{1}{L^2} = \frac{2\left(1 - \mu(3 + e^2)\right)}{l r_S} \tag{7.140}$$

$$\frac{E^2 - 1}{L^2} = \frac{(1 - 4\mu)(e^2 - 1)}{l^2} \tag{7.141}$$

It is a fact that

$$1 - \mu(3 + e^2) > 0 \tag{7.142}$$

as well as

$$\mu leq \frac{1}{4} \tag{7.143}$$

When

$$2\mu(3 + e) = 1 \tag{7.144}$$

That is, when $u_2 = u_3$, those become

$$\frac{4L^2}{r_S^2} = 4\frac{(3 + e)^2}{(3 - e)(e + 1)} \tag{7.145}$$

$$E^2 - 1 = \frac{e^2 - 1}{9 - e^2} \tag{7.146}$$

For these special orbits

$$1 \leq e < 3 \tag{7.147}$$

and the perihelion distances obey

$$\frac{3}{2}r_S < r_p \leq 2r_S \tag{7.148}$$

The impact parameter is given by

$$D^2 = \frac{L^2}{V^2} = \frac{L^2E^2}{E^2-1} \tag{7.149}$$

$$E^2 \equiv \frac{1}{1-V^2} \tag{7.150}$$

Making the substitution

$$u \equiv \frac{1 + e\ \cos\chi}{l} \tag{7.151}$$

Now when $u = 0$ then $\chi = \cos^{-1}\left(-\frac{1}{e}\right) \equiv \chi_\infty$ The perihelion still ocuurs at $\chi = 0$. Therefore

$$0 \leq \chi < \chi_\infty \tag{7.152}$$

Then

$$\phi = \frac{2\left(K(k) - F\left(\frac{\pi}{2} - \frac{\chi}{2}, k\right)\right)}{\sqrt{1-6\mu+2\mu e}} \tag{7.153}$$

The trajectory goes off at infinity asymptotically along the direction

$$\phi = \phi_\infty \equiv \frac{2\left(K(k) - F(\psi_\infty, k)\right)}{\sqrt{1-6\mu+2\mu e}} \tag{7.154}$$

where

$$\psi_\infty \equiv \frac{1}{2}\cos^{-1}\frac{1}{e} \tag{7.155}$$

- **The orbits with imaginary eccentricities** We have

$$\frac{1}{L^2} = \frac{2\left(1-\mu(3-e^2)\right)}{lr_S} \tag{7.156}$$

and

$$\frac{E^2-1}{L^2} = \frac{(4\mu-1)(1+e^2)}{l^2} \tag{7.157}$$

Now we have

$$\mu \geq \frac{1}{4} \tag{7.158}$$

as well as

$$1 - 3\mu + \mu e^2 > 0 \tag{7.159}$$

When $\mu > \frac{1}{3}$ is neccessary to impose

$$e^2 > 3 - \frac{1}{\mu} \tag{7.160}$$

The range of ξ nust be terminated at ξ_∞ where

$$\text{tg}\ \frac{\xi_\infty}{2} = -\frac{1}{e} \tag{7.161}$$

that is

$$\begin{aligned} \sin\frac{\xi_\infty}{2} &= -\frac{1}{\sqrt{1+e^2}} \\ \cos\frac{\xi_\infty}{2} &= \frac{e}{\sqrt{1+e^2}} \end{aligned} \tag{7.162}$$

When $\xi < \xi_\infty$, u becomes negative. The range of ξ is therefore

$$\xi_\infty < \xi \leq \pi \tag{7.163}$$

The solution still reads

$$\phi = \frac{K(k) - F(\psi, k)}{\sqrt{\Delta}} \tag{7.164}$$

The origin of ϕ is at the singularity, where $\xi = \pi$ and $\psi = \frac{\pi}{2}$. The lower limit of ψ, ψ_∞ reads

$$\sin^2\psi_\infty = \frac{1}{\Delta + 6\mu - 1}\left(\Delta + 6\mu - 1 - 2e^2\frac{4\mu - 1}{e^2 + 1}\right) \tag{7.165}$$

7.2 Null Geodesics

We have now

$$\frac{E^2}{1 - \frac{r_S}{r}} - \frac{\dot{r}^2}{1 - \frac{r_S}{r}} - \frac{L^2}{r^2} = 0 \tag{7.166}$$

that is

$$\left(\frac{dr}{d\tau}\right)^2 + \frac{L^2}{r^2}\left(1 - \frac{r_S}{r}\right) = E^2 \tag{7.167}$$

which must be considered together with

$$\begin{aligned} \left(1 - \frac{r_S}{r}\right)\frac{dt}{d\tau} &= E \\ \frac{d\phi}{d\tau} &= \frac{L}{r^2} \end{aligned} \tag{7.168}$$

All this boils down to

$$\left(\frac{du}{d\phi}\right)^2 = r_S u^3 - u^2 + \frac{1}{D^2} = f(u) \tag{7.169}$$

where the *impact parameter* is defined as

$$D \equiv \frac{D}{E} \tag{7.170}$$

- **The radial geodesics** Then

$$\frac{dr}{d\tau} = \pm E \tag{7.171}$$

and

$$\left(1 - \frac{r_S}{r}\right) \frac{dt}{d\tau} = E \tag{7.172}$$

That is

$$\frac{dr}{dt} = \pm \left(1 - \frac{r_S}{r}\right) \tag{7.173}$$

which means that

$$t = \pm r_* + C_\pm \tag{7.174}$$

in terms of the *tortoise coordinate*

$$r_* \equiv r + r_S \log \left(\frac{r}{r_S} - 1\right) \tag{7.175}$$

It is a fact of life that

$$r_* \to \infty \tag{7.176}$$

when

$$r \to r_S^+ \tag{7.177}$$

as well as

$$r_* \to r \tag{7.178}$$

whenever

$$r \to \infty \tag{7.179}$$

On the other hand,

$$r = \pm E\tau + K_\pm \tag{7.180}$$

whic means that radial geodesics cross the horizon in finite *proper parameter*.

The tangent vectors are

$$\begin{aligned}\frac{dt}{d\tau} &= \frac{r^2}{\Delta} E \\ \frac{dr}{d\tau} &= \pm E \\ \frac{d\theta}{d\tau} &= 0 \\ \frac{d\phi}{d\tau} &= 0\end{aligned} \tag{7.181}$$

- **The critical orbits**

  The roots of the equation $f(u) = 0$ obey

$$u_1 + u_2 + u_3 = \frac{1}{r_S}$$
$$u_1 u_2 u_3 = -\frac{1}{r_S D^2} \tag{7.182}$$

This means that there is at least one negative real root; the other two can be either real or o complex-conjugate pair. Let us begin with the case in which the two other real roots are degenerate. Indeed

$$u_2 = u_3 = \frac{2}{3r_S} \tag{7.183}$$

is a double root whenever

$$D^2 = \frac{27}{4} r_S^2 \tag{7.184}$$

In this case, moreover, the negative root is given by

$$u_1 = -\frac{1}{3r_S} \tag{7.185}$$

For such a D

$$\left.\frac{du}{d\phi}\right|_{u=u_2} = 0 \tag{7.186}$$

This means that a circular orbit of radius

$$r = \frac{3}{2} r_S \tag{7.187}$$

is an allowed null geodesic (however unstable). Indeed

$$\left(\frac{du}{d\phi}\right)^2 = r_S \left(u + \frac{1}{3r_S}\right)\left(u - \frac{2}{3r_S}\right)^2 \tag{7.188}$$

is satisfied by the substitution

$$u = -\frac{1}{3r_S} + \frac{1}{r_S} \operatorname{th}^2 \frac{\phi - \phi_0}{2} \tag{7.189}$$

Let us choose ϕ_0 such that

$$\operatorname{th}^2 \frac{\phi_0}{2} = \frac{1}{3} \tag{7.190}$$

Then, when $\phi = 0$,

$$u = 0 \tag{7.191}$$

and besides, when $\phi \to \infty$

$$u = \frac{2}{3r_S} \tag{7.192}$$

This means that a null geodesic approaching from infinity with impact parameter

$$D = \frac{3\sqrt{3}}{2} r_S \tag{7.193}$$

approaches a circle of radius $\frac{3}{2} r_S$ spiralling around it. Asspociated to this orbit, there must be another one that originates at the singularity at approaches frpom the opposite side the same asymptotic circle. This is given buy the substitution

$$u = \frac{2}{3r_S} + \frac{1}{r_S} \operatorname{tg}^2 \frac{\xi}{2} \tag{7.194}$$

Then the ODE reduces to

$$\left(\frac{d\xi}{d\phi} \right)^2 = \sin^2 \frac{\xi}{2} \tag{7.195}$$

so that

$$\operatorname{tg} \frac{\xi}{4} = e^{\frac{\phi}{2}} \tag{7.196}$$

and

$$u = \frac{2}{3r_S} + \frac{4e^\phi}{r_S \left(e^\phi - 1 \right)^2} \tag{7.197}$$

Along this orbit when $\phi = 0$

$$u \to \infty \tag{7.198}$$

and when $\phi \to \infty$

$$u \to \frac{2}{3r_S} \tag{7.199}$$

It is useful to define the *cone of avoidance* generated by those null rays passing through that point, since light rays inside the cone must necessarily cross the horizon and get trapped. Denoting by ψ the half-angle of the cone

$$\cot \psi = \frac{1}{r} \frac{dr}{d\phi} \frac{1}{1 - \frac{r_S}{r}} = -\frac{1}{u\sqrt{1 - r_S u}} \frac{du}{d\phi} = \frac{1}{\sqrt{\frac{r}{r_S} - 1}} \left(1 - \frac{2r}{3r_S} \right) \sqrt{1 + \frac{r}{3r_S}} \tag{7.200}$$

This means that when $r \to \infty$, then

$$\psi \sim \frac{3\sqrt{3}}{2} \frac{r_S}{r} \tag{7.201}$$

and when $r = \frac{3}{2}r_S$ then

$$\psi = \frac{\pi}{2} \tag{7.202}$$

Finally, at the horizon $r = r_S$

$$\psi = \pi \tag{7.203}$$

The cone of avoidance narrows as we approach the horizon.

- **Geodesics of the first kind**

  Consider now the case when all the roots of the cubic equation $f(u) = 0$ are real and the two positive roots are different.

$$\begin{aligned} u_1 &= \frac{P - r_S - Q}{2r_S P} \\ u_2 &= \frac{1}{P} \\ u_3 &= \frac{P - r_S + Q}{2r_S P} \end{aligned} \tag{7.204}$$

where P denotes the perihelion distance. Q will be determined in a moment.

The ordering iof the roots requires that

$$Q + P - 3r_S > 0 \tag{7.205}$$

Evaluating

$$f(u) = r_S\,(u - u_1)\,(u - u_2)\,(u - u_3) \tag{7.206}$$

we learn that

$$Q^2 = (P - r_S)\,(P + 3r_S) \tag{7.207}$$

and

$$\frac{1}{D^2} = \frac{Q^2 - (P - r_S)^2}{4r_S P^3} \tag{7.208}$$

so that

$$D^2 = \frac{P^3}{P - r_S} \tag{7.209}$$

The former inequality now implies

$$(P - r_S)\,(P + 3r_S) > (P - 3r_S)^2 \tag{7.210}$$

that is

$$P > \frac{3}{2}r_S \tag{7.211}$$

as well as

$$D > \frac{3\sqrt{3}}{2}r_S \equiv D_c \tag{7.212}$$

The orbits lie entirely outside the circle $r = \frac{3}{2} r_S$. Let us now make the substitution

$$u - \frac{1}{P} = -\frac{Q - P + 3r_S}{4r_S P}\,(1 + \cos\,\chi) \tag{7.213}$$

in such a way that the perihelion corresponds to $\chi = \pi$

$$u = \frac{1}{P} \tag{7.214}$$

and $u = 0$ whenever $\chi = \chi_\infty$

$$\sin^2\,\frac{\chi_\infty}{2} \equiv \frac{Q - P + r_S}{Q - P + 3r_S} \tag{7.215}$$

and the ODE reduces to

$$\left(\frac{d\chi}{d\phi}\right)^2 = \frac{Q\,(1 - k^2\,\sin^2\,\frac{\chi}{2})}{2Q} \tag{7.216}$$

with

$$k^2 = \frac{Q - P + 3r_S}{2Q} \tag{7.217}$$

This means that

$$\phi = 2\sqrt{\frac{P}{Q}}\left(K(k) - F\left(\frac{\chi}{2}, k\right)\right) \tag{7.218}$$

where the origin of ϕ has been chosen at perohelion passage when $\chi = \pi$. The asymptotic value of $\phi, at \to \infty$, is given by

$$\phi_\infty = 2\sqrt{\frac{P}{Q}}\left(K(k) - F\left(\frac{\chi_\infty}{2}, k\right)\right) \tag{7.219}$$

Let us compute the asymptotic value of ϕ_∞ when $P \to \frac{3}{2} r_S$ as well as for $\frac{P}{r_S} >> 1$. It is a fact that when $P = \frac{3}{2} r_S$ then

$$\begin{aligned}
Q &= \frac{3}{2} r_S \\
D &= \frac{3\sqrt{3}}{2} r_S \\
k^2 &= 1 \\
\sin^2\,\frac{\chi_\infty}{2} &= \frac{1}{3} \\
F\left(\frac{\chi_\infty}{2}, 1\right) &= \frac{1}{2}\,\log\,\frac{\sqrt{3}+1}{\sqrt{3}-1}
\end{aligned} \tag{7.220}$$

We next find that if

$$
\begin{aligned}
P &= \frac{r_S}{2}\,(3+\epsilon) \\
Q &= \frac{r_S}{2}\left(3+\frac{5}{3}\epsilon\right) \\
(k')^2 &= 1-k^2 = \frac{4}{9}\epsilon
\end{aligned}
\tag{7.221}
$$

And the asymptotic relation

$$K(k) \to \log\frac{4}{k'} = \log 6 - \frac{1}{2}\log\epsilon \tag{7.222}$$

Using those, we learn that

$$\phi_\infty = \frac{1}{2}\,\log\,\frac{6^4\sqrt{3}(\sqrt{3}-1)^2}{2(\sqrt{3}+1)^2} - \frac{1}{2}\,\log\frac{2\epsilon D}{r_S} \tag{7.223}$$

that is

$$\frac{2\epsilon D}{r_S} \to \frac{6^4\sqrt{3}(\sqrt{3}-1)^2}{2(\sqrt{3}+1)^2}\;e^{-2\phi_\infty} \tag{7.224}$$

So that if we write

$$\phi_\infty \equiv \frac{1}{2}\,(\pi+\Theta)\,asny \tag{7.225}$$

we learn that

$$\frac{2\epsilon D}{r_S} = 648\sqrt{3}\frac{(\sqrt{3}-1)^2}{\sqrt{3}-1)^2}\;e^{-\pi}\;e^{-\Theta} = 3.4823\;e^{-\Theta} \tag{7.226}$$

The geodesics which have been deflected by $\Theta + 2\pi n$ have impact parameters

$$D_n \equiv D_c + 3.4823\;\frac{r_S}{2}\;e^{-(\Theta+2\pi n)} \tag{7.227}$$

Similar arguments show that for $P >> r_S$ the deflection is given by

$$\Theta \sim \frac{2r_S}{D} \tag{7.228}$$

(this yields the celebrated *deflection of the light rays*) and

$$D \sim P\left(1+\frac{r_S}{2P}\right) \tag{7.229}$$

- **The geodesics of the second kind**

  Now the range is

$$u_3 \le u < \infty \tag{7.230}$$

Let us make the substitution

$$u = \frac{1}{P} + \frac{Q + P - 3r_S}{2r_S P} \sec^2 \frac{\chi}{2} \tag{7.231}$$

This means that u is at aphelion when

$$u = u_3 = \frac{Q + P - r_S}{2Pr_S} \tag{7.232}$$

and $\chi = 0$, and that $u \to \infty$ when $\chi = \pi$. Then

$$\phi = 2\sqrt{\frac{P}{Q}}\, F\left(\frac{\chi}{2}, k\right) \tag{7.233}$$

where the origin of ϕ is now at aphelion passage.

- **The orbits with imaginary eccentricities and impact parameters less that $\frac{3\sqrt{3}}{2} r_S$**

  Let us consider now the case when the equation $f(u) = 0$ has a pair of complex conjugate roots (besides a negative real toot)

$$f(u) = r_S \left(u - \frac{1}{r_S} + \frac{2}{l}\right)\left((U - \frac{1}{l})^2 + \frac{e^2}{l^2}\right) \tag{7.234}$$

so that

$$\begin{aligned} l - \frac{r_S}{2}\left(3 - e^2\right) &= 0 \\ \frac{1}{r_S D^2} &= \left(\frac{2}{l} - \frac{1}{r_S}\right)\frac{1 + e^2}{l^2} \end{aligned} \tag{7.235}$$

or in terms of $\mu \equiv \frac{r_S}{2l}$

$$\begin{aligned} e^2 &= \frac{3\mu - 1}{\mu} \\ \frac{4D^2}{r_S^2} &= \frac{1}{\mu(4\mu - 1)^2} \end{aligned} \tag{7.236}$$

This requires

$$\mu > \frac{1}{2} \tag{7.237}$$

and

$$D < \frac{3\sqrt{3}}{2} r_S \tag{7.238}$$

The solution to the ODE yields

$$\phi_\infty = \frac{K(k) - F(\psi_\infty, k)}{\sqrt{\Delta}} \tag{7.239}$$

where now

$$\sin^2\,\psi_\infty = \frac{\Delta + 1}{\Delta + 6\mu + 1} \tag{7.240}$$

and where

$$\Delta \equiv \sqrt{48\mu^2 - 16\mu + 1} \tag{7.241}$$

7.3 Rindler space.

Let us consider an accelerated observer in two dimensional flat space

$$
\begin{aligned}
t &= \frac{1}{a} sinh\, a\tau \\
x &= \frac{1}{a} cosh\, a\tau
\end{aligned} \tag{7.242}
$$

This is such that the four-velocity is given by

$$
u = \left(cosh\, a\tau \quad , \; sinh\, a\tau\right) \tag{7.243}
$$

normalized to

$$
u^2 = 1 \tag{7.244}
$$

and the acceleration

$$
\dot{u} \equiv a \left(sinh\, a\,\tau \quad , \; cosh\, a\tau\right) \tag{7.245}
$$

obeys

$$
\begin{aligned}
a^2 &= -1 \\
a.u &= 0
\end{aligned} \tag{7.246}
$$

In *comoving* coordinates, id est, adapted to the four-velocity,

$$
u = \frac{\partial}{\partial \xi^0} \tag{7.247}
$$

the worldline of the accelerated observer is

$$
\begin{aligned}
\xi^0(\tau) &= \tau \\
\xi^1(\tau) &= 0
\end{aligned} \tag{7.248}
$$

In general

$$
\begin{aligned}
t &= \frac{e^{a\xi^1}}{a}\, sinh\, a\xi^0 \equiv \rho \sinh\, \omega \\
x &= \frac{e^{a\xi^1}}{a} cosh\, a\xi^0 \equiv \rho\, \cosh\, \omega
\end{aligned} \tag{7.249}
$$

so that the value of the coordinate ξ^1 (or ρ) tells us which hyperbola we are talking about

$$
t^2 - x^2 = -\frac{e^{2a\xi^1}}{a^2} = -\rho^2 \tag{7.250}
$$

In terms of these coordinates the Minkowski metric reads

$$
ds^2 = dt^2 - dx^2 = e^{2a\xi^1}\left(d\xi_0^2 - d\xi_1^2\right) = \rho^2 d\omega^2 - d\rho^2 - dx_\perp^2 \tag{7.251}
$$

When

$$-\infty \leq \xi^0 \leq \infty$$
$$-\infty \leq \xi^1 \leq \infty \tag{7.252}$$

only one quarter of the original Minkowski space has been covered, namely the one corresponding to

$$|t| \leq x \tag{7.253}$$

This is called *Rindler's wedge* or *Rindler space*. The lightcone plays the role of the *event horizon*.

It is easy to realize that the behavior of (a piece of) Schwarzschild when $r \sim r_S$ is similar to the behavior of Rindler when $x \sim 0$.

7.4 Painlevé-Gullstrand coordinates.

It is possible to find coordinates such that Schwarzschil's metric read

$$ds^2 = dT^2 - \left(dr + \sqrt{\frac{r_S}{r}}dT\right)^2 - r^2 d\Omega_2^2 \tag{7.254}$$

which is manifestly regular at the horizon.

Besides, the spacelike hypersurfaces

$$T = \text{constant} \tag{7.255}$$

are flat. Time like radial geodesics can be obtained from the action principle

$$L = \dot{T}^2 - \left(\dot{r} + \sqrt{\frac{r_S}{r}}\dot{T}\right)^2 \tag{7.256}$$

with the first integral

$$\dot{T}^2 - \left(\dot{r} + \sqrt{\frac{r_S}{r}}\dot{T}\right)^2 = 1 = \dot{T}^2\left(1 - \frac{r_S}{r}\right) - \dot{r}^2 - 2\dot{r}\dot{T}\sqrt{\frac{r_S}{r}} \tag{7.257}$$

This action principle immediatly tells us that

$$\dot{T} - \left(\dot{r} + \sqrt{\frac{r_S}{r}}\dot{T}\right)\sqrt{\frac{r_S}{r}} = \text{constant} \tag{7.258}$$

It is clar that a particular solution is given by

$$\dot{T} = 1$$
$$\dot{r} + \sqrt{\frac{r_S}{r}} = 0 \tag{7.259}$$

This means that it is possible to identify the coordinate T with the proper time along such geodesics. Then

$$\dot{r} = -\sqrt{\frac{r_S}{r}} \tag{7.260}$$

reaches the velocity of light at the horizon $r = r_S$. The four velocity remains normalized all the way.

The metric on constant radius hypersurfaces $r = R$ reads

$$ds^2 = \left(1 - \frac{r_S}{R}\right) dT^2 - R^2\, d\Omega_2^2 \tag{7.261}$$

When $R = r_S$ the metric degenerates and becomes two-dimensional. The time translation vector

$$\frac{\partial}{\partial T} \tag{7.262}$$

then becomes null.

On the other hand, it is possible to see directly that no signal can travel at infinity from the region inside the horizon, $r < r_S$. The coefficient of dT^2 is negative, so that the only way the interval gets of avoiding being spacelike is through the cross-term

$$dTdr < 0 \tag{7.263}$$

This means that the future ($dT > 0$) lies entirely within the horizon and leads eventually towards the singularity.

Null radial curves obey

$$\frac{dr}{dT} + \sqrt{\frac{r_S}{r}} = \pm 1 \tag{7.264}$$

When the above sign is -1,then the whole range of values of the coordinate r is covered as t varies.

For those null curves governed by the +1 sign, this is not so. At large distances from the center

$$r > r_S \ \Rightarrow\ \frac{dr}{dT} > 0\ \Rightarrow r_S < r < \infty \tag{7.265}$$

Inside the horizon,

$$r < r_S \ \Rightarrow\ \frac{dr}{dT} < 0\ \Rightarrow 0 < r < r_S \tag{7.266}$$

Exactly at the horizon the derivative vanishes

$$r = r_S \ \Rightarrow\ \frac{dr}{dT} = 0 \tag{7.267}$$

Timelike radial geodesics when parameterized by the proper time obey

$$\dot{T}^2 - \left(\dot{r} + \sqrt{\frac{r_S}{r}}\dot{T}\right)^2 = 1 \tag{7.268}$$

7.5 The maximal analytic extension and black holes.

There is a maximally analytic extension of Schwarzschild's geometry discovered by the american scientist Martin Kruskal in 1959. It is believed to represent the black hole metric.

$$\begin{aligned} u &\equiv t - r^* \\ v &\equiv t + r^* \\ r^* &\equiv r + r_S \log\left(\frac{r}{r_S} - 1\right) \end{aligned} \tag{7.269}$$

The coordinate r_* is the Regge-Wheeler *tortoise* coordinate. The metric is

$$ds^2 \equiv \left(1 - \frac{r_S}{r}\right) du dv - r^2 d\Omega_2^2 \tag{7.270}$$

where the function $r(u, v)$ is defined through

$$r + r_S \log\left(\frac{r}{r_S} - 1\right) = \frac{v - u}{2} \tag{7.271}$$

It is a fact that

$$dr_* = \frac{dr}{1 - \frac{r_S}{r}} \tag{7.272}$$

There is a further change of coordinates

$$\begin{aligned} U &\equiv e^{-\frac{u}{2r_S}} \\ V &\equiv e^{\frac{v}{2r_S}} \end{aligned} \tag{7.273}$$

The function $r(U, V)$ is to be obtained from

$$e^{\frac{r}{r_S}}\left(\frac{r}{r_S} - 1\right) = UV = \frac{r}{r_S} e^{\frac{r}{r_S}}\left(1 - \frac{r_S}{r}\right) \tag{7.274}$$

The singularity, $r = 0$ corresponds to

$$UV = -1 \tag{7.275}$$

and the horizon, $r = r_S$ to

$$UV = 0 \tag{7.276}$$

Then

$$dUdV = \frac{1}{4r_S^2} dvdu \; \frac{r}{r_S} \; r^{\frac{r}{r_S}} \left(1 - \frac{r_S}{r}\right) \tag{7.277}$$

Finally, Kruskal's metric is

$$ds^2 \equiv -\frac{4r_S^3}{r(U, V)} e^{-\frac{r}{r_S}} dUdV - r^2(U, V) d\Omega_2^2 \tag{7.278}$$

Once this point has been reached, we can extend the values of the coordinates to the whole real line

$$-\infty \leq U, V \leq \infty \tag{7.279}$$

Region I corresponds to $U \geq 0$ $V \geq 0$. Region II to $U \leq 0$ $V \geq 0$. Region III to $U \geq 0$ $V \leq 0$. In region I of an eternal BH ($U > 0 \& V > 0$)

$$\frac{U}{V} = e^{-\frac{t}{r_S}} \tag{7.280}$$

This means that constant t surfaces are straight lines through the origin in Kruskal spacetime. They have a piece in region I and another piece in region IV. Actually

$$P : (U, V) \rightarrow (-U, -V) \tag{7.281}$$

is an isometry, so that region IV is isometric to region I.

If we rewrite the metric in *isotropic coordinates*

$$r \equiv \left(1 + \frac{r_S}{4\rho}\right)^2 \rho \tag{7.282}$$

They cover regions I and IV, because ρ becomes complex for $r < r_S$.Actually

$$2\rho = r - \frac{r_S}{2} \pm \sqrt{\left(r - \frac{r_S}{2}\right)^2 - \frac{r_S^2}{4}} \tag{7.283}$$

Note that to each value of the radial coordinate r there are two values of the coordinate ρ. These are related by the isometry

$$\rho \rightarrow \frac{r_S^2}{16\rho} \tag{7.284}$$

whose fixed point is

$$\rho = \frac{r_S}{4} \tag{7.285}$$

(This is nothing but the old isometry P). The metric reads

$$ds^2 = \left(\frac{1 - \frac{r_S}{4\rho}}{1 + \frac{r_S}{4\rho}}\right)^2 dt^2 - \left(1 + \frac{r_S}{4\rho}\right)^4 \left(d\rho^2 + \rho^2 \; d\Omega_2^2\right) \tag{7.286}$$

The constant time surfaces are conformally flat. When $\rho \rightarrow \frac{r_S}{4}$ from either side the radius of a 2-sphere of constant ρ on a constant time surface decreases to a minimum of r_S when $\rho = \frac{r_S}{4}$, which corresponds to a minimal 2-sphere. It is the midpoint of the *Einstein-Rosen bridge* connecting spatial chapters of regions I and IV. This is one of the simplest instances of a *wormhole* connecting two asymptotically flat regions of space-time

The Killing associated to the time translation, $\xi = \partial_t$ in Kruskal coordinates reads

$$\xi \equiv \frac{1}{2r_S}\left(-V\frac{\partial}{\partial V} + U\frac{\partial}{\partial U}\right) \tag{7.287}$$

Its square is

$$\xi^2 = \frac{UV}{4r_S^2} \tag{7.288}$$

It is timelike in regions I and IV; spacelike in regions II and III and null in $r = r_S$, that is when $U = 0$ or $V = 0$. This last set is the fixed set on k. It is easy to check that when $U = 0$, $\xi = \frac{\partial}{\partial v}$ whereas when $V = 0$, $\xi = \frac{\partial}{\partial u}$. This means that v is the natural parameter on $U = 0$. The *Boyer-Kruskal axis*, $U = V = 0$ (which is a 2-sphere) is a fixed point of the Killing vector.

Let us now consider null surfaces, that is

$$S(x) = C \tag{7.289}$$

For example, in Kruskal spacetime,

$$\mathcal{N} \equiv \{U = 0\} \cup \{V = 0\} \tag{7.290}$$

This means that the normal vector, which is proportional to

$$l \equiv g^{\mu\nu}\partial_\nu S \partial_\mu \tag{7.291}$$

is null, $l^2 = 0$. Null hyperfurfaces have a curious property. *tangent vectors* are by definition those orthogonal to the normal vector.

$$t.l = 0 \tag{7.292}$$

In Kruskal, the normal to $U = 0$ is

$$l \sim \frac{\partial}{\partial V} \tag{7.293}$$

and the normal to $V = 0$

$$l \sim \frac{\partial}{\partial U} \tag{7.294}$$

This means that l itself is also a tangent vector, so that there must exist a null curve $x^\mu = x^\mu(\lambda)$

$$t^\mu = \frac{dx^\mu}{d\lambda} \tag{7.295}$$

It is a fact of life that these curves are geodesic.

$$l^\lambda \nabla_\lambda l^\mu = g^{\mu\nu} l^\lambda \nabla_\lambda \partial_\nu S = g^{\mu\nu} l^\lambda \nabla_\nu \partial_\lambda S = \nabla^\mu l^2 \tag{7.296}$$

Now we all know that l^2 is constant on the surface. This means that the derivative in the direction of any tangent vector must vanish

$$t^\lambda \nabla_\lambda l^2 = 0 \tag{7.297}$$

which in turn tells us that

$$\nabla_\lambda l^2 \sim l_\lambda \tag{7.298}$$

and

$$l^\lambda \nabla_\lambda l^\mu \sim l^\mu \tag{7.299}$$

and it is possible to nomalize in such a way that the parameter is an affine one

$$l^\lambda \nabla_\lambda l^\mu = 0 \tag{7.300}$$

Thse null geodesics are called the *generators* of the null surface.

A *Killing horizon* is a null surface $\mathcal{N}$ such that the Killing vector ξ is normal to $\mathcal{N}$ on $\mathcal{N}$. This means that there exists a function f such that on $\mathcal{N}$

$$\xi = f(x) l \tag{7.301}$$

which in turn conveys the fact that

$$\xi.\nabla\xi^\mu = \xi^\lambda \nabla_\lambda \log\, f\ \xi^\mu \equiv \kappa \xi^\mu \tag{7.302}$$

The quantity κ is called the *surface gravity*.

Then on $U = 0$ in terms of the affine parameter of the geodesic

$$\xi \equiv f l = \frac{1}{2r_S} V \frac{\partial}{\partial V} \tag{7.303}$$

This means that

$$f(x) = \frac{1}{2r_S} V \tag{7.304}$$

and the surface gravity

$$\kappa \equiv \xi^\lambda \nabla_\lambda \log\, V = \frac{\partial_t V}{V} = \frac{1}{2r_S} \tag{7.305}$$

Otherwise in $V = 0$

$$\xi \equiv f l = -\frac{1}{2r_S} U \frac{\partial}{\partial U} \tag{7.306}$$

Indeed, let us condider the proper acceleration of a FIDO. Its velocity is given by

$$u^\mu \equiv \frac{\xi^\mu}{A} \tag{7.307}$$

where

$$A^2 \equiv \xi^2 \tag{7.308}$$

First of all, lets us show that A is time independent.

$$\dot{A}^2 \equiv u^\lambda \nabla_\lambda \xi^2 = 2\xi^\lambda \xi^\alpha \nabla_\lambda \xi_\alpha = 0 \tag{7.309}$$

because of KIlling's equation.

Now it is plain that

$$\dot{u}^\mu = u^\lambda \nabla_\lambda u^\mu = \xi^\lambda \frac{1}{A^2} \nabla_\lambda \xi^\mu = -\frac{1}{A^2} \xi^\lambda \nabla^\mu \xi_\lambda = -\frac{1}{2A^2} \nabla^\mu \xi^2 = -\nabla^\mu \log\, A^2 \tag{7.310}$$

There is a theorem by Frobenius that guarantees that

$$\xi_{[\mu} \nabla_\nu \xi_{\rho]} \Big|_{\mathcal{N}} = 0 \tag{7.311}$$

The theorem is simplest in the notation of differential forms, and it simply states that

$$\xi \wedge d\xi = 0 \tag{7.312}$$

whenever ξ is hypersurface normal. Taking into account that Killingness means that

$$\nabla_\mu \xi_\nu = \nabla_{[\mu} \xi_{\nu]} \tag{7.313}$$

Frobenius can be written as

$$\xi_\rho \nabla_\mu \xi_\nu + \xi_\mu \nabla_\nu \xi_\rho - \xi_\nu \nabla_\mu \xi_\rho |_{\mathcal{N}} = 0 \tag{7.314}$$

Multiplying by $\nabla^\mu \xi^\nu$ we learn that

$$\xi_\rho \nabla^\mu \xi^\nu \nabla_\mu \xi_\nu |_{\mathcal{N}} = -2\ \nabla^\mu \xi^\nu \xi_\mu \nabla_\nu \xi_\rho |_{\mathcal{N}} = -2\ \kappa \xi . \nabla \xi_\rho |_{\mathcal{N}} = -2\kappa^2 \xi_\rho \Big|_{\mathcal{N}} \tag{7.315}$$

so that the formula for the surface gravity reads

$$\kappa^2 = -\frac{1}{2} \nabla^\mu \xi^\nu \nabla_\mu \xi_\nu \Big|_{\mathcal{N}} \tag{7.316}$$

Coming back to Kruskal's spacetime, since $l.\mathcal{N} = 0$, $\mathcal{N}$ is a Killing horizon. Besides, $l.\nabla l = 0$, so that the surface gravity is, on $U = 0$,

$$\kappa \equiv k.\nabla \log\, f = \frac{1}{2r_S} V \frac{\partial}{\partial V} \log\, |V| = \frac{1}{2r_S} \tag{7.317}$$

And on $V = 0$

$$\kappa \equiv k.\nabla \log\, f = -\frac{1}{2r_S} U \frac{\partial}{\partial U} \log\, |U| = -\frac{1}{2r_S} \tag{7.318}$$

It is also easy to show that the surface gravity is constant on orbits of ξ. Consider a tangent to $\mathcal{N}$.

$$t.\nabla \kappa^2 = -\ \nabla^\mu \xi^\nu t^\lambda \nabla_\lambda \nabla_\mu \xi_\nu \Big|_{\mathcal{N}} = -\nabla^\mu \xi^\nu t^\lambda R_{\nu\mu\rho\sigma} \xi^\sigma \tag{7.319}$$

Choosing now $t = \xi$, we are done:

$$\xi.\nabla \kappa^2 = 0 \tag{7.320}$$

The surface gravity is essentially Hawking's temperature, of which more later

$$\kappa = 2\pi T_h \tag{7.321}$$

It is said that the horizon is *nondegenerate* when the surface gravity is nonvanishing. Otherwise it is *degenerate*. Let us assume that $\kappa \neq 0$ on one orbit of ξ in $\mathcal{N}$. Then this orbit coincides with only part of a null generator of $\mathcal{N}$. We define the *group parameter*, α as such that

$$\xi = \frac{\partial}{\partial \alpha} \tag{7.322}$$

This means that the relationship between the affine parameter and the group parameter we just defined is given by the old function f

$$f = \frac{d\lambda}{d\alpha} \tag{7.323}$$

Then

$$\lambda = \pm e^{\kappa\alpha} \tag{7.324}$$

so that when $-\infty \leq \alpha \leq \infty$, we cover only thone of the pieces of the generator of $\mathcal{N}$, either $\lambda > 0$ or else $\lambda < 0$. The bifurcation point $\lambda = 0$ is a fixed point of ξ, which can be shown to be a 2-sphere, the *bifurcation two-sphere*, $\mathcal{B}$. This is the BK axis in the Kruskal case. This is a *bifurcate Killing horizon.*

It is also a fact that in this case κ is constant on $\mathcal{N}$.

Surface gravity is not a property of $\mathcal{N}$ alone; it also depends on the normalization of ξ.

In the asymptotically flat case there is a natural normalization, namely

$$lim_{r\rightarrow\infty}\ \xi^2 = 1 \tag{7.325}$$

This is the one we have been using in the Kruskal example.

7.6 The Kerr metric and the Newman-Janis transformation.

The Newman-Janis transformation is a complex change of coordinates from Schwarzschild metric to the Kerr solution, which represents the metric sourced by a stationary (but not static) object with non-vanishing angular momentum. We start with the contravariant form of the metric in tortoise coordinates

$$u \equiv t - r - r_s \log\left(r - r_S\right) \tag{7.326}$$

namely

$$ds^2 = \left(1 - \frac{r_S}{r}\right) du^2 + 2du dr - r^2 d\Omega_2^2 \tag{7.327}$$

$$g^{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -(1-\frac{r_S}{r}) & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{r^2 \sin^2\theta} \end{pmatrix} \tag{7.328}$$

in a Newman-Penrose tetrad

$$g^{\mu\nu} = l^\mu n^\nu + l^\nu n^\mu - m^\mu \bar{m}^\nu - m^\nu \bar{m}^\mu \tag{7.329}$$

with

$$\begin{aligned} l &= \frac{\partial}{\partial r} \\ n &= \frac{\partial}{\partial u} - \frac{1}{2}\left(1 - \frac{r_s}{r}\right)\frac{\partial}{\partial r} \\ m &= \frac{1}{r\sqrt{2}}\left(\frac{\partial}{\partial \theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial \phi}\right) \end{aligned} \tag{7.330}$$

Now we assume that the radial coordinate can take complex values and we rewite

$$\begin{aligned} l &= \frac{\partial}{\partial r} \\ n &= \frac{\partial}{\partial t} - \frac{1}{2}\left(1 - \frac{r_S}{2}\left(\frac{1}{r} + \frac{1}{\bar{r}}\right)\right)\frac{\partial}{\partial r} \\ m &= \frac{1}{\bar{r}\sqrt{2}}\left(\frac{\partial}{\partial \theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial \phi}\right) \end{aligned} \tag{7.331}$$

Now we perform the change of coordinates

$$\begin{aligned} r' &= r + ia\cos\theta \\ u' &= u - ia\cos\theta \end{aligned} \tag{7.332}$$

Now let us insist in r' as well as u' to be real. Then

$$\begin{aligned} l' &= \frac{\partial}{\partial r'} \\ n' &= \frac{\partial}{\partial t} - \frac{1}{2}\left(1 - r_S\left(\frac{r'}{(r')^2 + a^2\cos^2\theta}\right)\right)\frac{\partial}{\partial r'} \\ m &= \frac{1}{(r' + ia\cos\theta)\sqrt{2}}\left(ia\sin\theta\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial \theta}\right) + \frac{i}{\sin\theta}\frac{\partial}{\partial \phi}\right) \end{aligned} \tag{7.333}$$

which can be shown to be equivalent to the Kerr metric.

This somewhat mysterious transformation ins related to the fact that Kerr's metric can be written as

$$g^{\mu\nu} = g_0^{\mu\nu} + \lambda^2 l^\mu l^\nu \tag{7.334}$$

whete $l \equiv \frac{\partial}{\partial r}$ and

$$\lambda^2 = \frac{r_S r}{r^2 + a^2 \cos^2\theta} \tag{7.335}$$

where $g_0^{\mu\nu}$ is flat.

7.7 Hawking's temperature.

Let us consider the *euclidean* Schwarzschild's metric

$$ds^2 = \left(1 - \frac{r_S}{r}\right) dx_4^2 + \frac{1}{1 + \frac{r_S}{r}} dr^2 + r^2 d\Omega_2^2 \tag{7.336}$$

Close to

$$r \equiv r_S + x \tag{7.337}$$

the metric reads

$$ds^2 = r_S^2 \left(\frac{x}{r_S^3} dx_4^2 + \frac{1}{x r_S} dx^2 + d\Omega_2^2\right) \tag{7.338}$$

Put

$$x \equiv r_S \frac{R^2}{4} \tag{7.339}$$

Then

$$ds^2 = r_S^2 \left(\frac{R^2}{4 r_S^2} dx_4^2 + dR^2 + d\Omega_2^2\right) \tag{7.340}$$

In order that this metric is regular ($S_2 \times S_2$) we need

$$\frac{1}{2 r_S} x_4 \tag{7.341}$$

to be an angle. It is well known on the other hand that a quantum field theory at finite temperature can be represented as an euclidean theory with periodic euclidean time coordinate. The period of the euclidean time coordinate is the inverse temperature. Then

$$\beta = 4\pi r_S \tag{7.342}$$

This suggests that were such an interpretation possible, this would be the associated temperature. With all factors included

$$T_H = \frac{\hbar c}{8\pi G M} \tag{7.343}$$

It is outside the bounds of the present course to show that vacuum fluctuations with

$$\Delta E \Delta t \sim \hbar \tag{7.344}$$

allow one of the components of the particle-antiparticle pair to materialize and fall inside tthe horizon, in such a way that the other component of the pair can escape the hole. This process constitutes the Hawking radiation, which is a black body radiation at $T = T_H$.

8

Spaces of constant curvature.

Let us first consider the simpler case of ordinary spheres embedded in euclidean space.

The sphere $\mathbb{S}_n$ of radius l embedded in $\mathbb{R}_{n+1}$ is defined thtough the equations

$$\sum_{A=1}^{A=n+1} X_A^2 = l^2 \tag{8.1}$$

where a point in $\mathbb{R}^{n+1}$ is represented by the (n+1) coordinates $(X_1, X_1, \ldots X_{n+1})$. We are all used to *polar coordinates*, a generalization of the polar angles (θ, ϕ) for the two-sphere S_2. We need n angles to define a point in the n sphere. We shall call these angles, $\theta_1 \ldots \theta_n$, and to be specific,

$$\begin{aligned} X_{n+1} &= r\, cos\, \theta_n \\ X_n &= r\, sin\, \theta_n\, cos\, \theta_{n-1} \\ &\ldots \\ X_2 &= r\, sin\, \theta_n\, sin\, \theta_{n-1} \ldots cos\, \theta_1 \\ X_1 &= r\, sin\, \theta_n\, sin\, \theta_{n-1} \ldots sin\, \theta_1 \end{aligned} \tag{8.2}$$

(were we to use r itself as the radial coordinate, those would be polar coordinates in $\mathbb{R}_{n+1}$, in them the equation of the sphere is simply

$$r = l = \text{constant} \tag{8.3}$$

The X_{n+1} axis is special in those coordinates; any axis however can be taken as the X_{n+1} axis. The metric induced on S^n by the euclidean metric in $\mathbb{R}_{n+1}$ is

$$ds_n^2 = \delta_{AB} dX^A(\theta) dX^B(\theta) = d\theta_n^2 + sin^2\, \theta_n d\theta_{n-1}^2 + \ldots + sin^2\, \theta_n\, sin^2\, \theta_{n-1} \ldots sin^2\, \theta_2 d\theta_1^2 \tag{8.4}$$

id est, in a recurrent form

$$ds_1^2 = d\theta_1^2$$
$$ds_n^2 = d\theta_n^2 + sin^2\theta_n \, ds_{n-1}^2 \tag{8.5}$$

The tangent space is a vector space T_n with the same dimension as the manifold itself. It can be defined as the set of vectors orthogonal to the normal vector

$$n_A = X_A \tag{8.6}$$

In general, given a surface in $\mathbb{R}_{n+1}$ defined by the equation

$$f(X_A) = 0 \tag{8.7}$$

the normal vector is given gy the gradient

$$n_A \equiv \partial_A f \tag{8.8}$$

To come back to the sphere, the tangent space is defined as those vectors that obey

$$\sum_A x_A t_A = 0 \tag{8.9}$$

Particularizing to the two-dimensional sphere, the tangent space is now the tangent plane, that is, the set of vector in $\mathbb{R}_3$ such that

$$n_1.\sin\,\theta\,\cos\,\phi + n_2.\sin\,\theta\,\sin\,\phi + n_0\cos\,\theta = 0 \tag{8.10}$$

In the North or South pole ($\theta = 0, \pi$) the tangent plane is just the plane

$$X_0 = \pm l \tag{8.11}$$

that is, the set of vectors

$$(0, n_1, n_2) \tag{8.12}$$

and in the equator ($\theta = \frac{\pi}{2}$)

$$n_1\,\cos\,\phi + n_2\sin\,\phi = 0 \tag{8.13}$$

Polar coordinates do not cover the whole sphere (neither do they cover euclidean space). They are not well defined at the two poles. It is interesting to study other set of coordinates, which are actually close to what cartographers do when drawing maps. The stereographic coordinates are defined out of one of the poles (either North or South) Northern pole stereographic projection

$$x_S^\mu \equiv \frac{2l}{X_0 + l} X^\mu \equiv \frac{X^\mu}{\Omega_S} \tag{8.14}$$

($\mu = 1 \dots n$). Let us choose cartesian coordinates in $\mathbb{R}_{n+1}$ with origin in the South pole itself. This meags tgeat the South pole is represented by $X^A = 0$, and the norh pole by $X_A = (l, 0, \dots, 0)$. One can imagine that one is projecting a point $P(X_A) \in \mathbb{S}_n$ from the South pole into into a point x^μ_S that one van view as living on the tangent plane at the North pole.

$$X_0 = l \frac{1 - \frac{x_S^2}{4l^2}}{1 + \frac{x_S^2}{4l^2}} = l(2\Omega_S - 1) = l(2\Omega_N + 1) \tag{8.15}$$

$$\Omega_S \equiv \frac{1}{1 + \frac{x_S^2}{4l^2}} \tag{8.16}$$

$$\frac{x_S^2}{4l^2} = \frac{l - X_0}{l + X_0} \tag{8.17}$$

This means that when $X_0 = l$ (the North pole) then

$$\frac{x_S^2}{4l^2} = 0 \tag{8.18}$$

and when $X_0 = -l$ (the South pole) then

$$X_S^2 = \infty \tag{8.19}$$

The jacobians of the embedding is

$$\begin{aligned} \partial_\mu X^0 &= -\Omega_S^2 \frac{x^\mu}{l} \\ \partial_\mu X^\alpha &= \Omega_S \delta^\alpha_\mu - \Omega_S^2 \frac{x^\alpha x_\mu}{2l^2} \end{aligned} \tag{8.20}$$

The induced metric

$$ds^2 = \delta_{AB} \partial_\mu X^A \partial_\nu X^B dx^\mu dx^\nu = \Omega_S^2 \delta_{\mu\nu} dx^\mu dx^\nu \tag{8.21}$$

Performing the North pole projection, uniqueness of X_0 means that

$$2\Omega_N + 1 = 2\Omega_S - 1 \tag{8.22}$$

and uniquenedd of X^μ

$$x^\mu_N = \frac{\Omega_S}{\Omega_N} x^\mu_S = -\frac{4l^2}{x_S^2} x^\mu_S \tag{8.23}$$

This leads to

$$\Omega_N = -\frac{1}{1 + \frac{x_N^2}{4l^2}} \tag{8.24}$$

The antipodal map

$$X^A \leftrightarrow -X^A \tag{8.25}$$

corresponds in stereographic coordinates to

$$\frac{x_N^2}{4l^2} = \frac{4l^2}{x_S^2} \tag{8.26}$$

and the jacobian is

$$\frac{\partial x_N^\mu}{\partial x_S^\nu} = -\frac{4l^2}{x_S^2}\left(\delta_\nu^\mu - 2\frac{x_S^\mu x_S^\nu}{x_S^2}\right) \tag{8.27}$$

Only functions which are invariant under the exchange of North and South pole stereographic coordinates are well defined on the sphere. The induced metric on the sphere reads

$$ds^2 = \frac{dx_S^2}{(1+\frac{x_S^2}{4l^2})^2} = \frac{dx_N^2}{(1+\frac{x_N^2}{4l^2})^2} \tag{8.28}$$

which is conformally flat. This is the main virtue of these coordinates, and the reason why cartographers are fond of them, We shall call a *frame* a basis on the tangent space to the sphere as a manifold. Let us define a frame through

$$\delta_{ab} e_\mu^a e_\nu^b = g_{\mu\nu} \tag{8.29}$$

The frames are given by

$$(e_S)_\mu^a = \delta_a^\mu \frac{1}{1+\frac{x_S^2}{4l^2}} \tag{8.30}$$

$$(e_N)_\mu^a = -\delta_a^\mu \frac{1}{1+\frac{x_N^2}{4l^2}} \tag{8.31}$$

It is easy to check that

$$L_b^a(x)\ \ (e_S)_\mu^b \equiv \frac{\delta_\mu^a - 2\frac{x_S^\mu x_a^S}{x_S^2}}{1+\frac{x_S^2}{4l^2}} = \frac{\partial x_N^\nu}{\partial x_S^\mu}\ (e_N)_\nu^a \tag{8.32}$$

where the position dependent rotation is given by

$$L_b^a \equiv \delta_b^a - 2\frac{x^a x_b}{x^2} \tag{8.33}$$

In fact this was the reason for the apparently arbitrary minus sign in front of the definition of e_N, which is unneccessary to reproduce the metric.

There are many reasons to be drawn from this example. First of all, it is *never* possible to cover a non trivial manifold with a single coordinate system. In this case we need at least two, namely North and South stereographic coordinates. Second, at each coordinate system, there is a frame in the tangent space, and if we refer all quantities to this frame formal operations are similar to the flat space ones.

When there is a nonvanishing cosmological constant the flat Minkowski space is *not* a solution of Einstein's equations. There are however two privileged spacetimes which do satisfy Einstein's equations. They are privileges in the sense that they have an isometru group which is *as big as* the Poincaré group, $IO(1, n-1)$ (also they both are Petrov Type O, as is the case with all Friedmann-Robertson-Walker spaces) . There are de Sitter space, with symnmetry group $O(1, n)$, and anti-de Sitter space, with isometry group $O(2, n-1)$. Both are symmetric spaces in the mathematical sense: all points are related by an isometry. Besides, the observational fact that the universe is accelerating, means that it resembles *grosso modo* de Sitter space. Were the hypothesis of inflation true in some sense, then the Univeser really underwent a phase of de Sitter expansion. On the other hand anti-de Sitter is very interesting from the point of view of other speculatve theories, such as supersymmetry and strings.

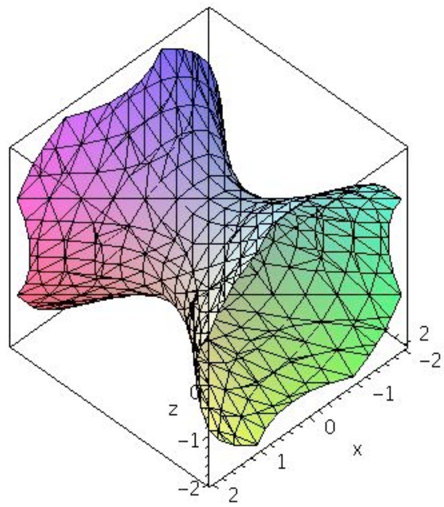

Figure 8.1: A pictorial representation of Anti de Sitter ($X_0^2 + X_1^2 = l^2 + \vec{X}^2$ in $\mathbb{R}^n$).

The real chapters of the complex sphere can be treated in an unified way. Let us choose coordinates in the embedding space in such a way that in the defining equation we have

$$X^2 = \sum_{A=0}^{n} \epsilon_A X_A^2 \equiv \eta_{AB} dX^A dX^B = \pm l^2 \tag{8.34}$$

on a flat space with metric $ds^2 = \eta_{AB} dX^A dX^B$. If we change in an arbitrary manifold $g_{AB} \to -g_{AB}$, then both Christoffels and Riemann tensor remain

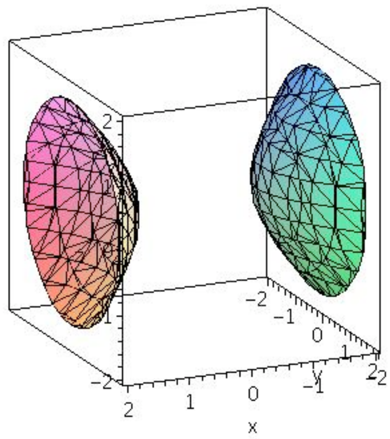

Figure 8.2: A pictorial representation of Euclidean Anti de Sitter (or Euclidean de Sitter) $(X_0^2 - X_1^2 = l^2 + \vec{X}^2$ in $\mathbb{R}^n$

invariant, but the scalar curvature flips sign $R \to -R$. We can furthermore group together times and spaces, in such a way that

$$\eta_{AB} = (1^t, (-1)^s) \tag{8.35}$$

If we call $n + 1 \equiv t + s$, then this ambient space is Wolf's $\mathbb{R}_s^{n+1}$ where the subindex indicates the number of *spaces*.

The standard nomenclature in Wolf's book [**?**] is

$$\begin{aligned} S_s^n &: X \in \mathbb{R}_s^{n+1},\ X^2 = l^2 \\ H_s^n &: X \in \mathbb{R}_{s+1}^{n+1},\ X^2 = -l^2 \end{aligned} \tag{8.36}$$

The curvature scalar is given by:

$$R = \pm \frac{n(n-1)}{l^2} \tag{8.37}$$

and

$$\begin{aligned} R_{\mu\nu} &= \pm \frac{n-1}{l^2} g_{\mu\nu} \\ R_{\mu\nu\rho\sigma} &= \pm \frac{1}{l^2} \left(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\rho} g_{\nu\sigma} \right) \end{aligned} \tag{8.38}$$

Please note that the curvature only depends on the sign on the second member, and not on the signs ϵ_A themselves.

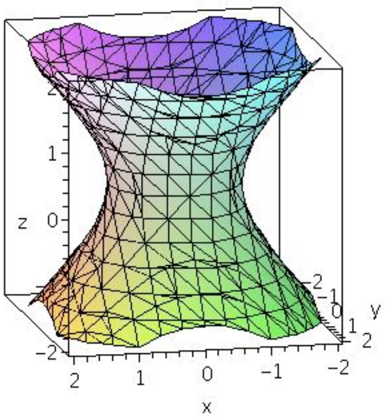

Figure 8.3: A pictorial representation of de Sitter ($X_0^2 - X_1^2 = -l^2 + \vec{X}^2$) in $\mathbb{R}^n$).

It is clear, on the other hand, that the isometry group of the corresponding manifold is one of the real forms of the complex algebra $SO(n+1)$. The Killing vector fields are explicitly given (no sum in the definition) by

$$L_{AB} \equiv \epsilon_A X^A \partial_B - \epsilon_B X^B \partial_A \equiv X_A \partial_B - X_B \partial_A \tag{8.39}$$

The square of the corresponding Killing vector is

$$L^2 = \epsilon_B X_A^2 + \epsilon_A X_B^2 \tag{8.40}$$

Our interest is concentrated on the euclidean and minkowskian cases:

- The sphere $S_n \equiv S_0^n \sim H_n^n$ is defined by $\vec{X}^2 = l^2$, with isometry group $SO(n+1)$.
- The euclidean Anti de Sitter (or euclidean de Sitter) $EAdS_n \equiv S_n^n \sim H_0^n$ is defined by $(X^0)^2 - \vec{X}^2 = l^2$, with isometry group $SO(1,n)$.
- The de Sitter space $dS_n \equiv H_{n-1}^n \sim S_1^n$ is defined by $(X^0)^2 - \vec{X}^2 = -l^2$, with isometry group $SO(1,n)$. In our conventions de Sitter has negative curvature, but positive cosmological constant.
- The Anti de Sitter space $AdS_n \equiv S_{n-1}^n \equiv H_1^n$ is defined by $(X^0)^2 + (X^1)^2 - \vec{X}^2 = l^2$, with isometry group $SO(2,n-1)$. For us AdS_n has positive curvature and negative cosmological constant.

8.1 Global coordinates

A very useful coordinate chart for these spaces is the one called *global* coordinates, wich nevertheless do not cover the full space in any case:

$$(X^A) = l\ \left(\cosh\tau\,\vec{u}_t(\Omega), \sinh\tau\,\vec{n}_s(\Omega')\right) \tag{8.41}$$

where $\vec{u}$ and $\vec{n}$ are unit vectors of both $t-1$ and $s-1$ dimensional spheres. This is for S^n_s spaces. For H^n_s spaces is simply:

$$(X^A) = l\ \left(\sinh\tau\,\vec{u}_{t-1}(\Omega), \cosh\tau\,\vec{n}_{s+1}(\Omega')\right) \tag{8.42}$$

Our convention for a unit vector of a $(n-1)$-dimensional sphere is:

$$\vec{u}_n(\Omega) = (\cos\theta_1, \sin\theta_1\cos\theta_2, \ldots, \sin\theta_1\ldots\sin\theta_{n-1}) \tag{8.43}$$

so that our convention for the "north pole" is:

$$S^n_s\colon\ \ N = (l, 0, \ldots)\ ;\ H^n_s\colon\ \ N = (\underbrace{0, \ldots,}_{t-1} l, 0, \ldots) \tag{8.44}$$

The invariant distance, that we call z, is defined as

$$z(X, Y) = \pm\frac{X\cdot Y}{l^2} \tag{8.45}$$

, where the sign is chosen to make $z(X, X) = 1$ in every space. In our cases of interest:

- Sphere: $X = l\,\vec{u}_n(\Omega)$, $z = \cos\theta_1$
- Euclidean Anti de Sitter: $X = l(\cosh\tau, \sinh\tau\vec{u}_{n-1}(\Omega))$, $z = \cosh\tau$
- de Sitter: $X = l(\sinh\tau, \cosh\tau\,\vec{u}_{n-1}(\Omega))$, $z = \cosh\tau\cos\theta_1$
- Anti de Sitter: $X = l(\cosh\tau\cos\theta, \cosh\tau\sin\theta, \sinh\tau\vec{u}_{n-2}(\Omega'))$, $z = \cosh\tau\cos\theta$

8.2 Projective coordinates

We shall further assume that $\epsilon_k = \pm 1$, that is, the choosen coordinate has the same sign for the metric as the second member in (8.36). We then define the south pole (i.e. $X^k = -l$) stereographic projection for $\mu \neq k$, as

$$x^\mu_S \equiv \frac{2l}{X^k + l}X^\mu \equiv \frac{X^\mu}{\Omega_S} \tag{8.46}$$

The equation of the surface then leads to

$$X^k = l(2\Omega_S - 1)\ ;\ \Omega_S = \frac{1}{1 \pm \frac{x^2_S}{4l^2}}\ ;\ x^2_S \equiv \sum_{\mu\neq k}\epsilon_\mu\,(x^\mu_S)^2 \tag{8.47}$$

The metric in these coordinates is conformally flat:

$$ds^2 = \Omega_S^2 \eta_{\mu\nu} dx_S^\mu dx_S^\nu \tag{8.48}$$

We could have done projection from the North pole (for that we need that $X^k \neq l$). Uniqueness of the definition of X^k needs

$$\Omega_N + \Omega_S = 1 \tag{8.49}$$

and uniqueness of the definition of X^μ

$$x_N^\mu = \frac{\Omega_S}{\Omega_N} x_S^\mu = \pm \frac{4l^2}{x_S^2} x_S^\mu \tag{8.50}$$

The antipodal $\mathbb{Z}_2$ map $X^A \rightarrow -X^A$ is equivalent to a change of the reference pole in stereographic coordinates

$$x_N^\mu \leftrightarrow x_S^\mu \tag{8.51}$$

8.3 The Poincaré metric

A generalization of Poincaré's metric for the half-plane can easily be obtained by introducing the horospheric coordinates [**?**]. It will always be assumed that $\epsilon_0 = +1$, that is that X^0 is a time, and also that $\epsilon_n = -1$, that is X^n is a space, in our conventions. Otherwise (**like in the all-important case of the sphere** S_n) it it not possible to construct these coordinates.

$$\begin{aligned} \frac{l}{z} &\equiv X^- \\ y^i &\equiv z X^i \end{aligned} \tag{8.52}$$

where

$$X^- \equiv X^n - X^0 \tag{8.53}$$

$1 \leq i, j \ldots \leq n-1$. The promised generalization of the Poincaré metric is:

$$ds^2 = \frac{\sum_1^{n-1} \epsilon_i dy_i^2 \mp l^2 dz^2}{z^2} \tag{8.54}$$

where the signs are correlated with the ones defined in (**??**), and the surfaces $z = const$ are sometimes called *horospheres*. This form of the metric is *conformally flat* in a manifest way.

The curvature scalar is given by:

$$R = \pm \frac{n(n-1)}{l^2} \tag{8.55}$$

For any constant curvature space,

$$R_{\mu\nu} = \frac{R}{n} g_{\mu\nu}$$
$$R_{\mu\nu\rho\sigma} = \frac{R}{n(n-1)} \left(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\rho} g_{\nu\sigma}\right) \tag{8.56}$$

In our case this yields

$$R_{\mu\nu} = \pm \frac{n-1}{l^2} g_{\mu\nu}$$
$$R_{\mu\nu\rho\sigma} = \pm \frac{1}{l^2} \left(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\rho} g_{\nu\sigma}\right) \tag{8.57}$$

Please note that the curvature only depends on the sign on the second member, and not on the signs ϵ_A themselves.

It is clear, on the other hand, that the isometry group of the corresponding manifold is one of the real forms of the complex algebra $SO(n+1)$. The Killing vector fields are explicitly given (no sum in the definition) by

$$L_{AB} \equiv \epsilon_A X^A \partial_B - \epsilon_B X^B \partial_A \equiv X_A \partial_B - X_B \partial_A \tag{8.58}$$

The square of the corresponding Killing vector is

$$L^2 = \epsilon_B X_A^2 + \epsilon_A X_B^2 \tag{8.59}$$

8.4 Euclidean de Sitter

To be specific, when the metric is given by:

$$ds^2 = \frac{\sum^{n-1} \delta_{ij} dy^i dy^j \mp l^2 dz^2}{z^2} \tag{8.60}$$

i.,e., $C^{\mp}_{1^n,-1}$, then the isometry group is $SO(n,1)$. This is the case for what could be called **euclidean de Sitter,** $EdS_n = H_0^n \equiv C^{-}_{1^n,-1}$, which in our conventions has got all coordinates timelike, and *negative* [1] curvature. This is the version of Lobatchevsky upper half plane used by Witten [**?**] to analyze the AdS/CFT correspondence. Witten refers to ot as "euclidean AdS".

The metric of EdS_n in Poincar'e coordinates reads:

$$ds^2_{EdS_n} = \frac{\sum^{n-1} \delta_{ij} dy^i dy^j + l^2 dz^2}{z^2} \tag{8.62}$$

[1] We use the Landau-Lifshitz Spacelike conventions (LLSC) and we define the Cosmological Constant in such a way that for a space of constant curvature,

$$R_{\mu\nu} = -\frac{2}{d-2} \lambda g_{\mu\nu} \tag{8.61}$$

8.5 de Sitter

The related situation where

$$ds^2 = \frac{-\sum^{n-1}\delta_{ij}dy^i dy^j \mp l^2 dz^2}{z^2} \tag{8.63}$$

i.e., $C^{\pm}_{1,-1^n}$ enjoys $SO(1,n)$ as isometry group, and includes **de Sitter space,** dS_n when z is a timelike coordinate, $dS_n = H^n_{n-1} \equiv C^{-}_{1,-1^n}$. Its metric reads

$$ds^2_{dS_n} = \frac{-\sum^{n-1}\delta_{ij}dy^i dy^j + l^2 dz^2}{z^2} \tag{8.64}$$

In our conventions de Sitter has negative curvature, but positive cosmological constant. Globally, dS_n is given by:

$$X_0^2 - X_1^2 - \ldots - X_n^2 = -l^2 \tag{8.65}$$

The square of the Killing vectors M_{0a} (candidates to be timelike) are

$$M_{0a}^2 = X_a^2 - X_0^2 = \sum_{b\neq a} X_b^2 - l^2 \tag{8.66}$$

so they are timelike only outside the *horizon* defined as

$$H_{0a} \equiv \sum_{b\neq a} X_b^2 = l^2 \tag{8.67}$$

For example, the horizon corresponding to H_{0n} is

$$\sum y_i^2 = l^2 z^2 \tag{8.68}$$

This means that de Sitter space, dS_n is not globally static.

To go from Poincaré coordinates to FRW, we need

$$\frac{dz}{z} = e^{-H\tau}\ H\ d\tau \tag{8.69}$$

so that

$$z = -e^{-H\tau} \tag{8.70}$$

It seems to be a convention that as

$$-\infty \leq \tau\infty \leftrightarrow -\infty \leq z \leq 0 \tag{8.71}$$

It is interesting to study *static coordinates*

$$\begin{aligned} X^0 &= \sqrt{l^2 - r^2}\sinh\,\frac{t}{l} \\ X^1 &= \sqrt{l^2 - r^2}\cosh\,\frac{t}{l} \\ X^i &= rn^i \quad (1 = 2\ldots n) \end{aligned} \tag{8.72}$$

8.6 Euclidean anti de Sitter

What one would want to call **Euclidean anti de Sitter** , $EAdS_n = S_n^n \equiv C^+_{1,-1^n}$, has got all its coordinates spacelike, and *positive* curvature. To be specific

$$ds^2_{EAdS_n} = \frac{-\sum^{n-1} \delta_{ij} dy^i dy^j - l^2 dz^2}{z^2} \tag{8.73}$$

Pleate note that the metric is just the one corresponding to EdS_n, with a change of sign. This explains the change of sign in the scalar curvature.

Globally,

$$X_0^2 - X_1^2 - \ldots - X_n^2 = l^2 \tag{8.74}$$

(That is, de Sitter with imaginary radius).

8.7 Anti de Sitter

Finally, when the metric is given by

$$ds^2 = \frac{\sum^{n-1} \eta_{ij} dy^i dy^j \mp l^2 dz^2}{z^2} \tag{8.75}$$

(where as usual, $\eta_{ij} \equiv diag(1, (-1)^{n-2})$), i.e. $C^{\pm}_{1^2,-1^{n-1}}$ then the isometry group is $SO(2, n-1)$. This includes the regular **Anti de Sitter**, $AdS_n = S^n_{n-1} \equiv C^+_{1^2,-1^{n-1}}$ when the z coordinate is spacelike. For us AdS_n has positive curvature and negative cosmological constant.

$$ds^2_{AdS_n} = \frac{\sum^{n-1} \eta_{ij} dy^i dy^j - l^2 dz^2}{z^2} \tag{8.76}$$

Globally, AdS_n is

$$X_0^2 + X_1^2 - X_2^2 - \ldots - X_n^2 = l^2 \tag{8.77}$$

In this case there is a globally defined timelike Killing vector field, namely M_{01} Indeed, $M_{01}^2 = x_0^2 + x_1^2 = l^2 + \sum_{a\neq 1} x_a^2$ is everywhere positive. This means that anti de Sitter space, AdS_n is globally static, as opposed to de Sitter.

Actually there is a host of admissible foliations [?]. AdS_n can be foliated by AdS_{n-1}, by dS_{n-1} or by M_{n-1}. In contrast, dS_n can only be foliated by M_{n-1} or by S_{n-1}. Curiously enough, M_n can also be foliated by dS_{n-1}.

8.8 Isometries, conformal invariance and Conformal structure

To be specific, let us denote

$$x^2 \equiv y^2 \mp l^2 z^2 \equiv \sum_i \epsilon_i y_i^2 \mp l^2 z^2 \tag{8.78}$$

Then we define

$$
\begin{aligned}
X^0 &\equiv \frac{l^2 - x^2}{2lz} \\
X^n &\equiv -\frac{l^2 + x^2}{2lz} \\
x^i &= \frac{y^i}{z} \quad (i = 1 \ldots n-1)
\end{aligned}
\tag{8.79}
$$

This is legitimate change of coordinates as long as we keep the radius l itself as one of the coordinates. Conversely

$$
\begin{aligned}
y^i &= \frac{X^i}{X^0 - X^n} l \\
z &= \frac{l}{X^0 - X^n} \\
l^2 &= \mp \left(X_0^2 - X_n^2 + \sum \epsilon_i X_i^2 \right)
\end{aligned}
\tag{8.80}
$$

Some useful formulas

$$
\begin{aligned}
\frac{\partial}{\partial X_0} &= -\frac{z}{l} y^i \partial_i - \frac{z^2}{l} \partial_z \mp \frac{l^2 - x^2}{lz} \partial_{l^2} \\
\frac{\partial}{\partial X_n} &= \frac{z}{l} y^i \partial_i + \frac{z^2}{l} \partial_z \mp \frac{l^2 + x^2}{lz} \partial_{l^2} \\
\frac{\partial}{\partial X_i} &= z \partial_i \mp 2 \frac{\epsilon_i y^i}{z} \partial_{l^2}
\end{aligned}
\tag{8.81}
$$

8.9 Conformal invariance.

The full isometry group is some non-compact real form of $SO(n+1)$. In Poincaré coordinates there is a manifest $ISO(n-1)$ isometry subgroup not involving the horographic coordinate. It is important to understand all isometries in Poincaré coordinates. Let us work out the non-explicit generators

$$
\begin{aligned}
L_{0n} &\equiv X^0 \partial_n + X_n \partial_0 = y^i \partial_i + z \partial_z \\
L_{0i} &\equiv X^0 \partial_i - \epsilon_i X_i \partial_0 = \sum_j \frac{(l^2 - x^2)\delta_{ij} + 2\epsilon_i y_i y_j}{2l} \partial_j + \epsilon_i y^i \frac{z}{l} \partial_z \\
L_{ni} &\equiv -X^n \partial_i - \epsilon_i X_i \partial_n = \sum_j \frac{(l^2 + x^2)\delta_{ij} - 2\epsilon_i y_i y_j}{2l} \partial_j - \epsilon_i y^i \frac{z}{l} \partial_z
\end{aligned}
\tag{8.82}
$$

Translations of the y^i correspond to the combination

$$
k_i \equiv l \frac{\partial}{\partial y^i} = -(L_{ni} + L_{0i}) \tag{8.83}
$$

All spaces considered here whose Poincaré metric reads

$$ds^2 = \frac{\sum_{i=1}^{i=n-1} \epsilon_i dy_i^2 \mp l^2 dz^2}{z^2} \tag{8.84}$$

are obviously *scale invariant*

$$\begin{aligned} y_i' &= \lambda y_i \\ z' &= \lambda z \end{aligned} \tag{8.85}$$

This corresponds in Weirestrass coordinates to the lorentz transformation on the plane $(X^0 X^n)$

$$\begin{aligned} X_0' &= \frac{(\lambda^2+1)X^0 + (\lambda^2-1)X^n}{2\lambda} \\ X_n' &= \frac{(\lambda^2-1)X^0 + (\lambda^2+1)X^n}{2\lambda} \end{aligned} \tag{8.86}$$

id est,

$$\begin{aligned} X^- &\rightarrow \lambda X^- \\ X^+ &\rightarrow \frac{X^+}{\lambda} \end{aligned} \tag{8.87}$$

(which should be plain from the previous formula for the generator L_{0n}.)

They also enjoy invariance under *inversions*, that is

$$\begin{aligned} y_i &\rightarrow \frac{y_i}{\sum \epsilon_i y_i^2 \mp l^2 z^2} \\ z &\rightarrow \frac{z}{\sum \epsilon_i y_i^2 \mp l^2 z^2} \end{aligned} \tag{8.88}$$

In Weierstrass coordinates they correspond to the swap of the two lightcone coordinates in the aforementioned plane $(X^0 X^n)$

$$X^+ \leftrightarrow X^- \tag{8.89}$$

The remaining isometries are the somewhat nasty combinations

$$L_{0i} - L_{ni} = \sum_j \frac{(-x^2)\delta_{ij} + 2\epsilon_i y_i y_j}{l} \partial_j + 2\epsilon_i y^i \frac{z}{l} \partial_z \tag{8.90}$$

We are now in a position to study the little group H of a given point (which can always be rotated to a fiducial one, P)

$$P \equiv (\vec{y} = \vec{0}, z = l) \tag{8.91}$$

because general theorems then ensure thatr the whole space will then be isomorphic to $SO(n+1)/H$. The translational isometries must be generated by the n generators

$$\begin{aligned} &L_{ni} + L_{0i} \\ &L_{0n} \end{aligned} \tag{8.92}$$

It seems then that

$$\begin{aligned} H^{+} &= \{L_{ij}, L_{ni}\} \\ H^{-} &= \{L_{ij}, L_{0i}\} \end{aligned} \tag{8.93}$$

The number of non-compact generators is equal to the number of timelike coordinates amongst the y^i in the + case, and equal to this same number plus une unit in the − case. This implies

$$\begin{aligned} &AdS_n = SO(2, n-1)/SO(1, n-1) \\ &EAdS_n = SO(1, n)/SO(n) \\ &dS_n = SO(1, n)/SO(1, n-1) \\ &EdS_n = SO(n, 1)/SO(n) \end{aligned} \tag{8.94}$$

Eunclidean anti de Sitter $EAdS_n$ is just de Sitter dS_n with imaginary radius. Euclidean deSitter EdS_n is Euclidean anti de Sitter with negative ambient metric.

8.10 Asymptotic Behavior.

8.11 de Sitter

$\mathbf{dS_n}$

The n-dimensional de Sitter space can be globally coordinatized by

$$\begin{aligned} X^0 &= l\, sinh\, \tau \\ X^i &= l\, n^i\, cosh\, \tau \ (i = 1 \ldots n) \end{aligned} \tag{8.95}$$

where $\sum_{i=1}^{i=n} n_i^2 = 1$ and $-\infty \leq \tau \leq \infty$. This gives

$$ds^2 = l^2 \left(d\tau^2 - cosh^2\, \tau\, d\Omega_{n-1}^2 \right) \tag{8.96}$$

A further change of coordinates, namely $cos\, T = \frac{1}{cosh\, \tau}$ where $-\pi/2 \leq T \leq \pi/2$ yields

$$ds^2 = \frac{l^2}{cos^2\, T} \left(dT^2 - d\Omega_{n-1}^2 \right) \tag{8.97}$$

which is conformal to a piece of $\mathbb{R} \times S^{n-1}$, which is the Einstein static universe, the template used by Hawking and Ellis [?] to study conformal

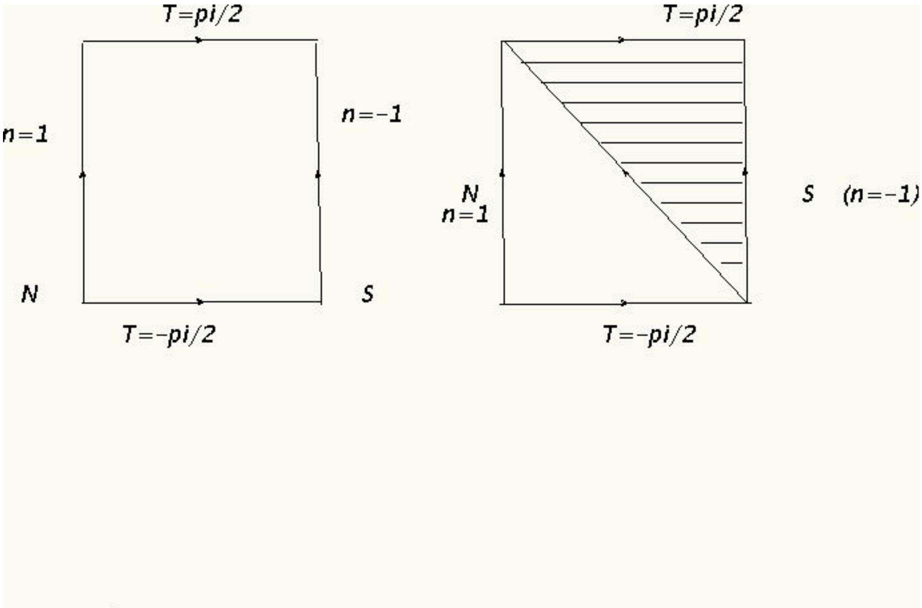

Figure 8.4: Conformal structure of dS_n. In the figure on the right it is represented the portion of the space covered by Poincaré coordinates.

structure. The piece is a slab in the timelike direction, but otherwise including the full three-sphere at each time. The fact that conformal infinity is spacelike means that there are both particle and event horizons.

The piece of space covered by the static patch is one quarter of it, namely just the left wedge containing the center and the corbers at minus and plus infinity.

8.12 Anti de Sitter $\mathbf{AdS_n}$.

The fact that in this case there are two times suggests:

$$
\begin{aligned}
X^0 &= l\frac{\cos\tau}{\cos\rho}\\
X^4 &= l\frac{\sin\tau}{\cos\rho}\\
X^i &= l\, n^i\, tg\,\rho \ (i = 1\ldots n-1)
\end{aligned}
\tag{8.98}
$$

where $\sum_{i=1}^{i=n-1} n_i^2 = 1$ and $-\pi \leq \tau \leq \pi$, $0 \leq \rho \leq \pi/2$. The space is again conformal to a piece of half Einstein' s static universe:

$$
ds^2 = \frac{l^2}{cos^2\rho}\left(d\tau^2 - d\rho^2 - sin^2\rho d\Omega_{n-2}^2\right) = \frac{l^2}{cos^2\rho}\left(d\tau^2 - d\Omega_{n-1}^2\right) \tag{8.99}
$$

If we want to eliminate the closed timelike lines, one can consider the covering space $-\infty \leq \tau \leq \infty$. The slab of $\mathbb{R} \times S^{n-1}$ to which AdS_n is conformal to includes now the full timelike direction, but only an hemisphere at each particular time. Null and spacelike infinity can be considered as the timelike surfaces $\rho = 0$ and $\rho = \pi/2$. This implies that there are no Cauchy surfaces. Consider for instance the null geodesic

$$\tau = \rho \tag{8.100}$$

It propagates from the prigin $\rho = 0$ to spatial infinity at $\rho = \frac{\pi}{2}$ in finite time

$$\Delta\tau = \frac{\pi/}{2} \tag{8.101}$$

Conversely information leaking in from spatial infinity reaches the origin in finite time.

8.13 Euclidean anti de Sitter space EAdS$_n$

We write

$$\begin{aligned} X^\mu &= l n^\mu \, sinh\, \tau \\ X^n &= l cosh\, \tau \end{aligned} \tag{8.102}$$

with $\sum_{\mu=0}^{n-1} \epsilon_\mu n_\mu^2 = 1$, so that the metric reads

$$ds^2 = l^2 \left(d\tau^2 + sinh\, \tau^2 d\Omega_{n-1}^2 \right) \tag{8.103}$$

The change of variables

$$e^T = th\, \tau/2 \tag{8.104}$$

yields

$$ds^2 = l^2 \frac{e^{2T}}{1 - e^{2T}} \left(dT^2 + d\Omega_{n-1}^2 \right) \tag{8.105}$$

(the other half of the global space would be covered by another copy of the above metric).

In this metric, $X_n \geq X_0$ always, which means that in Poincaré coordinates $z \geq 0$, and $z \to 0$ when $\tau \to \infty$, which is equivalent to $T \to \infty$, and represents the boundary of the space, a S_{n-1} sphere.

8.14 What portion of Weiersstrass coordinates do Poincaré coordinates cover?

- **dS$_n$**

  If we call n the n-th component of the unit vector $\vec{n}$, then there is a critical value of the parameter τ such that

$$tanh\, \tau(n) = n \tag{8.106}$$

which is such that

$$\tau < \tau(n) \Rightarrow z > 0 \tag{8.107}$$

and

$$z \to \pm\infty \Leftrightarrow \tau \to \tau(n)^{\mp} \tag{8.108}$$

This means that at any given value of τ only those points on the sphere that obey

$$n \leq tanh\, \tau \tag{8.109}$$

can be represented in Poincaré coordinates. For example, when $\tau = -\infty$, that is $T = -\pi/2$, $tanh\, \tau = -1$, so that only the South pole ($n = -1$) can be covered. At the other extreme, when, $\tau = \infty$, that is $T = +\pi/2$, $tanh\, \tau = 1$, we can cover the full sphere.

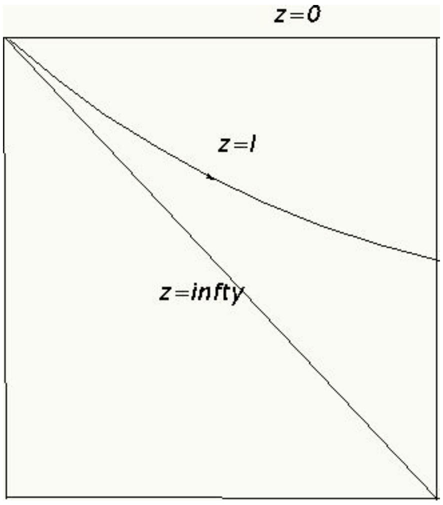

Figure 8.5: Surfaces of constant z in Poincaré coordinates in dS_n.

On the other hand, it is clear that

$$z \to 0^{\pm} \Leftrightarrow \tau \to \mp\infty \tag{8.110}$$

There is a discontinuity at $\tau(n)$ which depends on the point in de Sitter space.

- $\mathbf{AdS_n}$

It is clear that the region $z \geq 0$ corresponds to the patch

$$\pi/4 \leq \tau \leq \pi \tag{8.111}$$

and the region $0 \geq z$ to

$$-\pi \leq \tau \leq -3\pi/4 \tag{8.112}$$

The region

$$z = 0 \tag{8.113}$$

is dubbed the *boundary* (of the Poincaré patch) of AdS and corresponds to

$$\rho = \pi/2 \tag{8.114}$$

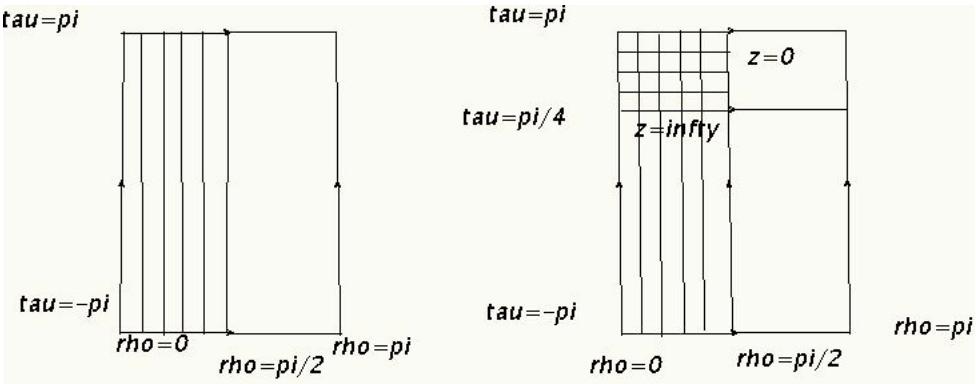

Figure 8.6: Conformal structure of AdS_n. In the figure on the right it is represented the portion of the space covered by Poincaré coordinates.

Finally

$$z = \infty \tag{8.115}$$

is usually called the *horizon* and corresponds to $X^n = X^0$, that is,

$$\tau = \pi/4 \tag{8.116}$$

or else

$$\tau = -3\pi/4 \tag{8.117}$$

(assuming $\rho \neq \pi/2$).

When $\rho = \pi/2 - \epsilon$ and $\tau = \pi/4 \pm \delta$,

$$z = \pm\frac{\sqrt{2}}{2}\frac{\epsilon}{\delta} \tag{8.118}$$

and the limit depends on how the limit point $\epsilon = \delta = 0$ is reached.

The same thing happens when $\rho = \pi/2 - \epsilon$ and $\tau = -3\pi/4 \pm \delta$,

$$z = \mp\frac{\sqrt{2}}{2}\frac{\epsilon}{\delta} \tag{8.119}$$

9

Friedmann-Robertson-Walker Cosmological Models.

The so called *cosmological principle* assumes that the universe is spatially homogeneous and isotropic and filled with a materia content which can be properly approximated by a perfect fluid.

A manifold is said to be *spatially homogeneous* in a mathematical sense when there are a uniparametric family of hypersurfaces, Σ_t such that given two points, P and Q in the same hypersurface, there is an isometry that carries P into Q (it is said in learned language, that the isometry group acts *transitively*). The group of isometries has maximum dimension

$$D \equiv \frac{n(n+1)}{2} \tag{9.1}$$

in which case, it is a space of constant curvature. This implies, D=10 in n=4 (this is the case of Minkowski, de Sitter and anti de Sitter spaces, an only those). In three dimensions, n=3, the maximum D=6, and this is the case we are interested with .

The *perfect cosmological principle* which was the basis for the *steady state* cosmological model assumed full homogeneity of the four-dimensional space-time manifold. This restrict the form of the metric to one of the three maximally symmetric four-dimensional spaces: de Sitter, anti-de Sitter or Minkowsli. It is not easy to accommodate the CMB data on these models without many epycicles.

This means that the three-dimensional Riemann tensor must be of constant curvature

$$^{(3)}R_{ijkl} = \kappa h_{k[i}h_{j]l} \tag{9.2}$$

Positive curvature ($\kappa = +1$) corresponds spaces isometric to the three-dimensional sphere

$$x^2 + y^2 + z^2 + w^2 = R^2 \tag{9.3}$$

Negative curvature ($\kappa = -1$) corresponds to spaces isometric to the hyperboloid

$$t^2 - x^2 - y^2 - z^2 = R^2 \tag{9.4}$$

Coordinates can be defined in such a way that

$$ds^2_{\kappa=+1} = d\psi^2 + \sin^2\,\psi\left(d\theta^2 + \sin^2\theta\;d\phi^2\right) \tag{9.5}$$

and

$$ds^2_{\kappa=-1} = d\psi^2 + \sinh^2\,\psi\left(d\theta^2 + \sin^2\theta\;d\phi^2\right) \tag{9.6}$$

It is also assumed that there is a uniparametric family of *isotropic observers* characterized by a vector field u, such that

$$u.\Sigma_t = 0 \tag{9.7}$$

9.1 The cosmological fluid of fundamental observers

The general form of the metric of an homogeneous and isotropic universe was shown by FRW to be

$$ds^2 = dt^2 - a(t)^2 ds_3^2 \tag{9.8}$$

where ds_3^2 is the metric of a three space of constant curvature such as the ones we have just seen. When the curvature of the three space vanishes, the FRW metric reduces to

$$ds^2 = dt^2 - a^2\delta_{ij}dx^i dx^j \tag{9.9}$$

FRW models are conformally flat. When $\kappa = 0$ this immediate using the *conformal time*

$$dt \equiv a(\eta)\;d\eta \tag{9.10}$$

so that

$$ds^2 = a^2(\eta)\left(d\eta^2 - d\vec{x}^2\right) \tag{9.11}$$

For the closed model ($\kappa = +1$) and in terms of the conformal time ($f' \equiv \frac{d}{d\eta}f$)

$$\begin{aligned} R_{00} &= \frac{3}{a^4}\left((a')^2 - aa''\right) \\ R_i^j &= -\frac{2}{a^2}\delta_i^j \\ R &= -\frac{6}{a^3}\left(a + a''\right) \end{aligned} \tag{9.12}$$

Then

$$\frac{8\pi G}{c^4}\rho = \frac{3}{a^4}\left(a^2 + (a')^2\right) \tag{9.13}$$

Assuming an energy-momentun tensor to be of the perfect fluid form, Einstein's equations can be easily shown to reduce in this case to

$$\dot{a}^2 - \frac{8\pi G}{3}\left(\rho + \rho_\lambda\right)a^2 - \frac{\kappa}{2} \tag{9.14}$$

where we have now represented by ρ_m the energy density of matter and by

$$\rho_\lambda \equiv -\frac{\lambda}{8\pi G} \tag{9.15}$$

the equivalent quantiy corresponding to the cosmological constant, which is dubbed *dark energy*.

The covariant conservation of the energy momentum tensor tells us that

$$\dot{p}a^3 = \frac{d}{dt}\left(a^3\left(\rho + p\right)\right) \tag{9.16}$$

From all this can be deduced that

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}\left(\rho + 3p\right) \tag{9.17}$$

This means that if the (negative) dark pressure is big enough then cosmic acceleration may result. The fact that this is observed by cosmologists was the first indication that there is a nonvanishing cosmological constant in the universe.

Assuming now the equation of state corresponding to radiation

$$p = \frac{1}{3}\rho \tag{9.18}$$

this yields

$$\rho a^4 \sim \text{constant} \tag{9.19}$$

When the pressure is negligible (which is the case for nonrelativistic matter); this type of matter is traditionally called *dust* by cosmologists, then

$$\rho a^3 \sim \text{constant} \tag{9.20}$$

9.2 Cosmological redshift

It is quite easy to check that the $\mu = 0$ component of the FRW geodesic equation is given by

$$\frac{d^2 t}{d\lambda^2} + a\dot{a}\delta_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda} = 0 \tag{9.21}$$

For null geodesics in the x-direction

$$\frac{dt}{d\lambda} = a\frac{dx}{d\lambda} \tag{9.22}$$

This means that in this case

$$\frac{d^2t}{d\lambda^2} + \frac{\dot{a}}{a}\left(\frac{dt}{d\lambda}\right)^2 = 0 \tag{9.23}$$

which in turn implies

$$\frac{dt}{d\lambda} \sim \frac{1}{a} \tag{9.24}$$

The four momentum of such a photon would then be

$$p = \hbar\,(\omega, \omega, 0, 0) \tag{9.25}$$

Now the energy of a photon measured by a comoving observer is

$$E \equiv p.u = \frac{dt}{d\lambda} = \frac{\omega_0 a_0}{a} \tag{9.26}$$

This formula yields the *cosmological redshift*. It could also be derived by using the constancy of the product

$$\xi.k \tag{9.27}$$

along a geodesic with tangent vector k, ξ being any Killing vector (not necessarily timelike). The frequancy is

$$\omega \equiv k.u \tag{9.28}$$

Owing to the fact that k is null, this is the same as the projection on the three-space orthogonal

$$k.\Sigma_t \tag{9.29}$$

But there always an specific spacelike killing ξ_s such that

$$k.\Sigma_t = k\frac{\xi_s}{\sqrt{\xi_s^2}} \sim \frac{1}{a} \tag{9.30}$$

Let us reming that in general the redshift is defined as

$$1 + z \equiv \frac{\omega}{\omega + \Delta\omega} \equiv \frac{\lambda + \Delta\lambda}{\lambda} = 1 + \frac{\Delta\lambda}{\lambda} = \frac{a_0}{a} \tag{9.31}$$

(this would become blueshift in case $\Delta\lambda < 0$),

The observation of the cosmological redshift then indicates that

$$\frac{a_0}{a} > 1 \tag{9.32}$$

that is, that we are in a universe in expansion. To put is in an equivalent way, the *Hubble parameter* is defined as

$$H(t) \equiv \frac{\dot{a}}{a} \tag{9.33}$$

When a numerical value is quoted it usually refers to its value at the present time, $t = t_0$, like in

$$H_0 \sim 71 \pm 4\, Kms^{-1}Mpc^{-1} \sim 2\times 10^{-18}\, s^{-1} \sim 10^{-10} y^{-1} \sim 2\times 10^{-42}\, GeV \sim 2\times 10^{-61}\, M_P \tag{9.34}$$

Ay any rate we can write to linear order

$$a(t) = a_0\left(1 + H_0\left(t - t_0\right)\right) \tag{9.35}$$

so that

$$z \sim H_0\left(t_0 - t\right) \tag{9.36}$$

In this approximation, the cosmological redshift is proportional to the distance of the object in question. It is also customary to define the *deceleration parameter* as

$$q_0 \equiv -\frac{a_0 \ddot{a}_0}{\dot{a}_0^2} \tag{9.37}$$

The observation favors negative values of this parameter indicating that the universe is in a period of acceleration.

9.3 Cosmological Horizons

One property of many models is the following. At a given instant of time, a given particle P has has time to interact only with a portion of the spacetime; the rest had not yet time to teach the particle with any signal; it is outside tha past light cone of the particle. This happens for example if we consider half of Minkowski space without the piece corresponding to negative times, $t \leq 0$.

$$M_4^+ \equiv \{x \in M_4 \quad \& \quad t > 0\} \tag{9.38}$$

This is a repectable spacetime with boundary. Consider now an event P at a given time $t = t_0$. It is plain that all events that at time $t = 0$ were not closer to the event P that ct_0 did not yet have time to send any signal to P; they have not been in causal contact.

Let us consider for simplicity the flat FRW model. We can define the *conformal time*

$$\eta \equiv \int \frac{dt}{a(t)} \tag{9.39}$$

so that the metric is formally conformal to Minkowski space

$$ds^2 = a(\tau^2)\left(d\tau^2 - dx^2 - dy^2 - dz^2\right) \tag{9.40}$$

Now there is the possibility that the range of η does not go from

$$-\infty \leq \eta \leq \infty \tag{9.41}$$

but instead from

$$\tau_0 \leq \eta \leq \infty \tag{9.42}$$

In that case there is a *particle horizon.* This happens when

$$\lim_{t\to 0} \int \frac{dt}{a(t)} > -\infty \tag{9.43}$$

This is what happens for example in de Sitter Universe, where

$$a(\eta) = \frac{1}{\eta} \tag{9.44}$$

so that

$$\eta = \pm\tau_0 \; e^{t-t_0} \tag{9.45}$$

So that the whole real line

$$-\infty \leq t \leq \infty \tag{9.46}$$

is mapper either into

$$0 \leq \eta \leq \infty \tag{9.47}$$

for the plus sign; or else into

$$-\infty \leq \eta \leq 0 \tag{9.48}$$

for the minus option.

9.4 Cosmological parameters

Let us define the associated energy density to the Hubble expansio

$$\rho_H \equiv \frac{3H^2}{8\pi G} \tag{9.49}$$

as well as the equivalent energy density to the curvature, namely

$$\rho_\kappa \equiv -\frac{3\kappa}{8\pi G a^2} \tag{9.50}$$

The *critical density* is by definition

$$\rho_c \equiv \rho_{H_0} \tag{9.51}$$

Taking quotients with the critical density, we get a sum rule

$$\Omega_\lambda + \Omega_m + \Omega_k = 1 \tag{9.52}$$

The recent observational results point towards

$$\begin{aligned} &\Omega_\kappa \sim 0 \\ &\Omega_m \sim 0.3 \\ &\Omega_\lambda \sim 0.7 \end{aligned} \tag{9.53}$$

10

Gravitational energy.

The energy of a gravitational field is not really well defined, owing to the equivalence principle, that says precisely that for a free-falling FREFO there is no local gravitational field. This shows that the notion of gravitational energy does depend on the frame, and so it can not be the timelike component of a geometrical vector. If it were, its vanishing would have had an intrinsic meaning. In a certain sense it could be said that Einstein's equations equate the energy-momentum of the matter to the analogous quantity for gravitation. This would indicate that the gravitational energy-momentum tensor would be

$$T^{\text{grav}}_{\mu\nu} = -\frac{c^3}{16\pi G}\left(R_{\mu\nu} - \frac{1}{2}(R + 2\lambda)\; g_{\mu\nu}\right) \tag{10.1}$$

in such a wat that the total energy density vanishes

$$T^{\text{grav}}_{\mu\nu} + T_{\mu\nu} = 0 \tag{10.2}$$

This is sensible, except that this construct is covariantly conserved (as is the energy-momentum tensor of the matter) and does not give rise to any conserved quantity different from zero. Let us see that in detail.

10.1 Energy of matter in the presense of a background gravitational field

Let us assume that there is a timelike Killing vector in our spacetime, that is, a vector ξ such that

$$\pounds(\xi)\; g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0 \tag{10.3}$$

Then there is a covariantly conserved four vector

$$P^\mu \equiv T^\mu{}_\lambda \xi^\lambda \tag{10.4}$$

In fact

$$\nabla_\mu P^\mu = (\nabla_\mu T^\mu{}_\rho)\,\xi^\rho + T^\mu{}_\rho \nabla_\mu \xi^\rho = 0 \tag{10.5}$$

where the second term vanishes because the symmetric part of the covariant derivative of a Kiling vector also vanishes.

The covariant energy-momentum conservation equation could be written as a covariant Killing current conservation and this in turn is equivalent to asserting that a certain one form is co-closed

$$\delta\tau = *^{-1} d * \tau = -\nabla^\mu \tau_\mu = 0 \tag{10.6}$$

where τ is the one-form

$$\tau \equiv T_{\mu\nu}\xi^\nu dx^\mu \tag{10.7}$$

Now (assuming the first Betti number $b_1 = 0$) this means that

$$d * \tau = 0 \tag{10.8}$$

Now $*\tau$ is a three-form

$$* \tau \equiv \eta_{\mu\nu\rho\sigma} dx^\nu \wedge dx^\rho \wedge dx^\sigma T^{\mu\lambda}\xi_\lambda \tag{10.9}$$

Stokes' theorem guarantees that the integral over any four-dimensional domain

$$0 = \int_M d * \tau = \int_{\partial M} *\tau \tag{10.10}$$

Let us take by M the four-dimensional cylinder bounded by two caps at $t = t_0$ and $t = t_1$. We get, refered to the particular static case where $\xi = \frac{\partial}{\partial_t}$

$$0 = \int_{\partial M} *\tau = \int_{t=t_1} \sqrt{|g|} T^0_0 - \int_{t=t_0} \sqrt{|g|} T^0_0 \tag{10.11}$$

this shows that Killing energy

$$E \equiv \int_{\Sigma_t} \eta_{\mu\nu\rho\sigma} dx^\nu \wedge dx^\rho \wedge dx^\sigma T^{\mu\lambda}\xi_\lambda n_\mu \tag{10.12}$$

where $n_\mu dx^\mu = dt.$ is time independent in the static case.

Energy can also be defined in GR in some additional cases, in particular for fields that correspond to compact sources, so that they are asymptotically flat. In this case it transforms covariantly under the asymptotic symmetry group. Can we do this also for the gravitational field? Yes, we can but Einstein's equations tell us that the result is exactly the same with opposite sign.

- Linear Bianchi Starting from

$$\nabla_\mu G^{\mu\nu} = 0$$

and expanding

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$

we easily get

$$\bar{\nabla}_\mu G^{\mu\nu}_{(1)} = -\Gamma_{(1)}{}^{\rho}_{\rho\lambda}\bar{G}^{\lambda\nu} - \Gamma_{(1)}{}^{\nu}_{\rho\lambda}\bar{G}^{\rho\lambda}$$

so that to the extent that

$$\bar{G}^{\rho\lambda} = 0$$

we get

$$\bar{\nabla}_\mu G^{\mu\nu}_{(1)} = 0$$

- Flat linear transversality It is a fact that

  $$4\,(D.h)_{\rho\sigma} = k^2 h_{\rho\sigma} - k^\lambda k_\rho h_{\lambda\sigma} - k^\lambda k_\sigma h_{\lambda\rho} + k_\rho k_\sigma h - \eta_{\rho\sigma}\left(k^2 h - k^\lambda k^\delta h_{\lambda\delta}\right)$$

  In terms of the new variables

  $$\bar{h}_{\alpha\beta} \equiv h_{\alpha\beta} - \frac{1}{2} h \eta_{\alpha\beta}$$

  it reads

  $$4\bar{D}_{\rho\sigma} = k^2 \bar{h}_{\rho\sigma} - k^\lambda k_\rho \bar{h}_{\lambda\sigma} - k^\lambda k_\sigma \bar{h}_{\lambda\rho} + \eta_{\rho\sigma} k^\lambda k^\delta \bar{h}_{\lambda\delta}$$

  Introducing the superpotential with the symmetries of the Riemann tensor

  $$2K_{\mu\nu\rho\sigma} \equiv \eta_{\mu\sigma}\bar{h}_{\nu\rho} + \eta_{\nu\rho}\bar{h}_{\mu\sigma} - \eta_{\mu\rho}\bar{h}_{\nu\sigma} - \eta_{n\sigma}\bar{h}_{\mu\rho}$$

  then the following is true

  $$-2k^\nu k^\sigma K_{\mu\nu\rho\sigma} = 4\bar{D}_{\mu\rho}$$

- General linear transversality

  Expanding Einstein Hilbert one gets

$$S = \frac{1}{2}\int d^n x\sqrt{\bar{g}}\, h^{\alpha\beta}\left(\frac{1}{4}\bar{g}_{\alpha\beta}\bar{g}_{\mu\nu}\bar{\nabla}^2 - \frac{1}{4}\bar{g}_{\alpha\mu}\bar{g}_{\beta\nu}\bar{\nabla}^2 + \frac{1}{2}\bar{g}_{\alpha\mu}\bar{\nabla}_\beta\bar{\nabla}_\nu - \frac{1}{2}\bar{g}_{\mu\nu}\bar{\nabla}_\alpha\bar{\nabla}_\beta\right.$$
$$\left.+\frac{1}{2}\bar{g}_{\alpha\beta}\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\alpha\mu}\bar{R}_{\beta\nu} - \frac{1}{2}\bar{R}_{\alpha\mu\beta\nu} - \frac{1}{8}\left(\bar{R}+2\lambda\right)\left(\bar{g}_{\alpha\beta}\bar{g}_{\mu\nu} - 2\bar{g}_{\alpha\mu}\bar{g}_{\beta\nu}\right)\right) h^{\mu\nu} \quad (10.13)$$

The third term can be shuffled using

$$\bar{\nabla}_\nu\bar{\nabla}_\beta h^{\mu\nu} = \bar{\nabla}_\beta\bar{\nabla}_\nu h^{\mu\nu} - h^{\lambda\nu}\bar{R}^\mu{}_{,\lambda\beta\nu} + h^{\lambda\mu}\bar{R}_{\lambda\beta}$$

For a constant curvature background,

$$\bar{R}_{\alpha\mu\beta\nu} = -\frac{2}{(n-1)(n-2)}\lambda\left(\bar{g}_{\alpha\beta}\bar{g}_{\mu\nu} - \bar{g}_{\alpha\mu}\bar{g}_{\beta\nu}\right)$$

$$\bar{R}_{\mu\nu} \equiv -\frac{2}{n-2}\lambda g_{\mu\nu}$$

This is the same as in TO, with

$$\lambda = -4\frac{n-1}{n+3}\Lambda_{TO}$$

The corresponding EM are

$$\left(\frac{1}{4}\bar{g}_{\alpha\beta}\bar{g}_{\mu\nu}\bar{\nabla}^2 - \frac{1}{4}\bar{g}_{\alpha\mu}\bar{g}_{\beta\nu}\bar{\nabla}^2 + \frac{1}{2}\bar{g}_{\alpha\mu}\bar{\nabla}_\beta\bar{\nabla}_\nu - \frac{1}{2}\bar{g}_{\mu\nu}\bar{\nabla}_\alpha\bar{\nabla}_\beta - \frac{(n+3)\lambda}{4(n-1)}\left(\bar{g}_{\alpha\beta}\bar{g}_{\mu\nu} - 2\bar{g}_{\alpha\mu}\bar{g}_{\beta\nu}\right)\right) h^{\mu\nu} = 0$$

raising indices with the background metric, that is

$$\bar{g}_{\alpha\beta}\bar{\nabla}^2 h - \bar{\nabla}^2 h_{\alpha\beta} + 2\bar{\nabla}_\beta\bar{\nabla}_\nu h^\nu_\alpha - 2\bar{\nabla}_\alpha\bar{\nabla}_\beta h - \frac{(n+3)\lambda}{(n-1)}\left(\bar{g}_{\alpha\beta}h - 2h_{\alpha\beta}\right) = 0$$

This is presumably equivalent to TO's operator which reads

$$\bar{D}^{\alpha\beta} \equiv \bar{\nabla}^2 h^{\alpha\beta} - \bar{\nabla}^\lambda\bar{\nabla}^\alpha h_\lambda{}^\beta - \bar{\nabla}^\lambda\bar{\nabla}^\beta h_\lambda{}^\alpha + \bar{\nabla}^\alpha\bar{\nabla}^\beta h - \bar{g}^{\alpha\beta}\left(\bar{\nabla}^2 h - \bar{\nabla}^\mu\bar{\nabla}^\nu h_{\mu\nu}\right) - 2\Lambda\left(h^{\alpha\beta} - \frac{1}{2}\bar{g}^{\alpha\beta}h\right)$$

This can be rewritten using

$$h_{\alpha\beta} \equiv \bar{h}_{\alpha\beta} - \frac{1}{n-2}\bar{h}\bar{g}_{\alpha\beta}$$

as

$$\bar{D}^{\alpha\beta}(\bar{h}) \equiv \bar{\nabla}^2\bar{h}^{\alpha\beta} - \bar{\nabla}^\lambda\bar{\nabla}^\alpha\bar{h}_\lambda{}^\beta - \bar{\nabla}^\lambda\bar{\nabla}^\beta\bar{h}_\lambda{}^\alpha + \bar{g}^{\alpha\beta}\bar{\nabla}^\mu\bar{\nabla}^\nu h_{\mu\nu} - 2\Lambda\bar{h}^{\alpha\beta}$$

This is nothing more than our old $G^{\alpha\beta}_{(1)}$, so that it also obeys on shell

$$\bar{\nabla}_\alpha\bar{D}^{\alpha\beta} = 0$$

Let us consider the current obtained by contracting with a Killing vector

$$j^\mu \equiv \bar{D}^{\mu\nu} k_\nu$$

It is an easily proved fact that

$$\bar{\nabla}_\mu j^\mu = 0$$

Let is define the superpotential

$$2K^{\mu\alpha\nu\beta} \equiv \bar{g}^{\mu\beta}\bar{h}^{\nu\alpha} + \bar{g}^{\nu\alpha}\bar{h}^{\nu\beta} - \bar{g}^{\mu\nu}\bar{h}^{\alpha\beta} - \bar{g}^{\alpha\beta}\bar{h}^{\mu\nu}$$

Let us compute

$$Y^{\mu\nu} \equiv \bar{\nabla}_\alpha \bar{\nabla}_\beta K^{\mu\alpha\nu\beta} = \frac{1}{2}\bar{D}^{\mu\nu} + \frac{1}{2}[\bar{\nabla}_\lambda, \bar{\nabla}^\nu]\bar{h}^\lambda_\mu + \Lambda \bar{h}^{\mu\nu} \equiv \frac{1}{2}\bar{D}^{\mu\nu} - X^{\mu\nu}$$

$$X^{\mu\nu} \equiv \frac{1}{2}[\bar{\nabla}_\lambda, \bar{\nabla}^\nu]\bar{h}^\lambda_\mu + \Lambda \bar{h}^{\mu\nu} = -\frac{1}{2}\bar{h}^{\lambda\rho}\bar{R}^\nu{}_{\lambda\mu\rho} + \frac{1}{2}\bar{h}^{\lambda\nu}\bar{R}_{\lambda\mu} + \Lambda \bar{h}^{\mu\nu}$$

$$\bar{R}^\nu{}_{\alpha\beta\gamma} K^{\mu\alpha\beta\gamma} = -2\bar{R}^\nu{}_\alpha{}^\mu{}_\gamma \bar{h}^{\alpha\gamma} - 2\bar{R}^\nu{}_\gamma \bar{h}^{\mu\gamma}$$

10.2 The energy-momentum pseudotensor of the gravitational field.

In agreement with our previous observations, it is not possible to define the gravitational energy in a geometrical way, but it is certainly possible to define it as a quantity that is only tensorial under certain restricted set of transformations. This is traditionally called a pseudotensor, and there are many of them, associated to the names of Einstein, Moller, Landau and Lifshitz, etc.

It seems appropiate to begin by investigating what happens in a free-falling frame. In normal coordinates appropiate to such a FREFO the Ricci tensor at the origin, where the metric tensor coincides with the Minkowskian one, and its first derivativa vanishes (but not the second derivatives)

$$\begin{aligned} & g_{\alpha\beta} = \eta_{\alpha\beta} \\ & \partial_\alpha g_{\beta\gamma} = 0 \\ & \partial_\alpha \partial_\beta g_{\beta\gamma} \neq 0 \end{aligned} \tag{10.14}$$

(we emphatize that thios is only true at one point, here taken as the origin of our reference system) reads

$$R^{\mu\nu} = \frac{1}{2} g^{\mu\lambda} g^{\nu\rho} g^{\delta\sigma} \left(\partial_\lambda \partial_\sigma g_{\delta\rho} + \partial_\delta \partial_\rho g_{\lambda\sigma} - \partial_\lambda \partial_\rho g_{\delta\sigma} - \partial_\delta \partial_\sigma g_{\lambda\rho}\right) \tag{10.15}$$

We have already pointed out that a natural definition of gravitational energy-momentum tensor is

$$T^{\mu\nu} \equiv \frac{c^4}{8\pi G}\left(R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}\right) = \frac{c^4}{16\pi G}\left(g^{\mu\lambda}g^{\nu\rho} - \frac{1}{2}g^{\mu\nu}g^{\lambda\rho}\right)g^{\delta\sigma} \times$$
$$(\partial_\lambda\partial_\sigma g_{\delta\rho} + \partial_\delta\partial_\rho g_{\lambda\sigma} - \partial_\lambda\partial_\rho g_{\delta\sigma} - \partial_\delta\partial_\sigma g_{\lambda\rho}) = \frac{c^4}{16\pi G}(\partial^\mu\partial_\sigma g^{\sigma\nu} - \partial_\sigma\partial^\nu g^{\mu\sigma} - \partial^\mu\partial^\nu g -$$
$$-\Box g^{\mu\nu} - g^{\mu\nu}\left(\partial_\sigma\partial_\lambda g^{\sigma\lambda} - \Box g\right)) \tag{10.16}$$

It so happens that this expression can be written in terms of a *superpotental*, $H^{\mu\nu\lambda}$

$$T^{\mu\nu} = \frac{1}{|g|}\partial_\lambda H^{\mu\nu\lambda} \tag{10.17}$$

where (the factors of $|g|$ are immaterial in this frame, in which $g = -1$, tpo the extent that they are undifferentiated)

$$H^{\mu\nu\lambda} \equiv \frac{c^4}{16\pi G}\,\partial_\sigma\left(|g|\left(g^{\mu\lambda}g^{\nu\sigma} - g^{\mu\nu}g^{\lambda\sigma}\right)\right) \tag{10.18}$$

The key to this result is that the derivative of the determinant

$$\partial_\alpha g = g g^{\rho\sigma}\partial_\alpha g_{\rho\sigma} \tag{10.19}$$

vanishes in a free falling system. The second derivative

$$\partial_\alpha\partial_\beta g = g g^{\rho\sigma}\partial_\alpha\partial_\beta g_{\rho\sigma} = \eta^{\rho\sigma}\partial_\alpha\partial_\beta g_{\rho\sigma} \tag{10.20}$$

For a FREFO the derivative of the determinant of the metric tensor is the same thing as the trace of the derivative of the same metric tensor. The superpotential is antisymmetric in the two last indices

$$H^{\mu\nu\lambda} = -H^{\mu\lambda\nu} \tag{10.21}$$

The conservation of our candidate for gravitational energy is then immediate (it was at any rate guaranteed by Bianchi identities). Now if we consider a general observer these formulas need modification; we shall call $t^{\mu\nu}$ the necessary addenda so that we have *exactly*

$$\sqrt{|g|}\,(T^{\mu\nu} + t^{\mu\nu}) = \partial_\lambda H^{\mu\nu\lambda} \tag{10.22}$$

LL found an exact expression for the pseudotensor

$$t^{\mu\nu} = \frac{c^4}{16\pi G}\Big\{\left(g^{\mu\lambda}g^{\nu\sigma} - g^{\mu\nu}g^{\lambda\sigma}\right)\left(2\Gamma^\delta_{\lambda\sigma}\Gamma^\sigma_{\delta\sigma} - \Gamma^\rho_{\gamma\lambda}\Gamma^\gamma_{\rho\sigma} - \Gamma^\gamma_{\lambda\gamma}\Gamma^\delta_{\sigma\delta}\right) +$$
$$+g^{\mu\lambda}g^{\rho\sigma}\left(\Gamma^\nu_{\delta\lambda}\Gamma^\delta_{\rho\sigma} + \Gamma^\nu_{\rho\sigma}\Gamma^\delta_{\delta\lambda} - \Gamma^\nu_{\delta\sigma}\Gamma^\delta_{\lambda\rho} - \Gamma^\nu_{\lambda\rho}\Gamma^\delta_{\delta\sigma}\right)$$
$$+g^{\nu\lambda}g^{\rho\sigma}\left(\Gamma^\mu_{\delta\lambda}\Gamma^\delta_{\rho\sigma} + \Gamma^\mu_{\rho\sigma}\Gamma^\delta_{\delta\lambda} - \Gamma^\mu_{\delta\sigma}\Gamma^\delta_{\lambda\rho} - \Gamma^\mu_{\lambda\rho}\Gamma^\delta_{\delta\sigma}\right)$$
$$+g^{\lambda\sigma}g^{\rho\delta}\Gamma^\mu_{\lambda\rho}\left(\Gamma^\nu_{\sigma\delta} - \Gamma^\mu_{\lambda\sigma}\Gamma^\nu_{\rho\delta}\right)\Big\} \tag{10.23}$$

This means that there is an ordinary conservation law for the quantities

$$P^\mu \equiv \frac{1}{c}\int_\Sigma \sqrt{|g|}\ dS\ n_\lambda \left(T^{\mu\lambda} + t^{\mu\lambda}\right) = \frac{1}{c}\int n_\lambda\ dS\ \partial_\sigma H^{\mu\lambda\sigma} \tag{10.24}$$

If we take

$$\Sigma \equiv \quad \{t = t_0\} \cup \{r \leq L\} \tag{10.25}$$

then the normal vector is dual to the one-form

$$n = dt \tag{10.26}$$

that is,

$$n_\mu = \delta^0_\mu \tag{10.27}$$

It follows

$$P^\mu = \frac{1}{c}\int_\Sigma\ d^{n-1}x\ \partial_\sigma H^{\mu 0\sigma} \tag{10.28}$$

The boundary is the timelike surface

$$\partial\Sigma \equiv \{r = R\} \tag{10.29}$$

whose normal one-form is

$$n = dr \tag{10.30}$$

This means that

$$P^\mu = \frac{1}{c}\int_{\partial\Sigma} d^{n-2}x\ n_k\ H^{\mu 0k} \tag{10.31}$$

It is necessary to introduce asymptotically flat coordinates

$$r^2 \equiv \sum_i x_i^2$$
$$r dr = \sum x_i dx^i \tag{10.32}$$

The spatial components (assuming $d\theta = d\phi = 0$) read

$$g_{ij} = -\delta_{ij} - \frac{r_S}{r}\frac{x_i x_j}{r^2} \tag{10.33}$$

$$H^{00k} = \frac{c^4}{16\pi G}\partial_j\left(g^{00}g^{jk}\right) = \frac{c^4}{16\pi G}\partial_j\left(\left(1 + \frac{r_S}{r}\right)\left(\delta^{jk} + \frac{r_S}{r}\frac{x_k x_j}{r^2}\right)\right) = \frac{M}{8\pi}\partial_j\left(-\frac{\delta^{jk}}{r} + \frac{x_k x_j}{r^3}\right.$$
$$= \frac{M}{4\pi}\frac{x^k}{r^2} \quad (10$$

This yields the LL pseudoenergy

$$E \equiv \int_{r=R}\frac{M}{4\pi r^2}r^2\ d\theta \wedge d\phi = M \tag{10.35}$$

The formula in a general reference system is easily computed by restoring all terms involving first derivatives of the metric tensor. For example

$$\partial_\lambda g = 2\Gamma_\lambda g \tag{10.36}$$

as well as

$$\partial_\lambda g^{\mu\nu} = -g^{\mu\alpha}\Gamma^{\nu}_{\lambda\alpha} - g^{\nu\beta}\Gamma^{\mu}_{\beta\lambda} \tag{10.37}$$

and

$$\partial_\rho \Gamma^{\sigma}_{\alpha\beta} = -\frac{1}{2}\left(\Gamma^{\sigma}_{\delta\rho} + \Gamma^{\lambda}_{\omega\rho} g^{\sigma\omega} g_{\gamma\lambda}\right)\Gamma^{\delta}_{\alpha\beta} \tag{10.38}$$

Define

$$T^{\mu\nu} \equiv \frac{c^4}{8\pi G}\left(R^{\mu\nu} - \frac{1}{2}R g^{\mu\nu}\right) \tag{10.39}$$

then

$$t^{\mu\nu} \equiv \frac{1}{g}\partial_\lambda H^{\mu\nu\lambda} - T^{\mu\nu} \tag{10.40}$$

yielding

$$\begin{aligned}
&\frac{8\pi G}{c^4} t^{\mu\nu} = 2\left(g^{\mu\alpha}\Gamma^{\nu}_{\rho\alpha} + g^{\nu\alpha}\Gamma^{\mu}_{\rho\alpha}\right)\left(g^{\rho\beta}\Gamma_\beta + \Gamma^\rho\right) - \\
&-\left(g^{\mu\beta}\Gamma^{\lambda}_{\rho\beta} + g^{\lambda\beta}\Gamma^{\mu}_{\rho\beta}\right)\left(g^{\nu\alpha}\Gamma^{\rho}_{\lambda\alpha} + g^{\rho\alpha}\Gamma^{\nu}_{\lambda\alpha}\right) - \left(g^{\mu\alpha}\Gamma_\alpha + \Gamma^\mu\right)\left(g^{\nu\beta}\Gamma_\beta + \Gamma^\nu\right) - \\
&-2g^{\rho\alpha}\left(g^{\mu\alpha}\Gamma^{\nu}_{\alpha\rho} + g^{\nu\alpha}\Gamma^{\mu}_{\alpha\rho}\right) - 2g^{\mu\nu}\left(\Gamma^\lambda + \Gamma_\alpha g^{\lambda\alpha}\right) \\
&+2\left(\Gamma_\rho + \Gamma_\lambda\right)\left(g^{\nu\rho}\left(g^{\mu\alpha}\Gamma^{\lambda}_{\rho\alpha} + g^{\lambda\alpha}\Gamma^{\mu}_{\rho\alpha}\right) + g^{\mu\lambda}\left(g^{\nu\alpha}\Gamma_\alpha + \Gamma^\nu\right)\right) + \\
&+4\left(g^{\mu\nu}g^{\rho\lambda} - g^{\mu\lambda}g^{\nu\rho}\right)\Gamma_\rho\Gamma_\lambda - \left(g^{\mu\nu}g^{\rho\lambda} - g^{\mu\lambda}g^{\nu\rho}\right)\left(\Gamma^{\beta}_{\delta\lambda} + \Gamma_\pi g^{\beta\pi} g_{\delta\lambda}\right)\Gamma^{\delta}_{\rho\beta} - \\
&-g^{\mu\gamma}g^{\nu\omega}\left(-\frac{1}{2}\left(\Gamma_\delta + \Gamma^\lambda g_{\delta\lambda}\right)\Gamma^{\delta}_{\gamma\omega} + \frac{1}{2}\left(\Gamma^{\beta}_{\omega\delta} + \Gamma^{\lambda}_{\omega\epsilon} g^{\beta\epsilon} g_{\delta\lambda}\right)\Gamma^{\delta}_{\gamma\beta} + \Gamma^{\lambda}_{\gamma\omega}\Gamma_\lambda - \Gamma^{\rho}_{\gamma\lambda}\Gamma^{\lambda}_{\omega\rho}\right) + \\
&+\frac{1}{2}g^{\mu\nu}\Big[-\frac{1}{2}\left(\Gamma_\delta + \Gamma^\lambda g_{\delta\lambda}\right)\Gamma^\delta + \frac{1}{2}\Gamma^{\lambda}_{\rho\sigma}\left(g^{\rho\alpha}\Gamma^{\sigma}_{\lambda\alpha} + g^{\sigma\alpha}\Gamma^{\rho}_{\lambda\alpha}\right) + \\
&+\frac{1}{2}g^{\lambda\sigma}\left(\Gamma^{\beta}_{\sigma\delta} + \Gamma^{\lambda}_{\sigma\sigma'} g^{\beta\sigma'} g_{\delta\lambda}\right)\Gamma^{\delta}_{\lambda\beta} + \Gamma^\lambda\Gamma_\lambda - g^{\alpha\beta}\Gamma^{\rho}_{\alpha\lambda}\Gamma^{\lambda}_{\beta\rho}\Big]
\end{aligned} \tag{10.41}$$

10.3 Hamiltonians on curved surfaces.

When defining canonical momenta in gauge theories there are always relationahips between coordinates and momenta, which Dirac dubbed *primary constraints*.

$$\phi_i(q,p) = 0 \quad (1 = 1 \ldots P) \tag{10.42}$$

Dirac [8] defines equations that are true when the constraints are used *weak equations*. Let us consider the electromagnetic lagrangian as a primary example

$$S = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \sum_{i,j}\left(\frac{1}{2}\dot{A}_i^2 - \dot{A}_i\partial_i A_0 + \frac{1}{2}\left(\vec{\nabla}A_0\right)^2 - \frac{1}{4}F_{ij}F^{ij}\right) \tag{10.43}$$

It is clear that the canonical momentum conjugate to the variables $A_0(t, \vec{x})$ vanishes.

$$\pi_0(t, \vec{x}) \sim 0 \tag{10.44}$$

There are then ∞^3 (one for each point $\vec{x} \in \mathbb{R}^3$) primary constraints.

$$\pi_i = -\partial_i A_0 + \dot{A}_i \tag{10.45}$$

The hamiltonian is not uniquely defined; we can always add any combination of constraints to the naïve H:

$$H \equiv H_{\text{naive}} + \sum_{i=1}^{P} c_i \phi_i \tag{10.46}$$

For electrodynamics, it reads

$$H \equiv \int d^3x \left(\frac{\vec{\pi}^2}{2} + \frac{\vec{B}^2}{2} + \vec{\pi}.\vec{\nabla} A_0 + c_1 \pi^0 \right) \tag{10.47}$$

By consistency, we must demand that the primary constraints stay weakly zero after baing propagated in time, that is,

$$\dot{\phi}_i(q, p) \equiv \{\phi_i, H\} = 0 \tag{10.48}$$

In order to make sure of that, it might be neccessary to introduce new constraints; those are called *secondary constraints.* Coming back again to electrodynamics, we have

$$\dot{\pi}_0\ (x) = \{\pi_0(x), H\} = \vec{\nabla}.\vec{\pi}(x) \sim 0 \tag{10.49}$$

This is Gauss'law, which in this language appears as a secondary constraint.

Altogether we have a set of constraints both primery and secondary

$$\phi_I = 0 \quad I = 1 \ldots N) \tag{10.50}$$

For example, at this stage in electrodynamics we have two constraints per point, namely

$$\begin{aligned} &\phi_1 \equiv \pi_0 \sim 0 \\ &\phi_2 \equiv \vec{\nabla}.\vec{\pi} \sim 0 \end{aligned} \tag{10.51}$$

Let us determine the arbitrary functions in the hamiltonian. Let us consider the case where we can ensure that

$$\{\phi_I, H\} + \sum_j u_i \{\phi_I, \phi_j\} \sim 0 \tag{10.52}$$

without introducing any new constraints; that is, by determining appropiate functions $u_i = U_i(p,q)$. There is always an ambiguity. Namely, given a solution U_i of the above system, let us consider the independent solutions of

$$\sum_j V_{aj}\ (q,p)\{\phi_I,\phi_j\} \sim 0 \tag{10.53}$$

(and this for all constraints, $\forall I$. Assume there are $a = 1\ldots A$ of them (it is of course possible that $A = 0$). Then the total hamiltonian Dirac writes is

$$H_T \equiv H + \sum U_i\phi_i + v_a\phi_a \tag{10.54}$$

where

$$\phi_a \equiv \sum V_{ak}\phi_k \tag{10.55}$$

and we recall that the coefficients v_a are arbitrary.

This constraints ϕ_a are such that

$$\{\phi_a,\phi_k\} = \sum_j V_{aj}\ (q,p)\{\phi_j,\phi_k\} \sim 0 \tag{10.56}$$

Vanishes $\forall\phi_k$. It is useful to define *first class* functions of the phase space as those that have vanishing bracket with all constraints. Otherwise we say the quantity is *second class*. It is not difficult to show that the Poisson bracket of two first class quantities is another first class quantity.

Independent first class primary constraints are the generating functions of contact transformations that leave invariant the physical state. There is a redundant description of physical states in our variables; any element of the gauge orbit is a good representative of the physical state.

In the case of electromagnetism

$$\{\phi_1,\phi_2\} \sim 0 \tag{10.57}$$

so that both ϕ_1 and ϕ_2 are first class constraints. There are $2\times\infty^3$ first class constraints and no second class ones. In general, the number of independent *degrees of freedom* is given by

$$n_{DOF} = n - n_F - \frac{1}{2}n_S \tag{10.58}$$

(where n_F is the number of first class constraints, and n_S the number of second class ones.

Let us represent the remaining second class constraints as

$$\chi_i(q,p) \sim 0 \quad i = 1\ldots n_S \tag{10.59}$$

Let us form the matrix

$$A_{ij} \equiv \{\chi_i,\chi_j\} \tag{10.60}$$

It is a nonvanishing antisymmetric matrix (which tells us by the way, that S is an even number). Now Dirac showed that by replacing Poisson brackets with what we now call Dirac brackets, namely

$$\{f, g\}^* \equiv \{f, g\} - \sum \{f, \chi_i\} \left(A^{-1}\right)^{ij} \{\chi_j, g\} \tag{10.61}$$

This changes nothing for first class quantitites, including the total hamiltonian H_T. On the other hand, the Diract bracket of any function with a second class constraint vanishes.

$$\{f, \chi_l\}^* = 0 \tag{10.62}$$

By using Dirac brackets we can implement second class constraints as strong equations. One way of dealing with gauge systems is to *gauge fix*; that is to introduce a geuge condition, so that only a unique representative is chosen for each gauge orbit. In the present language this means that all constraints are now second class. For example, in electrodynamics, we can [15] chose the *radiation gauge*

$$\begin{aligned}
\phi_1 (t, \vec{x}) &\equiv \pi_0 (t, \vec{x}) \sim 0 \\
\phi_2 (t, \vec{x}) &\equiv \vec{\nabla} \vec{\pi} (t, \vec{x}) \sim 0 \\
\phi_3 (t, \vec{x}) &\equiv A_0 (t, \vec{x}) \sim 0 \\
\phi_4 (t, \vec{x}) &\equiv \vec{\nabla} \vec{A} (t, \vec{x}) \sim 0
\end{aligned} \tag{10.63}$$

This gauge is admissible because, first of all, the secondary constraint implies

$$\vec{\nabla} \frac{\partial}{\partial t} \vec{A}(t, \vec{x}) - \Delta A_0(t, \vec{x}) = 0 \tag{10.64}$$

This means that the equations

$$\delta A_0 = \partial_0 \Lambda = 0 \tag{10.65}$$

and

$$\delta \vec{\nabla} \vec{A} = \Delta \Lambda = 0 \tag{10.66}$$

are compatible. It is quite easy to show that

$$A (\vec{x} - \vec{y}) \equiv \begin{pmatrix} 0 & 0 & \delta^3 (\vec{x} - \vec{y}) & 0 \\ 0 & 0 & 0 & -\Delta \delta^3 (\vec{x} - \vec{y}) \\ -\delta^3 (\vec{x} - \vec{y}) & 0 & 0 & 0 \\ 0 & -\Delta \delta^3 (\vec{x} - \vec{y}) & 0 & 0 \end{pmatrix} \tag{10.67}$$

We define the inverse matrix in the functional sense, that is

$$\sum_j \int d^3 y \; A_{ij} (\vec{x} - \vec{y}) \; A^{-1}_{jk} (\vec{y} - \vec{z}) \; \equiv \delta_{ik} \; \delta^3 (\vec{x} - \vec{z}) \tag{10.68}$$

We get

$$A^{-1} = \begin{pmatrix} 0 & 0 & -\delta^3(\vec{y}-\vec{z}) & 0 \\ 0 & 0 & 0 & \frac{1}{4\pi|\vec{x}-\vec{y}|} \\ \delta^3(\vec{x}-\vec{y}) & 0 & 0 & 0 \\ 0 & -\frac{1}{4\pi|\vec{x}-\vec{y}|} & 0 & 0 \end{pmatrix} \tag{10.69}$$

Dirac brackets now read

$$\begin{aligned} \{\pi_\mu(t,\vec{x}), A^\nu(t,\vec{y})\} &= \left(\eta_\mu^\nu - \eta_\mu^0\eta_0^\nu\right)\delta^3(\vec{y}-\vec{z}) + \partial_\mu\partial^\nu\frac{1}{4\pi|\vec{x}-\vec{y}|} \\ \{\pi_\mu,(t,\vec{x}),\pi_\nu(t,\vec{y})\} &= \{A^\mu,(t,\vec{x}),A^\nu(t,\vec{y})\} = 0 \end{aligned} \tag{10.70}$$

and the hamiltonian is

$$H = \int d^3x \left(\frac{\vec{\pi}^2}{2} + \frac{\vec{B}^2}{2}\right) \tag{10.71}$$

10.3.1 Lagrangians homogeneous of the first degree.

- In order to get a feeling of what Diff_0 invariance means in the present context, let us consider now the case of homogeneous of first degree lagrangians, that is, those that obey

$$\sum \dot{q}^i\frac{\partial L}{\partial\dot{q}^i} = L \tag{10.72}$$

This naively means that the hamiltonian is zero.

First of all, let us note that *any* system can be put in such a way by introducing new phantom degrees of freedom. Assume that

$$L = \sum_{ij}\frac{1}{2}g_{ij}(q)\dot{q}^i\dot{q}^j - V(q) \tag{10.73}$$

This system is equivalent to

$$L = \sum_{ij} g_{ij}(q)\left(b^i\dot{q}^j - \frac{1}{2}b^ib^j\right) - V(q) \tag{10.74}$$

Tha algebraic EM for b^i read

$$\dot{q}^j = b^i \tag{10.75}$$

and substituting them in the first order lagrangian, it yields the second order one.

In fact let us consider the time variable as an extra coordinate (more about this in the next paragraph)

$$q^0 \equiv t \tag{10.76}$$

The new lagrangian for our system is defined as

$$L^* \equiv L\left(q, \frac{\frac{dq}{d\tau}}{\frac{dq^0}{d\tau}}\right) \equiv L^*\left(q, \frac{dq}{d\tau}\right) \tag{10.77}$$

and is such that in a precise sense,

$$\int L^* \, d\tau = \int L dt \tag{10.78}$$

Let us elaborate. A general variation of the action is defined as

$$\begin{aligned} \delta \int_{t_1}^{t_2} dt \left(p_i \dot{q}^i - H\right) &= \int_{t_1+\delta t_1}^{t_2+\delta t_2} dt \left[(p_i + \delta p_i)(\dot{q}^i + \delta \dot{q}^i) - (H + \delta H)\right] - \\ &- \int_{t_1}^{t_2} dt \left(p_i \dot{q}^i - H\right) = \\ = \int_{t_1}^{t_2} dt \left(p_i \delta \dot{q}^i + \delta p_i \dot{q}^i - \delta H\right) &+ \int_{t_2}^{t_2+\delta t_2} dt \left(p_i \dot{q}^i - H\right) - \int_{t_1}^{t_1+\delta t_1} dt \left(p_i \dot{q}^i - H\right) = \\ = \int_{t_1}^{t_2} dt \left(\frac{d}{dt}\left(p_i \delta q^i\right) - \dot{p}_i \delta q^i + \dot{q}^i \delta p_i - \delta H\right) &+ \delta t_2 \left. \left(p_i \dot{q}^i - H\right)\right|_{t_2} - \delta t_1 \left. \left(p_i \dot{q}^i - H\right)\right|_{t_1} = \\ &= G(t_2) - G(t_1) \end{aligned} \tag{10.79}$$

where

$$G(t) \equiv \left(p_i \dot{q}^i - H\right) \delta t + F(t) + p_i \delta q^i \tag{10.80}$$

and

$$F(t) \equiv \int^t dt' \left(-\dot{p}_i \delta q^i + \dot{q}^i \delta p_i - \delta H\right) \tag{10.81}$$

The EM are recovered when the variations satisfy the conditions of the action principle, that is, when

$$\begin{aligned} &\delta t = 0 \\ &\delta q_i|_{t_i} = \delta q_i|_{t_f} = 0 \end{aligned} \tag{10.82}$$

by demanding that

$$F(t) = 0 \tag{10.83}$$

we recover the standard EM.

- It is always possible to rewrite the action principle in *parametrized form* by enlarging the configuration space

$$S = \int_{t_1}^{t_2} dt \left(\sum_{i=1}^{n} p_i \dot{q}^i - H\right) = \int_{\tau_1}^{\tau_2} d\tau \sum_{i=1}^{n+1} p_i q'_i \tag{10.84}$$

with

$$\begin{aligned} &q'_i \equiv \frac{dq^i}{d\tau} \\ &R(p, q) \equiv p_{n+1} + H = 0 \\ &q^{n+1} = t \end{aligned} \tag{10.85}$$

The constraint can be implemented by a Lagrange multiplier

$$S = \int_{\tau_1}^{\tau_2} d\tau \left(\sum_1^{n+1} p_i q_i' - N R(p.q) \right) \tag{10.86}$$

The EM for N

$$\frac{\delta S}{\delta N} = R = 0 \tag{10.87}$$

yields the constraint. The theory is now invariant under arbitrary reparametrizations

$$\tau \rightarrow \tau' \tag{10.88}$$

provided

$$N(\tau) d\tau \tag{10.89}$$

is interpreted as a 1-form.

To reduce a parametrized action to canonical form we insert the solution of the constraint equations

$$S = \int d\tau \left(\sum_1^{n} p_i q_i' - H q_{n+1}' \right) = \int dq_{n+1} \left(\sum_1^{n} p_i \frac{dq^i}{dq_{n+1}} - H \right) \tag{10.90}$$

so that q_{n+1} plays the rôle of the time coordinate. The EM only tell us that

$$- N(\tau) \frac{\partial R}{\partial p_{n+1}} = \frac{dp_{n+1}}{d\tau} = - \frac{\partial H}{\partial \tau} \equiv - \sum_1^{n+1} \frac{\partial H}{\partial q^i} \dot{q}^i \tag{10.91}$$

but given the fact that $N(\tau)$ is undetermined, this means that it is possible to choose $q_{n+1}(\tau)$ also at will.

10.4 Dirac universal brackets

In this section it will prove convenient to reserve tha label y^α for the spacetime coordinates to tell them apart from the $3+1$ coordinates (t, x^i) to be defined in the sequel.

Let us now consider a foliation of space-time given by the function

$$t(y^\alpha) = C \tag{10.92}$$

The level hypersurfaces are spacelike; that is, the normal vector

$$n_\alpha \equiv N \partial_\alpha t \tag{10.93}$$

is timelike, and will always be normalized

$$n^2 = 1 \tag{10.94}$$

so that

$$\frac{1}{N^2} \equiv g^{\alpha\beta}\partial_\alpha t \partial_\beta t \tag{10.95}$$

We shall also assume the existence of a congruence of curves

$$y^\alpha \equiv \sigma^\alpha(x^i, t) \tag{10.96}$$

Each curve in the congruence is thepresented by

$$x^i = C^i \tag{10.97}$$

Tangent vectors on the hypersurface are given by

$$\xi_i^\alpha \equiv \partial_i \sigma^\alpha \quad i = 1 \ldots n-1 \tag{10.98}$$

The normal vector n is such that

$$g_{\alpha\beta}\xi_i^\alpha n^\beta = 0 \tag{10.99}$$

The vector tangent to the congruence can be expanded as

$$N^\alpha \equiv \frac{\partial \sigma^\alpha}{\partial t} \equiv N n^\alpha + N^i \xi_i^\alpha \equiv N n^\alpha + \mathcal{N}^\alpha \tag{10.100}$$

where

$$N \equiv N^\alpha n_\alpha = \frac{\partial \sigma^\alpha}{\partial t} N \ \partial_\alpha t \tag{10.101}$$

is the *lapse* and N^i is the *shift* in ADM's (Arnowitt-Deser-Misner) notation.

This means that the vector that goes from $(t, x) \in \Sigma_t$ to the point $(t + dt, x) \in \Sigma_{t+dt}$ does not lie necessarily in the direction of the normal to the hypersurface.

Actually from the very definition follows that

$$N^\alpha \partial_\alpha t = 1 \tag{10.102}$$

Also our parametrization of the curves of the congruence as $x^i = C^i$ imply that

$$\pounds(N^\alpha)\xi_i^\beta = 0 \tag{10.103}$$

Finally, the fact that the coordinates x^i on each surface are independent means that, considered as spacetime vectors,

$$[\xi_i, \xi_j] = 0 \tag{10.104}$$

The ∞^3 dymamical variables σ^α will have some canonically conjugate momenta

$$\{\sigma^\mu(t, x), \pi_\nu(t, y)\} = \delta^\mu_\nu \delta^{n-1}(x - y) \tag{10.105}$$

If the generalized coordinates σ are to vary, the constraints have to involve the conjugate momenta, so that it must be possible to write the constraints as

$$H_T = \int d^{n-1}x \quad c^\alpha(t,x)\,(\pi_\alpha + K_\alpha) \tag{10.106}$$

Then it is a fact that

$$\dot{\sigma}^\mu \equiv \{\sigma^\mu, H_T\} = c^\mu \tag{10.107}$$

It is useful to decompose any vector index into normal and tangential components

$$\begin{aligned} V_n &\equiv V.n \\ V_i &\equiv V.\xi_i \end{aligned} \tag{10.108}$$

It is always the danger of taking V_i such defined as the space components of the n-dimensional quantity V, but we shall try not to do so in the future. Actually,

$$V^\mu = V_n.n^\mu + V_i h^{ij} \xi_j^\mu \tag{10.109}$$

where

$$h_{ij} \equiv \xi_i^\mu g_{\mu\nu} \xi_j^\nu \tag{10.110}$$

and the inverse matrix

$$h^{ij} h_{jk} \equiv \delta^i_k \tag{10.111}$$

(Please notice that

$$h^{ij} \neq g^{ik} g^{lm} h_{lm}) \tag{10.112}$$

The spacetime metric reads

$$\begin{aligned} ds^2 = g_{\mu\nu} dy^\mu dy^\nu = g_{\mu\nu}\left(N^\mu dt + \xi_i^\mu dx^i\right)\left(N^\nu dt + \xi_i^\nu dx^i\right) = \\ = N^2 dt^2 + h_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right) \end{aligned} \tag{10.113}$$

This in turn implies

$$g_{\mu\nu} = n_\mu n_\nu + \xi^i_\mu \xi_{\nu i} \tag{10.114}$$

Then Dirac showed that for all these systems there is an universal set of Poisson brackets, namely,

$$\begin{aligned} \{\pi_r(t,x), \pi_s(t,x')\} &= \pi_s(t,x)\partial_r\delta(x-x') + \pi_r(t,x')\partial_s\delta(x-x') \\ \{\pi_n(t,x), \pi_r(t,x')\} &= \pi_n(t;x')\partial_r\delta(x-x') \\ \{\pi_n(t,x), \pi_n(t,x')\} &= -2\pi^r(t,x)\partial_r(x-x') - \Delta\pi(t,x)\delta(x-x') \end{aligned} \tag{10.115}$$

Let us work this out in some detail.

$$0 = \{n_\mu \xi^\mu_i, \pi_{\nu'}\} = \{n_\mu, \pi_{\nu'}\}\xi^\mu_i + n_\mu\{\xi^\mu_i, \pi_{\nu'}\} =$$
$$= \{n_\mu, \pi_{\nu'}\}\xi^\mu_i + n_\nu \partial_i \delta(x - x') \qquad (10.116)$$

We learn that

$$\{n_\mu, \pi_{\nu'}\}\xi^\mu_i = -n_\nu \partial_i \delta(x - x') \qquad (10.117)$$

Again,

$$0 = \frac{1}{2}\{n^2, \pi_{\mu'}\} = \{n_\mu, \pi_{\mu'}\}n^\mu \qquad (10.118)$$

Then

$$\{n_\lambda, \pi_{\nu'}\} = \{n_\rho, \pi_{\nu'}\}\left(n^\rho n_\lambda + \xi^\rho_j \xi_{\lambda k} h^{jk}\right) =$$
$$= -n_\nu \xi^j_\lambda \partial_j \delta(x - x') \equiv -n_\nu \delta_{-\lambda}(x - x') \qquad (10.119)$$

so that

$$\{n_\lambda, \pi_{n'}\} = \{n_\lambda, \pi_{\mu'} n^{\mu'}\} = -n^{\mu'} n_\mu \delta_{-\lambda} =$$
$$= -\xi^i_\lambda \partial_i \left(n^{\mu'} n_\mu \delta(x - x')\right) + \xi^i_\lambda \partial_i \left(n^{\mu'} n_\mu\right)\delta = -\xi^i_\lambda \partial_i \delta(x - x')$$

Before going on, let us show an elementary relationship. It is plain that

$$\partial_j n_\mu \xi^\mu_i + n_\mu \partial_j \xi^\mu_i = 0 \qquad (10.120)$$

as well as

$$n_\mu \partial_i n^\mu = 0 \qquad (10.121)$$

as well as

$$\partial_i n_\mu \xi^\mu_j = -n_\mu \partial_i \xi^\mu_j = -n_\mu \partial_j \xi^\mu_i = \partial_j n_\mu \xi^\mu_i \qquad (10.122)$$

and multiplying by $\xi^{j\rho}$

$$\partial_i n^\rho = \xi^{j\rho} \partial_j n_\mu \xi^\mu_i \qquad (10.123)$$

Now

$$n_{\mu-\nu} \equiv \partial_i n_\mu \xi^i_\nu = \xi^i_\nu \xi^{j\mu} \partial_j n_\lambda \xi^\lambda_i = \xi^{j\mu} \partial_j n_\nu \equiv n_{\nu-\mu} \qquad (10.124)$$

It follows that

$$\{n_\lambda, \pi_{j'}\} = \{n_\lambda, \pi_{\nu'} \xi^{\nu'}_{j'}\} = -n_\nu \xi^k_\lambda \partial_k \delta(x - x')\xi^{\nu'}_{j'} =$$
$$= -\partial_k \left(n_\nu \xi^{\nu' j'} \xi^k_\lambda \delta(x - x')\right) + \partial_k \left(n_\nu \xi^k_\lambda \xi^{\nu'}_{j'}\right)\delta(x - x') = \partial_k n_\nu \xi^k_\lambda \xi^\nu_j \delta(x - x') =$$
$$= \partial_k n_\lambda \xi^k_\nu \xi^\nu_j (x - x') = \partial_j n_\lambda \delta(x - x')$$

Finally

$$\{\pi_n, \pi_{n'}\} = \{\pi_\lambda n^\lambda, \pi_{\mu'} n^{\mu'}\} = \pi_\lambda n^{\mu'}\{n^\lambda, \pi_{\mu'}\} + n^\lambda \pi_{\mu'}\{\pi_\lambda, n^{\mu'}\} = \pi_\lambda\{n^\lambda, \pi_{n'}\} + \pi_{\mu'}\{\pi_n, n^{\mu'}\}$$
$$-\pi_\lambda \xi^{\lambda i} \partial_i \delta(x - x') + \pi_{\mu'} \xi^{\mu' i} \partial_i \delta(x - x') \qquad (10.12$$

10.5 The Arnowitt-Deser-Misner (ADM) formalism.

Let us apply Dirac's ideas to the gravitational field. We shall assume that there is a foliation as before. We shall need the components of the spacetime metric in terms of the lapse and shift functions.

$$\begin{aligned} g_{00} &= N^2 \\ g_{0i} &= h_{ij} N^j \\ g_{ij} &= h_{ij} \end{aligned} \tag{10.126}$$

whose inverse reads

$$\begin{aligned} g^{00} &= N^{-2} \\ g^{0i} &= -\frac{N^i}{N^2} \\ g^{ij} &= h^{ij} + \frac{N^i N^j}{N^2} \end{aligned} \tag{10.127}$$

Let us denote by D_i the covariant derivative with respect to the $(n-1)$-dimensional Levi-Civita connection associated to the induced metric, h_{ij}.

It can be easily checked that

$$D_j A_i = \nabla_\beta A_\alpha \xi^\alpha_i \xi^\beta_j \tag{10.128}$$

From the definition itself of the induced metric follows

$$\partial_\rho g_{\alpha\beta} D_k \sigma^\rho \partial_i \sigma^\alpha \partial_j \sigma^b + g_{\alpha\beta} D_k(\partial_i \sigma^\alpha)\partial_j \sigma^\beta + g_{\alpha\beta}\partial_i \sigma^\alpha D_k(\partial_j \sigma^\beta) = 0 \tag{10.129}$$

Cyclic permutations

$$\begin{aligned} \partial_\rho g_{\alpha\beta} D_j \sigma^\rho \partial_k \sigma^\alpha \partial_i \sigma^\beta + g_{\alpha\beta} D_j(\partial_k \sigma^\alpha)\partial_i \sigma^\beta + g_{\alpha\beta}\partial_k \sigma^\alpha D_j(\partial_i \sigma^\beta) = 0 \\ \partial_\rho g_{\alpha\beta} D_i \sigma^\rho \partial_j \sigma^\alpha \partial_k \sigma^\beta + g_{\alpha\beta} D_i(\partial_j \sigma^\alpha)\partial_i \sigma^\beta + g_{\alpha\beta}\partial_j \sigma^\alpha D_i(\partial_k \sigma^\beta) = 0 \end{aligned}$$

Adding 1+2-3 yields

$$\begin{aligned} 0 = g_{\alpha\beta} D_j D_k \sigma^\alpha D_i \sigma^\beta + D_k \sigma^\rho D_i \sigma^\alpha D_j \sigma^\beta \frac{1}{2}\left(\partial_\rho g_{\alpha\beta} + \partial_\beta g_{\rho\alpha} - \partial_\alpha g_{\beta\rho}\right) = \\ g_{\alpha\beta} D_j D_k \sigma^\alpha D_i \sigma^\beta + D_k \sigma^\rho D_i \sigma^\alpha D_j \sigma^\beta \{\alpha, \beta\rho\} = \\ = g_{\alpha\beta} D_i \sigma^\beta \left(D_k D_j \sigma^\alpha + \{{}^{\alpha}_{\beta\rho}\} D_j \sigma^\beta D_k \sigma^\rho\right) \end{aligned} \tag{10.130}$$

This means that

$$D_k D_j \sigma^\alpha = -\{{}^{\alpha}_{\beta\rho}\} D_j \sigma^\beta D_k \sigma^\rho + K_{jk} n^\alpha \tag{10.131}$$

where the normal component reads

$$K_{jk} \equiv n_\alpha \left(D_k D_j \sigma^\alpha + \{{}^{\alpha}_{\beta\rho}\} D_j \sigma^\beta D_k \sigma^\rho\right) \tag{10.132}$$

Taking the D_j

$$0 = D_j\left(g_{\alpha\beta}\partial_i\sigma^\alpha n^\beta\right) = D_j g_{\alpha\beta}\partial_i\sigma^\alpha n^\beta + g_{\alpha\beta}D_j D_i\sigma^\alpha n^\beta + g_{\alpha\beta}D_i\sigma^\alpha D_j n^\beta \tag{10.133}$$

On the other hand,

$$D_j g_{\alpha\beta} = D_j\sigma^\nu\partial_\nu g_{\alpha\beta} = D_j\sigma^\nu\left(\{\alpha\nu;\beta\} + \{\beta\nu;\alpha\}\right) \tag{10.134}$$

so that

$$K_{jk} = n_\alpha\{{}^{\alpha}_{\beta\rho}\}D_j\sigma^\beta D_k\sigma^\rho - g_{\alpha\beta}D_j\sigma^\alpha D_k n^\beta - n^\beta D_k\sigma^\rho\left(\{\alpha\rho;\beta\} + \{\beta\rho;\alpha\}\right)D_j\sigma^\alpha =$$
$$-g_{\alpha\beta}D_j\sigma^\alpha D_k n^\beta - n^\beta D_k\sigma^\rho\{\beta\rho;\alpha\}D_j\sigma^\alpha = -\xi^\alpha_i\nabla_\rho n_\alpha\xi^\rho_j \tag{10.135}$$

This tensor is called the *extrinsic curvature*, and represents the derivative of the normal vector, projected on the surface.

Our purpose in life is now to relate the Riemann tensor on the hypersurface (computed with the induced metric) with the corresponding Riemann tensor of the spacetime manifold. Those are the famous Gauss-Codazzi equations, which we purport now to derive. They were one of the pillars of Gauss' *theorema egregium*, [25] which asserts that *If a curved surface is developed upon any other surface whatever the measure of curvature in each point remains unchanged.*

We start with

$$0 = D_j\left(g_{\alpha\beta}n^\alpha n^\beta\right) = D_j\sigma^\rho\left(\{\alpha\rho;\beta\} + \{\rho\beta;\alpha\}\right)n^\alpha n^\beta + g_{\alpha\beta}D_j n^\alpha n^\beta + g_{\alpha\beta}n^\alpha D_j n^\beta =$$
$$g_{\alpha\beta}n^\beta\left(D_j n^\alpha + \{{}^{\alpha}_{\mu\nu}\}D_j\sigma^{(\mu}n^{\nu)}\right) = g_{\alpha\beta}n^\beta\nabla_\mu n^\alpha D_j\sigma^\mu = n_\alpha\nabla_\mu n^\alpha\xi^\mu_j \tag{10.136}$$

On the other hand, the explicit expression for the extrinsic curvature reads

$$K_{ij} = -\xi^\alpha_i\nabla_\rho n_\alpha\xi^\rho_j \tag{10.137}$$

First of all let us derive some properties of the extrinsic curvature. It is symmetric, $K_{ij} = K_{ji}$.

$$-K_{ij} = \nabla_\beta n_\alpha\xi^\alpha_i\xi^\beta_j = -n_\alpha\nabla_\beta\xi^\alpha_i\xi^\beta_j \tag{10.138}$$

But

$$\left[\xi^\beta_j, \xi^\alpha_i\right] = 0 \tag{10.139}$$

so that

$$-K_{ij} = -n_\alpha\xi^\alpha_i\nabla_\beta\xi^\alpha_j = \nabla_\beta n_\alpha\xi^\beta_i\xi^\alpha_l = K_{ji} \tag{10.140}$$

This symmetry implies a very useful formula for the extrinsic curvature, namely

$$-K_{ij} = \nabla_{(\beta}n_{\alpha)}\xi^\alpha_i\xi^\beta_j = \pounds(n)g_{\alpha\beta}\xi^\alpha_i\xi^\beta_j \tag{10.141}$$

By the way, in the physics jargon when $K_{ij} = 0$ it is said that it is a *moment of time symmetry*.

On the other hand, remembering that

$$\xi^\alpha_i \xi^i_\beta = g^\alpha_\beta - n^\alpha n_\beta \tag{10.142}$$

we deduce that

$$-K_{ij}\xi^i_\mu = -\left(g^\alpha_\mu - n^\alpha n_\mu\right)\nabla_\rho n_\alpha \xi^\rho_j = -\nabla_\rho n_\mu \xi^\rho_j \tag{10.143}$$

(because of [10.136]).

Let us analyze the definition of extrinsic curvature in even more detail.

$$\begin{aligned}
&(D_k D_j D_i - D_j D_k D_i)\,\sigma^\alpha = \xi^\alpha_m h^{mh} R_{hijk} = D_k\left(-\{^{\ \alpha}_{\beta\rho}\}\xi^\beta_i\xi^\rho_j + K_{ij}n^\alpha\right) - \\
&-D_j\left(-\{^{\ \alpha}_{\beta\rho}\}\xi^\beta_i\xi^\rho_k + K_{ik}n^\alpha\right) = \partial_k\{^{\ \alpha}_{\beta\rho}\}\xi^\beta_i\xi^\rho_j - \{^{\ \alpha}_{\beta\rho}\}D_k\xi^\beta_i\xi^\rho_j - \{^{\ \alpha}_{\beta\rho}\}\xi^\beta_i D_k\xi^\rho_j + \\
&D_k K_{ij}n^\alpha + K_{ij}D_k n^\alpha + \partial_j\{^{\ \alpha}_{\beta\rho}\}\xi^\beta_i\xi^\rho_k - \{^{\ \alpha}_{\beta\rho}\}D_j\xi^\beta_i\xi^\rho_k + \{^{\ \alpha}_{\beta\rho}\xi^\beta_i D_j\xi^\rho_k - D_j K_{ik} - K_{ik}D_j n^\alpha
\end{aligned}$$

and using again the defnition of the extrinsic curvature to eliminate the term with two derivatives,

$$\begin{aligned}
&\xi^\alpha_m h^{mr} R_{rijk}[h] = -\partial_k\{^{\ \alpha}_{\beta\rho}\}\xi^\beta_i\xi^\rho_j - \{^{\ \alpha}_{\beta\rho}\}\xi^\rho_j\left(-\{^{\ \beta}_{\mu\nu}\}\xi^\mu_i\xi^\nu_k + K_{ik}n^\beta\right) + D_k K_{ij}n^\alpha + K_{ij}D_k n^\alpha + \\
&\partial_j\{^{\ \alpha}_{\beta\rho}\}\xi^\beta_i\xi^\rho_k + \{^{\ \alpha}_{\beta\rho}\}\xi^\rho_k\left(-\{^{\ \beta}_{\mu\nu}\}\xi^\mu_i\xi^\nu_j + K_{ij}n^\beta\right) - D_j K_{ik}n^\alpha - K_{ik}D_j n^\alpha = \\
&n^\alpha\left(D_k K_{ij} - D_j K_{ik}\right) + K_{ij}\left(D_k n^\alpha + \{^{\ \alpha}_{\beta\rho}\}n^\beta\xi^\rho_k\right) - K_{ik}\left(D_j n^\alpha + \{^{\ \alpha}_{\beta\rho}n^\beta\xi^\rho_j\right) - \\
&-\xi^\beta_i\xi^\rho_j\xi^\sigma_k\left(\partial_\sigma\{^{\ \alpha}_{\beta\rho}\} - \partial_\rho\{^{\ \alpha}_{\beta\sigma}\} - \{^{\ \alpha}_{\lambda\rho}\}\{^{\ \lambda}_{\beta\sigma} + \{^{\ \alpha}_{\lambda\sigma}\}\{^{\ \lambda}_{\beta\rho}\right)
\end{aligned} \tag{10.144}$$

Using again the definition of the extrinsic curvature, as well as the one of the full Riemann tensor, we get

$$\xi^\alpha_m h^{mr}\left(R_{rijk}[h] + K_{ij}K_{rk} - K_{ik}K_{rj}\right) - n^\alpha\left(D_k K_{ij} - D_j K_{ik}\right) = -\xi^\beta_i\xi^\rho_j\xi^\sigma_k R^\alpha{}_{\beta\sigma\rho}[g]$$

This projects into the famous Gauss-Codazzi equations

$$R_{lijk}[h] + K_{il}K_{jk} - K_{ik}K_{lj} = \xi^\alpha_l\xi^\beta_i\xi^\rho_j\xi^\sigma_k R_{\alpha\beta\rho\sigma}[g] \tag{10.145}$$

as well as

$$D_j K_{ik} - D_k K_{ij} = -n^\alpha\xi^\beta_i\xi^\rho_j\xi^\sigma_k R_{\alpha\beta\sigma\rho}[g] \tag{10.146}$$

Please note that not all components of the full Riemann tensor can be recovered from the knowledge of the Riemann tensor computed on the hypersurface plus the extrinsic curvature. As a matter of fact,

$${}^{(n)}R = {}^{(n-1)}R^{ij}{}_{ij} + 2\,{}^{(n)}R^i{}_{nin} = {}^{(n-1)}R + K^2 - K_{ij}k^{ij} + 2\,{}^{(n)}R^i{}_{nin} \tag{10.147}$$

This means that an explicit computation of ${}^{(n)}R^{i}{}_{nin}$ is needed before the Einstein-Hilbert term could be written in the 1+(n-1) decomposition. To do that, consider Ricci's identity

$$\nabla_\gamma \nabla_\beta n_\alpha - \nabla_\beta \nabla_\gamma n_\alpha = R^\rho{}_{\alpha\beta\gamma} n_\rho \tag{10.148}$$

Now

$$n^\beta \left(\nabla_\gamma \nabla_\beta n^\gamma - \nabla_\beta \nabla_\gamma n^\gamma\right) = n^\beta g^{\alpha\gamma} R^\rho{}_{\alpha\beta\gamma} n^\rho \equiv R^{n\alpha}{}_{n\alpha} \tag{10.149}$$

Besides,

$$\begin{aligned} &\nabla_\gamma n^\beta \nabla_\beta n^\gamma = \nabla_\gamma n_\beta \left(n^\beta n^\mu + \xi^\beta_i \xi^{\mu i}\right)\left(n^\gamma n^\nu + \xi^\gamma_j \xi^{j\nu}\right) \nabla_\mu n_\nu = \\ &\nabla_\gamma n_\beta \xi^\beta_i \xi^{\mu i} \xi^\gamma_j \xi^{j\nu} \nabla_\mu n_\nu = -K_{ij} K^{ij} \end{aligned} \tag{10.150}$$

Summarizing,

$$\begin{aligned} R^{n\alpha}{}_{n\alpha} = n^\beta \nabla_\gamma \nabla_\beta n^\gamma - n^\beta \nabla_\beta \nabla_\gamma n^\gamma = \nabla_\gamma \left(n^\beta \nabla_\beta n^\gamma\right) - \nabla_\gamma n^\beta \nabla_\beta n^\gamma - \nabla_\beta \left(n^\beta \nabla_\gamma n^\gamma\right) + \\ + \nabla_\beta n^\beta \nabla_\gamma n^\gamma = \\ = \nabla_\gamma \left(n^\beta \nabla_\beta n^\gamma - n^\gamma \nabla_\beta n^\beta\right) + K_{ij} K^{ij} - K^2 \end{aligned} \tag{10.151}$$

Besides,

$$\sqrt{{}^{(n)}\, g} = N \sqrt{{}^{(n-1)}\, g} \tag{10.152}$$

The EH lagrangian can then be written as follows

$$L_{EH} = N \sqrt{{}^{(n-1)}g} \left({}^{(n-1)}R + K_{ij} K^{ij} - K^2\right) - \partial_\alpha V^\alpha \equiv L'_{EH} - \partial_\alpha V^\alpha \tag{10.153}$$

where

$$V^\alpha \equiv 2\sqrt{{}^{(n)}g} \left(n^\beta \nabla_\beta n^\gamma - n^\gamma \nabla_\beta n^\beta\right) \tag{10.154}$$

The resulting lagrangian, L'_{EH} does not contain $\dot{N}$ or $\dot{N}^i$, and does containg only first time derivatives of g_{ij}. This is the starting point of the ADM hamiltonian formalism. There are the primary constraints

$$\pi^\mu = \frac{\delta L}{\dot{N}^\mu} \tag{10.155}$$

In order to compute the spacelike momenta, consider

$$\begin{aligned} &\dot{h}_{ij} = \pounds(N^\alpha) h_{ij} = \xi^\alpha_i \xi^\beta_j \pounds(N^\alpha) g_{\alpha\beta} = \\ &\text{(Remembering that this Lie derivative of the spacelike basis vectors vanishes)} = \\ &= \xi^\alpha_i \xi^\beta_j \left(\nabla_\alpha N_\beta + \nabla_\beta N_\alpha\right) = \xi^\alpha_i \xi^\beta_j \left(\nabla_\alpha \left(N n_\beta + \mathcal{N}_\beta\right) + \nabla_\beta \left(N n_\alpha + \mathcal{N}_\alpha\right)\right) = \\ &= 2N K_{ij} + D_i N_j + D_j N_i \end{aligned} \tag{10.156}$$

Then

$$\frac{\delta K_{ij}}{\delta \dot{h}_{kl}} = \frac{1}{4N}\left(\delta_i^k \delta_j^l + \delta_j^k \delta_i^l\right) \tag{10.157}$$

and

$$\frac{\delta K}{\delta \dot{k}_{kl}} = \frac{1}{2N} h^{kl} \tag{10.158}$$

and

$$\pi^{ij} = \sqrt{h}\left(K^{ij} - K h^{ij}\right) \tag{10.159}$$

Let us compute now the hamiltonian

$$H \equiv \int d^3x \left(\pi_\mu \dot{N}^\mu + \pi_{ij}\dot{h}^{ij}\right) - L \tag{10.160}$$

where

$$L = N\sqrt{|h|}\left(R[h] + K_{ij}K^{ij} - K^2\right) \tag{10.161}$$

Now, it is clear that

$$\sqrt{h}\left(K_{ij}K^{ij} - K^2\right) = \frac{1}{\sqrt{h}}\left(\pi_{ij}\pi^{ij} - \frac{\pi^2}{2}\right) \tag{10.162}$$

We just derived

$$\dot{h}_{ij} = 2NK_{ij} + D_i N_j + D_j N_i \tag{10.163}$$

Summarizing,

$$\begin{aligned} H &= \pi_{ij}\dot{h}^{ij} - L = \\ &= \pi^{ij}\left(\frac{2N}{\sqrt{|h|}}\left(\pi_{ij} - \frac{1}{2}\pi h_{ij}\right) + D_i N_j + D_j N_i\right) - N\left(\sqrt{|h|}R[h] + \frac{1}{\sqrt{|h|}}\left(\pi_{ij}\pi^{ij} - \frac{1}{2}\pi^2\right)\right) = \\ &= \int d^3x \left(N\mathcal{H} + N^i \mathcal{H}_i\right) \end{aligned} \tag{10.164}$$

where (dropping surface terms)

$$\begin{aligned} \mathcal{H} &= \frac{1}{2\sqrt{h}}\left(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl}\right)\pi^{ik}\pi^{kl} - \sqrt{h}R[h] = \frac{1}{\sqrt{h}}\left(\pi_{ij}\pi^{ij} - \frac{1}{2}\pi^2\right) - \sqrt{h}R[h] \\ \mathcal{H}_i &\equiv -2D_j \pi_i^j \end{aligned} \tag{10.165}$$

Now we have the following constraints

$$\begin{aligned} &\pi_\mu \sim 0 \\ &N^\mu - C^\mu \sim 0 \\ &\mathcal{H} \sim 0 \\ &\mathcal{H}_i \sim 0 \end{aligned} \tag{10.166}$$

and the corresponding brackets

$$\begin{aligned}
&\{\pi^\mu, N^\rho - C^\rho\} \sim g^{\mu\rho}\delta(x-x')\\
&\{\pi_\mu, \mathcal{H}\} \sim 0\\
&\{\pi_\mu, \mathcal{H}_i\} \sim 0\\
&\{N^\mu - C^\mu, \mathcal{H}\} \sim 0\\
&\{N^\mu - C^\mu, \mathcal{H}_i\} \sim 0
\end{aligned} \tag{10.167}$$

The other brackets got the universal Dirac-Schwinger form, which is valid for any diffeomorphism invariant field theory

$$\begin{aligned}
&\{\mathcal{H}(x), \mathcal{H}(x') = \left(\mathcal{H}^i(x) + \mathcal{H}^i(x')\right)\partial_i\delta(x-x')\} \sim\\
&\{\mathcal{H}_i(x), \mathcal{H}(x')\} \sim \mathcal{H}(x)\partial_i\delta(x-x')\\
&\{\mathcal{H}_i(x), \mathcal{H}_j(x')\} \sim \mathcal{H}_i(x')\partial_j\delta(x-x') + \mathcal{H}_j(x)\partial_i\delta(x-x')
\end{aligned} \tag{10.168}$$

10.6 Boundary terms

The purpose of this section is to give a detailed treatment of boundary terms following Brown and York [2].

Consider a tubular domain D of spacetime, whose boundary has three different pieces: The two caps at the initial and final times, Σ_{t_1} and Σ_{t_2}. Those are spacelike, codimension one hypersurfaces (that is $d = n-1$). Then there is the "boundary at infinity", $r = R \to \infty$, which is the surface of a cylinder, also of codimension one, but timelike instead of spacelike. We shall call it $B \equiv \partial D$. Now this boundary can be understood as generated by the union of all the codimension two boundaries of the constant time hypersurfaces

$$(B = \partial D) \equiv \cup_t \left(S_t = \partial\Sigma_t\right) \tag{10.169}$$

- An intuitive grasp of the general situation can stem from the trivial example in flat space, to which we are going to refer all the time.

  $$D \equiv \{r \leq R \quad t_1 \leq t \leq t_2\} \tag{10.170}$$

  In this way the caps are defined by the solid balls

  $$\Sigma_t \equiv \{r \leq R \cup t = \text{constant}\} \tag{10.171}$$

  The embedding in spacetime is simply

  $$\begin{aligned}
  &y^0 = t\\
  &y^i = x^i
  \end{aligned} \tag{10.172}$$

so that the induced tangent vectors is

$$\xi_i^\alpha \equiv \frac{\partial y^\alpha}{\partial x^i} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{10.173}$$

and the normal vector

$$n^\alpha \equiv (1,0,0,0) \tag{10.174}$$

The induced metric reads

$$h_{ij} \equiv \eta_{\alpha\beta}\xi_i^\alpha \xi_j^\beta = -\delta_{ij} \tag{10.175}$$

The normal to Σ_t in Minkowski space is

$$n^\alpha = (1,0,0,0) \tag{10.176}$$

The boundary of such caps are the two-spheres

$$S_t \equiv \{r = R \cup t = \text{constant}\} \tag{10.177}$$

We can choose polar coordinates $\theta_a \equiv (\theta, \phi)$. The imbedding matrix of the boundary in Σ_t using these is

$$\xi_a^i \equiv \frac{\partial y^i}{\partial \theta^a} = \begin{pmatrix} \cos\theta \cos\phi & -\sin\theta \sin\phi \\ \cos\theta \sin\phi & \sin\theta \cos\phi \\ -\sin\theta & 0 \end{pmatrix} \tag{10.178}$$

It is equivalent to use

$$\begin{aligned} \theta_1 &\equiv \frac{x}{R} \\ \theta_2 &\equiv \frac{y}{R} \end{aligned} \tag{10.179}$$

then

$$z \equiv \theta_3 R = R\sqrt{1-\theta_1^2-\theta_2^2} \tag{10.180}$$

The embedding matrix is now

$$\xi_a^i = \begin{pmatrix} R & 0 \\ 0 & R \\ -R\frac{\theta_1}{\sqrt{1-\theta_1^2-\theta_2^2}} & -R\frac{\theta_2}{\sqrt{1-\theta_1^2-\theta_2^2}} \end{pmatrix} \tag{10.181}$$

The induced metric reads

$$\sigma_{ab} \equiv \xi_a^i h_{ij} \xi_b^j = -\frac{R^2}{1-\theta_1^2-\theta_2^2} \begin{pmatrix} 1-\theta_2^2 & \theta_1\theta_2 \\ \theta_1\theta_2 & 1-\theta_1^2 \end{pmatrix} \tag{10.182}$$

The contravariant metric reads

$$\sigma^{ab} = -\frac{1}{R^2}\begin{pmatrix} 1-\theta_1^2 & -\theta_1\theta_2 \\ -\theta_1\theta_2 & 1-\theta_2^2 \end{pmatrix} \tag{10.183}$$

Out of the two embedding matrices we can draw the composition

$$\xi^\alpha_a \equiv e^\alpha_j e^j_a \equiv \begin{pmatrix} 0 & 0 \\ R & 0 \\ 0 & R \\ -R\frac{\theta_1}{\sqrt{1-\theta_1^2-\theta_2^2}} & -R\frac{\theta_2}{\sqrt{1-\theta_1^2-\theta_2^2}} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \cos\theta\cos\phi & -\sin\theta\sin\phi \\ \cos\theta\sin\phi & \sin\theta\cos\phi \\ -\sin\theta & 0 \end{pmatrix} \tag{10.184}$$

The normal to the boundary in Σ_t is

$$\nu^i = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) = \left(\frac{x}{R}, \frac{y}{R}, \frac{z}{R}\right) = \left(\theta_1, \theta_2, \sqrt{1-\theta_1^2-\theta_2^2}\right) \tag{10.185}$$

The extrinsic curvature of $S_t \hookrightarrow \Sigma_t$ reads

$$k_{ab} \equiv \nabla_j \nu_i \xi^j_a \xi^i_b = \frac{1}{R}\ \delta_{ij}\ \xi^j_a \xi^i_b = -\sigma_{ab} \tag{10.186}$$

Let us now examine the constructs

$$\nu^i\nu^j e^\alpha_i e^\beta_j = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \theta_1^2 & \theta_1\theta_2 & \theta_1\sqrt{1-\theta_1^2-\theta_2^2} \\ 0 & \theta_2\theta_1 & \theta_2^2 & \theta_2\sqrt{1-\theta_1^2-\theta_2^2} \\ 0 & \theta_1\sqrt{1-\theta_1^2-\theta_2^2} & \theta_2\sqrt{1-\theta_1^2-\theta_2^2} & 1-\theta_1^2-\theta_2^2 \end{pmatrix} \tag{10.187}$$

$$\sigma^{ab} e^\alpha_a e^\beta_b = -R^2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1-\theta_1^2 & -\theta_1\theta_2 & -\theta_1\sqrt{1-\theta_1^2-\theta_2^2} \\ 0 & -\theta_1\theta_2 & 1-\theta_2^2 & -\theta_2\sqrt{1-\theta_1^2-\theta_2^2} \\ 0 & -\theta_1\sqrt{1-\theta_1^2-\theta_2^2} & -\theta_2\sqrt{1-\theta_1^2-\theta_2^2} & \theta_1^2+\theta_2^2 \end{pmatrix} \tag{10.188}$$

All this explicitly checks that

$$n^\alpha n^\beta - \nu^\alpha\nu^\beta + \sigma^{ab} e^\alpha_a e^\beta_b = \eta^{\alpha\beta} \tag{10.189}$$

The timelike boundary is just $S_2 \times \mathbb{R}$

$$B \equiv \cup_t S_t \tag{10.190}$$

Its three cordinates are just $x^m = (t, x, y)$ (they could equally well be chosen as (t, θ, ϕ)). The embeddig matrix reads

$$\xi^\alpha_m \equiv R \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\frac{\theta_1}{\theta_3} & -\frac{\theta_2}{\theta_3} \end{pmatrix} \tag{10.191}$$

so that the induced metric is just

$$ds^2 = dt^2 - \frac{R^2}{\theta_3^2}\left((1-\theta_2^2)d\theta_1^2 + 2\theta_1\theta_2 d\theta_1 d\theta_2 + (1-\theta_1^2)d\theta_2^2\right) \quad (10.192)$$

The normal vector is

$$n^\alpha = \frac{1}{R}(0, x, y, z) \quad (10.193)$$

so that the extrinsic curvature of $B \hookrightarrow M$ reads

$$\kappa_{mn} \equiv \nabla_\alpha n_\beta \xi^\alpha_m \xi^\beta_n = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_{ab} \end{pmatrix} \quad (10.194)$$

- Let us now draw from the example to the general case. The surfaces $S_t^{n-2} \equiv \partial\Sigma_t^{n-1}$ provide a foliation of the timelike boundary $B_{n-1} \hookrightarrow V_n$ of the domain of spacetime under consideration. The coordinates in $S_t^{(n-2)}$ will be denoted by $\theta_a \quad a = 1 \ldots n-2$. The imbedding $S_{n-2} \hookrightarrow \Sigma_{n-1}$ is described by

$$\theta \in S_{n-2} \hookrightarrow \; y^i(\theta^a) \in \Sigma_{n-1} \quad (i = 1 \ldots n-1) \quad (a = 1 \ldots n-2) \quad (10.195)$$

The imbedding of S in Σ defines in a natural way $(n-2)$ tangent space vectors

$$\xi^i_a \equiv \frac{\partial y^i}{\partial \theta^a} \quad (10.196)$$

The unit normal to S_{n-2} in Σ_{n-1} will be denoted by ν^i, and out of it we construct a vector

$$\nu^\alpha \equiv \nu^i \xi^\alpha_i \in T(S) \quad (10.197)$$

which is such that it is unitary $\nu.\nu = 1$ and is tangent to Σ_{n-1}, that is, $\nu.n = 0$. There are also $n-2$ spacetime vectors obtained by combining the two imbeddings $S \hookrightarrow \Sigma$ and $\Sigma \hookrightarrow M$:

$$\xi^\alpha_a \equiv \xi^\alpha_i \xi^i_a \quad (10.198)$$

The induced metric in $S_{n-2} \equiv \partial\Sigma_{n-1}$ is

$$ds^2 \equiv \sum_{a,b=1}^{n-2} \sigma_{ab} d\theta^a d\theta^b = h_{ij}\xi^i_a \xi^j_b d\theta^a d\theta^b = g_{\alpha\beta} e^\alpha_a e^\beta_b d\theta^a d\theta^b \quad (10.199)$$

The spacetime metric can be recovered from

$$g^{\alpha\beta} = -\nu^\alpha \nu^\beta + n^\alpha n^\beta + \sigma^{ab} e^\alpha_a e^\beta_b \quad (10.200)$$

The extrinsic curvature of $S_{n-2} \hookrightarrow \Sigma_{n-1}$ is defined as usual

$$k_{ab} \equiv \nabla_j \nu_i \xi^j_a \xi^i_b \tag{10.201}$$

It is possible to choose the coordinates θ^a in such a way that they intersect $S^{n-2}_t \equiv \partial \Sigma^{n-1}_t$ orthogonally.

$\therefore$ the vector n^α is the tangent vector

$$N n^\alpha = \left(\frac{\partial x^\alpha}{\partial t}\right)_\theta \tag{10.202}$$

The set of all S^{n-1}_t for varying t do foliate the timelike boundary of spacetime $B_{n-1} \equiv \partial V_n$. In this boundary B_{n-1} we can also introduce coordinates $z^m \quad m = 1 \ldots n-1$ (one of which is timelike), ans the corresponding $(n-1)$ vectors

$$\xi^\alpha_m \equiv \frac{\partial x^\alpha}{\partial z^m} \tag{10.203}$$

The induced metric is

$$\gamma_{mn} = g_{\alpha\beta} \xi^\alpha_m \xi^\beta_n \tag{10.204}$$

and we can write the *completeness relation*

$$g_{\alpha\beta} = -\nu_\alpha \nu_\beta + \gamma_{mn} \xi^m_\alpha \xi^n_\beta \tag{10.205}$$

It is simplest to choose (as we did in our explicit example)

$$z^m \equiv (t, \theta^a) \tag{10.206}$$

then

$$dx^\alpha = \left(\frac{\partial x^\alpha}{\partial t}\right)_\theta dt + \left(\frac{\partial x^\alpha}{\partial \theta^a}\right)_t d\theta^a = N n^\alpha dt + \xi^\alpha_a d\theta^a \tag{10.207}$$

in such a way that

$$ds^2\Big|_B = \gamma_{mn} dz^m dz^n = N^2 dt^2 + \sigma_{ab} d\theta^a d\theta^b \tag{10.208}$$

and the determinant obeys

$$|\gamma| = N^2 \sigma \tag{10.209}$$

Finally, the extrinsic curvature of $B_{n-1} \hookrightarrow V_n$ is

$$\kappa_{mn} = \nabla_\beta \nu_\alpha \xi^\alpha_m \xi^\beta_n \tag{10.210}$$

- Let us apply all this mathematics to the Einstein-Hilbert action. We consider a tubular region of the full spacetime bounded by two space-like hypersurfaces of constant time, Σ_2 and Σ_1, and the surface of the asymptotic cylinder, B

$$\partial V_n = \Sigma_2 - \Sigma_1 + B \tag{10.211}$$

This is the generalization to an arbitrary spacetime of the construction made in the example.

The full EH action, including the boundary term as well as the total derivative neglected when constructing the hamiltonian is given by

$$S_{EH} = \frac{c^3}{16\pi G}\int_{t_1}^{t_2} dt \int_{\Sigma_t} N\sqrt{|h|}d^{n-1}y \left(R[h] + K_{ij}K^{ij} - K^2 - \right.$$
$$\left. -2\nabla_\alpha \left(n^\mu \nabla_\mu n^\alpha - n^\alpha \nabla_\lambda n^\lambda\right)\right) + \frac{1}{8\pi G}\int_{\Sigma_{t_1}} K - \frac{1}{8\pi G}\int_{\Sigma_{t_2}} K - \frac{1}{8\pi G}\int_B K$$

The total derivative piece in the expansion of R which yields a boundary piece

$$-2\int_{\partial V}\left(n^\beta \nabla_\beta n^\alpha - n^\alpha \nabla_\beta n^\beta\right) n_\alpha \sqrt{|h|}d^{n-1}y = -2\int K\sqrt{|h|}d^{n-1}y \tag{10.212}$$

This precisely cancel the boundary term in the action coming from Σ_t. The only surviving contribution comes from the timelike boundary, B, that is

$$-2\int_B \left(n^\beta \nabla_\beta n^\alpha - n^\alpha \nabla_\beta n^\beta\right)\nu_\alpha \sqrt{|\gamma|}d^{n-1}z = 2\int_B \nabla_\beta \nu_\alpha n^\alpha n^\beta \sqrt{|\gamma|}d^{n-1}z \tag{10.213}$$

Summarizing

$$\frac{16\pi G}{c^3}S_{EH} = \int_{t_1}^{t_2} + 2\int_B \left(\kappa + \nabla_\beta \nu_\alpha n^\alpha n^\beta\right)\sqrt{|\gamma|}d^{n-1}z \tag{10.214}$$

Let us use now the fact that the timelike boundary B is foliated by S_t

$$\kappa = \gamma^{ij}\kappa_{ij} = \gamma^{ij}\nabla_\beta \nu_\alpha e_i^\alpha e_j^\beta = \nabla_\beta \nu_\alpha \left(g^{\alpha\beta} - \nu^\alpha \nu^\beta\right) \tag{10.215}$$

$$\therefore \quad \kappa + \nabla_\beta \nu_\alpha n^\alpha n^\beta = \nabla_\beta \nu_\alpha \left(g^{\alpha\beta} - \nu^\alpha\nu^\beta + n^\alpha n^\beta\right) = \nabla_\beta \nu_\alpha e_a^\alpha e_b^\beta = \sigma^{ab}\kappa_{ab} = k \tag{10.216}$$

$$\therefore \quad \int_B = 2\int_{S_t^{(n-2)}} kN\sqrt{|\sigma|}d^{n-2}\theta \tag{10.217}$$

As was already clear from the explicit example, this integral diverges even $R \to \infty$ even in flat space. In order to refer all expressions to this value, so that the action in flat space vanishes, it is often substracted a term in the action

$$\Delta D \equiv -\frac{2}{16\pi G}\int_B k_0 N \tag{10.218}$$

where k_0 represents the extrinsic curvature of S_{n-2} embedded in flat space.

The boundary terms in the hamiltonian read

$$H_{\text{boundary}} = -2\frac{1}{16\pi G}\int_{S_{n-2}}\left(N\left(k-k_0\right) - N_i\left(K^{ij} - Kh^{ij}\right)\nu_j\right)\sqrt{|s|}\,d^{n-2}\,\theta \tag{10.219}$$

(where K_{ij} is to be understood as a functional of the hamiltonian variables h_{ij} and π_{kl}. To be specific,

$$K^{ij} \equiv \frac{16\pi G}{\sqrt{|h|}}\left(\pi^{ij} - \frac{1}{2}\pi h^{ij}\right) \tag{10.220}$$

This boundary term yields the value of the energy for the gravitational field. It depends of the foliation chosen as well as on the lapse and shift which are arbitrary. When the space is asymptotically flat, representing flat asymptotic coordinates as (T, X^i), it is possible to choose Σ_t so that goes into $T =$ costant. It is clear that

$$N^\alpha \to N\left(\frac{\partial x^\alpha}{\partial T}\right) + N^i\left(\frac{\partial x^\alpha}{\partial X^i}\right) \tag{10.221}$$

It is then natural to define the *ADM mass* associated to a given solution by choosing a FIDO at rest at infinity, that is, $N = 1$, $N^i = 0$, so that

$$N^\alpha \to \left(\frac{\partial x^\alpha}{\partial T}\right) \tag{10.222}$$

and the flow generates a time translation at infinity. Then

$$M \equiv -\lim_{R\to\infty}\frac{1}{8\pi G}\int_{S_{n-2}}\left(k-k_0\right)\sqrt{|\sigma|}d^{n-2}\theta \tag{10.223}$$

- Let us compute the ADM mass for the four-dimensional Schwarzschild's spacetime,

$$ds^2 = \left(1 - \frac{r_S}{r}\right)dt^2 - \frac{dr^2}{1-\frac{r_S}{r}} - r^2 d\Omega_2^2 \tag{10.224}$$

Let us choose Σ_t to be really the surfaces of constant Schwarzschild time. Then the unit normal is given by

$$n^\alpha = \frac{1}{\sqrt{1-\frac{r_S}{r}}}\left(1,0,0,0\right) \tag{10.225}$$

The induced metric in Σ_t is

$$h_{ij}dx^i dx^j = \frac{1}{1-\frac{r_S}{r}}dr^2 + r^2 d\Omega_2^2 \tag{10.226}$$

The boundary $S \equiv \partial\Sigma$ is again the two-sphere $r = R$, and the unit normal is

$$\nu = \sqrt{1-\frac{r_S}{r}}\frac{\partial}{\partial r} \tag{10.227}$$

The induced metric is

$$\sigma_{ab}d\theta^a d\theta^b = R^2 d\Omega_2^2 \tag{10.228}$$

The extrinsic curvature reads

$$k = \nabla_a n^a = \frac{1}{\sqrt{h}}\partial_i\left(\sqrt{|h|}\nu^i\right) = \frac{\sqrt{1-\frac{r_S}{r}}}{r^2}\partial_r\left[\frac{r^2}{\sqrt{1-\frac{r_S}{r}}}\sqrt{1-\frac{r_S}{r}}\right] = \frac{2}{R}\sqrt{1-\frac{r_S}{R}} \tag{10.229}$$

On the other hand

$$k_0 = \frac{1}{r^2}\partial_r r^2 = \frac{2}{R} \tag{10.230}$$

It is then a fact that

$$k - k_0 \sim -\frac{r_S}{R^2} \tag{10.231}$$

so that

$$M_{ADM} = \frac{1}{8\pi G}.4\pi r_S = M \tag{10.232}$$

This is actually the reason why we have defined $r_S \equiv 2GM$.

- The ADM mass does not capture the mass loss due to radiation. In order to do that, it is necessary to choose the boundaru at null infinity, instead of at spatial infinity. The corresponding mass is called the *Bondi mass*

$$M_{\text{bondi}} \equiv -\frac{1}{8\pi G}\int_{v\to\infty}(k - k_0) \tag{10.233}$$

where the retarded time has been defined as usual

$$u \equiv t - r \tag{10.234}$$

and the advanced time

$$v \equiv t + r \tag{10.235}$$

Let us work this out explicitly in an example [23] . Consider the source

$$T_{\mu\nu} \equiv -\frac{1}{4\pi G r^2}\frac{dM(u)}{du}\; l_\alpha l_\beta \tag{10.236}$$

where now u is the null Schwarzschild coordinate

$$u \equiv t - r - r_S \log\left(\frac{r}{r_S} - 1\right) \tag{10.237}$$

and the mass (and also $r_S(u) \equiv 2GM(u)$) depend on u. The null vector

$$l \equiv \partial_u \tag{10.238}$$

The matter represented by the energy-momentum tensor as above is refered to as *null dust*. The solution of Einstein's equations is called the Vaidya metric and reads

$$ds^2 = \left(1 - \frac{2GM(u)}{r}\right) du^2 + 2dudr - r^2 d\Omega^2 \tag{10.239}$$

The contravariant metric in the sector (u,r) reads

$$g^{\mu\nu} = \begin{pmatrix} 0 & 1 \\ 1 & -\left(1 - \frac{2GM(u)}{r}\right) \end{pmatrix} \tag{10.240}$$

Let us consider again the surface Σ_t where

$$u + r = \text{constant} \tag{10.241}$$

Its covariant normal reads

$$n_\alpha \sim (1,1) \tag{10.242}$$

so that the normal vector

$$n \sim (g^{uu} + g^{ur}, g^{ru} + g^{rr}) = \left(1, \frac{2GM(u)}{r}\right) \tag{10.243}$$

Normalizing

$$n = \frac{1}{\sqrt{1 + \frac{2GM(u)}{r}}}\left(\partial_u + \frac{2GM(u)}{r}\partial_r\right) \tag{10.244}$$

The induced metric in Σ is obtained by substituting $du = -dr$, so that

$$ds^2 = -\left(1 + \frac{2GM(u)}{r}\right) dr^2 - r^2 d\Omega^2 \tag{10.245}$$

The boundary $\partial\Sigma$ is just the sphere $r = R$. The normal is

$$\nu \equiv \frac{1}{\sqrt{1+\frac{2GM(u)}{r}}}\frac{\partial}{\partial r} \tag{10.246}$$

The extrinsic curvature reads

$$k \equiv \nabla_a \nu^a = \frac{2}{R\sqrt{1+\frac{2GM(u)}{R}}} \sim \frac{2}{R}\left(1 - \frac{GM(u)}{R} + \ldots\right) \tag{10.247}$$

The indiced metric on the boundary is just

$$ds^2 = -R^2 d\Omega^2 \tag{10.248}$$

The extrinsic curvature of a surface of the same intrincic geometry, only that embedded in flat space is

$$k_0 = \frac{1}{r^2}\partial_r(r^2) = \frac{2}{R} \tag{10.249}$$

so that

$$k - k_0 = -\frac{2GM(u)}{R^2} \tag{10.250}$$

If we integrate now on spatial infinity $R \to \infty$, this means that we keep $t \equiv u + r$ constant, so that $u \sim -R \to -\infty$. This means that

$$M_{ADM} = M(u = -R) \tag{10.251}$$

If we integrate now on null infinity $v \to \infty$, while u is kept fixed, then

$$M_B = M(u) \tag{10.252}$$

the mass function.

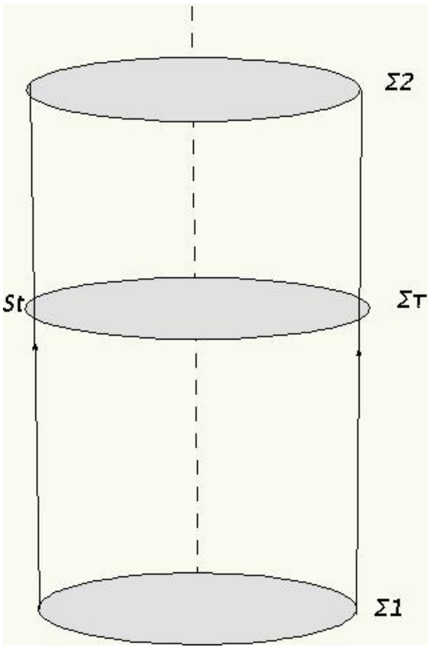

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Figure 10.1: The spacetime cylinder with the two caps.

11

Conformal Infinity.

We have alredy witnessed some concrete examples of Weyl mappings when studying spaces of constant curvature. The main idea of Penrose's conformal infinity is to perform a Weyl transformation of the metric in such a way thet $\Omega \sim 0$ at physical infinity. In that way, in a sense, infinity is brought up at finite distance. Let us work this out in detail for flat Minkowski spacetime. In cartesian coordinates and in terms of proper time the geodesics are linear functions

$$x^\mu - x_0^\mu = u^\mu s \tag{11.1}$$

where

$$u^2 = \pm 1, 0 \tag{11.2}$$

for timelike, spacelike or null geodesics. This means that

$$r = |\vec{u}| s \equiv u s \tag{11.3}$$

Now for future-directed timelike geodesics

$$\frac{u^0}{u} > 1 \tag{11.4}$$

whereas for past-directed timelike geodesics

$$\frac{u^0}{u} < -1 \tag{11.5}$$

We start from

$$ds^2 = dt^2 - dr^2 - r^2 d\Omega_2^2 = du dv - \frac{(v-u)^2}{4} d\Omega_2^2 \tag{11.6}$$

where we have defined the null coordinates

$$\begin{aligned} u &\equiv t - r \\ v &\equiv t + r \end{aligned} \tag{11.7}$$

with

$$\begin{aligned} &-\infty \leq u, v \leq +\infty \\ &v - u \geq 0 \end{aligned} \tag{11.8}$$

Let us now perform a Weyl rescaling with

$$\Omega^2 \equiv \frac{4}{(1+u^2)(1+v^2)} \tag{11.9}$$

Then

$$\tilde{ds}^2 = \frac{4}{(1+u^2)(1+v^2)} dudv - \frac{(u-v)^2}{(1+u^2)(1+v^2)} d\Omega_2^2 \tag{11.10}$$

Now define

$$\begin{aligned} u &\equiv \text{tg } p \\ v &\equiv \text{tg } q \end{aligned} \tag{11.11}$$

with

$$-\frac{\pi}{2} \leq p \leq q \leq \frac{\pi}{2} \tag{11.12}$$

Then

$$\tilde{ds}^2 == 4dpdq - \sin^2 (p-q)\ d\Omega_2^2 = dT^2 - dR^2 - \sin^2 R\ d\Omega_2^2 \tag{11.13}$$

with

$$\begin{aligned} T &\equiv p + q \\ R &\equiv q - p \end{aligned} \tag{11.14}$$

and that range of the new coordinates is

$$\begin{aligned} &-\pi \leq T \leq \pi \\ &0 \leq R \leq \pi \\ &-\pi \leq T - R \leq \pi \end{aligned} \tag{11.15}$$

Minkowski space in these coordinates is just a piece of Eisntein Static Universe (ESU) which is just $\mathbb{R} \times S_3$, and the coordinates there

$$\begin{aligned} &-\infty \leq T \leq \infty \\ &0 \leq R \leq 2\pi \end{aligned} \tag{11.16}$$

All future-pointing timelike geodesics are such that

$$t \pm r \to \infty \quad \therefore \quad u, v \to \infty \quad \therefore \quad p, q \to \frac{\pi}{2} \quad \therefore \quad T \to \pi\ \&\ R \to 0 \tag{11.17}$$

The point $(T, 0)$ is then the *future timelike infinity*, I^+. The limit when $r \to \infty$ of future-directed timelike geodesics obey

$$(u, v) = t \pm r = (\frac{u^0}{u} \pm 1)r \to \infty \tag{11.18}$$

This means that $(p, q) \sim \frac{\pi}{2}$ or else that $T \sim \pi$, $R \sim 0$.

In a similar way, we define the *past timelike infinity*, I^- as the point $(T = -\pi, R = 0)$. In this case

$$(u, v) = (\frac{u^0}{u} \pm 1)r \sim -\infty \tag{11.19}$$

so that $(p, q) \sim -\frac{\pi}{2}$, that is, $T \sim -\pi$ and $R \sim 0$. The *spacelike infinity* I^0 is the point $(T = 0, R = \pi)$. Spacelike geodesics are such that $|u_0| < u$; then

$$t \pm r = (\frac{u_0}{u} \pm 1)r \sim \pm\infty \tag{11.20}$$

then

$$p \sim \frac{-\pi}{2} \quad q \sim \frac{\pi}{2} \tag{11.21}$$

so that

$$T \sim 0 \quad R \sim \pi \tag{11.22}$$

Finally, null geodesics obey $|u^0| = u$. Then for future directed geodesics

$$t + r \sim \infty \tag{11.23}$$

and for past directed ones

$$t - r \sim -\infty \tag{11.24}$$

On the other hand, for future directed

$$t - r = t_0 - r_0 \tag{11.25}$$

and for past directed ones

$$t + r = t_0 + r_0 \tag{11.26}$$

Then in the future-directed case

$$q = \frac{\pi}{2} \quad p = p_0 \tag{11.27}$$

so that

$$T?\frac{\pi}{2} + p_0 \quad R = \frac{\pi}{2} - p_0 \quad T * T = \pi \tag{11.28}$$

and in the past-directed case

$$p = -\frac{\pi}{2} \quad q = q_0 \tag{11.29}$$

so that

$$T = -\frac{\pi}{2} + q_0 \quad R = \frac{\pi}{2} + q_0 \quad R - T = \pi \tag{11.30}$$

The line $T + R = \pi$ is the *future null infinity*, $\mathcal{J}^+$, and the line $R - T = \pi$ is the *past null infinity*, $\mathcal{J}^-$.

Hawking and Ellis characterize a spacetime (M, g) as *asymptotically simple* provided there is another manifold $(\overline{M}, \overline{g})$ such that

- The physical manifold M is an open submanifold of $\overline{M}$ with smooth boundary, ∂M.
- There exists a real function Ω on $\overline{M}$ such that $\overline{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ on M. Besides, $\Omega = 0$ and $\nabla_\mu \Omega \neq 0$ on ∂M.
- Every null geodesic has two endpoints on ∂M
- $R_{\mu\nu} = 0$ near ∂M.

This definition is however too restrictive. It is conveniene to define *weakly asymptotically simple* spaces when these conditions hold in a neighborhood only.

11.0.1 Spinor approach

It is a fact of life that

$$\begin{aligned}
&\Psi_{ABCD} = \tilde{\Psi}_{ABCD} \\
&\Lambda = \Omega^2 \tilde{\Lambda} - \frac{1}{4} \Omega \nabla_{CC'} \nabla^{CC'} \Omega + \frac{1}{2} \nabla_{CC'} \Omega \nabla^{CC'} \Omega \\
&\Phi_{ABA'B'} = \tilde{\Phi}_{ABA'B'} + \Omega^{-1} \nabla_{A(A'} \nabla_{B')B} \Omega
\end{aligned} \tag{11.31}$$

It can be also shown through Einstein's equations [26] that if the matter fields fall off fast enough near ∂M, namely

$$T_{\mu\nu} = O(\Omega^3) \tag{11.32}$$

then the conformal boundary ∂M is null.

Another easily proved fact is the following. If $T_{\mu\nu}$ vanishes in a neighborhood U of $\mathcal{J}$, then

$$\Psi_{ABCD} = O(\Omega) \tag{11.33}$$

The so-called *peeling theorem* gives information on the rate of falling-off (at infinity) of different physical quantities when reaching the boundary.

Consider a null geodesic $\tilde{\gamma}$ in $\tilde{M}$ reaching $\mathcal{J}$ at a point p. Call γ the corresponding geodesic in the physical space-time M. Choose a spin basis $(\circ, \iota)$ at a given point of γ, so that l is tangent to γ at this point. Then we propagate the spin basis in such a way that

$$D \circ^A = D \iota^A = 0 \tag{11.34}$$

Let us further define the parameter r in such a way that

$$D = o^A \bar{o}^{A'} \nabla_{AA'} \equiv \frac{d}{dr} \tag{11.35}$$

There is a cartain latitude in choosing the Weyl transformation of the spin basis. It can be used [26] to set at the point p

$$\tilde{r} = 0 \tag{11.36}$$

as well as

$$\tilde{D}\Omega = \frac{d\Omega}{dr} = 1 \tag{11.37}$$

The fact that

$$\tilde{D} = \Omega^{-2} D \tag{11.38}$$

implies that near $\mathcal{J}$

$$r \sim \Omega^{-1} \tag{11.39}$$

The peeling theorem proper states that the gravitational field far from an isolated source can be written as

$$\Psi_{ABCD} = \frac{[N]}{r} + \frac{[III]}{r^2} + \frac{[II]}{r^3} + \frac{[I]}{r^4} + O(r^{-5}) \tag{11.40}$$

If we assume that the energy-momentum tensor falls off even faster, namely

$$T_{\mu\nu} = O(\Omega^4) \tag{11.41}$$

then it is possible to choose the residual gauge freedom in Ω in such a way that on $\mathcal{J}$

$$\nabla_\alpha n_\beta = 0 \tag{11.42}$$

and besides, ι be covariantly consnat, again on $\mathcal{J}$

$$\nabla_\alpha \iota_A = 0 \tag{11.43}$$

To summarize, all the NP connection scalarsa can be made to vanish on $\mathcal{J}$, except σ, which then contains all the dynamical information on gravitational radiation. It can be shown to obey

$$D\sigma = 0 \tag{11.44}$$

The Einstein's equations imply

$$\begin{aligned} \Phi_{00} &= -\sigma\bar{\sigma} \\ \Phi_{01} &= -\bar{\delta}\sigma \\ \Phi_{02} &= -\Delta\sigma \\ \Phi_{11} &= \Lambda = 0 \end{aligned} \tag{11.45}$$

The component $\Phi_{20} \equiv -\Delta\bar{\sigma}$ is the famous *Bondi news function.*

Let S be a *cut* of $\mathcal{J}$. This means that S is a spacelike 2-surface in $\mathcal{J}$ orthogonal to the generators of $\mathcal{J}$. Then there exists another null surface Σ_S which contains S and whose generators are orthogonal to S. Then we can define a spinor field $\circ$ on S such that $l^\mu \equiv \circ^A \bar{\circ}^{A'}$ is tangent to the generators of Σ_S, and, besides, $\circ_A \iota^A = 1$. This is the *Bondi system* based on that cut. Althoug the cuts are locally euclidean, they are topologically S_2. From

$$ds^2 = -dx^2 - dy^2 = -dz d\bar{z} \tag{11.46}$$

Making the stereographic projection

$$z \equiv e^{i\phi} \cot \frac{\theta}{2} \tag{11.47}$$

reads

$$ds^2 = -\frac{(1 + z\bar{z})^2}{4} \, d\Omega_2^2 \tag{11.48}$$

Under a Möbius transformation

$$\hat{z} \equiv \frac{az + b}{cz + d} \tag{11.49}$$

(with $ad - bc = 1$).

It follows

$$\begin{aligned} d\hat{z}d\bar{\hat{z}} &= \frac{1}{|cz + d|^4} dz d\bar{z} = \frac{(1 + z\bar{z})^2}{4|cz + d|^4} d\Omega_2^2 = \\ &= \frac{\left(1 + \hat{z}\bar{\hat{z}}\right)^2}{4} d\hat{\Omega}_2^2 = \frac{\left(|cz + d|^2 + |az + b|^2\right)^2}{4|cz + d|^4} \, d\hat{\Omega}_2^2 \end{aligned} \tag{11.50}$$

Ergo

$$d\hat{\Omega}_2^2 = \frac{(1 + z\bar{z})^2}{\left(|cz + d|^2 + |az + b|^2\right)^2} d\Omega_2^2 \equiv K^2 d\Omega_2^2 \tag{11.51}$$

If we want the theory to remain scale invariant, we better compensate this by imposing

$$d\hat{u} = K du \tag{11.52}$$

that is

$$\hat{u} = K \left(u + \alpha\left(z, \bar{z}\right)\right) \tag{11.53}$$

This set of transformations constitute the *Bondi-Metzner-Sachs (BMS) group.* The largest proper normal (invariant under conjugation) subgroup is the *supertranslation group*, $\mathcal{S}$

$$\begin{aligned} \hat{u} &= u + \alpha\left(z, \bar{z}\right) \\ \hat{z} &= z \end{aligned} \tag{11.54}$$

This is so that

$$\text{Möbius} \sim SL\left(2,\mathbb{C}\right) = \text{BMS}/\mathcal{S} \tag{11.55}$$

An interesting subset of the supertranslations is the *translation sobgroup*, $\mathcal{T}$ where

$$\alpha \equiv \frac{A + Bz + \bar{B}\bar{z} + Cz\bar{z}}{1 + z\bar{z}} \tag{11.56}$$

In Minkowski space-time, defining

$$\zeta \equiv e^{i\phi} \cot \frac{\theta}{2} \tag{11.57}$$

we find

$$\begin{aligned}
x &= r\frac{\left(\zeta + \bar{\zeta}\right)}{1 + \zeta\bar{\zeta}}\\
y &= -ir\frac{\zeta - \bar{\zeta}}{1 + \zeta\bar{\zeta}}\\
z &= r\frac{\zeta\bar{\zeta} - 1}{1 + \zeta\bar{\zeta}}\\
\frac{\zeta}{\left(1 + \zeta\bar{\zeta}\right)^2} &= (x + iy)\,\frac{1 - \frac{z}{r}}{4r}
\end{aligned} \tag{11.58}$$

Under an ordinary translation

$$x^\mu \to x^\mu + a^\mu \tag{11.59}$$

$$r \to r + \frac{\vec{a}.\vec{x}}{r} + \ldots \tag{11.60}$$

This means that

$$\begin{aligned}
&u \to u + a_0 - \frac{\vec{a}.\vec{x}}{r} + \ldots = u + a_0 - \frac{1}{1 + \zeta\bar{\zeta}} \times\\
&\times \left(a_1\left(\zeta + \bar{\zeta}\right) + ia_2\left(\zeta - \bar{\zeta}\right) + \left(\zeta\bar{\zeta} - 1\right)a_3\right) + \ldots
\end{aligned} \tag{11.61}$$

This belongs to $\mathcal{T}$ with

$$\begin{aligned}
A &\equiv a_0 + a_3\\
B &\equiv a_1 - ia_2\\
C &\equiv a_0 - a_3
\end{aligned} \tag{11.62}$$

To summarize, there are many Poincare groups at $\mathcal{J}$ contained in the BMS group, namely one for each supertranslation which is not a translation.

A spinor τ is said to ba *strongly asymptotically constant* in $\tilde{M}$ whenever both $\tau_A o^A$ and $\tau_A \iota^A$ are regular at $\mathcal{J}$ and besides, it obeys at $\mathcal{J}$ the *asymptotic twistor equation*

$$\nabla^{\alpha'}{}_{(A}\tau_{B)} = 0 \tag{11.63}$$

where $\hat{\tau} = \tau$

Besides determining a null vector $t^a = \tau^A \tau^{A'}$ this spinor determines a symmetric two-spinor

$$\phi_{AB} \equiv \frac{1}{2}\left(\tau_{(A}\nabla^{C'}{}_{B)}\bar{\tau}_{C'} - \bar{\tau}_{C'}\nabla^{C'}{}_{(A}\tau_{B)}\right) \tag{11.64}$$

which in turn determines a two-form

$$F_{\mu\nu} \equiv \epsilon_{A'B'}\phi_{AB} + \epsilon_{AB}\bar{\phi}_{A'B'} \tag{11.65}$$

Consider now a null hypersurface Σ extending to $\mathcal{J}$, and call $S_\Omega \subset \Sigma$ the two-dimensional ð surface $\Omega = C$. Define a null tetrad such that l^μ is tangent to the generators of Σ, n^μ is orthogonal to S_Ω and m and $\bar{m}$ span $T(S_\Omega)$. Then

$$I(\Omega) \equiv \int_{S_\Omega} F_{\mu\nu} l^\mu n^\nu dS \tag{11.66}$$

It can be shown that

$$\lim_{\Omega\to 0} I(\Omega) \tag{11.67}$$

exists and defines the *Bondi energy* through

$$I = P^\mu t_\mu|_{\mathcal{J}} \tag{11.68}$$

It can also be shown that the Bondi mass is given by

$$M_B = -\frac{1}{2}\int \left(\Psi_2^0 + s\Delta\bar{s}\right) dS \tag{11.69}$$

where

$$\Psi_2 \sim \Psi_2^0 \Omega^3 + \dots \tag{11.70}$$

and

$$\sigma \sim s\Omega^2 + \dots \tag{11.71}$$

Using that, it follows that M_B is positive and non-decreasing.

12

Gravitation and quantum field theory: The Big Picture

There are many obvious issues when considering quantum gravity, by which we mean some unknown quantum theory that in the classical limit reduces to GR.

For example, one of the basis of quantum field theory (QFT) is *microcausality*, the statement that field variables defined at points spacelike separated should commute. Also the canonical commutators are defined at *equal time*. It is plain that these concepts make sense in a fixed gravitational background at best; and even then with caveats when horizons are present.

In a similar vein, any attempt to write a Schrödinger equation for the gravitational field must face the fact that there is no natural notion of time in GR, even classically. The Wheeler-DeWitt equation is obtained by interpreting the hamiltonian constraint as an operator equation by substituting the canonical momenta by functional derivatives. It is similar to the Schrödinger equation, except precisely for the absemce of time. It has been repeteadly conjectured ([**?**] cor a review) that such a time cam appear when a WKB type of semiclassical approximation is performed on the Wheeler-DeWitt equation, but this has not been properly substantiated.

Some people try to apply the canonical approach to a clever set of variables introduced by Ashtekar. Those variables are related to the spacetime metric in a complicated way. It is unclear how this approach is related to the classical regime at all. This whole approach is dubbed *loop quantum gravity*, because a loop representation is useful to understand some aspects of the corresponding Hilbert space.

It think it is fair to say that the results obtained so far from the canonical approach are quite modest.

Were we inclined to use the functional integral to define transition amplitudes from one three-dimensional metric on a given three-dimensional manifold Σ_i, say h_i to another three dimensional surface Σ_f with its corresponding metric h_f, something like

$$K\left[h_i, h_f\right] \equiv \int \mathcal{D}g e^{iS_{EH}(g)} \tag{12.1}$$

where the integration is performed over all metrics defined on a four-dimensional domain D such that

$$\partial D = \Sigma_f - \Sigma_i \tag{12.2}$$

We have to face several ugly facts. First of all, the gravitational action is not positive definite, even with the euclidean signature. The loop expansion is not then justified by any sort of saddle point expansion. Even worse, the set of four dimensional manifolds is a complicated one. Kolmogorov has shown that the problem of classifying four-dimensional geometries is an undecidable one. Given any two four-dimensional manifolds, there is no set of topological invariants the can decide when the two manifolds are diffeomorphic. The problem lies mainly with the first fundamental group, $\pi_1(M)$. It does not seem the case that there exists any justification for restricting the functional integral to any subset of manifolds.

Were the spacetime geometry to fluctuate we would have to build anew all our ideas about QFT, which we understand when defined in flat space only, and even there we miss non-perturbative effects known to be important.

Another issue is the following. Assuming that the symmetry group of the quantum theory is still Diff invariance, what are the observables? There are not many of those. For a *fixed* manifold, integrals of n-forms are Diff invariant objects, but there are not many of those.

The preceding difficulties did not deter physicist to work on quantum aspects of gravitation. Besides many long and inconclusive discussions of the basic points, to be discussed later, such as what are the observables of the theory, the manin breakthrough was made by 't Hooft and Veltman employing techniques invented by deWitt and Feynman. What is computed are the quantum fluctuations around an *arbitrary* background, $\bar{g}_{\alpha\beta}(x)$, which can be any solution of Einstein's EM. General relativity is considered in this treatment as an ordinary gauge theory, forgetting about all questions of principle. Actually the calculation is usually done with euclidean signature, making an appropiate analytical continuation at the end of the procedure. Particularly easy is the computation of the divergences of the effective action, which must be eliminated in the renormalized theory. In this computation beautiful mathematical techniques can be employed. The propagator is assumed to be however well-defined for a generic background metric, which is a delicate assumption in the presence of horizons and/or singularities.

It is doubtful whether we can assert some proposition about quantum [1] gravity with some confidence.

[1] In order to understand the sequel, the reader is assumed a working knowledge of quantum field theory (QFT) at a graduate level, up to and including, Feynman's path integral approach.

The tree level estimate for the cross section for production of gravitons in particle-antiparticle annihilation is of the order of the inverse of the mass scale associated to this problem whichjust by sheer dimensional analysis is Planck's mass, which is given in terms of Newton's constant, G by

$$m_p \equiv \sqrt{\frac{\hbar c}{8\pi G}} \sim 10^{19} GeV \tag{12.3}$$

If we remember that $1\, GeV (= 10^3\, MeV)$ is the rough scale of hadronic physics (the mass and inverse Comptom wavelength of a proton, for example), this means that quantum gravity effects will only be apparent when we are able to explore concentrated energy roughly 10^{19} times bigger (or an scale distance correspondingly smaller; these two statements are supposed to be equivalent owing to Heisenberg's principle). To set the scale, the Large Hadron Collider works roughly at the $TeV (= 10^3\, GeV)$ scale, so there is a long way to go before reaching expected quantum gravity effects in accelerators.

In terms of the cross section, this yields up to numerical factors of order unity

$$\sigma \sim l_p^2 \sim 10^{-66}\ cm^2 \sim 10^{-40}\ fm^2 \tag{12.4}$$

This is more or less 40 orders of magnitude smaller than typical nuclear reactions.

There are however some interesting experimental facts such as the ones reported in [5]. Free fall of neutros has been reported there. Also there interference effects due to the Earth's classical gravitational field on a neutron's wave function are reported. The experimental apparatus is a neutron interferometer. The phase shift between the two different paths is given by

$$\Delta\phi \sim \frac{2\pi m_p^2 l g \Delta h}{h^2} \tag{12.5}$$

where l is the common horizontal span of the paths and Δh is the difference in height. There are some more contributions in the actual experiment and the precision is not too big. Nevertheless the effect seems clear. It is not clear however what is its meaning with respect to the relationship between gravitation and quantum mechanics. More recently [17] experimental evidence for gravitational quantum bound states of neutrons has been claimed.

If we want to get direct experimental information of quantum gravitational effects, we have to turn our attention towards Cosmology, or perhaps look for some clever precision experiment in the laboratory. Lacking any experimental clue, the only thing we can do is to think and try to look for logical (in)consistencies.

It has been repeatedly argued by many particle physicists that the practical utility of the answer to this question will not presumably be great.

How would we know for sure beforehand?. There has always been a recurrent dream, exposed vehemently by Salam [?] that the inclusion of the gravitational interaction would cure many of the diseases and divergences of quantum field theory, through the inclusion in the propagator of terms of the type

$$e^{-\frac{1}{m_p^2 x^2}}$$

So that for example, the sum of tree graphs that leads to the Schwarzschild solution as worked out by Duff [?]

$$\frac{1}{r} + \frac{2M}{m_p^2 r^2} + \frac{4M^2}{m_p^4 r^3} + \ldots$$

would get modified to

$$\frac{1}{r} e^{-\frac{1}{m_p r}} + \frac{2M}{m_p^2 r^2} e^{-\frac{2}{m_p r}} + \ldots \sim \frac{1}{r e^{-\frac{1}{m_p r}} - 2\frac{M}{m_p^2}}$$

shifting the location of the horizon and eliminating the singularity at $r = 0$. Nobody has been able to substantiate this dream so far.

Ay ant rate quantum gravity is nevertheless a topic which has fascinated whole generations of physicists, just because it is so difficult. There seems to be a strong tension between the beautiful, geometrical world of General Relativity and the no less marvelous, less geometrical, somewhat mysterious, but very well tested experimentally, world of Quantum Mechanics.

As with all matters of principle we can hope to better understand both quantum mechanics and gravitation if we are able to clarify the issue.

The most conservative approach is of course to start from what is already known with great precision about the standard model of elementary particles associated to the names of Glashow, Weinberg and Salam. This can be called the *bottom-up approach* to the problem. In this way of thinking Wilson taught us that there is a working low energy effective theory, and some quantum effects in gravity can be reliably computed for energies much smaller than Planck mass. There are two caveats to this. First of all, we do not understand why the observed cosmological constant is so small: the natural value from the low energy effective lagrangian point of view ought to be much bigger. The second point is that one has to rethink again the lore of effective theories in the presence of horizons. We shall comment on both issues in due time.

There is not a universal consensus even on the most promising avenues of research from the opposite *top-down* viewpoint. Many people think that strings [?] are the best buy (I sort of agree with this); but it is true that after more than two decades of intense effort nothing substantial has come out of them. Others [?] try to quantize directly the Einstein-Hilbert lagrangian, something that is at variance with our experience in effective field theories.

But it is also true that as we have already remarked, the smallish value of the observed cosmological constant also cries out of the standard effective theories lore.

It is generally accepted that General Relativity, a generally covariant theory, is akin to a gauge theory, in the sense that the diffeomorphism group of the apace-time manifold, $Diff(M)$ plays a role similar to the compact gauge group in the standard model of particle physics. There are some differences though. To begin with, the group, $Diff(M)$ is too large; is not even a Lie group. Besides, its detailed structure depends on the manifold, which is a dynamical object not given a priori. Other distinguished subgroups (such as the area-preserving diffeomorphisms) are perhaps also arguable for. Those leave invariant a given measure, such as the Lebesgue measure, $d^n x$.

It also seems clear that when there is a boundary of space-time, then the gauge group is restricted to the subgroup consisting on those diffeomorphisms that act trivially on the boundary. The subgroup that act not-trivially is related to the set of conserved charges. In the asympotically flat case this is precisely the Poincaré group, $SO(1,3)$ that gives rise to the ADM mass.

In the asympotically anti-de Sitter case, this is presumably related to the conformal group $SO(2,3)$.

It is nevertheless not clear what is the physical meaning of keeping constant the boundary of spacetime (or keeping constant some set of boundary conditions) in a functional integral of some sort.

A related issue is that it is very difficult to define what could be *observables* in a diffeomorphism invariant theory, other than global ones defined as integrals of scalar composite operators $O(\phi_a(x))$ (where $\phi_a, a = 1 \ldots N$ parametrizes all physical fields) with the peudo-riemannian measure

$$\mathcal{O} \equiv \int \sqrt{|g|} d^4 x O(\phi_a(x))$$

Some people claim that there are no local observables whatsoever, but only *pseudolocal* ones; the fact is that we do not know. Again, the exception to this stems from keeping the boundary conditions fixed; in this case it is possible to define an S-matrix in the asymptotically flat case, and a conformal quantum field theory (CFT) in the asymptotically anti-de Sitter case. Unfortunatelly, the most interesting case from the cosmological point of view, which is when the space-time is asymptotically de Sitter is not well understood.

Incidentally, it is well known that the equivalence problem in four-dimensional geometries is undecidable [**?**]. In three dimensions Thurston's geometrization conjecture has recently been put on a firmer basis by Hamilton and Perelman, but it is still not clear whether it can be somehow implemented in a functional integral without some drastic restrictions. Those caveats should be kept in mind when reading the sequel.

A radically different viewpoint has recently been advocated by Gerardus 't Hooft by insisting in causality to be well-defined, so that the conformal class of the space-time metric should be determined by the physics, but not necessarily the precise point in a given conformal orbit. If we write the spacetime metric in terms of a unimodular metric and a conformal factor

$$g_{\mu\nu} = \omega^2(x)\hat{g}_{\mu\nu}$$

with

$$det\, \hat{g}_{\mu\nu} = 1$$

then the unimodular metric is in some sense intrisic and determines causality, whereas the conformal factor depends on the observer in a way dictated by *black hole complementarity.*

Finally, there is always the (in a sense, opposite) possibility that spacetime (and thus diffeomorphism invariance) is not a fundamental physical entity in such a way that the appropiate variables for studying short distances are non geometrical. Something like that could happen in string theory, but our understanding of it is still in its infancy.

On the other hand, it has been speculated that quantum gravitational effects can tame the infinities that appear in QFT yielding a finite theory eventually. Some arguments in favor of this (first proposed by the inventors of ADM) are as follows. The self-energy of a body of radius ϵ and mass m and charge e which in newtonian theory reads

$$m_\epsilon = m + \frac{e^2}{8\pi\epsilon} - \frac{Gm^2}{2\epsilon} \tag{12.6}$$

It diverges in the pointlike linit $\epsilon \to 0$. The only modification borne out by GR was shown by ADM to be the replacement in the second member of m_0 by m_ϵ

$$m_\epsilon = m + \frac{e^2}{8\pi\epsilon} - \frac{Gm_\epsilon^2}{2\epsilon} \tag{12.7}$$

Solving the quadratic equation yields

$$m_\epsilon = \frac{\epsilon}{G}\left[-1 \pm \sqrt{1 + \frac{2G}{\epsilon}\left(m + \frac{e^2}{8\pi\epsilon}\right)}\right] \tag{12.8}$$

which has a finite limit when $\epsilon \to 0$ namely

$$m_0 = \sqrt{\frac{e^2}{4\pi G}} \tag{12.9}$$

12.1 The Unruh effect

Before entering the subject matter as such it seems then only appropiate to dwell for a while in a subtle effects due to the non-inertial character of a observer, still in a flat background. By the equivalence principle, this ought to be related to a gravitational field. We are talking of the Unruh effect [?][27] that although was discovered after Hawking predicted the black hole thermal emission, is in fact logically simpler and independent.

Let us consider the trajectory of an accelerated observer in two dimensional flat space

$$
\begin{aligned}
t &= \frac{1}{a} sinh\, a\tau \\
x &= \frac{1}{a} cosh\, a\tau
\end{aligned}
\tag{12.10}
$$

This is such that the four-velocity is given by

$$
u = \Big(cosh\, a\tau \quad , \, sinh\, a\tau\Big) \tag{12.11}
$$

normalized to

$$
u^2 = 1 \tag{12.12}
$$

and the acceleration

$$
\dot{u} \equiv a \Big(sinh\, a\, \tau \quad , \, cosh\, a\tau\Big) \tag{12.13}
$$

obeys

$$
\begin{aligned}
a^2 &= -1 \\
a.u &= 0
\end{aligned}
\tag{12.14}
$$

In *comoving* coordinates, id est, adapted to the four-velocity,

$$
u = \frac{\partial}{\partial \xi^0} \tag{12.15}
$$

the worldline of the accelerated observer is

$$
\begin{aligned}
\xi^0(\tau) &= \tau \\
\xi^1(\tau) &= 0
\end{aligned}
\tag{12.16}
$$

In general

$$
\begin{aligned}
t &= \frac{e^{a\xi^1}}{a}\, sinh\, a\xi^0 \equiv \rho \sinh\, \omega \\
x &= \frac{e^{a\xi^1}}{a} cosh\, a\xi^0 \equiv \rho\, \cosh\, \omega
\end{aligned}
\tag{12.17}
$$

so that the value of the coordinate ξ^1 (or ρ) tells us which hyperbola we are talking about

$$t^2 - x^2 = -\frac{e^{2a\xi^1}}{a^2} = -\rho^2 \tag{12.18}$$

In terms of these coordinates the Minkowski metric reads

$$ds^2 = dt^2 - dx^2 = e^{2a\xi^1}\left(d\xi_0^2 - d\xi_1^2\right) = \rho^2 d\omega^2 - d\rho^2 - dx_\perp^2 \tag{12.19}$$

When

$$\begin{aligned} -\infty \leq \xi^0 \leq \infty \\ -\infty \leq \xi^1 \leq \infty \end{aligned} \tag{12.20}$$

only one quarter of the original Minkowski space has been covered, namely the one corresponding to

$$|t| \leq x \tag{12.21}$$

This is called *Rindler's wedge* or *Rindler space.* The lightcone plays the role of the *event horizon.*

Let us now consider an scalar field

$$S = \frac{1}{2}\int dt \wedge dx \left(\left(\frac{\partial\phi}{\partial t}\right)^2 - \left(\frac{\partial\phi}{\partial x}\right)^2\right) = \frac{1}{2}\int d\xi^0 \wedge d\xi^1 \left(\left(\frac{\partial\phi}{\partial \xi^0}\right)^2 - \left(\frac{\partial\phi}{\partial \xi^1}\right)^2\right) = \frac{1}{2}\int \rho d\rho d\omega dx_\perp \left(\frac{1}{\rho^2}(\partial_\omega\phi)^2 - (\partial_\rho\phi)^2 - (\partial_\perp\phi)^2\right) \tag{12.22}$$

The hamiltonial reads

$$H = \frac{1}{2}\int \rho d\rho d\omega dx_\perp \left(\pi^2 + (\partial_\rho\phi)^2 + (\partial_\perp\phi)^2\right) \tag{12.23}$$

We can use lightcone coordinates

$$x_\pm \equiv t \pm x \tag{12.24}$$

as well as

$$X\pm \equiv \xi^0 \pm \xi^1 \tag{12.25}$$

The full solution of the classical equations of motion

$$\frac{\partial^2}{\partial x^+ \partial x^-}\phi = \frac{\partial^2}{\partial X^+ \partial X^-}\phi = 0 \tag{12.26}$$

is a combination of rightmoving, positive frequency modes such as

$$f_R^+(\omega) \equiv e^{-i\omega x^-} = e^{-i\omega(t-x)} \tag{12.27}$$

and their complex conjugates, wich are negative energy left movers.

It is worthwile to stop a while to think on the reason why we say that it is rightmoving. It is because

$$\hat{k} f_R^+ = -i \frac{\partial}{\partial x} f_R^+ = \omega f_R^+ \tag{12.28}$$

The reason why we say that it also enjoys positive frequency is because

$$\hat{H} f_R^+ = i \frac{\partial}{\partial t} f_R^+ = \omega f_R^+ . \tag{12.29}$$

The plane waves

$$g_L^+(x^+) \equiv e^{-i\omega x^+} \tag{12.30}$$

are left-moving, positive energy solutions.

The general classical solution can be expanded in a sum of a Fourier series for the left movers and a corresponding series for the right movers. We split the series in f_R, f_R^*, g_L, g_L^* considering that

$$0 \leq \omega \leq \infty \tag{12.31}$$

We could as well suppress the complex conjugate basis functions and integrate from

$$-\infty \leq \omega \leq \infty \tag{12.32}$$

$$\phi = \int_0^\infty \frac{d\omega}{\sqrt{4\pi\omega}} \left(\left(a_R^-(\omega) e^{-i\omega x^-} + a_R^+(\omega) e^{i\omega x^-} \right) + \left(a_L^-(\omega) e^{-i\omega x^+} + a_L^+(\omega) e^{i\omega x^+} \right) \right) \tag{12.33}$$

We could also say the corresponding solutions

$$F_R^+(\Omega) \equiv e^{-i\Omega X^-} \tag{12.34}$$

are right-moving positive frequency with respect to the new space and time coordinates (ξ^0, ξ^1)

The relationship between the two light cone coordinates is given by:

$$\begin{aligned} x^- &= -\frac{1}{a} e^{-aX^-} \\ x^+ &= \frac{1}{a} e^{aX^+} \end{aligned} \tag{12.35}$$

We then have a different expansion

$$\phi = \int_0^\infty \frac{d\Omega}{\sqrt{4\pi\Omega}} \left(\left(b_R^-(\Omega) e^{-i\Omega X^-} + b_R^+(\Omega) e^{i\Omega X^-} \right) + \left(b_L^-(\Omega) e^{-i\Omega X^+} + b_L^+(\Omega) e^{i\Omega X^+} \right) \right) \tag{12.36}$$

We are then tempted to write the field operator

$$\begin{aligned} \hat{\phi} &= \int_0^\infty \frac{d\omega}{\sqrt{4\pi\omega}} \left(\left(\hat{a}_R(\omega) e^{-i\omega x^-} + \hat{a}_R^+(\omega) e^{i\omega x^-} \right) + \left(\hat{a}_L(\omega) e^{-i\omega x^+} + \hat{a}_L^+(\omega) e^{i\omega x^+} \right) \right) = \\ &\int_0^\infty \frac{d\Omega}{\sqrt{4\pi\Omega}} \left(\left(\hat{b}_R(\Omega) e^{-i\Omega X^-} + \hat{b}_R^+(\Omega) e^{i\Omega X^-} \right) + \left(\hat{b}_L(\Omega) e^{-i\Omega X^+} + \hat{b}_L^+(\Omega) e^{i\Omega X^+} \right) \right) \end{aligned} \tag{12.37}$$

where the operators obey canonical commutation relations

$$[\hat{a}(\omega)_R, \hat{a}^+(\omega')_R] = \delta(\omega - \omega')$$
$$[\hat{b}(\Omega)_R, \hat{b}^+(\Omega')_R] = \delta(\Omega - \Omega') \tag{12.38}$$

and so on.

- We now *define* the Minkowski vacuum state by the condition

$$\hat{a}_R(\omega)|0_M\rangle = 0 \tag{12.39}$$

It is clear that this is the vacuum whose excitations would measure an inertial observer. The Rindler vacuum instead will be defined by

$$\hat{b}_R(\omega)|0_R\rangle = 0 \tag{12.40}$$

and this is the ground state for excitations measured by the accelerated observer.

- Assuming that the Minkowski vacuum is a physical state, the Rindler state requires an infinite energy to be prepared: It can be checked from the expansions that

$$\langle 0|T_{x^-x^-}|0\rangle \sim \langle 0_M|\frac{\partial\hat{\phi}}{\partial x^-}\frac{\partial\hat{\phi}}{\partial x^-}|0_M\rangle = \langle 0_R|\frac{\partial\hat{\phi}}{\partial X^-}\frac{\partial\hat{\phi}}{\partial X^-}|0_R\rangle \tag{12.41}$$

This yields

$$\langle 0_R|\frac{\partial\hat{\phi}}{\partial x^-}\frac{\partial\hat{\phi}}{\partial x^-}|0_R\rangle = \left(\frac{\partial X^-}{\partial x^-}\right)^2\langle 0_R|\frac{\partial\hat{\phi}}{\partial X^-}\frac{\partial\hat{\phi}}{\partial X^-}|0_R\rangle = \frac{1}{a^2\,(x^-)^2}\langle 0_R|\frac{\partial\hat{\phi}}{\partial X^-}\frac{\partial\hat{\phi}}{\partial X^-}|0_R\rangle \tag{12.42}$$

which is expected to diverge at the future horizon $x^- = 0$.

In a completely analogous way we would have shown that

$$\langle 0|T_{x^+x^+}|0\rangle \tag{12.43}$$

are expected to diverge at the past horizon, $x^+ = 0$.

- It is clear that we can Fourier expand one set of modes in terms of the other:

$$F_R^+(\Omega) = e^{-i\Omega X^-} = \int_{-\infty}^{\infty} d\omega\rho(\omega)e^{-i\omega x^-} = \int_0^{\infty} d\omega\left(\rho(\omega)f_R^+(\omega) + \rho(-\omega)f_R^*(\omega)\right) \tag{12.44}$$

with

$$\rho(\omega) = \int\frac{dx^-}{2\pi}e^{-i\Omega X^-}e^{i\omega x^-} = \int\frac{dx^-}{2\pi}e^{i\omega x^-}\left(-ax^-\right)^{\frac{i\Omega}{a}} =$$
$$\frac{i}{2\pi\omega}\left(\frac{a}{i\omega}\right)^{i\frac{\Omega}{a}}\Gamma\left(1 + i\frac{\Omega}{a}\right) = -\frac{1}{2\pi\omega}e^{\frac{\pi\Omega}{2a}}e^{i\frac{\Omega}{a}log\,\frac{a}{\omega}}\Gamma\left(i\frac{\Omega}{a}\right) \tag{12.45}$$

We also have

$$f_R(\omega) = e^{-i\omega x^-} = \int_{-\infty}^{\infty} d\Omega\, \gamma(\Omega) e^{-i\Omega X^-} = \int_0^\infty d\Omega \left(\gamma(\Omega)\, F_R(\Omega) + \gamma(-\Omega)\, F_R^*(\Omega)\right) =$$
$$\int_0^\infty d\Omega \sqrt{\frac{\omega}{\Omega}} \left(\alpha(\Omega) F_R - \beta^*(\Omega) F_R^*\right) \tag{12.46}$$

where this last notation has been introduced with an eye for the Bogoliubov transformation that will appear in a moment, and

$$\gamma(\Omega) = \int_{-\infty}^{\infty} \frac{dX^-}{2\pi}\, e^{-i\omega x^-} e^{i\Omega X^-} = -\int_{-\infty}^{0} \frac{dx^-}{2\pi} \frac{1}{ax^-} e^{-i\omega x^-} (-ax^-)^{\frac{-i\Omega}{a}} =$$
$$\int_0^\infty \frac{dy}{2\pi} \frac{1}{ay} e^{i\omega y} (ay)^{-i\frac{\Omega}{a}} = \int_0^\infty \frac{dt}{2\pi a} \frac{1}{t}\, e^{-t} \left(\frac{ita}{\omega}\right)^{-i\frac{\Omega}{a}} =$$
$$\frac{1}{2\pi a}\, e^{\frac{\pi\Omega}{2a}} e^{-i\frac{\Omega}{a}\, log\, \frac{a}{\omega}} \Gamma\left(-i\frac{\Omega}{a}\right) \tag{12.47}$$

This clearly implies that

$$|\alpha(\Omega)|^2 = e^{\frac{2\pi\Omega}{a}} |\beta(\Omega)|^2 \tag{12.48}$$

- There is a Bogolyubov transformation relating both sets of creation and destruction operators. Symbolically, the change of basis we have just done yields

$$\phi \sim \sum \hat{a}_R \left(\alpha F - \beta^* F^*\right) + \hat{a}_R^+ \left(\alpha^* F^* - \beta F\right) + left =$$
$$\sum \hat{b}_R F + \hat{b}_R^+ F^* + left \tag{12.49}$$

In gory detail,

$$\hat{b}_R(\Omega) = \int_0^\infty d\omega \left(\alpha_{\Omega\omega} \hat{a}_R(\omega) - \beta_{\Omega\omega} \hat{a}_R^+(\omega)\right) \tag{12.50}$$

The canonical commutation relations do imply that (suppressing carets over operators from now on)

$$\left[\int_0^\infty d\omega_1 \left(\alpha_{\Omega_1,\omega_1} a(\omega_1) - \beta_{\Omega_1,\omega_1} a^+(\omega_1)\right), \int d\omega_2 \left(\alpha^*_{\Omega_2,\omega_2} a(\omega_2) - \beta^*_{\Omega_2,\omega_2} a^+(\omega_2)\right)\right] = \delta\left(\Omega_1 - \Omega_2\right) =$$
$$\int d\omega \left(\alpha_{\Omega_1\omega} \alpha^*_{\Omega_2\omega} - \beta_{\Omega_1\omega} \beta^*_{\Omega_2\omega}\right) \tag{12.51}$$

which is a normalization condition for Bogoliubov's coefficients. It implies, in particular, that

$$\int d\omega \left(|\alpha_{\Omega\omega}|^2 - |\beta_{\Omega\omega}|^2\right) = \delta(0) = \int d\omega \left(e^{\frac{2\pi\Omega}{a}} - 1\right) |\beta_{\Omega\omega}|^2 \tag{12.52}$$

The expectation value of b-particles in the Minkowski vacuum will be

$$\begin{aligned}&\langle 0_M|N_\Omega \equiv b^+_\Omega b_\Omega|0_M\rangle =\\&\langle 0_M|\int d\omega_1\left(\alpha^*_{\Omega\omega_1}a^+_\omega - \beta_{\Omega\omega_1}a_\omega\right)\int d\omega_2\left(\alpha_{\Omega\omega_2}a_\omega - \beta_{\Omega\omega_2}a^+_\omega\right)|0_M\rangle = \int d\omega|\beta_{\Omega\omega}|^2 =\\&\frac{1}{e^{\frac{2\pi\Omega}{a}}-1}\delta(0) \sim \frac{1}{e^{\frac{2\pi\Omega}{a}}-1}V\end{aligned} \tag{12.53}$$

where V has to be interpreted as the volume of space. These massless particles detected by the accelerated oberved in the Minkowski vacuum obey the Bose-Einstein distribution at a temperature

$$T = \frac{a}{2\pi} \tag{12.54}$$

This is the *Unruh temperature*. In order to get to a temperature of

$$T = 1 \sim 10^{-16} erg \sim 10^{-10} MeV \tag{12.55}$$

and given the fact that the gravitational acceleration at earth is

$$g \sim 10ms^{-2} \sim 10^{-29} MeV \tag{12.56}$$

the corresponding acceleration necessary to raise the temperature a miserable degree is

$$a \sim 10^{19} g \tag{12.57}$$

The possibillity of its detection in storage rings has been advanced by Bell and Leinaas. More recently, a proposal was put forward by Chen and Tajima [6] of detecting Unruh radiation with the help of ultra-intense lasers. It seems however that we have to wait somewhat before getting experimental confirmation of such an effect.

In the full Minkowski space there is a correlation for example, between

$$\langle\phi(t,|x|)\phi(t,-|x|)\rangle \sim \frac{1}{|x|^2} \tag{12.58}$$

which is not observable bt the Rindler observer. Let us now define, again in Minkowski space at $t = 0$

$$\begin{aligned}\phi_R(t,x) &\equiv \phi(t,x) \quad (x > 0)\\ \phi_L(t,x) &\equiv \phi(t,x) \quad (x < 0)\end{aligned} \tag{12.59}$$

This a different meaning of the subscripts L and R than the one formerly used. Clearly a general wave function depends on both ϕ_L and ϕ_R.

$$\Psi(\phi_L, \phi_R) \tag{12.60}$$

The Rindler observer will associate a density matrix, ρ_R to the Minkowski vacuum. We know that

$$[\rho_R, p_y] = [\rho_R, p_z] = [\rho_R, H_R] = 0 \tag{12.61}$$

We can write

$$\Psi(\phi_L, \phi_R) \equiv \int \mathcal{D}\phi e^{-S_E} \tag{12.62}$$

where we integrate over all fields with $t > 0$ sith that

$$\phi(t = 0) = (\phi_L, \phi_R) \tag{12.63}$$

Invariance under time translations (boosts)

$$\delta\omega = C \tag{12.64}$$

becomes rotation invariance in the euclidean action. The generator of such rotations is precisely the Rindler hamiltonian. The wave function can be computed by performing a full rotation of π radians; that is determining the transfer matrix from ϕ_R to ϕ_R, which is proportional to

$$\Psi(\phi_L, \phi_R) \equiv \int \mathcal{D}\phi e^{-S_E} = \langle\phi_L|e^{-\pi H_R}|\phi_R\rangle \tag{12.65}$$

Now

$$\rho_R(\phi_R, \phi'_R) = \int \mathcal{D}\phi_L \Psi(\phi_L, \phi_R)\Psi^*(\phi_L, \phi_R)\Psi(\phi_L, \phi'_R) =$$
$$\int \mathcal{D}\phi_L \langle\phi_R|e^{-\pi H_R}|\phi_L\rangle\langle\phi_L|e^{-\pi H_R}|\phi'_R\rangle =$$
$$= \langle\phi_R|e^{-2\pi H_R}|\phi'_R\rangle \tag{12.66}$$

so that

$$\rho_R \sim e^{-2\pi H_R} \tag{12.67}$$

and the Unruh temperature is given by

$$T_R = \frac{1}{2\pi} \tag{12.68}$$

Clearly a FIDO would observe a temperaure

$$T_R = \frac{a}{2\pi} = \frac{1}{2\pi\rho} \tag{12.69}$$

although a FREFO would observe no temperature at all.

Imagine a vacuum fluctuation around the origin in Minkowski. From the Fido point of view the fluctuation appears at $\omega = -\infty$, $\rho \sim 0$ and disappears at $\omega = +\infty$, $\rho \sim= 0$, so that it lasts an infinite time and it is not virtal at all. The horizon behaves as a hot membrane emitting and absorbing thermal energy.

The correct boundary condition to be imposed in QFT on Rindler space, which has a boundary at $\rho = 0$, is at a given cutoff, $\rho = \epsilon$, an effective *stretched horizon* is kept at a constant temperature

$$T = \frac{1}{2\pi\epsilon} \tag{12.70}$$

by a heat reservoir.

There is a host of possible vacua in Schwarzschild as classified by Candelas [**?**]. Define Kruskal coordinates as

$$\begin{aligned} v &\equiv \sqrt{\frac{r}{r_s} - 1}\; e^{\frac{r}{2r_s}} \; \sinh \frac{t}{2r_s} \\ u &\equiv \sqrt{\frac{r}{r_s} - 1}\; e^{\frac{r}{2r_s}} \; \cosh \frac{t}{2r_s} \\ U &\equiv v - u \\ V &\equiv v + u \end{aligned} \tag{12.71}$$

- The *Boulware vacuum* $|B\rangle$. It is defined by requiring normal modes to be positive frequency with respect to $\frac{\partial}{\partial t}$. It is pathological at the horizon, in the sense that the expectation value of the energy-momentum tensor evaluated in a freely falling frame diverges as $r \rightarrow r_s$.

- The *Unruh vacuum* $|U\rangle$. Defined by taking modes that are incoming from $\mathcal{J}^-$ to be of positive frequency with respect to $\frac{\partial}{\partial t}$, while those that emanate from the past horizon are taken to be positive frequency with respect to U. $\langle T \rangle$ is regular in the future horizon but not on the past horizon. At infinity this corresponds to an outgoing flux of BB radiation at T_H.

- The *Hartle-Hawking vacuum*, $|HH\rangle$. Defined by taking incoming modes to be positive frequency with respect to V and outgoing modes to be positive frequancy with respect to U. It corresponds to an unstable equilibrium with an infinite sea of BB radiation.

13

Gravitation and Quantum Field Theory: Poor man's approach.

13.1 The Effective Lagrangian Approach to Quantum Gravity

But if our previous experience with the other interactions is to be of any relevance here, there ought to be a regime, experimentally accessible in the not too distant future, in which gravitons propagating in flat spacetime can be isolated. This is more or less unavoidable, provided gravitational waves are discovered experimentally, and the road towards gravitons should not be too different from the road that lead from the discovery of electromagnetic waves to the identifications of photons as the quanta of the corresponding interaction, a road that led from Hertz to Planck.

Any quantum gravity theory that avoids identifying gravitational radiation as consisting of large numbers of gravitons in a semiclassical state would be at variance with all we believe to know about quantum mechanics.

What we expect instead to be confirmed by observations somewhere in the future is that the number of gravitons per unit volume with frequencies between ω and $\omega + d\omega$ is given by Planck's formula

$$n(\omega)d\omega = \frac{\omega^2}{\pi^2}\frac{1}{e^{\frac{\hbar\omega}{kT}} - 1}d\omega$$

It is natural to keep an open mind for surprises here, because it can be argued that gravitational interaction is not alike any other fundamental interaction in the sense that the whole structure of space-time ought presumably be affected, but it cannot be denied that this is the most conservative approach and as such it should be explored first, up to its very limits, which could hopefully indicate further avenues of research.

From our experience then with the standard model of elementary particles, and assuming we have full knowledge of the fundamental symmetries of our problem, we know that we can parametrize our ignorance on the *fundamental* ultraviolet (UV) physics by writing down all local operators in the low energy fields $\phi_i(x)$ compatible with the basic symmetries we have assumed.

$$L = \sum_{n=0}^{\infty} \frac{\lambda_n(\Lambda)^n}{\Lambda^n} \mathcal{O}^{(n+4)}\left(\phi_i\right)$$

Here Λ is an ultraviolet (UV) cutoff, which restricts the contributions of large euclidean momenta (or small euclidean distances) and $\lambda_n(\Lambda)$ is an infinite set of dimensionless bare couplings.

Standard Wilsonian arguments imply that *irrelevant operators*, corresponding to $n > 4$, are less and less important as we are interested in deeper and deeper infrared (IR) (*low energy*) variables. The opposite occurs wuth *relevant operators*, corresponding to $n < 4$, like the masses, that become more and more important as we approach the IR. The intermediate role is played by the *marginal operators*, corresponding to precisely $n = 4$, and whose relevance in the I R is not determined solely by dimensional analysis, but rather by quantum corrections. The range of validity of any finite number of terms in the expansion is roughly

$$\frac{E}{\Lambda} << 1$$

where E is a characteristic energy of the process under consideration.

In the case of gravitation, we assume that general covariance (or diffeomorphism invariance) is the basic symmetry characterizing the interaction. We can then write

$$\begin{aligned} L_{eff} = \lambda_0\Lambda^4\sqrt{|g|} + \lambda_1\Lambda^2 R\sqrt{|g|} + \lambda_2 R^2 + \frac{1}{2}g^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi\sqrt{|g|} + \\ +\lambda_3\frac{1}{\Lambda^2}R^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi\sqrt{|g|} + \lambda_4\frac{1}{\Lambda^2}R^3\sqrt{|g|} + \lambda_5\phi^4\sqrt{|g|} + \\ +\bar{\psi}\left(e^\mu_a\gamma^a\left(\partial_\mu - \omega_\mu\right)\psi - m\right)\psi + \frac{\lambda_5}{\Lambda^2}\bar{\psi}e^\mu_a\gamma^a R\left(\partial_\mu - \omega_\mu\right)\psi + \ldots \end{aligned} \tag{13.1}$$

where e^μ_a is the tetrad, such that

$$e^\mu_a e^\nu_\beta \eta^{\alpha\beta} = g^{\mu\nu}$$

$\eta^{\alpha\beta}$ being Minkowski's metric. The quantities ω_μ are the spin connection.

The need to recover General Relativity in the classical IR limit means

$$\lambda_1\Lambda^2 = -\frac{c^3}{16\pi G} \equiv -2M_p^2$$

This in turn, means that if

$$\lambda_0 \Lambda^4$$

is to yield the observed value for the cosmological constant (which is of the order of magnitude of Hubble's constant, H_0^4, which is a very tiny figure when expressed in particle physics units, $H_0 \sim 10^{-33}\, eV$) then

$$\lambda_0 \sim 10^{-244}$$

This is one aspect of the cosmological constant problem; it seems most unnatural that the cosmological constant is observationally so small from the effective lagrangian point of view. I do not have anything new to say on this.

This expansion is fine as long as it is considered a low energy expansion. As Donoghue [**?**] has emphasized, even if it is true that each time that a renormalization is made there is a finite arbitrariness, there are physical predictions stemming from the non-local finite parts.

The problem is when energies are reached that are comparable to Planck's mass,

$$E \sim M_p.$$

Then all couplings in the effective Lagrangian become of order unity, and there is no *decoupling limit* in which gravitation can be considered by itself in isolation from all other interactions. This then seems the minimum prize one has to pay for being interested in quantum gravity; all couplings in the derivative expansion become important simultaneously. No significant differences appear when supergravity is considered.

In conclusion, it does not seem likely that much progress can be made by somehow quantizing Einstein-Hilbert's Lagrangian in isolation. To study quantum gravity means to study all other interactions as well.

On the other hand, are there any reasons to go beyond the standard model (SM)?

Yes there are some, both theoretical, and experimental. From the latter, and most important, side, both the existence of neutrino masses and dark matter do not fit into the SM. And from the former, abelian sectors suffer from Landau poles and are not believed to be UV complete; likewise the self-interactions in the Higgs sector appear to be a trivial theory. Also the experimental values of the particle masses in the SM are not natural from the effective lagrangian point of view.

The particle physics community has looked thoroughly for such extensions since the eighties: extra dimensions (Kaluza-Klein), supersymmetry and supergravity, technicolor, etc. From a given point of view, the natural culmination of this road is string theory

A related issue is the understanding of the so-called *semiclassical gravity*, in which the second memnber of Einstein's equations is taken as the expectation value of some quantum energy-momentum operator. It can be proved that this is the dominant $1/N$ approximation in case there are N identical matter fields (confer [**?**]). In spite of tremendous effort, there is not yet a full understanding of Hawking's emission of a black hole from the effective theory point of view . Another topic in which this approach has been extensively studied is Cosmology. Novel effects (or rather old ones on which no emphasis was put until recently) came from lack of momentum conservation and sem to point towards some sort of instability [**?**]; again the low energy theory is not fully understood; this could perhaps have something to do with the presence of horizons.

Coming back to our theme, and closing the loop, what are the prospects to make progress in quantum gravity?

Insofar as effective lagrangians are a good guide to the physics there are only two doors open: either there is a ultraviolet (UV) attractive fixed point in coupling space, such as in Weinbergś *asymptotic safety* or else new degrees of freedom, like in string theory exist in the UV. Even if Weinbergś approach is vindicated, the fact that the fixed point most likely lies at strong coupling combined with our present inability to perform analitically other than perturbative computations, mean that lattice simulations should be able to cope with the integration over (a subclass of) geometries before physical predictions could be made with the techniques at hand at the present moment.

It is to be remarked that sometimes theories harbor the seeds of their own destruction. Strings for example, begin as theories in flat spacetime, but there are indications that space itself should be a derived, not fundamental concept. It is hoped that a simpler formulation of string theory exists bypassing the roundabouts of its historical development. This is far from being the case at present.

Finally, it is perhaps worth pointing out that to the extent that a purely gravitational canonical approach, as the ones based upon the use of Ashtekar variables makes contact with the classical limit (which is an open problem from this point of view) the preceding line of argument should still carry on.

It seems *unavoidable* with our present understanding, that any theory of quantum gravity should recover, for example, the prediction that there are quantum corrections to the gravitational potential given by [**?**]

$$V(r) = -\frac{Gm_1m_2}{r}\left(1 + 3\frac{G\,(m_1+m_2)}{r} + \frac{41}{10\pi}\frac{G\hbar}{r^2}\right)$$

(the second term is also a loop effect, in spite of the conspicuous absence of $\hbar$.) Similarly, and although this has been the subject of some controversy, it seems now established that there are gravitational corrections to the running of gauge couplings, first uncovered by Robinson and Wilczek and given in standard notation by

$$\beta(g,E) = -\frac{b_0}{(4\pi)^2}g^3 - 3\frac{16\pi G}{(4\pi)^2\hbar c^3}gE^2$$

Sometimes these effects are dismissed as perturbative, and therefore trivial. This is not a healthy attitude.

Something that can be done is to ignore most of the conceptual problems of quantum gravity, and treat it as a gauge theory. This is possible because the action of diffeomorphisms is formally similar to the one of the symmetry group of an ordinary gauge theory. Locally the fact that the group of dimmeomorphisms of a given manifold, Diff(M) is not a fixed entity, but rather depends in a complicated way on the specific manifold considered, this problem we say if of no concern for our perturbative analysis. All we aim at is to compute the quantum corrections to the gravitational action to first order in the coupling constant, κ. This was first done in a classic paper by 't Hooft and Veltman in 1973, as a byproduct of their analysis of one-loop amplitudes in non-abelian gauge theories. An essential tool of their analysis is the background field technique, first devised by deWitt, to which we now turn.

13.2 The background field approach in quantum field theory.

The main problem in quantum field theory is the computation of the partition function, which is nothing else than Schwinger's *vacuum persistence amplitude* in the presence of an external source, $J(x)$. It is useful to represent it as a functional integral

$$Z[J] \equiv e^{iW[J]} \equiv \int \mathcal{D}\phi\ e^{iS[\phi]+i\int J(x)\phi(x)} \tag{13.2}$$

Where in this formal analysis we represent all fields (including the gravitational field) by $\phi(x)$, and we add a coupling to an arbitrary external source as a technical device to compute Green functions out of it by taking functional derivatives of $Z[J]$ and then putting the sources equal to zero. This trick was also invented by Schwinger. The partition funtion generates all Green functions, connected and disconnected. Its logarithm, $W[J]$ sometimes dubbed the *free energy* generates connected functions only. These names come from a direct analogy with similar quantities in statistical physics.

It is possible to give an intuitive meaning to the path integral in quantum mechanics as a transition amplitude from an initial state to a final state. This is actually the way Feynman derived it.

In QFT the integration measure is not mathematically well-defined. For loop calculations, however, it is enough to *formally define* the gaussian path integral as a functional determinant, that is

$$\int \mathcal{D}\phi e^{i(\phi K \phi)} = \det\, K^{-\frac{1}{2}} \tag{13.3}$$

where the scalar product is defined as

$$(\phi, K\phi) \equiv \int d(vol)\ \phi\ K\ \phi \tag{13.4}$$

and K is a differential operator, usually

$$K = \nabla^2 + \text{something} \tag{13.5}$$

There are implicit indices in the operator to pair the (also implicit) components of the field ϕ.

The only extra postulate needed is translation invariance of the measure, in the sense that

$$\int \mathcal{D}\phi e^{i\ ((\phi+\chi)\ K,\ (\phi+\chi))} = \int \mathcal{D}\phi e^{i(\phi\ ,K\ \phi)} \tag{13.6}$$

This is the crucial property that allows the computation of integrals in the presence of external sources by completing the square.

It is quite useful to introduce a generating function for one-particle irreducible (1-PI) Green functions. This is usually called the *effective action* and is obtained through a Legendre transform, quite analogous to the one performed when passing from the Lagrangian to the hamiltonian ion classical mechanics.

One defines the *classical field* as a functional of the external current by

$$\phi_c[J] \equiv \frac{1}{i}\ \frac{\delta W[J]}{\delta J(x)} \tag{13.7}$$

The Legendre transform then reads

$$\Gamma[\phi_c] \equiv W[J] - i\int d^n x J(x)\phi_c(x) \tag{13.8}$$

It is a fact that

$$\frac{\delta\Gamma}{\delta\phi_c(x)} = \int d^n z\ \frac{\delta W}{\delta J(z)}\ \frac{\delta J(z)}{\delta\phi_c(x)} - iJ(x) - i\int d^n z\phi_c(z)\ \frac{\delta J(z)}{\delta\phi_c(x)} = -iJ(x) \tag{13.9}$$

The background field technique was invented by Bryce Dewitt as a clever device to keep track of divergent terms in theories (such as gravity) with a complicated algebraic structure. The main idea is to split the integration fields into a *classical* and a *quantum* piece:

$$W_\mu \equiv \bar{A}_\mu + A_\mu \tag{13.10}$$

The functional integral is performed over the quantum fields only. where for an ordinary gauge theory the action has three pieces. First the gauge invariant piece

$$L_{\text{gauge}} \equiv -\frac{1}{4} F_{\mu\nu}[W]^2 \tag{13.11}$$

with

$$F^a_{\mu\nu}[W] \equiv \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g f_{abc} W^b_\mu W^c_\nu \tag{13.12}$$

The gauge transformations are

$$\delta W^a_\mu \equiv \delta \left(\bar{A}^a_\mu + A^a_\mu \right) \equiv -f_{abc}\omega^b W^c_\mu + \frac{1}{g}\partial_\mu \omega^a = -f_{abc}\omega^b \left(\bar{A}^a_\mu + A^a_\mu \right) + \frac{1}{g}\partial_\mu \omega^a \tag{13.13}$$

This can be implemented in two ways. First letting the background field be inert. Those are the *quantum gauge*

$$\begin{aligned} &\delta_Q \bar{A}^a_\mu \equiv 0 \\ &\delta_Q A^a_\mu = -f_{abc}\omega^b \left(\bar{A}^a_\mu + A^a_\mu \right) + \frac{1}{g}\partial_\mu \omega^a \end{aligned} \tag{13.14}$$

Those are the transfomations in need of gauge fixing. It is to be remarked thet gauge symmetry is realized non-linearly on the quantum fields.

It is also possible to reproduce the total gauge transformations through the *classical* transformations

$$\begin{aligned} &\delta_C \bar{A}^a_\mu = -f_{abc}\omega^b \bar{A}^a_\mu + \frac{1}{g}\partial_\mu \omega^a \\ &\delta_C A^a_\mu = -f_{abc}\omega^b A^c_\mu \end{aligned} \tag{13.15}$$

under which the quantum field transforms as an adjoint vector field.

Currents transform in such a way that

$$\delta_C \int J^\mu_a A^a_\mu = 0 \tag{13.16}$$

that is

$$\delta J^a_\mu = -f_{abc}\omega^b J^c_\mu \tag{13.17}$$

The beauty of the background field method is that it is possible to gauge fix the quantum symmetry while preserving the classical gauge symmetry. All computations are then invariant under gauge transformations of the clasical field, and so are the counterterms. This quite simplifies the heavy work involved in computing with gravity.

The simplest background gauge is

$$\bar{F}^a[A] \equiv \partial_\mu A^\mu_a + g f_{abc} \bar{A}^b_\mu A^\mu_c \equiv \left(\bar{D}_\mu A^\mu \right)^a \tag{13.18}$$

L. Abbott was able to prove a beautiful theorem to the effect that the effective action computed by the background field methos is simply related to the ordinary effective action

$$\Gamma_{BF}[A_c^{BF}, \bar{A}] = \Gamma[A_c]|_{A_c = A_c^{BF} + \bar{A}} \tag{13.19}$$

This means in particular, that

$$\Gamma[A_c] = \Gamma_{BF}[0, \bar{A} = A_c] \tag{13.20}$$

At the one loop order all this simplifies enomoursly. Working in euclidean space

$$e^{-W[\bar{\phi}]} \equiv \int \mathcal{D}\phi e^{-S[\bar{\phi}] - \int \phi K[\bar{\phi}]\phi - \int J\phi} = \\ e^{-S[\bar{\phi}] - \frac{1}{2}\log \det\, K[\bar{\phi}] - \frac{1}{2}\int JK^{-1}[\bar{\phi}]J} \tag{13.21}$$

This means that

$$\phi_c = -K^{-1}\;[\bar{\phi}]J \tag{13.22}$$

so that

$$J = -K[\bar{\phi}]\;\phi_c \tag{13.23}$$

and

$$\Gamma^{BF}[\phi_c, \bar{\phi}] = W[J(\phi_c)] - \int J\phi_c = \\ = S[\bar{\phi}] + \frac{1}{2}\log \det\, K[\bar{\phi}] + \frac{1}{2}\int K\phi_c K^{-1}[\bar{\phi}]K\phi_c - \int K\phi_c\phi_c = \\ = S[\bar{\phi}] + \frac{1}{2}\log \det\, K[\bar{\phi}] - \frac{1}{2}\int \phi_c K\phi_c \tag{13.24}$$

Then by Abbott's theorem

$$\Gamma(\phi_c) = \Gamma^{BF}[0, \bar{\phi} = \phi_c] = W[\bar{\phi}] \equiv S[\bar{\phi}] + \frac{1}{2}\log \det\, K[\bar{\phi}] \tag{13.25}$$

The one loop effective action is equal to the background field free energy, and the background field can be equated to the classical field.

13.3 Geometric computation of the one loop effective action.

To one loop order all functional integral computations reduce to gaussian integrals, which can in turn be formally represented as functional determinants. This is hardly of any advantage when computing finite parts. Contrasting with that, a geometric approach for computing the *divergent piece* of the effective action exist. This approach was pioneered by Julian Schwinger and Bryce DeWitt.

When breaking the total gravitational field $g_{\mu\nu}(x)$ into a *background part*, $\bar{g}_{\mu\nu}(x)$ and a quantum fluctuation, $h_{\mu\nu}(x)$, we are working in a *background manifold*, $\overline{M}$, with metric $\bar{g}_{\mu\nu}(x)$, and thereby avoiding most of the problems of principle of quantum gravity. Quantum gravitational fluctuations are treated as ordinary gauge fluctuations. This approach was culminated by the brilliant work of 't Hooft and Veltman, where it was shown that pure quantum gravity is one loop finite on shell. This is not true any more as soon as some matter is added. Even pure quantum gravity at two-loops is divergent on shell, as was shown by Goroff and Sagnotti.

The formalism is such that in order to compute the divergent piece of the effective action, background gauge invariance can be maintained, so that we do not commit to any specific background, although we assume that some such background always exists.

Were we to compute correlators, then the particular Green function appropiate to each background is needed, and then all subtle points associated witk background horizons and singularities will reappear. The Unruh radiation is the simplest manifestation of these.

It is to be emphasized that quantum Diff invariance is spontaneously broken in this approach. The background gauge transformations read

$$\begin{aligned} \delta\bar{g}_{\mu\nu} &= \xi^\lambda\partial_\lambda\bar{g}_{\mu\nu} + \partial_\mu\xi^\lambda\bar{g}_{\lambda\nu} + \partial_\nu\xi^\lambda\bar{g}_{\mu\lambda} = \bar{\nabla}_\mu\xi_\nu + \bar{\nabla}_\nu\xi_\mu \\ \delta h_{\mu\nu} &= \xi^\lambda\partial_\lambda h_{\mu\nu} + \partial_\mu\xi^\lambda h_{\lambda\nu} + \partial_\nu\xi^\lambda h_{\mu\lambda} \end{aligned} \tag{13.26}$$

and the quantum gauge transformations read

$$\begin{aligned} \delta\bar{g}_{\mu\nu} &= \xi^\lambda\partial_\lambda\bar{g}_{\mu\nu} \\ \delta h_{\mu\nu} &= \xi^\lambda\partial_\lambda h_{\mu\nu} + \partial_\mu\xi^\lambda\left(\bar{g}_{\lambda\nu} + h_{\lambda\nu}\right) + \partial_\nu\xi^\lambda\left(\bar{g}_{\mu\lambda} + h_{\mu\lambda}\right) \end{aligned} \tag{13.27}$$

Working to one loop order, they simplify to

$$\begin{aligned} \delta\bar{g}_{\mu\nu} &= \xi^\lambda\partial_\lambda\bar{g}_{\mu\nu} + \partial_\mu\xi^\lambda\bar{g}_{\lambda\nu} + \partial_\nu\xi^\lambda\bar{g}_{\mu\lambda} = \bar{\nabla}_\mu\xi_\nu + \bar{\nabla}_\nu\xi_\mu \\ \delta h_{\mu\nu} &= \xi^\lambda\partial_\lambda h_{\mu\nu} \end{aligned} \tag{13.28}$$

and to

$$\begin{aligned} \delta\bar{g}_{\mu\nu} &= \xi^\lambda\partial_\lambda\bar{g}_{\mu\nu} \\ \delta h_{\mu\nu} &= \xi^\lambda\partial_\lambda h_{\mu\nu} + \partial_\mu\xi^\lambda\bar{g}_{\lambda\nu} + \partial_\nu\xi^\lambda\bar{g}_{\mu\lambda} \end{aligned} \tag{13.29}$$

They still act nonlinearly of the quantum fluctuations owing to the inhomogeneous term. This physically means that the quantum fluctuations behave as goldstone bosons of broken Diff invariance.

To study the Diff invariant phase would mean to compute with

$$\eta_{\mu\nu} = 0 \tag{13.30}$$

which is not possible, bacause there is then no background geometry. For starters, it is not possible to define even the inverse metrix, $\bar{g}^{\mu\nu}$, neither the Christoffels, etc.

In some cases, and using the first order formalism, it is possible to functionally integrate without the restriction that the determinant of the metric does not vanish

$$\bar{g} \neq 0 \tag{13.31}$$

An example is Witten treatment of three-dimensional quantum gravity as a gauge theory.

It is not clear what are the conclusions to draw for the four-dimensional case.

13.4 Zeta function

Consider the partition function in euclidean signature

$$Z \equiv \int \mathcal{D}\phi \; e^{-\frac{1}{2}\int \sqrt{|g|}\; d^n x \; \phi \; A \; \phi}$$

This means that the dimension of the fields ϕ must be $\frac{n-d_A}{2}$, where d_A is trhe mass dimension of the operator A; usually $d_A = 2$. The eigenvalues equation for this operator is

$$A\phi_n = \lambda_n \phi_n$$

The dimension of λ_n must necessarily be that of the operator A. We can fool around with the dimension of ϕ_n, or fix it through normalization:

$$\langle \phi_n | \phi_m \rangle \equiv \int \sqrt{|g|}\; d^n x \; \phi_n^* \; \phi_m = \delta_{mn}$$

The dimension of ϕ_m is then $\frac{n}{2}$ in the Kronecker case, or 0 in the continuous case when the Kronecker delta is replaced by a Dirac delta of momentum $\delta^n(k)$.

If the set of eigenfunctions is complete in the functional space, it is possible to formally expand

$$\phi \equiv \sum a_n \; \phi_n$$

The dimensions of the expansion coefficients a_n is $\frac{n-d_A}{2} - \frac{n}{2} = -\frac{d_A}{2}$ with the discrete normalization.

It is tempting to *define the functional measure* as the dimensionless quantity

$$\mathcal{D}\phi \equiv \prod_n \mu^{\frac{d_A}{2}} \; da_n$$

Then the gaussain integral is represented by the infinite product

$$Z = \prod_n \mu^{\frac{d_A}{2}} \sqrt{\frac{2\pi}{\lambda_n}}$$

The zeta-function associated to the operator A is now defined by analogy with Riemann's zeta function

$$\zeta(s) \equiv \sum_n \left(\frac{\lambda_n}{\mu^{d_A}}\right)^{-s}$$

and find

$$\zeta'(s) = -\sum_n \, \log \, \left(\frac{\lambda_n}{\mu^{d_A}}\right) \, \left(\frac{\lambda_n}{\mu^{d_A}}\right)^{-s}$$

so that

$$-\zeta'(0) = \sum_n \, \log \, \left(\frac{\lambda_n}{\mu^{d_A}}\right) \; = \log \, \det \, A$$

then the determinant of the operator itself is defined by analytic continuation as

$$\det \, A \equiv e^{-\zeta'(0)} \tag{13.32}$$

Let us work in detail the most basic of all determinants, the one of the flat space d'Alembertian. The dimensionless eigenfunctions ara plane waves

$$\phi_k \equiv \frac{1}{(2\pi)^{\frac{n}{2}}} \; e^{ikx} \tag{13.33}$$

and are normalized in such a way that

$$\int d^n x \; \phi^*_k(x)\phi_{k'}(x) = \delta^n \;\; (k - k') \tag{13.34}$$

The eigenvectors are simply

$$\lambda_k = -k^2 \tag{13.35}$$

The continuum normalization means that fields are expanded as

$$\phi(x) = \int d^n k \; a_k \; \phi_k(x) \tag{13.36}$$

This means that the dimension of the expansion coefficients is now

$$[a_k] = -\frac{n + d_A}{2} \tag{13.37}$$

The zeta function is given by

$$\zeta(s) = \int \frac{d^n k}{\mu^n} \left(\frac{-k^2}{\mu^2}\right)^{-s} = \int \frac{d^n k}{\mu^n}\; e^{-s\,\log\left(\frac{-k^2}{\mu^2}\right)} \tag{13.38}$$

This leads to the expression for the determinant of the ordinary d'Alembert operator

$$\log\,\det\,\Box = \int \frac{d^n\;k}{\mu^n}\log\left(\frac{-k^2}{\mu^2}\right)$$

13.5 Heat kernel

Let us now follow a slightly different route which is however intimately related. We begin, following Schwinger, by considering the divergent integral which naively is independent of λ

$$I(\lambda) \equiv \int_0^\infty \frac{dx}{x}\; e^{-x\lambda} \tag{13.39}$$

The integral is actually divergent, so before speaking about is it has to be regularized. It can be defined through

$$I(\lambda) \equiv \lim_{\epsilon\to 0} I(\epsilon,\lambda) \equiv \lim_{\epsilon\to 0}\int_\epsilon^\infty \frac{dx}{x} e^{-x\lambda} \tag{13.40}$$

such that

$$\lim_{\epsilon\to 0}\frac{\partial I(\epsilon,\lambda)}{\partial\lambda} = -\frac{1}{\lambda} \tag{13.41}$$

It follows

$$\therefore \quad I(\lambda) = -\log\lambda + C \tag{13.42}$$

It is natural to define (for trace class [1]) operators

$$\log\det\Delta = \mathrm{tr}\log\Delta \equiv \sum_n \log\lambda_n \tag{13.43}$$

Now given an operator (with purely discrete, positive spectrum) we could generalize the above idea (Schwinger)

$$\log\det\Delta \equiv -\int_0^\infty \frac{d\tau}{\tau}\mathrm{tr}\; e^{-\tau\Delta} \tag{13.44}$$

The trace here encompasses not only discrete indices, but also includes an space-time integral. Let is define now the *heat kernel* associated to that operator as the operator

$$K(\tau) \equiv e^{-\tau\Delta} \tag{13.45}$$

[1] In the physical Lorentzian signatura, all quantities will be computed from analytic continuations from riemannian configurations where they are better defined. This procesire is not always unambiguous when gravity is present.

Formally the inverse operator is given through

$$\Delta^{-1} \equiv \int_0^\infty d\tau \; K(\tau) \tag{13.46}$$

where the kernel obeys the heat equation

$$\left(\frac{\partial}{\partial\tau} + \Delta\right) K(\tau) = 0 \tag{13.47}$$

In all case that will interest us, the operator Δ will be a differential operator. Then the heat equation is a parabolic equation

$$\left(\frac{\partial}{\partial\tau} + \Delta\right) K(\tau; x, y) = 0 \tag{13.48}$$

which need to be solved with the boundary condition

$$K(x, y, 0) = \delta^{(n)}(x - y) \tag{13.49}$$

The mathematicians have studied operators which are deformations of the laplacian of the type

$$\Delta \equiv D^\mu D_\mu + Y \tag{13.50}$$

where D_μ is a gauge covariant derivative

$$D_\mu \equiv \nabla_\mu + X_\mu \tag{13.51}$$

and ∇_μ is the usual covariant space-time derivative.

In the simplest case $X = Y = 0$ and $\nabla_\mu = \partial_\mu$, the flat space solution corresponding to the euclidean laplacian is given by

$$K_0(x, y; \tau) = \frac{1}{(4\pi\tau)^{n/2}} e^{-\frac{\sigma(x,y)}{2\tau}} \tag{13.52}$$

where the world function in flat space is simply

$$\sigma(x, y) \equiv \frac{1}{2}(x - y)^2 \tag{13.53}$$

This can be easily checked by direct computation. It is unfortunately quite difficult to get explicit solutions of the heat equation except in very simple cases. This limits the applicability of the method for computing finite determinants. These determinants are however divergent in all cases of interest in QFT, , and their divergence is due to the lower limit of the proper time integral. It we were able to know the solution close to the lower limit, we could get at least some information on the structure of the divergences. This is exactly how far it is possible to go.

The small proper time expansion of Schwinger and DeWitt is given by a Taylor expansion

$$K(\tau; x, y) = K_0(\tau; x, y) \sum_{p=0}^{\infty} a_p(x, y)\, \tau^p \tag{13.54}$$

with

$$a_0(x, x) = 1 \tag{13.55}$$

The coefficients $a_p(x, y)$ characterize the operator whose determinant is to be computed. Actually, for the purpose at hand, only their diagonal part, $a_n(x, y)$ is relevant.

The integrated diagonal coefficients will be denoted by capital letters

$$A_n \equiv \int \sqrt{|g|}\; d^n x\; a_n(x, x) \tag{13.56}$$

in such a way that

$$A_0 = vol \equiv \int_M \sqrt{|g|}\; d^n x \tag{13.57}$$

The determinant of the operator is then given by an still divergent integral. The short time expansion did not arrange anything in that respect. This integral has to be regularized by some procedure. One of the possibilities is to keep $x \neq y$ in the exponent, so that

$$\log \det \Delta \equiv -\int_0^{\infty} \frac{d\tau}{\tau} \mathrm{tr}\, K(\tau) \equiv -\lim_{\sigma \to 0} \int_0^{\infty} \frac{d\tau}{\tau} \frac{1}{(4\pi\tau)^{n/2}} \sum_{p=0}^{\infty} \tau^p \mathrm{tr}\, a_p(x, y)\, e^{-\frac{\sigma^2}{4\tau}} \tag{13.58}$$

We have regularized the determinant by point-splitting. For consistency, also the off-diagonal part of the short-time coefficient ought to be kept.

All ultraviolet divergences are given by the behavior in the $\tau \sim 0$ endpoint. Changing the order of integration, and performing first the proper time integral, the Schwinger-de Witt expansion leads to

$$\log \det \Delta = -\int d(vol)\, \lim_{\sigma \to 0} \sum_{p=0}^{\infty} \frac{\sigma^{2p-n}}{4^p \pi^{\nu/2}}\, \Gamma(n/2 - p)\; \mathrm{tr}\, a_p(x, y) \tag{13.59}$$

Here it has not been not included the σ dependence of

$$\lim_{\sigma \to 0} a_n(x, y) \tag{13.60}$$

In flat space this corresponds to

$$(x - y)^2 = 2\sigma \to 0 \tag{13.61}$$

Assuming this dependence is analytic, this could only yield higher powers of σ, as will become plain in a moment.

The term $p = 0$ diverges in four dimensions when $\sigma \to 0$ as

$$\frac{1}{\sigma^4} \tag{13.62}$$

but this divergence is common to all operators and can be absorbed by a counterterm proportional to the total volume of the space-time manifold. This renormalizes the the cosmological constant.

The next term corresponds to $p = 2$, and is independent on σ.In order to pinpoint the divergences, When $n = 4 - \epsilon$ it is given by

$$\log\det\Delta|_{n=4} \equiv \frac{1}{156\pi^2\,(4-n)}\;A_2 \tag{13.63}$$

From this term on, the limit $\sigma \to 0$ kills everything.

A different way to proceed is to take $\sigma = 0$ from the beginning and put explicit IR (μ) and UV (Λ) proper time cutoffs, such that $\frac{\Lambda}{\mu} >> 1$. It should be emphasized that these cutoffs are not cutoffs in momentum space; they respect in particular all gauge symmetries the theory may enjoy.

$$\log\det\Delta \equiv -\int\frac{d\tau}{\tau}\text{tr }K(\tau) \equiv -\int_{\frac{1}{\Lambda^2}}^{\frac{1}{\mu^2}}\frac{d\tau}{\tau}\frac{1}{(4\pi\tau)^{n/2}}\sum_{p=0}\tau^p\text{tr }A_p\left[\Delta\right] \tag{13.64}$$

This yields, for example in $n = 4$ dimensions

$$\log\det\Delta = \frac{1}{(4\pi)^2}\left(\frac{1}{2}\Lambda^4\;Vol + A_1\left[\Delta\right]\;\Lambda^2 + A_2\left[\Delta\right]\;\log\frac{\Lambda^2}{\mu^2}\right) \tag{13.65}$$

There are finite contributions that are not captured by the small proper time expansion; those are much more difficult to compute and the heat kernel method is not particularly helpful in that respect.

13.6 Flat space determinants

Let us see in detail how the heat equation can be iterated to get the coefficients of the short time expansion for operators pertaining to flat space gauge theories.

The small proper time expansion of the heat kernel should be substituted into the heat equation for the gauge operator as above, It follows

$$\frac{\partial}{\partial\tau}\ K\left(\tau;x,y\right)=\frac{1}{(4\pi)^{\frac{n}{2}}}\ e^{-\frac{(x-y)^2}{4\tau}}\ \sum_{p=0}\left(a_p\frac{(x-y)^2}{4}+(p-\frac{n}{2}-1)a_{p-1}\right)\tau^{p-2-\frac{n}{2}}$$

$$\partial_\mu K=\frac{1}{(4\pi\tau)^{\frac{n}{2}}}e^{\frac{-(x-y)^2}{4\tau}}\sum_p\left(-\frac{\sigma_\mu}{2\tau}a_p+\partial_\mu a_p\right)\tau^p$$

$$D_\mu K\left(\tau;x,y\right)=\frac{1}{(4\pi\tau)^{\frac{n}{2}}}e^{\frac{-(x-y)^2}{4\tau}}\sum_p\left(-\frac{\sigma_\mu}{2\tau}a_p+D_\mu a_p\right)a_p$$

$$\sum_\mu D^2_\mu K\left(\tau;x,y\right)=$$

$$=\frac{1}{(4\pi)^{\frac{n}{2}}}\ e^{-\frac{(x-y)^2}{4\tau}}\ \sum_p\left(-\frac{n}{2\tau}a_p+\frac{(x-y)^2}{4\tau^2}a_p-\sum_\mu\frac{\sigma^\mu}{\tau}D_\mu a_p+D^2a_p\right)\tau^{p-\frac{n}{2}}=$$

$$=\frac{1}{(4\pi)^{\frac{n}{2}}}\ e^{-\frac{(x-y)^2}{4\tau}}\ \sum_p\left(-\frac{n}{2}\ a_{p-1}+\frac{(x-y)^2}{4}a_p-\sum_\mu\ \sigma^\mu\ D_\mu a_{p-1}+D^2a_{p-2}\right)\tau^{p-2-\frac{n}{2}}$$

$$\Delta K\left(\tau;x,y\right)=\left(D^2_\mu-Y\right)K=$$

$$=\frac{1}{(4\pi)^{\frac{n}{2}}}\ e^{-\frac{(x-y)^2}{4\tau}}\ \sum_p\left(-\frac{n}{2}\ a_{p-1}+\frac{(x-y)^2}{4}a_p-\sum_\mu\ \sigma^\mu\ D_\mu a_{p-1}+\Delta a_{p-2}\right)\tau^{p-2-\frac{n}{2}}$$

(where $\sigma_\mu\equiv(x_\mu-y_\mu)$).

The more divergent terms are those in $\tau^{-2-\frac{n}{2}}$, but they do not give anything new

$$\frac{a_0}{4}(x-y)^2=\frac{a_0}{4}(x-y)^2 \tag{13.66}$$

The next divergent term (only even p contribute to the expansion without boundaries) is $\tau^{1-\frac{n}{2}}$

$$-\frac{n}{2}a_0=-\frac{n}{2}a_0-\sigma.Da_0 \tag{13.67}$$

ao that we learn that

$$\sigma.Da_0=0 \tag{13.68}$$

Generically,

$$\left(p-\frac{n}{2}-1\right)a_{p-1}=-\frac{n}{2}a_{p-1}+\sigma.Da_{p-1}+\Delta a_{p-2} \tag{13.69}$$

which is equivalent to

$$(p+1)a_{p+1}=-\sigma.Da_{p+1}+D^2a_p \tag{13.70}$$

Taking the covariant derivative of the first equation,

$$D_\lambda(\sigma^\mu D_\mu a_0)=0=D_\lambda a_0+\sigma^\mu D_\lambda D_\mu a_0 \tag{13.71}$$

the first *coincidence limit* follows

$$[D_\mu a_0] = 0 \tag{13.72}$$

(please note that $[a_0] = 1$ which we knew already, does not imply the result.) Taking a further derivative, we get

$$[(D_\mu D_\nu + D_\nu D_\mu)a_0] = 0 \tag{13.73}$$

whose trace reads

$$\left[D^2 a_0\right] = 0 \tag{13.74}$$

The usual definition

$$W_{\mu\nu} \equiv [D_\mu, D_\nu] \tag{13.75}$$

implies

$$[D_\mu D_\nu \; a_0] = \frac{1}{2}\left[([D_\mu, D_\nu]_- + \{D_\mu, D_\nu\})a_0\right] = \frac{1}{2} W_{\mu\nu} \tag{13.76}$$

where the fact has been used that

$$[a_0] = 1 \tag{13.77}$$

Taking $p = 0$ in (13.69)

$$-\, a_1 = D^2 a_0 + \sigma.D a_1 \tag{13.78}$$

so that

$$[a_1] = -\left[\Delta a_0\right] = -Y \tag{13.79}$$

(since $D^2 = \Delta - Y$). When $p = 1$ in (13.69)

$$-\, 2a_2 = \Delta a_1 + \sigma.D a_2 \tag{13.80}$$

so that

$$[a_2] = -\frac{1}{2}\left[\Delta a_1\right] \tag{13.81}$$

Let us derive again the $p = 0$ expression before the coincidence limit:

$$-\, D_\mu a_1 = D_\mu D^2 a_0 + D_\mu\left(\sigma.D a_1\right) = D_\mu D^2 a_0 + D_\mu a_1 + \sigma^\lambda D_\mu D_\lambda a_1 \tag{13.82}$$

Then

$$-\, 2D_\mu a_1 = D_\mu \Delta a_0 + \sigma^\lambda D_\mu D_\lambda a_1 \tag{13.83}$$

which implies at the coincidence limit

$$-\, 2\left[D^2 a_1\right] = \left[D^2 \Delta a_0\right] + \left[D^2 a_1\right] \tag{13.84}$$

that is

$$[\Delta a_1] \equiv \left[D^2 a_1\right] + [Y a_1] = -\frac{1}{3}\left[D^2 D^2 a_0\right] - Y^2 - \frac{1}{3} D^2 Y \tag{13.85}$$

Now deriving three times the equation (13.68)

$$(D_\delta D_\sigma D_\rho D_\mu + D_\delta D_\sigma D_\mu D_\rho + D_\delta D_\rho D_\mu D_\sigma + D_\sigma D_\rho D_\mu D_\delta + \sigma^\lambda D_\delta D_\sigma D_\rho D_\mu D_\lambda) a_0 = 0 \tag{13.86}$$

Contracting with $\eta^{\delta\sigma}\eta^{\rho\mu}$

$$\left[(D^2 D^2 + D^\mu D^2 D_\mu) a_0\right] = 0 \tag{13.87}$$

and contracting instead with $\eta^{\delta\rho}\eta^{\sigma\mu}$

$$[(D^\mu D^\nu D_\mu D_\nu) a_0] = 0 \tag{13.88}$$

Now

$$[(D^\sigma D^\mu D_\mu D_\sigma) a_0] = [(D^\mu D^\sigma D_\mu D_\sigma a_0 + W^{\sigma\mu} D_\mu D_\sigma a_0] \tag{13.89}$$

It follows that

$$\left[D^\alpha D^2 D_\alpha a_0\right] = 0 + W^{\sigma\mu}\left[D_\mu D_\sigma a_0\right] = -\frac{1}{2} W^2 \tag{13.90}$$

so that

$$\left[D^2 D^2 a_0\right] = \frac{1}{2} W^2 \tag{13.91}$$

and finally

$$[a_2] = -\frac{1}{2}\left[\Delta a_1\right] = \frac{1}{6}\left[D^2 D^2 a_0\right] + \frac{1}{2} Y^2 + \frac{1}{6} D^2 Y = \frac{1}{12} W^2 + \frac{1}{2} Y^2 + \frac{1}{6} D^2 Y \tag{13.92}$$

The final expression for the divergent piece of the determinant of the flat space gauge operator reads

$$\log \ \det \Delta = -\frac{2}{(4-n)} \frac{i}{(4\pi)^2} \int d^n x \ \mathrm{tr} \ \left(\frac{1}{12} W^{\mu\nu} W_{\mu\nu} + \frac{1}{2} Y^2\right) \tag{13.93}$$

(the term in $D^2 Y$ vanishes as a surface term).

13.7 The deWitt computation of gravitational determinants

When the gravitational interaction is physically relevant, things are much more complicated. First of all, the space-time manifold is not flat, so that the flat space free solution has got to be generalized. All computations should be covariant. It is precisely at this point that all computations already done with the world fiunction will become handy. On the other hand it is when dealing with this sort of problems that the real power of the heat kernel technique is visible.

The relevant expansion has been worked out by Bryce DeWitt. Let us proceed in a pedestrian way, by writing

$$K\left(\tau;x,x'\right)=\frac{1}{(4\pi\tau)^{\frac{n}{2}}}N(x,x')\ e^{-\frac{\sigma(x,x')}{2\tau}}\ \sum_{p=0}^{\infty}a_p(x,x')\tau^p \tag{13.94}$$

where we have left an arbitrary global coefficient, to be determined later, $N(x,x')$ in front of the Taylor expansion. The purpose here is to show that it should be equal to the van Vleck determinant. In order to do that, let us now substitute the short time expansion into the heat equation

$$\frac{\partial}{\partial\tau}\ K\left(\tau;x,y\right)=\frac{1}{(4\pi)^{\frac{n}{2}}}\ N\ e^{-\frac{\sigma}{2\tau}}\ \sum_{p=0}\left(a_p\frac{\sigma}{2}+\left(p-1-\frac{n}{2}\right)a_{p-1}\right)\tau^{p-2-\frac{n}{2}}$$

Let us do the computation for (minus) the ordinary laplacian

$$\nabla_\mu K\left(\tau;x,y\right)=\frac{1}{(4\pi\tau)^{\frac{n}{2}}}\ e^{-\frac{\sigma}{2\tau}}\ \sum_p\left(\nabla_\mu\ N\ a_p+N\ \left(\nabla_\mu a_p-\frac{\sigma_\mu}{2\tau}a_p\right)\right)\tau^p$$

$$\begin{aligned}\nabla^2K\left(\tau;x,y\right)=&\ \frac{1}{(4\pi\tau)^{\frac{n}{2}}}\ e^{-\frac{\sigma}{2\tau}}\ \sum_p\tau^p\Big\{\nabla^2N\ a_p+2N^\mu\nabla_\mu a_p-\frac{1}{2\tau}\ \sigma^\mu N_\mu\ a_p-\\
&-\frac{1}{2\tau}\ N\ \sigma^\mu\nabla_\mu a_p-\frac{1}{2\tau}N\nabla^2\sigma a_p+N\nabla^2 a_p-\frac{1}{2\tau}\sigma^\mu\left(N_\mu a_p-\frac{1}{2\tau}N\sigma_\mu a_p+N\nabla_\mu a_p\right)\Big\}=\\
&=\ \frac{1}{(4\pi\tau)^{\frac{n}{2}}}\ e^{-\frac{\sigma}{2\tau}}\ \sum_{p=0}\tau^{p-2}\ \Big\{\nabla^2Na_{p-2}+2N^\mu\nabla_\mu a_{p-2}-\sigma^\mu N_\mu a_{p-1}+\\
&+N\left(\frac{1}{4}\sigma_\mu\sigma^\mu a_p-\sigma^\mu\nabla_\mu a_{p-1}-\frac{1}{2}\nabla^2\sigma a_{p-1}+\nabla^2 a_{p-2}\right)\Big\}\end{aligned} \tag{13.95}$$

We have defined

$$a_p=0 \tag{13.96}$$

for negative values of the index p.

The generic recursion relation is

$$\nabla^2 N a_{p-2} + 2N^\mu \nabla_\mu a_{p-2} - \sigma^\mu N_\mu a_{p-1} + N\left(\frac{1}{4}\sigma_\mu\sigma^\mu a_p - \sigma^\mu\nabla_\mu a_{p-1} - \frac{1}{2}\nabla^2\sigma a_{p-1} + \nabla^2 a_{p-2} - a_p\frac{\sigma}{2} - \left(p - 1 - \frac{n}{2}\right)a_{p-1}\right) = 0 \tag{13.97}$$

Let us work it out for low values of the index p.

- $p = 0$. This yields the equation

$$\frac{1}{4}\left(\nabla_\mu\sigma\nabla^\mu\sigma - 2\sigma\right)\; N\; a_0\left(x, x'\right) = 0 \tag{13.98}$$

The prefactor here is an identity for the world function. Let us prove it.

Start for the action for a free particle

$$S \equiv \int_{x,\tau}^{x',\tau'} d\tau\; \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu \equiv \frac{\sigma(x, x']}{\tau' - \tau} \tag{13.99}$$

where the integral is taken over the geodesic $x^\mu = x^\mu(\tau)$ that goes from the *base point* x' at value τ' of the parameter to the *field point* x' at value τ' of the same parameter. This defines the square of the geodesic distance between the points x' and x. It is a scalar for independent Einstein transformations of the base and field points. This was called by Synge *world function*. He used the notation Ω for it, but nowadays the notation σ is much more common. When the geodesic is timelike and parametrizing with the proper time

$$\sigma(x, x') = \frac{(\tau - \tau')^2}{2} \tag{13.100}$$

The canonical momentum is given by

$$p_\mu \equiv \partial_\mu S = \frac{\nabla_\mu\sigma}{\tau' - \tau} \tag{13.101}$$

The Hamilton-Jacobi equation for the free particle reads

$$0 = \frac{\partial S}{\partial\tau} + H = -\frac{\sigma}{(\tau' - \tau)^2} + \frac{1}{2}\frac{\sigma_\mu\sigma^\mu}{(\tau' - \tau)^2} \tag{13.102}$$

and it leads to the basic equation obeyed by the world function

$$\nabla_\mu\sigma\nabla^\mu\sigma = 2\sigma \tag{13.103}$$

It is instructive to study a more pedestrian derivation. Consider a variation of the world function

$$\delta\sigma \equiv \sigma(x + \delta x, x') - \sigma(x, x') \tag{13.104}$$

where we rescale the parameters in such a way that (λ_0, λ_1) label the ends of the new geodesic.

The variation can be computed in an standard way

$$\delta\sigma = (\lambda_1 - \lambda_0) \int_{\lambda_0}^{\lambda_1} d\lambda \left(g_{\mu\nu} \dot{z}^\mu \delta \dot{z}^\nu + \frac{1}{2} \partial_\lambda g_{\mu\nu} \dot{z}^\mu \dot{z}^\nu \delta z^\lambda \right) = (\lambda_1 - \lambda_0)\; g_{\alpha\beta} \dot{z}^\alpha \delta z^\beta \Big|_{\lambda_0}^{\lambda_1} -$$
$$-(\lambda_1 - \lambda_0) \int_{\lambda_0}^{\lambda_1} \left(g_{\alpha\beta} \ddot{z}^\beta + \Gamma_{\alpha\beta\gamma} \dot{z}^\beta \dot{z}^\gamma \right) \delta z^\alpha d\lambda \tag{13.105}$$

Inserting the information that ine path integral is taken over a geodesic, we learn that

$$\delta\sigma = (\lambda_1 - \lambda_0) g_{\alpha\beta} u^\alpha \delta x^\beta \tag{13.106}$$

This means that the derivative of the world function is proportional to the tangent vector

$$\sigma_\alpha = (\lambda_1 - \lambda_0) u_\alpha = +\sqrt{2\sigma}\; u_\alpha \tag{13.107}$$

Also

$$\sigma_{\alpha'} = -(\lambda_1 - \lambda_0) u_{\alpha'} \tag{13.108}$$

and it is now obvious that

$$\sigma_\mu \sigma^\mu = \sigma_{\mu'} \sigma^{\mu'} = 2\sigma \tag{13.109}$$

This implies that the equation of parallel transport of any quantity T can be written as

$$u^\mu \nabla_\mu T = \sigma^\mu \nabla_\mu T = 0 \tag{13.110}$$

also, taking derivatives, the d'Alembertian of the world function is related to the expansion of the geodesic congruence

$$\Box\sigma = 1 + \sqrt{2\sigma}\; \theta \tag{13.111}$$

- Let us examine the $p = 1$ equation.

$$-\sigma^\mu N_\mu a_0 + \frac{1}{4} N \sigma_\mu \sigma^\mu a_1 - \sigma^\mu \nabla_\mu a_0 - \frac{1}{2} \Box\sigma a_0 - N \frac{\sigma}{2}\; a_1 + \frac{n}{2} N a_0 = 0 \tag{13.112}$$

It is plain that the coefficient of a_1 is

$$N \left(\frac{1}{4} \sigma_\mu \sigma^\mu - \frac{1}{2} \sigma \right) \tag{13.113}$$

which vanishes identically. The coefficient a_0 obeys the equation for the parallel propagator

$$\sigma^\mu \nabla_\mu a_0(x, y) = 0 \tag{13.114}$$

The vanishing of the term that algebraically multiplies a_0 imposes the following equation on the prefactor N

$$-\sigma^\mu \nabla_\mu N - \frac{1}{2}\Box\sigma N + \frac{n}{2}N = 0 \tag{13.115}$$

This can also be written in terms of $\Delta \equiv N^2$ as

$$\nabla_\mu \left(\Delta\sigma^\mu\right) = n\Delta \tag{13.116}$$

and this equation in turn, identifies Δ as the van Vleck determinant. Let us explain this.

The van Vleck determinant is defined by

$$\Delta(x,x') \equiv \det\, \Delta^{\alpha'}{}_{\beta'}(x,x') \equiv \det\, \left(-g^{\alpha'}_{\alpha}(x',x)\sigma^{\alpha}_{\beta'}(x,x')\right) \tag{13.117}$$

The parallel propagator is defined as

$$g^{\alpha'}{}_{\alpha}(x,x') \equiv e^{\alpha'}_{a}(x')e^{a}_{\alpha}(x) \tag{13.118}$$

so that

$$\det\, \left(g^{\alpha'}{}_{\alpha}(x,x')\right) = \frac{e(x)}{e'(x')} \tag{13.119}$$

Taking determinants in the definition of the van Vleck determinant yields

$$\Delta(x,x') = -\frac{\det\, \left(-\sigma_{\alpha\beta'}(x,x')\right)}{ee'} \equiv -\frac{\mathcal{D}(x,x')}{ee'} \tag{13.120}$$

It is plain that

$$\left[\Delta^{\alpha'}_{\beta'}\right] = \delta^{\alpha'}_{\beta'} \tag{13.121}$$

$$[\Delta] = 1 \tag{13.122}$$

It is a fact [23] that it obeys the fundamental equation

$$\nabla_\alpha \left(\Delta\ \sigma^\alpha\right) = n\Delta = \sigma^\alpha\nabla_\alpha\Delta + \Delta\Box\sigma = n\Delta = \sigma^\alpha\nabla_\alpha\Delta + \left(1+\sqrt{2\sigma}\ \theta\right)\Delta \tag{13.123}$$

The failure of the van Vleck determinant to be parallel propageted is measured by the expansion of the geodesic congruence

$$\sigma^\alpha\nabla_\alpha \left(\log\,\Delta\right) = (n-1) - \sqrt{2\sigma}\ \theta \tag{13.124}$$

Indeed, starting from

$$\Delta^{\alpha'}{}_{\beta'} = -g^{\alpha'\alpha}\left(\sigma^{\gamma}{}_{\alpha}\sigma_{\gamma\beta'} + \sigma^{\gamma}\sigma_{\alpha\beta'\gamma}\right) = g^{\alpha'}{}_{\alpha}g^{\gamma}_{\gamma'}\sigma^{\alpha}{}_{\gamma}\Delta^{\gamma'}{}_{\beta'} + \nabla_\gamma\Delta^{\alpha'}_{\beta'}\sigma^{\gamma} \tag{13.125}$$

multiplying by the inverse matrix Δ^{-1} and taking the trace

$$n = \Box\sigma + (\Delta^{-1})^{\beta'}{}_{\alpha'}\sigma^{\gamma}\nabla_{\gamma}\Delta^{\alpha'}_{\beta'} \tag{13.126}$$

which implies the desired identity.

The coindidence limit of the fundamental equation holds trivially. We shall indeed denote the *coincidence limit* of any bi-scalar function by

$$[W] \equiv \lim_{x\to x'} W(x, x') \tag{13.127}$$

There is a general rule, called *Synge's rule* for computing such limits. The rule as applied to the world function states that

$$[\nabla_{\alpha'}\sigma_{\ldots}] = \nabla_{\alpha'}[\sigma_{\ldots}] - [\nabla_{\alpha}\sigma_{\ldots}] \tag{13.128}$$

where the dots indicate further derivations. Let us prove it. Given any bi-scalar

$$\Omega_{AB'}(x, x') \tag{13.129}$$

where $A, B, \ldots$ are multi-indexes. Further consider a physical quantity $P^A(x)$ with the same multi-index structure as A; and another one $Q^{B'}(x')$ with the same multi-index structure as B'. Both objects are parallel propagated

$$u^{\alpha}\nabla_{\alpha}P^A(x) = u^{\alpha'}\nabla_{\alpha'}Q^{B'}(x') = 0 \tag{13.130}$$

The bi-scalar

$$H(x, x') \equiv \Omega_{AB'}(x, x')P^A(x)Q^{B'}(x') \tag{13.131}$$

can be Taylor expanded in two different ways

$$\begin{aligned} H(\lambda_1, \lambda_0) &= H(\lambda_0, \lambda_0) + (\lambda_1 - \lambda_0)\left.\frac{\partial H}{\partial \lambda_1}\right|_{\lambda_1=\lambda_0} + \ldots = \\ &= H(\lambda_1, \lambda_1) - (\lambda_1 - \lambda_0)\left.\frac{\partial H}{\partial \lambda_0}\right|_{\lambda_0=\lambda_1} + \ldots \end{aligned} \tag{13.132}$$

Then

$$H(\lambda_0, \lambda_0) \equiv [\Omega_{AB'}]\, P^A Q^{B'} \tag{13.133}$$

obeys

$$\begin{aligned} &\frac{d}{d\lambda_0}H(\lambda_0, \lambda_0) \equiv \lim_{\lambda_1\to\lambda_0}\frac{H(\lambda_1, \lambda_1) - H(\lambda_0, \lambda_0)}{\lambda_1 - \lambda_0} = \left.\frac{\partial H}{\partial \lambda_0}\right|_{\lambda_0=\lambda_1} + \left.\frac{\partial H}{\partial \lambda_1}\right|_{\lambda_1=\lambda_0} = \\ &= u^{\alpha'}[\nabla_{\alpha'}\Omega_{AB'}]\, P^A Q^{B'} + u^{\mu}[\nabla_{\mu}\Omega_{AB'}]\, P^A Q^{B'} \end{aligned} \tag{13.134}$$

Finally

$$\nabla_{\alpha'}[\Omega_{AB'}] = [\nabla_{\alpha'}\Omega_{AB'}] + [\nabla_{\alpha}\Omega_{AB'}] \tag{13.135}$$

Let us now rewrite the recursion relation taking into account all information gathered until now.

$$\Delta^{-\frac{1}{2}}\Box\left(\Delta^{\frac{1}{2}}a_p\right)=\sigma^{\mu}\nabla_{\mu}a_{p+1}+(p+1)\,a_{p+1}=-\frac{1}{4}\Delta^{-2}\Delta_{\mu}^{2}a_p+$$
$$+\frac{1}{2}\Delta^{-1}\Box\Delta a_p+\Delta^{-1}\Delta^{\mu}\nabla_{\mu}a_p+\Box a_p \qquad (13.136)$$

- $p=2$. Let us finally write

$$\sigma^{\mu}\nabla_{\mu}a_2+2a_2=-\frac{1}{4}\Delta^{-2}\Delta_{\mu}^{2}a_1+\frac{1}{2}\Delta^{-1}\Box\Delta a_1+\Delta^{-1}\Delta^{\mu}\nabla_{\mu}a_1+\Box a_1 \qquad (13.137)$$

Our aim is to find the coincidence limit

$$[a_2]=-\frac{1}{4}\Delta^{-2}\Delta_{\mu}^{2}a_1+\frac{1}{2}\Delta^{-1}\Box\Delta a_1+\Delta^{-1}\Delta^{\mu}\nabla_{\mu}a_1+\Box a_1 \qquad (13.138)$$

We need $[\Box a_1]$. For that we can start with the first equation (we shall need the off diagonal one)

$$\sigma^{\mu}\nabla_{\mu}a_1+a_1=-\frac{1}{4}\Delta^{-2}\Delta_{\mu}^{2}a_0+\frac{1}{2}\Delta^{-1}\Box\Delta a_0+\Delta^{-1}\Delta^{\mu}\nabla_{\mu}a_0+\Box a_0$$

At coindicence, this determines $[a_1]$

$$[a_1]=\frac{1}{6}\;R+[\Box a_0] \qquad (13.139)$$

We derive once

$$\sigma^{\mu}{}_{\nu}a^{1}_{\mu}+\sigma^{\mu}a^{1}_{\mu\nu}+a^{1}_{\nu}=\frac{1}{2}\Delta^{-3}\Delta_{\nu}\Delta_{\mu}^{2}a_0-\frac{1}{2}\Delta^{-2}\Delta_{\mu}\Delta^{\mu\nu}a_0-$$
$$-\frac{1}{4}\Delta^{-2}\Delta_{\mu}^{2}a^{0}_{\nu}-\frac{1}{2}\Delta^{-2}\Delta_{\nu}\Box\Delta a_0-\frac{1}{2}\Delta^{-1}\nabla_{\nu}\Box\Delta a_0-\frac{1}{2}\Delta^{-1}\Box\Delta a^{0}_{\mu}$$
$$-\Delta^{-2}\Delta_{\nu}\Delta^{\mu}a^{0}_{\mu}+\Delta^{-1}\Delta^{\mu\nu}a^{0}_{\mu}+\Delta^{-1}\Delta_{\mu}a^{0}_{\mu\nu}+\nabla_{\nu}\Box a_0 \qquad (13.140)$$

Take a deep breath and derive again

$$\sigma^{\mu\nu\lambda}a^1_\mu + \sigma^{\mu\nu}a^1_{\mu\lambda} + \sigma^{\mu\lambda}a^1_{\mu\nu} + \sigma^{\mu}a^1_{\mu\nu\lambda} + a^1_{\nu\lambda} =$$
$$= -\frac{3}{2}\Delta^{-4}\Delta_\lambda\Delta_\nu\Delta_\nu\Delta^2_\mu a_0 + \frac{1}{2}\Delta^{-3}\Delta_{\nu\lambda}\Delta^2_\mu a_0 + \Delta^{-3}\Delta_\nu\Delta_\mu\Delta_{\mu\lambda}a_0 +$$
$$+\frac{1}{2}\Delta^{-3}\Delta_\nu\Delta^2_\mu a^0_\lambda + \Delta^{-3}\Delta_\lambda\Delta_\mu\Delta^{\mu\nu}a_0 - \frac{1}{2}\Delta^{-2}\Delta_{\mu\lambda}\Delta^{\mu\nu}a_0 -$$
$$-\frac{1}{2}\Delta^{-2}\Delta_\mu\Delta^{\mu\nu\lambda}a_0 - \frac{1}{2}\Delta^{-2}\Delta_\mu\Delta^{\mu\nu}a^\lambda_0 + \frac{1}{2}\Delta^{-3}\Delta_\lambda\Delta^2_\mu a^\nu_0 -$$
$$-\frac{1}{2}\Delta^{-2}\Delta_\mu\Delta^{\mu\lambda}a^\nu_0 - \frac{1}{4}\Delta^{-2}\Delta^2_\mu a^{\nu\lambda}_0 + \Delta^{-3}\Delta_\lambda\Delta_\nu\Box\Delta a_0 -$$
$$-\frac{1}{2}\Delta^{-2}\Delta_{\nu\lambda}\Box\Delta a_0 - \frac{1}{2}\Delta^{-2}\Delta_\nu\nabla_\lambda\Box\Delta a_0 - \frac{1}{2}\Delta^{-2}\Delta_\nu\Box\Delta a^\lambda_0 +$$
$$+\frac{1}{2}\Delta^{-2}\Delta_\lambda\nabla_\nu\Box\Delta a_0 - \frac{1}{2}\Delta^{-1}\nabla_\lambda\nabla_\nu\Box\Delta a_0 - \frac{1}{2}\Delta^{-1}\nabla_\nu\Box\Delta a^0_\lambda + \frac{1}{2}\Delta^{-2}\Delta_\lambda\Box\Delta a^0_\mu -$$
$$-\frac{1}{2}\Delta^{-1}\nabla_\lambda\Box\Delta a^0_\mu - \frac{1}{2}\Delta^{-1}\Box\Delta a^0_{\mu\lambda} + 2\Delta^{-3}\Delta_\lambda\Delta_\nu\Delta^\mu a^\mu_0 - \Delta^{-2}\Delta_{\nu\lambda}\Delta^\mu a^\mu_0 -$$
$$-\Delta^{-2}\Delta_\nu\Delta^{\mu\lambda}a^\mu_0 - \Delta^{-2}\Delta_\nu\Delta^\mu a^{\mu\lambda}_0 - \Delta^{-2}\Delta_\lambda\Delta^{\mu\nu}a^\mu_0 + \Delta^{-1}\Delta^{\mu\nu\lambda}a^\mu_0 +$$
$$+\Delta^{-1}\Delta^{\mu\nu}a^{\mu\lambda}_0 - \Delta^{-2}\Delta_\lambda\Delta_\mu a^{\mu\nu}_0 + \Delta^{-1}\Delta_{\mu\lambda}a^{\mu\nu}_0 + \Delta^{-1}\Delta_\mu a^{\mu\nu\lambda}_0 + \nabla_\lambda\nabla_\nu\Box a_0$$

This means that everything starts with coincidence limits of covariant derivatives of a_0. Let us proceed carefully to work out coincidence limits of covariant derivatives of the world function.

- It is plain that

$$[\sigma] = [\sigma_\mu] = 0 \tag{13.141}$$

(The second equation is true because there is no prefered vector in the manifold.) Deriving several times the equation (13.103)

$$\sigma_\mu\sigma^{\mu\alpha} = \sigma^\alpha \tag{13.142}$$

Once again

$$\sigma_{\mu\beta}\sigma^{\mu\alpha} + \sigma_\mu\sigma^{\mu\alpha\beta} = \sigma^{\alpha\beta} \tag{13.143}$$

$$\therefore\ \Box\sigma = \sigma_{\mu\nu}\sigma^{\mu\nu} + \sigma_\mu\Box\sigma^{\mu\nu} \tag{13.144}$$

It follows that

$$[\sigma_{\mu\nu}] = g_{\mu\nu} \tag{13.145}$$

as well as its trace

$$[\Box\sigma] = n \tag{13.146}$$

Another derivative

$$\sigma_{\mu\beta\gamma}\sigma^{\mu\alpha} + \sigma_{\mu\beta}\sigma^{\mu\alpha\gamma} + \sigma_{\mu\gamma}\sigma^{\mu\alpha\beta} + \sigma_\mu\sigma^{\mu\alpha\beta\gamma} = \sigma^{\alpha\beta\gamma} \tag{13.147}$$

In an obvious notation

$$[\sigma_{123} + \sigma_{213} + \sigma_{312}] = [\sigma_{123}] \tag{13.148}$$

$$\therefore [\sigma_{213}] = -[\sigma_{312}] \tag{13.149}$$

But even outside coincidence

$$\sigma_{123} = \sigma_{213} \tag{13.150}$$

so that

$$[\sigma_{123}] = -[\sigma_{132}] \tag{13.151}$$

The Ricci identity implies that

$$\sigma_{\alpha\beta\gamma} = \sigma_{\alpha\gamma\beta} - \sigma_\mu R^\mu_{\alpha\gamma\beta} \tag{13.152}$$

(our conventions are different than [23]). Then the coincidence limit of three derivatives vanishes.

$$[\sigma_{\mu\nu\lambda}] = 0 \tag{13.153}$$

The fourth derivative reads

$$\begin{aligned}&\sigma_{\mu\beta\gamma\delta}\sigma^{\mu\alpha} + \sigma_{\mu\beta\gamma}\sigma^{\mu\alpha\delta} + \sigma_{\mu\beta\delta}\sigma^{\mu\alpha\gamma} + \sigma_{\mu\beta}\sigma^{\mu\alpha\gamma\delta} + \sigma_{\mu\beta}\sigma^{\mu\alpha\gamma\delta} + \\ &+\sigma_{\mu\gamma\delta}\sigma^{\mu\alpha\beta} + \sigma_{\mu\gamma}\sigma^{\mu\alpha\beta\delta} + \sigma_{\mu\delta}\sigma^{\mu\alpha\beta\gamma} + \sigma_\mu\sigma^{\mu\alpha\beta\gamma\delta} = \sigma^{\alpha\beta\gamma\delta}\end{aligned} \tag{13.154}$$

On the other hand, from the last equation follows

$$[\sigma_{\alpha_3\alpha_1\alpha_2\alpha_4}] + [\sigma_{\alpha_4\alpha_1\alpha_2\alpha_3}] + [\sigma_{\alpha_4\alpha_1\alpha_2\alpha_3}] = 0 \tag{13.155}$$

First of all,

$$\sigma_{\alpha_2\alpha_1\alpha_3\alpha_4} = \sigma_{\alpha_1\alpha_2\alpha_3\alpha_4} \tag{13.156}$$

Deriving it once we learn that

$$\sigma_{\alpha_1\alpha_2\alpha_3\alpha_4} = \sigma_{\alpha_1\alpha_3\alpha_2\alpha_4} - \sigma_{\mu\alpha_4} R^\mu_{\alpha_1\alpha_3\alpha 2} - \sigma_\mu \nabla_{\alpha_4} R^\mu_{\alpha_1\alpha_3\alpha_2} \tag{13.157}$$

This means that

$$2\,[\sigma_{\alpha_3\alpha_1\alpha_2\alpha_4}] + R_{\alpha_4\alpha_1\alpha_3\alpha_2} + [\sigma_{\alpha_1\alpha_4\alpha_2\alpha_3}] = 0 \tag{13.158}$$

But again the Ricci identity implies that

$$[\sigma_{\alpha_1\alpha_2\alpha_3\alpha_4}] - [\sigma_{\alpha_1\alpha_2\alpha_4\alpha_3}] = -R_{\alpha_1\alpha_2\alpha_3\alpha_4} - R_{\alpha_2\alpha_1\alpha_3\alpha_4} = 0 \tag{13.159}$$

This implies in turn that the coincidence limit

$$[\sigma_{\mu\nu\rho\sigma}] = \frac{1}{3}\left(R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu}\right) \tag{13.160}$$

In particular

$$[\nabla_\alpha \nabla_\beta \Box\sigma] = -\frac{1}{3}\, R_{\alpha\beta} \tag{13.161}$$

The expression with five dervatives is also needed.

$$\begin{aligned}
&\sigma_{\mu\beta\gamma\delta\epsilon}\sigma^{\mu\alpha} + \sigma_{\mu\beta\gamma\delta}\sigma^{\mu\alpha\epsilon} + \sigma_{\mu\beta\gamma\epsilon}\sigma^{\mu\alpha\delta} + \sigma_{\mu\beta\gamma}\sigma^{\mu\alpha\delta\epsilon} + \\
&+\sigma_{\mu\beta\delta\epsilon}\sigma^{\mu\alpha\gamma} + \sigma_{\mu\beta\delta}\sigma^{\mu\alpha\gamma\epsilon} + \sigma_{\mu\beta\epsilon}\sigma^{\mu\alpha\gamma\delta} + \sigma_{\mu\beta}\sigma^{\mu\alpha\gamma\delta\epsilon} + \\
&+\sigma_{\mu\beta\epsilon}\sigma^{\mu\alpha\gamma\delta} + \sigma_{\mu\beta}\sigma^{\mu\alpha\gamma\delta\epsilon} + \sigma_{\mu\gamma\delta\epsilon}\sigma^{\mu\alpha\beta} + \sigma_{\mu\gamma\delta}\sigma^{\mu\alpha\beta\epsilon} + \\
&+\sigma_{\mu\gamma\epsilon}\sigma^{\mu\alpha\beta\delta} + \sigma_{\mu\gamma}\sigma^{\mu\alpha\beta\delta\epsilon} + \sigma_{\mu\delta\epsilon}\sigma^{\mu\alpha\beta\gamma} + \sigma_{\mu\delta}\sigma^{\mu\alpha\beta\gamma\epsilon} + \\
&+\sigma_{\mu\epsilon}\sigma^{\mu\alpha\beta\gamma\delta} + \sigma_{\mu}\sigma^{\mu\alpha\beta\gamma\delta\epsilon} = \sigma^{\alpha\beta\gamma\delta\epsilon} \qquad (13.162)
\end{aligned}$$

At coincidence

$$[\sigma_{\alpha\beta\gamma\delta\epsilon}] + [\sigma_{\beta\alpha\gamma\delta\epsilon}] + [\sigma_{\beta\alpha\gamma\delta\epsilon}] + [\sigma_{\gamma\alpha\beta\delta\epsilon}] + [\sigma_{\delta\alpha\beta\gamma\epsilon}] + [\sigma_{\epsilon\alpha\beta\gamma\delta}] = [\sigma_{\alpha\beta\gamma\delta\epsilon}] \qquad (13.163)$$

We need Ricci's help

$$\sigma_{\gamma\alpha\beta\delta\epsilon} = \sigma_{\alpha\gamma\beta\delta\epsilon} = \nabla_\epsilon\nabla_\sigma\left(\nabla_\beta\nabla_\gamma\sigma^\alpha\right) = \sigma_{\alpha\beta\gamma\delta\epsilon} + \nabla_\epsilon\nabla_\delta\left(R_{\beta\gamma\alpha\lambda}\sigma^\lambda\right) \qquad (13.164)$$

$$\sigma_{\alpha\delta\beta\gamma\epsilon} \equiv \nabla_\epsilon\nabla_\gamma\left(\nabla_\beta\nabla_\delta\sigma^\alpha\right) = \sigma_{\alpha\beta\delta\gamma\epsilon} + \nabla_\epsilon\nabla_\gamma\left(R_{\beta\delta\alpha\lambda}\sigma^\lambda\right) \qquad (13.165)$$

$$\sigma_{\alpha\beta\delta\gamma\epsilon} \equiv \nabla_\epsilon\nabla_\gamma\nabla_\delta\sigma^{\alpha\beta} = \sigma_{\alpha\beta\gamma\delta\epsilon} + \nabla_\epsilon\left(R_{\gamma\delta\beta\sigma}\sigma^{\sigma\alpha} + R_{\gamma\delta\alpha\sigma}\sigma^{\sigma\beta}\right) \qquad (13.166)$$

$$\sigma_{\alpha\epsilon\beta\gamma\delta} \equiv \nabla_\delta\nabla_\gamma\nabla_\beta\nabla_\epsilon\sigma^\alpha = \sigma_{\alpha\beta\epsilon\gamma\delta} + \nabla_\delta\nabla_\gamma\left(R_{\beta\epsilon\alpha\sigma}\sigma^\sigma\right) \qquad (13.167)$$

$$\sigma_{\alpha\beta\epsilon\gamma\delta} \equiv \nabla_\delta\nabla_\gamma\nabla_\epsilon\sigma^{\alpha\beta} = \sigma_{\alpha\beta\gamma\epsilon\delta} + \nabla_\delta\left(R_{\gamma\epsilon\alpha\sigma}\sigma^{\sigma\beta} + R_{\gamma\epsilon\beta\sigma}\sigma^{\alpha\sigma}\right) \qquad (13.168)$$

$$\sigma_{\alpha\beta\gamma\epsilon\delta} \equiv \nabla_\delta\nabla_\epsilon\sigma^{\alpha\beta\gamma} = \sigma_{\alpha\beta\gamma\delta\epsilon} + R_{\delta\epsilon\alpha\lambda}\sigma_{\lambda\beta\gamma} + R_{\delta\epsilon\beta\lambda}\sigma^{\alpha\lambda\gamma} + R_{\delta\epsilon\gamma\lambda}\sigma^{\alpha\beta\lambda} \qquad (13.169)$$

Putting all together,

$$\begin{aligned}
&3\sigma_{\alpha\beta\gamma\delta\epsilon} + \nabla_\epsilon\nabla_\delta\left(R_{\beta\gamma\alpha\lambda}\sigma^\lambda\right) + \nabla_\epsilon\nabla_\gamma\left(R_{\beta\delta\alpha\lambda}\sigma^\lambda\right) + \\
&+\nabla_\epsilon\left(R_{\gamma\delta\beta\sigma}\sigma^{\sigma\alpha} + R_{\gamma\delta\alpha\sigma}\sigma^{\sigma\beta}\right) + \nabla_\delta\nabla_\gamma\left(R_{\beta\epsilon\alpha\sigma}\sigma^\sigma\right) + \nabla_\delta\left(R_{\gamma\epsilon\alpha\sigma}\sigma^{\sigma\beta} + R_{\gamma\epsilon\beta\sigma}\sigma^{\alpha\sigma}\right) + \\
&+R_{\delta\epsilon\alpha\lambda}\sigma^{\alpha\beta\gamma} + R_{\delta\epsilon\beta\lambda}\sigma^{\alpha\lambda\gamma} + R_{\delta\epsilon\gamma\lambda}\sigma^{\alpha\beta\lambda} = 0 \qquad (13.170)
\end{aligned}$$

At coincidence

$$\begin{aligned}
&-3\left[\sigma_{\alpha\beta\gamma\delta\epsilon}\right] = \nabla_\epsilon R_{\beta\gamma\alpha\delta} + \nabla_\delta R_{\beta\gamma\alpha\epsilon} + \nabla_\epsilon R_{\beta\delta\alpha\gamma} + \nabla_\gamma R_{\beta\delta\alpha\epsilon} + \nabla_\delta R_{\beta\epsilon\alpha\gamma} + \nabla_\gamma R_{\beta\epsilon\alpha\delta} = \\
&\nabla_\epsilon\left(R_{\beta\gamma\alpha\delta} + R_{\beta\delta\alpha\gamma}\right) + \nabla_\delta\left(R_{\beta\gamma\alpha\epsilon} + R_{\beta\epsilon\alpha\gamma}\right) + \nabla_\gamma\left(R_{\beta\delta\alpha\epsilon} + R_{\beta\epsilon\alpha\delta}\right) \qquad (13.171)
\end{aligned}$$

In particular

$$\left[\sigma^\alpha_{\alpha\gamma\delta\epsilon}\right] = -\frac{1}{6}\left(\nabla_\delta R_{\gamma\epsilon} + \nabla_\epsilon R_{\gamma\delta} + \nabla_\gamma R_{\epsilon\delta}\right) \qquad (13.172)$$

Six derivatives

$$
\begin{aligned}
&\sigma_{\mu\beta\gamma\delta\epsilon\sigma}\sigma^{\mu\alpha} + \sigma_{\mu\beta\gamma\delta\epsilon}\sigma^{\mu\alpha\sigma} + \sigma_{\mu\beta\gamma\delta\sigma}\sigma^{\mu\alpha\epsilon} + \sigma_{\mu\beta\gamma\delta}\sigma^{\mu\alpha\epsilon\sigma} + \\
&+\sigma_{\mu\beta\gamma\epsilon\sigma}\sigma^{\mu\alpha\delta} + \sigma_{\mu\beta\gamma\epsilon}\sigma^{\mu\alpha\delta\sigma} + \sigma_{\mu\beta\gamma\sigma}\sigma^{\mu\alpha\delta\epsilon} + \sigma_{\mu\beta\gamma}\sigma^{\mu\alpha\delta\epsilon\sigma} + \\
&+\sigma_{\mu\beta\delta\epsilon\sigma}\sigma^{\mu\alpha\gamma} + \sigma_{\mu\beta\delta\epsilon}\sigma^{\mu\alpha\gamma\sigma} + \sigma_{\mu\beta\delta\sigma}\sigma^{\mu\alpha\gamma\epsilon} + \sigma_{\mu\beta\delta}\sigma^{\mu\alpha\gamma\epsilon\sigma} + \\
&+\sigma_{\mu\beta\epsilon\sigma}\sigma^{\mu\alpha\gamma\delta} + \sigma_{\mu\beta\epsilon}\sigma^{\mu\alpha\gamma\delta\sigma} + \sigma_{\mu\beta\sigma}\sigma^{\mu\alpha\gamma\delta\epsilon} + \sigma_{\mu\beta}\sigma^{\mu\alpha\gamma\delta\epsilon\sigma} + \\
&+\sigma_{\mu\gamma\delta\epsilon\sigma}\sigma^{\mu\alpha\beta} + \sigma_{\mu\gamma\delta\epsilon}\sigma^{\mu\alpha\beta\sigma} + \sigma_{\mu\gamma\delta\sigma}\sigma^{\mu\alpha\beta\epsilon} + \sigma_{\mu\gamma\delta}\sigma^{\mu\alpha\beta\epsilon\sigma} + \\
&+\sigma_{\mu\gamma\epsilon\sigma}\sigma^{\mu\alpha\beta\delta} + \sigma_{\mu\gamma\epsilon}\sigma^{\mu\alpha\beta\delta\sigma} + s_{\mu\gamma\sigma}\sigma^{\mu\alpha\beta\delta\epsilon} + \sigma_{\mu\gamma}\sigma^{\mu\alpha\beta\delta\epsilon\sigma} + \\
&+\sigma_{\mu\delta\epsilon\sigma}\sigma^{\mu\alpha\beta\gamma} + \sigma_{\mu\delta\epsilon}\sigma^{\mu\alpha\beta\gamma\sigma} + \sigma_{\mu\delta\sigma}\sigma^{\mu\alpha\beta\gamma\epsilon} + \sigma_{\mu\delta}\sigma^{\mu\alpha\beta\gamma\epsilon\sigma} + \\
&+\sigma_{\mu\epsilon\sigma}\sigma^{\mu\alpha\beta\gamma\delta} + \sigma_{\mu\epsilon}\sigma^{\mu\alpha\beta\gamma\delta\sigma} + \sigma_{\mu\sigma}\sigma^{\mu\alpha\beta\gamma\delta\epsilon} + \sigma_{\mu}\sigma^{\mu\alpha\beta\gamma\delta\epsilon\sigma} = \sigma^{\alpha\beta\gamma\delta\epsilon\sigma}
\end{aligned}
$$

At coincidence

$$
\begin{aligned}
&\left[\sigma_{\mu\beta\gamma\delta}\sigma^{\mu\alpha\epsilon\sigma} + \sigma_{\mu\beta\gamma\epsilon}\sigma^{\mu\alpha\delta\sigma} + \sigma_{\mu\beta\gamma\sigma}\sigma^{\mu\alpha\delta\epsilon} + \sigma_{\mu\beta\delta\epsilon}\sigma^{\mu\alpha\gamma\sigma} + \sigma_{\mu\beta\delta\sigma}\sigma^{\mu\alpha\gamma\epsilon} + \right.\\
&+\sigma_{\mu\beta\epsilon\sigma}\sigma^{\mu\alpha\gamma\delta} + \sigma_{\mu\beta}\sigma^{\mu\alpha\gamma\delta\epsilon\sigma} + \sigma_{\mu\gamma\delta\epsilon}\sigma^{\mu\alpha\beta\sigma} + \\
&+\sigma_{\mu\gamma\delta\sigma}\sigma^{\mu\alpha\beta\epsilon} + \sigma_{\mu\gamma\delta}\sigma^{\mu\alpha\beta\epsilon\sigma} + \sigma_{\mu\gamma\epsilon\sigma}\sigma^{\mu\alpha\beta\delta} + \sigma_{\mu\gamma}\sigma^{\mu\alpha\beta\delta\epsilon\sigma} + \sigma_{\mu\delta\epsilon\sigma}\sigma^{\mu\alpha\beta\gamma} + \\
&\left.+\sigma_{\mu\delta}\sigma^{\mu\alpha\beta\gamma\epsilon\sigma} + \sigma_{\mu\epsilon}\sigma^{\mu\alpha\beta\gamma\delta\sigma} + \sigma_{\mu\sigma}\sigma^{\mu\alpha\beta\gamma\delta\epsilon}\right] = 0 \qquad (13.173)
\end{aligned}
$$

Ricci tells us that

$$
\begin{aligned}
&\sigma_{\alpha\sigma\beta\gamma\delta\epsilon} = \sigma_{\alpha\beta\gamma\delta\epsilon\sigma} + \nabla_{\epsilon\delta\gamma}\left(R_{\beta\sigma\alpha\lambda}\sigma^{\lambda}\right) + \nabla_{\epsilon\delta}\left(R_{\gamma\sigma\alpha\lambda}\sigma^{\lambda\beta} + R_{\gamma\sigma\beta\lambda}\sigma^{\alpha\lambda}\right) + \\
&+\nabla_{\epsilon}\left(R_{\delta\sigma\alpha\lambda}\sigma^{\alpha\beta\gamma} + R_{\delta\sigma\beta\lambda}\sigma^{\alpha\lambda\gamma} + R_{\delta\sigma\gamma\lambda}\sigma^{\alpha\beta\lambda}\right) + \\
&+R_{\sigma\epsilon\alpha\lambda}\sigma^{\alpha\beta\gamma\delta} + R_{\sigma\epsilon\beta\lambda}\sigma^{\alpha\lambda\gamma\delta} + R_{\sigma\epsilon\gamma\lambda}\sigma_{\alpha\beta\lambda\delta} + R_{\sigma\epsilon\delta\lambda}\sigma_{\alpha\beta\gamma\lambda} \qquad (13.174)
\end{aligned}
$$

$$
\begin{aligned}
&\sigma_{\epsilon\alpha\beta\gamma\delta\sigma} = \sigma_{\alpha\beta\gamma\delta\epsilon\sigma} + \nabla_{\sigma\delta\gamma}\left(R_{\beta\epsilon\alpha\lambda}\sigma^{\lambda}\right) + \nabla_{\sigma\delta}\left(R_{\gamma\epsilon\alpha\lambda}\sigma^{\lambda\beta} + R_{\gamma\epsilon\beta\lambda}\sigma^{\alpha\lambda}\right) + \\
&+\nabla_{\sigma}\left(R_{\delta\epsilon\alpha\lambda}\sigma^{\lambda\beta\gamma} + R_{\delta\epsilon\beta\lambda}\sigma^{\alpha\lambda\gamma} + R_{\delta\epsilon\gamma\lambda}\sigma^{\alpha\beta\lambda}\right) \qquad (13.175)
\end{aligned}
$$

$$
\sigma_{\alpha\delta\beta\gamma\epsilon\sigma} = \sigma_{\alpha\beta\gamma\delta\epsilon\sigma} + \nabla_{\epsilon\sigma\gamma}\left(R_{\beta\delta\alpha\lambda}\sigma^{\lambda}\right) + \nabla_{\sigma\epsilon}\left(R_{\gamma\delta\alpha\lambda}\sigma^{\lambda\beta} + R_{\gamma\delta\beta\lambda}\sigma^{\alpha\lambda}\right) \qquad (13.176)
$$

$$
\sigma_{\alpha\gamma\beta\delta\epsilon\sigma} = \sigma_{\alpha\beta\gamma\delta\epsilon\sigma} + \nabla_{\sigma\epsilon\delta}\left(R_{\beta\gamma\alpha\lambda}\sigma^{\lambda}\right) \qquad (13.177)
$$

Then the equation at coincidence

$$
\begin{aligned}
&6[\sigma_{\alpha\beta\gamma\delta\epsilon\sigma} + \nabla_\epsilon\nabla_\delta R_{\beta\sigma\alpha\gamma} + \nabla_\epsilon\nabla_\gamma R_{\beta\sigma\alpha\delta} + \nabla_\delta\nabla_\gamma R_{\beta\sigma\alpha\epsilon} + \frac{1}{3}R_{\beta\sigma\alpha\lambda}\left(R_{\lambda\gamma\delta\epsilon} + R_{\lambda\delta\epsilon\gamma}\right) + \\
&+\nabla_\epsilon\nabla_\delta R_{\gamma\sigma\alpha\beta} + \nabla_\epsilon\nabla_\delta R_{\gamma\sigma\beta\alpha} + \frac{1}{3}R_{\gamma\sigma\alpha\lambda}\left(R_{\lambda\beta\delta\epsilon} + R_{\lambda\delta\epsilon\beta}\right) + \frac{1}{3}R_{\gamma\sigma\beta\lambda}\left(R_{\alpha\lambda\delta\epsilon} + R_{\alpha\delta\epsilon\lambda}\right) + \\
&+\frac{1}{3}R_{\delta\sigma\alpha\lambda}\left(R_{\lambda\beta\gamma\epsilon} + R_{\lambda\gamma\epsilon\beta}\right) + \frac{1}{3}R_{\delta\sigma\beta\lambda}\left(R_{\alpha\lambda\gamma\epsilon} + R_{\alpha\gamma\epsilon\lambda}\right) + \frac{1}{3}R_{\delta\sigma\gamma\lambda}\left(R_{\alpha\beta\lambda\epsilon} + R_{\alpha\lambda\epsilon\beta}\right) * \\
&+\frac{1}{3}R_{\sigma\epsilon\alpha\lambda}\left(R_{\alpha\beta\gamma\delta} + R_{\alpha\gamma\delta\beta}\right) + \frac{1}{3}R_{\sigma\epsilon\beta\lambda}\left(R_{\alpha\lambda\gamma\delta} + R_{\alpha\gamma\delta\lambda}\right) + \\
&+\frac{1}{3}R_{\sigma\epsilon\gamma\lambda}\left(R_{\alpha\beta\lambda\delta} + R_{\alpha\lambda\delta\beta}\right) + \frac{1}{3}R_{\sigma\epsilon\delta\lambda}\left(R_{\alpha\beta\gamma\lambda} + R_{\alpha\gamma\lambda\beta}\right) + \\
&+\nabla_\sigma\nabla_\delta R_{\beta\epsilon\alpha\gamma} + \nabla_\sigma\nabla_\delta R_{\beta\epsilon\alpha\delta} + \nabla_\delta\nabla_\gamma R_{\beta\epsilon\alpha\sigma} + \frac{1}{3}R_{\beta\epsilon\alpha\lambda}\left(R_{\lambda\gamma\delta\sigma} + R_{\lambda\delta\sigma\gamma}\right) + \\
&+\nabla_\sigma\nabla_\delta R_{\gamma\epsilon\alpha\beta} + \nabla_\sigma\nabla_\delta R_{\gamma\epsilon\beta\alpha} + \\
&+\frac{1}{3}R_{\gamma\epsilon\alpha\lambda}\left(R_{\lambda\beta\delta\sigma} + R_{\lambda\delta\sigma\beta}\right) + \frac{1}{3}R_{\gamma\epsilon\beta\lambda}\left(R_{\alpha\lambda\delta\sigma} + R_{\alpha\delta\sigma\lambda}\right) + \frac{1}{3}R_{\delta\epsilon\alpha\lambda}\left(R_{\lambda\beta\gamma\sigma} + R_{\lambda\gamma\sigma\beta}\right) + \\
&+\frac{1}{3}R_{\delta\epsilon\beta\lambda}\left(R_{\alpha\lambda\gamma\sigma} + R_{\alpha\gamma\sigma\lambda}\right) + \frac{1}{3}R_{\delta\epsilon\gamma\lambda}\left(R_{\alpha\beta\lambda\sigma} + R_{\alpha\lambda\sigma\beta}\right) + \\
&+\nabla_\epsilon\nabla_\sigma R_{\beta\delta\alpha\gamma} + \nabla_\sigma\nabla_\gamma R_{\beta\delta\alpha\epsilon} + \nabla_\epsilon\nabla_\gamma R_{\beta\delta\alpha\sigma} + \nabla_\sigma\nabla_\epsilon R_{\gamma\delta\alpha\beta} + \nabla_\sigma\nabla_\epsilon R_{\gamma\delta\beta\alpha} + \\
&+\frac{1}{3}R_{\beta\delta\alpha\lambda}\left(R_{\lambda\gamma\sigma\epsilon} + R_{\lambda\sigma\epsilon\gamma}\right) + \frac{1}{3}R_{\gamma\delta\alpha\lambda}\left(R_{\lambda\beta\epsilon\sigma} + R_{\lambda\epsilon\sigma\beta}\right) + \frac{1}{3}R_{\gamma\delta\beta\lambda}\left(R_{\alpha\lambda\epsilon\sigma} + R_{\alpha\epsilon\sigma\lambda}\right) + \\
&+\nabla_\sigma\nabla_\epsilon R_{\beta\gamma\alpha\delta} + \nabla_\sigma\nabla_\delta R_{\beta\gamma\alpha\epsilon} + \nabla_\epsilon\nabla_\delta R_{\beta\gamma\alpha\sigma} + \frac{1}{3}R_{\beta\gamma\alpha\lambda}\left(R_{\lambda\delta\epsilon\sigma} + R_{\lambda\epsilon\sigma\delta}\right) + \\
&+\frac{1}{9}\left(R_{\mu\beta\gamma\delta} + R_{\mu\gamma\delta\beta}\right)\left(R_{\mu\alpha\epsilon\sigma} + R_{\mu\epsilon\sigma\alpha}\right) + \frac{1}{9}\left(R_{\mu\beta\gamma\epsilon} + R_{\mu\gamma\epsilon\beta}\right)\left(R_{\mu\alpha\delta\sigma} + R_{\mu\delta\sigma\alpha}\right) + \\
&+\frac{1}{9}\left(R_{\mu\beta\gamma\sigma} + R_{\mu\gamma\sigma\beta}\right)\left(R_{\mu\alpha\delta\epsilon} + R_{\mu\delta\epsilon\alpha}\right) + \frac{1}{9}\left(R_{\mu\beta\delta\epsilon} + R_{\mu\delta\epsilon\beta}\right)\left(R_{\mu\alpha\gamma\sigma} + R_{\mu\gamma\sigma\alpha}\right) + \\
&+\frac{1}{9}\left(R_{\mu\beta\delta\sigma} + R_{\mu\delta\sigma\beta}\right)\left(R_{\mu\alpha\gamma\epsilon} + R_{\mu\gamma\epsilon\alpha}\right) + \frac{1}{9}\left(R_{\mu\beta\epsilon\sigma} + R_{\mu\epsilon\sigma\beta}\right)\left(R_{\mu\alpha\gamma\delta} + R_{\mu\gamma\delta\alpha}\right) + \\
&+\frac{1}{9}\left(R_{\mu\gamma\delta\epsilon} + R_{\mu\delta\epsilon\gamma}\right)\left(R_{\mu\alpha\beta\sigma} + R_{\mu\beta\sigma\alpha}\right) + \frac{1}{9}\left(R_{\mu\gamma\delta\sigma} + R_{\mu\delta\sigma\gamma}\right)\left(R_{\mu\alpha\beta\epsilon} + R_{\mu\beta\epsilon\alpha}\right) + \\
&+\frac{1}{9}\left(R_{\mu\gamma\epsilon\sigma} + R_{\mu\epsilon\sigma\gamma}\right)\left(R_{\mu\alpha\beta\delta} + R_{\mu\beta\delta\alpha}\right) + \frac{1}{9}\left(R_{\mu\delta\epsilon\sigma} + R_{\mu\epsilon\sigma\delta}\right)\left(R_{\mu\alpha\beta\gamma} + R_{\mu\beta\gamma\alpha}\right)] = 0
\end{aligned}
$$

Putting $\alpha = \beta$,$\gamma = \delta$,$\epsilon = \sigma$

$$
\begin{aligned}
&-6\left[\Box\Box\Box\sigma\right] = \nabla_\sigma \nabla_\gamma R_{\sigma\gamma} + \nabla_\sigma \nabla_\gamma R_{\sigma\gamma} + \Box R + \frac{1}{3} R_{\gamma\sigma\alpha\lambda}\left(R_{\lambda\alpha\gamma\sigma} + R_{\lambda\gamma\sigma\beta}\right) + \\
&+\frac{1}{3} R_{\gamma\sigma\alpha\lambda}\left(R_{\alpha\lambda\gamma\sigma} + R_{\alpha\gamma\sigma\lambda}\right) + \frac{1}{3} R_{\gamma\sigma\alpha\lambda}\left(R_{\lambda\alpha\gamma\sigma} + R_{\alpha\gamma\sigma\lambda}\right) - \frac{1}{3} R_{\sigma\lambda} R^{\sigma\lambda} + \\
&+\nabla_\sigma \nabla_\gamma R_{\sigma\gamma} + \Box R + \frac{1}{3} R_{\gamma\sigma\alpha\lambda}\left(R_{\lambda\alpha\gamma\sigma} + R_{\lambda\gamma\sigma\alpha}\right) + \frac{1}{3} R_{\gamma\sigma\alpha\lambda}\left(R_{\alpha\lambda\gamma\sigma} + R_{\alpha\gamma\sigma\lambda}\right) + \\
&\frac{1}{3} R_{\gamma\sigma\alpha\lambda}\left(R_{\alpha\lambda\gamma\sigma} + R_{\alpha\sigma\lambda\gamma}\right) - \\
&-\frac{1}{3} R_{\sigma\lambda} R^{\sigma\lambda} + \Box R + \nabla_\sigma \nabla_\gamma R^{\gamma\sigma} + \nabla_\sigma \nabla_\gamma R^{\gamma\sigma} - \\
&-\frac{1}{3} R_{\lambda\gamma} R^{\lambda\gamma} + \Box R + \nabla_\sigma \nabla_\gamma R^{\gamma\sigma} + \nabla_\sigma \nabla_\gamma R^{\gamma\sigma} + \frac{1}{3} R_{\gamma\lambda} R^{\gamma\lambda} + \frac{1}{9} R_{\mu\alpha} R^{\mu\alpha} + \\
&+\frac{1}{9}\left(R_{\mu\alpha\gamma\sigma} + R_{\mu\gamma\sigma\alpha}\right)\left(R_{\mu\alpha\gamma\sigma} + R_{\mu\gamma\sigma\alpha}\right) + \\
&+\frac{4}{9}\left(R_{\mu\alpha\gamma\sigma} + R_{\mu\gamma\sigma\alpha}\right)\left(R_{\mu\alpha\gamma\sigma} + R_{\mu\gamma\sigma\alpha}\right) + \frac{1}{9} R_{\mu\alpha} R^{\mu\alpha} + \frac{1}{9} R_{\mu\alpha} R^{\mu\alpha}
\end{aligned}
\tag{13.178}
$$

How many scalars are there of the form Riemann^2?. Let us denote

$$
\begin{aligned}
I_1 &\equiv R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \\
I_2 &\equiv R_{\mu\nu\rho\sigma} R^{\mu\rho\sigma\nu} \\
I_3 &\equiv R_{\mu\nu\rho\sigma} R^{\mu\sigma\nu\rho}
\end{aligned}
\tag{13.179}
$$

It is plain that

$$
R_{\mu\nu\rho\sigma} R^{\mu\rho\sigma\nu} = R^{\mu\nu\rho\sigma} R_{\mu\sigma\nu\rho} \tag{13.180}
$$

so that

$$
I_2 = I_3 \tag{13.181}
$$

Bianchi tells us that

$$
R_{\mu\nu\rho\sigma} + R_{\mu\sigma\nu\rho} + R_{\mu\rho\sigma\nu} = 0 \tag{13.182}
$$

Then contracting with $R^{\mu\nu\rho\sigma}$

$$
I_1 + I_2 + I_3 = 0 \tag{13.183}
$$

Bianchi squared on the other hand implies

$$
3I_1 + 2I_3 + 2I_2 + 2I_3 = 0 \tag{13.184}
$$

This means that only one of the three possible contractions are independent

$$
I_2 = I_3 = -\frac{1}{2} I_1 \tag{13.185}
$$

Tis yields

$$-6\left[\Box\Box\Box\sigma\right]=7\nabla_\sigma\nabla_\gamma R^{\sigma\gamma}+4\Box R+\frac{7}{6}R_{\gamma\sigma\alpha\lambda}R^{\gamma\sigma\alpha\lambda} \tag{13.186}$$

We need also other scalar combination of six derivatives of σ to wit

$$\left[\nabla^\lambda\Box\nabla_\lambda\Box\sigma\right]=\left[\Box\Box\Box\sigma\right]+\frac{1}{3}R_{\alpha\beta}R^{\alpha\beta} \tag{13.187}$$

because

$$\nabla_\lambda\Box\nabla_\lambda\Box\sigma=\Box^3\sigma+\nabla^\mu\left(R_{\mu\sigma}\nabla_\sigma\Box\sigma\right) \tag{13.188}$$

- Let us now draw some consequences for the coincidence limits of derivatives of the van Vleck determinant. Taking covariant derivatives of the fundamental equation

$$\nabla_\mu\left(\Delta\sigma^\mu\right)=n\Delta \tag{13.189}$$

it results

$$\nabla_\rho\Delta\nabla_\alpha\sigma^\alpha+\Delta\nabla_\rho\nabla_\alpha\sigma^\alpha+\nabla_\rho\sigma^\alpha\nabla_\alpha\Delta+\sigma^\alpha\nabla_\rho\nabla_\alpha\Delta=n\nabla_\rho\Delta \tag{13.190}$$

At coincidence it follows

$$\Delta_\rho+\left[\nabla_\rho\Box\sigma\right]+\Delta_\rho=n\Delta_\rho \tag{13.191}$$

which implies

$$\left[\Delta_\rho\right]=0 \tag{13.192}$$

One more derivative leads to

$$\Delta_{\rho\delta}\sigma^\alpha_\alpha+\Delta\sigma^\alpha_{\alpha\delta}+\Delta_\delta\sigma^\alpha_{\alpha\rho}+\Delta\sigma^\alpha_{\alpha\rho\delta}+\sigma^\alpha_{\rho\delta}\Delta_\alpha+$$
$$\sigma^\alpha_\rho\Delta_{\alpha\delta}+\sigma^\alpha_\delta\Delta_{\alpha\rho}+\sigma^\alpha\Delta_{\alpha\rho\delta}=n\Delta_{\rho\delta}$$

At coincidence

$$n\left[\Delta_{\rho\sigma}\right]+\left[\nabla_\rho\nabla_\sigma\Box\sigma\right]+2\left[\Delta_{\rho\delta}\right]=n\left[\Delta_{\rho\delta}\right] \tag{13.193}$$

so that

$$\left[\Delta_{\alpha\beta}\right]=-\frac{1}{6}R_{\alpha\beta} \tag{13.194}$$

Its trace

$$\left[\Box\Delta\right]=-\frac{1}{6}R \tag{13.195}$$

One more derivative leads to

$$\Delta_{\rho\delta\lambda}\Box\sigma+\Delta_{\rho\delta}\nabla_\lambda\Box\sigma+\Delta_{\rho\lambda}\nabla_\delta\Box\sigma+\Delta_\rho\sigma_{\alpha\alpha\rho\delta\lambda}+\sigma^{\alpha\rho\delta\lambda}\Delta_\alpha+\sigma^{\alpha\rho\delta}\Delta_{\alpha\lambda}+$$
$$+\Delta_{\delta\lambda}\sigma^{\alpha\alpha\rho}+\Delta_\lambda\sigma^{\alpha\alpha\rho\delta}+\Delta\sigma^{\alpha\alpha\rho\delta\lambda}+$$
$$+\sigma^{\alpha\rho\lambda}\Delta_{\alpha\delta}+\sigma^{\alpha\rho}\Delta_{\alpha\delta\lambda}+\sigma^{\alpha\delta\lambda}\Delta_{\alpha\rho}+\sigma^{\alpha\delta}\Delta_{\alpha\rho\lambda}+\sigma^{\alpha\lambda}\Delta_{\alpha\rho\delta}+\sigma^\alpha\Delta_{\alpha\rho\delta\lambda}=n\Delta_{\rho\delta\lambda}$$

At coincidence this yields

$$[\nabla_\lambda \nabla_\delta \nabla_\rho \Box\sigma] + 3\,[\Delta_{\rho\delta\lambda}] = 0 \tag{13.196}$$

so that

$$[\Delta_{\rho\delta\lambda}] = -\left(\nabla_\delta R_{\rho\lambda} + 2\nabla_\lambda R_{\rho\delta} + 2\gamma_\rho R_{\lambda\delta}\right) \tag{13.197}$$

Yet one more

$$\begin{aligned}
&\Delta_{\rho\delta\lambda\epsilon}\Box\sigma + \Delta_{\rho\delta\lambda}\nabla_\epsilon\Box\sigma + \Delta_{\rho\delta\epsilon}\nabla_\lambda\Box\sigma + \Delta_{\rho\delta}\nabla_\epsilon\nabla_\lambda\Box\sigma + \\
&\Delta_{\rho\lambda\epsilon}\nabla_\delta\Box\sigma + \Delta_{\rho\lambda}\nabla_\epsilon\nabla_\delta\Box\sigma + \Delta_{\rho\epsilon}\sigma_{\alpha\alpha\rho\delta\lambda} + \Delta_\rho\sigma_{\alpha\alpha\rho\delta\lambda\epsilon} + \\
&+\Delta_{\delta\lambda\epsilon}\sigma^{\alpha\alpha\rho} + \Delta_{\delta\lambda}\sigma^{\alpha\alpha\rho\epsilon} + \Delta_{\lambda\epsilon}\sigma^{\alpha\alpha\rho\delta} + \Delta_\lambda\sigma^{\alpha\alpha\rho\delta\epsilon} + \Delta_\epsilon\sigma^{\alpha\alpha\rho\delta\lambda} + \Delta\sigma^{\alpha\alpha\rho\delta\lambda\epsilon} + \\
&\sigma^{\alpha\rho\delta\lambda\epsilon}\Delta_\alpha + \sigma^{\alpha\rho\delta\lambda}\Delta_{\alpha\epsilon} + \sigma^{\alpha\rho\delta\epsilon}\Delta_{\alpha\lambda} + \sigma^{\alpha\rho\delta}\Delta_{\alpha\lambda\epsilon} + \\
&+\sigma^{\alpha\rho\lambda\epsilon}\Delta_{\alpha\delta} + \sigma^{\alpha\rho\lambda}\Delta_{\alpha\delta\epsilon} + \sigma^{\alpha\rho\epsilon}\Delta_{\alpha\delta\lambda} + \sigma^{\alpha\rho}\Delta_{\alpha\delta\lambda\epsilon} + \sigma^{\alpha\delta\lambda\epsilon}\Delta_{\alpha\rho} + \sigma^{\alpha\delta\lambda}\Delta_{\alpha\rho\epsilon} + \\
&\sigma^{\alpha\delta\epsilon}\Delta_{\alpha\rho\lambda} + \sigma^{\alpha\delta}\Delta_{\alpha\rho\lambda\epsilon} + \sigma^{\alpha\lambda\epsilon}\Delta_{\alpha\rho\delta} + \sigma^{\alpha\lambda}\Delta_{\alpha\rho\delta\epsilon} + \sigma^{\alpha\epsilon}\Delta_{\alpha\rho\delta\lambda} + \sigma^{\alpha}\Delta_{\alpha\rho\delta\lambda\epsilon} = n\Delta_{\rho\delta\lambda\epsilon}
\end{aligned}$$

At coincidence

$$\begin{aligned}
&\Delta_{\rho\delta}\nabla_\epsilon\nabla_\lambda\Box\sigma + \Delta_{\rho\lambda}\nabla_\epsilon\nabla_\delta\Box\sigma + \Delta_{\rho\epsilon}\sigma_{\alpha\alpha\rho\delta\lambda} + \Delta_\rho\sigma_{\alpha\alpha\rho\delta\lambda\epsilon} + \\
&+\Delta_{\delta\lambda}\sigma^{\alpha\alpha\rho\epsilon} + \Delta_{\lambda\epsilon}\sigma^{\alpha\alpha\rho\delta} + \Delta\sigma^{\alpha\alpha\rho\delta\lambda\epsilon} + \sigma^{\alpha\rho\delta\lambda}\Delta_{\alpha\epsilon} + \sigma^{\alpha\rho\delta\epsilon}\Delta_{\alpha\lambda} + \\
&+\sigma^{\alpha\rho\lambda\epsilon}\Delta_{\alpha\delta} + +\sigma^{\alpha\rho}\Delta_{\alpha\delta\lambda\epsilon} + \sigma^{\alpha\delta\lambda\epsilon}\Delta_{\alpha\rho} + \sigma^{\alpha\delta}\Delta_{\alpha\rho\lambda\epsilon} + \sigma^{\alpha\lambda}\Delta_{\alpha\rho\delta\epsilon} + \\
&\sigma^{\alpha\epsilon}\Delta_{\alpha\rho\delta\lambda} = 0
\end{aligned} \tag{13.198}$$

Ricci implies

$$\Delta_{\rho\epsilon\delta\lambda} = \Delta_{\delta\rho\lambda\epsilon} + \nabla_\lambda\left(R_{\delta\epsilon\rho\sigma}\Delta_\sigma\right) + R_{\lambda\epsilon\delta\sigma}\Delta^{\rho\sigma} + R_{\lambda\epsilon\rho\sigma}\Delta^{\sigma\delta} \tag{13.199}$$

$$\Delta_{\rho\epsilon\delta\lambda} = \Delta_{\delta\rho\lambda\epsilon} + \nabla_\lambda\left(R_{\delta\epsilon\rho\delta}\Delta_\sigma\right) + R_{\lambda\epsilon\delta\sigma}\Delta^{\sigma\rho} + R_{\lambda\epsilon\rho\sigma}\Delta^{\sigma\delta} \tag{13.200}$$

$$\Delta_{\lambda\rho\delta\epsilon} = \Delta_{\delta\lambda\rho\epsilon} + \nabla_\epsilon\left(R_{\delta\rho\lambda\sigma}\Delta^\sigma\right) + \nabla_\epsilon\left(R_{\rho\lambda\delta\sigma}\Delta^\sigma\right) \tag{13.201}$$

Making $\epsilon = \rho$ and $\delta = \lambda$

$$\begin{aligned}
&-4\,[\Box\Box\Delta] = -\frac{3}{6}R^{\rho\lambda}\left[\nabla_\rho\nabla_\lambda\Box\sigma\right] - \frac{2}{6}R\left[\Box\Box\sigma\right] + \left[\nabla^\rho\Box\nabla^\rho\Box\sigma\right] - \frac{1}{6}R^{\alpha\rho}\left[\Box\nabla_\rho\nabla_\alpha\sigma\right] - \\
&-\frac{1}{6}R^{\alpha\lambda}\left[\nabla_\rho\nabla_\lambda\nabla^\rho\nabla_\alpha\sigma\right] - \frac{1}{6}R^{\alpha\lambda}\left[\nabla_\rho\nabla_\lambda\nabla^\rho\nabla_\alpha\sigma\right] - \frac{1}{6}R^{\alpha\rho}\left[\nabla_\rho\Box\nabla_\alpha\sigma\right] = \\
&-\frac{1}{6}R_{\alpha\beta}R^{\alpha\beta} - \frac{1}{9}R^2 + \frac{1}{3}R_{\alpha\beta}R^{\alpha\beta} - \frac{7}{6}\nabla_\alpha\nabla_\beta R^{\alpha\beta} - \frac{2}{3}\Box R - \frac{7}{36}R^2_{\mu\nu\rho\sigma} - \\
&-\frac{1}{18}R_{\alpha\beta}R^{\alpha\beta} - \frac{1}{9}R_{\alpha\beta}R^{\alpha\beta} - \frac{1}{18}R_{\alpha\beta}R^{\alpha\beta}
\end{aligned} \tag{13.202}$$

We then get

$$\begin{aligned}
&[\Box\Box\Delta] = \frac{1}{72}R_{\alpha\beta}R^{\alpha\beta} + \frac{7}{144}R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} + \frac{1}{36}R^2 + \\
&\frac{1}{6}\Box R + \frac{7}{24}\nabla^\alpha\nabla^\beta R_{\alpha\beta}
\end{aligned} \tag{13.203}$$

- Let us now come back towards coincidence limits of the DeWitt-Schwinger coefficients themselves. From the equation

$$\sigma^{\mu}\nabla_{\mu}a_0 = 0 \tag{13.204}$$

by deriving once

$$\sigma^{\mu}{}_{\nu}\nabla_{\mu}a_0 + \sigma^{\mu}\nabla_{\nu}\nabla_{\mu}a_0 = 0 \tag{13.205}$$

Then

$$[\nabla_{\mu}a_0] = 0 \tag{13.206}$$

Deriving again

$$\sigma^{\mu}{}_{\nu\lambda}\nabla_{\mu}a_0+\sigma^{\mu}{}_{\nu}\nabla_{\lambda}\nabla_{\mu}a_0+\sigma^{\mu}{}_{\lambda}\nabla_{\nu}\nabla_{\mu}a_0+\sigma^{\mu}\nabla_{\lambda}\nabla_{\nu}\nabla_{\mu}a_0 = 0 \tag{13.207}$$

Then

$$[(\nabla_{\mu}\nabla_{\nu} + \nabla_{\nu}\nabla_{\mu})\, a_0] = 0 \tag{13.208}$$

as well as

$$[\Box a_0] = 0 \tag{13.209}$$

This implies that

$$[a_1] = \frac{1}{6}\, R \tag{13.210}$$

In general the fields will have got indices. Then

$$[\nabla_{\mu}, \nabla_{\nu}]\, \phi^A = \mathcal{R}_{\mu\nu}{}^{A}{}_{B}\phi^B \tag{13.211}$$

Then

$$[(\nabla_{\mu}\nabla_{\nu} - \nabla_{\nu}\nabla_{\mu})\, a_0] = \mathcal{R}_{\mu\nu} \tag{13.212}$$

and

$$[\nabla_{\mu}\nabla_{\nu}a_0] = \frac{1}{2}\mathcal{R}_{\mu\nu} \tag{13.213}$$

Deriving once more

$$\sigma^{\mu\nu\lambda\delta}a^0_{\mu}+\sigma^{\mu\nu\lambda}a^0_{\mu\delta}+\sigma^{\mu\nu\delta}a^0_{\mu\lambda}+\sigma^{\mu\nu}a^0_{\mu\lambda\delta}+\sigma^{\mu\lambda\delta}a^0_{\mu\nu}+\sigma^{\mu\lambda}a^0_{\mu\nu\delta}+\sigma^{\mu\delta}a^0_{\mu\nu\lambda}+\sigma^{\mu}a^0_{\mu\nu\lambda\delta} = 0 \tag{13.214}$$

It is got to be derived again

$$\begin{aligned}
&\sigma^{\mu\nu\lambda\delta\epsilon}a^0_{\mu} + \sigma^{\mu\nu\lambda\delta}a^0_{\mu\epsilon} + \sigma^{\mu\nu\lambda\epsilon}a^0_{\mu\delta} + \sigma^{\mu\nu\lambda}a^0_{\mu\delta\epsilon} + \sigma^{\mu\nu\delta\epsilon}a^0_{\mu\lambda} + \sigma^{\mu\nu\delta}a^0_{\mu\lambda\epsilon} + \\
&\sigma^{\mu\nu\epsilon}a^0_{\mu\lambda\delta} + \sigma^{\mu\nu}a^0_{\mu\lambda\delta\epsilon} + \sigma^{\mu\lambda\delta\epsilon}a^0_{\mu\nu} + \sigma^{\mu\lambda\delta}a^0_{\mu\nu\epsilon} + \sigma^{\mu\lambda\epsilon}a^0_{\mu\nu\delta} + \sigma^{\mu\lambda}a^0_{\mu\nu\delta\epsilon} + \\
&+\sigma^{\mu\delta\epsilon}a^0_{\mu\nu\lambda} + \sigma^{\mu\delta}a^0_{\mu\nu\lambda\epsilon} + \sigma^{\mu\epsilon}a^0_{\mu\nu\lambda\delta} + \sigma^{\mu}a^0_{\mu\nu\lambda\delta\epsilon} = 0
\end{aligned} \tag{13.215}$$

Taking traces and using the Ricci identity we learn that

$$[\Box\Box a_0] = 0 \tag{13.216}$$

Our former equation [13.140] tells us that

$$[a_1] = \frac{1}{6} R \tag{13.217}$$

as well as

$$[\nabla_\mu a_1] = \frac{1}{24}\ \nabla_\mu R \tag{13.218}$$

The long equation [13.141] at coincidence limit reads

$$3\,[\nabla_\lambda \nabla_\nu a_1] = -\frac{1}{18} R_{\mu\lambda} R^\mu_\nu - \frac{1}{18} R_{\nu\lambda} R + \frac{3}{2}\,[\nabla_\lambda \nabla_\nu \Box \Delta] \tag{13.219}$$

Which implies

$$[\Box a_1] = \frac{1}{54} R^2_{\mu\nu} - \frac{1}{54} R^2 - \frac{1}{2}\,[\Box\Box\Delta] = \frac{1}{54} R^2_{\mu\nu} - \frac{1}{54} R^2 - \frac{1}{144} R_{\alpha\beta} R^{\alpha\beta} - \\ -\frac{1}{72} R^2 - \frac{7}{288} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} - \frac{1}{12} \Box R - \frac{7}{48} \nabla^\alpha \nabla^\beta R^{\alpha\beta} \tag{13.220}$$

Finally

$$[a_2] = \frac{1}{2} R\,[a_1] + [\Box a_1] = \frac{1}{12} R^2 - \frac{1}{144} R_{\alpha\beta} R^{\alpha\beta} - \frac{1}{72} R^2 - \\ \frac{7}{288} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} - \frac{1}{12} \Box R - \frac{7}{48} \nabla^\alpha \nabla^\beta R^{\alpha\beta} \tag{13.221}$$

13.8 Recursion relations for the coefficients of the short time expansion of the heat kernel.

Let us here consider a system of recursion relations obtained by Gilkey [12] in a remarkable paper. They greatly simpify the computation of the deWitt-Schwinger coefficients. We are refering to an operator on a riemannian manifold, M,

$$D \equiv -\left(g^{\mu\nu}\partial_\mu\partial_\nu + P^\mu\partial_\mu + Q\right) \tag{13.222}$$

P^μ and Q are square matrices acting on a vector bundle over M, V, where the fibers are isomorphic to $\mathbb{R}^k$. When the fibers are isomorphic to a Lie group, we talk of a primcipal bundle. What mathematicians call a *section* of a pbundle is what we call a matter field, and when they talk of a connection on a principal bundle, we talk of a gauge field.

What this means in practice is that the matrices P^μ and Q are defined in the vector space defined by the finite dimensional representations of the matter fields (including the graviton itself). To be specific, Gilkey's theory includes operators of the form

$$\hat{D} \equiv -\left(K^A{}_B g^{\mu\nu}\partial_\mu\partial_\nu + \left(\hat{P}^\mu\right)^A{}_B\partial_\mu + \hat{Q}^A{}_B\right) = K^A{}_C\, D^C{}_B \tag{13.223}$$

where

$$\begin{aligned} P^A{}_B &= \left(K^{-1}\right)^A{}_C\, \hat{P}^C{}_B \\ Q^A{}_B &= \left(K^{-1}\right)^A{}_C\, \hat{Q}^C{}_B \end{aligned} \tag{13.224}$$

Gilkey defines the heat kernel $K(\tau, x, y; D)$ precisely as the kernel of

$$e^{-\tau D} \tag{13.225}$$

The heat kernel is considered as an operator acting on the fibers

$$V_y \to V_x \tag{13.226}$$

Then

$$e^{-\tau D} K = 0 \tag{13.227}$$

It is plain that the HK also obeys the heat equation

$$\left(\frac{\partial}{\partial\tau} + D\right) K = 0 \tag{13.228}$$

Defined in that way, the heat kernel vanishes to infinite order for $x \neq y$. When $x = y$ it has an asymptotic expansion when $\tau \to 0^+$ of the form

$$K(\tau, x, y; D) \sim (4\pi\tau)^{-n/2} \sum_{p=0}^{\infty} \tau^n E_n(x, D) \tag{13.229}$$

In the self-adjoint case, and if $(\lambda_i, \phi_i(x))$ is a spectral resolution of D, the heat kernel is given by

$$K(\tau, x, y; D) = \sum_i e^{-\tau\lambda_i} \phi_i(x) \otimes \phi_i^*(y) \tag{13.230}$$

This means that

$$\text{Tr } e^{-\tau D} = \sum_i e^{-\tau\lambda_i} = \int_M \text{Tr } K(\tau, x, x; D) \; d(vol) \sim$$

$$\sim (4\pi\tau)^{-n/2} \sum_{p=0}^{\infty} \tau^p \int_M E_p(x, D) \; d(vol) \equiv (4\pi\tau)^{-n/2} \sum_{p=0}^{\infty} a_p(D) \; \tau^p$$

Where we still denote

$$a_p(x, D) \equiv \text{Tr } E_p(x, D) \tag{13.231}$$

and

$$a_p(D) \equiv \int_M a_p(x, D) \; d(vol) \tag{13.232}$$

which is a spectral invariant of the operator D.

There is a useful scaling relation which stems from the very definition, since $e^{-\tau\lambda^2 D} = e^{-\lambda^2 \tau D}$

$$K\left(\tau, x, y; \lambda^2 D\right) d(Vol)_{\lambda^{-2}G} = K\left(\lambda^2\tau, x, y; D\right) d(Vol)_G \tag{13.233}$$

Where G is the metric induced by the leading symbol of G (that is $g_{\mu\nu}\xi^\mu\xi^\nu$). Now

$$d(Vol)_{\lambda^{-2}G} = \lambda^{-n} \; d(Vol)_G = \tag{13.234}$$

It follows that

$$(4\pi\tau)^{-n/2} \sum_{p=0} \tau^p E_p\left(x, \lambda^2 D\right) \sim (4\pi\lambda^2\tau)^{-n/2} \sum_{p=0} \left(\lambda^2\tau\right)^p E_p(x, D) \tag{13.235}$$

so that

$$E_p\left(x, \lambda^2 D\right) = \lambda^{2p} \; E_p(x, D) \tag{13.236}$$

as well as the integrated version of it

$$a_p\left(\lambda^2 D\right) = \lambda^{n+2p} \; a_p(D) \tag{13.237}$$

We have now, besides the Levi-Civita connection in $T^*(M)$ another connection $A \equiv dx^\mu \otimes A_\mu$ in V. The reduced laplacian is defined as the operator

$$D_A \equiv -\left(\nabla^\mu + A^\mu\right)\left(\nabla_\mu + A_\mu\right) = -\Big(g^{\mu\nu}\partial_\mu\partial_\nu + \left(2A^\mu - g^{\alpha\beta}\Gamma^\mu_{\alpha\beta}\right)\partial_\mu +$$

$$+\partial_\mu A^\mu + A_\mu A^\mu - A_\lambda \Gamma^\lambda_{\alpha\beta} g^{\alpha\beta}\Big)$$

We define the matrix (endomorphism)

$$E \equiv D_A - D \tag{13.238}$$

The gauge field is univocally defined as

$$A_\mu = \frac{1}{2}\left(P^\mu + g^{\alpha\beta}\Gamma^\mu_{\alpha\beta}\right) \tag{13.239}$$

and the matrix E is given by

$$E = Q - \partial_\mu A^\mu - A_\mu A^\mu + A_\lambda \Gamma^\lambda_{\alpha\beta} g^{\alpha\beta} \tag{13.240}$$

It is not difficult to show that

$$E_0 = 1 \tag{13.241}$$

Dimensional analysis tells us that the general expression for $E : 1$ is given by

$$E_1 = \frac{1}{6}\left(a_1\mathcal{E} + b_1 R\right) \tag{13.242}$$

and the most general possible expression for E_2 reads

$$E_2(x, D) = \frac{1}{360}\Big\{d_1\Box R + c_1 R^2 + c_2 R^2_{\mu\nu} + c_3 R^2_{\mu\nu\rho\sigma} + c_4\mathcal{E}R + d_2\Box\mathcal{E} + \\ + c_5\mathcal{E}^2 + c_6 F^2_{\mu\nu}\Big\} \tag{13.243}$$

Lets work out all coefficients by a clever use of consistency requirements.

- It is also the case that when the manifold is a direct product

  $$M_n = M_{n_1} \times M_{n_2} \tag{13.244}$$

  and the operator D is

  $$D = D_1 \otimes 1 + 1 \otimes D_2 \tag{13.245}$$

  then

  $$E_{m,n}(x_1, x_2) = \sum_{m_1+m_2=m} E_{m_1,n_1}(x_1, D_1) \otimes E_{m_2,n_2}(x_2, D_2) \tag{13.246}$$

- Another useful relation easily proved just by expanding both members,

  $$E_m(x, D - \epsilon I) = \sum \epsilon^k E_{n-k}(x, D) \tag{13.247}$$

- This shows, in particular, that

  $$c_5 = 180 \tag{13.248}$$

- In order to proceed, let us define a first order Dirac-like operator acting in a n-dimensional conformally flat space with metric

$$ds^2 = e^{-h(x)}\delta_{\mu\nu}\ dx^\mu dx^\nu \tag{13.249}$$

It is easy to check that

$$g^{\alpha\beta}\Gamma^\lambda_{\alpha\beta} = \frac{n-2}{2}e^h\ \partial_\lambda h \tag{13.250}$$

With my conventions,

$$R_{\beta\gamma} = \frac{n-2}{2}\left(\partial_\beta\partial_\gamma\sigma + \frac{1}{2}\partial_\gamma\sigma\partial_\beta\sigma\right)+\left(\frac{1}{2}\sum\partial^2_\lambda\sigma - \frac{n-2}{4}\sum(\partial_\mu\sigma)^2\right)\delta_{\beta\gamma} \tag{13.251}$$

$$R = e^\sigma\left((n-1)\sum_\lambda\partial^2_\lambda\sigma - \frac{(n-2)(n-1)}{4}\sum_\lambda(\partial_\lambda\sigma)^2\right) \tag{13.252}$$

The operator in question is

$$A \equiv e^{\frac{nh}{4}}\ \sum_\mu\gamma_\mu\partial_\mu\ e^{\frac{(2-n)h}{4}} \tag{13.253}$$

Its adjoit is easily obtained from

$$\begin{aligned}\langle f, Ag\rangle &\equiv \int d^n x\ e^{-nh}\ f^*\ e^{\frac{nh}{4}}\ \sum_\mu\gamma_\mu\partial_\mu\ e^{\frac{(2-n)h}{4}}g = -\int d^n x\ e^{-n\frac{h}{2}}\ g\ e^{\frac{n+2}{4}h}\partial_\mu f^*\gamma_\mu e^{-\frac{nh}{4}} \\ &= -\left[\int d^n x\ e^{-n\frac{h}{2}}\ g^*\ e^{\frac{n+2}{4}h}\sum_\mu\partial_\mu f\gamma^*_\mu e^{-\frac{nh}{4}}\right]^* = -\left[\int d^n x\ e^{-n\frac{h}{2}}\ g^*\ e^{\frac{n+2}{4}h}\sum_\mu\gamma^+_\mu\partial_\mu f e^{-} \right. \\ &= \langle g, A^+ f\rangle^* \end{aligned}$$

with

$$A^+ = e^{\frac{n+2}{2}h}\sum_\mu\gamma_\mu\partial_\mu e^{-\frac{nh}{4}} \tag{13.255}$$

where we have assumed that

$$\{\gamma_\mu, \gamma_\nu\} = -2\delta_{\mu\nu} \tag{13.256}$$

as well as

$$\gamma^+_\mu = -\gamma_\mu \tag{13.257}$$

A basis of the Clifford algebra is given by the 2^n matrices

$$\gamma_\Lambda \equiv \left\{\gamma_{\mu_1\ldots\mu_j}\Big|_{\mu_1<\ldots<\mu_j}\right\} \tag{13.258}$$

There are 2^n of them because

$$\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = (1+1)^n \tag{13.259}$$

We shall assume that the set γ_Λ forms an orthonormal basis.

We now define two self-adjoint modifications of the Laplace operator

$$D_1 \equiv A^+ A = -e^h \left\{ \Box - \frac{n-2}{2} \sum h_\mu \partial_\mu + \left(\frac{n-2}{2}\right)^2 - \frac{n-2}{4} \Box h \right\} \tag{13.260}$$

where

$$h_\mu \equiv \partial_\mu h. \tag{13.261}$$

From the definition itself it follows that

$$\begin{aligned} A_\mu &= F_{\mu\nu} = 0 \\ \mathcal{E} &= e^h \left(\left(\frac{n-2}{4}\right)^2 \sum h_\mu^2 - \frac{n-2}{4} \Box h \right) \end{aligned} \tag{13.262}$$

The second operator reads

$$D_2 \equiv A A^+ = -e^h \left\{ \Box - \frac{n-2}{2} \sum h_\mu \partial_\mu - \sum \gamma_{\mu\nu} h_\mu \partial_\nu + \frac{n^2 - 4n}{16} \sum h_\mu^2 - \frac{n}{4} \Box h \right\} \tag{13.263}$$

where

$$\gamma_{\mu\nu} \equiv \frac{1}{2} \left[\gamma_\mu, \gamma_\nu\right] \tag{13.264}$$

It follows

$$\begin{aligned} A_\mu &= -\frac{1}{2} \sum_\lambda \gamma_{\lambda\mu} h_\lambda \\ F_{\mu\nu} &= -\gamma_{\lambda[\mu} \partial_{\nu]} h_\lambda + \frac{1}{2} \left(\gamma_{\nu\mu} \sum h_\lambda^2 + \sum_\sigma \left(\gamma_{\mu\sigma} h_\nu - \gamma_{\nu\sigma} h_\mu\right) \right) \\ \mathcal{E} &= e^h \left(\frac{n^2 - 4}{16} \sum h_\lambda^2 - \frac{n}{4} \Box h \right) \end{aligned} \tag{13.265}$$

where we have used

$$\gamma_{\lambda\nu} \gamma_{\sigma\mu} = -\delta_{\lambda\sigma} \delta_{\nu\mu} + \delta_{\lambda\mu} \delta_{\nu\sigma} - \delta_{\nu\sigma} \gamma_{\lambda\mu} + \delta_{\nu\mu} \gamma_{\lambda\sigma} + \delta_{\lambda\sigma} \gamma_{\nu\mu} - \delta_{\lambda\mu} \gamma_{\nu\sigma} \tag{13.266}$$

The operators are such that

$$A D_1 = D_2 A \tag{13.267}$$

This means that if we diagonalize D_1 (which is isomorphic to 2^n copies of the associated scalar operator)

$$D_1 \gamma_\Lambda \phi_i = \lambda_i \gamma_\Lambda \phi_i \tag{13.268}$$

then

$$AD_1\gamma_\Lambda\phi_i = \lambda_i\left(A\gamma_\Lambda\phi_i\right) = D_2\left(A\gamma_\Lambda\phi_i\right) \tag{13.269}$$

so that $A\phi_i$ is a set of eingenvectors for D_2 with eigenvalues λ_i. In order that they are normalized in agreement with the initial basis ϕ_i we have to dinite by $\frac{1}{\sqrt{\lambda_i}}$ because

$$\langle A\phi|A\phi\rangle = \langle A^+A\phi|\phi\rangle = \lambda\langle\phi|\phi\rangle \tag{13.270}$$

Then

$$\begin{aligned}
&\frac{d}{dt}\left(\text{tr } K\left(t,x,x,D_1\right) - \text{tr } K\left(t,x,x,D_2\right)\right) = \sum_{i,\Lambda} e^{-t\lambda_i}\left(-\lambda_i\langle\gamma_\Lambda\phi_i,\gamma_\Lambda\phi_i\rangle + \right.\\
&\left.+\lambda_i\left\langle\frac{A\gamma_\Lambda\phi_i}{\sqrt{\lambda_i}},\frac{A\gamma_\Lambda\phi_i}{\sqrt{\lambda_i}}\right\rangle\right) = \sum_{i,\Lambda} e^{-t\lambda_i}\left(-\langle D_1\gamma_\Lambda\phi_i,\gamma_\Lambda\phi_i\rangle + \langle A\gamma_\Lambda\phi_i, A\gamma_\Lambda\phi_i\rangle\right) = \\
&= e^h\sum_i\left(\phi_i\Box\phi_i - \frac{n-2}{4}\Box h\phi_i^2 + \sum_\mu \phi_\mu^i\phi_\mu^i - (n-2)\phi_i h_\mu\phi_\mu^i + \right.\\
&\left.+\frac{(n-2)^2}{8}h_\mu^2\phi_i^2 - \frac{n-2}{4}\Box h\phi_i^2\right) = \frac{1}{2}e^{\frac{nh}{2}}\sum_\mu\partial_\mu\partial_\mu e^{-\frac{n-2}{2}h}\sum_i\phi_i^2
\end{aligned} \tag{13.271}$$

It follows that

$$(2p-n)\left\{\text{tr } E_n\left(x,D_1\right) - \text{tr}E_n\left(x,D_2\right)\right\} = \sum_\mu e^{\frac{n}{2}h}\partial_\mu\partial_\mu e^{-\frac{n-2}{2}h}\text{ tr } E_{p-1}\left(x,D_1\right) \tag{13.272}$$

- We need also the following fact. We can write

$$E_p(x,D) = a_p\Box^{p-1}\mathcal{E} + b_p\Box^{p-1}\ R + \ldots + \text{less derivatives} \tag{13.273}$$

with

$$a_p = \frac{(n-1)!}{(2p-1)!}. \tag{13.274}$$

It is not difficult to prove that theorem in the one-dimensional case, $M \equiv S^1$, where

$$E_p(x,D) = a_p\frac{d^{2p-2}}{dx^{2p-2}}\mathcal{E} + \text{less derivatives} \tag{13.275}$$

and then use the fact that the coefficients of the short time expansion are dimension-independent.

In order to do that, we can again define

$$A \equiv \frac{\partial}{\partial x} + f(x) \tag{13.276}$$

whose adjoint reads

$$A^{+} = -\frac{\partial}{\partial x} + f(x) \tag{13.277}$$

and define the two self-adjoint operators

$$D_1 \equiv A^{+}A = -\left(\frac{\partial^2}{\partial x^2} + f' - f^2\right) \tag{13.278}$$

$$D_2 \equiv AA^{+} = -\left(\frac{\partial^2}{\partial x^2} - f' - f^2\right) \tag{13.279}$$

Then explicit calculation shows that

$$E_p(x, D_1) - E_p(x, D_2) = \frac{1}{2p-1}\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} + 2f(x)\right) E_{p-1}(x, D_1) \tag{13.280}$$

Then it is a fact that

$$E_p(x, D_1) - E_p(x, D_2) = 2a_p f^{(2p-1)} + \ldots = \frac{1}{2p-1}\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} + 2f(x)\right) E_{n-1}(x, D_1) = \\ = a_{p-1}\frac{1}{2p-1} f^{(2p-1)} + \ldots \tag{13.281}$$

This yields the recurrence

$$a_p = \frac{a_{p-1}}{4p-2} \tag{13.282}$$

We known that

$$a_1 = 1 \tag{13.283}$$

Then it follows that

$$a_p = \frac{(p-1)!}{(2p-1)!} \tag{13.284}$$

- Let us now come back to the general dimensional setting. Applying the general theorem in the case where

$$2p + 2 = n \tag{13.285}$$

we learn that

$$e^{\frac{nh}{2}} \sum \partial_\mu^2 \, e^{\frac{(2-n)h}{2}} \text{ tr } E_p(x, D_1) = 0 \tag{13.286}$$

This means that

$$\text{tr } E - p(x, D_1) = 0 \tag{13.287}$$

Computing it yields

$$\text{tr } E - p(x, D_1) = a_p \, \Box^{n-1} \, \mathcal{E} + b_p \, \Box^{n-1} \, R + \text{less} \tag{13.288}$$

Now

$$\mathcal{E}(D_1) = \frac{2-n}{4} e^h \sum \partial_\mu \partial_\mu h + \text{less} \tag{13.289}$$

and

$$R = (n-1) e^h \sum \partial_\mu \partial_\mu h + \text{less} \tag{13.290}$$

(Gilkey's conventions have a minus sign here). Then

$$\frac{2-2p-2}{4} a_p + (1-2p-2)\, b_p = 0 \tag{13.291}$$

which means that

$$b_p = -\frac{2p}{4(2p+1)} a_p = -\frac{p}{2(2p-1)} \frac{(p-1)!}{(2p-1)!} \tag{13.292}$$

- In particular, $a_1 = 1$ and $b_1 = -\frac{1}{6}$, so that ,

$$E_1 = a_1 \mathcal{E} + b_1 R = \mathcal{E} - \frac{1}{6} \, R \tag{13.293}$$

- Let us now apply the former theorem that asserts

$$E_p(\mathcal{E} + \epsilon) = \sum \frac{\epsilon^k}{k!} E_{p-k}(\mathcal{E}) \tag{13.294}$$

to the case $p = 2$ and $k = 1$. Then the only term linear in ϵ is c_4.

$$\frac{1}{360} \epsilon \left(c_4 R + 2 c_5 \mathcal{E} \right) = \epsilon \left(\mathcal{E} - \frac{1}{6} \, R \right) \tag{13.295}$$

Then

$$c_4 = -60 \tag{13.296}$$

(It was already known that $c_5 = 180$).

- Now let us consider a product manifold, $M \equiv M_1 \times M_2$. The scalar curvature is additive,

$$R(x_1, x_2) = R(x_1) + R(x_2) \tag{13.297}$$

The operator we wish to consider is

$$D \equiv D_1 \otimes 1 \oplus 1 \otimes D_2 \tag{13.298}$$

Then, using the results obtained up to now,

$$E_2(x_1, x_2; D) = E_2(x_1, D_1) + E_2(x_2, D_2) + \frac{2 c_1}{360} R(x_1) R(x_2) + \text{other} \tag{13.299}$$

Our general formula for product manifolds on the other hand, tells us that

$$E_2(x_1, x_2; D) = E_2(x_1, D_1) + E_2(x_2, D_2) + E_1(x_1, D_1)E_1(x_2, D_2) + \text{other} \tag{13.300}$$

We learn that

$$\frac{c_1}{180} = \frac{1}{36} \tag{13.301}$$

That is,

$$c_1 = 5 \tag{13.302}$$

- Let us apply the ur-theorem to the two-dimensional case, $n = 2$. Then

$$\begin{aligned} \mathcal{E}(D_1) &= 0 \\ F_{\mu\nu}(D_1) &= 0 \end{aligned} \tag{13.303}$$

$$\begin{aligned} \mathcal{E}(D_2) &= -\frac{1}{2} e^h \Box h \\ F_{12}(D_2) &= -\frac{1}{2} \gamma_{12} \Box h \end{aligned} \tag{13.304}$$

Taking into account the dimension of the γ matrices (4)

$$\text{tr } E_1(x, D_1) = \frac{4}{6} e^h \ \Box h \tag{13.305}$$

so that the rhs becomes

$$\text{rhs} = \frac{4}{6} \ e^{2h} \ (\partial_1 h)^2 + \text{more} \tag{13.306}$$

As for the lhs,

$$\begin{aligned} \text{lhs} &= -\text{tr}\frac{1}{360} \left(-60R \ \mathcal{E} + 60\Box\mathcal{E} + 180\mathcal{E}^2 + c_6 F_{\mu\nu}^2 \right)(x, D_2) = \\ &= \frac{4}{360} e^{2h} \ \left(30 + 30 - 45 + \frac{1}{2} c_6 \right) (\partial_1^2 h)^2 + \text{more} \end{aligned} \tag{13.307}$$

The theorem then implies that

$$\frac{8}{360} \left(15 + \frac{1}{2} c_6 \right) = \frac{4}{6} \tag{13.308}$$

We obtain

$$c_6 = 30 \tag{13.309}$$

- Finally, apply the ur-theorem to the six-dimensional case, $n = 6$ with $p = 3$, so that again

$$2p - n = 0 \tag{13.310}$$

and

$$\text{tr } E_2(x, D_1) = 0 \tag{13.311}$$

Taking into account that

$$\mathcal{E}(D_1) = e^h \left(-\Box h + \sum h_\mu^2\right) \tag{13.312}$$

and still

$$F_{\mu\nu}(D_1) = 0 \tag{13.313}$$

we learn that

$$-12\Box R + 60\Box\mathcal{E} = 0 \tag{13.314}$$

as well as (always for the operator D_1)

$$6R^2 - 60R\mathcal{E} + 180\mathcal{E}^2 = 5\mathcal{E}^2 \tag{13.315}$$

$$\begin{aligned}
5\mathcal{E}^2 &= e^{2h}\left(5h_{11}^2 + 0h_{12}\right) + \text{more} \\
c_2 R_{\mu\nu}^2 &= e^{2h}\left(\frac{15}{2}c_2 h_{11}^2 + 8c_2 h_{12}^2\right) + \text{more} \\
c_3 R_{\mu\nu\rho\sigma}^2 &= e^{2h}\left(5c_3 h_{11}^2 + 8c_3 h_{12}^2\right) + \text{more}
\end{aligned} \tag{13.316}$$

Then

$$\begin{aligned}
&c_2 + c_3 = 0 \\
&10 + 15c_2 + 10c_3 = 0
\end{aligned} \tag{13.317}$$

so that

$$\begin{aligned}
c_2 &= -2 \\
c_3 &= 2
\end{aligned} \tag{13.318}$$

13.9 The one-loop effective action of quantum gravity.

Let us consider the Einstein-Hilbert lagrangian with a scalar field minimally coupled to the gravitational field

$$S_g = -\frac{1}{2\kappa^2}\int d^n x\,\sqrt{g}\,\left[R+\lambda\right] + \int d^n x\,\sqrt{g}\,\frac{1}{2}\,g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi. \qquad (13.319)$$

This computation was first performed by 't Hooft and Veltman [?] in an epoch-making paper using the background-field method, but *without* employing heat-kernel techniques.

The computation will be performed here using deWitt's backgrund field technique as well as heat-kernel methods. Both the metric and the scalar field in the action (13.319) are expanded in a background field and a perturbation

$$\begin{aligned} g_{\mu\nu} &= \bar{g}_{\mu\nu} + \kappa\, h_{\mu\nu} \\ g^{\mu\nu} &= \bar{g}^{\mu\nu} - \kappa\, h^{\mu\nu} + \kappa^2\, h^{\mu}{}_{\alpha} h^{\alpha\nu} + O(\kappa^3) \\ \phi &= \bar{\phi} + \kappa\,\phi. \end{aligned} \qquad (13.320)$$

Where indices are raised with the background metric and geometric quantities (curvature tensors, covariant derivatives...) calculated with respect to this metric wear a bar. To take into account one-loop effects it is enough to expand the action up to quadratic order in the perturbations. After expanding, the term linear in the coupling cancels due to the background equations of motion, namely

$$\begin{aligned} \bar{\nabla}^2\bar{\phi} &= 0 \\ \bar{R}_{\mu\nu} - \frac{1}{2}\bar{R}\bar{g}_{\mu\nu} - \lambda\bar{g}_{\mu\nu} - \frac{1}{2}\bar{\nabla}_\mu\bar{\phi}\bar{\nabla}_\nu\bar{\phi} + \frac{1}{4}\bar{g}_{\mu\nu}\bar{g}^{\alpha\beta}\bar{\nabla}_\alpha\bar{\phi}\bar{\nabla}_\beta\bar{\phi} &= 0 \end{aligned} \qquad (13.321)$$

Using the known expansion for the scalar curvature the quadratic order operator is

$$\begin{aligned} S_g = \frac{1}{2}\int d^n x\,\sqrt{\bar{g}}\,\Big[& h^{\alpha\beta}\Big(\frac{1}{4}\bar{g}_{\alpha\beta}\bar{g}_{\mu\nu}\bar{\nabla}^2 - \frac{1}{4}\bar{g}_{\alpha\mu}\bar{g}_{\beta\nu}\bar{\nabla}^2 + \frac{1}{2}\bar{g}_{\alpha\mu}\bar{\nabla}_\beta\bar{\nabla}_\nu - \frac{1}{2}\bar{g}_{\mu\nu}\bar{\nabla}_\alpha\bar{\nabla}_\beta \\ & + \frac{1}{2}\bar{g}_{\alpha\beta}\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\alpha\mu}\bar{R}_{\beta\nu} - \frac{1}{2}\bar{R}_{\alpha\mu\beta\nu} + \frac{1}{2}\bar{g}_{\alpha\mu}\partial_\beta\bar{\phi}\partial_\nu\bar{\phi} - \frac{1}{4}\bar{g}_{\alpha\beta}\partial_\mu\bar{\phi}\partial_\nu\bar{\phi} \\ & - \frac{1}{8}\left(\bar{R} + 2\lambda - \frac{1}{2}\bar{g}^{\rho\sigma}\partial_\rho\bar{\phi}\partial_\sigma\bar{\phi}\right)\left(\bar{g}_{\alpha\beta}\bar{g}_{\mu\nu} - 2\bar{g}_{\alpha\mu}\bar{g}_{\beta\nu}\right)\Big) h^{\mu\nu} + \\ & h^{\alpha\beta}\left(\frac{1}{2}\bar{g}_{\alpha\beta}\bar{g}^{\rho\sigma}\partial_\rho\bar{\phi}\partial_\sigma - \partial_\alpha\bar{\phi}\partial_\beta\right)\phi - \frac{1}{2}\phi\bar{\nabla}^2\phi\Big] \end{aligned} \qquad (13.322)$$

At this stage the operator is very cumbersome, but we still have the freedom to fix the gauge in a way that simplifies the computation. The gauge fixing term will be

$$S_{gf} = \frac{1}{2}\int d^n x\, \sqrt{\bar{g}}\, \frac{1}{2\xi}\, \bar{g}^{\mu\nu}\chi_\mu\chi_\nu \tag{13.323}$$

where the function characterizing the harmonic or de Donder gauge is

$$\chi_\nu = \bar{\nabla}^\mu h_{\mu\nu} - \frac{1}{2}\bar{\nabla}_\nu h - \phi\partial_\nu\bar{\phi} \tag{13.324}$$

After expanding can be expressed in the form

$$\begin{aligned} S_{gf} &= \frac{1}{2}\int d^n x\, \sqrt{\bar{g}}\, \frac{1}{2\xi}\, \Big[h^{\alpha\beta}\left(\bar{g}_{\mu\nu}\bar{\nabla}_\alpha\bar{\nabla}_\beta - \bar{g}_{\alpha\mu}\bar{\nabla}_\beta\bar{\nabla}_\nu - \frac{1}{4}\bar{g}_{\alpha\beta}\bar{g}_{\mu\nu}\bar{\nabla}^2\right)h^{\mu\nu} \\ &+2h^{\alpha\beta}\left(\partial_\alpha\bar{\phi}\partial_\beta + \bar{\nabla}_\alpha\bar{\nabla}_\beta\bar{\phi} - \frac{1}{2}\bar{g}_{\alpha\beta}\bar{g}^{\rho\sigma}\partial_\rho\bar{\phi}\partial_\sigma - \frac{1}{2}\bar{g}_{\alpha\beta}\bar{g}^{\rho\sigma}\bar{\nabla}_\rho\bar{\nabla}_\sigma\bar{\phi}\right)\phi + \\ &+\phi\left(\bar{g}^{\alpha\beta}\partial_\alpha\bar{\phi}\partial_\beta\bar{\phi}\right)\phi\Big] \end{aligned} \tag{13.325}$$

Let us define the following tensor with the desired symmetry properties, i.e., symmetric in $(\mu\nu)$, $(\alpha\beta)$ and under the interchange $(\mu\nu) \leftrightarrow (\alpha\beta)$

$$\begin{aligned} C_{\alpha\beta\mu\nu} &= \frac{1}{4}\left(\bar{g}_{\alpha\mu}\bar{g}_{\beta\nu} + \bar{g}_{\alpha\nu}\bar{g}_{\beta\mu} - \bar{g}_{\alpha\beta}\bar{g}_{\mu\nu}\right) \\ C^{\alpha\beta\mu\nu} &= \bar{g}^{\alpha\mu}\bar{g}^{\beta\nu} + \bar{g}^{\alpha\nu}\bar{g}^{\beta\mu} - \frac{2}{n-2}\bar{g}^{\alpha\beta}\bar{g}^{\mu\nu} \\ \delta^{\alpha\beta}_{\mu\nu} &= \delta^{(\alpha}_\mu\delta^{\beta)}_\nu \end{aligned} \tag{13.326}$$

the full action can then be written as a quadratic form in the quantum fields, $h_{\mu\nu}$ and ϕ.

$$S_g + S_{gf} = \frac{1}{2}\int d^n x\, \sqrt{\bar{g}}\, \frac{1}{2}\, \left[h^{\alpha\beta}M_{\alpha\beta\mu\nu}h^{\mu\nu} + h^{\alpha\beta}D_{\alpha\beta}\phi + \phi E_{\mu\nu}h^{\mu\nu} + \phi F\phi\right] \tag{13.327}$$

where the operators are

$$\begin{aligned} M_{\alpha\beta\mu\nu} &= C_{\alpha\beta\rho\sigma}\left(-\delta^{\rho\sigma}_{\mu\nu}\bar{\nabla}^2 + \frac{1-\xi}{\xi}\bar{g}_{\mu\nu}\bar{\nabla}^{(\rho}\bar{\nabla}^{\sigma)} + \frac{2(\xi-1)}{\xi}\delta^{(\rho}_{(\mu}\bar{\nabla}^{\sigma)}\bar{\nabla}_{\nu)} + P^{\rho\sigma}_{\mu\nu}\right) \\ P^{\rho\sigma}_{\mu\nu} &= -2\bar{R}^{(\rho}{}_\mu{}^{\sigma)}{}_\nu - 2\delta^{(\rho}_{(\mu}\bar{R}^{\sigma)}_{\nu)} + \left(\bar{R} + 2\lambda - \frac{1}{2}\bar{g}^{\alpha\beta}\partial_\alpha\bar{\phi}\partial_\beta\bar{\phi}\right)\delta^{\rho\sigma}_{\mu\nu} + \bar{g}^{\rho\sigma}\bar{R}_{\mu\nu} \\ &+\frac{2}{(n-2)}\bar{g}_{\mu\nu}\bar{R}^{\rho\sigma} - \frac{1}{(n-2)}\bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}\bar{R} + 2\delta^{(\rho}_\mu\partial_{\nu)}\bar{\phi}\partial^{\sigma)}\bar{\phi} - \frac{1}{2}\bar{g}_{\mu\nu}\partial^\rho\bar{\phi}\partial^\sigma\bar{\phi} - \\ &-\frac{1}{n-2}\bar{g}^{\rho\sigma}\partial_\mu\bar{\phi}\partial_\nu\bar{\phi} + \frac{1}{2(n-2)}\bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}\partial_\lambda\bar{\phi}\partial^\lambda\bar{\phi} \\ D_{\alpha\beta} &= \frac{2(1-\xi)}{\xi}C_{\alpha\beta\rho\sigma}\,\bar{\nabla}^\rho\bar{\phi}\bar{\nabla}^\sigma + \frac{\xi+1}{\xi}C_{\alpha\beta\rho\sigma}\bar{\nabla}^\rho\bar{\nabla}^\sigma\bar{\phi} \\ &E_{\mu\nu} \equiv \frac{2(\xi-1)}{\xi}C_{\mu\nu\rho\sigma}\bar{\nabla}^\rho\bar{\phi}\bar{\nabla}^\sigma + \frac{\xi+1}{\xi}C_{\mu\nu\rho\sigma}\bar{\nabla}^\rho\bar{\nabla}^\sigma\bar{\phi} \\ &F \equiv -\bar{\nabla}^2 + \frac{1}{\xi}\bar{g}^{\rho\sigma}\partial_\rho\bar{\phi}\partial_\sigma\bar{\phi} \end{aligned} \tag{13.328}$$

In terms of the combined field

$$\psi^A \equiv \begin{pmatrix} h^{\mu\nu} \\ \phi \end{pmatrix} \tag{13.329}$$

and in the Feynman gauge, corresponding to $\xi = 1$, the operator

$$S = \frac{1}{2} \int d^n x \sqrt{\bar{g}} \, \frac{1}{2} \psi^A \Delta_{AB} \psi^B \tag{13.330}$$

is minimal, in the sense that it takes a Laplacian form

$$\Delta_{AB} = -g_{AB} \bar{\nabla}^2 + Y_{AB} \tag{13.331}$$

with the metric

$$g_{AB} = \begin{pmatrix} C_{\alpha\beta\mu\nu} & 0 \\ 0 & 1 \end{pmatrix} \tag{13.332}$$

the inverse metric

$$g^{AB} = \begin{pmatrix} C^{\alpha\beta\mu\nu} & 0 \\ 0 & 1 \end{pmatrix} \tag{13.333}$$

and the term without derivatives

$$Y_{AB} = \begin{pmatrix} C_{\alpha\beta\rho\sigma} P^{\rho\sigma}_{\mu\nu} & 2C_{\alpha\beta\rho\sigma} \bar{\nabla}^\rho \bar{\nabla}^\sigma \bar{\phi} \\ 2C_{\mu\nu\rho\sigma} \bar{\nabla}^\rho \bar{\nabla}^\sigma \bar{\phi} & \bar{g}^{\rho\sigma} \bar{\nabla}_\rho \bar{\phi} \bar{\nabla}_\sigma \bar{\phi} \end{pmatrix} \tag{13.334}$$

The short time expansion coefficients can be found in the literature for such quadratic minimal operators [1], to wit

$$\begin{aligned} a_2 \;\; &= \;\; \frac{1}{(4\pi)^{\frac{n}{2}}} \frac{1}{360} \int d^n x \sqrt{\bar{g}} \; \mathrm{tr} \; \Big(180 Y^2 - 60 \bar{R} Y + 5 \bar{R}^2 - \\ & \quad -2 \bar{R}_{\mu\nu} \bar{R}^{\mu\nu} + 2 \bar{R}_{\mu\nu\rho\sigma} \bar{R}^{\mu\nu\rho\sigma} + 30 W_{\mu\nu} W^{\mu\nu} \Big) \end{aligned} \tag{13.335}$$

where the field strength is defined through

$$[\bar{\nabla}_\mu, \bar{\nabla}_\nu] \psi^A = W^A_{B\mu\nu} \psi^B \tag{13.336}$$

Therefore, in order to find the explicit value counterterm we will need a few traces

$$
\begin{aligned}
&\text{tr}\ \mathbb{I} = \delta^{\alpha\beta}_{\alpha\beta} + 1 = \frac{n(n+1)+2}{2}\\
&\text{tr}\ Y = g^{AB} Y_{AB} = \delta^{\mu\nu}_{\alpha\beta} P^{\alpha\beta}_{\mu\nu} + \bar{g}^{\rho\sigma}\partial_\rho\bar{\phi}\partial_\sigma\bar{\phi}\\
&= \frac{n(n+1)}{2}\left(\bar{R} + 2\lambda - \frac{1}{2}\bar{g}^{\rho\sigma}\partial_\rho\bar{\phi}\partial_\sigma\bar{\phi}\right) - n\bar{R} + (n-2)\bar{g}^{\rho\sigma}\partial_\rho\bar{\phi}\partial_\sigma\bar{\phi}\\
&\text{tr}\ Y^2 = Y_{AB}\, g^{BC}\, Y_{CD}\, g^{DA} = P^{\alpha\beta}_{\mu\nu} P^{\mu\nu}_{\alpha\beta} + 2D_{\alpha\beta}E_{\mu\nu}C^{\mu\nu\alpha\beta} + \left(\bar{g}^{\rho\sigma}\partial_\rho\bar{\phi}\partial_\sigma\bar{\phi}\right)^2\\
&= 3\bar{R}_{\mu\nu\rho\sigma}\bar{R}^{\mu\nu\rho\sigma} + \frac{n^2-8n+4}{n-2}\bar{R}_{\mu\nu}\bar{R}^{\mu\nu} + \frac{n+2}{n-2}\bar{R}^2 - 2n\bar{R}\left(\bar{R} + 2\lambda\right.\\
&\left.-\frac{1}{2}\bar{g}^{\rho\sigma}\partial_\rho\bar{\phi}\partial_\sigma\bar{\phi}\right) + \frac{n\,(n+1)}{2}\left(\bar{R} + 2\lambda - \frac{1}{2}\bar{g}^{\rho\sigma}\partial_\rho\bar{\phi}\partial_\sigma\bar{\phi}\right)^2 + 2\bar{\nabla}^2\bar{\phi}\bar{\nabla}^2\bar{\phi}\\
&+\frac{n^2-5}{n-2}\left(\bar{g}^{\rho\sigma}\partial_\rho\bar{\phi}\partial_\sigma\bar{\phi}\right)^2 + \frac{n(4-n)(3n-8)-4(n-2)^2}{(n-2)^2}\bar{R}^{\mu\nu}\partial_\mu\bar{\phi}\bar{\nabla}_\nu\bar{\phi} -\\
&-\frac{n^2+4n-16}{(n-2)^2}\bar{R}\bar{g}^{\rho\sigma}\partial_\rho\bar{\phi}\partial_\sigma\bar{\phi} + 2(n-1)\left(\bar{R} + 2\lambda - \frac{1}{2}\bar{g}^{\rho\sigma}\partial_\rho\bar{\phi}\partial_\sigma\bar{\phi}\right)\bar{g}^{\gamma\delta}\partial_\gamma\bar{\phi}\partial_\delta\bar{\phi} -\\
&-\frac{n^2+4n-16}{(n-2)^2}\bar{R}\bar{g}^{\rho\sigma}\partial_\rho\bar{\phi}\partial_\sigma\bar{\phi}\\
&\text{tr}\ W_{\mu\nu}W^{\mu\nu} = -(n+2)\bar{R}_{\mu\nu\rho\sigma}\bar{R}^{\mu\nu\rho\sigma}
\end{aligned}
\tag{13.337}
$$

Using the known expression (13.335) of the second heat kernel coefficient

$$
\begin{aligned}
&a_2 = \frac{1}{(4\pi)^{\frac{n}{2}}}\ \frac{1}{360}\int d^n x\ \sqrt{\bar{g}}\ \Big\{ \left(542 + n(n+1) - 30(n+2)\right)\bar{R}_{\mu\nu\rho\sigma}\bar{R}^{\mu\nu\rho\sigma}\\
&+\left[180\frac{n^2-8n+4}{n-2} - n(n+1) - 2\right]\bar{R}_{\mu\nu}\bar{R}^{\mu\nu} + \left[180\frac{n+2}{n-2} + 60n + \frac{5n(n+1)+10}{2}\right]\bar{R}^2\\
&-30n(n+13)\bar{R}\left(\bar{R} + 2\lambda - \frac{1}{2}\bar{g}^{\rho\sigma}\partial_\rho\bar{\phi}\partial_\sigma\bar{\phi}\right) + 90n(n+1)\left(\bar{R} + 2\lambda - \frac{1}{2}\bar{g}^{\rho\sigma}\partial_\rho\bar{\phi}\partial_\sigma\bar{\phi}\right)^2 +\\
&+180\frac{n(4-n)(3n-8)-4(n-2)^2}{(n-2)^2}\bar{R}^{\mu\nu}\partial_\mu\bar{\phi}\partial_\nu\bar{\phi} - 60\left[\frac{3(n^2+4n-16}{(n-2)^2} + n - 2\right]\bar{R}\bar{g}^{\rho\sigma}\partial_\rho\bar{\phi}\partial_\sigma\bar{\phi} -\\
&360(n+1)\left(\bar{R} + 2\lambda - \frac{1}{2}\bar{g}^{\rho\sigma}\partial_\rho\bar{\phi}\partial_\sigma\bar{\phi}\right)\bar{g}^{\gamma\delta}\partial_\gamma\bar{\phi}\partial_\delta\bar{\phi} + 180\frac{n^2+n-7}{n-2}\left(\bar{g}^{\rho\sigma}\partial_\rho\bar{\phi}\partial_\sigma\bar{\phi}\right)^2 +\\
&360\bar{\nabla}^2\bar{\phi}\bar{\nabla}^2\bar{\phi}\Big\}
\end{aligned}
\tag{13.338}
$$

This yields, for example in $n = 4$ dimensions

$$
\log\det\Delta = \frac{1}{(4\pi)^2}\left(\Lambda^2 + A_1\Lambda + A_2\log\frac{\Lambda}{\mu}\right) \tag{13.339}
$$

The contribution coming from ghost loops is also needed. The gauge fixing term mantains background invariance, under which the background

$\bar{g}_{\mu\nu}$ transforms as a metric and the fluctuation $h_{\mu\nu}$ as a tensor. On the other hand it has to break the quantum symmetry

$$\begin{aligned}
&\delta\bar{g}_{\mu\nu} = 0\\
&\delta h_{\mu\nu} = \frac{2}{\kappa}\bar{\nabla}_{(\mu}\xi_{\nu)} + \mathcal{L}_{\xi}h_{\mu\nu}\\
&\delta\bar{\phi} = 0\\
&\delta\phi = \frac{1}{\kappa}\xi^{\mu}\bar{\nabla}_{\mu}\left(\bar{\phi} + \kappa\phi\right)
\end{aligned} \tag{13.340}$$

The ghost Lagrangian is obtained performing a variation on the gauge fixing term

$$\delta\chi_{\nu} = \frac{1}{\kappa}\left(\bar{\nabla}^2\bar{g}_{\mu\nu} + \bar{R}_{\mu\nu} - \bar{\nabla}_{\mu}\bar{\phi}\bar{\nabla}_{\nu}\bar{\phi}\right)\xi^{\mu} \tag{13.341}$$

plus terms that give operators cubic in fluctuations and therefore are irrelevant at one loop (please remember that the ghosts being quantum fields they do not appear as external states). The ghost Lagrangian then reads

$$S_{gh} = \frac{1}{2}\int d^n x\,\sqrt{\bar{g}}\,\frac{1}{2}V^{*}_{\mu}\left(-\bar{\nabla}^2\bar{g}^{\mu\nu} - \bar{R}^{\mu\nu} + \bar{\nabla}^{\mu}\bar{\phi}\bar{\nabla}^{\nu}\bar{\phi}\right)V_{\nu} \tag{13.342}$$

The relevant ghostly traces are

$$\begin{aligned}
&\text{tr}\;\mathbb{I} = n\\
&\text{tr}\;Y = -\bar{R} + \bar{g}^{\rho\sigma}\partial_{\rho}\bar{\phi}\partial_{\sigma}\bar{\phi}\\
&\text{tr}\;Y^2 = \bar{R}_{\mu\nu}\bar{R}^{\mu\nu} - 2\bar{g}^{\mu\nu}\partial_{\mu}\bar{\phi}\partial_{\nu}\bar{\phi} + \left(\bar{g}^{\rho\sigma}\partial_{\rho}\bar{\phi}\partial_{\sigma}\bar{\phi}\right)^2\\
&\text{tr}\;W_{\mu\nu}W^{\mu\nu} = -\bar{R}_{\mu\nu\rho\sigma}\bar{R}^{\mu\nu\rho\sigma}
\end{aligned} \tag{13.343}$$

and the ghostly heat kernel coefficient

$$\begin{aligned}
a_a^{gh} \;=\; &\frac{1}{(4\pi)^{\frac{n}{2}}}\,\frac{1}{360}\int d^n x\,\sqrt{\bar{g}}\,\Big\{[2n-30]\,\bar{R}_{\mu\nu\rho\sigma}\bar{R}^{\mu\nu\rho\sigma} + [180-2n]\,\bar{R}_{\mu\nu}\bar{R}^{\mu\nu}\\
&+ [60+5n]\,\bar{R}^2 - 360\bar{R}^{\mu\nu}\partial_{\mu}\bar{\phi}\partial_{\nu}\bar{\phi} - 60\bar{R}\bar{g}^{\rho\sigma}\partial_{\rho}\bar{\phi}\partial_{\sigma}\bar{\phi} + 180\left(\bar{g}^{\rho\sigma}\partial_{\rho}\bar{\phi}\partial_{\sigma}\bar{\phi}\right)^2\Big\}
\end{aligned} \tag{13.344}$$

Adding the two pieces together and particularizing to the physical dimension $n = 4$ o the one-loop counterterm is obtained (please note the factor and the sign of the ghost contribution)

$$\begin{aligned}
\Delta S \;=\; &\frac{1}{\epsilon}\left(a_4 - 2a_4^{gh}\right) = \frac{1}{\epsilon}\,\frac{1}{(4\pi)^2}\int d^4 x\,\sqrt{\bar{g}}\,\Big\{\frac{71}{60}\bar{R}_{\mu\nu\rho\sigma}\bar{R}^{\mu\nu\rho\sigma} - \frac{241}{60}\bar{R}_{\mu\nu}\bar{R}^{\mu\nu} + \frac{15}{8}\bar{R}^2\\
&-\frac{17}{3}\bar{R}\left(\bar{R} + 2\lambda - \frac{1}{2}\bar{g}^{\rho\sigma}\partial_{\rho}\bar{\phi}\partial_{\sigma}\bar{\phi}\right) + 5\left(\bar{R} + 2\lambda - \frac{1}{2}\bar{g}^{\rho\sigma}\partial_{\rho}\bar{\phi}\partial_{\sigma}\bar{\phi}\right)^2 -\\
&-\frac{8}{3}\bar{R}\bar{g}^{\rho\sigma}\partial_{\rho}\bar{\phi}\partial_{\sigma}\bar{\phi} + 5\left(\bar{R} + 2\lambda - \frac{1}{2}\bar{g}^{\rho\sigma}\partial_{\rho}\bar{\phi}\partial_{\sigma}\bar{\phi}\right)\bar{g}^{\gamma\delta}\partial_{\gamma}\bar{\phi}\partial_{\delta}\bar{\phi} +\\
&+\frac{9}{4}\left(\bar{g}^{\rho\sigma}\partial_{\rho}\bar{\phi}\partial_{\sigma}\bar{\phi}\right)^2 + \bar{\nabla}^2\bar{\phi}\bar{\nabla}^2\bar{\phi}\Big\}
\end{aligned} \tag{13.345}$$

There is in the literature a completely equivalent computation by Barvinsky and collaborators expressed in a different gauge, but using the background equations of motion one can go from one to the other. This a consequence of the old theorem asserting that the pieces in the counterterm that do not vanish on-shell are gauge invariant (Kallosh).

Let us prove that the piece of L_∞ that vanishes on shell is irrelevant. What this means is that the path integral is invariant under field redefinitions

$$\phi'_i = f_i(\phi_j) \tag{13.346}$$

Now it must be the case that

$$L_\infty = \sum \frac{\delta S}{\delta \phi_i} F_i(\phi) \tag{13.347}$$

for some functions $F_i(\phi)$ depending on all the fields. This is the same as a field redefinition

$$\phi_i \rightarrow \phi_i + F_i(\phi) \tag{13.348}$$

In case that the cosmological constant vanishes the final result is

$$\begin{aligned}
\Delta S &= \frac{1}{\epsilon}\frac{1}{(4\pi)^2}\frac{1}{360}\int d^4x\,\sqrt{\bar{g}}\,\Big\{426\bar{R}_{\mu\nu\rho\sigma}\bar{R}^{\mu\nu\rho\sigma} - 1446\bar{R}_{\mu\nu}\bar{R}^{\mu\nu} + 435\bar{R}^2 + \\
&\quad +60\bar{R}\bar{g}^{\rho\sigma}\partial_\rho\bar{\phi}\partial_\sigma\bar{\phi} + 360\left(\bar{g}^{\rho\sigma}\partial_\rho\bar{\phi}\partial_\sigma\bar{\phi}\right)^2 + 360\bar{\nabla}^2\bar{\phi}\bar{\nabla}^2\bar{\phi} = \Big\} \\
&= \frac{1}{\epsilon}\frac{1}{(4\pi)^2}\int d^n x\,\sqrt{\bar{g}}\Big\{\frac{43}{60}\bar{R}_{\mu\nu}\bar{R}^{\mu\nu} + \frac{1}{40}\bar{R}^2 + \frac{1}{6}\bar{R}\bar{g}^{\rho\sigma}\partial_\rho\bar{\phi}\partial_\sigma\bar{\phi} + \left(\bar{g}^{\rho\sigma}\partial_\rho\bar{\phi}\partial_\sigma\bar{\phi}\right)^2 + \\
&\quad +\bar{\nabla}^2\bar{\phi}\bar{\nabla}^2\bar{\phi}\Big\}
\end{aligned} \tag{13.349}$$

which coincides with the result of 't Hooft and Veltman except for the last term. That term is however irrelevant in this case since it vanishes due to the background equations of motion. Using them the counterterm can be written in the form

$$\Delta S = \frac{1}{\epsilon}\frac{1}{(4\pi)^2}\int d^4x\,\sqrt{\bar{g}}\,\frac{203}{40}\,\bar{R}^2 \tag{13.350}$$

14

Windows onto the future.

15

Exercises.

15.1

Consider the totally antisymmetric Levi-Civita tensor in $\mathbb{R}^3$.

- Compute

$$\sum_{ijk} \epsilon_{ijk}\epsilon_{ijk} \tag{15.1}$$

$$\sum_{ij} \epsilon_{ijk}\epsilon_{ijl} \tag{15.2}$$

$$\sum_{i} \epsilon_{ijk}\epsilon_{ilm} \tag{15.3}$$

- Define, given a vector $v \in T(\mathbb{R}^3)$,

$$(\text{rot } v)_i \equiv \sum_{jk} \epsilon_{ijk}\partial_i v_k \tag{15.4}$$

$$\text{div } v \equiv \sum \partial_i v_i \tag{15.5}$$

  Compute

$$\text{div rot } v \tag{15.6}$$

- Define , for $v, w \in T(\mathbb{R}^3)$

$$(v \times w)_i \equiv \sum_{jk} \epsilon_{ijk}\ v_j w_k \tag{15.7}$$

  Compute the scalar product

$$(v_1 \times v_2)\,.\,(v_3 \times v_4) \tag{15.8}$$

- Compute

$$(v_1 \times v_2) \times (v_3 \times v_4) \tag{15.9}$$

  Compare with

$$v_1 \times (v_2 \times (v_3 \times v_4)) \tag{15.10}$$

15.2

Consider the totally antisymmetric Levi-Civita tensor in the Minkowski space M_4. We choose

$$\epsilon_{0123} \equiv +1 \tag{15.11}$$

($\epsilon^{\mu\nu\rho\sigma}$ is defined through Minkowski metric, $\eta^{\mu\nu}$)

- Compute

$$e^{\mu\nu\rho\sigma} \tag{15.12}$$

$$\epsilon^{\mu}{}_{\nu\rho\sigma} \tag{15.13}$$

- Compute

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\rho\sigma} \tag{15.14}$$

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\lambda\delta} \tag{15.15}$$

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\tau\lambda\delta} \tag{15.16}$$

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\pi\tau\lambda\delta} \tag{15.17}$$

- Define the Kronecker tensor

$$\epsilon^{\lambda_1\ldots\lambda_n}_{\mu_1\ldots\mu_n} \equiv p!\delta^{[\lambda_1}_{[\mu_1}\ldots\delta^{\lambda_n]}_{\mu_n]} \tag{15.18}$$

Compute

$$\epsilon^{\lambda_1\ldots\lambda_q}_{\mu_1\ldots\mu_q}\epsilon^{\mu_1\ldots\mu_q\sigma_1\ldots\sigma_p}_{\nu_1\ldots\nu_{p+1}\ldots\nu_{p+q}} \tag{15.19}$$

$$\epsilon^{\lambda_1\ldots\lambda_q\rho_1\ldots\rho_p}_{\mu_1\ldots\mu_q\rho_1\ldots\rho_p} \tag{15.20}$$

15.3

- Demostrar que el espacio bidimensional

$$ds^2 = dv^2 - v^2 du^2 \tag{15.21}$$

es realmente el espacio de Minkowski bidimensional, M_2 escrito en otras coordenadas (Milne) Demostrar que para una partícula libre p_u es constante, pero p_v no lo es.

- La métrica en la superficie terrestre viene dada por

$$ds^2 = a^2 \left(dl^2 + \cos^2 l \; dL^2 \right) \tag{15.22}$$

siendo L la latitud y l la longitud. La metrica de un mapa plano es, naturalmente,

$$ds^2 = dx^2 + dy^2 \tag{15.23}$$

Expresar la métrica de la superficie de la Tierra en las coordenadas (x, y) para la proyección ciulíndrica.

- Definamos la proyección de Mercator como aquella que hace correspoonder una línea recta en el mapa a una línea de demora constante. Demostrar que está definida por

$$\begin{aligned} x &= \phi \\ y &= \log \cot \frac{\theta}{2} \end{aligned} \tag{15.24}$$

siendo (θ, ϕ) coordenadas polares. Escribir la métrica de la superficie terrestre en las coordenadas (x, y). Demostrar que los círculos máximos están dados por la fórmula

$$\text{sh } y = \alpha \sin (x + \beta) \tag{15.25}$$

15.4

- Consider the following lagrangian

$$L = \frac{1}{2}(\partial_\mu A_\nu)(\partial^\nu A^\mu) - \frac{1}{2}(\partial_\mu A^\mu)^2 \tag{15.26}$$

- Is it Lorentz invariant?
- Is it gauge invariant?
- Compute the equations of motion.
- How are the results you have got mutually compatible?

15.5

Show that if $h^{(0)}_{\mu\nu}$ is an arbitrary configuration, there exists a Fierz-Pauli gauge parameter

$$\xi_\mu[h^{(0)}] \tag{15.27}$$

such that the gauge transformed field

$$h^{(0)}_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \tag{15.28}$$

obeys the harmonic (de Donder) gauge

$$\partial_\mu h^{\mu\nu} = \frac{1}{2}\partial^\nu h \tag{15.29}$$

15.6

Consider the interaction energy between two symmetric, conserved sources,

$$W \equiv T^{\mu\nu}_{(1)} T^{(2)}_{\mu\nu} - \xi T_{(1)} T_{(2)} \tag{15.30}$$

where ξ is an arbitrary parameter. Use conservation of the source in momentum space, in the Lorentz frame where

$$k^{\mu} = (E, 0, 0, k) \tag{15.31}$$

($E \equiv \sqrt{k^2 + m^2}$) to eliminate T_{0i} and T_{00} in terms of T_{ij}. Take now the massless limit of the expression above. Check that for $\xi = \frac{1}{2}$ (and only for this value), the energy can be written as

$$W = \frac{1}{2}\left(T^{11}_{(1)} - T^{22}_{(1)}\right)\left(T^{11}_{(2)} - T^{22}_{(2)}\right) + 2T^{12}_{(1)} T^{12}_{(2)} \tag{15.32}$$

which would mean that the corresponding particle carries helicity ± 2 only.(Zee)

15.7

Consider the flat metric in $\mathbb{R}^3$ in polar coordinates

$$ds^2 = dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right) \tag{15.33}$$

Write a natural dreibein in terms of differential forms and the corresponding dual basis of vectors in the tangent space. Compute the Hodge dual of this basis.

15.8

Do the same things for the Minkowski metric

$$ds^2 = dt^2 - dr^2 - r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right) \tag{15.34}$$

Beware of the signs!

15.9

- Consider the 1-form
$$A \equiv \frac{q}{r} dt \tag{15.35}$$
Compute
$$d * F \tag{15.36}$$

- Consider a different field strength such that now
$$G = \frac{q}{2} \sin\theta d\theta \wedge d\phi = \frac{q}{2r^3} \left(xdy \wedge dz + ydz \wedge dx + zdx \wedge dy \right) \tag{15.37}$$
What sort of field does it represent? Show that the gauge potential reads
$$A_+ \equiv \frac{1}{2r} \frac{1}{z+r} (xdy - ydx) = \frac{1 - \cos\theta}{2} d\phi \tag{15.38}$$
whenever
$$0 < \theta < \theta_1 \tag{15.39}$$
$$A_- \equiv \frac{1}{2r} \frac{1}{z-r} (xdy - ydx) = -\frac{1 + \cos\theta}{2} d\phi \tag{15.40}$$
whenever
$$\theta_0 < \theta < \pi \tag{15.41}$$
(It is assumed that
$$\theta_0 < \theta_1) \tag{15.42}$$

- Evaluate the 1-form J such that
$$* J \equiv d * G \tag{15.43}$$

- Compute the electric and magnetic fields corresponding to this field strength.

- Compute the fluxes
$$\int_{B(R)} d * F \tag{15.44}$$
$$\int_{B(R)} d * G \tag{15.45}$$
where $B(R)$ is the surface of a 2-sphere around the origin.

15.10

Define an adequate vierbein for the metric

$$ds^2 = \left(dt - A_i dx^i\right)^2 - a^2(t)\delta_{ij}dx^i dx^j \tag{15.46}$$

Using it, compute the spin connection and the two-form of curvature, (no need to go back to Christoffel symbols)

What is the condition for a new coordinate T to exist such that

$$dT = dt - A_i dx^i? \tag{15.47}$$

15.11

Same question for the metric

$$ds^2 = dt^2 - \left(dr + \beta\frac{dt}{r}\right)^2 - r^2 d\Omega_2^2 \tag{15.48}$$

15.12

- Compute the equations of motion for the general lagrangian quadratic in curvature

$$L = \alpha R_{\mu\nu} R^{\mu\nu} + \beta R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + \gamma R^2 \tag{15.49}$$

where $\alpha, \beta\gamma$ are dimensionless constants.

- A particle of negative gravitational mass $-|m|$ is released from rest at a distance $l >> M\kappa^2$ from another fixed particle of equal positive mass, $|m|$. As seen by a static FIDO observer, what is the magnitude and direction of the acceleration of each particle?

15.13

- Consider the Einstein-Rosen metric.

$$g = \begin{pmatrix} \left(\frac{1-\frac{r_S}{4\rho}}{1+\frac{r_S}{4\rho}}\right)^2 & 0 & 0 & 0 \\ 0 & -\left(1+\frac{r_S}{4\rho}\right)^2 & 0 & 0 \\ 0 & 0 & -\rho^2 & 0 \\ 0 & 0 & 0 & -\sin^2\theta \end{pmatrix} \tag{15.50}$$

By using an appropriate tetrad, compute Riemann's tensor.

- What is the physical meaning of it?

15.14

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15.27 Exam 2013. Einstein-Rosen.

- The original Einstein-Rosen metric is obtained from Schwarzschild by the change
$$u^2 \equiv r - r_S \tag{15.51}$$
What is the advantage of this change over the seemingly more natural one
$$u \equiv r - r_S? \tag{15.52}$$

- Answer. The metric
$$ds^2 = \frac{u}{r_S + u} dt^2 - \frac{u}{u + r_S}\ du^2 - (u + r_S)^2 d\Omega^2 \tag{15.53}$$
behaves badly when $u = 0$. For example
$$R_{tuut} = -\frac{r_S}{u(r_S + u)^2} \tag{15.54}$$
The Einstein-Rosen metric, instead
$$ds^2 = \frac{u^2}{r_S + u^2} dt^2 - 4(u^2 + r_S)\ du^2 - (u^2 + r_S)^2 d\Omega^2 \tag{15.55}$$
is well-behaved at $u = 0$. For example,
$$R_{tuut} = -\frac{4 r_S}{(r_S + u^2)^2} \tag{15.56}$$

- Compute the time to reach u=0 in the Einstein-Rosen space starting at rest at infinity trough a radial geodesic, measured by a FIDO at infinity and by a FREFO.

- . Answer.
$$L = \dot{t}^2 \frac{u^2}{u^2 + r_S} - 4\dot{u}^2 (u^2 + r_S) \tag{15.57}$$
$$E \equiv \dot{t} \frac{u^2}{u^2 + r_S} \tag{15.58}$$
Normalization:
$$L = 1 = \dot{t}^2 \frac{u^2}{u^2 + r_S} - 4\dot{u}^2 (u^2 + r_S) = \frac{E^2 (u^2 + r_S)^2}{u^4} \left(\frac{u^2}{u^2 + r_S} - 4 \left(\frac{du}{dt} \right)^2 (u^2 + r_S) \right) \tag{15.59}$$
When $u = \infty$
$$1 = E^2 \tag{15.60}$$

The FIDO time

$$dt = 2\left(u^2 + r_S\right)^{\frac{3}{2}} \frac{du}{ur_S} \tag{15.61}$$

The integrand diverges when $u = 0$. The FREFO time

$$d\tau = \frac{u^2}{u^2 + r_S} dt \tag{15.62}$$

is clearly finite.

- Define "radius" as

$$R \equiv 3\frac{Vol(S_2)}{A(S_2)} \tag{15.63}$$

where Vol is the volume of a two-dimensional sphere of constant u, and A is its surface. Compute the radius in terms of u.

- Answer.

$$A = 4\pi(u^2 + r_S)^2 \tag{15.64}$$

$$V = 4\pi \int_0^U du(u^2 + r_S)^{\frac{5}{2}} = \frac{\pi}{6}\left[U\sqrt{U^2 + r_S}\left(33r_S^2 + 26r_S U^2 + 8U^4\right) + 15r_S^3 \log\left(2\left(U + \sqrt{U^2 + r_S}\right)\right)\right] \tag{15.65}$$

Then

$$R = \frac{u\sqrt{U^2 + r_S}\,(33r_S^2 - 26r_S U^2 + 8U^4) + 15r_S^3 \log\left(2\left(U + \sqrt{U^2 + r_S}\right)\right)}{8(U^2 + r_S)^2} \tag{15.66}$$

15.28 Exam 2014. Gödel.

In 1949 Kurt Gödel discovered a solution of Einstein's equations that in some local coordinate patch reads

$$ds^2 = a^2\left(dt^2 - dx^2 + \frac{1}{2}e^{2x}dy^2 - dz^2 + 2e^2 dtdy\right) \tag{15.67}$$

where a is a constant. The nonvanishing Christoffels are easily calculated

$$\begin{aligned} \Gamma^0_{01} &= 1 \\ \Gamma^0_{12} &= \Gamma^1_{02} = \frac{1}{2}e^x \\ \Gamma^1_{22} &= \frac{1}{2}e^{2x} \\ \Gamma^2_{01} &= -e^{-x} \end{aligned} \tag{15.68}$$

- Consider the congruence defined by

$$u \equiv \frac{1}{a}\frac{\partial}{\partial t} \tag{15.69}$$

It is a fact of life that

$$R_{\mu\nu} = \frac{1}{a^2} u_\mu u_\nu \tag{15.70}$$

(Please notice the positioning of the indices). There are two ways to interpret this solution: either with vanishing cosmological constant and a perfect fluid source with a certain equation of state (what is it?) or else as a pressureless fluid with a cosmological constant (which is?)

- **Solution** Start from

$$R_{\alpha\beta} - \frac{1}{2}(R + 2\lambda)\, g_{\alpha\beta} = \kappa^2 T_{\alpha\beta} \tag{15.71}$$

which is the same as

$$R_{\mu\nu} = \frac{1}{a^2} u_\alpha u_\beta = \kappa^2 T_{\alpha\beta} - \left(2\lambda + \frac{1}{2}\kappa^2 T\right) g_{\alpha\beta} = \kappa^2 (\rho + p)\, u_\alpha u_\beta - \left(2\lambda + \frac{\kappa^2}{2}(\rho - p)\right) g_{\alpha\beta} \tag{15.72}$$

This means that

$$\begin{aligned} \rho + p &= \frac{1}{a^2} \\ 2\lambda + \frac{\kappa^2}{2}(\rho - p) &= 0 \end{aligned} \tag{15.73}$$

Either

$$\begin{aligned} \lambda &= 0 \\ \rho &= \frac{1}{2\kappa^2}\frac{1}{a^2} \\ p &= \rho \end{aligned} \tag{15.74}$$

or else

$$\begin{aligned} p &= 0 \\ \rho &= \frac{1}{a^2\kappa^2} \\ \lambda &= -\frac{\kappa^2}{4}\rho \end{aligned} \tag{15.75}$$

- The congruence as above. Is it geodesic? Is it expanding? Is it rotating? How does the expansion depend on time (Raychaudhuri)?

- **Solution** The optical scalars of the geodesic congruence

$$\begin{aligned}
&\dot{u} = 0\\
&\theta = 0\\
&\omega_{12} = \frac{1}{2} a\ e^x\\
&\omega_{\mu\nu}\omega^{\mu\nu} = \frac{1}{a^2}\\
&R_{\mu\nu}u^\mu u^\nu = \frac{1}{a^2}\\
&\dot{\theta} = 0
\end{aligned} \tag{15.76}$$

- In the same paper Gödel gives another coordinate patch

$$\begin{aligned}
&e^x = \cosh\, 2r + \cos\, \phi\, \sinh\, 2r\\
&ye^x = \sqrt{2}\, \sin\, \phi\, \sinh\, 2r\\
&\mathrm{tg}\, \left(\frac{\phi}{2} + \frac{t - 2\tau}{2\sqrt{2}}\right) = e^{-2r} \mathrm{tg}\, \frac{\phi}{2}\\
&z = 2\eta
\end{aligned} \tag{15.77}$$

In this patch the metric reads

$$ds^2 = 4a^2 \left(d\tau^2 - dr^2 - d\eta^2 + \left(\sinh^4\, r - \sinh^2\, r\right) d\phi^2 + 2\sqrt{2}\, \sinh^2\, r d\phi d\tau\right) \tag{15.78}$$

(no need to check this; trust Gödel).

- Consider the curve

$$\begin{aligned}
&r = R\\
&\eta = 0\\
&\tau = -\alpha\, \phi
\end{aligned} \tag{15.79}$$

$(0 \leq \phi \leq 2\pi)$. This curve. Is it timelike? Is it geodesic? Is it there some remarkable property after one turn on the axis?

- **Solution** It is timelike for R big enough, but not a geodesic. It is a closed timelike curve. After a full turn

$$\tau = -2\pi\alpha \tag{15.80}$$

we end up before we started, in spite of the fact that the curve is everywhere timelike.

By the way, essentially the *same* curve

$$\begin{aligned} r &= R \\ \theta &= \frac{\pi}{2} \\ \tau &= -\alpha\ \phi \end{aligned} \tag{15.81}$$

in Minkowski space is also timelike when $|\alpha| > R$

$$\dot{x}^2 = \left(\alpha^2 - R^2\right) \dot{\phi}^2 \tag{15.82}$$

What happens is that it is *past time oriented* with respect to

$$u \equiv (1, 0, 0, 0) \tag{15.83}$$

$$u.\dot{x} = -\alpha\dot{\phi} \tag{15.84}$$

In Gödel spacetime, however the CTC is not only timelike, but also *future oriented* for R big enough

$$u.\dot{x} = \left(2\sqrt{2}\sinh^2\ R - \alpha\right) \dot{\phi} \tag{15.85}$$

This is Gödel at his best.

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