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Gauge Anomalies.

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Abstract:

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Contents

1 The singlet chiral anomaly.

Consiser a set of left and right fermions in an external gauge field

$$L = \bar{\psi} i \not{D}(A) \psi = \bar{\psi}_L i \not{D}(A) \psi_L + \bar{\psi}_R i \not{D}(A) \psi_R \tag{1.1}$$

When necessary, we shall use Weyl's representation of Dirac's gamma matrices

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1.2}$$

and

$$\gamma^i = \begin{pmatrix} 0 & -\vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \tag{1.3}$$

In this form, the operator $i\not{\partial}$ reads

$$\not{\partial} = \begin{pmatrix} 0 & i\partial_0 + i\vec{\sigma}\vec{\nabla} \\ i\partial_0 - i\vec{\sigma}\vec{\nabla} & 0 \end{pmatrix} \tag{1.4}$$

Finally

$$\gamma_5 \equiv i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.5}$$

In that way the left and right projectors

$$P_L \equiv \frac{1}{2}(1 + \gamma_5) \tag{1.6}$$

as well as $P_R \equiv 1 - P_L$. To be specific

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \tag{1.7}$$

It is plain that

$$\bar{\psi}_L \equiv (P_L\psi)^+\gamma_0 = \psi^+ P_L \gamma_0 = \bar{\psi} P_R \tag{1.8}$$

Kinetic energy terms do not mix chiralities

$$L = \bar{\psi} i \not{D} \psi = \bar{\psi}_L i \not{D} \psi_L + \bar{\psi}_R i \not{D} \psi_R \tag{1.9}$$

which is not the case with either masses or Yulawa couplings

$$L_m \equiv \bar{\psi} m \psi = \bar{\psi}_L m \psi_R + \bar{\psi}_R m \psi_L \tag{1.10}$$

Charge conjugates are defined by

$$\psi^c = -\gamma_2 \psi^* = \begin{pmatrix} \sigma_2 \psi_R^* \\ -\sigma_2 \psi_L^* \end{pmatrix} \tag{1.11}$$

Let us also recall the well-known fact that the whole action cal be expressed in terms of left-handed fields

$$(\bar{\psi}_c)_L = (0, \psi_R^T \sigma_2) \tag{1.12}$$

Also

$$(\bar{\psi}_c)_R = (-\psi_L^T \sigma_2, 0) \tag{1.13}$$

In fact

$$\begin{aligned}(\bar{\psi}_c)_L i \not{D} (\psi_c)_L = \psi_R \sigma_2 (i\partial_0 - i\vec{\sigma}\vec{\nabla})\sigma_2 \psi_R^* = \\ \psi_R^T (i\partial_0 + i\vec{\sigma}^* \vec{\nabla})\psi_R^* = -i\partial_0 \psi_R^+ \psi_R - i\vec{\nabla}\psi_R^+ \vec{\sigma}\psi_R\end{aligned} \tag{1.14}$$

Integrating by parts this yields

$$i\psi_R^+ \partial_0 \psi_R + i\psi_R^+ \vec{\sigma}\vec{\nabla}\psi_R \tag{1.15}$$

which is precisely

$$\bar{\psi}_R i \not{D} \psi_R \tag{1.16}$$

All this holds independently of the structure of any non-spinorial indices the fermions may have

For example, if we have a Dirac spinor with a flavor index $i = 1 \ldots N$, we can always define a $2N$ left component spinor

$$\Psi \equiv \begin{pmatrix} \psi_L \\ \psi_L^c \end{pmatrix} \tag{1.17}$$

The kinetic energy piece reads

$$L = \bar{\Psi} i \not{D} \Psi \tag{1.18}$$

and is $U(2N)$ invariant under

$$\delta\Psi = iU\Psi \tag{1.19}$$

Majorana spinors ara self-conjugate $\psi = \psi^c$. Then

$$\psi^M = \begin{pmatrix} \psi_L \\ -\sigma_2 \psi_L^* \end{pmatrix} \tag{1.20}$$

Both Weyl and Majorana spinors have only two complex independent components, which is half those of a Dirac spinor.

Majorana masses are couplings of the form

$$M\bar{\psi}_M \psi_M \tag{1.21}$$

and they violate fermion number conservation.

This lagrangian is invariant under two different global transformations. This first is the *vector* one

$$\delta\psi = i\epsilon\psi \tag{1.22}$$

that is

$$\begin{aligned} \delta\psi_L &= i\epsilon\psi_L \\ \delta\psi_R &= i\epsilon\psi_R \end{aligned} \tag{1.23}$$

The corresponding Noether current is fermion number conservation

$$j_\mu = \bar{\psi}\gamma_\mu\psi \tag{1.24}$$

The second symmetry is the *axial* or *chiral*

$$\delta\psi = i\epsilon\gamma_5\psi \tag{1.25}$$

that is

$$\delta\psi_L = i\epsilon\psi_L \tag{1.26}$$

$$\delta\psi_R = -i\epsilon\psi_R \tag{1.27}$$

The corresponding Noether current reads

$$j_5^\mu \equiv \bar{\psi}\gamma_5\gamma^\mu\psi \tag{1.28}$$

It is plain that

$$\begin{aligned} \bar{\psi}\gamma^\mu\psi &= \psi_L\gamma^\mu\psi_L + \bar{\psi}_R\gamma^\mu\psi_R \\ \bar{\psi}\gamma_5\gamma^\mu\psi &= \psi_L\gamma^\mu\psi_L - \bar{\psi}_R\gamma^\mu\psi_R \end{aligned} \tag{1.29}$$

What happens is that in quantum mechanics there is an anomaly in the latter current.

$$\partial_\mu j_5^\mu \equiv \mathcal{A} = \frac{g^2}{16\pi^2}\mathrm{tr}\ \epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu} \tag{1.30}$$

(the trace is irrelevant in the abelian case).

The fact that we keep conservation of the vector current implies that the left anomaly is equal and opposite in sign from the right anomaly.

$$\partial_\mu j_L^\mu \equiv \partial_\mu(\bar{\psi_L}\gamma^\mu\psi_L) = -\partial_\mu j_R^\mu \equiv -\partial_\mu(\bar{\psi_R}\gamma^\mu\psi_R) = \frac{1}{2}\mathcal{A} \tag{1.31}$$

Is this simple fact that allows for cancellation of anomalies between different species of fermions to be possible at all.

1.1 Anomalies as due to non-invariante of the functional integral measure.

This way of looking to the anomaly is due to Fujikawa [25]. The starting point is the formal definition of Berezin's functional integral measure

$$\prod_x \mathcal{D}\bar{\psi}(x)\mathcal{D}\psi(x) \tag{1.32}$$

Giving the fact that

$$\int d\psi\psi = 1, \tag{1.33}$$

then

$$\int d(\lambda\psi)\lambda\psi = 1 \tag{1.34}$$

which implies

$$d(\lambda\psi) = \frac{1}{\lambda}d\psi. \tag{1.35}$$

The infinitesimal version of the jacobian of the transformation (1.26)

$$\begin{aligned}
&\psi'(x) = e^{i\epsilon\gamma_5}\psi(x) \\
&\bar{\psi}' = \bar{\psi}e^{i\epsilon\gamma_5} \\
&\mathcal{D}\psi'\mathcal{D}\bar{\psi}' = e^{-2i\epsilon\gamma_5}\mathcal{D}\psi\mathcal{D}\bar{\psi}
\end{aligned} \tag{1.36}$$

will then be

$$J \equiv det\,(1 - 2i\epsilon(x)\gamma_5) \tag{1.37}$$

id est

$$\log\, J = -2\, i\, \mathrm{tr}\, \epsilon(x)\gamma_5\delta(x-y) \tag{1.38}$$

The only thing that remains is to give some precise sense to the above expression. In order to perform the trace, we shall use a complete set of eigenfunctions of Dirac's operator

$$\not{D}\phi_n(x) \equiv (\not{\partial} - ig\not{A})\phi_n(x) = \lambda_n\phi_n(x). \tag{1.39}$$

Let us regularize as follows

$$\begin{aligned}
&\frac{i}{2}\log\, J = \sum_n \int d^4x\; d^4y\; \phi_n(x)^+\; \epsilon(x)\; \gamma_5\; \delta_{xy}\phi_n(y) \equiv \\
&\lim_{\Lambda\to\infty}\int d^4x\; \epsilon(x)\; \sum_n \phi_n^+(x)\; \gamma_5\; e^{-\frac{\lambda_n^2}{\Lambda^2}}\; \phi_n(x) = \\
&\lim_{\Lambda\to\infty}\int d^4x\;\; \epsilon(x)\; \sum_n \phi_n^+(x)\; \gamma_5\; e^{-\frac{\not{D}^2}{\Lambda^2}}\; \phi_n(x) = \\
&\lim_{\Lambda\to\infty}\int d^4x\; \epsilon(x)\; \mathrm{tr}\int \frac{d^4k}{(2\pi)^4}\; e^{-ikx}\; \gamma_5\; e^{-\frac{\not{D}^2}{\Lambda^2}} e^{ikx}
\end{aligned} \tag{1.40}$$

where in the last line we have changed to a plane wave basis.

It is not difficult to check the following facts

$$\frac{\not{D}^2}{\Lambda^2} = \frac{1}{\Lambda^2}\left(D^\mu D_\mu + \frac{1}{4}[\gamma^\mu,\gamma^\nu][D_\mu, D_\nu]\right) \tag{1.41}$$

$$D_\mu e^{ikx} = (\partial_\mu - igA_\mu)e^{ikx} = (ik_\mu - igA_\mu)e^{ikx} \tag{1.42}$$

$$D^\mu D_\mu e^{ikx} = (-k^2 - ig\partial.A + 2gk.A - g^2 A_\alpha A^\alpha)e^{ikx} \tag{1.43}$$

$$[D_\mu, D_\nu]e^{ikx} = igF_{\mu\nu}e^{ikx} \tag{1.44}$$

What is left to compute is precisely

$$\lim_{\Lambda\to\infty} \mathrm{Tr} \int \frac{d^4k}{(2\pi)^4}\; e^{-ikx}\; \gamma_5\; e^{\frac{1}{\Lambda^2}[-k^2-ig\partial.A+2gk.A-g^2A_\alpha A^\alpha+\frac{i}{4}g\gamma^{\mu\nu}F_{\mu\nu}]}e^{ikx}. \tag{1.45}$$

where

$$\gamma^{\mu\nu} \equiv [\gamma^\mu, \gamma^\nu]. \tag{1.46}$$

Rescaling $k = p\Lambda$ and keeping the exponential of momenta in the integral, the only surviving term after tracing and regulating is

$$\int \frac{d^4p}{(2\pi)^4} e^{-p^2}\frac{1}{2!}\mathrm{Tr}\;\frac{-g^2}{16}(F_{\mu\nu}\gamma^{\mu\nu})^2 = \frac{ig^2}{32\pi^2}\mathrm{Tr}\;\epsilon^{\mu\nu\rho\sigma}\;F_{\mu\nu}\;F_{\rho\sigma} \tag{1.47}$$

given that the volume of the unit three-sphere is $V(S_3) = 2\pi^2$, the integral $\int_0^\infty p^3 dp e^{-p^2} = \frac{1}{2}$ and Tr $\gamma_5\gamma_{\mu\nu}\gamma_{\rho\sigma} = -16i\epsilon_{\mu\nu\rho\sigma}$.

All this means that taking into account the jacobian, the axial current is not conserved anymore, but rather

$$\partial_\mu\langle\bar{\psi}\gamma^\mu\gamma_5\psi\rangle = \frac{g^2}{8\pi^2}\mathrm{Tr}\;\int d^4x * F^{\mu\nu}\;F_{\mu\nu} \tag{1.48}$$

where the dual field strength has been defined

$$* F_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} \tag{1.49}$$

The preceding analysis is related to the index theorem (cf. [22]). What we are evaluating is actually con la cantidad

$$\sum \phi_n^+\gamma_5\phi_n = n_{(+)} - n_{(-)} \tag{1.50}$$

namely the difference between the number of positive and negative chirality eigenmodes of Dirac's operator. Neverteless it is fact that only zero modes can be chiral, because,

$$\not{D}\phi_n^{(+)} = \lambda_n\phi_n^{(+)} \tag{1.51}$$

acting with γ_5

$$-\lambda_n = \lambda_n \tag{1.52}$$

In a similar way, starting from

$$\not{D}\ \phi_n^{(-)} = \lambda_n \phi_n^{(-)} \tag{1.53}$$

and acting again with γ_5,

$$\lambda_n = -\lambda_n \tag{1.54}$$

This means that nonvanishing eigenvalues just come in pairs with opposite sign, and the only difference can only stem from the zero modes, for which our arguments do not apply. The quantity (1.50) is exactly what mathematicians call the *index* of the Dirac operator, and what Fujikawa just proved with physicist's techniques, is that

$$\mathrm{ind}\not{D} = -\frac{1}{16\pi^2}\mathrm{Tr}\int d^4x * F^{\mu\nu}\ F_{\mu\nu} \tag{1.55}$$

1.2 The Adler-Bell-Jackiw computation.

Let us now perform a perturbative calculation in QED with external vector and axial sources. Define

$$\Delta_{\lambda\mu\nu}(k_1,k_2) \equiv \mathcal{F}\langle 0|T J^5_\lambda(0) J_\mu(x_1) J_\nu(x_2)|0\rangle \tag{1.56}$$

$$\Delta_{\mu\nu}(k_1,k_2) \equiv \mathcal{F}\langle 0|T J_5(0) J_\mu(x_1) J_\nu(x_2)|0\rangle \tag{1.57}$$

The diagrams give

$$\begin{aligned}\Delta_{\lambda\mu\nu}(k_1,k_2) = i\int \frac{d^4p}{(2\pi)^4}\mathrm{Tr}\ \gamma_\lambda\gamma_5 \frac{1}{\not{p}-\not{k}_1-\not{k}_2}\gamma_\nu\frac{1}{\not{p}-\not{k}_1}\gamma_\mu\frac{1}{\not{p}} + \\ +\gamma_\lambda\gamma_5\frac{1}{\not{p}-\not{k}_1-\not{k}_2}\gamma_\mu\frac{1}{\not{p}-\not{k}_2}\gamma_\nu\frac{1}{\not{p}}\end{aligned} \tag{1.58}$$

and

$$\begin{aligned}\Delta_{\mu\nu}(k_1,k_2) = i\int \frac{d^4p}{(2\pi)^4}\mathrm{Tr}\ \gamma_5 \frac{1}{\not{p}-\not{k}_1-\not{k}_2}\gamma_\nu\frac{1}{\not{p}-\not{k}_1}\gamma_\mu\frac{1}{\not{p}} + \\ +\gamma_\lambda\gamma_5\frac{1}{\not{p}-\not{k}_1-\not{k}_2}\gamma_\mu\frac{1}{\not{p}-\not{k}_2}\gamma_\nu\frac{1}{\not{p}}\end{aligned} \tag{1.59}$$

Ward's identity for the vector current implies

$$k_1^\mu \Delta_{\lambda\mu\nu} = k_2^\nu \Delta_{\lambda\mu\nu} = 0 \tag{1.60}$$

and the axial Ward identity

$$q^\mu \Delta_{\lambda\mu\nu} \equiv (k_1+k_2)^\mu \Delta_{\lambda\mu\nu} = 0 \tag{1.61}$$

Using the expression just derived for the correlators, we can write

$$k_1^\mu \Delta_{\lambda\mu\nu} = i \int \frac{d^4p}{(2\pi)^4} \text{Tr } \gamma_\lambda\gamma_5 \frac{1}{\not{p} - \not{k}_1 - \not{k}_2} \gamma_\nu \frac{1}{\not{p} - \not{k}_1} \not{k}_1 \frac{1}{\not{p}} +$$
$$\gamma_\lambda\gamma_5 \frac{1}{\not{p} - \not{k}_1 - \not{k}_2} \not{k}_1 \frac{1}{\not{p} - \not{k}_1} \gamma_\nu \frac{1}{\not{p}} \tag{1.62}$$

We can now make in the first integral the change

$$\not{k}_1 = \not{p} - (\not{p} - \not{k}_1) \tag{1.63}$$

and in the second integral

$$\not{k}_2 = (\not{p} - \not{k}_2) - (\not{p} - \not{q}) \tag{1.64}$$

This yields

$$k_1^\mu \Delta_{\lambda\mu\nu} = i \int \frac{d^4p}{(2\pi)^4} \text{Tr } \gamma_\lambda\gamma_5 \frac{1}{\not{p} - \not{q}} \gamma_\nu \left(\frac{1}{\not{p} - \not{k}_1} - \frac{1}{\not{p}} \right) +$$
$$\left(\frac{1}{\not{p} - \not{q}} - \frac{1}{\not{p} - \not{k}_2} \right) \gamma_\nu \frac{1}{\not{p}} =$$
$$i \int \frac{d^4p}{(2\pi)^4} \text{Tr } \gamma_\lambda\gamma_5 \frac{1}{\not{p} - \not{q}} \gamma_\nu \frac{1}{\not{p} - \not{k}_1} - \frac{1}{\not{p} - \not{k}_2} \gamma_\nu \frac{1}{\not{p}} \tag{1.65}$$

This would vanish if we could make the change of integration variable

$$p \rightarrow p - k_1 \tag{1.66}$$

But this is not kosher, because the integral does not converge. Let us be careful and define

$$k_1^\mu \Delta_{\lambda\mu\nu}(a, k_1, k_2) = i \int \frac{d^4p}{(2\pi)^4} tr\, \gamma_\lambda\gamma_5 \frac{1}{\not{p} + \not{a} - \not{q}} \gamma_\nu \frac{1}{\not{p} + \not{a} - \not{k}_1} - \frac{1}{\not{p} + \not{a} - \not{k}_2} \gamma_\nu \frac{1}{\not{p} + \not{a}}$$

We shall compute the difference

$$\delta \left[k_1 \Delta \right]_{\lambda\nu} \equiv k_1^\mu \Big(\Delta_{\lambda\mu\nu}(a, k_1, k_2) - \Delta_{\lambda\mu\nu}(a = 0, k_1, k_2) \Big) \tag{1.67}$$

Using Stokes'theorem ,

$$\int d^n p \big(f(p + a) - f(p) \big) = \int d^n p \; a^\mu \partial_\mu f(p) + \ldots = \lim_{k \to \infty} a^\mu \frac{k_\mu}{k} \; f(k) \; S_{n-1}(k) \tag{1.68}$$

Our function is given by

$$f(p) \equiv \text{Tr } \left(\gamma_\lambda\gamma_5 \frac{1}{\not{p} - \not{k}_2} \gamma_\nu \frac{1}{\not{p}} \right) \tag{1.69}$$

What we actually have to compute is the difference between doing that with $a_1 = a$ and doing it with $a_2 = a - k_1$

This is

$$k_\mu^1 \lim_{k\to\infty} \frac{k^\mu}{k}(2\pi^2 k^3)\frac{\gamma_\lambda\gamma_5(\not k - \not k_2)\gamma_\nu \not k}{(k-k_2)^2 k^2} = -i2\pi^2 k_\mu^1 \lim_{k\to\infty}\frac{k^\mu}{k}k^3\frac{1}{k^4}\epsilon_{\lambda\rho\nu\sigma}k_2^\rho k^\sigma =$$
$$= -i2\pi^2 k_\mu^1 \epsilon_{\lambda\rho\nu\sigma}k_2^\rho \eta^{\mu\sigma} \tag{1.70}$$

This yields a nonvanishing value for the vector Ward identity.

$$k_1^\mu \Delta_{\lambda\mu\nu} = 2i\pi^2\frac{1}{(2\pi)^4}k_1^\mu k_2^\rho \epsilon_{\lambda\rho\nu\mu} \tag{1.71}$$

It looks that there is no possible way to keep vector symmetry in the quantum theory. In order to clarify the issue, let us go back to basics and define yet another correlator

$$\Delta_{\lambda\mu\nu}(a,k_1,k_2) = i\int\frac{d^4p}{(2\pi)^4}\mathrm{Tr}\ \gamma_\lambda\gamma_5\frac{1}{\not p+\not a-\not k_1-\not k_2}\gamma_\nu\frac{1}{\not p+\not a-\not k_1}\gamma_\mu\frac{1}{\not p+\not a}+$$
$$\gamma_\lambda\gamma_5\frac{1}{\not p+\not a-\not k_1-\not k_2}\gamma_\mu\frac{1}{\not p+\not a-\not k_2}\gamma_\nu\frac{1}{\not p+\not a}+(\mu\nu)(12) \tag{1.72}$$

Here $(\mu\nu)(12)$ means the result of exchanging the two indices $\mu\nu$ as well as the labels (12). Let us compute again the object $\delta\Delta$. The function we have to analyze is now

$$f(p) \equiv \mathrm{Tr}\ \gamma_\lambda\gamma_5\frac{1}{\not p-\not q}\gamma_\nu\frac{1}{\not p-\not k_1}\gamma_\mu\frac{1}{\not p} = \mathrm{Tr}\ \frac{\gamma_\lambda\gamma_5(\not p-\not q)\gamma_\nu(\not p-\not k_1)\gamma_\mu\not p}{(p-q)^2(p-k_1)^2p^2} \tag{1.73}$$

Traces are easily computed

$$\mathrm{Tr}\ \left(\gamma_\lambda\gamma_5\not p\gamma_\nu\not p\gamma_\mu\not p\right) = \mathrm{Tr}\ \gamma_\lambda\gamma_5(2\eta_{\nu\mu_1}-\gamma_\nu\gamma_{\mu_1})\gamma_{\mu_2}(2\eta_{\mu_3\nu}-\gamma_{\mu_3}\gamma_\nu)p^{\mu_1}p^{\mu_2}p^{\mu_3} =$$
$$= \mathrm{Tr}\ \gamma_\lambda\gamma_5\gamma_\nu p^2\gamma_{\mu_3}\gamma_\mu p^{\mu_3} \tag{1.74}$$

This leads to the expression

$$-4i\frac{p^\alpha}{p^4}\epsilon_{a\nu\mu\lambda} \tag{1.75}$$

$$\delta\Delta = \frac{4i}{8\pi^2}\lim\frac{k_\alpha k_\beta}{k^2}\epsilon^{\beta\nu\mu\lambda}+(\mu,k_1\to\nu k_2) = \frac{i}{8\pi^2}a_\alpha\epsilon^{\alpha\nu\mu\lambda}+(\mu,k_1\to\nu k_2) \tag{1.76}$$

Let us write a linear combination of the momenta k_1 and k_2

$$a \equiv x(k_1+k_2)+y(k_1-k_2) \tag{1.77}$$

leading to

$$\nabla\delta = \frac{iy}{4\pi^2}\epsilon_{\lambda\mu\nu\sigma}(k_1-k_2)^\sigma \tag{1.78}$$

(the piece linear in q disappears when symmetrizing)-

Imposing now conservation of the vector current we get

$$k_1^\alpha\Delta_{\lambda\alpha\nu}(k_1,k_2) = \frac{i}{8\pi^2}\epsilon_{\lambda\nu\tau\sigma}k_1^\tau k_2^\sigma \tag{1.79}$$

namely

$$y = -\frac{1}{2} \tag{1.80}$$

Reexamine now the axial current

$$(k_1 + k_2)_\lambda \Delta^{\lambda\mu\nu}(a, k_1, k_2) = (k_1 + k_2)_\lambda \delta^{\lambda\mu\nu}(k_1, k_2) + \frac{i}{4\pi^2}\epsilon_{\mu\nu\lambda\sigma} k_1^\lambda k_2^\sigma \tag{1.81}$$

The last part of the computation reads

$$(k_1 + k_2)_\lambda \delta^{\lambda\mu\nu}(k_1, k_2) = i\int \frac{d^4p}{(2\pi)^4} tr\, \not{a}\gamma_5 \frac{1}{\not{p} - \not{k}_1 - \not{k}_2}\gamma_\nu \frac{1}{\not{p} - \not{k}_1}\gamma_\mu \frac{1}{\not{p}} +$$
$$\not{a}\gamma_5 \frac{1}{\not{p} - \not{k}_1 - \not{k}_2}\gamma_\mu \frac{1}{\not{p} - \not{k}_2}\gamma_\nu \frac{1}{\not{p}} = i\int \frac{d^4p}{(2\pi)^4} tr\, \gamma_5 \frac{1}{\not{p} - \not{k}_1 - \not{k}_2}\gamma_\nu \frac{1}{\not{p} - \not{k}_1}\gamma_\mu -$$
$$\gamma_5 \frac{1}{\not{p} - \not{k}_2}\gamma_\nu \frac{1}{\not{p}}\gamma_\mu + (\mu, k_1 \to \nu k_2) = \frac{i}{4\pi^2}\epsilon_{\mu\nu\lambda\sigma} k_1^\lambda k_2^\sigma \tag{1.82}$$

namely exactly the same trace we got already when checking the vector Ward identity.

$$(k_1 + k_2)_\lambda \delta^{\lambda\mu\nu}(a, k_1, k_2) = \frac{i}{2\pi^2}\epsilon_{\mu\nu\lambda\sigma} k_1^\lambda k_2^\sigma \tag{1.83}$$

The anomaly can be written as a total derivative:

$$\mathcal{A} = \frac{1}{4\pi^2}\epsilon^{\mu\nu\rho\sigma}\partial_\mu \mathrm{Tr}\left(A_\nu \partial_\rho A_\sigma - \frac{2}{3} i A_\nu A_\rho A_\sigma\right) \tag{1.84}$$

1.3 Pauli-Villars regularization.

The Pauli-Villars regularization is a gauge invariant way of introducing a cutoff. The main idea stems from the fact that the difference of two propagators behaves much better at infinity than each one separately.

$$\frac{1}{p^2 - m^2} - \frac{1}{p^2 - M^2} = \frac{m^2 - M^2}{(p^2 - m^2)(p^2 - M^2)} \tag{1.85}$$

Of course the minus sign in from of the propagator is not physical, and indicates that the corresponding particle is a ghost. One must make sure that all unwanted ghostly efects are gone when $M \to \infty$. This regularization works best with fermion loops (like the one appearing in the abelian vacuum polarization diagram), which can be understood as the determinant of Dirac's operator

$$\det\, i\not{D}_m \tag{1.86}$$

where

$$i\not{D}_m \equiv i\not{\partial} - e\not{A} - m \tag{1.87}$$

Then we substitute instead of the determinant the quantity

$$\det\, i\not{D}_m \prod_{i=i}^{i=n} \left(\det\, i\not{D}_{M_i}\right)^{c_i} \tag{1.88}$$

or what is the same,

$$\text{Tr}\,\log\, i\not{D}_m + \sum_{i=i}^{i=n} c_i \text{Tr}\,\log\left(\,i\not{D}_{M_i}\right) \tag{1.89}$$

The coefficients c_i cannot be all positive, because they have to obey

$$\begin{aligned} &\sum c_i + 1 = 0 \\ &\sum_i c_i M_i^2 + m^2 = 0 \end{aligned} \tag{1.90}$$

This means that in general the regulators will violate the spin-statistics theorem,*id est* , they are ghosts.

In order to compute the j-th determinant we write

$$\begin{aligned} &\text{Tr}\,\log\, i\not{D}_{M_j} = \text{Tr}\,\log\ \left(i\not{\partial} - M_j\right)\left(1 - e(i\not{\partial} - M_j)^{-1}\not{A}\right) = \\ &= \text{Tr}\,\log\ \left(i\not{\partial} - M_j\right) + \text{Tr}\,\log\left(1 - e(i\not{\partial} - M_j)^{-1}\not{A}\right) = \\ &N + \sum_{n_1}^{\infty} \frac{(-e)^n}{n} \text{Tr} \int d^4x_1 d^4x_2 \ldots d^4x_n\ \not{A}(x_1) S_j(x_1 - x_2) \not{A}(x_2) \ldots \not{A}(x_n) S_j(x_n - x_1) \end{aligned}$$

where N is a divergent constant and

$$\left(i\not{\partial} - M_j\right)^{-1} \equiv S_j(x-y) \tag{1.91}$$

(we can include as well the physical mass as $M_0 \equiv m$). The Pauli-Villars'regulator loop in momentum space is proportional to

$$\begin{aligned} &\int d^4k_1 \ldots d^4k_n \int d^4p \frac{\text{Tr}\ \left(\gamma_{\mu_1}(\not{p} + M_j)\gamma_{\mu_2}(\not{p} + \not{k}_1 + M_j) \ldots \gamma_{\mu_n}(\not{p} + \not{k}_{n-1} + M_j)\right)}{(p^2 - m^2)((p+k_1)^2 - M_j^2) \ldots ((p+k_{n-1})^2 - M_j^2)} \times \\ &\times A^{\mu_1}(k_1) \ldots A^{\mu_n}(k_n)\ \delta^{(4)}(k_1 + k_2 + \ldots k_n) \end{aligned} \tag{1.92}$$

Given the fact that the numerator of the integrand has mass dimension n whereas the denominator has mass dinension $2n$, the superficial degree of divergence of this diagram is

$$D = 4 - n \tag{1.93}$$

This means that all terms with $n \leq 4$ will be divergent. We can represent the integrand as a power series in the masses ($P_\lambda(p)$ represents a polynomial in p of degree λ).

$$\begin{aligned} &\frac{P_n(p) + M_j^2 P_{n-2}(p) + \ldots + M_j^n}{P_{2n}(p) + M_j^2 P_{2n-2}(p) + \ldots + M_j^{2n}} = \frac{P_n(p)\left(1 + M_j^2 \frac{P_{n-2}(p)}{P_n(p)} + \ldots + M_j^n \frac{1}{P_n(p)}\right)}{P_{2n}(p)\left(1 + M_j^2 \frac{P_{2n-2}(p)}{P_{2n}(p)} + \ldots + M_j^{2n} \frac{1}{P_{2n}(p)}\right)} = \\ &\frac{P_n(p)}{P_{2n}(p)}\left(M_j^2\left(\frac{P_{n-2}(p)}{P_n(p)} - \frac{P_{2n-2}(p)}{P_{2n}(p)}\right) + \ldots\right) \end{aligned} \tag{1.94}$$

The net contribution of the regulators is the sum of all this terms weighted with c_j. The coefficient of M_j^λ behaves at large momenta as $p^{-n-\lambda}$. If the weights are chosen to obey the conditions as above, this cancels the terms in M_j^0 (behaving as Λ^{4-n}) and M_j^2 (behaving as Λ^{2-n}). This is enough in or case. In other situations, we maight have to impose extra conditions to the coefficients c_j.

For our purposes, it is enough to consider a single regulator of mass M. The physical limit will be $m \to 0$ and $M \to \infty$. In the regularized theory, with finite M, we can safely perform changes of variables in the finite integrals

$$\Delta^{PV}_{\lambda\mu\nu}(k_1,k_2) \equiv \Delta_{\lambda\mu\nu}(m) - \Delta_{\lambda\mu\nu}(M) \tag{1.95}$$

The axial Ward identity reads

$$q^\lambda \Delta_{\lambda\mu\nu} \equiv \lim_{M\to\infty}\left[2m\Delta_{\mu\nu}(m) - 2M\Delta_{\mu\nu}(M)\right] \tag{1.96}$$

Let us compute the diagram corresponding to the regulator

$$\Delta_{\mu\nu}(M) = -i\int \frac{d^4p}{(2\pi)^4} tr\left(\frac{i}{\not p - M + i\epsilon}\gamma_5 \frac{i}{\not p - \not q - M + i\epsilon}\gamma_\nu \frac{i}{\not p - \not k_1 - M + i\epsilon}\gamma_\mu - \right.$$
$$\left.\frac{i}{\not p - M + i\epsilon}\gamma_5 \frac{i}{\not p - \not q - M + i\epsilon}\gamma_\mu \frac{i}{\not p - \not k_2 - M + i\epsilon}\gamma_\nu\right) \tag{1.97}$$

Introducing Feynman parameters,

$$\delta_{\mu\nu}(M) = -\int \frac{d^4p}{(2\pi)^4} 2\int_0^1 dx_1 \int_0^{1-x_1} dx_2$$
$$\frac{\text{Tr }(\not p + M)\gamma_5(\not p - \not q + M)\gamma_\nu(\not p - \not k_1 + M)\gamma_\mu}{[(p^2-M^2)x_2 + ((p-q)^2 - M^2)(1-x_1-x_2) + ((p-k_1)^2 - M^2)x_1]^3} -$$
$$(k_1 \leftrightarrow k_2)(\mu \leftrightarrow \nu) \tag{1.98}$$

The only way we cab get a nonvanishing trace is with four Dirac matrices besides the γ_5. The full set of terms in the numerator reads

$$\not p \gamma_5 \not q \gamma_\nu M \gamma_\mu + \not p \gamma_5 M \gamma_\nu (\not p - \not k_1)\gamma_\mu + M\gamma_5(\not p - \not q)\gamma_\nu(\not p - \not k_1)\gamma_\mu - M\gamma_5 \not p \gamma_\nu \not k_1 \gamma_\mu \tag{1.99}$$

All those terms cancel but one.

$$M\text{Tr } \gamma_5 \not q \gamma_\nu \not k_1 \gamma_\mu = M4i\epsilon_{\beta\nu\alpha\mu} k_2^\mu k_1^\alpha + (k_1 \leftrightarrow k_2)(\mu \leftrightarrow \nu) \tag{1.100}$$

ending up with

$$\Delta_{\mu\nu} = \int \frac{d^4p}{(2\pi)^4} 2\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{2M4i\epsilon_{\mu\nu\alpha\beta}k_1^\alpha k_2^\beta}{[p^2 - 2pk - N^2]^3} \tag{1.101}$$

where

$$k \equiv q(1-x_1-x_2) + k_1 x_1 \tag{1.102}$$

and

$$N^2 \equiv M^2 - q^2(1-x_1-x_2) - k_1^2 x_1 \tag{1.103}$$

The momentum integral is a particular instance of

$$\int \frac{d^n p}{(p^2 - 2pk - N^2)^a} = i^{1-2a} \pi^{n/2} \frac{\Gamma(a - n/2)}{\Gamma(a)} \frac{1}{(k^2 + N^2)^{a-n/2}} \tag{1.104}$$

The final result is then

$$\lim_{M\to\infty} 2M\Delta_{\mu\nu}(M) = \lim_{M\to\infty} \frac{1}{(2\pi)^4} \frac{\pi^2}{2i} \frac{1}{M^2} 2M2M4i\epsilon_{\mu\nu\alpha\beta} k_1^\alpha k_2^\beta 2 \int_0^1 dx_1 \int_0^{1-x_1} dx_2 = \frac{1}{2\pi^2} \epsilon_{\mu\nu\alpha\beta} k_1^\alpha k_2^\beta \tag{1.105}$$

From this viewpoint, all the anomaly comes from the regulator.

1.4 Dimensional Regularization.

It is nowadays clear that the best definition of γ_5 in dimensional regularization is the one initially proposed by 't Hooft y Veltman [**?**]:

$$\begin{aligned} &\{\gamma_5, \gamma_\mu\} = 0 \quad (\mu = 0 \ldots 3) \\ &[\gamma_5, \gamma_\mu] = 0 \quad (\mu = 4 \ldots n - 1) \end{aligned} \tag{1.106}$$

The diagram we have to consider is

$$\begin{aligned} \delta_{\lambda\mu\nu} = &- \int \frac{d^4 p}{(2\pi)^4} \frac{d^{n-4} P}{(2\pi)^{n-4}} tr \frac{1}{\not p + \not P} \gamma_\lambda \gamma_5 \frac{1}{\not p + \not P - \not q} \gamma_\nu \frac{1}{\not p + \not P - \not k_1} \gamma_\mu \\ &-(k_1 \leftrightarrow k_2)(\mu \leftrightarrow \nu) \end{aligned} \tag{1.107}$$

where we have been careful in distinguishing

$$\not p \equiv \sum_0^3 \gamma_\mu p^\mu \tag{1.108}$$

from the extra components

$$\not P \equiv \sum_4^{n-1} P^\mu \gamma_\mu \tag{1.109}$$

Again, once the theory is regularized, we can translate the integration variables

$$p \to p + k_1 \tag{1.110}$$

The axial Ward identity reads

$$\begin{aligned} q^\lambda \Delta_{\lambda\mu\nu} = &- \int \frac{d^4 p}{(2\pi)^4} \frac{d^{n-4} P}{(2\pi)^{n-4}} \frac{tr\, (\not p + \not P + \not k_1) \not q \gamma_5 (\not p + \not P - \not k_2) \gamma_\nu (\not p + \not P) \gamma_\mu}{[(p + k_1)^2 - P^2][(p - k_2)^2 - P^2][p^2 - P^2]} \\ &+(k_1 \leftrightarrow k_2)(\mu \leftrightarrow \nu) \end{aligned} \tag{1.111}$$

The rules of the game mean that

$$\begin{aligned}
&\not{p}\not{p} = p^2 \\
&\not{P}\not{P} = -P^2 \\
&\not{p}\not{P} + \not{P}\not{p} = 0 \\
&(\not{p} + \not{P})(\not{q} + \not{P}) = p^2 - P^2
\end{aligned} \tag{1.112}$$

There are 18 different terms in the numerator. The computation simplifies using

$$\mathrm{Tr}\,\gamma_5\gamma_\alpha\gamma_\beta\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma = \frac{i}{2}\left(\eta_{\rho\sigma}\epsilon_{\alpha\beta\mu\nu} - \eta_{\nu\sigma}\epsilon_{\alpha\beta\mu\rho} + \eta_{\sigma\mu}\epsilon_{\alpha\beta\nu\rho} - \eta_{\beta\sigma}\epsilon_{\alpha\mu\nu\rho} + \eta_{\sigma\alpha}\epsilon_{\beta\mu\nu\rho}\right) \tag{1.113}$$

The only surviving terms after taking the trace are the ones proportional to $\not{P}\not{P}$:

$$4\ \mathrm{Tr}\ \gamma_5\gamma_\mu\gamma_\nu\not{k}_1\not{p}\int\frac{d^4p}{(2\pi)^4}\frac{d^{n-4}P}{(2\pi)^{n-4}}\frac{P^2}{[p^2-P^2][(p+k_1)^2-P^2][(p-k_2)^2-P^2]} \tag{1.114}$$

Introducing Feynman parameters and performing the momentum integral we get

$$16i\epsilon_{\mu\nu\alpha\beta}k_1^\alpha k_2^\beta\frac{1}{(2\pi)^4}\frac{\pi^2}{2i}2\int_0^1 dx_1\int_0^{1-x_1}dx_2\frac{d^{n-4}P}{(2\pi)^{n-4}}\frac{P^2}{P^2+f(x_1,x_2)} \tag{1.115}$$

The last integral is a particular instance of

$$\int\frac{d^nP}{(2\pi)^n}\frac{(P^2)^a}{(P^2+f)^b} = \frac{f^{a+b+n/2}}{(2\sqrt{\pi})^n}\frac{\Gamma(a+n/2)\Gamma(b-a-n/2)}{\Gamma(n/2)\Gamma(b)} \tag{1.116}$$

so that the physical four-dimensional limit

$$\lim_{n\to 4}\int\frac{d^{n-4}P}{(2\pi)^{n-4}}\frac{P^2}{P^2+f(x_1,x_2)} = -1 \tag{1.117}$$

where the finite value is the result of a cancellation

$$0\times\infty \tag{1.118}$$

due to the product

$$\frac{\Gamma(-\epsilon)}{\Gamma(\epsilon)} \tag{1.119}$$

These operators are often dubbed *evanescent operators*.

Finally we recover the result

$$q^\lambda\Delta_{\lambda\mu\nu} = -\frac{1}{2\pi^2}\epsilon_{\mu\nu\alpha\beta}k_1^\alpha k_2^\beta \tag{1.120}$$

1.5 Reminder of differential forms.

Consider an n-dimensional vector space $\vec{v} \in V$:

$$\vec{v}(f) \in \mathbb{R} \tag{1.121}$$

1-Forms live in the dual space, $\theta \in V^*$:

$$\underline{\theta}(\vec{v}) \in \mathbb{R} \tag{1.122}$$

To any function there is an canonical differential form defined as

$$\underline{df}(\vec{v}) \equiv \vec{v}(f) \tag{1.123}$$

To any local frame we define vectors as operators on functions, namely directional derivatives

$$\vec{e}_\alpha(f) \equiv \partial_\alpha f \tag{1.124}$$

The dual basis

$$\underline{\theta}^\beta(\vec{e}_\alpha) \equiv \delta^\beta_\alpha \tag{1.125}$$

In a local system of coordinates

$$dx^\alpha(\partial_\beta) = \partial_\beta x^\alpha = \delta^\alpha_\beta \tag{1.126}$$

Direct products are naturally defined

$$\Pi^r_s \equiv \underbrace{V^* \times \ldots \times V^*}_{r factors} \times \underbrace{V \times \ldots \times V}_{s factors} \tag{1.127}$$

Tensors are just multilineal applications

$$T : \Pi^r_s \to \mathbb{R} \tag{1.128}$$

We define the *tensor space* in a natural way

$$T^r_s \in \underbrace{V \otimes \ldots \otimes V}_{r factors} \times \underbrace{V^* \otimes \ldots \otimes V^*}_{s factors} \tag{1.129}$$

$$T^1_0 = V \tag{1.130}$$

$$T^0_1 = V^* \tag{1.131}$$

A basis in the tensor space is given by

$$\vec{v}_1 \otimes \ldots \vec{v}_r \otimes \underline{\omega}^1 \otimes \ldots \otimes \underline{\omega}^s(\underline{\eta}^1, \ldots \underline{\eta}^r, \vec{w}_1 \ldots \vec{w}_s) \equiv \underline{\eta}^1(\vec{v}_1) \ldots \underline{\eta}^s(\vec{v}_s) \tag{1.132}$$

A general tensor is defined as a linear combination of those objects,

$$T = T^{\alpha_1 \ldots \alpha_r}{}_{\beta_1 \ldots \beta_s} \vec{e}_{\alpha_1} \otimes \ldots \vec{e}_{\alpha_r} \otimes \underline{\theta}^{\beta_1} \ldots \otimes \underline{\theta}^{\beta_s} \tag{1.133}$$

It is useful to define Kronecker's tensor:

$$\delta^{\lambda_1\ldots\lambda_q}_{\mu_1\ldots\mu_q} \equiv q!\delta^{[\lambda_1}_{\mu_1}\ldots\delta^{\lambda_q]}_{\mu_q} \tag{1.134}$$

Differential forms are just covariant completely antisymmetric tensors

$$\underline{\alpha} = a_{\lambda_1\ldots\lambda_q}\underline{\theta}^{\lambda_1}\otimes\ldots\otimes\underline{\theta}^{\lambda_q} = \sum \alpha_{\iota_1\ldots\iota_q}\sum \delta^{\iota_1\ldots\iota_q}_{\lambda_1\ldots\lambda_q}\underline{\theta}^{\lambda_1}\otimes\ldots\otimes\underline{\theta}^{\lambda_q} \tag{1.135}$$

(with $\iota_1 < \ldots < \iota_q$).

It is a fact that the product of two Kronecker tensors is another Kronecker tensor

$$\delta^{\lambda_1\ldots\lambda_q}_{\mu_1\ldots\mu_q}\delta^{\mu_1\ldots\mu_q\nu_1\ldots\nu_{q'}}_{\sigma_1\ldots\sigma_{q+q'}} = q!\delta^{\lambda_1\ldots\lambda_q\nu_1\nu_{q'}}_{\sigma_1\ldots\sigma_{q+q'}} \tag{1.136}$$

The *exterior product* of a q-form and a q'-form is a $(q+q')$-form defined by:

$$(\underline{\alpha}\wedge\underline{\beta})_{\nu_1\ldots\nu_{q+q'}} \equiv \frac{1}{q!q'!}\delta^{\lambda_1\ldots\lambda_q\mu_1\ldots\mu_{q'}}_{\nu_1\ldots\nu_{q+q'}}\underline{\alpha}_{\lambda_1\ldots\lambda_q}\underline{b}_{\mu_1\ldots\mu_{q'}} \tag{1.137}$$

It is easy to prove that

$$\underline{\alpha}\wedge\underline{\beta} = (-1)^{qq'}\underline{\beta}\wedge\underline{\alpha} \tag{1.138}$$

as well as

$$\underline{\theta}^{\alpha}\wedge\underline{\theta}^{\beta} = \underline{\theta}^{\alpha}\otimes\underline{\theta}^{\beta} - \underline{\theta}^{\beta}\otimes\underline{\theta}^{\alpha} \tag{1.139}$$

We shall write

$$\underline{\alpha} = a_{\alpha\beta}\underline{\theta}^{\alpha}\otimes\underline{\theta}^{\beta} = \frac{1}{2}a_{\alpha\beta}\underline{\theta}^{\alpha}\wedge\underline{\theta}^{\beta} \tag{1.140}$$

or else in the natural basis associated to a local system of coordinates

$$\underline{\alpha} = \frac{1}{q!}a_{\alpha_1\ldots\alpha_q}dx^{\alpha_1}\wedge\ldots\wedge dx^{\alpha_q} \tag{1.141}$$

The *exterior differential* of a q-form is a $(q+1)$-form defined as:

$$d\underline{\alpha} = \frac{1}{(q+1)!}\frac{1}{q!}\delta^{\lambda_0\lambda_1\ldots\lambda_q}_{\mu_0\ldots\mu_q}\partial_{\lambda_0}\underline{\alpha}_{\lambda_1\ldots\lambda_q}dx^{\mu_0}\wedge\ldots dx^{\mu_q} \tag{1.142}$$

We shall denote

$$dx^{\alpha_1\ldots\alpha_p} \equiv dx^{\alpha_1}\wedge\ldots\wedge dx^{\alpha_p} \tag{1.143}$$

It can be easily shown that

$$d(\underline{\alpha}\wedge\underline{\beta}) = (d\underline{\alpha})\wedge\underline{\beta} + (-1)^p\underline{\alpha}\wedge(d\underline{\beta}) \tag{1.144}$$

as well as the fundamental nilpotency

$$d^2 = 0 \tag{1.145}$$

1.6 Gauge fields as one-forms.

Gauge fields can be considered as Lie algebra valued one-forms.

$$A \equiv -iA_\mu dx^\mu \tag{1.146}$$

and the field strencgth ris represented by the two-form

$$F = dA + A \wedge A \tag{1.147}$$

It is customary in this context to supress the wedge symbol, which is implicit in all products of forms. Let us check that

$$\text{Tr } A^{2n} = 0 \tag{1.148}$$

By cyclically permuting indices, we get a minus sign from the differentials, while the rest of the expression remains invariant. Let us spell this in detail in the case $n = 2$

$$\text{Tr } A^{abcd}_{\mu\nu\rho\sigma} T_{abcd} dx^{\mu\nu\rho\sigma} = tr A^{abcd}_{\mu\nu\rho\sigma} T_{dabc} dx^{\mu\nu\rho\sigma} = \text{Tr } A^{bcda}_{\mu\nu\rho\sigma} T_{abcd} dx^{\mu\nu\rho\sigma} = tr A^{abcd}_{\nu\rho\sigma\mu} T_{abcd} dx^{\nu\rho\sigma\mu} \tag{1.149}$$

which is equal to minus the original expression, which then has to vanish. The fact the the anomaly is a total derivative reads in the present language

$$\text{Tr } F \wedge F \equiv \text{Tr}\, F^2 = d\, Tr\, (A \wedge dA + \frac{2}{3} A \wedge A \wedge A) = d \text{ Tr } \left(AdA + \frac{2}{3} A^3 \right) \tag{1.150}$$

As a warmup, we can check that defining

$$\text{Tr } T_a T_b T_c \equiv T_{abc} \tag{1.151}$$

where for the fundamental of $SU(N)$

$$T_{abc} = \frac{1}{4} (i f_{abc} + d_{abc}) \tag{1.152}$$

$$\begin{aligned} &\text{Tr } BA^2 = T_{abc} B^a_\mu A^b_\nu A^c_\rho dx^{\mu\nu\rho} = B^a_\mu A^b_\nu A^c_\rho T_{abc} dx^{\mu\nu\rho} = \\ &= -B^a_\mu A^b_\nu A^c_\rho T_{abc} dx^{\mu\rho\nu} = -B^a_\mu A^b_\rho A^c_\nu T_{abc} dx^{\mu\nu\rho} = -B^a_\mu A^c_\nu A^b_\rho T_{abc} dx^{\mu\nu\rho} = \\ &= -B^a_\mu A^b_\nu A^c_\rho T_{acb} dx^{\mu\nu\rho} = B^a_\mu A^c_\nu A^b_\rho \frac{1}{2} (T_{abc} - T_{acb}) dx^{\mu\nu\rho} \end{aligned} \tag{1.153}$$

In arbitrary dimension n the equivalent of the four-dimensional triangle diagram is a diagram with $N = 1 + \frac{n}{2}$ legs, because in order to compite the divergence, we saturate one polarization with one momentum, which means that there are exactly $\frac{n}{2}$ momenta and polarizations to be contracted with the n-dimensional Levi-Civita tensor. In n=10 dimensions this yields the hexagon diagram.

The *abelian anomaly* is related with the *Chern character* (cf. [42],[22])

$$\text{ch } F \equiv \text{Tr } e^{\frac{i}{2\pi} F} \tag{1.154}$$

which is understood as a direct sum of dimension $2n$ forms proportional to

$$\text{Tr } F^n \equiv \text{Tr } \underbrace{F \wedge \ldots \wedge F}_{\text{n times}} \equiv \text{Tr } F^n \tag{1.155}$$

which can be integrated in a manifold of dimension $2n$.

For example, the four-dimensional abelian anomaly, in particular, is proportional to

$$\text{Tr } F^2 \tag{1.156}$$

All those forms are closed (and then locally exact, by Poincaré's theorem)

$$\text{Tr } F^n = d\omega_{2n-1} \tag{1.157}$$

where ω_{2n-1}, is the *formChern-Simons*, is given by ([42])

$$\omega_{2n-1} = n \int \text{Tr } dtA \wedge F_t^{n-1} \tag{1.158}$$

where

$$F_t = F(tA) \equiv tdA + t^2 A \wedge A. \tag{1.159}$$

Again in four dimensions $d = 4$ the Chern-Simons three-form reads

$$\omega_3 = tr(A \wedge F - \frac{1}{3} A^3). \tag{1.160}$$

$$\begin{aligned} d\omega_3 &= \text{Tr } \left(dAF - dAA^2\right) = \text{Tr } \left(\left(F - A^2\right) F - dAA^2\right) = \\ &= \text{Tr } \left(F^2 - A^2 dA - A^4 - dAA^2\right) = \text{Tr } F^2 \end{aligned} \tag{1.161}$$

(where we have used the facts that Tr $A^4 = 0$ and Tr $A^2 dA = -tr\, dAA^2$). This fact allows to express the anomaly as a total divergence. Indeed

$$\omega_3 \equiv \frac{1}{3!} \omega_{\mu\nu\rho} dx^{\mu\nu\rho} \tag{1.162}$$

implies

$$d\omega_3 = \frac{1}{4!} \frac{1}{3|} \delta^{\lambda_0\lambda_1\lambda_2\lambda_3}_{\mu_0\mu_1\mu_2\mu_3} \partial_{\lambda_0} \omega_{\lambda_1\lambda_2\lambda_3} dx^{\mu_0\mu_1\mu_2\mu_3} \tag{1.163}$$

and using the fact that

$$dx^{\lambda_0\lambda_1\lambda_2\lambda_3} = \epsilon^{\lambda_0\lambda_1\lambda_2\lambda_3}\ d(vol) \tag{1.164}$$

where the *volume element four form* is represented as

$$d(vol) \equiv dx^{0123} \tag{1.165}$$

Then

$$\begin{aligned} d\omega_3 &= \frac{1}{4!} \frac{1}{3!} \delta^{\lambda_0\lambda_1\lambda_2\lambda_3}_{\alpha_0\alpha_1\alpha_2\alpha_3} \epsilon^{\alpha_0\alpha_1\alpha_2\alpha_3} \partial_{\lambda_0} \omega_{\lambda_1\lambda_2\lambda_3} = \\ &= \frac{1}{3!} \delta^{\lambda_0\lambda_1\lambda_2\lambda_3} \partial_{\lambda_0} \omega_{\lambda_1\lambda_2\lambda_3} \tau = \partial_{\lambda_0} (*\omega_3)^{\lambda_0} \tau \end{aligned} \tag{1.166}$$

where

$$(*\omega_3)^{\lambda_0} \equiv \frac{1}{3!}\epsilon^{\lambda_0\lambda_1\lambda_2\lambda_3}\omega_{\lambda_1\lambda_2\lambda_3} \tag{1.167}$$

Recall now that the anomaly reads

$$\partial_\mu j_5^\mu = \frac{g^2}{16\pi^2}\mathrm{Tr}\ \epsilon^{\mu\nu\rho\sigma}\ F_{\mu\nu}\ F_{\rho\sigma} \tag{1.168}$$

as well as

$$F\wedge F = \frac{1}{4!}\frac{1}{2}\frac{1}{2}\delta^{\alpha\beta\gamma\lambda}_{\mu\nu\rho\sigma}\ F_{\alpha\beta}\ F_{\gamma\lambda}dx^{\mu\nu\rho\sigma} = \frac{1}{4!}\frac{1}{4}4!\epsilon^{\alpha\beta\gamma\lambda}\ F_{\alpha\beta}\ F_{\gamma\lambda}d(vol) \tag{1.169}$$

we are ledto write

$$\partial_\mu j_5^\mu\tau = \frac{g^2}{4\pi^2}\mathrm{Tr}\ F\wedge F = \frac{g^2}{4\pi^2}\partial_\mu(*\omega_3)^\mu\ d(vol) \tag{1.170}$$

Let us allow for possible higher order contributions by defining

$$a(g) \equiv \frac{1}{4\pi^2} \tag{1.171}$$

This function is such that to one loop order, $a(g)=1$. Then

$$\partial_\mu j_5^\mu = a(g)g^2\partial_\mu\left(*\omega_3\right)^\mu \tag{1.172}$$

It will be seen in due moment that the Chern-Simons current is not gauge invariant, but rather picks a total derivative under a gauge transformation

$$\delta(*\omega_3)^\mu = 2\epsilon^{\mu\nu\rho\sigma}\partial_\nu\ \mathrm{Tr}\ (g^2\Lambda\partial_\rho A_\sigma) \tag{1.173}$$

This fact in turn means that besides the gauge invariant but anomalous current j_5^μ we have been considering up to now, there is a conserved, but not gauge invariant modified current, to wit

$$k_5^\mu \equiv j_5^\mu - a(g)\ g^2(*\omega_3)^\mu \tag{1.174}$$

On the other hand, (cf.[35]) in four dimensiona there are self-dual solutions of the euclidean Yang-Mills equations such that the integral of the abelian anomaly is non vanishing (in spite of it being locally a total derivative). Properly normalized, this integral is an integer, dubbed the *instanton number* by physicists, and the *second Chern number* by mathematicians (cf. [22]).

$$\int d^4xF^2 = 8\pi^2 k = -8\pi^2 C_2 \tag{1.175}$$

This euclidean configurations generate transition amplitudes between states with different fermion number. In the WKB approximation, those are proportional to

$$e^{-S_{clas}} \sim e^{-\frac{8\pi^2 k}{g^2}} \tag{1.176}$$

which is very small, although the fact that it is nonvanishing is quite important physically.

1.7 The Adler-Bardeen theorem of the non-renormalization of the anomaly.

The formula (1.30) due to Adler-Bell-Jackiw is exact in the sense that it is possible to renormalize the theory in such a way that it is preserved by quantum corrections.

- Let us study an approach due to Breitenlohner, Maison and Stelle [10]. We have to study the renormalization of the composite operator k_5^μ. Owing to the fact that the current j_5^μ is gauge invariant, the matrix of anomalous dimensions must be of the form

$$\gamma = \begin{pmatrix} j_5^\mu \\ k_5^\mu \end{pmatrix} = \begin{pmatrix} \gamma_j & 0 \\ \gamma_{KJ} & \gamma_K \end{pmatrix} \begin{pmatrix} j_5^\mu \\ k_5^\mu \end{pmatrix} \tag{1.177}$$

Let us start from the anomalous Ward identity

$$\partial_\mu \langle 0|T\Big(j_5^\mu(x) - a(g)(*\omega_3)^\mu(x)\Big)X|0\rangle + \langle 0|T\frac{\delta X}{\delta\psi(x)}\gamma_5\psi(x)|0\rangle + \langle 0|T\bar\psi(x)\gamma_5\frac{\delta X}{\delta\bar\psi(x)}|0\rangle = 0 \tag{1.178}$$

where

$$X \equiv \prod_i gA_{\mu_i}(x_i)\prod_j \psi(y_j)\prod_k \bar\psi(z_k) \tag{1.179}$$

The renormalization group operator

$$\mathcal{D} \equiv \mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g} + \delta\frac{\partial}{\partial\alpha} + \gamma_\psi N_\psi \tag{1.180}$$

(where N_ψ counts the total number of factors ψ or $\bar\psi$) leaves invariant the Green functions, so that

$$\mathcal{D}\langle 0|TX|0\rangle = 0 \tag{1.181}$$

When there are current insertions

$$(\mathcal{D} + \gamma_J)\,\langle 0|Tj_5^\mu(x)X|0\rangle = 0 \tag{1.182}$$

as well as

$$(\mathcal{D} + \gamma_K)\,\langle 0|(*\omega_3)^\mu(x)X|0\rangle + \gamma_{KJ}\langle 0|j_5^\mu(x)X|0\rangle = 0 \tag{1.183}$$

Choose now $X \equiv g^2A_\mu(y)A_\nu(z)$. Then

$$\begin{aligned}
&\partial_\mu\langle 0|T\left(j_5^\mu(x) - a(g)(*\omega_3)^\mu(x)\right)g^2A_\mu(y)A_\nu(z)|0\rangle = 0 = \\
&\partial_\mu\left(-\gamma_J\langle 0|Tj_5^\mu g^2AA|0\rangle - \mathcal{D}a(g)\langle 0|T(*\omega_3)^\mu g^2AA|0\rangle\right. \\
&\left.-a(g)[-\gamma_K\langle 0|T(*\omega_3)^\mu g^2AA|0\rangle - \gamma_{KJ}\langle 0|Tj_5^\mu g^2AA|0\rangle]\right)
\end{aligned} \tag{1.184}$$

which can also be written as

$$(a(g)\gamma_{KJ} - \gamma_J)\langle 0|T\partial_\mu j_5^\mu g^2AA|0\rangle + (a(g)\gamma_K - \mathcal{D}a(g))\langle 0|T\partial_\mu(*\omega_3)^\mu g^2AA|0\rangle = 0 \tag{1.185}$$

Choosing instead $X = \psi(y)\bar{\psi}(z)$ we get

$$\partial_\mu \langle 0|T\left(j_5^\mu(x) - a(g)(*\omega_3)^\mu(x)\right)\psi(y)\bar{\psi}(z)|\rangle + \nabla(x-y)\langle 0|\bar{\psi}\gamma_5\psi|0\rangle + \nabla(x-z)\langle 0|\bar{\psi}\gamma_5\psi|0\rangle = 0 =$$
$$\partial_\mu\left(-\gamma_J\langle 0|Tj_5^\mu\psi(y)\bar{\psi}(z)|0\rangle - \mathcal{D}a(g)\langle 0|T(*\omega_3)^\mu\psi(y)\bar{\psi}(z)\right.$$
$$\left.|0\rangle - a(g)[-\gamma_K\langle 0|T(*\omega_3)^\mu g^2 AA|0\rangle - \gamma_{KJ}\langle 0|Tj_5^\mu\psi(y)\bar{\psi}(z)|0\rangle]\right) \tag{1.186}$$

we conclude that

$$\gamma_J - a(g)\gamma_{KJ} = 0 \tag{1.187}$$

and

$$(\beta\frac{\partial}{\partial g} - \gamma_K)a(g) = 0 \tag{1.188}$$

Owing to the topological meaning of the Chern-Simons current, it is natural to renormalize in such a way that

$$\gamma_K = 0 \tag{1.189}$$

implying that the coefficient $a(g)$ is actually independent of the coupling constant.

- It is also useful to recall the original proof (cf. [3]). Regularizing with a Pauli-Villars field A_μ^R of mass M the lagrangian reads

$$L = \bar{\psi}(i\not{D} - m_0)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{4}F^R_{\mu\nu}F_R^{\mu\nu} - \frac{M^2}{2}A^R_\mu A^R_\mu$$
$$-e_0\bar{\psi}\gamma_\mu\psi(A^\mu + A^{R\,\mu}) + C[F_{\mu\nu} + F_{R\,\mu\nu}][F^{R\,\mu\nu} + F^{\mu\nu}] \tag{1.190}$$

In the original reference [3] it was found useful to distinguish two different cases. The first is when the axial current hooks outside a loop. Those terms are not anomalous.

The second case, which is the real origin of the anomaly, is when the axial current hooks to a fermion loop of the type

$$T^{\mu_1\ldots\mu_k\mu\mu_{k+1}\ldots\mu_{2n}} \equiv \int d^4p\,\mathrm{Tr}\sum_{k=1}^{2n}\prod_{j=1}^{k-1}\gamma^{\mu_j}\frac{1}{(\not{p}+\not{p}_j) - m_0}.$$
$$\gamma^{\mu_k}\frac{1}{(\not{p}+\not{p}_k) - m_0}\gamma_5\gamma^\mu\frac{1}{(\not{p}+\not{p}_k - \not{q}) - m_0}\prod_{j=k+1}^{2n}\gamma^{\mu_j}\frac{1}{(\not{p}+\not{p}_j - \not{q}) - m_0} \tag{1.191}$$

where the gauge insertion carries momentum q. Compute now the divergence

$$q_\mu T^{\mu_1\ldots\mu_k\mu\mu_{k+1}\ldots\mu_{2n}} \tag{1.192}$$

by using the identity

$$\frac{1}{(\not{p}+\not{p}_k) - m_0}\not{q}\gamma_5\frac{1}{(\not{p}+\not{p}_k - \not{q}) - m_0} = \frac{1}{(\not{p}+\not{p}_k) - m_0}2m_0\gamma_5\frac{1}{(\not{p}+\not{p}_k - \not{q}) - m_0} +$$
$$\frac{1}{(\not{p}+\not{p}_k) - m_0}\gamma_5 + \gamma_5\frac{1}{(\not{p}+\not{p}_k - \not{q}) - m_0} \tag{1.193}$$

This identity actuay stems from the obvious one

$$\not{q}\gamma_5 = 2m_0\gamma_5 + \gamma_5((\not{p}+\not{p}_k-\not{q})-m_0)+((\not{p}+\not{p}_k)-m_0)\gamma_5 \tag{1.194}$$

Let us forget all terms proportional to m_0, because we are going to be interested anyway in the massless chiral limit

$$\begin{aligned} q_\mu T^{\mu_1\ldots\mu_k\mu\mu_{k+1}\ldots\mu_{2n}} &= \int d^4p\,\mathrm{Tr}\sum_{k=1}^{2n}\prod_{j=1}^{k-1}\gamma^{\mu_j}\frac{1}{\gamma.(p+p_j)-m_0}. \\ &\gamma^{\mu_k}[\frac{1}{(\not{p}+\not{p}_k)-m_0}\gamma_5+\gamma_5\frac{1}{(\not{p}+\not{p}_k-\not{q})-m_0}] \\ &\prod_{j=k+1}^{2n}\gamma^{\mu_j}\frac{1}{(\not{p}+\not{p}_j-\not{q})-m_0} \end{aligned} \tag{1.195}$$

In this expression we have terms of the type

$$\gamma_5\gamma_{k+1}+\gamma_{k+1}\gamma_5 = 0 \tag{1.196}$$

for all values of the index k except the first $k=1$ and the last, $k=2n$. Using the cyclic property the result is proportional to

$$\begin{aligned} q_\mu T^{\mu_1\ldots\mu_k\mu\mu_{k+1}\ldots\mu_{2n}} &\sim \int d^4p\,\mathrm{Tr}\;\gamma_5\prod_{j=1}^{2n}\gamma^{\mu_j}\frac{1}{(\not{p}+\not{p}_j)-m_0}. \\ &-\gamma_5\prod_{j=1}^{2n}\gamma^{\mu_j}\frac{1}{(\not{p}+\not{p}_j-\not{q})-m_0} \end{aligned} \tag{1.197}$$

These integrals cancel if $n\geq 2$ (when they are convergent); whereas when $n=2$ they correspond to the one loop anomaly already computed.

2 The Wess-Zumino consistency conditions and the gauge anomaly

$\Gamma[A]$ is the generator of 1PI Green functions. This fact implies some consistency conditions, which are almost enough to determine the form of the anomalies. The proper setup for those consistency conditions is the algebraic BRST symmetry as worked out by Stora and Zumino.

Write first the gauge variations as

$$T_\Lambda A^a_\mu = -(D_\mu\Lambda)^a = -(\partial_\mu\Lambda - i[A_\mu,\Lambda]) = -(\partial_\mu\Lambda^a + f_{abc}A^b_\mu\Lambda^c) \tag{2.1}$$

as well as

$$T_\Lambda F^a_{\mu\nu} = -i([F_{\mu\nu},\Lambda])^a \tag{2.2}$$

They in turn determine the variation of all functionals of gauge fields

$$T_\Lambda \Gamma[A] = \int d^n x \frac{\delta \Gamma}{\delta A^a_\mu} T_\Lambda A^a_\mu \tag{2.3}$$

Denoting $J^\mu_a \equiv \frac{\delta\Gamma}{\delta A^a_\mu}$ we can write

$$T_\Lambda \Gamma[A] = -\int d^n x J^\mu_a (D_\mu \Lambda)^a = \int_x D_\mu J^\mu_a \Lambda^a = \int_x \Lambda_a(x) G^a(x) \tag{2.4}$$

where

$$G_a(x) \equiv X_a(x)\Gamma[A] \tag{2.5}$$

where

$$X_a \equiv D_\mu \frac{\delta}{\delta A^a_\mu} = \partial_\mu \frac{\delta}{\delta A^a_\mu} + f_{abc} A^b_\mu \frac{\delta}{\delta A^c_\mu} \tag{2.6}$$

These operators represent the Lie algebra of the gauge group

$$[X_a(x), X_b(y)] = i f^c_{ab} X_c(x) \delta^n(x-y) \tag{2.7}$$

This is in fact equivalent to

$$\left(T_\Lambda T_{\Lambda'} - T_{\Lambda'} T_\Lambda\right)\Gamma[A] = T_{[\Lambda,\Lambda']}\Gamma[A] \tag{2.8}$$

The consistency conditions for the quantities G_a stem from the fact that they come from a single generating functional $\Gamma[A]$, and can be expressed as

$$X_a(x)G_b(y) - X_b(y)G_a(x) = (X_a X_b - X_b X_a)\Gamma = i f^c_{ab} G_c \delta_{xy} \tag{2.9}$$

It is plain than any expression of the type

$$G_a(x) = X_a(x) F[A] \tag{2.10}$$

where $F[A]$ is an arbitrary local functional is a solution of (2.9). Those are the so called *trivial solutions*. They represent physically the addition to the action of finite local counterterms, that is, a change in the renormalization conditions.

We define the *comsistent anomalies* as any nontrivial solution of the consistency conditions. The fact that they cannoy be eliminated by a local coumnterterm means that they are independent of the ultraviolet physics.

Let us rewrite all this in BRST language. Define

$$cX \equiv \sum_a \int d^n x c_a(x) X_a(x) \tag{2.11}$$

Then

$$(cX)(cG) - \frac{1}{2} f_{abc} c^b c^c G_a = 0 \tag{2.12}$$

Even more consise is Slavnov's form

$$s\,(cG) = 0 \tag{2.13}$$

Just emember that BRST variations can be expressed as

$$sA_\mu = (cX)A_\mu \tag{2.14}$$

and

$$sc_a = -\frac{1}{2}f_{abc}c^b c^c \tag{2.15}$$

What has to vanish is really the integral, so we can allow the variation of the integrand to become an exact differential

$$s\,\mathrm{Tr}\; cG = d\chi \tag{2.16}$$

The anomaly that appears in the chiral current

$$j_a^\mu \equiv i\bar{\psi}\gamma_5\gamma^\mu T_a\psi \tag{2.17}$$

is usually called the *gauge anomaly* because this current is the one that couples to the non-abelian gauge field. It is easier to work with the chiral currents

$$j^\mu_{a(L)} \equiv i\bar{\psi}_L\gamma^\mu T_a\psi_L \tag{2.18}$$

$$j^\mu_{a(R)} \equiv i\bar{\psi}_R\gamma^\mu T_a\psi_R \tag{2.19}$$

The result is

$$D^L_\mu J^\mu_{a(L)} = C_a(A_L) = \frac{1}{24\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}\; T_a\partial_\mu\left(A^L_\nu\partial_\rho A^L_\sigma - \frac{1}{2}iA^L_\nu A^L_\rho A^L_\sigma\right) \tag{2.20}$$

and with a change of sign

$$D^R_\mu J^\mu_{a(R)} = -C_a(A_R) = -\frac{1}{24\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}\; T_a\partial_\mu\left(A^R_\nu\partial_\rho A^R_\sigma - \frac{1}{2}iA^R_\nu A^R_\rho A^R_\sigma\right) \tag{2.21}$$

The non-abelian anomaly is also a total derivative

$$C_a(A) = d\,\mathrm{Tr}\; T_a(A\wedge dA + \frac{1}{2}A\wedge A\wedge A) \tag{2.22}$$

The covariant form of the anomaly reads

$$D_\mu\tilde{J}^\mu_a = \frac{3}{2}A(R)d_{abc}F^b_{\mu\nu}F^c_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma} \tag{2.23}$$

where the representation-dependent coefficients are defined from

$$\mathrm{Tr}\;\{T_a, T_b\}T_c \equiv A(R)d_{abc} \tag{2.24}$$

2.1 The Stora-Zumino algebraic formalism.

The cohomological analysis of Stora and Zumino (cf. [42]) is by far the simplest way to find solutions to the consistency conditions. The starting point is a *characteristic polynomial*, that is, an *invariant symmetric polynomial* which by definition obeys

$$P(F_1, \ldots, F_n) \tag{2.25}$$

such that

$$P(F_{p(1)}, \ldots, F_{p(n)}) = P(F_1, \ldots, F_n) \tag{2.26}$$

for any permutation $p \in S_n$, and

$$P(g^{-1}F_1 g, \ldots, g^{-1}F_n g) = P(F_1, \ldots, F_n) \tag{2.27}$$

This imput must be external; that is, it is not provided by the algebraic analysis itself. The simplest example is of course the abelian anomaly

$$P \equiv \text{Tr } F^n \tag{2.28}$$

We shall see in a moment that we can get the non-abelian anomaly in dimension $d = 2n-2$ out of the abelian anomaly in dimension $d = 2n$.

Define the exterior differential acting on polynomia containing A and F, $P(A, F)$ such that $P(0, 0) = 0$ starting with

$$\begin{aligned} dA &= F - A^2 \\ dF &= FA - AF \end{aligned} \tag{2.29}$$

It is plain that

$$d^2 = 0. \tag{2.30}$$

Actually,

$$\begin{aligned} d^2 A &= FA - AF - (F - A^2)A + A(F - A^2) = 0 \\ d^2 F &= (FA - AF)A + F(F - A^2) - (F - A^2)F + A(FA - AF) \end{aligned} \tag{2.31}$$

Every invariant polynomial is closed

$$dP\,(F^n) = 0 \tag{2.32}$$

and locally exact. For example, in the case of the abelian anomaly

$$\text{Tr}\, d\,(F^n) = n\text{Tr}\,\left(dF F^{n-1}\right) = n\text{Tr}\,\left((FA - AF)\, F^{n-1}\right) = 0 \tag{2.33}$$

owing to the cyclic property of the trace.

There is a generalization of the Chern-Simons form

$$P(F^n) = d\omega_{2n-1}(A, F) \tag{2.34}$$

namely

$$\omega_{2n-1}(A, F) = n \int_0^1 dt P(A, F_t^{n-1}) \tag{2.35}$$

Let us see how this comes about.

In order to do that, let us define formally ([6]) three operators l, δ and the BRST s through

$$\begin{array}{ccc} lA = 0 & \delta A = B & sA = -Dc \equiv -dc - Ac - cA \\ lF = B & \delta F = dB + BA + AB & sF = Fc - cF \\ lc = 0 & \delta c = 0 & sc = -c^2 \\ lB = 0 & \delta B = 0 & sB = -cB - Bc \end{array} \tag{2.36}$$

The derivations s, l and d are odd, whereas δ is even. It is easy to check that

$$\begin{aligned} & l^2 = \delta^2 = s^2 = 0 \\ & \{l, d\} = \delta \\ & \{s, d\} = \{l, s\} = 0. \end{aligned} \tag{2.37}$$

Let us check for example the middle property. Acting on A,

$$(ld + dl)A = ldA = l(F - A^2) = B = \delta A. \tag{2.38}$$

Acting on F

$$(ld + dl)F = l(FA - AF) + d\delta A = \delta AA + A\delta A + d\delta A \equiv \delta F. \tag{2.39}$$

Here we have used that $[d, \delta] = 0$.

It is easy to define families of fields that interpolate between $A = 0$ and a given configuration; for example

$$A_t \equiv tA \tag{2.40}$$

for $0 \leq t \leq 1$.

To that it corresponds a field strength

$$F_t = tdA + t^2A^2 = tF + (t^2 - t)A^2 \tag{2.41}$$

For this type of one-parameter families we define in an analogous way

$$\begin{aligned} & l_t A_t = 0 \\ & l_t F_t = \delta A_t \equiv \delta t \frac{\partial A_t}{\partial t} \end{aligned} \tag{2.42}$$

We shall now state a particular instance of the *extended Cartan homotopy formula*

$$\{d, l_t\} = \delta \equiv \delta t \frac{\partial}{\partial t} \tag{2.43}$$

(In the sequel we shall always define the operator δ in this explicit way, from the derivative with respect to whatever parameters the second member depends on.)

Let us check this last result. Indeed, acting on A_t,

$$(l_t d + d l_t) A_t = l_t d A_t = l_t (F_t - A_t^2) = \delta A_t \tag{2.44}$$

Acting on F_t,

$$(l_t d + d l_t) F_t = l_t (F_t A_t - A_t F_t) + d\delta A_t = \delta A_t A_t + A_t \delta A_t + d\delta A_t \equiv \delta F_t \tag{2.45}$$

as a result of our previous definition [2.37].

Now we introduce yet another operator, the *homotopy operator* through

$$kP \equiv \int_0^1 l_t P_t \tag{2.46}$$

Let us show that in the particular case when $P = d(AF)$, then

$$k\ d(AF) = \frac{1}{2}(AF + FA) \tag{2.47}$$

This follows from

$$\begin{aligned} l_t d(A_t F_t) &= l_t (dA_t F_t - A_t dF_t) = l_t \big((F_t - A_t^2) F_t - A_t (F_t A_t - A_t F_t) \big) = \\ &= \delta A_t F_t + F_t \delta A_t - A_t^2 \delta A_t + A_t \delta A_t A_t + A_t^2 \delta A_t = \delta t \left(A F_t + F_t A + t^2 A^3 \right) \end{aligned} \tag{2.48}$$

in such a way that

$$\begin{aligned} \int_0^1 l_t d(A_t F_t) &= \int_0^1 \delta t \Big\{ A \left(tF + (t^2 - t) A^2 \right) + \left(tF + (t^2 - t) A^2 \right) A + t^2 A^3 \Big\} = \\ &= \frac{1}{2} AF + \left(\frac{1}{3} - \frac{1}{2} \right) A^3 + \frac{1}{2} FA + \left(\frac{1}{3} - \frac{1}{2} \right) A^3 + \frac{1}{3} A^3 = \frac{1}{2}(AF + FA) \end{aligned} \tag{2.49}$$

Owing to (2.43), this operator obeys

$$\{k, d\} = 1 \tag{2.50}$$

Let us check this fact for the particular polynomial $P = AF$. Indeed $l(AF) = -A\delta A$ so that

$$l_t (A_t F_t) = -A_t \delta A_t = -\delta t t A^2 \tag{2.51}$$

and

$$k\ \ (AF) = \int_0^1 l_t (A_t F_t) = -A^2/2 \tag{2.52}$$

Then

$$d\ k(AF) = -\frac{1}{2}dAA + \frac{1}{2}AdA = -\frac{1}{2}(F - A^2)A + \frac{1}{2}A(F - A^2) = -\frac{1}{2}(FA - AF) \quad (2.53)$$

We can also check on the same polynomial that

$$k^2 = 0. \quad (2.54)$$

Indeed

$$k^2\ (AF) = k\left(-\frac{A^2}{2}\right) = 0 \quad (2.55)$$

This is a useful fact, because

$$\text{Tr}\ F^n = (dk + kd)\ \text{Tr}\ F^n = d\left(k\text{Tr}\ F^n\right) \quad (2.56)$$

which yields an explicit formula for the Chern-Simons form in arbitrary dimension

$$\omega^0_{2n-1} = k\,\text{Tr}\ F^n = \int_0^1\ dt\, l_t\,\text{Tr}\ F^n_t = \int_0^1\ dt\,\text{Tr}\ \left(AF_t^{n-1} + \ldots + F_t^{n-1}A\right) = n\int_0^1\ dt\,\text{Tr}\ \left(AF_t^{n-1}\right) \quad (2.57)$$

When $n = 2$ we recover the well-known formula

$$\omega_3 = 2\int_0^1 dt\ \text{Tr}\ \ A\left(tdA + t^2A^2\right) = \text{Tr}\ \left(AdA + \frac{2}{3}A^3\right) \quad (2.58)$$

It is interesting to study the behavior of the Chern-Simons forms under gauge transformations

$$A_g = g^{-1}Ag + g^{-1}dg \quad (2.59)$$

and

$$F_g = g^{-1}Fg \quad (2.60)$$

We shall prove that the desired behavior is

$$\omega^0_{2n-1}(A_g, F_g) = \omega^0_{2n-1}(A, F) + d\alpha_{2n-2} + \omega^0_{2n-1}(g^{-1}dg, 0) \quad (2.61)$$

Let us denote

$$V \equiv dgg^{-1} \quad (2.62)$$

the right invariant one-form, in such a way that

$$dV = V^2 \quad (2.63)$$

By construction we have

$$\omega^0_{2n-1}(A_g, F_g) = \omega^0_{2n-1}(g^{-1}Ag + g^{-1}dg, g^{-1}Fg) = n\int_0^1\ dt\ \text{Tr}\ \left(g^{-1}Agg^{-1}F_t^{n-1}g + g^{-1}dgg^{-1}F_t^{n-1}g\right) = \\ = \omega^0_{2n-1}(A + V, F) \quad (2.64)$$

Consider now the object

$$\Omega \equiv \omega^0_{2n-1}(A+V,F) - \omega^0_{2n-1}(V,0) - \omega^0_{2n-1}(A,F) \tag{2.65}$$

It is possible to check in general that is a closed form

$$d\Omega = 0 \tag{2.66}$$

Let us verify this fact in ther particular case when $n = 2$:

$$\begin{aligned}
\Omega_3 &= \mathrm{Tr}\left((A+V)F - \frac{1}{3}(A+V)^3 + \frac{1}{3}V^3 - AF + \frac{1}{3}A^3 \right) = \\
&= \mathrm{Tr}\left(VF - \frac{1}{3}(A^2V + AVA + AV^2 + VA^2 + VAV + V^2A) \right) = \\
&= \mathrm{Tr}\left(VF - A^2V - AV^2 \right)
\end{aligned} \tag{2.67}$$

In order to take the exterior differential, we take into account that

$$\begin{aligned}
dA &= F - A^2 \\
dV &= V^2 \\
dF &= FA - AF
\end{aligned} \tag{2.68}$$

It follows that

$$\begin{aligned}
d\Omega = \mathrm{Tr}\Big(& V^2F - V\left(FA - AF\right) - \left(F - A^2\right)AV + A\left(F - A^2\right)V - A^2V^2 - \left(F - A^2\right)V^2 + \\
& + A\left(V^3 - V^3\right)\Big) = 0
\end{aligned} \tag{2.69}$$

We can repeat now the same trick used before

$$(kd + dk)\ \Omega = \Omega = d\ (k\Omega) \equiv d\ \alpha_{2n-2} \tag{2.70}$$

where to be specific

$$\alpha_{2n-2} = k\omega^0_{2n-1}(A+V,F) \tag{2.71}$$

which proves the desired formula.

Another useful formula which we shall not prove is

$$\alpha_{2n-2} = -n(n-1)\int_0^1 \delta\lambda \int_0^{1-\lambda} \delta\mu\ \mathrm{Str}\left(VA\mathcal{F}^{n-2}_{\lambda,\mu}\right) \tag{2.72}$$

where we have used the notation

$$\mathcal{F}_{\lambda,\mu} \equiv \lambda A - \mu V \tag{2.73}$$

and denoted the *symmetrized trace* as

$$\mathrm{Str}\, M_1 \ldots M_p \equiv \sum_{\pi \in S_p} \mathrm{Tr}\, \left(M_{\pi(1)} \ldots M_{\pi(p)} \right) \tag{2.74}$$

The proof is simple although some labor is needed. It can be easily checked that

$$\alpha_2 = -\mathrm{Str}\,(VA) \tag{2.75}$$

as well as

$$\alpha_4 = -\mathrm{Str}\,(VAdA) + \frac{1}{2}\mathrm{Tr}\, \left(VA^3 - V^3 A - \frac{1}{2} VAVA \right) \tag{2.76}$$

All this certainly means that the Chern-Simons form is not gauge invariant, but rather its variation is an exact differential that we shall compute in a more explicit way in the next paragraph.

2.2 The Stora-Zumino descent equations.

This is the algebraic way to recover the anomaly, given the appropiate starting point, that is, the invariant polynomial. It is duly credited to Stora and Zumino, although they never published a joint paper on this. It is to be found however in Zumino's les Houches lectures [**?**]. Let us dubb

$$\mathcal{A} = A + c \tag{2.77}$$

and define the field strength

$$\mathcal{F} = (d+s)\mathcal{A} + \mathcal{A}^2 \tag{2.78}$$

It is plain that

$$\mathcal{F} = (d+s)(A+c) + (A+c)^2 = dA - Dc + dc - c^2 + A^2 + cA + Ac + c^2 = F \tag{2.79}$$

This is the famous *Russian formula.* This means that the relationship of the operator $d+s$ with the gauge field $A+c$ is exactly the same as the one of the exterior differential d with the ordinary gauge field, A.

It follows that we can write

$$\mathrm{Tr}\; F^n = \mathrm{Tr}\; \mathcal{F}^n = d\omega^0_{2n-1}(A,F) = (d+s)\omega^0_{2n-1}(A+c,F) \tag{2.80}$$

The Chern-Simons $\omega^0_{2n-1}(A+c,F)$ can now be expanded in powers of the ghost field

We shall use such a notation that the superindex indicates the ghost number, whereas the subindex indicates the degree as a differential form.

$$\omega^0_{2n-1}(A+c,F) = \omega^0_{2n-1}(A,F) + \omega^1_{2n-2}(A,F) + \ldots + \omega^{2n-1}_0(A,F) \tag{2.81}$$

Now, owing to the equality (2.80)

$$(d+s)\left(\omega^0_{2n-1} + \ldots + \omega^{2n-1}_0\right) = d\omega^0_{2n-1} \tag{2.82}$$

Just by identifying terms with the same ghost number we bet the *Stora-Zumino descent equations*

$$\begin{aligned}
&s\omega^0_{2n-1} + d\omega^1_{2n-2} = 0\\
&s\omega^1_{2n-2} + d\omega^2_{2n-3} = 0\\
&\ldots\\
&s\omega^{2n-2}_1 + d\omega^{2n-1}_0 = 0\\
&s\omega^{2n-1}_0 = 0
\end{aligned} \tag{2.83}$$

To start with, these equations imply the gauge variation of the Chern-Simons For example,

$$s\omega^0_3 = -d\omega^1_2 \tag{2.84}$$

Not only that, but we also obtain a solution to the consistency relations; that is a consistent anomaly. In particular, something linear in ghosts, ω^1_{2n-2}, obeys the equations (2.16),with

$$\chi = -\omega^2_{2n-3} \tag{2.85}$$

Let us summarize. We started with the abelian anomaly in $d = 2n$ dimensions

$$\mathrm{Tr}\, F^n \tag{2.86}$$

and we expressed it as a differential of the corresponding Chern-Simons form

$$\mathrm{Tr}\ F^n = d\omega^0_{2n-1} \tag{2.87}$$

The BRST variation of the Chern-Simons is the differential of the non-abelian anomaly in dimension $d = 2n - 2$; we have descended two dimensiona.

$$s\omega^0_{2n-1} = -d\ \omega^1_{2n-2} \tag{2.88}$$

An explicit calculation leads to [42])

$$\omega^1_{2n-2} = n(n-1)\int\ \delta t\ (1-t)\mathrm{Str}\, c\, d\left(\mathcal{A}\mathcal{F}_t^{n-2}\right). \tag{2.89}$$

In other circumstances, the integrand needs to be replaced by an invariant symmetric polynomial to be determined by a diagrammatic calculation, or else by the index theorem.

It is possible to check that

$$\omega^1_2 = I_R\ c^a\ \partial_\mu A^a_\nu \epsilon^{\mu\nu} d(vol). \tag{2.90}$$

as well as the formula for the four-dimensional gauge anomaly

$$\omega^1_4 = \epsilon^{\mu\nu\rho\sigma}\mathrm{Str}\, c\ \partial_\mu\left(A_\nu\partial_\rho A_\sigma + \frac{1}{2}A_\nu A_\rho A_\sigma\right)\ d(vol) \tag{2.91}$$

Although it might seem from the formula that the anomaly depends on the symmetrized trace of the product of three and four gauge generators, this is a delusion, as shown in the basic formula (2.89).

In general the quantity

$$C_3(R) \equiv g_{abc} T^a T^b T^c \tag{2.92}$$

where the tensor g_{abc} is defined out of the symmetrized traces in the fundamental representation

$$g_{abc} \equiv \mathrm{Str}\, T_a^F T_b^F T_c^F = \frac{1}{6} \mathrm{Tr}\, \left(T_a^F T_b^F T_c^F + T_c^F T_a^F T_b^F + T_b^F T_c^F T_a^F + T_b^F T_a^F T_c^F + T_a^F T_c^F T_b^F + T_c^F T_b^F T_a^F \right)$$

is a Casimir operator, which is constant in every representation. Actually, all Casimir operators can be obtained by expending the expression

$$D(\lambda) \equiv \det \left(\lambda - D_R \left(X^a T_a \right) \right) \tag{2.93}$$

(cf. [38]). Let us define an index

$$\frac{1}{2} tr\, T_a \{T_b, T_c\} \equiv I_3(R) g_{abc} \tag{2.94}$$

Consistency with our previous definition implies that

$$g_{abc} = 2 I_3(F) g_{abc} \tag{2.95}$$

which means that we have normalized

$$I_3(F) = \frac{1}{2} \tag{2.96}$$

The knowledge of this index leads directly to the anomaly, because

$$C_3(R) d_R = g^{abc} I_3(R) g_{abc} \tag{2.97}$$

It is a fact that

$$I_3(R_1 \oplus R_2) = I_3(R_1) + I_3(R_2) \tag{2.98}$$

as well as

$$I_3(R_1 \otimes R_2) = d_{R_1} I_3(R_2) + d_{R_2} I_3(R_1) \tag{2.99}$$

which implies that

$$I_3(\bar{R}) = -I_3(R) \tag{2.100}$$

Those formulas are very useful for explicit calculations.

When working in general space-time dimension, and remembering that the anomaly changes sign for left versus right fermions it is clear that the condition for a theory to be anomaly free (which is necessary for consistency of the $d = 2n - 2$ dimensional quantum theory) is just

$$\sum_L \mathrm{Str}\,(T_{a_1} \ldots T_{a_n}) = \sum_R Str\ (T_{a_1} \ldots T_{a_n}) \tag{2.101}$$

where the sum runs over all fermions that couple to the gauge field

In d=4 dimensional this leads to (cf. [27])

$$\sum_L A(R_L) = \sum_R A(R_R) \tag{2.102}$$

Now, given some matrix representation of a group G

$$g \to D(g) \tag{2.103}$$

its complex conjugate is also a representation

$$g \to D^*(g) \tag{2.104}$$

This looks in the Lie algebra

$$-T_a^* = -T_a^T \tag{2.105}$$

The representation is *pseudoreal* if it is equivalent to its complex conjugate

$$-T_a^T = S T_a S^{-1} \tag{2.106}$$

In this case

$$A_{abc} \equiv \mathrm{Tr}\ \big(\{T_a, T_b\} T_c\big) = \mathrm{Tr}\ \big(T_c^T \{T_a^T, T_b^T\}\big) = -A_{abc} = 0 \tag{2.107}$$

For the abelian anomaly this is the same as demanding that

$$\sum Q_L^3 = \sum Q_R^3 \tag{2.108}$$

2.3 Anomaly cancellation in the Standard Model.

When there is an anomaly in a current that couples to a gauge field, gauge invariance is spoiled and the resulting theory is non renormalizable.

In the standard model there is a curious anomaly cancellation which takes place generation by generation.

Quarks *up left* are a $SU(2)$ doublet,

$$q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix} \tag{2.109}$$

and they transform with the representation of the gauge group

$$G \equiv SU(3) \times SU(2) \times U(1) \tag{2.110}$$

as

$$q_i \in (3 \otimes 2 \otimes 1/6) \tag{2.111}$$

Here each representation is characterized by its dimension, except in the case of hypercharge $U(1)_Y$, where we indicate the abelian charge, which is half of the value of the hypercharge. This is normalized by

$$Q = T_3 + \frac{Y}{2} \tag{2.112}$$

so that for example

$$\begin{aligned} Q(u_L) &= \frac{1}{2} + \frac{1}{6} = \frac{2}{3} \\ Q(d_L) &= -\frac{1}{2} + \frac{1}{6} = -\frac{1}{3}. \end{aligned} \tag{2.113}$$

Turning now to quarks *right*

$$u_R \in (\bar{3} \otimes 1 \otimes 2/3) \tag{2.114}$$

Then

$$Q(u_R) = 0 + \frac{2}{3} = \frac{2}{3}. \tag{2.115}$$

$$d_R \in (\bar{3} \otimes 1 \otimes -1/3) \tag{2.116}$$

The lepton doublet reads

$$l_i \equiv \begin{pmatrix} \bar{\nu}_L \\ e_L \end{pmatrix} \tag{2.117}$$

The electric charges read

$$\begin{aligned} Q(\bar{\nu} &= \frac{1}{2} - \frac{1}{2} = 0 \\ Q(e) &= -\frac{1}{2} - \frac{1}{2} = -1 \end{aligned} \tag{2.118}$$

$$l_L \in (1 \otimes 2 \otimes -1/2) \tag{2.119}$$

Finally there is the singlet

$$e_R \in (1 \otimes 1 \otimes -1) \tag{2.120}$$

The charge is

$$Q(e) = 0 - 1 = -1 \tag{2.121}$$

Let us see in detail how anomaly cancellation proceeds in this case. We have to take different signs for left and right fermions. In gory detail

$$\begin{aligned}
& SU(3)^3 : real \\
& SU(3)^2 \times U(1) : A = \sum_{color} Y = 6(-1/6) + 3(2/3) - 3(1/3) = 0 \\
& SU(2)^3 : pseudoreal \\
& SU(2)^2 \times U(1) : A = \sum_{weak} Y = 6(-1/6) + 2(1/2) = 0 \\
& U(1)^3 : A = \sum_{all} Y^3 = 6(-1/216) + 3(8/27) - 3(1/27) + 2(1/8) - 1 = 0 \\
& E^2 \times U(1) : A = \sum_{all} Y = 6(-1/6) + 3(2/3) + 3(-1/3) + 2(1/2) - 1 = 0
\end{aligned} \tag{2.122}$$

The last line corresponds to the Einstein anomaly, whose cancellation is a bonus, since is not necessary for renormalizability. As has been already mensioned, the fact that this cancellation takes place in each family independently of the others means that anomaluy cancellation is not likely to be the reason for the existence of different families.

2.4 Consistent versus covariant anomalies.

It is a fact of life that consistent anomalies (those that satisfy the consistency conditions) are not always Lorentz invariant. The gauge current is then not a Lorentz vector. Now there is a counterterm that transforms the consistent anomaly in a covariant one, where the gauge current is a true vector. The covariant anomaly however does not fulfill the consistency conditions.

To begin with, let us study some properties of the non-abelian current

$$J^a_\mu = \frac{\delta\Gamma}{\delta A^\mu_a} \tag{2.123}$$

Recall the definition of the operator δ en (2.36)

$$\delta A^a_\mu = B^a_\mu \tag{2.124}$$

so that from now on

$$\delta\Gamma[A] = \int_x \frac{\delta\Gamma}{\delta A^\mu_a}\delta A^a_\mu \equiv J^\mu_a . B^a_\mu \tag{2.125}$$

Recall that

$$T_\Lambda A^c_\lambda(x) = -\partial_\lambda \Lambda^c + \Lambda^a f_{abc} A^b_\lambda \tag{2.126}$$

whereas we do not yet know what

$$T_\Lambda B^c_\lambda(x) \tag{2.127}$$

stands for.

It is s a fact that

$$\delta_B T_\Lambda - T_\Lambda \delta_B = \delta_{[B,\Lambda]-T_\Lambda B} \tag{2.128}$$

Let us work this out in detail, acting on the left on the effective action

$$\begin{aligned}
&\delta_B \int \frac{\delta\Gamma}{\delta A^a_\mu} T_\Lambda A^a_\mu - \int T_\Lambda \frac{\delta\Gamma}{\delta A^c_\sigma} B^c_\sigma = \\
&\int \frac{\delta^2\Gamma}{\delta A^a_\mu \delta A^c_\sigma} B^c_\sigma T_\Lambda A^a_\mu + \int \frac{\delta\Gamma}{\delta A^a_\mu}\frac{\delta(T_\Lambda A^a_\mu)}{\delta A^c_\sigma} B^c_\sigma - \int \frac{\delta^2\Gamma}{\delta A^c_\sigma \delta A^a_\mu} T_\Lambda A^a_\mu B^c_\sigma - \int \frac{\delta\Gamma}{\delta A^c_\sigma} T_\Lambda B^c_\sigma = \\
&\int \frac{\delta\Gamma}{\delta A^a_\mu} f_{acd}\Lambda^b B^c_\mu - \int \frac{\delta\Gamma}{\delta A^c_\sigma} T_\Lambda B^c_\sigma = -\delta_{[\Lambda,B]+T_\Lambda B}
\end{aligned} \tag{2.129}$$

QED. Using this property, when acting on $\Gamma[A]$, and taking into account that $\delta_B\Lambda = 0$, and independently of what is going to be the value of $T_\Lambda B$

$$\begin{aligned}
&(\delta_B T_\Lambda - T_\Lambda \delta_B) W = \delta_{[B,\Lambda]-T_\Lambda B}\Gamma = \\
&\delta_B \int \Lambda^a G_a - T_\Lambda \int J^a_\mu B^\mu_a = \int J^\mu_a [B,\Lambda]^a - \int J^\mu T_\Lambda B = \\
&\int (\delta_B G_a)\Lambda^a - \int (T_\Lambda J^\mu_a) B^a_\mu - \int J^\mu T_\Lambda B
\end{aligned} \tag{2.130}$$

that is to say, a relationship between the gauge transform of the current in terms of the anomaly. Actually the second term of the second member would be the adjoint transformation would it not be for the gauge anomaly.

$$(T_\Lambda J^\mu_a).B^a_\mu = (\delta_B G_a).\Lambda^a - J^\mu_a.[B,\Lambda]^a = (\delta_B G_a).\Lambda^a - [\Lambda, J_\mu]^a B^\mu_a \tag{2.131}$$

If now we decide that B transforms with the adjoint, that is $T_\Lambda B = [B,\Lambda]$ then this is equivalent to

$$T_\Lambda \int (B.J) = \delta_B \int (\Lambda.G) \tag{2.132}$$

It is then clear that in order to find the covariant current we have to find a local polynomial on the gauge fields, say Q such that

$$(T_\Lambda Q^\mu_a).B^a_\mu = -(\delta_B G_a).\Lambda^a - [\Lambda, Q^\mu]_a B^a_\mu \tag{2.133}$$

or what is the same thing,

$$T_\Lambda \int (B.Q) = -\delta \int (\Lambda.G) \tag{2.134}$$

If we ever suceed, then

$$J^\mu_{a(cov)} \equiv J^\mu_a + Q^\mu_a \tag{2.135}$$

indeed would transform as a vector in the adjoint

$$T_\Lambda J^\mu_{a(cov)} = -[\Lambda, J^\mu_{(cov)}]_a \tag{2.136}$$

Let us examine this condition (2.134) a bit more closely. Remembering that the anomaly is $\delta_B G,\, v.G[A;F] = \int \omega^1_{2n-2}(v,A,F))$ we can rewrite it as

$$s(B.Q) = \delta_B \int \omega^1_{2n-2}(\Lambda, A, F) \tag{2.137}$$

Recall now the algebra (2.36) and the first descent equation $s\omega^0_{2n-1} = -d\omega^1_{2n-2}$,

$$\delta\omega^1_{2n-2} = (dl + ld)\,\omega^1_{2n-2} = d(l\omega^1_{2n-2}) - l\, s\omega^0_{2n-1} \tag{2.138}$$

and using now $\{s,l\} = 0$ we get up to a total derivative

$$\delta \int \omega^1_{2n-2} = s\int l\omega^0_{2n-1} \tag{2.139}$$

This means that we have found a quantity that has the property (2.134)

$$B.Q = \int l\,\omega^0_{2n-1} = n(n-1)\int\int_0^1 dt\ t\ P(A,B,F_t^{n-2}) \tag{2.140}$$

where we have used the expression for the Chern-Simons term as well as $l\,F_t = l\,\left(tF + (t^2-t)A^2\right) = tB$. This means that the Bardeen-Zumino counterterm is given by the operator l acting on the abelian anomaly in a space with two extra dimensions.

Starting in four dimensions $d = 4$ with $P(F^3) = \lambda \mathrm{Tr}\ F^3$, we can show that

$$Q = \lambda \left(dAA + AdA + \frac{3}{2} A^3 \right) \tag{2.141}$$

This is plain, because

$$\omega_5 = \lambda \mathrm{Tr}\ \left(F^2 A - \frac{1}{2} F A^3 + \frac{1}{10} A^5 \right) \tag{2.142}$$

Then

$$l\omega_5 = \lambda \mathrm{Tr}\ B \left(FA + AF - \frac{1}{2} A^3 \right) = \lambda \mathrm{Tr}\ B \left(dAA + AdA + \frac{3}{2} A^3 \right) \tag{2.143}$$

Since the covariant current is a true vector, then its divergence must be a true scalar. This can be written as (using integration by parts because this is going to be integrated) (2.140):

$$c.DQ = Dc.Q = n(n-1) \int_0^1 t\ dt\ \int P \left(dc + \{A, c\}, A, F_t^{n-2} \right) \tag{2.144}$$

On the other hand

$$c.DJ = Dc.J = sW[A] = c.G = \int \omega_{2n-2}^1 = n(n-1) \int_0^1\ dt\ (1-t) \int P \left(dc, A, F_t^{n-2} \right) \tag{2.145}$$

Adding both expressions

$$c.DJ_{cov} = n(n-1) \int_0^1\ dt\ P \left(dc + t\{A, c\}, A, F_t^{n-2} \right) \tag{2.146}$$

which is equivalent to

$$c.DJ_{cov} = n(n-1) \int_0^1\ dt\, P \left(c, dA + t\{A, A\}, F_t^{n-2} \right) = n \int_0^1\ dt\ \frac{d}{dt} P \left(c, F_t^{n-1} \right) = n \int P \left(c, F^{n-1} \right) \tag{2.147}$$

This means that the dominant terms in the covariant anomaly is n times bigger than the dominant term on the consistent anomaly in $d = 2n - 2$ dimensions.

In $d = 4$ dimensions the covariant anomaly is given by

$$D_\mu J^\mu_{a\,(cov)} = \frac{3}{2} A(R) d_{abc} F^b_{\mu\nu} F^c_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma} \tag{2.148}$$

2.5 The Wess-Zumino effective lagrangian

The fermion-gauge boson coupling in $d = 2n - 2$ dimensions reads

$$L = i\bar{\psi} \left(\partial\!\!\!/ - iT_a \left(V\!\!\!/^a + \gamma_5 A\!\!\!/^a \right) \right) \psi \tag{2.149}$$

Where the gauge fields acting on chiral fermions are

$$\begin{aligned} A_L &\equiv V + A \\ A_R &\equiv V - A \end{aligned} \tag{2.150}$$

where as usual $P_L \equiv \frac{(1-\gamma_5)}{2}$. We shall also denote

$$\begin{aligned} V_+ &= V + \gamma_5 A \\ V_- &= V - \gamma_5 A \end{aligned} \tag{2.151}$$

The action is invariant under

$$\psi \to e^{iT_a(\beta^a + \alpha^a \gamma_5)} \psi \tag{2.152}$$

We shall denote as *vector transformations* the subset where $\alpha = 0$; and as *axial transformations* when $\beta = 0$.

Elements of the group $g \in G_L \times G_R$ will be denoted by

$$g = e^v = e^{\beta + \alpha\gamma_5} = e^{P_R(\beta+\alpha) + P_L(\beta-\alpha)} \equiv P_R g_R + P_L g_L \tag{2.153}$$

That is, vector transformations do not distinguish left from right $g_L = g_R$, whereas axial ones do, $g_L = -g_R$.

The effective lagrangian of Wess-Zumino is to write a local lagrangian such that its variation is precisely the anomaly. If the anomaly is non trivial, this means that in order for this to be possible at all, we have to include extra fields in the physical lagrangian, namely the *goldstone bosons.*

A trivial solution would have been to write

$$\Gamma'[A] = C_n \int_{D_{2n-1}} \omega^0_{2n-1}(A) \tag{2.154}$$

as long as the integral runs over a $(2n-1)$-dimensional disk whose boundary is an $(2n-2)$-sphare

$$\partial D_{2n-1} = S_{2n-2}, \tag{2.155}$$

and C_n be a dimension dependent constant, namely $C_n \equiv \frac{1}{n!} \frac{i^n}{(2\pi)^{n-1}}$.

This is so because the descent equations tell us that

$$\delta\Gamma' = C_n \int_{D_{2n-1}} s\omega^0_{2n-1} = C_n \int_{D_{2n-1}} d\omega^1_{2n-2}(A, v) = C_n \int_{S_{2n-2}} \omega^1_{2n-2}(A, v) = \int_{S_{2n-2}} v^a G_a \tag{2.156}$$

This is not satisfactory though because even though Γ' is local as an integral over the disk D_{2n-1}, it becomes nonlocal when expressed in the physical euclidean compactified space S_{2n-2}.

This expression can be much improved by recalling the transfomation law of the Chern-Simons term

$$T_g\omega^0_{2n-1}(A,F) \equiv \omega^0_{2n-1}(A_g,F_g) = \omega^0_{2n-1}(A,F) + \omega^0_{2n-1}(g^{-1}dg,0) + d\alpha_{2n-2}(A,g) \quad (2.157)$$

From the group property $T_gT_h = T_{hg}$ (because $T_gA_h = A_{hg}$) we learn

$$T_g\omega^0_{2n-1}(A) = T_{h^{-1}g}T_h\omega^0_{2n-1}(A) = T_{h^{-1}g}\omega^0_{2n-1}(A_h) \quad (2.158)$$

that is the quantity $T_g\omega^0_{2n-1}(A)$ is gauge invariant under the transformation $A \to A_h$ and $g \to h^{-1}g$. It is then natural to define

$$\Gamma[A,g] = C_n\int_{D_{2n-1}}\left(\omega^0_{2n-1}(A) - T_g\omega^0_{2n-1}(A)\right) \quad (2.159)$$

in such a way that, as a consequence of the fresh gauge invariance we have just discovered

$$\delta\Gamma = \delta\Gamma' = v_aG^a(A) \quad (2.160)$$

To be specific,

$$\begin{aligned}\Gamma[A,g] &= -C_n\int_{D_{2n-1}}\left(\Lambda_{2n-1}(g) + d\alpha_{2n-2}(A,g)\right) = \\ &= -C_n\int_{D_{2n-1}}\Lambda_{2n-1}(g) - C_n\int_{S_{2n-2}}\alpha_{2n-2}(A,g)\end{aligned} \quad (2.161)$$

where we have denoted $\Lambda(g) \equiv \omega^0_{2n-1}(g^{-1}dg,0)$.

This solves our problem. It is often useful however, to treat left and right fields independently.

$$\Gamma[A_R,A_L,g_R,g_L] = \Gamma[A_R,g_R] - \Gamma[A_L,g_L] \quad (2.162)$$

It is possible to add a counterterm (the *Bardeen counterterm*) which makes this invariant under vector transformations.

What we want is a modified Chern-Simons with the addition of a total derivative dS_{2n-2},

$$\tilde{\omega}^0_{2n-1}(A_R,A_L) = \omega^0_{2n-1}(A_R,A_L) + dS_{2n-2}(A_R,A_L) \quad (2.163)$$

in such a way that

$$\delta_\beta\tilde{\omega}^0_{2n-1}(A_R,A_L) = 0 \quad (2.164)$$

Besides, the Bardeen counterterm should be a finite polynomial in the gauge fields. This leads to

$$\tilde{\Gamma}(A_R, A_L, g_R, g_L) = C_n \int_{D_{2n-1}} \left\{ \omega^0_{2n-1}(A_R, A_L) - T_{g_R g_L} \tilde{\omega}^0_{2n-1}(A_R, A_L) \right\} =$$
$$C_n \int_{D_{2n-1}} \left\{ \omega^0_{2n-1}(A_R, A_L) - T_{g_R g_L} (\omega^0_{2n-1}(A_R, A_L) + dS_{2n-2}(A_R, A_L)) \right\} =$$
$$\Gamma(A_R, A_L, g_R, g_L) - C_n \int_{S_{2n-2}} T_{g_R g_L} S_{2n-2}(A_R, A_L) \tag{2.165}$$

The counterterm is clearly invariant under

$$\begin{aligned} A_R &\to A_R^{h_R} \\ A_L &\to A_L^{h_L} \\ g_R &\to h_R^{-1} g_R \\ g_L &\to h_L^{-1} g_L \end{aligned} \tag{2.166}$$

because then the gauge transformation factorizes as

$$T_{h_R^{-1} g_R h_L^{-1} g_L} = T_{g_L} T_{h_L^{-1}} . T_{g_R} . T_{h_R^{-1}} = T_{g_R g_L} . T_{h_R^{-1} h_L^{-1}} \tag{2.167}$$

This ensures that this new WZ reproduces the anomaly

$$\begin{aligned} \frac{\delta \tilde{\Gamma}}{\delta v_L} &= \frac{\delta W}{\delta v_L} = G_L \\ \frac{\delta \tilde{\Gamma}}{\delta v_R} &= \frac{\delta W}{\delta v_R} = G_R \end{aligned} \tag{2.168}$$

We can use now the group property to combine g_L y g_R in an unique field

$$T_{g_R|g_L} \tilde{\omega}^0_{2n-1}(A_R, A_L) = T_{g_R|g_R} T_{e|g_L g_R^{-1}} \tilde{\omega}^0_{2n-1}(A_R, A_L) = T_{g_R|g_R} \tilde{\omega}^0_{2n-1}(A_R, A_L^{g_L g_R^{-1}}) \tag{2.169}$$

This last transformation is a vector one (under which $\tilde{\omega}^0$ is invariant by construction). Then

$$\tilde{\omega}^0_{2n-1}(A_R, A_L^{g_L g_R^{-1}}) = T_{e|g_L g_R^{-1}} \tilde{\omega}^0_{2n-1}(A_R, A_L) \tag{2.170}$$

and defining

$$U \equiv g_L g_R^{-1} \in G \tag{2.171}$$

it is a fact that

$$T_{g_R|g_L} \tilde{\omega}^0_{2n-1}(A_R, A_L) = T_{e|U} \tilde{\omega}^0_{2n-1}(A_R, A_L) \tag{2.172}$$

The new U field transforms as

$$U \to h_L^{-1} g_L g_R^{-1} h_R = h_L^{-1} U h_R \tag{2.173}$$

On the other hand

$$T_{e|U} \tilde{\omega}^0_{2n-1}(A_R, A_L) = T_{e|U} \left\{ \omega^0_{2n-1}(A_R, A_L) + dS_{2n-2}(A_R, A_L) \right\} = \omega^0_{2n-1}(A_R, A_L) +$$
$$\Lambda_{2n-1}(e|U) + d\alpha_{2n-2}(A_R, A_L^U) + dS_{2n-2}(A_R, A_L^U)] \tag{2.174}$$

Now

$$\Lambda_{2n-1}(e|U) = 0 - \Lambda_{2n-1}(U) \tag{2.175}$$

and

$$d\alpha_{2n-2}(A_R|A_LU) = d\alpha_{2n-2}(A_R|e) - d\alpha_{2n-2}(A_LU) = -d\alpha_{2n-2}(A_LU) \tag{2.176}$$

because $d\alpha_{2n-2}(A_R|e) = 0$.

Summarizing, the final expression for the Wess-Zumino term is

$$W(A_R, A_L, U) = C_n\Big\{\int_{D_{2n-1}} \Lambda_{2n-1}(U) + \int_{S_{2n-2}} (\alpha_{2n-2}(A_L, U) - S_{2n-2}(A_R, A_L^U)\Big\} \tag{2.177}$$

where $U(x) \equiv e^{i\xi(x)} \in G$.

The first integral contains the WZ proper; the second is the result of the gauging we have made.

Adding now Bardeen's counterterm, then the vector current anomaly disappears. To be specific, by defining

$$\tilde{\Gamma}(A_R, A_L, U) \equiv \Gamma(A_R, A_L, U) + \int_{S^{2n-2}} S_{2n-2}(A_R, A_L) \tag{2.178}$$

then

$$\frac{\delta}{\delta\beta}\tilde{\Gamma}(A_R, A_L, U) = 0 \tag{2.179}$$

(where let us recall that $\beta \equiv \frac{\alpha_L+\alpha_R}{2}$).

All this is nicely proven in ([31]) where also an explicit formula for the counterterm can be found. The formula reads

$$S_{2n-2} = \frac{n(n-1)}{d} \int\int \mathrm{Str}\, \gamma_5 V_- V_+ F_{\lambda\mu}^{n-2} \tag{2.180}$$

where we consider a one-parameter family of connections

$$A_{\lambda\mu} \equiv \lambda V_+ + \mu V_- \tag{2.181}$$

with $0 \leq \lambda, \mu \leq 1$.

In can be proven that in d=4 dimensions.

$$\Gamma(A_R, A_L, U) = -\frac{i}{240\pi^2}\int_{D_5} (tr\, U^{-1}dU)^5 - \frac{i}{48\pi^2}\int_{S^4} Z \tag{2.182}$$

where

$$
\begin{aligned}
&Z = tr\,[-U_L(A_L dA_L + dA_L A_L + A_L^3) + U_L^3 A_L - U_R(A_R dA_R + dA_R A_R + A_R^3) + \\
&U_R^3 A_R - U^{-1}A_L U A_R^3 + U A_R U^{-1} A_L^3 + \frac{1}{2}U_L A_L U_L A_L - U^{-1}A_L U(A_R dA_R + dA_R A_R) - \\
&\frac{1}{2}U_R A_R U_R A_R + U A_R U^{-1}(A_L dA_L + dA_L A_L) - U A_R U^{-1} A_L U_L A_L - \\
&U^{-1}A_L U A_R U_R A_R + A_L U U_R^2 A_R U^{-1} - A_R U^{-1} U_L^2 A_L U - \\
&dA_L U_L U A_R U^{-1} - dA_R U_R U^{-1} A_L U + \frac{1}{2} A_R U^{-1} A_L U A_R U^{-1} A_L U]
\end{aligned}
\tag{2.183}
$$

This lagrangian was first obtained by Edward Witten. A systematic approach is to be found in cf. [31].

3 Gravitational anomalies.

Consider a *frame field* in spacetime

$$\eta_{ab} e^a{}_\mu e^b{}_\nu = g_{\mu\nu}, \tag{3.1}$$

We shall follow Bardeen and Zumino by calling latin indices $a, b, c \ldots$ *Lorentz*. They transform with the fundamental vector represesentation of the tangent group acting on a real frame with one timelike basis vector and three spacelike basis vactors namely, $SO(1,3)$:

$$e'^a{}_\mu(x) \equiv L^a{}_b e^b{}_\mu(x) \tag{3.2}$$

where

$$\eta_{ac} L^a{}_b L^c{}_d = \eta_{bd}. \tag{3.3}$$

Greek indices, on the other hand $\mu, \nu, \rho \ldots$, will be dubbed *Einstein*, and transform in a tensorial way under diffeomorphisms

$$e'^a{}_\mu(x') \equiv \frac{\partial x^\nu}{\partial x'^\mu} e^a{}_\nu(x) \tag{3.4}$$

Anomalies in both these transformations are related. They were discovered by Luis Álvarez-Gaumé y Edward Witten ([4]).

An Einstein transformation (a diffeomorphism) reads

$$\delta x^\mu \equiv x'^\mu - x^\mu \equiv -\xi^\mu(x) \tag{3.5}$$

Geoemetric objects transform with the Lie derivative. For example

$$E_\xi \phi \equiv \phi'(x) - \phi(x) = \pounds(\xi)\phi \equiv \xi^\alpha \partial_\alpha \phi \tag{3.6}$$

For the metric itself

$$E_\xi g_{\mu\nu} = \pounds(\xi) g_{\mu\nu} \equiv \xi^\alpha \partial_\alpha g_{\mu\nu} + \partial_\mu \xi^\alpha g_{\alpha\nu} + \partial_\nu \xi^\alpha g_{\mu\alpha} \tag{3.7}$$

The commutator of two Einstein transformations reads

$$\begin{aligned}
&T_1 x \equiv x' \equiv x - \xi_1(x) \\
&T_2 T_1 x \equiv x'' \equiv x' - \xi_2(x') = x - \xi_1(x) - \xi_2\left(x - \xi_1(x)\right) = x - \xi_1(x) - \xi_2(x) + \xi_1(x).\partial\xi_2(x) \\
&[T_2, T_1] x = \partial_\alpha \xi_2(x).\xi_1^\alpha(x) - \partial_\alpha \xi_1(x).\xi_2^\alpha(x) \equiv [\xi_1, \xi_2]
\end{aligned} \tag{3.8}$$

That is,

$$[E_{\xi_1}, E_{\xi_2}] = E_{[\xi_2, \xi_1]} \tag{3.9}$$

The Einstein tranformation of a connection reads

$$E_\xi \Gamma^\rho_{\lambda\mu} = \pounds_\xi \Gamma^\rho_{\lambda\mu} + \partial_\lambda \partial_\mu \xi^\rho \tag{3.10}$$

It is often useful to look at the connection as a matrix of one-forms

$$(\Gamma)_\mu{}^\rho \equiv \Gamma^\rho_{\lambda\mu} dx^\lambda \tag{3.11}$$

The curvature is then a two-form

$$(R)_\mu{}^\rho \equiv (d\Gamma + \Gamma^2)_\mu{}^\rho = \frac{1}{2} R_{\nu\lambda\mu}{}^\rho dx^\nu \wedge dx^\lambda \tag{3.12}$$

It is a fact that

$$E_\xi \Gamma = \pounds_\xi \Gamma + T_\Lambda \Gamma \tag{3.13}$$

where $\Lambda^\alpha_\beta \equiv \partial_\beta \xi^\alpha$ and $T_\Lambda \Gamma \equiv D\Lambda \equiv d\Lambda + [\Gamma, \Lambda]$ can be viewed as a gauge transformation of Γ with respect to the remaining indices on which the Lie derivative did not act. This transformation consists in adding the covariant derivative of Λ with respect to the connection Γ. On the other hand $\pounds(\xi)\Gamma$ refers to the Lie derivative considering the connection as a one-form, that is

$$\pounds(\xi)\Gamma^\rho_{\lambda\mu} = \xi^\alpha \partial_\alpha \Gamma^\rho_{\lambda\mu} + \partial_\lambda \xi^\alpha \Gamma^\rho_{\alpha\mu} \tag{3.14}$$

and

$$[T_\Lambda \Gamma]^\rho_{\lambda\mu} = \partial_\lambda \Lambda^\rho_\mu + \Lambda^\sigma_\mu (\Gamma_\lambda)^\rho_\sigma - (\Gamma_\lambda)^\sigma_\mu \Lambda^\rho_\sigma \tag{3.15}$$

This decomposition (3.13) might look unwieldly but it is useful. It is also easy to prove that

$$E_\xi \; R = \pounds(\xi) R + T_\Lambda R \tag{3.16}$$

Recall that the Lie derivative acting on differential forms can be written as

$$\pounds_\xi = d i_\xi + i_\xi d \tag{3.17}$$

where the *inner product* i_ξ is given by

$$i_\xi (\Gamma^\rho_{\lambda\mu} dx^\lambda) \equiv \Gamma^\rho_{\lambda\mu} \xi^\lambda \tag{3.18}$$

This means in particular that the integral of a Lie derivative of a form of the maximal rank (equal to the dimension of spacetime) should vanish

$$\int \pounds_\xi \omega_d = \int d(i_\xi \omega_d) = 0 \tag{3.19}$$

It is easy to check that indeed

$$\left[\pounds_\xi \omega_{(d)}\right]_{\alpha_1 \ldots \alpha_d} = \partial_\mu (\xi^\mu \omega_{(d)\alpha_1 \ldots \alpha_d}) \tag{3.20}$$

3.1 The consistent gravitational anomaly.

Consider now the connected generating functional $W[g_{\mu\nu}]$ representing the effective action of chiral fermions in an external nondynamical gravitational field. Under an Einstein transformation

$$E_\xi W[g_{\alpha\beta}] \equiv H_\xi \tag{3.21}$$

The algebra just discussed implies consistency conditions

$$E_{\xi_1} H_{\xi_2} - E_{\xi_2} H_{\xi_1} = H_{[\xi_2,\xi_1]} \tag{3.22}$$

Now the gauge anomaly yields immediatly a candidate for the consistent Einstein anomaly just by identifying $\mathcal{A} \equiv \Gamma$ and $\mathcal{F} \equiv R$. It reads

$$H_\xi = \Lambda G(\Gamma, R) = -\int \partial_\rho \xi^\mu G^\rho_\mu(\Gamma, R) \tag{3.23}$$

Let us check that it is indeed consistent

$$E_{\xi_1} H_{\xi_2} = (\pounds_{\xi_1} + T_{\Lambda_1})\Lambda_2.G = -\int \partial_\rho \xi_2^\nu \partial_\lambda(\xi_1^\lambda G^\rho_\nu) + T_{\Lambda_1}\Lambda_2 G \tag{3.24}$$

because the Lie derivative does not act on the parameter Λ_2. After integration by parts

$$E_{\xi_1} H_{\xi_2} - E_{\xi_2} H_{\xi_1} = \int \left(\xi_1^\lambda \partial_\lambda \partial_\rho \xi_2^\nu - \xi_2^\lambda \partial_\lambda \partial_\rho \xi_1^\nu\right)\; G^\rho_\nu + [\Lambda_1, \Lambda_2].G \tag{3.25}$$

taking into account that

$$[\Lambda_1, \Lambda_2]^\sigma_\rho = \partial_\rho \xi_1^\lambda \partial_\lambda \xi_2^\sigma - \partial_\rho \xi_2^\lambda \partial_\lambda \xi_1^\sigma \tag{3.26}$$

as well as

$$\xi_1^\lambda \partial_\lambda \partial_\rho \xi_2^\nu - \xi_2^\lambda \partial_\lambda \partial_\rho \xi_1^\nu = \partial_\rho \left(\xi_1^\lambda \partial_\lambda \xi_2^\nu - \xi_2^\lambda \partial_\lambda \xi_1^\nu\right) - \partial_\rho \xi_1^\lambda \partial_\lambda \xi_2^\nu + \partial_\rho \xi_2^\lambda \partial_\lambda \xi_1^\nu \tag{3.27}$$

we end up with

$$E_{\xi_1} H_{\xi_2} - E_{\xi_2} H_{\xi_1} = -\partial_\rho[\xi_2^\lambda \partial_\lambda \xi_1^\nu - \xi_1^\lambda \partial_\lambda \xi_2^\nu] G^\rho_\nu = H_{[\xi_2,\xi_1]} \tag{3.28}$$

QED.

It is a fact that this anomaly can only exist in dimensions such that $d \in 4\mathbb{Z}+2$, that is, $d = 2, 6, 10,$ etc. This follows from the fact that the curvature two-form is also antisymmetric in the two extra indices. Then the invariant symmetric polynomial, which in dimension $d = 2n-2$ corresponds to $P(F_1 \ldots F_n)$, vanishes whenever n is odd. This is necause

$$P\left(F_1 \ldots F_n\right) = P\left(F_1^T \ldots F_n^T\right) = (-1)^n\; P\left(F_1 \ldots F_n\right) \tag{3.29}$$

This forces $n \in 2\mathbb{Z}$, which is just what we wanted to prove.

To be specific, in two dimensions the consistent gravitational anomaly is given by

$$\partial_\rho \xi^\mu \partial_\nu \Gamma^\rho_{\lambda\mu} \epsilon^{\nu\lambda} \tag{3.30}$$

3.2 The physical meaning of the gravitational anomaly.

Let us define provisionally the *energy-momentum tensor* as

$$\theta_{\mu\nu} \equiv 2\frac{\delta W}{\delta g^{\mu\nu}} \tag{3.31}$$

It so happens that the Einstein anomaly as derived is equivalent to the non-conservation of the energy-momentum *tensor*. This in turn leads to an inconsistency in Einstein's equations.

Recall the definition of E_ξ

$$E_\xi \equiv \int dx (E_\xi g_{\mu\nu}) \frac{\delta}{\delta g_{\mu\nu}} \tag{3.32}$$

from where it follows (after partial integrations) that

$$E_\xi W = \frac{1}{2} \int E_\xi g_{\mu\nu} \theta^{\mu\nu} = -\int \xi_\rho D_\mu \theta^{\mu\rho} \tag{3.33}$$

Then

$$\int \xi_\alpha D_\beta \theta^{\beta\alpha} = -H_\xi = -\Lambda.G \tag{3.34}$$

The previous emphasis in the word tensor is due to the fact that the anomaly is the origin of another evil, namely that the quantity $\theta^{\mu\nu}$ does not transform as a tensor.

To see that, let us compute in two different ways the commutator of an Eisntein transformation with an arbitrary variation of the metric

$$\delta_\phi \equiv \int \phi_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}} \tag{3.35}$$

Write

$$\begin{aligned} &(E_\xi \delta_\phi - \delta_\phi E_\xi) W[g_{\mu\nu}] = E_\xi \int \phi_{\mu\nu} \frac{\delta W}{\delta g_{\mu\nu}} - \delta_\phi \int E_\xi g_{\mu\nu} \frac{\delta W}{\delta g_{\mu\nu}} = \\ &\int E_\xi g_{\alpha\beta} \frac{\delta}{\delta g_{\alpha\beta}} \left(\phi_{\mu\nu} \frac{\delta W}{\delta g_{\mu\nu}} \right) - \int \phi_{\alpha\beta} \frac{\delta}{\delta g_{\alpha\beta}} \left(E_\xi g_{\mu\nu} \frac{\delta W}{\delta g_{\mu\nu}} \right) = \int E_\xi \phi_{\alpha\beta} \frac{1}{2} \theta^{\alpha\beta} \end{aligned} \tag{3.36}$$

A different way of writing the commutator is

$$\int \phi_{\alpha\beta} E_\xi \frac{\delta W}{\delta g_{\alpha\beta}} - \delta_\phi H_\xi \tag{3.37}$$

They both must be equal, so that

$$E_\xi \int \theta^{\alpha\beta} \phi_{\alpha\beta} = 2 \delta_\phi H_\xi \tag{3.38}$$

so that the first member is not Einstein invariant.

It is natural to think that there must exist a counterterm such that the modified energy-momentum is a true tensor. That is we need to find a local symmetric tensor $Y^{\mu\nu}$, such that $\tilde{\theta}^{\mu\nu} \equiv \theta^{\mu\nu} + Y^{\mu\nu}$ behaves as a true tensor, that is,

$$E_\xi \int \phi_{\alpha\beta} \tilde{\theta}^{\alpha\beta} = 0 \tag{3.39}$$

The transformation rule (3.38) means that

$$E_\xi \int \phi_{\mu\nu} Y^{\mu\nu} = -2\delta_\phi H_\xi \tag{3.40}$$

It is not difficult to check that the work we did previously for the gauge anomaly in (2.140) yields a solution to our problem. Details can be found in Bardeen and Zumino's paper [?].

3.3 Lorentz anomalies.

Lorentz transformations live in $L \in SO(1, d-1)$. In our current euclidean setting, $L \in SO(d)$,

$$e'_{a\mu} \equiv L_a^b e_{b\mu} \tag{3.41}$$

We do not transform the point, neither the vector Einstein index.

Linealizing

$$L_{ab} = \delta_{ab} + \theta_{ab} \tag{3.42}$$

(with $\theta_{(ab)} = 0$)

$$L_\theta e_{\mu a} \equiv e'_{\mu a} - e_{\mu a} = e_{\mu b} \theta_{ba} \tag{3.43}$$

Under Einstein transformations,

$$E_\xi e_{\mu a} = \xi^\lambda \partial_\lambda e_{\mu a} + \partial_\mu \xi^\lambda e_{\lambda a} \tag{3.44}$$

It is possible to check that Einstein and Lorentz close

$$[L_{\theta_1}, L_{\theta_2}] = L_{[\theta_1, \theta_2]} \tag{3.45}$$

$$[L_\theta, E_\xi] = L_{\xi.\partial\theta} \tag{3.46}$$

If there were to exist any Lorentz anomaly, that is in case

$$L_\theta W = K_\theta \tag{3.47}$$

they necessarily have to obey the corresponding consistency conditions.

$$
\begin{aligned}
L_{\theta_1} K_{\theta_2} - L_{\theta_2} K_{\theta_1} &= K_{[\theta_1,\theta_2]} \\
L_\theta H_\xi - E_\xi K_\theta &= K_{\xi.\partial\theta}
\end{aligned}
\tag{3.48}
$$

It is consistent to assume $K_\theta = 0$; it is also consistent to assume $H_\xi = 0$. A consistent varion of the Lorentz anomaly stems immediatly from the gauge potential

$$
\omega_{ab} = -\omega_{ba} \equiv \omega_{\mu ab} dx^\mu \tag{3.49}
$$

and the field strength

$$
R_{ab} = -R_{ba} \equiv d\omega + \omega \wedge \omega = \frac{1}{2} R_{\mu\nu ab} dx^\mu \wedge dx^\nu \tag{3.50}
$$

In that way we reach the conclusion that

$$
K_\theta = \int \omega^1_{2n-2}(\theta, \omega, R) = \theta.G[\omega, R] \tag{3.51}
$$

Here we have a situation analogous to what was the case with Einstein's anomaly Given the fact that $R_{(ab)} = 0$, $P(R^n) = (-1)^n P(R^n)$, so that $P = 0$ unless $n \in 2\mathbb{Z}$, which in turn means that $d = 2n - 2 = 4m - 2$.

Bardeen y Zumino ([6]) found a functional denoted by S such that its Einstein variation is equal to the Einstein anomaly, whereas its Lorentz variation is the Lorentz anomaly.

$$
\begin{aligned}
E_\xi S &= -H_\xi \\
L_\theta S &= K_\theta
\end{aligned}
\tag{3.52}
$$

It is then clear that using this counterterm we can move from one anomaly to the other; by changing W by $W + S$ we cancel Einstein's anomaly; whereas by trading W by W_S is the Lorentz anomaly that is cancelled.

3.4 The Green-Schwartz mechanism.

This ten-dimensional exercise had a tremendous historical importance in the establishing the status of string theory as a candidate theory for all fundamental interactions. Either a diagrammatic calculation, or else through the index theorem, we can get the 12-form that characterizes the anomaly of $\mathcal{N} = 1$ supergravity coupled to a Yang-Mills supermultiplet in $d = 10$ dimensions.

$$
\begin{aligned}
\hat{I}_{12} = &-\frac{1}{720}\mathrm{Tr}_A F^6 + \frac{1}{24.48}\mathrm{Tr}_A F^4 \mathrm{Tr}_F R^2 - \frac{1}{256}\mathrm{Tr}_A F^2 \left[\frac{1}{45}\mathrm{Tr}_F R^4 + \frac{1}{36}\left(\mathrm{Tr}_F R^2\right)^2\right] \\
&+\frac{n-496}{64}\left[\frac{1}{2.2835}\mathrm{Tr}_F R^6 + \frac{1}{4.1080}\mathrm{Tr}_F R^2 \mathrm{Tr}_F R^4 + \frac{1}{8.1296}(\mathrm{Tr}_F R^2)^3\right] \\
&+\frac{1}{384}\mathrm{Tr}_F R^2 \mathrm{Tr}_F R^4 + \frac{1}{1536}(\mathrm{Tr}_F R^2)^3
\end{aligned}
\tag{3.53}
$$

Here traces over gauge fields (Tr_A) are traces in the adjoint, whereas traces over curvatures (Tr_F) are traces on the fundamental of $SO(1,9)$

The only possibility for the field V to cancel this anomaly is that there is some factorization of the type

$$I_{12} = \left(\mathrm{Tr}_F R^2 + k\mathrm{Tr}_A F^2\right)\ X_8 \tag{3.54}$$

We have to determine when this is possible at all.

The first condition is that the dimension of the group has to be

$$n = 496 \tag{3.55}$$

because $SO(10)$ has an independent sixth-order Casimir which contributes to $\mathrm{Tr}\ R^6$, and this term cannot be cancelled any other way

This means that already we are restricted to $G = SO(32)$, $G = E_8 \times E_8$, $G = U(1)^{496}$, or else $G = E_8 \times U(1)^{248}$.

Working in $SO(n)$, for example, it is not difficult to show that

$$\begin{aligned}
&\mathrm{Tr}_A\ F^2 = (n-2)\mathrm{Tr}_F F^2 \\
&\mathrm{Tr}_A F^4 = (n-8)\mathrm{Tr}_F\ F^4 + 3\left(\mathrm{Tr}_F\ F^2\right)^2 \\
&\mathrm{Tr}_A\ F^6 = (n-32)\mathrm{Tr}_F\ F^6 + 15\mathrm{Tr}_F\ F^2\mathrm{Tr}_F\ F^4
\end{aligned} \tag{3.56}$$

We also need that the trace over gauge fields can be rewritten as

$$\mathrm{Tr}\ F^6 = \frac{1}{48}\mathrm{tr}\, F^2 Tr\, F^4 - \frac{1}{14400}(\mathrm{tr}\, F^2)^3 \tag{3.57}$$

(which corresponds to $k = -\frac{1}{30}$), Then

$$X_8 = \frac{1}{24}\mathrm{tr}\, F^4 - \frac{1}{7200}(\mathrm{tr}\, F^2)^2 - \frac{1}{240}\mathrm{tr}\, F^2\mathrm{Tr}\ R^2 + \frac{1}{8}\mathrm{Tr}\ R^4 + \frac{1}{32}(\mathrm{Tr}\ R^2)^2 \tag{3.58}$$

Now, for the group $G = SO(32)$

$$\mathrm{Tr}_A\, F^2 = \frac{1}{30}\mathrm{Tr}_F\, F^2 \tag{3.59}$$

The fact that there is factorization

$$I_{12} = \left(\mathrm{Tr}\ R^2 - \mathrm{Tr}\ F^2\right)\ X_8 \tag{3.60}$$

where now all traces are in the fundamental has far reaching consequences. To begin with, given that

$$I_{12} = dI_{11} \tag{3.61}$$

necessarily X_8 must be exact, that is $X_8 = dX_7$. The most general possible choice is

$$I_{11} = \frac{1}{3} (\omega_{3L} - \omega_{3Y}) \; X_8 + \frac{2}{3} \left(\text{Tr } R^2 - \text{Tr } F^2\right) X_7 + \alpha d\Big((\omega_{3L} - \omega_{3Y}) \; X_7 \Big) \tag{3.62}$$

Define as usual

$$\delta I_{11} \equiv dI^1_{10} \tag{3.63}$$

and

$$\delta X_7 \equiv dX^1_6 \tag{3.64}$$

We get

$$I^1_{10} = (2/3 + \alpha)(\text{Tr } R^2 - \text{Tr } F^2) X^1_6 + (1/3 - \alpha) \left(\omega^1_{2L} - \omega^1_{2Y}\right) \; X_8 \tag{3.65}$$

which leads to the anomaly

$$G = (2/3 + \alpha) \int (\omega_{3L} - \omega_{3Y}) \; dX^1_6 + (1/3 - \alpha) \int \left(\omega^1_{2L} - \omega^1_{2Y}\right) \; X_8 \tag{3.66}$$

Green and Schwartz's idea [28]) is that if one is willing to change the transformation law of the two-forms

$$\delta B = \omega^1_{2Y} - \omega^1_{2L} \tag{3.67}$$

then the putative anomaly can be cancelled by a counterterm

$$\delta S = -\xi_1 \int B X_8 - \xi_2 \int (\omega_{3L} - \omega_{3Y}) \; X_7 \tag{3.68}$$

Given that

$$\begin{aligned} &\delta B = \omega^1_{23Y} - \omega^1_{2L} \\ &\delta X_8 = \delta(dX_7) = d(dX^1_6) = 0 \\ &\delta(\omega_{3L} - \omega_{3Y}) = d(\omega^1_{2L} - \omega^1_{2Y}) \\ &\delta X_7 = dX^1_6 \end{aligned} \tag{3.69}$$

we get

$$\delta(\delta S) = -\xi_1 \int (\omega^1_{2L} - \omega^1_{2Y}) X_8 + \xi_2 \int (\omega^1_{2L} - \omega^1_{2Y}) X_8 - \xi_2 \int (\omega_{3L} - \omega_{3Y}) dX^1_6 \tag{3.70}$$

which determines

$$\begin{aligned} &\xi_2 = 2/3 + \alpha \\ &\xi_1 - \xi_2 = 1/3 - \alpha \end{aligned} \tag{3.71}$$

The three form must then be redefined as

$$H = dB + \omega_{3L} - \omega_{3Y} \tag{3.72}$$

which means that

$$dH = \text{Tr } R^2 - \text{Tr } F^2 \tag{3.73}$$

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