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Lectures on Classical Mechanics.

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Abstract: Abstract...

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Contents

1 Newton's laws.

$$m\ddot{\vec{x}} = \vec{F} \tag{1.1}$$

Conservation of energy for conservative systems

$$T \equiv \frac{1}{2}m\dot{\vec{x}}^2 \tag{1.2}$$

$$\vec{F} = -\vec{\nabla}V \tag{1.3}$$

$$\frac{d}{dt}(T+V) = m\dot{\vec{x}}.\ddot{\vec{x}} + \dot{\vec{x}}.\vec{\nabla}V = \dot{\vec{x}}.\left(m\ddot{\vec{x}} - \vec{F}\right) = 0 \tag{1.4}$$

On the other hand

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}_i} = \frac{d}{dt}m\dot{x}_i = m\ddot{x}_i \tag{1.5}$$

$$\frac{\partial V}{\partial x_i} = -F_i \tag{1.6}$$

Taking into account that

$$\frac{\partial V}{\dot{x}_i} = 0 \tag{1.7}$$

Newton's law can be written

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = 0 \tag{1.8}$$

where the lagrangian has been defined

$$L \equiv T - V \tag{1.9}$$

(Please note the minus sign).

Let us now work out an example, namely the simple pendulum. This is the simp'lest example of an *harmonic oscillator*. The motion takes place in the (x,y) plane. Clearly there is only a degree of freedom, because

$$\begin{aligned} x &= l \sin\theta \\ y &= l \cos\theta \end{aligned} \tag{1.10}$$

Then

$$\begin{aligned} \dot{x} &= l \cos\theta\ \dot{\theta} \\ \dot{y} &= -l \sin\theta\ \dot{\theta} \end{aligned} \tag{1.11}$$

We shall call θ a *generalized coordinate.* We shall usally represent generalized coordinates by the letter q. The kinetic energy reads

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = \frac{1}{2}\ m\ l^2\dot{\theta}^2 \tag{1.12}$$

On the other hand

$$V = mgl\ \cos\theta \tag{1.13}$$

Newton's equations then read

$$m\ l^2\ \ddot{\theta} + mgl\ \sin\ \theta = 0 \tag{1.14}$$

For small oscillations (small angles) the ordinary differential equation (ODE) reads

$$l\ \ddot{\theta} + g\theta = 0 \tag{1.15}$$

The quotient

$$\omega^2 \equiv \frac{g}{l} \tag{1.16}$$

has dimension of $1/T^2$ (frequency) The solution of the ODE is

$$\theta(t) = C_1\ \cos\ \omega t + C_2\ \sin\ \omega\ t \tag{1.17}$$

We can impose initial (Cauchy) conditions

$$\begin{aligned} \theta_0 &\equiv \theta(t=0) = C_1 \\ \dot{\theta}_0 &\equiv \dot{\theta}(0) = C_2\omega \end{aligned} \tag{1.18}$$

The solution can then be written in more physical terms as

$$\theta(t) = \theta_0\ \cos\ \omega t + \frac{\dot{\theta}_0}{\omega}\ \sin\ \omega\ t \tag{1.19}$$

The solution can also be determined by boundary conditions.

$$\begin{aligned} \theta_0 &\equiv \theta(t=0) = C_1 \\ \theta_1 &\equiv \theta(t=T) = C_1\ \cos\ \omega T + C_2\ \sin\ \omega\ T \end{aligned} \tag{1.20}$$

namely

$$\theta(t) = \theta_0\ \cos\ \omega t + \frac{\theta_1 - \theta_0\ \cos\ \omega T}{\sin\ \omega T}\ \sin\ \omega\ t \tag{1.21}$$

When the frequency goes to zero, the trajectory reduces to a straight line as it should

$$x = \theta_0 + (\theta_1 - \theta_0)\,\frac{t}{T} \tag{1.22}$$

The integral of the lagrangian over time is called the *action*. It has dimensions of energy times time.

$$S[q(t) \equiv \int_{t_i}^{t_f} dt L(q, \dot{q}) \tag{1.23}$$

The action is a function of a space $\mathcal{F}$ of trajectories (to be specified precisely) into the field of real numbers $\mathbb{R}$.

$$S : q(t) \rightarrow S[q] \in \mathbb{R} \tag{1.24}$$

A particular $\mathcal{F}$ is defined as those functions $q(t)$ such that

$$\begin{aligned} q(t = t_i) &= q_i \\ q(t = t_f) &= q_f \end{aligned} \tag{1.25}$$

Given two such trajectories, $q_1 \in \mathcal{F}$ and $q_2 \in \mathcal{F}$, the difference

$$\delta q(t) \equiv q_1(t) - q_2(t) \tag{1.26}$$

obeys the boundary conditions

$$\delta q(t = t_i) = \delta q(t = t_f) = 0 \tag{1.27}$$

2 The Action Principle

We demand that the action is an extremun for all possible variations of the trajectory that do not change the boundary conditions. A neccessary condition for that is

$$\delta S = \int dt \left(\frac{\partial L}{\partial \dot{q}} \delta \dot{q} + \frac{\partial L}{\partial q} \delta q \right) = \int dt \left(-\frac{d}{dt} \left(-\frac{\partial L}{\partial \dot{q}} \right) + \frac{\partial L}{\partial q} \right) \delta q(t) = 0 \tag{2.1}$$

This is only true $\forall\ \delta q(t)$ if the Euler-Lagrange equations are satisfied.

$$-\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) + \frac{\partial L}{\partial q} = 0 \tag{2.2}$$

This generalizes in a trivial way for systems with many degrees of freedom, $q_a,\ a = 1 \ldots N$

$$-\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_a} \right) + \frac{\partial L}{\partial q_a} = 0 \tag{2.3}$$

Let us compute the value of the action for the harmonic oscillator, evaluated on the solution just found.

$$\begin{aligned} S &= \frac{ml^2}{2} \int_0^T dt \left(\dot{\theta}^2 - \theta^2 \right) = \frac{ml^2}{2} \int_0^T dt \left(\omega^2 \left(C_2^2 - C_1^2 \right) \cos 2\omega t - 2 C_1 C_2 \omega^2 \sin 2\omega t \right) = \\ &= \frac{ml^2}{2} \left(\frac{C_2^2 - C_1^2}{4} \omega \sin 2\omega T + \frac{C_1 C_2 \omega}{2} \left(\cos 2\omega T - 1 \right) \right) = \\ &= ml^2 \omega \frac{\left(\theta_1^2 + \theta_0^2 \right) \cos \omega T - 2\theta_0 \theta_1}{2 \sin \omega T} \end{aligned} \tag{2.4}$$

In the limit when the frequency $\omega \to 0$,

$$S = \frac{ml^2}{2T} \left(\theta_1 - \theta_0 \right)^2 \tag{2.5}$$

3 Noether's theorem.

When the lagrangian fails to depend explicitly on one coordinate, that is

$$\frac{\partial L}{\partial q_a} = 0 \tag{3.1}$$

we dubb this coordinate as *cyclic*, and Euler's equation immediately yield a conserved charge

$$\frac{\partial L}{\partial \dot{q}_a} = C \tag{3.2}$$

In some cases there are *symmetries* in the lagrangian. Those are specific transformations

$$\delta q_a \tag{3.3}$$

such that under those

$$\delta S = 0 \tag{3.4}$$

For example, consider

$$L = \frac{1}{2}\left(\sum_{i=1}^{3} m\left(\dot{q}^i\right)^2 - k^2\left(q^i\right)^2\right) \tag{3.5}$$

It is plain that under $\epsilon_{ij} = -\epsilon_{ji}$)

$$\delta q_i = \sum_{j=1}^{3} \epsilon_{ij} q^j \tag{3.6}$$

the variation of the lagrangian reads

$$\delta L = \frac{1}{2}\sum_i \left(\dot{q}_i \sum_j \epsilon_{ij}\dot{q}_j - k^2 q_i \sum_j \epsilon_{ij} q_j\right) = 0 \tag{3.7}$$

Because the trace of the product of a symmetric and a skew matrices vanishes.

$$\text{tr}\ (SA) = \text{tr}\ (SA)^T = \text{tr}\ \left(A^T S^T\right) = -\text{tr}\ (AS) = -\text{tr}\ (SA) = 0 \tag{3.8}$$

Please note that the transformation above does not vanbish at the boundary.

Let us now contiinue with the general argument. We know that the variation of the action vanishes. Then

$$0 = \delta S = \int_{t_i}^{t_f} dt \left(\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}\right)\delta q^i + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\delta q^i\right) \tag{3.9}$$

When on shell, then

$$0 = \delta S = \int_{t_i}^{t_f} dt \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\delta q^i\right) \tag{3.10}$$

This conveys the fact that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial q^i}\delta \dot{q}^i\right) = 0 \tag{3.11}$$

There is then a *first integral* or *constant of motion*

$$Q_N = \sum \frac{\partial L}{\partial \dot{q}^i}\delta q^i \tag{3.12}$$

$$\delta q^i = \epsilon T^i_j q^j \tag{3.13}$$

There is another proof, which is slightly more general. Perform a local transformation $\dot{\epsilon} \neq 0$. The action is not invariant, but the variation of the lagrangian must be proportional to $\dot{\epsilon}$ (because we know that it is actually invariant when $\dot{\epsilon} = 0$). Then

$$0 = \delta S = \int dt \dot{\epsilon} Q_N(q, \dot{q}) = \int dt \frac{d}{dt}\left(\epsilon Q_N(q, \dot{q})\right) - \epsilon \frac{d}{dt} Q_N(q, \dot{q}) \tag{3.14}$$

Now we choose the (until now arbitrary) function $\epsilon(t)$ such that

$$\epsilon(t_i) = \epsilon(t_f) = 0 \tag{3.15}$$

We deduce that there is a first integral of motion, namely

$$\frac{d}{dt} Q_N = 0 \tag{3.16}$$

Let us analyze the two-body problem from the viewpoint of the lagrangian approach.

$$L = \frac{1}{2} m_1 \dot{\vec{r}}_1^2 + \frac{1}{2} m_2 \dot{\vec{r}}_2^2 - V(|\vec{r}_1 - \vec{r}_2|) \tag{3.17}$$

Let us define the coordinates of the canter of mass (CDM):

$$(m_1 + m_2)\vec{R} \equiv m_1 \vec{r}_1 + m_2 \vec{r}_2 \tag{3.18}$$

as well as the relative distance between the two bodies:

$$\vec{r} \equiv \vec{r}_1 - \vec{r}_2 \tag{3.19}$$

It so happens that

$$\vec{r}_2 = \vec{R} - \frac{m_1}{M} \vec{r} \tag{3.20}$$

$$\vec{r}_1 = \vec{R} + \frac{m_2}{M} \vec{r} \tag{3.21}$$

and the lagrangian reads

$$L = \frac{1}{2} m_2 \left(\dot{\vec{R}} - \frac{m_1}{M} \dot{\vec{r}} \right)^2 + \frac{1}{2} m_1 \left(\dot{\vec{R}} + \frac{m_2}{M} \dot{\vec{r}} \right)^2 - V(r) = \frac{1}{2} M \left(\dot{\vec{R}} \right)^2 + \frac{1}{2} \frac{m_1 m_2}{M} \left(\dot{\vec{r}} \right)^2 - V(r) \tag{3.22}$$

The CDM motion decouples. Let us analyze the Noether charges.

$$Q_{ij} \equiv \sum_{ij} \dot{x}_i \epsilon_{ij} x_j \tag{3.23}$$

There are as many conserved charges as there are antysymmtric 3×3 matrices, that is, 3. This is nothing else that angular momentum conservation. The angular momentum (per unit mass) is defined as

$$\begin{aligned} J_x &\equiv y\dot{z} - z\dot{y} \\ J_y &\equiv z\dot{x} - x\dot{z} \\ J_z &\equiv x\dot{y} - y\dot{z} \end{aligned} \tag{3.24}$$

It is plain that it is conserved when the external forces are *central*, which means that they derive from a potential that only depends on r (that is, a spherically symmetric situation) Indeed

$$\frac{d}{dt} \vec{J} \equiv \frac{d}{dt} \left(\vec{r} \times \dot{\vec{r}} \right) = -\dot{r} \times \nabla V(r) = 0 \tag{3.25}$$

(because

$$\nabla V(r) = \frac{1}{r}\frac{\partial V(r)}{dr}\vec{r}) \tag{3.26}$$

This in turn implies that the motion is a planar one. We will assume later that this plane is just $z = 0$ or $\theta = \frac{\pi}{2}$ in polar coordinates).

This could be made explicit using polar coordinates

$$\begin{aligned} x &= r\,\sin\theta\,\cos\phi \\ y &= r\,\sin\theta\,\sin\phi \\ z &= r\,\cos\theta \end{aligned} \tag{3.27}$$

$$\begin{aligned} \dot{x} &= \dot{r}\,\sin\theta\,\cos\phi + r\,\dot{\theta}\,\cos\theta\,\cos\phi - r\,\dot{\phi}\,\sin\theta\,\sin\phi \\ \dot{y} &= \dot{r}\,\sin\theta\,\sin\phi + r\,\dot{\theta}\,\cos\theta\,\sin\phi + r\,\dot{\phi}\,\sin\theta\,\cos\phi \\ \dot{z} &= \dot{r}\,\cos\theta - r\,\dot{\theta}\,\sin\theta \end{aligned} \tag{3.28}$$

in such a way that

$$\left(\dot{\vec{r}}\right)^2 = \dot{r}^2 + r^2\,\dot{\theta}^2 + r^2\,\sin^2\theta\,\dot{\phi}^2 \tag{3.29}$$

It is plain that ϕ is cyclic, so that

$$\frac{\partial L}{\partial\dot{\phi}} = 2r^2\,\sin^2\theta\,\dot{\phi} = C \tag{3.30}$$

It is asy to check that this is again angular momentum in disguise. When $\theta = \frac{\pi}{2}$ the only non-vanishing component of the angular momentum is

$$J_z = r^2\dot{\phi} \tag{3.31}$$

Let us now check what happens if we intend to mimic this approach for the three-body problem. We try the same approcah that was so sucessful in the two-body system. define

$$M\vec{R} \equiv m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 \tag{3.32}$$

Then

$$\begin{aligned} &\vec{r}_{12} \equiv \vec{r}_1 - \vec{r}_2 \\ &\vec{r}_{13} \equiv \vec{r}_1 - \vec{r}_3 \\ &\vec{r}_1 = \vec{R} + \frac{m_2}{M}\vec{r}_{12} + \frac{m_3}{M}\vec{r}_{13} \\ &\vec{r}_2 = \vec{R} - \frac{m_1}{M}\vec{r}_{12} - \frac{m_3}{M}\vec{r}_{32} = \vec{R} - \frac{m_1 + m_3}{M}\vec{r}_{12} + \frac{m_3}{M}\vec{r}_{13} \\ &\vec{r}_3 = \vec{R} - \frac{m_1}{M}\vec{r}_{13} - \frac{m_2}{M}\vec{r}_{23} = \vec{R} + \frac{m_2}{M}\vec{r}_{12} - \frac{m_1 + m_2}{M}\vec{r}_{13} \end{aligned} \tag{3.33}$$

The COM still decouples

$$L = \frac{1}{2}M\dot{\vec{R}}^2 - \frac{m_2 m_3}{M}\dot{\vec{r}}_{12}\dot{\vec{r}}_{13} + \frac{m_2(m_1+m_3)}{2M}\dot{\vec{r}}_{12}^2 + \frac{m_3(m_1+m_2}{2M}\dot{\vec{r}}_{13}^2 - V(\vec{r}_{12}, \vec{r}_{13}) \tag{3.34}$$

In the system Sun, Moon, Earth the relevant masses are such that

$$\frac{m_2}{m_1} \sim 10^{-6}$$
$$\frac{m_3}{m_1} \sim 10^{-8} \tag{3.35}$$

It is clar that some expansion is called for.

Let us now briefly consider first order lagrangians. Condider, for example, the system with two degrees of freedom (q_1, q_2)

$$L = q_2 \dot{q}_1 - \frac{q_2^2}{2} - V(q_1) \tag{3.36}$$

The EM for the variable q_2 just tells us that

$$\dot{q}_1 - q_2 = 0 \tag{3.37}$$

Plugging this value of q_2 back in the original =lagrangian

$$L = \frac{\dot{q_1}^2}{2} - V(q_1) \tag{3.38}$$

All lagrangians can be put into first order form introducing new variables, if necessary.

Let us now check that the Euler-Lagrange are left invariant by *point transformations.* Those are arbitrary mappings

$$q_i \rightarrow q'_i(q_j) \tag{3.39}$$

such that the determinant of the jacobian matrix is nowhere vanishing

$$\det\left(\frac{\partial q'_i}{\partial q_j}\right) \neq 0 \tag{3.40}$$

There is then an induced mapping

$$\dot{q}'_i = \sum_k \frac{\partial q'_i}{\partial q_k}\dot{q}_k \tag{3.41}$$

It is fact that

$$\frac{\partial \dot{q}'_i}{\partial \dot{q}_j} = \frac{\partial q'_i}{\partial q_j} \tag{3.42}$$

as well as

$$\frac{\partial \dot{q}'_i}{\partial q_j} = \sum_{kl} \frac{\partial^2 q'_i}{\partial q_k \partial q_j}\dot{q}_k \tag{3.43}$$

Indeed

$$\frac{\partial L}{\partial q_i} = \sum_k \frac{\partial L}{\partial q'_k}\frac{\partial q'_k}{\partial q_i} + \sum_{kl} \frac{\partial L}{\partial \dot{q}'_k}\frac{\partial^2 q'_k}{\partial q_l \partial q_i}\dot{q}_l \tag{3.44}$$

$$\frac{\partial L}{\partial \dot{q}_i} = \sum_k \frac{\partial L}{\partial \dot{q}'_k}\frac{\partial \dot{q}'_k}{\partial \dot{q}_i} = \sum_k \frac{\partial L}{\partial \dot{q}'_k}\frac{\partial q'_k}{\partial q_i} \tag{3.45}$$

Noq

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \sum_k \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}'_k}\right)\frac{\partial q'_k}{\partial q_i} + \sum_{lk} \frac{\partial L}{\partial \dot{q}'_k}\frac{\partial^2 q'_k}{\partial q_i \partial q_l}\dot{q}_l \tag{3.46}$$

The EM in terms of the old coordinates read

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \sum_k \frac{\partial L}{\partial q'_k}\frac{\partial q'_k}{\partial q_i} + \sum_{kl} \frac{\partial L}{\partial \dot{q}'_k}\frac{\partial^2 q'_k}{\partial q_l \partial q_i}\dot{q}_l - \left(\sum_k \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}'_k}\right)\frac{\partial q'_k}{\partial q_i} + \sum_{lk} \frac{\partial L}{\partial \dot{q}'_k}\frac{\partial^2 q'_k}{\partial q_i \partial q_l}\dot{q}_l\right) =$$
$$= \sum_k \left(\frac{\partial L}{\partial q'_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}'_k}\right)\frac{\partial q'_k}{\partial q_i} \tag{3.47}$$

The vanishing of the EM in the unprime coordinate system is equivalent to the vanishing of the EM in trhe primed coordinate system, since the jacobian matrix is a nonsingular one.

Let us now check that two lagrangians which differ in a total derivative of a function of the coordinates yield the same EM. Indeed, all we have to show is that the contribution of

$$\frac{d}{dt}F(q) = \sum_k \frac{\partial F}{\partial q_k}\dot{q}_k \tag{3.48}$$

is trivial. Euler's tell us that

$$\sum_k \frac{\partial^2 F}{\partial q_k \partial q_i}\dot{q}_k = \frac{d}{dt}\left(\frac{\partial F}{\partial q_i}\right) \tag{3.49}$$

which holds identically.

4 The Legendre transform.

Let us consider a function of two sets of variables, denotes by x (*active*) and a (*passive*)-

$$f(x_1 \dots x_n; a_1 \dots a_n) \tag{4.1}$$

We shall define a mapping from the functional space of functions of $2n$ variables into itself.

$$f(x;a) \to g(y;a) \tag{4.2}$$

$g(y;a)$ is a *different* function that depends also on $2n$ variables, of which trhe first n are different $(x \to y)$, and the second n (a) are the same. This mapping is called *Legendre transform.*

We give a name to the derivarives with respect to the first set

$$y_i \equiv \frac{\partial f}{\partial x^i}(x,a) \tag{4.3}$$

This defines implicitly functions

$$x_i = f_i(y;a) \tag{4.4}$$

Using that information, define a *different* function

$$g(y,a) \equiv \sum_{i=1}^{n} x^i(y,a) y_i - f\left(x\left(y,a\right);a\right) \tag{4.5}$$

This dfinition implies

$$\frac{\partial g}{\partial y_i} = x^i + \sum_{k=1}^{n} y_k \frac{\partial x^k}{\partial y_i} - \sum_{l=1}^{n} \frac{\partial f}{\partial x^l}\frac{\partial x^l}{\partial y_i} = x^i \tag{4.6}$$

Derivatives with respect to the spectator variables change only sign.

$$\frac{\partial g}{\partial a^i} + \sum_k \frac{\partial g}{\partial y_k}\frac{\partial y_k}{\partial a^i} = \sum_l \left(x^l \frac{\partial y_l}{\partial a^i} + y_l \frac{\partial x^l}{\partial a^i}\right) - \sum_l \frac{\partial f}{\partial x^l}\frac{\partial x^l}{\partial a^i} - \frac{\partial f}{\partial a^i} \tag{4.7}$$

$$\frac{\partial g}{\partial a^i} = -\frac{\partial f}{\partial a^i} \tag{4.8}$$

Consider, for example a quadratic form

$$f(x,a) \equiv \sum_{i,j=1}^{n} A_{ij} x^i x^j - F(a) \tag{4.9}$$

Then

$$y_i = 2\sum_{j=1}^{n} A_{ij} x^j \tag{4.10}$$

Then

$$x^j = \frac{1}{2}\sum_{k=1}^{n} \left(A^{-1}\right)_{jk} y_k \tag{4.11}$$

When the matrix A has a vanishing determinant the Legendre transform is singular

The definition above tells us that

$$g(y,a) = \sum_{j=1}^{n} \frac{1}{2} y^j \sum_{k=1}^{n} \left(A^{-1}\right)_{jk} y_k - \frac{1}{4} \sum_{ij=1}^{n} A_{ij} \sum_{l=1}^{n} \left(A^{-1}\right)_{il} y^l \sum_{k=1}^{n} \left(A^{-1}\right)_{jk} y^k + F(a) =$$
$$= \frac{1}{4} \sum_{jk=1}^{n} y^j \left(A^{-1}\right)_{jk} y_k + F(a) \tag{4.12}$$

It is plain to check the general results in this example.

The mapping does not work for linear functions. Indeed

$$f(x,a) \equiv \sum_{i=1}^{n} C_i x_i - F(a) \tag{4.13}$$

Then

$$y_i = C_i \tag{4.14}$$

and we are unable to express x in terms of y.

There is a nice geometrical interpretation (Courant-Hilbert). Consider a surface

$$z = z(x,y) \tag{4.15}$$

Any displacement on the surface obeys

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy \tag{4.16}$$

Thois tells us that the normal vector is proportional to

$$\vec{n} \equiv \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1 \right) \tag{4.17}$$

so that the tangent plane at a point $(x_0, y_0, z_0 = z_0(x_0, y_0))$ is given by

$$(\vec{x} - \vec{x}_0)\, \vec{n} = 0 \Leftrightarrow \quad z - z_0 - (x - x_0) \left. \frac{\partial z}{\partial x} \right|_0 - (y - y_0) \left. \frac{\partial z}{\partial y} \right|_0 = 0 \tag{4.18}$$

For example, in the unit sphere

$$z = \sqrt{1 - x^2 - y^2} \tag{4.19}$$

the tangent plane is characterized by

$$z - z_0 + (x - x_0) \frac{x_0}{z_0} + (y - y_0) \frac{y_0}{z_0} = 0 \tag{4.20}$$

(this is just the equation

$$(\vec{x} - \vec{x}_0)\, \vec{x}_0 = 0 \Rightarrow \vec{x}.\vec{x}_0 = \vec{x}_0^2). \tag{4.21}$$

This last equation can be written in the form

$$(x, y, z).(\xi, \eta, 1) = \omega \tag{4.22}$$

just by defining

$$(\xi, \eta, 1) \equiv \left(-\left.\frac{\partial z}{\partial x}\right|_0, -\left.\frac{\partial z}{\partial y}\right|_0, 1) \right) \tag{4.23}$$

as well as

$$w \equiv z_0 - x_0 \left.\frac{\partial z}{\partial x}\right|_0 - y_0 \left.\frac{\partial z}{\partial y}\right|_0 = z_0 + x_0\xi + y_0\eta \tag{4.24}$$

In the same way that the family of tangent planes is fully characterized once the surface is given, the surface itself can also be characterized by the family the planes, that is, by the function

$$\omega = \omega(\xi, \eta) \equiv x_0\xi + y_0\eta + z_0 \tag{4.25}$$

Indeed

$$\frac{\partial \omega}{\partial \xi} = x_0 + \xi\frac{\partial x_0}{\partial \xi} + \frac{\partial y_0}{\partial \xi}\eta + \left.\frac{\partial z}{\partial x}\right|_0 \frac{\partial x_0}{\partial \xi} + \left.\frac{\partial z}{\partial y}\right|_0 \frac{\partial y_0}{\partial \xi} = x_0 \tag{4.26}$$

as well as

$$\frac{\partial \omega}{\partial \eta} = \xi\frac{\partial x_0}{\partial \eta} + y_0 - \eta\frac{\partial y_0}{\partial \eta} + \left.\frac{\partial z}{\partial x}\right|_0 \frac{\partial x_0}{\partial \eta} + \left.\frac{\partial z}{\partial y}\right|_0 \frac{\partial y_0}{\partial \eta} = y_0 \tag{4.27}$$

The dual character of the relationship is embodied in the formulas

$$\begin{aligned} &\omega(\xi, \eta) + z(x, y) = x\xi + y\eta \\ &\xi = -\frac{\partial z}{\partial x} \quad \eta = -\frac{\partial z}{\partial y} \\ &x_0 = \frac{\partial \omega}{\partial \xi} \quad y_0 = \frac{\partial \omega}{\partial \eta} \end{aligned} \tag{4.28}$$

Let is apply the Legendre transform to the lagrangian. The correspondence is

$$\begin{aligned} &L(\dot{q}, , q) \leftrightarrow f((x, a) \\ &\dot{q} \leftrightarrow x \\ &q \leftrightarrow a \\ &y \leftrightarrow p \end{aligned} \tag{4.29}$$

That is

$$y \leftrightarrow p_i = \frac{\partial L}{\partial \dot{q}^i} \tag{4.30}$$

The Legendre transform of the Lagrangian is called the Hamiltonian:

$$g \leftrightarrow H(p, q) = \sum_i p_i \dot{q}^i - L \tag{4.31}$$

The general results tell us that

$$\frac{\partial H}{\partial p_i} = \dot{q}^i \tag{4.32}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q^i} \tag{4.33}$$

It is customary to denote the n-dimensional space of the q_i *configuration space*, and the 2n-dimensional space of the (p_i, q^j) *phase space*.

Let us now compute the Hamiltonian for a particle in a central potential

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\,\dot{\theta}^2 + r^2\,\sin^2\theta\,\dot{\phi}^2\right) - V(r) \tag{4.34}$$

$$\begin{aligned} p_r &\equiv \frac{\partial L}{\partial \dot{r}} = m\dot{r} \\ p_\theta &= mr^2\dot{\theta} \\ p_\phi &= mr^2 \sin^2\theta\dot{\phi} \end{aligned} \tag{4.35}$$

$$H = \frac{p_r^2}{m} + \frac{p_\theta^2}{mr^2} + \frac{p_\phi^2}{mr^2\sin^2\theta} - L = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2\sin^2\theta} + V(r) \tag{4.36}$$

Let is define the Poisson brackets

$$\{f,g\} \equiv \sum_a \left(\frac{\partial f}{\partial q_a}\frac{\partial g}{\partial p_a} - \frac{\partial g}{\partial q_a}\frac{\partial f}{\partial p_a}\right) \tag{4.37}$$

Facts of life

$$\begin{aligned} &\{q^i, p_j\} = \delta^i_j \\ &\{f, g = -\{g, f\} \\ &\{Cf, g\} = C\{f, g\} \\ &\{fg, h\} = f\{g, h\} + \{f, h\}g \\ &\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0 \end{aligned} \tag{4.38}$$

The last identity is called the *Jacobi identity* and its verification needs a little bit of labor, quite convenient to strentghen character.

Hamilton's equations can now be written as

$$\begin{aligned} \dot{q}^a &= \{q^a, H\} \\ \dot{p}_i &= \{p_i, H\} \end{aligned} \tag{4.39}$$

It is plain that this makes evident the fact that

$$\frac{dH}{dt} = \sum_a \left(\frac{\partial H}{\partial q_a}\{q_a, H\} + \frac{\partial H}{\partial p_a}\{p_a, H\}\right) = \{H, H\} = 0 \tag{4.40}$$

The hamiltonian is a first integral of the EM, and its numerical value is the energy of the system.

We can then refine our definition of first integral as those functions that commute with the Hamiltonian

$$\frac{dI(p,q)}{dt} \equiv \sum_i \frac{\partial f}{\partial q_i}\{q_i, H\} + \frac{\partial f}{\partial p_i}\{p_i, H\} \equiv \{f, H\} = 0 \tag{4.41}$$

For example, the bracket

$$\{p_\phi . H\} = 0 \tag{4.42}$$

tells us that

$$\dot{p}_\phi = C \tag{4.43}$$

It is easy to check (using Jacobi's identity) that the bracket of two first integrals is another first integral

Indeed if

$$I_3 \equiv \{I_1, I_2\} \tag{4.44}$$

then

$$\{I_3, H\} = \{\{I_1, I_2\}, H\} = -\{\{I_2, H\}, I_1\} - \{\{H, I_1\}, I_2\} = 0 \tag{4.45}$$

That ism the set of first integrals is closed upon Poisson brackets.

5 Hamilton's equations from a variational principle

Let us consider the following action principle (Hamilton's).

It seems perverse to step back and consider now a Lagrangian defined *a posteriori* from the knowledge of the Hamiltonian, and consider besides both the coordinates as well as the momenta as new coordinates of another system with $2n$ degrees of freedom. Nevertheless, it is worth the effort. Let us then consider a system with $2n$ coordinates

$$Q \equiv \left(p_1 \ldots p_n, q^1 \ldots q^n\right) \tag{5.1}$$

$$S = \int \; dt \; L(Q, \dot{Q}) \equiv \int dt \; \left(\sum_i p_i \dot{q}^i - H(p,q)\right) \tag{5.2}$$

It is worth noticing that, by integrating by parts, ans in obvious matrix notation,

$$\int dt \sum_i p_i \dot{q}^i = \frac{1}{2} \int dt \begin{pmatrix} p & q \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} \tag{5.3}$$

The Euler-Lagrange equations read

$$\frac{\delta S}{\delta Q^i} \equiv \frac{\partial L}{\partial Q^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{Q}^i} = 0 \tag{5.4}$$

In gory detail the first n equations of the set read

$$\frac{\delta S}{\delta p_k} = \dot{q}^k - \frac{\partial H}{\partial p_k} = 0 \tag{5.5}$$

and the second half of the equations are

$$\frac{\delta S}{\delta q^k} = -\frac{\partial H}{\partial q^k} - \frac{d}{dt} \; p_k = 0 \tag{5.6}$$

It is then a fact that the Euler-Lagrange equations for this Lagrangian system with $2n$ coordinates (p, q) reproduce Hamilton's equations for the original dynamical system with n degrees of freedom.

6 Canonical transformations

We shall define canonical transformations as those transformations of the phase space $(p, q) \rightarrow (P, Q)$ (nothing to do with the preceding paragraph, now there are $2n$ (p, q) coordinates of the phase space and also $2n$ (P, Q) new coordinates in the same phase space; all we are doing now is a change of coordinates with certain convenient properties) that leave invariant hamilton'saction principle. This means that

$$\sum p_i \left(\dot{q}^i - H(p,q)\right) dt = \sum_j \left(P_j \dot{Q}^j - K(P,Q)\right) dT + dF \tag{6.1}$$

There will be now a new hamiltonian $K(P, Q)$ as well as a new time T. Hamilton's principle applied with the new coordinates implies

$$\begin{aligned} \frac{dP}{dT} &= -\frac{\partial K}{\partial Q} \\ \frac{dQ}{dT} &= \frac{\partial K}{\partial P} \end{aligned} \tag{6.2}$$

Let us call, following Gantmacher *free* canonical transformations those for which the n by n matrix

$$\frac{\partial Q^i}{\partial q^j} \tag{6.3}$$

is nonsingular. This means that both sets of coordinates can be taken as independent variables. It is then possible to choose the *generating function* as

$$F(t, q, Q) \tag{6.4}$$

It is then plain that

$$\begin{aligned} \frac{\partial F}{\partial q^i} &= p_i \\ \frac{\partial F}{\partial Q^i} &= -P^i \\ K &= H + \frac{\partial F}{\partial t} \end{aligned} \tag{6.5}$$

As a particular example of free canonical transformation, it is possible to interchange coordinates and momenta. The corresponding generating function reads

$$F \equiv \sum_{i=1}^{n} q^i Q^i \tag{6.6}$$

The formulas tell us that

$$\begin{aligned} p_i &= Q^i \\ P_i &= -q^i \end{aligned} \tag{6.7}$$

Let us consider the example of the harmonic oscillator

$$H \equiv \frac{p^2}{2m} + \frac{m\omega^2}{2} q^2 \tag{6.8}$$

after the canonical transformation,

$$K = \frac{Q^2}{2m} + \frac{m\omega^2}{2} P^2 \tag{6.9}$$

The new Hamilton's equations are

$$\begin{aligned} \dot{P} &= -\frac{Q}{m} \\ \dot{Q} &= m\omega^2 P \end{aligned} \tag{6.10}$$

Then

$$\ddot{Q} = m\omega^2 \left(-\frac{Q}{m} \right) = \omega^2 Q \tag{6.11}$$

In an analogous way it is possible to derive explicit finctions in cases where the independent variables is an arbitary 2n dimensional subset of the 4n-dimensional set $(p, q; P, Q)$. Let us work out for example the case when the independent variables are (p, Q). Then we can write

$$\sum_i p_i dq^i - H dt = \sum_i P_i dQ^i - K dt + \sum_i F_{p_i} dp^i + \sum_j F_{Q^j} dQ^j \tag{6.12}$$

It is plain that the first member can be rewritten as

$$d \sum_i p_i q^i - \sum q^i dp_i - H dt \tag{6.13}$$

which conveys the fact that

$$\begin{aligned} P_i &= -\frac{\partial F}{\partial Q^i} \\ q^i &= -\frac{\partial F}{\partial p_i} \\ K &= H - \frac{\partial F}{\partial t} \end{aligned} \tag{6.14}$$

The identity transformation can be expressed in this form; it corresponds to a generating function

$$F \equiv -\sum_i p_i Q^i \tag{6.15}$$

It is quite interesting to study in detail canonical transformations close to the identity. In order to do that, let us assume that the generating function is such that

$$F(p, Q) = -\sum_i p_i Q^i + \epsilon f(p, Q) \tag{6.16}$$

and we agree to work to first order in ϵ. Then

$$P_i = p_i - \epsilon \frac{\partial f}{Q^i} = p_i - \epsilon \{f, p_i\}$$
$$q^i = Q^i - \epsilon \frac{\partial f}{\partial p_i} = Q^i + \epsilon \{f, q^i\} \tag{6.17}$$

Me must realize thet the quantities q^i and Q^i are quite close; they differ in order ϵ. In any expression which is already of order ϵ they are indistinguishable (the contribution of the difference would appear at order ϵ^2 only). For example, the generating function can be written as $f(p, Q)$ or else as $f(p, q)$; both expressions yield the same results to first order. Actually

$$\delta p_i \equiv P_i - p_i = -\epsilon \{f, p_i\}$$
$$\delta q^i = Q^i - q^i = -\epsilon \{f, q^i\} \tag{6.18}$$

7 The Hamilton-Jacobi equation.

Let us perform a general variation of the action. We shall give some intuitive arguments first and only then the real computation. First imagine that we perform a variation such that the endpoint is not fixed. Then the boundary term does contribute, and we get

$$\delta S = \sum_i p_i \delta q^i \tag{7.1}$$

Then

$$\frac{\partial S}{\partial q^i} = p_i \tag{7.2}$$

Now assume that the endpoints are fixed, but that the time it takes to reach the final position is not fixed. Now the total derivative

$$\frac{dS}{dt} = L \tag{7.3}$$

On the other hand,

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \sum \frac{\partial S}{\partial q^i} \dot{q}^i = \frac{\partial S}{\partial t} + \sum_i p_i \dot{q}^i \tag{7.4}$$

It follows that

$$\frac{\partial S}{\partial t} = L - \sum_i p_i \dot{q}^i = -H \tag{7.5}$$

Then

$$\frac{\partial S}{\partial t} = -H \tag{7.6}$$

We conclude that

$$\delta S = \sum_i p_i \delta q^i - H \delta t \tag{7.7}$$

Now for the hard proof.

$$\delta S = \int_{t_i+\delta t_i}^{t_f+\delta t_f} dt\ L\left(q+\delta q, \dot{q}+\delta\dot{q}\right) - \int_{t_i}^{t_f} dt\ L(q,\dot{q}) =$$
$$= \left(\int_{t_i}^{t_f} - \int_{t_i}^{t_i+\delta t_i} + \int_{t_f}^{t_f+\delta t_f}\right) dt\ L\left(q+\delta q, \dot{q}+\delta\dot{q}\right) - \int_{t_i}^{t_f} dt\ L(q,\dot{q}) \tag{7.8}$$

This yields

$$\delta S = \int_i^f dt\ \frac{\delta S}{\delta q}\delta q + \sum p_i \delta^* q^i\Big|_i^f - L_i\delta t_i + L_f \delta t_f \tag{7.9}$$

We have now to realize that δq is not really well defined at the boundary. This is the reson why we have written a symbol $\delta^* q$. The trajectory $q+\delta q$ is not defined before $t_i+\delta t_i$ and the trajectory q is not defined after t_f. The best wat to understand this is to consider a family of trajectories

$$q = q(t,\lambda) \tag{7.10}$$

such that the boundary points also depend on the parameter:

$$\begin{aligned} t_i = t_i(\lambda) \quad q_i = q_i(t_i(\lambda),\lambda) \\ t_f = t_f(\lambda) \quad q_f \equiv t(t_f(\lambda),\lambda) \end{aligned} \tag{7.11}$$

In that way is clear that the total, variation, for example at the initial point, is given by

$$\delta q|_i \equiv \frac{dq(t_i(\lambda),\lambda)}{d\lambda}\Big|_i = \dot{q}\delta t_i + \frac{\partial q(t,\lambda)}{d\lambda}\Big|_i \equiv \dot{q}\delta t|_i + \delta^* q|_i \tag{7.12}$$

Clearly

$$\delta^* q \equiv \frac{\partial q(t,\lambda)}{d\lambda}\Big|_i \tag{7.13}$$

then

$$\delta^* q|_i = \delta q\big|_i - \dot{q}\big|_i \delta t_i \tag{7.14}$$

We then reach the conclusion that

$$\delta S = \int_i^f \sum_i \frac{\delta S}{\delta q^i}\delta q^i dt + \left(\sum_i p_i \delta q^i - H\delta t\right)\Bigg|_i^f \tag{7.15}$$

The Hemilton-Jacobi equation tells us that

$$\frac{\partial S}{\partial t} + H\left(\frac{\partial S}{\partial q}, q, t\right) = 0 \tag{7.16}$$

This is a nonlinear PDE for the function $S(q,t)$ which depends on the $n+1$ variables (q^i,t). Given any solution,

$$S(q,\alpha,t) \quad \alpha = \alpha_1 \ldots \alpha_m \tag{7.17}$$

then all the derivatives

$$I_k \equiv \frac{\partial S}{\partial \alpha_k} \tag{7.18}$$

are first integrals of the equations of motion.

Indeed HJ implies that

$$\frac{\partial^2 S}{\partial \alpha_k \partial t} + \sum_l \frac{\partial H}{\partial p_l} \frac{\partial^2 S}{\partial q_l \partial \alpha_k} = 0 \tag{7.19}$$

so that

$$\frac{dI_m}{dt} = \sum_j \frac{\partial^2 S}{\partial q^j \partial \alpha_k} \frac{\partial H}{\partial p_j} - \sum_j \frac{\partial^2 S}{\partial \alpha_k \partial p_j} \frac{\partial H}{\partial q_j} + \frac{\partial^2 S}{\partial \alpha_k \partial t} = 0 \tag{7.20}$$

(because $\frac{\partial^2 S}{\partial \alpha_k \partial p_j} = 0$).

There is another theorem by Jacobi that guarantees that if we find a solution of the HJ equation depending on precisely n-parameters,$S\left(t, q^i, \alpha_1, \ldots, \alpha_n\right)$ with

$$\det \frac{\partial^2 S}{\partial q^i \partial \alpha_j} \neq 0 \tag{7.21}$$

then

$$\begin{aligned} p_i &= \frac{\partial S}{\partial q^i} \\ \beta_i &= \frac{\partial S}{\partial \alpha_i} \end{aligned} \tag{7.22}$$

yields the full solution of the equations of motion. Indeed

$$0 = \frac{d}{dt} \frac{\partial S}{\partial \alpha_k} = \frac{\partial^2 S}{\partial \alpha_k \partial t} + \sum \frac{\partial^2 S}{\partial \alpha_k \partial q_l} \dot{q}_l = \sum \frac{\partial^2 S}{\partial q_i \partial \alpha_k} \left(\frac{\partial H}{\partial p_i} - \dot{q}_i \right) \tag{7.23}$$

and the first set of Hamilton's equations follow from the hypothesis.

Likewise, from the definition itself og the p_i

$$\dot{p}_i = \left(\frac{\partial}{\partial t} + \sum_j \dot{q}_l \frac{\partial}{\partial q_j} \right) \frac{\partial S}{\partial q_i} \tag{7.24}$$

But HJ implies

$$\frac{\partial^2 S}{\partial t \partial q_i} + \frac{\partial H}{\partial q_i} + \sum_j \frac{\partial H}{\partial p_j} \frac{\partial^2 S}{\partial q_j \partial q_i} = 0 \tag{7.25}$$

It follows

$$\dot{p}_i = -\left(\frac{\partial H}{\partial q_i} + \sum_j \frac{\partial H}{\partial p_j} \frac{\partial^2 S}{\partial q_j \partial q_i} \right) + \sum_j \dot{q}_l \frac{\partial^2 S}{\partial q \partial q_j} \tag{7.26}$$

The first set of Hamilton's equations implies that the second set of Hamilton's equations follow

$$\dot{p}_i + \frac{\partial H}{\partial q_i} = 0 \tag{7.27}$$

In the particular case when some coordinate, say q_1 and the corresponding derivative $\frac{\partial S}{\partial q_1}$ appear only in the combination

$$f\left(q_1, \frac{\partial S}{\partial q_1}\right) \tag{7.28}$$

(we shall say that this variable is *separable*) then, here are the solutions of the form

$$S = S_1(q_1) + S(q_2 \ldots q_n) \tag{7.29}$$

where the PDE for the function $S(q_2 \ldots q_n)$ sis obtained through

$$f\left(q_1, \frac{\partial S}{\partial q_1}\right) = \text{constant} \tag{7.30}$$

When the hamiltonian does not depend on time, it is possible to write

$$S(q^i, t) = W(q^i) - Et \tag{7.31}$$

where *Hamilton's characteristic function*, W obeys

$$H\left(q, \frac{\partial W}{\partial q}\right) = E \tag{7.32}$$

For example, for an harmonic oscillator

$$\frac{1}{2m}\left(\frac{\partial W}{\partial q}\right)^2 + m\frac{\omega^2}{2}q^2 = E \tag{7.33}$$

which leads to

$$\begin{aligned}
W &= \int dq\sqrt{2mE - m^2\omega^2 q^2} = \sqrt{2mE}\int dq\sqrt{1 - \frac{m\omega^2}{2E}\,q^2} = \\
&= \sqrt{2mE}\sqrt{\frac{2E}{m}\frac{1}{\omega}}\int dx\sqrt{1-x^2} = \quad (x = \sin\theta) \quad = \\
&= \frac{2E}{\omega}\int \cos^2\theta\; d\theta = \frac{2E}{\omega}\int \frac{1+\cos 2\theta}{2}\; d\theta = \\
&= \frac{2E}{\omega}\frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) = \frac{E}{\omega}\left(\sin^{-1}\left(\sqrt{\frac{m}{2E}}\omega q\right) + \sqrt{\frac{m}{2E}}\omega q\sqrt{1 - \frac{m\omega^2 q^2}{2E}}\right)
\end{aligned} \tag{7.34}$$

and the full integral of the system is found through

$$\frac{\partial S}{\partial E} = \frac{\partial W}{\partial E} - t = \beta \tag{7.35}$$

which, lo and behold yields

$$q = \sqrt{\frac{2E}{m\omega^2}}\,\sin(t+\beta) \tag{7.36}$$

We did all this in a clumsy way on purpose as an exercise. It would have been much clevered to write directly

$$\beta+t=\int dq\;\frac{m}{\sqrt{2mE-m^2\omega^2q^2}}=\frac{m}{\sqrt{2mE}}\int dq\;\frac{1}{\sqrt{1-\frac{m\omega^2q^2}{2E}}}=\qquad\left(x\equiv\sqrt{\frac{m}{2E}}\;\omega\;q\right)$$

$$=\frac{1}{\omega}\int dx\;\frac{1}{\sqrt{1-x^2}}=\frac{1}{\omega}\text{ arc sin }x=\frac{1}{\omega}\text{ arc sin }\sqrt{\frac{m}{2E}}\;\omega\;q \tag{7.37}$$

Then

$$q=\frac{1}{\omega}\sqrt{\frac{2E}{m}}\;\sin\,\omega(t+\beta) \tag{7.38}$$

A system is completely separable when it is possible to write Hamilton's function as

$$W=\sum_{i=1}^{n}W_i\left(q^i\right) \tag{7.39}$$

Let us now study a central potential (in the plane $\theta=\frac{\pi}{2}$) from this viewpoint

$$H=\frac{p_r^2}{2m}+\frac{p_\theta^2}{2mr^2}+V(r) \tag{7.40}$$

(where we have included the angular momentum contribution

$$\frac{J^2}{r^2} \tag{7.41}$$

in the radial potential.

Let us separate variables:

$$S=W_r(r)+W_\theta(\theta)-Et \tag{7.42}$$

Then

$$\frac{1}{2m}\left(\frac{\partial W_r}{\partial r}\right)^2+\frac{1}{2mr^2}\left(\frac{\partial W_\theta}{\partial\theta}\right)^2+V(r)=E \tag{7.43}$$

It is plain that

$$W_\theta=Q\;\theta \tag{7.44}$$

as well as

$$W_r=\int^r\sqrt{2m\left(E-V(x)\right)-\frac{Q^2}{x^2}} \tag{7.45}$$

The general trajectory can now be easily recovered.

Let us study now a somewhat nontrivial example, the planar *problem of two suns*, a restricted planar three body problem. This was first solved by Euler [9]

Let us call $2a$ the distance between the body of mass M placed at $(-a,0)$ and the second body pf equal mass placed at M en $(a,0)$. Call r_1 and r_2 the third body coordinates assumed of mass m with respect to the two other bodies.

$$L=\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right)-G\frac{Mm}{r_1}-G\frac{Mm}{r_2} \tag{7.46}$$

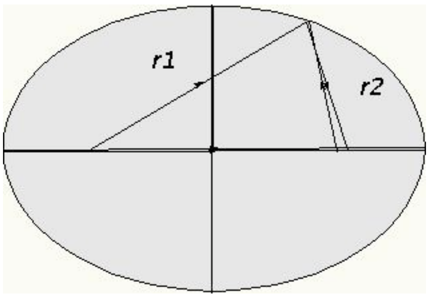

Figure 1. Three body problems.

$$r_1 \equiv \sqrt{(x+a)^2 + y^2}$$
$$r_2 \equiv \sqrt{(x-a)^2 + y^2} \tag{7.47}$$

Assume then the position of the third body at the point (x, y) and choose elliptic coordinates

$$\xi \equiv r_1 + r_2 = \sqrt{(x+a)^2 + y^2} + \sqrt{(x-a)^2 + y^2}$$
$$\eta \equiv r_1 - r_2 = \sqrt{(x+a)^2 + y^2} - \sqrt{(x-a)^2 + y^2} \tag{7.48}$$

The lines $\xi =$ const are elipses whereas the lines $\eta =$ const are hyperbolas Let us write the Euclidean metric in elliptic coordinates (ξ, η).

$$\xi\eta = r_1^2 - r_2^2$$
$$\xi^2 + \eta^2 = 2\left(r_1^2 + r_2^2\right)$$
$$\xi^2 - \eta^2 = 4r_1 r_2 \tag{7.49}$$

$$x = \frac{\xi\eta}{4a}$$
$$y = \frac{1}{4a}\sqrt{4a^2\left(\xi^2 + \eta^2\right) - \xi^2\eta^2 - 16a^4} = \frac{1}{4a}\sqrt{\left(4a^2 - \eta^2\right)\left(\xi^2 - 4a^2\right)} \tag{7.50}$$

$$4adx = \xi d\xi + \eta d\eta$$
$$4ady = \xi d\xi\sqrt{\frac{4a^2 - \eta^2}{\xi^2 - 4a^2}} - \eta d\eta\sqrt{\frac{\xi^2 - 4a^2}{4a^2 - \eta^2}} \tag{7.51}$$

$$ds^2 = \frac{\xi^2 - \eta^2}{4}\left(\frac{d\xi^2}{\xi^2 - 4a^2} + \frac{d\eta^2}{4a^2 - \eta^2}\right) \tag{7.52}$$

The kinetic energy of a test particle in elliptic coordinates reads

$$K = \frac{m}{2}\frac{\xi^2 - \eta^2}{4}\left(\frac{\dot{\xi}^2}{\xi^2 - 4a^2} + \frac{\dot{\eta}^2}{4a^2 - \eta^2}\right) \tag{7.53}$$

As a matter of fact, when the euclidean metrics reads in some coordinates

$$ds^2 = \sum_i g_i^2 dx_i^2, \tag{7.54}$$

The lagrangian for a free particle would then read

$$L = \frac{m}{2} \sum_i g_i \dot{q}_i^2 - V(q) \tag{7.55}$$

and the hamiltonian

$$H = \sum_i \frac{p_i^2}{2mg_i} + V \tag{7.56}$$

Let us now assume that the distance between the two big masses remains constant in value ($2a$), The third-body potential energy then reads

$$- G\frac{Mm}{r_1} - G\frac{Mm}{r_2} = -2GMm\frac{\xi}{\xi^2 - \eta^2} \tag{7.57}$$

The hamiltonian per unit mass

$$H = 2p_\xi^2 \frac{\xi^2 - 4a^2}{m(\xi^2 - \eta^2)} + 2p_\eta^2 \frac{4a^2 - \eta^2}{m(\xi^2 - \eta^2)} - \frac{4k\xi}{\xi^2 - \eta^2} \tag{7.58}$$

(here $k \equiv 2GMm$). Let us now write the Hamilton Jacobi equation

$$2\left(\frac{\partial W}{\partial \xi}\right)^2 \left(\xi^2 - 4a^2\right) - 4k\xi - Em\xi^2 = -2\left(\frac{\partial W}{\partial \eta}\right)^2 \left(4a^2 - \eta^2\right) - Em\eta^2 \tag{7.59}$$

The action then follows from quadratures (Arnold attributes this result to Charlier)

$$S = -Et + \int^\xi dx \sqrt{\frac{4kx + Emx^2 + C}{2\left(x^2 - 4a^2\right)}} + \int^\eta dx \sqrt{\frac{C + Emx^2}{2\left(x^2 - 4a^2\right)}} \tag{7.60}$$

The trajectory is determined by

$$\begin{aligned} \beta_1 &= \frac{\partial S}{\partial E} \\ \beta_2 &= \frac{\partial S}{\partial \alpha} \end{aligned} \tag{7.61}$$

8 Rigid bodies

A rigid body is a discrete of continuous system of particles such that the distance between any two points does not change with time. Rigid bodies do not exist in nature; they are idealizations. In practice all bodies are deformable. We shall assume nevertheless here that

$$\frac{\partial}{\partial t}|\vec{r}_1 - \vec{r}_2| \tag{8.1}$$

for every pair of points P_1 and P_2. A rigid body embodies six dof: three to determine the position of a given point, and another three to determine a rotation around that point. It is useful to consider two cartesians reference systems. One moving with the body, the *body frame*

$$\vec{e}_a(t) \tag{8.2}$$

with a=1,2,3. They are supposed to be kept orthonormalized for all time

$$\vec{e}_a(t).\vec{e}_b(t) = \delta_{ab} \tag{8.3}$$

The components of the vectors $\vec{e}_a$ can be grouped in a 3×3 matrix

$$e_a^i \tag{8.4}$$

Now it is a fact that the 3×3 matrix

$$M^{ij} \equiv \sum_a e_a^i e_a^j = \delta^{ij} \tag{8.5}$$

This can be checked easily by multiplying by an arbitrary vector e_c^j.

And another one fixed once and for all, the *space frame*

$$\vec{E}_a \equiv \delta_a^i \tag{8.6}$$

It is also othonormalized

$$\vec{E}_a \vec{E}_b = \delta_{ab} \tag{8.7}$$

The body frame is fully determines by its components in the space frame

$$\vec{e}_a\ (t) = R_a{}^b \vec{E}_b \tag{8.8}$$

That is

$$e_a^i = R_{ai} \tag{8.9}$$

It is a fact that the matrix $R_a{}^b$ is othogonal

$$R^T R = R R^T = 1 \tag{8.10}$$

This is immediate after

$$R_{ab}(t) \equiv \vec{e}_a.\vec{E}_b \tag{8.11}$$

Indeed

$$\sum_b R_{ab}R_{cb} = \sum_b \left(\vec{e}_a.\vec{E}_b\right)\left(\vec{e}_c.\vec{E}_b\right) = \delta_{ac} \tag{8.12}$$

As a consequence the matrix M is unimodular (det $M = 1$). This matrix $R(t)$ completely describes the motion of the rigid body. Such a matrix close to the identity reads

$$M = 1 + \omega \tag{8.13}$$

with the matrix ω antisymmetric

$$\omega^T = -\omega \tag{8.14}$$

That is, again six parameters.

Any given point P in the rigid body can be expanded as

$$\vec{x} = \sum_a x_a\ \vec{e}_a(t) \equiv \sum_b X_b(t)\ \vec{E}_b = \sum_{ab} x_a R_a{}^b(t)\vec{E}_b \tag{8.15}$$

It is plain that

$$\frac{d\vec{x}}{dt} = \sum_a x_a\ \frac{d\vec{e}_a(t)}{dt} \equiv \sum_b \frac{dX_b(t)}{dt}\ \vec{E}_b = \sum_{ab} x_a \frac{dR_a{}^b(t)}{dt}\vec{E}_b \tag{8.16}$$

The time variation of the body frame with respect to the body frame itself is given by

$$\frac{d\vec{e}_a(t)}{dt} = \sum_b \frac{dR_a{}^b(t)}{dt}\vec{E}_b = \sum_b \left(\frac{dR(t)}{dt}R^{-1}\right)_a{}^b\ \vec{e}_b(t) \equiv \omega_a{}^b\vec{e}_b(t) \tag{8.17}$$

It is a fact of life that

$$\omega_{ab} = -\omega_{ba} \tag{8.18}$$

As a matrix

$$\dot{R}R^{-1} = -R\dot{R}^{-1} = -R\dot{R}^T \tag{8.19}$$

Please note that

$$\omega^{-1} = R\dot{R}^T = -\dot{R}R^T = -\omega \tag{8.20}$$

so that ω is a so called *symplectic matrix*

$$\omega^2 = -1 \tag{8.21}$$

$$\omega^T = \left(\dot{R}R^T\right)^T = -\left(R\dot{R}^T\right)^T = -\omega \tag{8.22}$$

The dual vector is defined as

$$\omega_a \equiv \frac{1}{2}\sum_{bc} \epsilon_{abc}\omega_{bc} \tag{8.23}$$

Clearly

$$\omega_{ab} = \sum_c \epsilon_{abc}\omega_c \tag{8.24}$$

This defines a vector

$$\vec{\omega} \equiv \sum_a \omega_a\vec{e}_a \tag{8.25}$$

The evolution in time of this body frame is then given by

$$\frac{d\vec{e}_a(t)}{dt} = \sum \epsilon_{abc}\omega_c\vec{e}_b \tag{8.26}$$

On the other hand, let us work out

$$\vec{\omega} \times \vec{e}_a \equiv \sum_c \omega_c\vec{e}_c \times \vec{e}_a \equiv \sum_c \omega_c\epsilon_{cab}\vec{e}_b = \sum_c \epsilon_{abc}\omega_c\vec{e}_b \tag{8.27}$$

We learn

$$\frac{d\vec{e}_a(t)}{dt} = \vec{\omega} \times \vec{e}_a \tag{8.28}$$

The vector $\vec{\omega}$ then represents the *instantaneous angular velocity.*

The velocity of any given point in the body frame is given by

$$\frac{d\vec{x}}{dt} = \sum_a x_a\frac{d\vec{e}_a}{dt} = \sum_a x_a\vec{\omega} \times \vec{e}_a = \vec{\omega} \times \vec{x} \tag{8.29}$$

Please note that this is nothing but

$$\dot{x}^i = -\omega_{ik}x^k \tag{8.30}$$

The velocity in the space frame can be computed equally easily, reme4mbering that

$$\begin{aligned} X^i &= \sum_a x_a R_{ai} \\ x_a &= \sum_i R_{ai}X_i \end{aligned} \tag{8.31}$$

namely

$$\dot{X}_i = \sum_a x_a\dot{R}_{ai} = \sum_{aj} R_{aj}X_j\dot{R}_{ai} = \sum_j X_j\left(R^T\dot{R}\right)_{ji} = -\sum_j X^j\left(R^T\omega R\right)_{ji} \tag{8.32}$$

Indeed, let is call

$$M \equiv R^T\dot{R} \tag{8.33}$$

Now, remember that

$$\omega \equiv \dot{R}R^T = -R\dot{R}^T \tag{8.34}$$

Then indeed is the case that

$$M \equiv \omega(R^T) = -R^T\omega R \tag{8.35}$$

It is sometimes useful to reconstruct the rotation matrix out of the angular velocity. This needs the matrix ODE

$$\frac{dR(r)}{dt}R^{-1}(t) = \omega(t) \tag{8.36}$$

to be solved. This yields the *time ordered exponential*

$$R(t) = T\ e^{\int^t dt'\omega(t')} \equiv 1 + \int^t \omega(t') + \int^t dt'\omega(t')\int^{t'} dt''\omega(t'') + \ldots \tag{8.37}$$

8.1 The tensor of Inertia

It is possible to write the position of each particle in the body as

$$\vec{r} = \vec{R} + \vec{x} \tag{8.38}$$

where $\vec{R}$ is the position of the center of mass, defined (in discrete notation, $A = 1 \ldots N$) as

$$M\vec{R} = \sum m_A \vec{r}_A \equiv \int d^3x\ \rho(\vec{r})\ \vec{r} \tag{8.39}$$

Then it is a fact that

$$\sum_{A=1}^{N} m_A \vec{x}_A = \vec{0} \tag{8.40}$$

The kinetic energy reads

$$I = \frac{1}{2}\sum_A m_A \left(\dot{\vec{R}} + \dot{\vec{x}}_A\right)^2 = \frac{1}{2}M\dot{\vec{R}}^2 + \sum_A m_A \left(2\dot{\vec{R}}.\dot{\vec{x}}_A + \dot{\vec{x}}_A^2\right) \tag{8.41}$$

This is the sum of the CDM energy plus another term on which we shall concentrate from now on.

$$T = \frac{1}{2}M\dot{\vec{R}}^2 + K \tag{8.42}$$

$$\begin{aligned} K = \frac{1}{2}\sum_A m_A \dot{\vec{x}}_A^2 = \frac{1}{2}\sum_A m_A \left(\vec{\omega}\times\vec{x}_A\right)\left(\vec{\omega}\times\vec{x}_A\right) = \frac{1}{2}\sum_A m_A \left(\omega^2 \vec{x}_A^2 - \left(\vec{x}_a\cdot\vec{\omega}\right)^2\right) \equiv \\ \equiv \frac{1}{2}\sum_{ij} I_{ij}\omega^i\omega^j \end{aligned} \tag{8.43}$$

This is so because

$$\sum_i \epsilon_{jki}\epsilon_{lmi} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} \tag{8.44}$$

so that

$$\sum_i \epsilon_{jki}\epsilon_{lmi}\omega^j x^k \omega^l x^m = \omega^2 x^2 - (\omega\cdot x)^2 \tag{8.45}$$

The *tensor of inertia* is then defined as

$$I_{ij} \equiv \sum_A m_A \left(\vec{x}_A^2 \delta_{ij} - x_i^A x_j^A\right) \tag{8.46}$$

In the usual case that the continuum approximations is used, this is replaced by

$$I_{ij} \equiv \int d^3x \rho(\vec{x})\left(\vec{x}^2\delta_{ij} - x_i x_j\right) \tag{8.47}$$

It is always possible to find a frame in which the tensor of inertia is diagonal. This frame defines the *principal axes* of the body, and the diagonal elements are the *principal moments of inertia.* Those are positive, because

$$I_{ab}\lambda^a\lambda^b \sim x^2\lambda^2\left(1 - \cos^2\,\theta\right) \geq 0 \tag{8.48}$$

There is a simple relationship between the tensor of inertia computed with different fixed points. If the fixed point P has coordinates $\vec{p}$ with respect to the center of mass, then

$$I_{ij}^p = I^{CDM} + M\left(p^2\delta_{ij} - p_i p_j\right) \tag{8.49}$$

This is easy to prove, because

$$\begin{aligned}
&I_{ij}^p = \sum_A m_A\left(\vec{x}^A - \vec{p}\right)^2 \delta_{ij} - \left(x^A - p\right)_i \left(x^A - p\right)_j = \\
&= I^{CDM} + \sum_A m_A\left(-2\vec{x}\,\vec{p}\delta_{ij} - x_i^A p_j - x_j^A p_i\right) + \\
&+ m_A\left(p^2\delta_{ij} - p_i p_j\right)
\end{aligned} \tag{8.50}$$

and the extra terms vanish because in CDM frame

$$\sum m_A x_i^A = 0 \tag{8.51}$$

8.2 Angular momentum.

The angular momentum is defined as

$$\begin{aligned}
&\vec{L} = \sum_A m_A \vec{x}_A \times \dot{\vec{x}}_A = \sum_A m_A \vec{x}_A\left(\vec{\omega} \times \vec{x}_A\right) = \sum_A m_A\left(x_A^2\vec{\omega} - \left(\vec{\omega}.\vec{x}_A\right)\vec{x}_A\right) = \\
&= \sum_j I_{ij}\omega^j
\end{aligned} \tag{8.52}$$

That means that

$$L_i = \sum_j I_{ij}\omega_j \tag{8.53}$$

In general the vectors $\vec{L}$ and $\vec{\omega}$ are not even proportional.

9 The rotation group.

In our case we are interested in the matrix that relates both frames

$$\vec{e}_a = R_a{}^b \vec{E}_b \tag{9.1}$$

This a matrix $R \in SO(3)$. This means that

$$R R^T = R^T R = 1 \tag{9.2}$$

Put it into another form, this is the condition that

$$x^2 + y^2 + z^2 \tag{9.3}$$

remains invariant under such a linear transformation.

The groups $SO(3)$ and $SU(2)/\mathbb{Z}_2$ are intimately related. Indeed any unitary matrix can be parameterized as

$$u = \begin{pmatrix} \cos\,\alpha\; e^{i\beta} & \sin\,\alpha\; e^{i\gamma} \\ -\sin\,\alpha\; e^{-i\gamma} & \cos\alpha\; e^{-i\beta} \end{pmatrix} \tag{9.4}$$

Consider an arbitrary hermitian matrix

$$M \equiv \begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix} \tag{9.5}$$

Its determinant is

$$\det\, M = 1 - r^2 \tag{9.6}$$

It is plain that the transformation

$$M \rightarrow u M u^+ \tag{9.7}$$

leaves this determinant unchanged. Then there is a map

$$u \in SU(2) \rightarrow R \in SO(3) \tag{9.8}$$

It is plain that both $\pm u$ yield the same rotation; this is the reason for a factor $\mathbb{Z}_2$. To be specific, when $\beta = \gamma = 0$

$$u M u^+ = \begin{pmatrix} 1 + z\, \cos\, 2\alpha + x\, \sin\, 2\alpha & -i\; y + x\, \cos\, 2\alpha - z\, \sin\, 2\alpha \\ i\; y + x\, \cos\, 2\alpha - z\, \sin\, 2\alpha & 1 - z\, \cos\, 2\alpha - x\, \sin\, 2\alpha \end{pmatrix} \tag{9.9}$$

which means that

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\, 2\alpha & 0 & -\sin\, 2\alpha \\ 0 & 1 & 0 \\ \sin\, 2\alpha & 0 & \sin\, 2\alpha \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{9.10}$$

It represents a rotation of angle 2α around the y axis, $R_2(2\alpha)$. Also, when, $\alpha = 0$,

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\, 2\beta & \sin\, 2\beta & 0 \\ -\sin\, 2\beta & \cos\, 2\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{9.11}$$

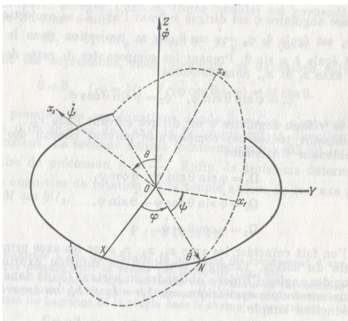

Figure 2. Euler angles

namely, $R_3(2\beta)$. It is curious that when

$$\alpha = \frac{\pi}{2}$$
$$\beta = 0 \tag{9.12}$$

we recover again a rotation $R_3(2\gamma)$.

In the general case,

$$uMu^+ = \begin{pmatrix} 1+z' & x'-iy' \\ x'+iy' & 1-z' \end{pmatrix} \tag{9.13}$$

$$\begin{aligned}
1+z' &\equiv 1+\ z\cos 2\alpha + \left(e^{i(\beta-\gamma)}(x-iy)+e^{i(\gamma-\beta)}(x+iy)\right)\sin 2\alpha \\
x'-iy' &\equiv e^{2i\beta}(x-iy)\cos^2\alpha - e^{2i\gamma}(x+iy)\sin^2\alpha - e^{i(\beta+\gamma)}z\sin 2\alpha \\
x'+iy' &\equiv e^{-2i\beta}(x+iy)\cos^2\alpha - e^{-2i\gamma}(x-iy)\sin^2\alpha - e^{-i(\beta+\gamma)}z\sin 2\alpha \\
1-z' &\equiv 1-z\cos 2\alpha - \left(e^{i(\beta-\gamma)}(x-iy)+e^{i(\gamma-\beta)}(x+iy)\right)\sin 2\alpha
\end{aligned} \tag{9.14}$$

Staring at this formula, we learn that when precisely

$$\beta = 0$$
$$\gamma = \frac{\pi}{2} \tag{9.15}$$

we recover a rotation around the first axis, $R_1(2\alpha)$

$$\begin{aligned}
x' &= x \\
y' &= y\cos 2\alpha + z\sin 2\alpha \\
z' &= -y\sin 2\alpha + z\cos 2\alpha
\end{aligned} \tag{9.16}$$

Euler showed that every rotation $R \in SO(3)$ can be written in the form

$$R = R_3(\psi)\ R_1(\theta)\ R_3(\phi) \tag{9.17}$$

In our $SU(2)$ language this is

$$u = \begin{pmatrix} e^{i\psi} & 0 \\ 0 & e^{-i\psi} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ i\sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix} \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} = \begin{pmatrix} e^{i(\phi+\psi)}\cos\frac{\theta}{2} & i\ e^{i(\psi-\phi)}\sin\frac{\theta}{2} \\ i\ e^{i(\phi-\psi)}\sin\frac{\theta}{2} & e^{-i(\phi+\psi)}\cos\frac{\theta}{2} \end{pmatrix}$$

It is plain that this covers the whole group manifold, just by identifying

$$\begin{aligned} &\psi + \phi = \beta \\ &\psi - \phi = \gamma - \frac{\pi}{2} \\ &\alpha = \frac{\theta}{2} \end{aligned} \tag{9.18}$$

In $SO(3)$ language this is

$$R = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & \sin\phi\cos\psi + \cos\theta\sin\psi\cos\phi & \sin\theta\sin\psi \\ -\cos\phi\sin\psi - \cos\theta\cos\psi\sin\phi & -\sin\psi\sin\phi + \cos\theta\cos\psi\cos\phi & \sin\theta\cos\psi \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix}$$

It ia good exercise to check that this matrix is orthogonal, that is,

$$R^T R = R R^T = 1 \tag{9.19}$$

It is clear that

$$\vec{x} = \sum_a x_a \vec{e}_a = \sum_a X_a \vec{E}_a = \sum_{ab} x_a R_a{}^b \vec{E}_b \tag{9.20}$$

so that

$$X_a = \sum_b x_b \; R_{ba} \tag{9.21}$$

10 Euler's equation.

Let us forget about the motion of the CDM. Conservation of angular momentum (in the absence of external torque) tells us that

$$\frac{d\vec{L}}{dt} = \vec{0} = \sum_a \left(\frac{dL_a}{dt}\vec{e}_a + L_a \frac{d\vec{e}_a}{dt} \right) = \sum_a \left(\frac{dL_a}{dt}\vec{e}_a + L_a \vec{\omega} \times \vec{e}_a \right) \tag{10.1}$$

Choosing the body frame in the direction of the principal axis,

$$L_a = I_a \omega_a \tag{10.2}$$

Recall that

$$\vec{\omega} \times \vec{e}_a = \sum_{bc} \epsilon_{abc} \omega_c \vec{e}_b \tag{10.3}$$

Then the equation of conservation of angular momentum tells us that

$$\frac{dL_b}{dt} + \sum_{ac} L_a \epsilon_{abc} \omega_c = 0 \tag{10.4}$$

In gory detail, Euler's equations read

$$\begin{aligned} I_1 \frac{d\omega_1}{dt} + (I_3 - I_2)\omega_2\omega_3 = 0 \\ I_2 \frac{d\omega_2}{dt} + (I_1 - I_3)\omega_1\omega_3 = 0 \\ I_3 \frac{d\omega_3}{dt} + (I_2 - I_1)\omega_2\omega_1 = 0 \end{aligned} \tag{10.5}$$

The *symmetric top* (for which it is meant any body with $I_1 = I_2 = I_3 = 0$) obeys

$$\dot{\omega}_a = 0 \tag{10.6}$$

so that the velocity is constant.

The next simpler object is the *symmetric top*, where

$$I_1 = I_2 \neq I_3 \tag{10.7}$$

Then

$$\begin{aligned} I_1 \frac{d\omega_1}{dt} + (I_3 - I_1)\omega_2\omega_3 = 0 \\ I_2 \frac{d\omega_2}{dt} + (I_1 - I_3)\omega_1\omega_3 = 0 \\ I_3 \frac{d\omega_3}{dt} = 0 \end{aligned} \tag{10.8}$$

This means that ω_3 is constant. The full equations collapse to

$$\begin{aligned} \dot{\omega}_1 &= \frac{I_1 - I_3}{I_1}\omega_3\omega_2 \equiv \Omega\omega_2 \\ \dot{\omega}_2 &= -\Omega\omega_1 \end{aligned} \tag{10.9}$$

Clearly

$$\begin{aligned} \omega_1 &= A \sin \Omega t \\ \omega_2 &= A \cos \Omega t \end{aligned} \tag{10.10}$$

which is *spin precession* in the body frame. The direction depends on the relative magnitude of I_1 and I_3.

In the space frame, $\vec{L}$ is constant. Also ω_3 as well as L_3 are constant. This means that $\vec{e}_3$ stays at a fixed angle with respect to $\vec{L}$ and $\vec{\omega}$. It rotates about the $\vec{L}$ axis. the spin precession looks then like a wobble.

The *asymmetric top* corresponds to $I_1 \neq I_2 \neq I_3$. A particular solution of Euler's equations is

$$\begin{aligned} \omega_1 &= \Omega \\ \omega_2 &= \omega_3 = 0 \end{aligned} \tag{10.11}$$

It is not difficult to analyze its stability. Write

$$\begin{aligned} \omega_1 &= \Omega + \epsilon\eta_1 \\ \omega_2 &= \eta_2 \\ \omega_3 &= \eta_3 \end{aligned} \tag{10.12}$$

Then, and to first order in the perturbation,

$$\begin{aligned} I_1\dot{\eta}_1 &= 0 \\ I_2\dot{\eta}_2 &= \Omega \ \eta_3 \ (I_3 - I_1) \\ I_3\dot{\eta}_3 &= \Omega \ \eta_2 \ (I_1 - I_2) \end{aligned} \tag{10.13}$$

This implies

$$I_2\ddot{\eta}_2 = \frac{\Omega^2(I_3 - I_1)(I_1 - I_2)}{I_3} \ \eta_2 \equiv C\eta_2 \tag{10.14}$$

This means that this solution is stable about the largest or smallest moment of inertia, but it is unstable about the intermediate one.

In the body frame there are two first integrals

$$2T \equiv I_1 \ \omega_1^2 + I_2 \ \omega_2^2 + I_3 \ \omega_3^2 \tag{10.15}$$

which defines the *quadric of inertia*, and

$$L^2 = I_1^2 \ \omega_1^2 + I_2^2 \ \omega_2^2 + I_3^2 \ \omega_3^2 \tag{10.16}$$

The *polhode* is the path that $\vec{\omega}$ traces on the quadric of inertia. The polhode curves are always closed.

In the space frame, given the fact that $\vec{L}$ is constant, as well as

$$2T \equiv \vec{L}.\vec{\omega} \tag{10.17}$$

the tip of the vector $\vec{\omega}$ lies on a fixed plane, called the *invariable plane*. This plane is tangent to the quadric at the point $\vec{\omega}$. The curve that this point describes on the invariable plane is called the *herpolhode*. This curve does not necessarily close.

The general form of the angular velocity matrix can be easily deduced in termns of Euler's angles,

$$\vec{e}_a(t) = R_a{}^b(t)\vec{E}_b \tag{10.18}$$

$$R = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & \sin\phi\cos\psi + \cos\theta\sin\psi\cos\phi & \sin\theta\sin\psi \\ -\cos\phi\sin\psi - \cos\theta\cos\psi\sin\phi & -\sin\psi\sin\phi + \cos\theta\cos\psi\cos\phi & \sin\theta\cos\psi \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix}$$

Then the angular velocity matrix is given by

$$\dot{M}M^T = \begin{pmatrix} 0 & \dot\psi + \dot\phi\cos\theta & \dot\theta\sin\psi - \dot\phi\cos\psi\sin\theta \\ -\dot\psi - \dot\phi\cos\theta & 0 & \dot\theta\cos\psi + \dot\phi\sin\psi\sin\theta \\ -\dot\theta\sin\psi + \dot\phi\cos\psi\sin\theta & -\dot\theta\cos\psi - \dot\phi\sin\psi\sin\theta & 0 \end{pmatrix} \tag{10.19}$$

which conveys the fact that the angular velocity vector reads

$$\begin{aligned} \omega_1 &= \omega_{23} = \dot\theta\cos\psi + \dot\phi\sin\psi\sin\theta \\ \omega_2 &= \omega_{31} = -\dot\theta\sin\psi + \dot\phi\cos\psi\sin\theta \\ \omega_3 &= \omega_{12} = \dot\psi + \dot\phi\cos\theta \end{aligned} \tag{10.20}$$

10.1 The heavy symmetric top.

It is now possible to write down the lagrangian for the heavy symmetric top. This solid body really resembles an ordinary top. We assume that the pin point is at a distance l from the center of mass. The angle between $\vec{e}_3$ and $\vec{E}_3$ is precisely $\cos\theta$, so that the lagrangian reads

$$\begin{aligned} L = \frac{1}{2}I_1(\omega_1^2 + \omega_2^2) + \frac{1}{2}I_3\omega_3^2 - Mgl\cos\theta = \frac{1}{2}I_1\left(\dot\theta^2 + \dot\phi^2\sin^2\theta\right) + \\ + \frac{1}{2}I_3\left(\dot\psi + \dot\phi\cos\theta\right)^2 - Mgl\cos\theta \end{aligned} \tag{10.21}$$

There are two angles which act as cyclic coordinates, namely ψ and ϕ. The corresponding momenta

$$p_\psi = I_3\left(\dot\psi + \dot\phi\cos\theta\right) = I_3\omega_3 \tag{10.22}$$

This is nothing else that the angular momentum about the symmetry axis of the top.

$$p_\phi = I_1\dot\phi\sin^2\theta + I_3\left(\dot\psi + \dot\phi\cos\theta\right)\cos\theta \tag{10.23}$$

Also the energy is a constant of motion

$$E = \frac{1}{2} I_1 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta\right) + \frac{1}{2} I_3 \left(\dot{\psi} + \dot{\phi} \cos \theta\right)^2 + Mgl \cos \theta \tag{10.24}$$

It is customary to define a couple of constants

$$\begin{aligned} a &\equiv \frac{I_3 \omega_3}{I_1} \\ b &\equiv \frac{p_\phi}{I_1} \end{aligned} \tag{10.25}$$

Then

$$\begin{aligned} \dot{\phi} &= \frac{b - a \cos \theta}{\sin^2 \theta} \\ \dot{\psi} &= \frac{I_1 a}{I_3} - \frac{(b - a \cos \theta) \cos \theta}{\sin^2 \theta} \end{aligned} \tag{10.26}$$

This means that once the function $\theta(t)$ is known, the other two angles can be also known algebraically. The energy reads

$$E - \frac{1}{2} I_3 \omega_3^2 = \frac{1}{2} I_1 \left(\dot{\theta}^2 + \left(\frac{b - a \cos \theta}{\sin^2 \theta}\right)^2 \sin^2 \theta\right) + Mgl \cos \theta \tag{10.27}$$

The effective potential for the angle θ then reads

$$V(\theta) = \frac{1}{2} I_1 \left(\frac{b - a \cos \theta}{\sin \theta}\right)^2 + Mgl \cos \theta \tag{10.28}$$

It is convenient to analyze it in termns of the variable

$$\mu \equiv \cos \theta \tag{10.29}$$

Beside, we define

$$\begin{aligned} \alpha &\equiv \frac{2}{I_1} \left(E - \frac{1}{2} I_3 \omega_3^2\right) \\ \beta &\equiv \frac{2Mgl}{I_1} \end{aligned} \tag{10.30}$$

Then the full system reads

$$\begin{aligned} \dot{\mu}^2 &= (1 - \mu^2)(\alpha - \beta\mu) - (b - a\mu)^2 \equiv f(\mu) \\ \dot{\phi} &= \frac{b - a\mu}{1 - \mu^2} \\ \dot{\psi} &= \frac{I_1 a}{I_3} - \frac{\mu(b - a\mu)}{1 - \mu^2} \end{aligned} \tag{10.31}$$

The motion in ϕ is called *precession* while the motion in θ is called *nutation.*

Given the fact that

$$-1 \leq \mu \leq 1 \tag{10.32}$$

the system is confined between two roots μ_1, μ_2 of

$$f(\mu) = 0 \tag{10.33}$$

There are three possibilities.

- $\dot{\phi} > 0$ at both μ_1 as well as μ_2.

  The body wobbles in the direction of $\dot{\phi}$.

- $\dot{\phi} > 0$ at μ_1 but instead $\dot{\phi} < 0$ at μ_2.

  The body wobbles in the direction of $\dot{\phi}$ until it reaches of $\theta_{\max}$ and then turns back for a while until it recovers the positive $\dot{\phi}$ direction.

- $\dot{\phi} > 0$ at μ_1 but instead $\dot{\phi} = 0$ at μ_2.

  There is then a cusp st $\theta_{\max}$

Let us examine some physical possibilities.

- The first one is letting the top go. This means that $\dot{\theta}_0 = 0$. Then $f(\mu_0) = 0$, which implies $\mu_0 = \mu_2$. We also assume $\dot{\phi}_0 = 0$, so that

$$b - a\mu_0 = 0 \tag{10.34}$$

so that we learn that

$$\mu_0 = \frac{b}{a} \tag{10.35}$$

On the other hand ,

$$p_\theta = I_1\dot{\phi}\ \sin^2\ \theta + I_3\omega_3\ \cos\ \theta = I_3\omega_3 \cos\ \theta_0 \tag{10.36}$$

First it starts to fall under the influence of gravity. Then θ increases, Then $\dot{\phi}$ must also increase to keep p_ϕ constant. Besides, the direction of the precession must be the same as the one of ω_3.

In the particular case when the function $f(\mu) = 0$ has a single root, we can have $\dot{\phi}$ constant with $\dot{\theta} = 0$.

We need

$$f(\mu_0) = f'(\mu_0) = 0 \tag{10.37}$$

that is

$$\begin{aligned}&(1-\mu_0^2)(\alpha - \beta\mu_0) - (b - a\mu_0)^2 = 0\\ &-2\mu_0(\alpha - \beta\mu_0) - \beta(1-\mu_0^2) + 2a(b - a\mu_0) = 0\end{aligned} \tag{10.38}$$

Besides, we know that

$$\dot{\phi}_0^2 = \frac{(b - a\mu_0)^2}{(1-\mu_0^2)^2} = \frac{\alpha - \beta\mu_0}{1-\mu_0^2} \tag{10.39}$$

which we write as

$$\alpha - \beta\mu_0 = \dot{\phi}^2(1-\mu_0^2) \tag{10.40}$$

Using that, we learn that

$$-2\mu_0\dot{\phi}^2 - \beta + 2a\dot{\phi} = 0 \tag{10.41}$$

Because this is the same thing as

$$-2\mu_0\frac{\alpha-\beta\mu_0}{1-\mu_0^2}-\beta+2a\frac{b-a\mu_0}{1-\mu_0^2}=\frac{f'(\mu_0)}{1-\mu_0^2}=0 \tag{10.42}$$

This translates into

$$\dot\phi_0\left(2a-2\mu_0\dot\phi_0\right)-\beta=0 \tag{10.43}$$

and using

$$\begin{aligned} a &= \frac{I_3\omega_3}{I_1} \\ \beta &= \frac{Mgl}{I_1} \end{aligned} \tag{10.44}$$

we learn that

$$Mgl=\dot\phi\left(I_3\omega_3-I_1\dot\phi\mu_0\right) \tag{10.45}$$

This yields the two values of $\dot\phi$ for which the top can spin without bobbing. This is possible only when

$$\frac{I_3\omega_3}{2}>\sqrt{MglI_1\cos\theta_0} \tag{10.46}$$

- Consider now the *sleeping top*

  $$\theta_0=\dot\theta_0=0 \tag{10.47}$$

  For this to be possible, we need

  $$f(\mu=1)=0 \tag{10.48}$$

  Then $a=b$ and $\alpha=\beta$, so that actually $f(\mu)$ has a double zero at $\mu=1$.

  $$f(\mu)=(1-\mu)^2(\alpha(1+\mu)-a^2) \tag{10.49}$$

  The second root is

  $$\mu_2=\frac{a^2}{\alpha-1} \tag{10.50}$$

  Then if $\mu_2>1$ (that is, $\omega_3^2>\frac{4I_1Mgl}{I_3^2}$) the motion is stable, whereas if $\mu_2<1$ (that is, $\omega_3^2<\frac{4I_1Mgl}{I_3^2}$) it is unstable.

References

[1] VI. Arnold, *Mathematical methods in classical mechanics (Springer); Méthodes mathématiques de la mécanique classique*
Equations differentielles ordinaires
Chapitres supplementaires de la théorie des equations differentielles ordinaires; (MIR,Moscou)

[2] V.I. Arnold, V.V. Kozlov, A.I. Neishtadt, *Mathematical aspects of classical and celestial mechanics* (Springer Verlag, 1991)

[3] PAM DIrac,*Lectures on quantum mechanics.* (Dover)

[4] F.R. Gantmajer, *Mecánica Analítica*, (URSS,Moscú)

[5] I.M. Gelfand and S.V. Fomin,*Calculus of variations* (Dover)

[6] H. Goldstein, *Mecánica clásica* (Revertè)

[7] A. Hanson, T. Regge and C. Teitelboim, *Constrained Hamiltonian systems*, (Accademia nazionale dei lincei, Roma 1976)

[8] L. Landau y E.M. Lifshitz, *Mechanics*

[9] E. Whittaker, *A treatise on the analytical dynamics of particles and rigid bodies* (Cambridge)