Advanced mathematical methods.

Enrique Alvarez

vi Maii MMXVIII

Index

1

Curves and surfaces in $\mathbb{R}^3$.

1.1 Curves.

$$\begin{aligned} x &= x(u) \\ y &= y(u) \\ z &= z(u) \\ u_1 \leqslant\ & u \leqslant u_2 \end{aligned} \tag{1.1}$$

or else

$$x^i = x^i(u) \quad i = 1, 2, 3. \tag{1.2}$$

One example is the circular helix

$$\begin{aligned} x &= a \cos\, u \\ y &= a \sin\, u \\ z &= b\, u \\ 0 \leqslant\ & u \leqslant \infty \end{aligned} \tag{1.3}$$

which for $b = 0$ reduces to a circle of radius $r = a$ in the $z = 0$ plane.
The arc length is given by Pythagoras' theorem

$$ds^2 \equiv dx^2 + dy^2 + dz^2 = \left(\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2 \right) du^2 \tag{1.4}$$

Then the arc between two points is given by

$$s_{01} \equiv \int_{u_0}^{u_1} \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, du \tag{1.5}$$

Let us compute it for a circle

$$y = \sqrt{R^2 - x^2} \tag{1.6}$$

$$s = \int dsa = \int \sqrt{dx^2 + \left(\frac{x}{\sqrt{R^2 - x^2}}\right)^2 dx^2} = \int \frac{dx}{\sqrt{R^2 - x^2}} =$$
$$= \int \frac{dt}{\sqrt{1-t^2}} = \sin^{-1}\ \frac{x}{R} \tag{1.7}$$

Then

$$\sin\ s = \frac{x}{R} \tag{1.8}$$

For example, for the helix,

$$s_{01} = \sqrt{a^2 + b^2}\,(u_1 - u_0) \tag{1.9}$$

The tangent vector is defined as

$$\vec{t} \equiv \frac{d\vec{x}}{ds} \equiv \frac{\frac{d\vec{x}}{du}}{\frac{ds}{du}} \equiv \frac{1}{\sqrt{\left(\frac{d\vec{x}}{du}\right)^2}} \frac{d\vec{x}}{du} \tag{1.10}$$

For the helix,

$$\vec{t} = \frac{1}{\sqrt{a^2 + b^2}}\,(-a \sin\ u, a\ \cos\ u, b) \tag{1.11}$$

The normal to the tangent at a given point,

$$\vec{t}.\vec{n} = 0 \tag{1.12}$$

and normalized such that

$$\vec{n}^2 = 1 \tag{1.13}$$

In our example

$$\vec{n} = \pm\,(\cos\ u, \sin\ u, 0) \tag{1.14}$$

It is clear that

$$\vec{t}^2 = 1 \tag{1.15}$$

which implies that

$$\vec{t}\dot{\vec{t}} = 0 \tag{1.16}$$

We can write

$$\frac{d\vec{t}}{ds} \equiv \kappa\,\vec{n} \tag{1.17}$$

where κ is called the *curvature* at a given point (there is a sign that must be fixed by some convention); and the *radius of curvature* is defined by

$$\kappa \equiv \frac{1}{R} \tag{1.18}$$

In the example

$$\frac{d\vec{t}}{ds} = \frac{a}{a^2 + b^2}\,(-\cos\ u, -\sin\ u, 0) \tag{1.19}$$

so that

$$R = \frac{a^2 + b^2}{a} \tag{1.20}$$

which reduces to a when $b = 0$.

It is clear that

$$\kappa^2 = \left(\dot{\vec{t}}\right)^2 \tag{1.21}$$

We define the *binormal* as

$$\vec{b} \equiv \vec{t} \times \vec{n} \tag{1.22}$$

In the example

$$\vec{b} = \frac{1}{\sqrt{a^2 + b^2}} \left(-b \text{ sin } u, b \text{ cos } u, -a\right) \tag{1.23}$$

It is clear that the vectors $(\vec{t}, \vec{n}, \vec{b})$ form a *moving trihedron* along the curve. Consider now

$$\dot{\vec{b}} \equiv \frac{d\vec{b}}{ds} = \dot{\vec{t}} \times \vec{n} + \vec{t} \times \dot{\vec{n}} = \vec{t} \times \dot{\vec{n}} \tag{1.24}$$

It is clear that this vector is orthognonal to both $\vec{t}$ as well as to $\vec{b}$, so that it must lie in the direction of $\vec{n}$

$$\frac{d\vec{b}}{ds} \equiv -\tau\, \vec{n} \tag{1.25}$$

ahere τ is called the *torsion* of the curve at the point considered. For the helix

$$\vec{b} = \frac{1}{\sqrt{a^2 + b^2}} \left(-b \sin\, u, b \cos\, u, a)\right) \tag{1.26}$$

and then

$$\frac{d\vec{b}}{ds} = \frac{1}{a^2 + b^2} \left(-b \cos\, u, -b \sin\, u, 0\right) \tag{1.27}$$

and then

$$\tau = \mp \frac{b}{a^2 + b^2} \tag{1.28}$$

which vanishes for $b = 0$ as it does for any plane curve. Finally, the derivative of the normal vector has to lie in the plane spanned by $(\vec{t}, \vec{b})$

$$\frac{d\vec{n}}{ds} = C_1 \vec{t} + C_2 \vec{b} \tag{1.29}$$

We find that

$$\begin{aligned} C_1 &= \vec{t}.\dot{\vec{n}} = -\vec{n}.\dot{\vec{t}} = -\kappa \\ C_2 &= \vec{b}.\dot{\vec{n}} = -\vec{n}.\dot{\vec{b}} = \tau \end{aligned} \tag{1.30}$$

conveying the fact that

$$\frac{d\vec{n}}{ds} = -\kappa \vec{t} + \tau\, \vec{b} \tag{1.31}$$

This is the last of *Frenet-Serret's formulas.*

Frenet-Serret's formulas also imply that the *acceleration* is given by

$$\begin{aligned} \frac{d^2\vec{t}}{ds^2} &= \kappa\frac{d\vec{n}}{ds} + \dot{\kappa}\vec{n} = -\kappa^2\vec{t} + \kappa\tau\vec{b} + \dot{\kappa}\vec{n} \\ \frac{d^2\vec{n}}{ds^2} &= -\kappa\frac{d\vec{t}}{ds} + \tau\frac{d\vec{b}}{ds} - \dot{\kappa}\vec{t} + \dot{\tau}\vec{b} = -\left(\kappa^2+\tau^2\right)\vec{n} - \dot{\kappa}\vec{t} + \dot{\tau}\vec{b} \end{aligned} \tag{1.32}$$

Neglecting the derivatives of the curvature and the torsion, this yields the familiar *centripetal acceleration* for plane curves, for which $\tau = 0$.

1.2 Surfaces.

$$x^i = x^i\,(u,v) \quad u_1 \leqslant u \leqslant u_2 \quad v_1 \leqslant v \leqslant v_2 \tag{1.33}$$

For example, the *circular cone* $z^2 = x^2 + y^2$

$$\begin{aligned} x &= u \sin v \\ y &= u \cos v \\ z &= u \end{aligned} \tag{1.34}$$

It has a singular point at $u = 0$. Another example is the cylinder

$$\begin{aligned} x &= \cos u \\ y &= \sin u \\ z &= v \end{aligned} \tag{1.35}$$

The *induced metric* on the surface by the euclidean metric in $\mathbb{R}^3$ is

$$ds^2 \equiv \sum_{a,b=1}^{a,b=2} \delta_{ij}\frac{\partial x^i}{\partial x^a}\frac{\partial x^j}{\partial x^b}dx^a dx^b \equiv Edu^2 + 2Fdudv + Gdv^2 \tag{1.36}$$

It used to be called the *first fundamental form* on the surface.

For the cone

$$ds^2 = 2du^2 + u^2dv^2 \tag{1.37}$$

and for the cylinder

$$ds^2 = du^2 + dv^2 \tag{1.38}$$

The *tangent plane* to the surface at a given point is generated by the two vectors

$$\vec{t_a} \equiv \partial_a\vec{x} \quad a = 1, 2. \tag{1.39}$$

and normalized in such a way that

$$\vec{t_a^2} = 1 \tag{1.40}$$

Foe the unit sphere

$$ds_1^2 = d\theta^2 + \sin^2\,\theta d\phi^2 \tag{1.41}$$

$$\vec{t}_\theta = (\cos\,\theta\,\cos\,\phi, \cos\,\theta\,\sin\,\phi, \sin\theta)$$
$$\vec{t}_\phi = (-\sin\,\phi, \sin\,\cos\,\phi, 0) \tag{1.42}$$

Again for the cone

$$\vec{t}_u \equiv \tfrac{1}{\sqrt{2}}(\sin\,v, \cos\,v, 1)$$
$$\vec{t}_v \equiv (\cos\,v, -\sin\,v, 0) \tag{1.43}$$

For the cylinder

$$\vec{t}_u \equiv (-\sin\,u, \cos\,u, 0)$$
$$\vec{t}_v \equiv (0,0,1) \tag{1.44}$$

The *normal vector* is uniquely defined as the unit vector proportional to

$$\vec{N} \equiv \frac{\vec{x}_u \times \vec{x}_v}{|\vec{x}_u \times \vec{x}_v|} \tag{1.45}$$

For the sphere it reads

$$\vec{N} = (\sin\,\theta\,\cos\,\phi, \sin\,\theta\,\sin\,\phi, \cos\,\theta) \equiv \vec{x} \tag{1.46}$$

For our cone

$$\vec{N} \equiv \frac{1}{\sqrt{2}}(\sin\,v, \cos\,v, -1) \tag{1.47}$$

And for the cylinder

$$\vec{N} \equiv (\cos\,u, \sin\,u, 0) \tag{1.48}$$

Consider now a curve on the surface; its tangent vector surely lies on the tangent plane. We can project the derivative of the tangent vector with respect to the arc (the *normal curvature vector*) on a tangential and a normal component.

$$\frac{d\vec{t}}{ds} \equiv \vec{k}_n + \vec{k}_t \tag{1.49}$$

where

$$\vec{k}_n \equiv \kappa_n \vec{N} \equiv \left(\vec{k}.\vec{N}\right)\vec{N} \tag{1.50}$$

and the *tangent or geodesic curvature vector* is $\vec{k}_t$.

Now, the fact that $\vec{N}.\vec{t} = 0$ implies that

$$\kappa_n \equiv \frac{d\vec{t}}{ds}.\vec{N} = -\vec{t}.\frac{d\vec{N}}{ds} = -\frac{d\vec{x}}{ds}.\frac{d\vec{N}}{ds} \tag{1.51}$$

The *second fundamental form* is defined as

$$-d\vec{x}.d\vec{N} \equiv edu^2 + 2fdudv + gdv^2 \tag{1.52}$$

This means that for the sphere the first and second fundamental forms are the same.

$$ds_1^2 = ds_2^2 \tag{1.53}$$

For the cone it gives

$$ds_2^2 = udv^2 \tag{1.54}$$

and for the cylinder

$$ds_2^2 = du^2 \tag{1.55}$$

For the sphere, the determinant

$$g_2 \equiv eg - f^2 = \sin^2 \theta \geqslant 0 \tag{1.56}$$

For the other two surfaces, however,

$$g_2 = 0 \tag{1.57}$$

It so happens that

$$\kappa_n = \frac{edu^2 + 2fdudv + gdv^2}{Edu^2 + 2Fdudv + Gdv^2} = \frac{e + 2f\lambda + g\lambda^2}{E + 2F\lambda + G\lambda^2} \tag{1.58}$$

where

$$\lambda \equiv \frac{dv}{du} \tag{1.59}$$

This defines a function $\kappa_n(\lambda)$. The extrema of this function ara the *directions of principal curvature*, κ_1 and κ_2. The condition of an extrema can be written as

$$[(E + F\lambda) + \lambda(F + G\lambda)]\,(f + g\lambda) = [(e + f\lambda + \lambda(f + g\lambda)]\,(F + G\lambda) \tag{1.60}$$

For those λ we can write

$$\kappa(\lambda) = \frac{e + f\lambda + \lambda(f + g\lambda)}{E + F\lambda + \lambda(F + G\lambda)} = \frac{f + g\lambda}{F + G\lambda} = \frac{e + f\lambda}{E + F\lambda} \tag{1.61}$$

Then

$$\begin{aligned}(e - \kappa E)du + (f - \kappa F)dv = 0\\ (f - \kappa F)du + (g - \kappa G)dv = 0\end{aligned} \tag{1.62}$$

and eliminating κ we get $det\, M = 0$ where

$$M \equiv \begin{pmatrix} dv^2 & -dudv & du^2 \\ E & F & G \\ e & f & g \end{pmatrix} \tag{1.63}$$

from which we get the two directions of principal curvature.

It is also easy to prove ([2]) that they are mutually orthogonal.

In terms of those, the *mean curvature* is defined as

$$M \equiv \frac{\kappa_1 + \kappa_2}{2} = \frac{Eg - 2fF + eG}{2\,(EG - F^2)} \tag{1.64}$$

and the *gaussian curvature* as

$$K \equiv \kappa_1.\kappa_2 = \frac{eg - f^2}{EG - F^2} \tag{1.65}$$

It is clear that when

$$g_2 \geqslant 0 \tag{1.66}$$

the normal chapters are all convex; those points are dubbed *elliptic points.* When

$$g_2 = 0 \tag{1.67}$$

there is one direction with $\kappa = 0$; those are *parabolic points.* Finally, when

$$g_2 \leqslant 0 \tag{1.68}$$

some normal chapters are convex and others are concave; those are *hyperbolic points.*

For the cone

$$\begin{aligned} M &= \tfrac{2u}{2u^2} = \tfrac{1}{u} \\ K &= 0 \end{aligned} \tag{1.69}$$

Clearly something special happens at the apex of the cone, $u = 0$, although the gaussian curvature does not see it.

The three vectors

$$\left(\vec{x}_u, \vec{x}_v, \vec{N}\right) \tag{1.70}$$

onstitute a *moving frame* (that is, a frame at each point of the surface). Consequently, we can expand

$$\begin{aligned} \vec{x}_{uu} &\equiv \Gamma^1_{11}\,\vec{x}_u + \Gamma^2_{11}\,\vec{x}_v + e\,\vec{N} \\ \vec{x}_{uv} &\equiv \gamma^1_{12}\,\vec{x}_u + \Gamma^2_{12}\,\vec{x}_v + f\,\vec{N} \\ \vec{x}_{vv} &\equiv \Gamma^1_{22}\,\vec{x}_u + \Gamma^2_{22}\,\vec{x}_v + g\,\vec{N} \end{aligned} \tag{1.71}$$

where the *Christoffel sumbols* are given by

$$\begin{aligned} \Gamma^1_{11} &= \tfrac{GE_u - 2FF_u + FE_u}{2(EG-F^2)} \\ \Gamma^1_{12} &= \tfrac{GE_v - FG_u}{2(EG-F^2)} \\ \Gamma^1_{22} &= \tfrac{2GF_v - GG_u - FG_v}{2(EG-F^2)} \\ \Gamma^2_{11} &= \tfrac{2EF_u - EE_v - FE_u}{2(EG-F^2)} \\ \Gamma^2_{12} &= \tfrac{EG_u - FE_v}{2(EG-F^2)} \\ \Gamma^2_{22} &= \tfrac{EG_v - 2FF_v + FG_u}{2(EG-F^2)} \end{aligned} \tag{1.72}$$

Also, from $\vec{N}^2 = 1$ we know that

$$\begin{aligned} \vec{N}_u &= p_1\,\vec{x}_u + p_2\,\vec{x}_v \\ \vec{N}_v &= q_1\,\vec{x}_u + q_2\,\vec{x}_v \end{aligned} \tag{1.73}$$

Gauss' *theorema egregium* states that the Gaussian curvature depends onlt on E, F, G, and their first derivatives. This shows that it is a *bending invariant*, in Struik's words. This means that those properties are intrinsic to the surface, and they do not depend on how the surface is imbedded in euclidean ambient space. The theorem can be proven by demanding that

$$\begin{aligned} \vec{x}_{uuv} &= \vec{x}_{uvu} \\ \vec{x}_{vvu} &= \vec{x}_{uvv} \end{aligned} \tag{1.74}$$

2

Tensor calculus in vector spaces

Consider a n-dimensional vector space, V with a basis

$$\forall v \in V \quad v = \sum_{i=1}^{n} v^i e_i \equiv v^i e_i \tag{2.1}$$

where we have introduced the summation convention. This only affects contravariant coupled with covariant indices. Given a nonsingular $n \times n$ matrix, we can change to a different basis, f_a

$$f_a \equiv A^i_a e_i \tag{2.2}$$

Then the vector v can be expressed in the new basis

$$v = v^a f_a = v^i e_i = v^a A^j_a e_j \tag{2.3}$$

and owing to the fact the the basis elements are linearly independent,

$$v^j = v^a A^j_a \quad \longrightarrow \quad v^a = B^a_i v^i \tag{2.4}$$

where the matrix $B = A^{-1}$

$$A^j_a B^a_i = \delta^j_i \tag{2.5}$$

For the time being, indices cannot be raised or lowered. Consider now the dual space, V^*.

$$\theta \in V \quad \theta(v) \in \mathbb{R} \tag{2.6}$$

We can define the *dual basis* of the basis of V through

$$E^i(e_j) \equiv \delta^i_j \tag{2.7}$$

Please note carefully the position of the indices in the Kronecker delta. Those are the only deltas that are allowed in this course. Any element $\omega \in V^*$ can be expanded in the dual basis

$$\omega \equiv \omega^i \epsilon_i \tag{2.8}$$

Under a change of basis in V

$$\omega^i \rightarrow A^i_a \omega^a \tag{2.9}$$

Everybody heard aboud some wild and ferocious animals called tensors. What are those? Consider bilinear mappings from

$$V \times V^* \quad \rightarrow \quad \mathbb{R} \tag{2.10}$$

$$T: \quad (v, \theta) \rightarrow T(v, \theta) \in \mathbb{R} \tag{2.11}$$

Owing to linearity, it is enough to know the values on the basis, because

$$T(v, \theta) = v^i\ \theta_j\ T(e_i, E^j) \equiv v^i\ \theta_j\ T_i{}^j \tag{2.12}$$

The space of those animals is called the tensor product of $V^* \otimes V$, and its elements are called (1-covariant 1-contravariant) tensors. Under a change of basis

$$T_i{}^j \rightarrow A^a_i\ B^j_b\ T_a{}^b \tag{2.13}$$

The set of all those (1,1) tensors is another vector space, which is calles the tensor product of $V^* \otimes V$

$$E^l \otimes e_i \ \in \ V^* \otimes V \tag{2.14}$$

Pleasee note carefully that

$$V \otimes V^* \neq V^* \otimes V \tag{2.15}$$

that is

$$T_i{}^j \neq T^j{}_i \tag{2.16}$$

Ordinary vectors and ordinary dual vectors are particular instances (0,1) and (1,0) respectiveli. The generalization to

$$T^{i_1 \ldots i_p}{}_{j_1 \ldots j_q}\ e_{i_1} \otimes \ldots e_{i_p} \otimes E^{j_1} \otimes E^{j_q} \ \in \ V \otimes \ldots (p) \ldots \otimes V \otimes V^* \otimes \ldots (q) \tag{2.17}$$

is immediate. The contravariant or covariant character of the indices is an absolute property. There is no in general a canonical way of raising or lowering indices. When there is a metric, there is such a canonical way.

But before introducing a metric, let us examine some particularly interesting tensors which are defined independently of the metric. As a matter of notation, let us define the *symmetrization operator*

$$T_{(a_1\ldots a_p)} \equiv \frac{1}{p!}\sum_{\pi\in S_p} T_{a_{\pi(1)}\ldots a_{\pi)n)}} \tag{2.18}$$

where the sum extends over all $p!$ elements of the permutation group S_p; as well as the *antisymmetrization operator*

$$T_{[a_1\ldots a_p]} \equiv \frac{1}{p!}\sum_{\pi\in S_p} (-1)^{P_\pi}\, T_{a_{\pi(1)}\ldots a_{\pi)n)}} \tag{2.19}$$

where P_π is the parity of the permutation π.

2.1 Differential forms.

Let us identify tangent vectors $\vec{v} \in T_x$ with directional derivatives of functions defined at a given point

$$\vec{v}(f) \equiv v^\mu \partial_\mu f \tag{2.20}$$

A particular basis is given by the vectors

$$\partial_\mu \tag{2.21}$$

Given an arbitrary function, its differential is defined as $df \in T_x^*$

$$df\,(\vec{v}) \equiv \vec{v}\,(f) \tag{2.22}$$

Differential forms are antisymmetric linear maps

$$\omega_1\ : v \in \mathbb{R}^n \rightarrow \omega(v) \in \mathbb{R} \tag{2.23}$$

A local basis is given by

$$dx^a(\partial_b) = \delta^a_b \tag{2.24}$$

Let us define a *p-form* $A \in \Lambda^p$ as a tensor with p cpvariant indix, totally antisymmetric

$$A_{a_1\ldots a_p} \equiv A_{[a_1\ldots a_p]} \tag{2.25}$$

Dor examplem $\omega \in \Lambda^2$:

$$\omega_2\,(v,w)\ \in \mathbb{R}^n \times \mathbb{R}^n \rightarrow \omega_2(v,w) \in \mathbb{R} \tag{2.26}$$

- **Exterior product.** The exterior product of two one-forms yields a two-form

$$(\omega_1 \wedge \alpha_1)(v_1, v_2) \equiv \det \begin{pmatrix} \omega_1(v_1) & \alpha_1(v_1) \\ \omega_1(v_2) & \alpha_1(v_2) \end{pmatrix} \tag{2.27}$$

In the general case, the product of a p-form and a q-form is a (p+q)-form

$$(\omega_k \wedge \omega_l)\,(v_1 \ldots v_{k+l}) \equiv \sum \pm \omega_k(v_{i_1} \ldots v_{i_k})\omega_l(v_{i_{k+1}} \ldots v_{i_{k+l}}) \tag{2.28}$$

The basic identity reads

$$\omega_p \wedge \omega_q = (-1)^{pq} \;\; \omega_q \wedge \omega_p \tag{2.29}$$

Sometimes we shall write

$$dx^{\mu_1 \ldots \mu_p} \equiv dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p} \tag{2.30}$$

This means that for every odd degree form

$$\omega_{2p+1} \wedge \omega_{2p+1} = 0 \tag{2.31}$$

- **Coordinate basis.**

In the basis of the tangent space associated to a local chart, (x^α),

$$\omega_k \equiv \sum_{\iota_1 < \ldots < \iota_k} \omega_{\iota_1 \ldots \iota_k} dx^{\iota_1} \wedge \ldots \wedge dx^{\iota_k} \tag{2.32}$$

$$dx^\mu \wedge dx^\nu = dx^\mu \otimes dx^\nu + dx^\nu \otimes dx^\mu \tag{2.33}$$

We shall write in local coordinates

$$\alpha \equiv \frac{1}{p!} \alpha_{\mu_1 \ldots \mu_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p} \tag{2.34}$$

It is exceedingly useful to introduce the Kronecker symbols

$$\epsilon^{\lambda_1 \ldots \lambda_p}_{\mu_1 \ldots \mu_p} \equiv p! \;\delta^{\lambda_1}_{[\mu_1} \ldots \delta^{\lambda_p}_{\mu_p]} \tag{2.35}$$

It is a good exercise to prove that

$$\begin{aligned} &\epsilon^{\rho_1 \ldots \rho_p}_{\mu_1 \ldots \mu_p} \alpha_{\rho_1 \ldots \mu_p} = p! \;\alpha_{\mu_1 \ldots \mu_p} \\ &\epsilon^{\lambda_1 \ldots \lambda_q}_{\mu_1 \ldots \mu_q} \epsilon^{\mu_1 \ldots \mu_q \sigma_1 \ldots \sigma_p}_{\nu_1 \ldots \nu_{p+q}} = q! \;\epsilon^{\lambda_1 \ldots \lambda_q \sigma_1 \ldots \sigma_p}_{\nu_1 \ldots \nu_{p+q}} \\ &\epsilon^{\lambda_1 \ldots \lambda_q \rho_1 \ldots \rho_p}_{\mu_1 \ldots \mu_q \rho_1 \ldots \rho_p} = p! \;\epsilon^{\lambda_1 \ldots \lambda_q}_{\mu_1 \ldots \mu_q} \end{aligned} \tag{2.36}$$

A general formula for the exterior product is given by

$$\alpha \wedge \beta = \frac{1}{p!}\frac{1}{q!} \;\alpha_{\lambda_1 \ldots \lambda_p} \beta_{\mu_1 \ldots \mu_q} dx^{\lambda_1} \wedge \ldots \wedge dx^{\lambda_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_q} \tag{2.37}$$

- **Exterior differential.** The differential of a function is given by a one-form

$$df \equiv \sum \partial_a f dx^a \tag{2.38}$$

In the general case, the differential of a p-form is a (p+1)-form

$$d\omega \equiv \sum_{\iota_1<\ldots<\iota_k} d\omega_{\iota_1\ldots\iota_k} \wedge dx^{\iota_1} \wedge \ldots \wedge dx^{\iota_k} \tag{2.39}$$

A general formula can also be given

$$(d\alpha)_{\mu_0\mu_1\ldots\mu_p} \equiv \frac{1}{(p+1)!}\, \epsilon^{\lambda_0\lambda_1\ldots\lambda_p}_{\mu_0\mu_1\ldots\mu_p} \partial_{\lambda_0}\alpha_{\lambda_1\ldots\lambda_p} \tag{2.40}$$

The uselfuness of exterior calculus stems essentially from the basic fact that

$$d^2 = 0 \tag{2.41}$$

It is also a fact that the graded Leibnitz rule holds, id est,

$$d\left(\alpha_p \wedge \beta_q\right) = d\alpha_p \wedge \beta_q + (-1)^p \alpha_p \wedge d\beta_q \tag{2.42}$$

- **Pullback.**

$$\phi : x \in M_p \rightarrow y \in N_q \tag{2.43}$$

$$\omega = a_i\, dy^i \in \Lambda(N) \rightarrow \phi^*\omega \equiv a_i(y(x))\, \frac{\partial y^i}{\partial x^a}\, dx^a \in \Lambda(M) \tag{2.44}$$

It is fact of life that

$$d\left(\phi^*\omega\right) = \phi^* d\omega \tag{2.45}$$

- **Poincaré.** Everybody knows that in $\mathbb{R}^3$

$$\vec{\nabla} \times \vec{v} = \vec{0} \Longrightarrow \vec{v} = \vec{\nabla}\phi \tag{2.46}$$

In fact Poncaré was able to show that in $\mathbb{R}^n$

$$d\omega = 0 \Longrightarrow \omega = d\alpha \tag{2.47}$$

This is not true in general, and the number of independent ω that fail to satisfy that is called the *Bettti number* of the manifold. Let us prove this theorem. Given a p-form,

$$\omega_p \in \Lambda(\mathbb{R}^n) \equiv \frac{1}{p!}\omega_{a_1\ldots a_p}\left(x_1, \ldots x_n\right) dx^{a_1} \wedge dx^{a_2} \ldots \wedge dx^{a_p} \tag{2.48}$$

, define the *homotopy operator*, K in two steps. First define a Λ_{p+1} form

$$(\phi * \omega) \equiv \frac{1}{p!}\omega_{a_1\ldots a_p}\left(tx_1, \ldots, tx_n\right)\left(x^{a_1}dt + tdx^{a_1}\right)\wedge\left(x^{a_2}dt + tdx^{a_2}\right)\wedge\ldots\wedge\left(x^{a_p}dt + tdx^{a_p}\right) \tag{2.49}$$

Now the operator K is defined in two steps. In those monomials of $\phi^*\omega$ not involving dt

$$K\omega = 0 \tag{2.50}$$

On monomials of $\phi^*\omega$ involving dt

$$(K\omega) \equiv \left(\int_0^1 (\phi^*\omega)_{\vec{a}}\,(tx)dt\right)\, dx^{\vec{a}} \tag{2.51}$$

Let us work out an example in $n = 3$ dimensions.

$$\omega = x dx \wedge dy + e^z dy \wedge dz \tag{2.52}$$

$$\begin{aligned}\phi^*\omega &= tx\,(tdx + xdt) \wedge (tdy + ydt) + e^{tz}\,(tdy + ydt) \wedge (tdz + zdt) = \\ &= -t^2 xy\, dt \wedge dx + \left(t^2x^2 - e^{tz}\, tz\right)\, dt \wedge dy + e^{tz} ty\, dt \wedge dz + \text{ no dt terms}\end{aligned} \tag{2.53}$$

$$\begin{aligned}\alpha \equiv K\omega &= \textstyle\int_0^1 -t^2xy\, dt \wedge dx + \left(t^2x^2 - e^{tz}\, tz\right)\, dt \wedge dy + e^{tz} ty\, dt \wedge dz = \\ &= -\tfrac{1}{3}xydx + \left(\tfrac{x^2}{3} - \tfrac{e^z(z-1)+1}{z}\right) dy + y\tfrac{e^z(z-1)+1}{z^2} dz\end{aligned} \tag{2.54}$$

since

$$\int u e^u du = e^u(u-1) \tag{2.55}$$

And lo and behold,

$$\omega = d\alpha \tag{2.56}$$

- **Hodge dual.** Let us introduce the so called *volume element* defined as

$$\eta_{\mu_1\ldots\mu_n} \equiv \sqrt{|g|}\ \epsilon^{1\ldots n}_{\mu_1\ldots\mu_n} \tag{2.57}$$

Actually, $\epsilon^{1\ldots n}_{\mu_1\ldots\mu_n}$ is not a tensor. Let us work it out in two dimensions. Denote the jacobian matrix

$$J^b_{a'} \equiv \frac{\partial x^b}{\partial x^{a'}} \tag{2.58}$$

and its determinant by $J \equiv \det J^b_{a'}$ Also the determinant of the metric itself does not transform as a true scalar, but rather

$$g'(x') \equiv J^2 g(x) \tag{2.59}$$

Then

$$\eta_{ab} dx^a \wedge dx^b = \sqrt{g}\epsilon_{ab} J^a_{a'} J^b_{b'} dx^{a'} \wedge dx^{b'} = \frac{1}{J}\sqrt{g'}J\epsilon_{a'b'} dx^{a'} \wedge dx^{b'} = \eta_{a'b'} dx^{a'} \wedge dx^{b'} \tag{2.60}$$

This means that

$$\sqrt{g}\,\epsilon_{ab} \tag{2.61}$$

is a true tensor. Some properties;

$$\begin{aligned}
&\eta^{\mu_1\ldots\mu_n} = \frac{1}{\sqrt{|g|}}\;\epsilon^{\mu_1\ldots\mu_n}_{1\ldots n}\\
&\eta_{\lambda_1\ldots\lambda_p\lambda_{p+1}\ldots\lambda_n}\eta^{\lambda_1\ldots\lambda_p\mu_{p+1}\ldots\mu_n} = p!\;\epsilon^{\mu_{p+1}\ldots\mu_n}_{\lambda_{p+1}\ldots\lambda_n}\\
&\nabla_\rho\eta_{\mu_1\ldots\mu_n} = 0\\
&d(vol) \equiv \eta_{\mu_1\ldots\mu_n}dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_n} = \sqrt{|g|}dx^1\wedge\ldots\wedge dx^n\\
&dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_n} = \eta^{\mu_1\ldots\mu_n}d(vol)
\end{aligned} \tag{2.62}$$

To verify these formulas is excellent gymnastics.

The Hodge operator maps p-forms into (n-p)-forms.

$$* : \Lambda^p \rightarrow \Lambda^{n-p} \tag{2.63}$$

It is defined by

$$(*A)_{\mu_{p+1}\ldots\mu_n} \equiv \frac{1}{p!}\;\eta_{\mu_1\ldots\mu_n}A^{\mu_1\ldots\mu_p} \tag{2.64}$$

It is clear that in $\mathbb{R}^3$

$$\begin{aligned}
*dz &= dx\wedge dy\\
*dy &= dz\wedge dx\\
*dx &= dy\wedge dz
\end{aligned} \tag{2.65}$$

Its square depends on the dimension of spacetime as well as on the degree of the form

$$*^2 : \Lambda^p \rightarrow \Lambda^p \tag{2.66}$$

First of all

$$(*A)_{a_1\ldots a_{n-p}} = \frac{1}{p!}\eta_{b_1\ldots b_p a_1\ldots a_{n-p}}A^{b_1\ldots b_p} = (-1)^{p(n-p)}\frac{1}{p!}\eta_{a_1\ldots a_{n-p}b_1\ldots b_p}A^{b_1\ldots b_p} \tag{2.67}$$

and

$$\begin{aligned}
(*^2A)_{c_1\ldots c_p} = \tfrac{1}{p!}\tfrac{1}{(n-p)!}\eta_{c_1\ldots c_p b_1\ldots b_{n-p}}\eta^{b_1\ldots b_{n-p}d_1\ldots d_p}A_{d_1\ldots d_p} =\\
= \tfrac{1}{p!}\epsilon^{d_1\ldots d_p}_{c_1\ldots c_p}A_{d_1\ldots d_p} = A_{c_1\ldots c_p}
\end{aligned} \tag{2.68}$$

$$*^2 = (-1)^{p(n-p)} \tag{2.69}$$

In four dimensions (actually, in any even dimension)

$$*^2 = (-1)^p \tag{2.70}$$

In $\mathbb{R}^4$

$$*dx \wedge dy = dz \wedge dw \tag{2.71}$$

There are then euclidean self-dual two-forms

$$\omega_2 \equiv dx \wedge dy + dz \wedge dw \tag{2.72}$$

In three-dimensions Hodge squared it is always +1

$$*^2 = +1. \tag{2.73}$$

The *exterior codifferential* is the adjoint of the exterior differential

$$(\alpha, \delta\beta) \equiv (d\alpha, \beta) \tag{2.74}$$

It is given by

$$\delta \equiv (-1)^p *^{-1} d* \tag{2.75}$$

It is possible to give a simple formula

$$(\delta\alpha)_{\rho_1\ldots\rho_{p-1}} = -\frac{1}{p!}\, \epsilon^{\mu_1\ldots\mu_p}_{\nu\rho_1\ldots\rho_{p-1}} \nabla^\nu \alpha_{\mu_1\ldots\mu_p} \tag{2.76}$$

The *interior product* of a p-form and a vector, X, is the (p-1)-form given by

$$(i(X)\omega)(v_1 \ldots v_{p-1}) \equiv \omega_p (X, v_1 \ldots v_{p-1}) \tag{2.77}$$

- **Stokes' theorem** We start from the properties of the volume defined by an elementary cell of $\mathbb{R}^3$

  - It vanishes if the vectors are linearly dependent.
  - It stays the same when we add to a given vector a linear combination of the other vectors.
  - Depends in a linear way on all vectors.

Al these properties are enjoyed by the elementary formula

$$V = \sum \epsilon_{ijk} v_1^i v_2^j v_3^k = \eta\,(\vec{v}_1, \vec{v}_2, \vec{v}_3) \tag{2.78}$$

where the *volume element* is defined by

$$\eta \equiv dx^1 \wedge dx^2 \wedge dx^3 \tag{2.79}$$

This leads in a natural way to define volumes through integration. For example, in $\mathbb{R}^4$,

– **Codimension-1 hypersurfaces** $dS_a \equiv \frac{1}{3!}\eta_{abcd}dx^{bcd}$ Consider for example the hypersurface

$$S \equiv \{x_4 = T\} \tag{2.80}$$

The normal is the vector

$$n = (0, 0, 0, 1) \tag{2.81}$$

The hyprersurface can be paremeterized by

$$x^i = \xi^i \tag{2.82}$$

so that

$$dS_a \equiv \frac{1}{3!}\, n \sqrt{g(x_4 = T)}\, d^3\xi \tag{2.83}$$

– **Codimension-2 hypersurfaces** $dV_{ab} \equiv \frac{1}{2!}\eta_{abcd}dx^{cd}$ Consider the two-sphere

$$S_2 \hookrightarrow \mathbb{R}^4 \tag{2.84}$$

$$\begin{aligned} x_4 &= T \\ x^2 + y^2 + z^2 &= R^2 \end{aligned} \tag{2.85}$$

It can be parameterized by polar coordinates

$$x^i = x^i(\theta, \phi) \tag{2.86}$$

There are two normal vectors, namely

$$\begin{aligned} n_1 &= (0, 0, 0, 1) \\ n_2 &= (x, y, z, 0) \end{aligned} \tag{2.87}$$

and the volume element is

$$dV_{ab} = n_a^1 n_b^2 \sqrt{g(T, \theta, \phi)}\, d\theta \wedge d\phi \tag{2.88}$$

– **Codimension-3 hypersurfaces** $dV_{abc} \equiv \frac{1}{3!}\eta_{abcd}dx^d$

For a trivial example, consider

$$x^i = x_0^i \tag{2.89}$$

which can be pameterized as

$$x_4 = \sigma \tag{2.90}$$

there are now three normals

$$n_i \equiv e_i \tag{2.91}$$

and the volume element reads

$$dV_{abc} = n^1_a n^2_b n^3_c \sqrt{g(x^i_0)}\, d\sigma \tag{2.92}$$

Stokes' theorem in general states that.

$$\int_{\partial V} \omega = \int_V d\omega \tag{2.93}$$

The classical theorems of Gauss, Stokes and the divergence are but particular instamces of this. For example

$$\int_{S_2} dA_1 = \int_{C_1 \equiv \partial S_2} A_1 \tag{2.94}$$

If A_1 is a 1-form of $\mathbb{R}^3$

$$A_1 \equiv A_i dx^i \tag{2.95}$$

then

$$dA_2 = \frac{1}{2}\ (\partial_i A_j - \partial_j A_i)\, dx^i \wedge dx^j \tag{2.96}$$

It is customary to define the *rotational* or *curl* as

$$(\text{rot}A)_i \equiv \epsilon_{ijk} \partial_j A_k \tag{2.97}$$

The surface integral

$$\int_S dA_2 = \int_S \frac{1}{2} (\partial_i A_j - \partial_j A_i)\, dx^i \wedge dx^j = \tag{2.98}$$

It is customary to define

$$n^i dS \equiv \frac{1}{2}\ \epsilon_{ijk} dx^j \wedge dx^k \tag{2.99}$$

so that

$$\sum_i (\text{rot}A)_i\, n_i dS = \sum_{jk} (\partial_j A_k - \partial_k A_j)\, dx^j \wedge dx^k \tag{2.100}$$

and we recover Stokes'original theorem

$$\int_S \vec{\text{rot}} \vec{A}\ \vec{n}\ dS = \int_{\partial S} \vec{A}\ d\vec{x} \tag{2.101}$$

Let us now apply it to

$$\int_{V_3} d\omega_2 = \int_{\partial V_3} \omega_2 \tag{2.102}$$

Write

$$\omega_2 \equiv \frac{1}{2}\ \omega_{ij} dx^i \wedge dx^j \tag{2.103}$$

so that

$$d\omega_2 \equiv \frac{1}{2}\partial_k \omega_{ij} dx^k \wedge dx^i \wedge dx^j = \frac{1}{2}\ \partial_k \omega_{ij} \epsilon^{kij} dV \tag{2.104}$$

Now we define the dual one-form

$$\Omega_i dx^i \equiv (*\omega_2)_1 \equiv \frac{1}{2}\ \epsilon_{ijk}\omega_{jk} \tag{2.105}$$

then

$$d\omega_2 = \partial_k \Omega_k \equiv \text{div}\vec{\Omega} \tag{2.106}$$

and we recover Gauss'divergence teorem

$$\int_V \text{div}\ \vec{\Omega}\ dV = \int_{\partial V} \vec{\Omega}\ \vec{n}\ dS \tag{2.107}$$

Let us work out the integral over a two-sphere

$$\begin{aligned} x &= R \sin\ \theta\ \cos\ \phi \\ y &= R\ \sin\ \theta\ \sin\ \phi \\ z &= R\ \cos\ \theta \end{aligned} \tag{2.108}$$

z is no an independent variable; rather,

$$z \equiv \sqrt{1 - x^2 - y^2} \tag{2.109}$$

$$\begin{aligned} \sin\ \phi &= \tfrac{y}{\sqrt{x^2+y^2}} \\ \sin\ \theta &= \sqrt{x^2 + y^2} \end{aligned} \tag{2.110}$$

$$\begin{aligned} \tfrac{\partial\theta}{\partial x} &= \tfrac{\cos\ \phi}{\cos\ \theta} \\ \tfrac{\partial\theta}{\partial y} &= \tfrac{\sin\ \phi}{\cos\ \theta} \\ \tfrac{\partial\phi}{\partial x} &= -\tfrac{y}{\sin^2\ \theta} \\ \tfrac{\partial\phi}{\partial y} &= \tfrac{x}{\sin^2\ \theta} \end{aligned} \tag{2.111}$$

Exterior normal

$$\begin{aligned} \vec{v}.\vec{ds} &= \tfrac{xv^x + yv^y + zv^z}{R}\ R^2 \sin\ \theta d\theta d\phi = \\ &= R\left(v^x \sin\theta\ \cos\ \phi + v^y \sin\ \theta\ \sin\ \phi + v^z\ \cos\ \theta\right) \sin\ \theta d\theta d\phi \end{aligned} \tag{2.112}$$

Assume, for example,

$$v = z\frac{\partial}{\partial z} \tag{2.113}$$

(such that

$$\vec{\nabla}\vec{v} = 1) \tag{2.114}$$

$$\int_{S^2} \vec{\nabla}\vec{v}\; d(vol) = \int_0^R r^2 dr \int \sin\; \theta d\theta d\phi = 2\pi\, \frac{r^3}{3}\bigg|_0^R = \frac{4}{3}\pi R^3 \tag{2.115}$$

This equals

$$R^3 \int \sin\; \theta \cos^2\; \theta d\phi = -2\pi R^3 \cos^3\; \theta\big|_0^\pi = \frac{4\pi}{3} R^3 \tag{2.116}$$

- **Lie derivative.** The Lie derivative of a function is defined as the directional derivative

$$\vec{v}(f) = \pounds(\vec{v})f \tag{2.117}$$

The Lie derivative of a one-form is defined in a natural way.

$$\pounds(\vec{v})df \equiv d\vec{v}(f) \tag{2.118}$$

This definition extends to a general case simply by postulating that Leibnitz' rule holds true

$$\pounds(\vec{v})a_a d\xi^a = (\pounds(\vec{v})a_a)d\xi^a + \alpha_a \pounds(\vec{v})d\xi^a \tag{2.119}$$

In the case of vectors we use the dual application

$$\pounds(\vec{v})\langle\alpha, \vec{X}\rangle = \langle\pounds(\vec{v})\alpha, \vec{X}\rangle + \langle\alpha, \pounds(\vec{v})\vec{X}\rangle \tag{2.120}$$

It is a fact that

$$\begin{aligned} &\pounds(\vec{X})\vec{Y} = [\vec{X}, \vec{Y}] \\ &\pounds(\vec{X}) = i(\vec{X})d + di(\vec{X}) \end{aligned} \tag{2.121}$$

- Diffeomorfisms An active diffeomorphism

$$\xi : x \in M \rightarrow y = \xi(x) \in M \tag{2.122}$$

Acting on vectors, given $g : y \rightarrow \mathbb{R}$, then $g \circ \xi : x \rightarrow \mathbb{R}$ and $v \in T_x$, we define a different vector $\xi_* v \in T_y$ through

$$\xi_*(v)(g) \equiv v(g \circ \xi) \tag{2.123}$$

In a local coordinate basis

$$(\xi_* v)^\mu(y) = v^\rho \partial_\rho \xi^\mu(x) \tag{2.124}$$

Given a one-form $\omega \in T_y^*$ we define another form $\xi^*\omega \in T_x$ through

$$\xi^*\omega(v) \equiv \omega(\xi_* v) \tag{2.125}$$

In a local coordinate basis

$$(\xi^*\omega)_\alpha(x) = \omega_\mu(y)\partial_\alpha \xi^\mu(x) \tag{2.126}$$

If it were a 2-form

$$(\xi^*\omega)(v, w) = \omega(v, w) \tag{2.127}$$

that is

$$(\xi^*\omega)_{\alpha\beta}(x) = \omega_{\mu\nu}(y)\partial_\alpha \xi^\mu \partial_\beta \xi^\nu \tag{2.128}$$

2.2 The metric tensor.

The metric tensor in $\mathbb{R}^n$ is defined through

$$ds^2 = g_{ab}(x)dx^a dx^b \tag{2.129}$$

with

$$g \equiv \det g_{ab} \neq 0 \tag{2.130}$$

so that there is the inverse matrix

$$g^{ac} g_{cb} = \delta^a_b \tag{2.131}$$

For example, in polar coordinates

$$ds^2 = dr^2 + r^2\ d\Omega^2 \tag{2.132}$$

where

$$d\Omega^2 \equiv d\theta^2 + \sin^2\ \theta d\phi^2 \tag{2.133}$$

Then, there is a canonical mapping from

$$V \rightarrow V^* \tag{2.134}$$

$$V^*(V) \equiv g(V, V) \tag{2.135}$$

This is

$$V_a = g_{ac} V^c \qquad\qquad V^a = g^{ab} V_b \tag{2.136}$$

Ordinary derivatives of any object more complicated than scalar (id est, a vector, or any higher rank tensor) are not tensors, not even in $\mathbb{R}^n$. This is because under the change of coordinates

$$x^a \rightarrow y^\alpha(x^a) \tag{2.137}$$

under which

$$V^\alpha(y) \equiv \frac{\partial y^\alpha}{\partial x^b} V^b(x) \tag{2.138}$$

derivatives transform as

$$\frac{\partial V^\alpha(y)}{\partial y^\gamma} = \frac{\partial^2 y^\alpha}{\partial x^b \partial x^c} \frac{\partial x^c}{\partial y^\gamma} V^b(x) + \frac{\partial y^\alpha}{\partial x^b} \frac{\partial V^b(x)}{\partial x^c} \frac{\partial x^c}{\partial y^\gamma} \tag{2.139}$$

Let us lighten the notation a little bit. First of all,

$$\partial_\alpha \equiv \frac{\partial}{\partial y^\alpha}$$
$$\partial_b \equiv \frac{\partial}{\partial x^b} \tag{2.140}$$

Er also introduce

$$J^\alpha_b \equiv \frac{\partial y^\alpha}{\partial x^b} \tag{2.141}$$

Then

$$\left(J^{-1}\right)^a_\beta \equiv \frac{\partial x^a}{\partial y^\beta} \tag{2.142}$$

is the inverse matrix

$$J \ \left(J^{-1}\right) = \left(J^{-1}\right) \ J = 1 \tag{2.143}$$

The previous equation reads

$$\partial_\gamma V^\alpha = \left(J^{-1}\right)^c_\gamma \partial_c J^\alpha_b V^b + J^\alpha_b \partial_c V^b \left(J^{-1}\right)^c_\gamma \tag{2.144}$$

It is conceptually much simpler if we imagine matrices with rows defined by the covariant indices and columns by the contravariant indices. The equation then reads

$$\partial V(y) = \left(J^{-1}\right) \partial J V + J^{-1} \partial V(x) J \tag{2.145}$$

Let us now ask the question: is it possible to modify the definition of derivative in such a way that

$$\nabla V = \left(J^{-1}\right) \nabla V J \tag{2.146}$$

Let us try the ansatz

$$\nabla V = \partial V + \Gamma V \tag{2.147}$$

(Γ is a three-index beast). In order for that to be true the transformed covariant derivative

$$\partial V(y) + \Gamma(y) V(y) = \left(J^{-1}\right) \partial J V + J^{-1} \partial V(x) J + \Gamma(x) V(x) \tag{2.148}$$

ought to be equal to

$$J^{-1} \Big(\partial V(x) + \Gamma(x) V(x) \Big) J \tag{2.149}$$

This would be true provided the Γ transform as

$$\Gamma(y) = J^{-1}\, \partial J + J^{-1} \Gamma J \tag{2.150}$$

When such an object exists, there is an invariant concept of derivative. This is what mathematicians call a *connection*. The surprising thing is that whenever there is a metric, there is such a connection, which is called the Levi-Civita one, and the coefficients, the *Christoffel symbols*,

$$\Gamma^{\alpha}_{\beta\gamma} \equiv \frac{1}{2} g^{\alpha\lambda} \left(-\partial_\lambda g_{\beta\gamma} + \partial_\beta g_{\lambda\gamma} + \partial_\gamma g_{\lambda\beta}\right) \tag{2.151}$$

Let us check that

$$\begin{aligned}
\Gamma^{\alpha}_{\beta\gamma} = & \frac{1}{2} J^{\alpha}_{a} J^{\lambda}_{l} g^{al} \Big\{ -\partial_\lambda \left(J^{b}_{\beta} J^{c}_{\gamma} g_{bc}\right) + \partial_\beta \left(J^{l}_{\lambda} J^{c}_{\gamma} g_{lc}\right) + \partial_\gamma \left(J^{b}_{\beta} J^{l}_{\lambda} g_{bl}\right) \Big\} = \\
= & \frac{1}{2} J^{\alpha}_{a} J^{\lambda}_{u} g^{au} \Big\{ -\partial_\lambda J^{b}_{\beta} J^{c}_{\gamma} g_{bc} - J^{b}_{\beta} \partial_\lambda J^{c}_{\gamma} g_{bc} - J^{b}_{\beta} J^{c}_{\gamma} \partial_\lambda g_{bc} + \\
& + \partial_\beta J^{l}_{\lambda} J^{c}_{\gamma} g_{lc} + J^{l}_{\lambda} \partial_\beta J^{c}_{\lambda} g_{lc} + J^{l}_{\lambda} J^{c}_{\gamma} \partial_\beta g_{lc} + \\
& + \partial_\gamma J^{b}_{\beta} J^{l}_{\lambda} g_{bl} + J^{b}_{\beta} \partial_\gamma J^{l}_{\lambda} g_{bl} + J^{b}_{\beta} J^{l}_{\lambda} \partial_\gamma g_{bl} \Big\}
\end{aligned} \tag{2.152}$$

The three terms in the right of the rows yield

$$J^{\alpha}_{a} J^{b}_{\beta} J^{c}_{\gamma} \Gamma^{a}_{bc} \tag{2.153}$$

If this were all, this would have been a true tensor. But there is more. Taking into account that

$$\partial_\alpha J^{a}_{\beta} = \partial_\beta J^{a}_{\alpha}, \tag{2.154}$$

the terms in the paces 11 and 21 cancel, as do the terms 12 and 32. The rest (22+31) yield

$$J^{\alpha}_{a} J^{\lambda}_{u} g^{au} \partial_\gamma J^{b}_{\beta} J^{l}_{\lambda} g_{bl} = g^{a\lambda} \partial_\gamma J^{b}_{\beta} J^{l}_{\lambda} g_{bl} \tag{2.155}$$

QED.

The two basic properties of the Levi-Civita connection are

$$\begin{aligned}
& \Gamma^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\gamma\beta} \\
& \nabla_\alpha\, g_{\beta\gamma} = 0
\end{aligned} \tag{2.156}$$

2.3 Winding numbers and such.

Spheres are defined as

$$S \equiv \sum_{a=1}^{n} x_a^2 = L^2 \tag{2.157}$$

The n-dimensional ball is defined as

$$B_n : \sum_{a=1}^{n} x_a^2 \leqslant L^2 \tag{2.158}$$

The normal vector is

$$n_a = c\partial_a S = c\, x_a \tag{2.159}$$

Let us study the one form in $\mathbb{R}^n$

$$rdr \equiv \sum_i x^i dx^i = x^1 dx^1 + x^2 dx^2 + \ldots + x^n dx^n \tag{2.160}$$

It is clear that

$$\begin{aligned} *rdr &= x^1 dx^2 \wedge \ldots \wedge dx^n - x^2 dx^1 \wedge dx^3 \wedge \ldots \wedge dx^n + \ldots = \\ &= \sum (-1)^{i-1} x^i dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^n \end{aligned} \tag{2.161}$$

It so happens that the measure on the S^{n-1} sphere is proportional to this (n-1)-form

$$dS_a = cn_a * rdr \tag{2.162}$$

On the other hand

$$d * rdr = \sum (-1)^{i-1} dx^i \wedge dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^n = n\eta \tag{2.163}$$

This shows that actually $c = 1$. You can show as an exercise that

$$\begin{aligned} V(B_n) &= \tfrac{L}{n} V(S_{n-1}) \\ V(B_n) &= \tfrac{\pi^{n/2}}{\Gamma(n/2+1)} \end{aligned} \tag{2.164}$$

In particular, we recover

$$\begin{aligned} V(B_3) &= \tfrac{4}{3}\pi L^3 \\ V(S_2) &= 4\pi L^2 \end{aligned} \tag{2.165}$$

Let us denote the volume element in euclidean space by

$$\omega_n \equiv d(vol) \equiv dx^1 \wedge \ldots \wedge dx^n \tag{2.166}$$

In ordinary euclidean space, $\mathbb{E}^3$

$$\omega_3 = r^2 \sin\theta\, dr \wedge d\theta \wedge d\phi \tag{2.167}$$

and by σ' the volume element on the codimension-one unit sphere

$$r = L \tag{2.168}$$

Consider

$$rdr = \sum x_i dx_i \tag{2.169}$$

It is plain that

$$*rdr = \sum (-1)^{i-1} x_i dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx_n \tag{2.170}$$

In $\mathbb{E}^2$

$$*rdr = xdy - ydx = r^2 d\theta \tag{2.171}$$

Again, in $\mathbb{E}^3$, it is easy to work out that

$$*rdr = xdy \wedge dz - ydx \wedge dz + zdx \wedge dy = r^3 \sin\theta \, d\theta \wedge d\phi \tag{2.172}$$

It is a fact that

$$\sigma' = *rdr \tag{2.173}$$

(on the sphere S_{n-1}), because

Demonstratio.

$$d\left(*rdr\right) = n\omega_n \tag{2.174}$$

and in particular in $\mathbb{E}^3$

$$d(*rdr) = \frac{1}{r^2}\omega_3 \tag{2.175}$$

□

Consider now the projection

$$\pi : \mathbb{E}_n \backslash 0 \longrightarrow S_{n-1} \tag{2.176}$$

$$\pi(\vec{x}) \equiv \frac{\vec{x}}{|x|} \tag{2.177}$$

We know that

$$d\left(\pi^* \sigma'\right) = \pi^* d\sigma' = 0 \tag{2.178}$$

(because there are no n-forms in S_{n-1}). Let us show that

$$\pi^* \sigma' = \frac{\sigma}{r^n} \equiv \tau \tag{2.179}$$

(σ' is the restriction of σ to S_{n-1}).

Demonstratio. First of all,

$$d\tau = \frac{1}{r^n} d\sigma - \frac{n}{r^{n+1}}(rdr) \wedge \sigma = \frac{n}{r^n}\omega - \frac{n}{r^n}\omega = 0 \tag{2.180}$$

Now, define

$$\begin{gathered}\pi^* \sigma' = \sum (-1)^{i-1} \pi_i d\pi_1 \wedge \ldots \wedge \widehat{d\pi_i} \wedge \ldots \wedge d\pi_n = \\ = \sum (-1)^{i-1} \tfrac{x_i}{r} \tfrac{1}{r^{2(n-1)}} \left(rdx_1 - x_1 dr\right) \wedge \ldots \wedge \widehat{d\pi_i} \wedge \ldots \wedge \left(rdx_n - x_n dr\right) =\end{gathered} \tag{2.181}$$

In $\mathbb{E}_2$

$$\pi^*\sigma = \frac{x}{r}d\frac{y}{r} - \frac{y}{r}d\frac{x}{r} = \frac{1}{r^2}\sigma = d\theta \tag{2.182}$$

In $\mathbb{E}_3$ is clear that it is going to give the same as σ without the radial coordinate. This defines the angular measure, τ. □

Given two closed and oriented manifolds, M and N, and a mapping

$$f: \quad M \longrightarrow N \tag{2.183}$$

then it is a fact that $f_* N$ is an integral multiple on M plus a boundary ([3]). This is dubbed the **degree of f**, $\deg f$. When $\Sigma \hookrightarrow \mathbb{E}^n \backslash 0$, we can deinne the projection as above

$$\pi: \quad \Sigma \rightarrow S_{n-1} \tag{2.184}$$

Then

$$\int_\Sigma \tau = \int_\Sigma \pi^*\sigma' = \int_{\pi(\Sigma)} \sigma' = \deg \pi \int_{S_{n-1}} \sigma' = \deg \pi\, A_{n-1} \tag{2.185}$$

We can generalize this a little bit. Consider a closed manifold

$$M_{n-1} \xrightarrow{f} E_n \backslash 0 \xrightarrow{\pi} S_{n-1} \tag{2.186}$$

The **winding number** of this hypersurface around the origin is given by

$$w \equiv \frac{1}{A_{n-1}} \int_M f^*\, \tau \tag{2.187}$$

In general, given

$$M_n \xrightarrow{f} N_n \tag{2.188}$$

and a volume form β in N normalized to 1

$$\int_N \beta = 1 \tag{2.189}$$

we have

$$\deg f = \int f^* \beta \tag{2.190}$$

This is essentially the mathematics behind the WZNW lagrangian.

Let us now study the **Hopf invariant**. Normalize

$$\int_{S_n} \sigma_n = 1 \tag{2.191}$$

Consider a map

$$S_3 \xrightarrow{f} S_2 \tag{2.192}$$

Then

$$d(f^*\sigma_2) = f^*(d\sigma_2) = 0 \tag{2.193}$$

Now, it is known that S_3 does not have nontrivial cycles, so that $\exists\, \alpha_1$

$$d\alpha_1 = f^*\sigma_2 \tag{2.194}$$

It so happens that the integral

$$\int \alpha_1 \wedge f^*\sigma_2 \tag{2.195}$$

is an integar dubbed the **Hopf invariant**. Represent the sphere S_3 as (z, w) $|z|^2 + |w|^2 = 1$, and the sphere S_2 as

$$z_S \equiv r_S e^{i\phi_S} \tag{2.196}$$

in such a way that

$$ds^2 \equiv \frac{1}{(1+|z_S|^2)^2} dz_S d\bar{z}_S = \frac{1}{\left(1+r_S^2\right)^2}\left(dr_S^2 + r_S^2 d\phi_S^2\right) \tag{2.197}$$

In order to get that we have to rescale

$$x_S \rightarrow \frac{x_S}{2L} \tag{2.198}$$

so that

$$\begin{array}{l} x_S \equiv \frac{1}{2L}\frac{2Lx}{L+x_n} \\ ds^2 \rightarrow 4L^2\, ds^2 \end{array} \tag{2.199}$$

is now dimensionless.

$$\sigma_2 = C\frac{4}{\left(1+r_S^2\right)^2} r_S\, dr_S \wedge d\phi_S = -2Cd\left(\frac{1}{1+r_S^2}\right) \wedge d\phi \tag{2.200}$$

The normalization is

$$\int \sigma_2 = 8\pi C \int_0^\infty \frac{rdr}{(1+r^2)^2} = -4\pi C\, \frac{1}{1+r_S^2}\bigg|_0^\infty = 4\pi C \tag{2.201}$$

Then the mapping

$$(z_1, z_2) \equiv \left(x^1 + ix^2, x^3 + ix^4\right) \equiv (r_1 e^{i\phi_1}, r_2 e^{i\phi_2}) \in S_3 \xrightarrow{f} z_S \equiv \frac{z_1}{z_2} \in S_2 \tag{2.202}$$

The condition

$$|z_1|^2 + |z_2|^2 \equiv x_1^2 + y_1^2 + x_2^2 + y_2^2 = 1 \Longleftrightarrow r_1 \equiv \cos\,\psi \quad r_2 \equiv \sin\,\psi \tag{2.203}$$

We shall see in due course that he standard parameterization in terms of Euler angles of SU(2) is

$$g(\theta,\phi,\chi) \equiv e^{\frac{i}{2}\chi\sigma_3} e^{\frac{i}{2}\theta\sigma_1} e^{\frac{i}{2}\phi\sigma_3} \tag{2.204}$$

where the range of the different angles is

$$\begin{aligned} 0 \leqslant \theta \leqslant \pi \\ 0 \leqslant \phi \leqslant 2\pi \\ 0 \leqslant \chi \leqslant 4\pi \end{aligned} \tag{2.205}$$

The left-invariant one-forms read

$$\begin{aligned} g^{-1}dg &= \tfrac{i}{2}\sigma_a\omega_{aL} \\ \omega_{1L} &= \cos\,\phi\, d\theta + \sin\,\theta\,\sin\,\phi d\chi \\ \omega_{2L} &= \sin\,\phi\, d\theta - \sin\,\theta\,\cos\,\phi\, d\chi \\ \omega_{3L} &= d\phi + \cos\,\theta\, d\chi \end{aligned} \tag{2.206}$$

It is convenient to define

$$\begin{aligned} z_1 \equiv e^{i\frac{\chi+\phi}{2}}\cos\,\tfrac{\theta}{2} \\ z_2 \equiv e^{i\frac{\chi-\phi}{2}}\sin\,\tfrac{\theta}{2} \end{aligned} \tag{2.207}$$

The round metric in S_3 then reads

$$ds^2 \equiv \sum \omega_{aL}^2 = d\theta^2 + d\phi^2 + 2\cos\theta\, d\phi\;\; d\chi + d\chi^2 \tag{2.208}$$

The *Hopf fibering* goes as follows. In the neighborhood $z_1 \neq 0$

$$p : S_3 \rightarrow S_2 \quad p(z_1,z_2) \equiv \frac{z_2}{z_1} \equiv z \tag{2.209}$$

and if $z_2 \neq 0$

$$p : S_3 \rightarrow S_2 \quad p(z_1,z_2) \equiv \frac{z_1}{z_2} \equiv \frac{1}{z} \tag{2.210}$$

Denoting by $H_\pm$ the two hemispheres of the two-sphere S_2,

$$\begin{aligned} H_+ : (z,u_+) \in S_2\times S_1 \rightarrow \left(\tfrac{u_+}{\sqrt{1+|z|^2}}, \tfrac{z u_+}{\sqrt{1+|z|^2}}\right) \equiv \left(e^{i\frac{\chi+\phi}{2}}\cos\,\tfrac{\theta}{2}, e^{i\frac{\chi-\phi}{2}}\sin\,\tfrac{\theta}{2}\right)_+ \\ H_- : (z,u_+) \in S_2\times S_1 \rightarrow \left(\tfrac{|z|u_-}{\sqrt{1+|z|^2}}, \tfrac{|z|u_-}{z\sqrt{1+|z|^2}}\right) \equiv \left(e^{i\frac{\chi+\phi}{2}}\cos\,\tfrac{\theta}{2}, e^{i\frac{\chi-\phi}{2}}\sin\,\tfrac{\theta}{2}\right)_- \end{aligned}$$

Now, start with

$$\sigma \equiv \frac{1}{4\pi}\left(u_1 du_2\wedge du_3 + u_2 du_3\wedge du_1 + u_3 du_1\wedge du_2\right) \tag{2.211}$$

in such a way that

$$dr \wedge \sigma \equiv \sum_i \frac{u_i du_i}{r} \wedge \sigma = \frac{r}{4\pi} du_1 \wedge du_2 \wedge du_3 \tag{2.212}$$

Using the relationship

$$\sum u_i du_i = 0 \tag{2.213}$$

we get

$$\sigma = \frac{1}{4\pi} \frac{du_1 \wedge du_2}{u_3} \tag{2.214}$$

Now the sterographic projection of $CP_1 \longrightarrow S_2 \subset \mathbb{R}_3$ reads

$$z \equiv z + iy \longrightarrow g(z) \equiv \left(u_1 = \frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{-1+x^2+y^2}{1+x^2+y^2} \right) \tag{2.215}$$

The form $g^*\sigma$ is given by

$$g^*\sigma = -\frac{i}{2\pi} \frac{dz \wedge d\bar{z}}{(1+|z|^2)} = \frac{i}{2\pi} \frac{(z_1 dz_0 - z_0 dz_1) \wedge (d\bar{z}_1 \bar{z}_0 - \bar{z}_0 d\bar{z}_1)}{(|z_0|^2 + |z_1|^2)} \tag{2.216}$$

Using now real coordinates

$$|z_0|^2 + |z_1|^2 = 1 = x_1^2 + x_2^2 + x_3^2 + x_4^2 \tag{2.217}$$

we get, after some calculation,

$$f^*\sigma = \frac{1}{\pi} (dx_1 \wedge dx_2 + dx_3 \wedge dx_4) = \frac{1}{\pi} d\alpha \tag{2.218}$$

where

$$\alpha \equiv \frac{1}{\pi} (x_1 dx_2 + x_3 dx_4) \tag{2.219}$$

and finally [**?**]

$$H(f) = \int_{S_3} \alpha d\alpha = \frac{2}{\pi^2} \int x_2 dx_2 \wedge dx_3 \wedge dx_4 = \frac{2}{\pi^2} \int_0^\pi d\theta \int_0^\pi d\phi \int_0^{2\pi} d\xi \sin^4 \xi \sin^3 \phi \cos^2 \theta = 1 \tag{2.220}$$

Assuming

$$z_S = r_S \, e^{i\phi_S}$$
$$u_+ = e^{i\alpha} \tag{2.221}$$

This implies on H_+

$$\begin{aligned} \cos\,\tfrac{\theta}{2} &\equiv \tfrac{1}{\sqrt{1+r^2}} \\ \sin\,\tfrac{\theta}{2} &\equiv \tfrac{r}{\sqrt{1+r^2}} \\ \chi &= 2\alpha + \phi_S \\ \phi &= -\phi_S \end{aligned} \tag{2.222}$$

On H_-

$$z_N \equiv \frac{1}{z_S} = \frac{1}{r}\,e^{-i\phi_S} \tag{2.223}$$

$$\begin{aligned} \sin\,\tfrac{\theta_-}{2} &\equiv \tfrac{1}{\sqrt{1+r_N^2}} = \tfrac{r}{\sqrt{1+r^2}} = \sin\,\tfrac{\theta_+}{2} \\ \cos\,\tfrac{\theta_-}{2} &\equiv \tfrac{r_N}{\sqrt{1+r_N^2}} = \tfrac{1}{\sqrt{1+r^2}} = \cos\,\tfrac{\theta_+}{2} \\ \chi &= 2\alpha + \phi_S = 2\alpha - \phi_N \\ \phi &= -\phi_S = \phi_N \end{aligned} \tag{2.224}$$

On the equator

$$u_+ \equiv e^{i\alpha} = \frac{|z|}{z}\,u_- \equiv e^{-i\phi_S}\,u_- \equiv e^{i(\alpha+\phi_S)}e^{-i\phi_S} \tag{2.225}$$

This is the twist that makes all the difference between the fiber bundle and the product space.

(2.226)

3

Gauss' integral

4

Surfaces revisited.

Choose a moving frame in such a way that $(\vec{e}_1, \vec{e}_2)$ are a basis for the tangent plane and $\vec{e}_3$ is the normal to the surface. On the surface itself

$$d\vec{x} = \sigma_1 \vec{e}_1 + \sigma_2 \vec{e}_2 \tag{4.1}$$

where σ_1 and σ_2 are a couple of 1-forms. For example, in the case of the two-sphere we can choose

$$\begin{aligned}
\vec{e}_1 = \frac{1}{L}\frac{\partial}{\partial\theta} = \frac{1}{L}\left(\frac{x\sqrt{L^2-x^2-y^2}}{\sqrt{x^2+y^2}}, \frac{y\sqrt{L^2-x^2-y^2}}{\sqrt{x^2+y^2}}, -\sqrt{x^2+y^2}\right) = \\
= \frac{1}{L}\left(\frac{xz}{\sqrt{L^2-z^2}}, \frac{yz}{\sqrt{L^2-z^2}}, -\sqrt{L^2-z^2}\right) \\
\vec{e}_2 = \frac{1}{L\sin\theta}\frac{\partial}{\partial\phi} = \frac{1}{\sqrt{L^2-z^2}}\left(-y, x, 0\right) \\
\vec{e}_3 = \frac{\partial}{\partial r} = \frac{1}{L}\left(x, y, z\right)
\end{aligned} \tag{4.2}$$

Then

$$\begin{aligned}
d\vec{x} = \left(dx, dy, -\frac{xdx+ydy}{z}\right) = \\
= \frac{Lx}{z\sqrt{L^2-z^2}}\left(\vec{e}_1 - \frac{yz}{Lx}\vec{e}_2\right)dx + \frac{Ly}{z\sqrt{L^2-z^2}}\left(\vec{e}_1 + \frac{xz}{Ly}\vec{e}_2\right)dy
\end{aligned} \tag{4.3}$$

This determines the one-forms

$$\begin{aligned}
\sigma_1 = \frac{L}{z\sqrt{L^2-z^2}}\left(xdx + ydy\right) \\
\sigma_2 = \frac{1}{\sqrt{L^2-z^2}}\left(-ydx + xdy\right)
\end{aligned} \tag{4.4}$$

It is also the case that

$$d\vec{e}_a = \sum_{b=1}^{b=3} \omega_{ab}\vec{e}_b \tag{4.5}$$

We find

$$d\vec{e}_1 = \frac{1}{L}\left(\frac{z}{\sqrt{L^2-z^2}}dx + \frac{xL^2}{(L^2-z^2)^{3/2}}dz, \frac{z}{\sqrt{L^2-z^2}}dy + \frac{yL^2}{(L^2-z^2)^{3/2}}dz, \frac{z}{\sqrt{L^2-z^2}}dz\right) =$$
$$= \frac{1}{L(L^2-z^2)^{3/2}}\left(z(L^2-z^2)dx + xL^2dz, z(L^2-z^2)dy + yL^2dz, +z(L^2-z^2)dz\right) =$$
$$= \frac{1}{L(L^2-z^2)^{3/2}}\left(z\left(x^2+y^2-\frac{L^2x^2}{z^2}\right)dx - \frac{L^2xy}{z}dy, z\left(L^2-z^2-\frac{L^2y^2}{z^2}\right)dy - \frac{L^2xy}{z}dx,\right.$$
$$\left., -\left(L^2-z^2\right)(xdx+ydy)\right) \tag{4.6}$$

$$d\vec{e}_2 = \left(-\frac{dy}{\sqrt{L^2-z^2}} - \frac{yz}{(L^2-z^2)^{3/2}}dz, \frac{dx}{\sqrt{L^2-z^2}} + \frac{xz}{(L^2-z^2)^{3/2}}dz, 0\right) =$$
$$= \frac{1}{(L^2-z^2)^{3/2}}\left(-(L^2-z^2)dy - yzdz, (L^2-z^2)dx + xzdz, 0\right) =$$
$$= \frac{1}{(L^2-z^2)^{3/2}}\left(xydx - x^2dy, y^2dx - xydy, 0\right) \tag{4.7}$$

$$d\vec{e}_3 = \frac{1}{L}(dx, dy, dz) \tag{4.8}$$

Then

$$\omega_{12} \equiv \vec{e}_2 d\vec{e}_1 = z\frac{xdy - ydx}{L(L^2-z^2)} \tag{4.9}$$

$$\omega_{13} = \vec{e}_3.d\vec{e}_1 = \frac{1}{L^2\sqrt{L^2-z^2}}\left(z(xdx+ydy) + (L^2+z^2)dz\right) = \frac{1}{L^2\sqrt{L^2-z^2}}\left(-z(xdx+ydy) + (L^2 -\right.$$
$$= -\frac{1}{z\sqrt{L^2-z^2}}(xdx+ydy)$$

(because on the sphere $xdx + ydy + zdz = 0$)

$$\omega_{23} = \vec{e}_3.d\vec{e}_2 = \frac{1}{L\sqrt{L^2-z^2}}(ydx - xdy) \tag{4.11}$$

But we have normalized in such a way that

$$\vec{e}_a.\vec{e}_b = \delta_{ab} \tag{4.12}$$

so that

$$d\vec{e}_a.\vec{e}_b + \vec{e}_a.d\vec{e}_b = 0 \tag{4.13}$$

that is that the matrix of one-forms

$$\Omega \equiv \omega_{ab} \tag{4.14}$$

is antisymmetric. In fact

$$\omega_{21} = \vec{e}_1.d\vec{e}_2 = \frac{z}{L(L^2-z^2)}(ydx - xdy) = -\omega_{12} \tag{4.15}$$

$$\Omega = -\Omega^T \equiv \begin{pmatrix} 0 & \varpi & -\omega_1 \\ -\varpi & 0 & -\omega_2 \\ \omega_1 & \omega_2 & 0 \end{pmatrix} \tag{4.16}$$

We can the write in gory detail

$$\begin{aligned} d\vec{e}_1 &= \varpi\vec{e}_2 - \omega_1\vec{e}_3 \\ d\vec{e}_2 &= -\varpi\vec{e}_1 - \omega_2\vec{e}_3 \\ d\vec{e}_3 &= \omega_1\vec{e}_1 + \omega_2\vec{e}_2 \end{aligned} \tag{4.17}$$

As a consequence of our definitions we have

$$0 = d^2\vec{x} = \sum_{A=1}^{A=2} d\sigma_A\vec{e}_A - \sigma_A d\vec{e}_A = 0 = \sum_{A=1}^{A=2} d\sigma_A\vec{e}_A - \sigma_A\omega_{Ab}\vec{e}_b \tag{4.18}$$

This implies

$$d\sigma = \sigma\Omega \tag{4.19}$$

That is

$$\begin{aligned} d\sigma_1 &= \varpi \wedge \sigma_2 \\ d\sigma_2 &= -\varpi \wedge \sigma_1 \\ d\sigma_3 \equiv 0 &= \sigma_1 \wedge \omega_1 + \sigma_2 \wedge \omega_2 \end{aligned} \tag{4.20}$$

We also deduce

$$0 = d^2\sigma = d\sigma\Omega - \sigma d\Omega = \sigma\Omega^2 - \sigma d\Omega \tag{4.21}$$

so that

$$d\Omega = \Omega^2 \tag{4.22}$$

$$d\begin{pmatrix} 0 & \varpi & -\omega_1 \\ -\varpi & 0 & -\omega_2 \\ \omega_1 & \omega_2 & 0 \end{pmatrix} = \begin{pmatrix} -\varpi^2 - \omega_1^2 & -\omega_1\omega_2 & -\varpi\omega_2 \\ -\omega_2\omega_1 & -\varpi^2 - \omega_2^2 & \varpi\omega_1 \\ -\omega_2\varpi & \omega_1\varpi & -\omega_1^2 - \omega_2^2 \end{pmatrix} \tag{4.23}$$

To be specific

$$\begin{aligned} d\varpi &= -\omega_1 \wedge \omega_2 \\ d\omega_1 &= \varpi \wedge \omega_2 \\ d\omega_2 &= -\varpi \wedge \omega_1 \end{aligned} \tag{4.24}$$

Let us recap.

$$\begin{aligned} \sigma_1 &= \tfrac{L}{z\sqrt{L^2-z^2}}\left(xdx + ydy\right) \\ \sigma_2 &= \tfrac{1}{\sqrt{L^2-z^2}}\left(-ydx + xdy\right) \\ \omega_1 &= \tfrac{1}{z\sqrt{L^2-z^2}}\left(xdx + ydy\right) = \tfrac{1}{L}\sigma_1 \\ \omega_2 &= -\tfrac{1}{L\sqrt{L^2-z^2}}\left(ydx - xdy\right) = \tfrac{1}{L}\sigma_2 \end{aligned} \tag{4.25}$$

Since there is only one linearly independent 2-form om the two-dimensional surface Σ, we have

$$\omega_1 \wedge \omega_2 = K\sigma_1 \wedge \sigma_2 \tag{4.26}$$

where K is our old friend the *Gaussian curvature*. For the sphere S_2 we find

$$K = \frac{1}{L^2} \tag{4.27}$$

It is actually independent on the choice of the basis vectors $(\vec{e}_1, \vec{e}_2)$. Exactly the same reasoning tells us that

$$\sigma_1 \wedge \omega_2 - \sigma_2 \wedge \omega_1 = 2H\sigma_1 \wedge \sigma_2 \tag{4.28}$$

Here H is the *mean curvature* of the surface Σ. For the sphere S_2 it reads $H = \frac{1}{L}$. In order to write the forms (ω_1, ω_2) in terms of the forms (σ_1, σ_2), we have to be consistent wit the equation

$$\sigma_1 \wedge \omega_1 + \sigma_2 \wedge \omega_2 = 0 \tag{4.29}$$

The general solution Flanders claims to be

$$\begin{aligned} \omega_1 &= p\sigma_1 + q\sigma_2 \\ \omega_2 &= q\sigma_1 + r\sigma_2 \end{aligned} \tag{4.30}$$

Actually, for the sphere, this is satisfied in a trivial way, namely

$$\begin{aligned} \omega_1 &= \tfrac{1}{L}\,\sigma_1 \\ \omega_2 &= \tfrac{1}{L}\,\sigma_2 \end{aligned} \tag{4.31}$$

(That is $q = 0$ and $p = r$.)

It follows that

$$\begin{aligned} H &= \tfrac{p+r}{2} \\ K &= pr - q^2 \end{aligned} \tag{4.32}$$

Now the relation

$$d\varpi + \omega_1 \wedge \omega_2 \Longrightarrow d\varpi + K\sigma_1 \wedge \sigma_2 = 0 \tag{4.33}$$

which determines K in terms of $(\varpi, \sigma_1, \sigma_2)$. But

$$\begin{aligned} d\sigma_1 &= \varpi \wedge \sigma_2 \\ d\sigma_2 &= -\varpi \wedge \sigma_1 \end{aligned} \tag{4.34}$$

determine ϖ in terms of (σ_1, σ_2); actually

$$\varpi = a\sigma_1 + b\sigma_2 \tag{4.35}$$

Then K is completely determined by (σ_1, σ_2). Lo and behold, this is Gauss' *theorema egregium*.

5

Differential Geometry

It is important to be able to pinpoint characteristics that are intrinsic, that is, independent of the coordinates used in overlaps of open sets in a covering. The two main ones are

- The contravariant vector interpreted a a *directional derivative*. Given the linear space $\mathcal{F}(M)$ of all functions $f : M \to \mathbb{R}$, and a local coordinate system $x^\mu : M \to U \subset \mathbb{R}^n$

$$V \in T : \mathcal{F}(M) \to \mathbb{R} \tag{5.1}$$

$$V(f) \equiv V^\mu \partial_\mu f \tag{5.2}$$

It follows that

$$V^\mu \frac{\partial}{\partial x^\mu} = V^{\mu'} \frac{\partial}{\partial x^{\mu'}} \tag{5.3}$$

- A covariant vector interpreted as the differential of a function

$$W \in T^* : f \in \mathcal{F}(M) \to d\mathcal{F}(M) \tag{5.4}$$

$$df \equiv \partial_\mu f \, dx^\mu \tag{5.5}$$

this means that

$$\frac{\partial f}{\partial x^\mu} dx^\mu = \frac{\partial f}{\partial x^{\mu'}} dx^{\mu'} \tag{5.6}$$

Starting with those elements, more complicated transformation laws can be easily derived.

5.1 Geodesics

The integral given the distance is extended over the parameterized curve γ

$$x^\mu = x^\mu(\lambda) \tag{5.7}$$

and we have denoted by

$$\dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda} \tag{5.8}$$

We can normalize the tangent vector

$$u^\mu \equiv \frac{\dot{x}^\mu}{\sqrt{\dot{x}^2}} \tag{5.9}$$

The extrema of the action are by definition the *geodesics* of the manifold. We get

$$\delta S = -mc \int d\lambda \left\{\partial_\rho g_{\mu\nu} \delta x^\rho \dot{x}^\mu \dot{x}^\nu + \ 2 g_{\mu\nu} \dot{x}^\mu \delta \dot{x}^\nu\right\} = -mc \int d\lambda \ \delta x^\rho \Big\{$$
$$\partial_\rho g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu - \left(\partial_\lambda g_{\mu\rho} + \partial_\mu g_{\lambda\rho}\right) \dot{x}^\lambda \dot{x}^\mu - 2 g_{\mu\rho} \ddot{x}^\mu \Big\} \tag{5.10}$$

Expressed in the form of four ordinary differential equations for the four functions of one variable $x^\mu(s)$ they read

$$\frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0 \tag{5.11}$$

Here the Christoffel symbols are given by

$$\Gamma_{\lambda,\mu\nu} \equiv g_{\lambda\rho} \Gamma^\rho_{\mu\nu} \equiv g_{\lambda\rho} \ \frac{1}{2} g^{\rho\sigma} \left(-\partial_\sigma g_{\mu\nu} + \partial_\mu g_{\nu\sigma} + \partial_\mu g_{\nu\sigma}\right) \tag{5.12}$$

This is true insofar as we are parametrizing the curve using the arc length (which is anyway possible only when the tangent to the curve is everywhere timelike or spacelike). Assume now we use another parameter,

$$\lambda = \lambda(s) \tag{5.13}$$

Then

$$\begin{aligned} u^\alpha &\equiv \tfrac{dx^\alpha}{ds} = \tfrac{dx^\alpha}{d\lambda} \tfrac{d\lambda}{ds} \\ \tfrac{du^\alpha}{ds} &= \tfrac{d^2 x^\alpha}{d\lambda^2} \left(\tfrac{d\lambda}{ds}\right)^2 + \tfrac{dx^\alpha}{d\lambda} \tfrac{d^2\lambda}{ds^2} \end{aligned} \tag{5.14}$$

This means that the geodesic equations now read

$$\frac{d^2 x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{d\lambda} \frac{dx^\gamma}{d\lambda} = f(\lambda) \frac{dx^\alpha}{d\lambda} \tag{5.15}$$

where the function $f(\lambda)$ is given by

$$c(\lambda) \equiv -\frac{\frac{d^2\lambda}{ds^2}}{\left(\frac{d\lambda}{ds}\right)^2} \tag{5.16}$$

The Christoffel symbols are in a sense the gauge field associated to diffeomorphisms. An element $V \in T$ behaves as

$$V^{\mu'}(x') \equiv \frac{\partial x^{\mu'}}{\partial x^{\rho}} \, V^{\rho} \, (x) \tag{5.17}$$

or in matrix motation

$$V' = J.V \tag{5.18}$$

It is obvious that

$$\partial_\alpha V^\mu \tag{5.19}$$

does not transform as a tensor unless the diffeomorphism is a linear one, because

$$dV' = dJ.V + JdV \tag{5.20}$$

The idea on a connection ($\equiv gaugefield$) is to modify the ordinary detivative into a *covariant derivative*

$$DV \equiv dV + \Gamma V \tag{5.21}$$

To be specific, the gauge fields are defined in such a way that

$$\nabla_\rho V^\mu \equiv \partial_\rho V^\mu + \Gamma^\mu_{\rho\sigma} \, V^\sigma \tag{5.22}$$

does transform as a tensor, that is

$$\nabla_{\rho'} V^{\mu'}(x') = \frac{\partial x^\sigma}{\partial x^{\rho'}} \frac{\partial x^{\mu'}}{\partial x^\lambda} \nabla_\sigma V^\lambda \tag{5.23}$$

It is easier to vidualize all this in matrix notation

$$\begin{aligned}(DV)' = dV' + \Gamma' V' = d(J.V) + \Gamma'.J.V = dJ.V + J.dV + \Gamma'.J.V = \\ = JDV = J\left(dV + \Gamma V\right)\end{aligned} \tag{5.24}$$

It is plain that for this to be true, it is enough that

$$\Gamma' J + dJ = J\Gamma \tag{5.25}$$

that is

$$\Gamma' = J\Gamma J^{-1} - dJ.J^{-1} \tag{5.26}$$

In gory detail

$$\Gamma^{\mu'}_{\rho'\lambda'} \frac{\partial x^{\lambda'}}{\partial x^\lambda} + \frac{\partial^2 x^{\mu'}}{\partial x^\lambda \partial x^\sigma} \frac{\partial x^\sigma}{\partial x^{\rho'}} = \frac{\partial x^\sigma}{\partial x^{\rho'}} \frac{\partial x^{\mu'}}{\partial x^\delta} \Gamma^\delta_{\sigma\lambda} \tag{5.27}$$

Because of the inhomogeneous term the Christoffel symbols are NOT tensors. Thay are connections, that is, gauge fields. It is useful exercise to check at least that the Christoffel symbols are a solution of these equations. Actually they are the *unique* solution involving the metric tensor alone.

It is also useful to check that for covariant tensors.

$$\nabla_\mu \omega_\nu \equiv \partial_\mu \omega_\nu - \Gamma^\lambda_{\mu\nu} \omega_\lambda \tag{5.28}$$

Using this formula, it is plain to check that the metric is covariantly constant

$$\nabla_\alpha g_{\beta\gamma} = \partial_\alpha g_{\beta\gamma} - \Gamma^\lambda_{\alpha\beta} g_{\lambda\gamma} - \Gamma^\lambda_{\alpha\gamma} g_{\lambda\beta} = 0 \tag{5.29}$$

- Let us begin by computing geodesics on the plane

$$ds^2 = dr^2 + r^2 d\theta^2 \tag{5.30}$$

$$\begin{aligned} \Gamma^r_{\theta\theta} &= \tfrac{1}{2} g^{rr} \left(-\partial_r g_{\theta\theta}\right) = -r \\ \Gamma^\theta_{r\theta} &= \tfrac{1}{2} g^{\theta\theta} \left(\partial_r g_{\theta\theta}\right) = \tfrac{1}{r} \end{aligned} \tag{5.31}$$

$$\begin{aligned} &\ddot{r} - r\dot{\theta}^2 = 0 \\ &\ddot{\theta} + \tfrac{1}{r}\dot{\theta}\dot{r} = 0 \end{aligned} \tag{5.32}$$

First integral

$$\dot{r}^2 + r^2\dot{\theta}^2 = 1 \tag{5.33}$$

It easier to start from

$$L = \int dr \sqrt{r^2 \left(\frac{d\theta}{dr}\right)^2 + 1} \tag{5.34}$$

Euler-Lagrange

$$\frac{d}{dr}\left(r^2 \frac{\theta'}{\sqrt{r^2 \left(\frac{d\theta}{dr}\right)^2 + 1}} \right) = 0 \tag{5.35}$$

$$\therefore r^2 \frac{\theta'}{\sqrt{r^2 \left(\frac{d\theta}{dr}\right)^2 + 1}} = C \tag{5.36}$$

It is not difficult to check that the equation of a general planar straight line

$$r \sin(\theta + \theta_0) = r_0 \sin\, \theta_0\, \cos\, \theta_0 \tag{5.37}$$

is a solution of the first integral.

- Let us compute now the geodesics on the ordinary two-sphere. The arc distance is

$$S \equiv \int \sqrt{d\theta^2 + \sin^2\theta\, d\phi^2} \tag{5.38}$$

Let us describe the curve as

$$\phi = \phi(\theta) \tag{5.39}$$

Then

$$S \equiv \int d\theta \sqrt{1 + \sin^2\theta \left(\frac{d\phi}{d\theta}\right)^2} \tag{5.40}$$

Euler-Lagrange

$$\frac{d}{d\theta}\left(\frac{\partial}{\partial\phi'}\sqrt{1 + \sin^2\theta\,(\phi')^2}\right) = \frac{\partial}{\partial\phi}\sqrt{1 + \sin^2\theta\,(\phi')^2} = 0 \tag{5.41}$$

This means that

$$\frac{\sin^2\theta\,\phi'}{\sqrt{1 + \sin^2\theta\,(\phi')^2}} = C \tag{5.42}$$

That is

$$\phi' = \frac{C}{\sin\theta\sqrt{\sin^2\theta - C^2}} \tag{5.43}$$

$$\phi = \int d\theta \frac{C}{\sin\theta\sqrt{\sin^2\theta - C^2}} \tag{5.44}$$

Let us change variables

$$u = \cot\theta, \quad du = -\frac{d\theta}{\sin^2\theta} d\theta \tag{5.45}$$

$$\phi = -C\int \frac{du}{\sqrt{1 - C^2(1+u^2)}} = -\int \frac{du}{\sqrt{a^2 - u^2}} = \cos^{-1}\frac{u}{a} + \phi_0; \quad a \equiv \frac{\sqrt{1-C^2}}{C} \tag{5.46}$$

$$\cot\theta = a\cos(\phi - \phi_0) \tag{5.47}$$

This equation has got a nice geometric interpretation. Consider a fixed unit vector defined by (θ_0, ϕ_0). The plane orthogonal to it is generated by the vectors such that

$$\sin\theta \sin\theta_0 \cos(\phi - \phi_0) + \cos\theta\cos\theta_0 = 0 \tag{5.48}$$

The interchapter of this plane with unit sphere yields the desired geodesic.

The geodesic equation can be written as

$$\nabla_u u = f u \tag{5.49}$$

that is, the tangent vector to the curve propagates parallel to itself, in the sense that the tangent component of the covariant derivative is proportional to the tangent vector itself.

5.2 Covariant derivative and curvature.

In fact the metric connection (Christoffels) is the unique symmetric connection such that the coveriant derivative of the metric vanishes.

$$\begin{aligned}
&\partial_\alpha g_{\mu\nu} - \Gamma^\lambda_{\alpha\mu} g_{\lambda\nu} - \Gamma^\lambda_{\alpha\nu} g_{\lambda\mu} = 0\\
&\partial_\mu g_{\alpha\nu} - \Gamma^\lambda_{\nu\mu} g_{\lambda\alpha} - \Gamma^\lambda_{\alpha\mu} g_{\lambda\nu} = 0\\
&\partial_\nu g_{\mu\alpha} - \Gamma^\lambda_{\alpha\nu} g_{\lambda\mu} - \Gamma^\lambda_{\alpha\mu} g_{\lambda\alpha} = 0
\end{aligned} \tag{5.50}$$

1-2+3 yields

$$-2\,\Gamma^\lambda_{\alpha\nu} g_{\lambda\mu} = \partial_\alpha + \partial_\nu - \partial_\mu \tag{5.51}$$

The commutator of two vectors, $X, Y \in T$ is defined as the vector

$$T \times T \to T \tag{5.52}$$

$$[X,Y]^\alpha \equiv X^\mu \partial_\mu Y^\alpha - Y^\mu \partial_\mu X^\alpha = X^\mu \nabla_\mu Y^\alpha - Y^\mu \nabla_\mu X^\alpha \tag{5.53}$$

Ths covariant derivative in the direction of a vector V is

$$\nabla_V : T \to T \tag{5.54}$$

$$(\nabla_V W)^\alpha \equiv V^\mu \nabla_\mu W^\alpha \equiv V^\mu \left(\partial_\mu W^\alpha + \Gamma^\alpha_{\mu\lambda} W^\lambda\right) \tag{5.55}$$

The curvature of the connection ∇ is defines as the operator

$$R : T^3 \equiv T \times T \times T \to T \tag{5.56}$$

$$Z \to R_{XY} Z \equiv [\nabla_X, \nabla_Y]\, Z - \nabla_{[X,Y]}\, Z \tag{5.57}$$

Let us slowly work this out

$$(\nabla_Y Z)^\alpha = Y^\lambda \left(\partial_\lambda Z^\alpha + \Gamma^\alpha_{\lambda\sigma} Z^\sigma\right) \tag{5.58}$$

The commmutator of two covariant derivatives reads

$$\begin{aligned}
(\nabla_X \nabla_Y Z)^\alpha &\equiv X^\rho \left(\partial_\rho (\nabla_Y Z)^\alpha + \Gamma^\alpha_{\rho\delta} (\nabla_Y Z)^\delta\right) =\\
&= X^\rho \left\{\partial_\rho \left(Y^\lambda \partial_\lambda Z^\alpha + Y^\lambda \Gamma^\alpha_{\lambda\sigma} Z^\sigma\right) + \Gamma^\alpha_{\rho\delta} \left(Y^\sigma \partial_\sigma Z^\delta + Y^\sigma \Gamma^\delta_{\sigma\beta} Z^\beta\right)\right\} =\\
&= X^\rho \Big\{\partial_\rho Y^\lambda \partial_\lambda Z^\alpha + Y^\lambda \partial_{\rho\lambda} Z^\alpha + \partial_\rho Y^\lambda \Gamma^\alpha_{\lambda\sigma} Z^\sigma + Y^\lambda \partial_\rho \Gamma^\alpha_{\lambda\sigma} Z^\sigma + Y^\lambda \Gamma^\alpha_{\lambda\sigma} \partial_\rho Z^\sigma +\\
&\quad + \Gamma^\alpha_{\rho\delta} \left(Y^\sigma \partial_\sigma Z^\delta + Y^\sigma \Gamma^\delta_{\sigma\beta} Z^\beta\right)\Big\}
\end{aligned} \tag{5.59}$$

In the opposite order

$$\begin{aligned}
(\nabla_Y \nabla_X Z)^\alpha \equiv Y^\rho \Big\{&\partial_\rho X^\lambda \partial_\lambda Z^\alpha + X^\lambda \partial_{\rho\lambda} Z^\alpha + \partial_\rho X^\lambda \Gamma^\alpha_{\lambda\sigma} Z^\sigma + X^\lambda \partial_\rho \Gamma^\alpha_{\lambda\sigma} Z^\sigma + X^\lambda \Gamma^\alpha_{\lambda\sigma} \partial_\rho Z^\sigma +\\
&+ \Gamma^\alpha_{\rho\delta} \left(X^\sigma \partial_\sigma Z^\delta + X^\sigma \Gamma^\delta_{\sigma\beta} Z^\beta\right)\Big\}
\end{aligned} \tag{5.60}$$

On the other hand, the covariant derivative in the direction of the commutator reads

$$\left(\nabla_{[X,Y]} Z\right)^{\alpha} = \left(X^{\mu}\nabla_{\mu}Y^{\lambda} - Y^{\mu}\nabla_{\mu}X^{\lambda}\right)\left(\partial_{\lambda}Z^{\alpha} + \Gamma^{\alpha}_{\lambda\sigma}Z^{\sigma}\right) \tag{5.61}$$

In the commutator all terms proportional to derivatives of the vector we are mapping, ∂Z disappear, what is left out is a linear mapping

$$(X, Y, Z) \in T^3 \to (R_{XY}Z)^{\alpha} \equiv R^{\alpha}{}_{\sigma\rho\lambda}Z^{\sigma}X^{\rho}Y^{\lambda} \tag{5.62}$$

The tensor $R^{\alpha}{}_{\sigma\rho\lambda}$ is called the Riemann tensor ans is by construction antisymmetric in the last two indices

$$R^{\alpha}{}_{\sigma\rho\lambda} = -R^{\alpha}{}_{\sigma\lambda\rho} \tag{5.63}$$

Its value can be easily read out from the preceding formulas

$$R^{\alpha}{}_{\sigma\rho\lambda} \equiv \partial_{\rho}\Gamma^{\alpha}_{\lambda\sigma} - \partial_{\lambda}\Gamma^{\alpha}_{\rho\sigma} + \Gamma^{\alpha}_{\delta\rho}\Gamma^{\delta}_{\sigma\lambda} - \Gamma^{\alpha}_{\delta\lambda}\Gamma^{\delta}_{\sigma\rho} \tag{5.64}$$

Lets work out the two-sphere as an example. We shall actually consider a rugby ball.

5.3 Differential manifolds

Differential manifolds are smoooth objects that locally are similar to $\mathbb{R}^n$, but globally are different. Instead of giving the general theory we shall content ourselves here with a detailed study of the simplest non-trivial example. Let us first consider the simpler case of ordinary spheres embedded in euclidean space.

The sphere $\mathbb{S}_n$ of radius l embedded in $\mathbb{R}_{n+1}$ is defined thtough the equations

$$\sum_{A=1}^{A=n+1} X_A^2 = l^2 \tag{5.65}$$

where a point in $\mathbb{R}^{n+1}$ is represented by the (n+1) coordinates $(X_1, X_1, \ldots X_{n+1})$. We are all used to *polar coordinates*, a generalization of the polar angles (θ, ϕ) for the two-sphere S_2. We need n angles to define a point in the n sphere. We shall call these angles, $\theta_1 \ldots \theta_n$, and to be specific,

$$\begin{aligned}
X_{n+1} &= r\, cos\, \theta_n \\
X_n &= r\, sin\, \theta_n\, cos\, \theta_{n-1} \\
&\ldots \\
X_2 &= r\, sin\, \theta_n\, sin\, \theta_{n-1} \ldots cos\, \theta_1 \\
X_1 &= r\, sin\, \theta_n\, sin\, \theta_{n-1} \ldots sin\, \theta_1
\end{aligned} \tag{5.66}$$

(were we to use r itself as the radial coordinate, those would be polar coordinates in $\mathbb{R}_{n+1}$, in them the equation of the sphere is simply

$$r = l = \text{constant} \tag{5.67}$$

here

$$\begin{aligned} 0 \leqslant \theta_1 \leqslant 2\pi \\ 0 \leqslant \theta_j \leqslant \pi \quad \text{for } j \neq 1 \end{aligned} \tag{5.68}$$

The X_{n+1} axis is special in those coordinates; any axis however can be taken as the X_{n+1} axis. The metric induced on S^n by the euclidean metric in $\mathbb{R}_{n+1}$ is

$$\begin{aligned} ds_n^2 = \delta_{AB} dX^A(\theta) dX^B(\theta) = \\ = d\theta_n^2 + sin^2\,\theta_n d\theta_{n-1}^2 + \ldots + sin^2\,\theta_n\, sin^2\,\theta_{n-1} \ldots sin^2\,\theta_2 d\theta_1^2 \end{aligned} \tag{5.69}$$

id est, in a recurrent form

$$\begin{aligned} ds_1^2 &= d\theta_1^2 \\ ds_n^2 &= d\theta_n^2 + sin^2\theta_n\, ds_{n-1}^2 \end{aligned} \tag{5.70}$$

The tangent space is a vector space T_n with the same dimension as the manifold itself. It can be defined as the set of vectors orthogonal to the normal vector

$$n_A = X_A \tag{5.71}$$

In general, given a surface in $\mathbb{R}_{n+1}$ defined by the equation

$$f(X_A) = 0 \tag{5.72}$$

the normal vector is given by the gradient

$$n_A \equiv \partial_A f \tag{5.73}$$

To come back to the sphere, the tangent space is defined as those vectors that obey

$$\sum_A x_A t_A = 0 \tag{5.74}$$

Particularizing to the two-dimensional sphere, the tangent space is now the tangent plane, that is, the set of vector in $\mathbb{R}_3$ such that

$$n_1.\sin\,\theta\,\cos\,\phi + n_2.\sin\,\theta\,\sin\,\phi + n_0\cos\,\theta = 0 \tag{5.75}$$

In the North or South pole ($\theta = 0, \pi$) the tangent plane is just the plane

$$X_0 = \pm l \tag{5.76}$$

that is, the set of vectors

$$(0, n_1, n_2) \tag{5.77}$$

and in the equator ($\theta = \frac{\pi}{2}$)

$$n_1 \cos\phi + n_2 \sin\phi = 0 \tag{5.78}$$

Polar coordinates do not cover the whole sphere (neither do they cover euclidean space). They are not well defined at the two poles. It is interesting to study other set of coordinates, which are actually close to what cartographers do when drawing maps. The stereographic coordinates are defined out of one of the poles (either North or South) Northern pole stereographic projection

$$x_S^\mu \equiv \frac{2l}{X_0 + l} X^\mu \equiv \frac{X^\mu}{\Omega_S} \tag{5.79}$$

($\mu = 1 \ldots n$). Let us choose cartesian coordinates in $\mathbb{R}_{n+1}$ with origin in the South pole itself. This meags tgeat the South pole is represented by $X^A = 0$, and the norh pole by $X_A = (l, 0, \ldots, 0)$. One can imagine that one is projecting a point $P(X_A) \in \mathbb{S}_n$ from the South pole into into a point x_S^μ that one van view as living on the tangent plane at the North pole.

$$X_0 = l \frac{1 - \frac{x_S^2}{4l^2}}{1 + \frac{x_S^2}{4l^2}} = l(2\Omega_S - 1) = l(2\Omega_N + 1) \tag{5.80}$$

$$\Omega_S \equiv \frac{1}{1 + \frac{x_S^2}{4l^2}} \tag{5.81}$$

$$\frac{x_S^2}{4l^2} = \frac{l - X_0}{l + X_0} \tag{5.82}$$

This means that when $X_0 = l$ (the North pole) then

$$\frac{x_S^2}{4l^2} = 0 \tag{5.83}$$

and when $X_0 = -l$ (the South pole) then

$$X_S^2 = \infty \tag{5.84}$$

The jacobians of the embedding is

$$\begin{aligned} \partial_\mu X^0 &= -\Omega_S^2 \frac{x^\mu}{l} \\ \partial_\mu X^\alpha &= \Omega_S \delta_\mu^\alpha - \Omega_S^2 \frac{x^\alpha x_\mu}{2l^2} \end{aligned} \tag{5.85}$$

The induced metric

$$ds^2 = \delta_{AB} \partial_\mu X^A \partial_\nu X^B dx^\mu dx^\nu = \Omega_S^2 \delta_{\mu\nu} dx^\mu dx^\nu \tag{5.86}$$

Performing the North pole projection

$$x_N^\mu \equiv \frac{2l}{X_0 - l} X^\mu \tag{5.87}$$

Uniqueness of X_0 means that

$$2\Omega_N + 1 = 2\Omega_S - 1 \tag{5.88}$$

and uniqueness of X^μ

$$x_N^\mu = \frac{\Omega_S}{\Omega_N} x_S^\mu = -\frac{4l^2}{x_S^2} x_S^\mu \tag{5.89}$$

Conversely,

$$x_S^\mu = -\frac{4l^2}{x_N^2} x_N^\mu \tag{5.90}$$

This leads to

$$\Omega_N = -\frac{1}{1 + \frac{x_N^2}{4l^2}} \tag{5.91}$$

The antipodal map

$$X^A \leftrightarrow -X^A \tag{5.92}$$

is equivalent to

$$x_S^\mu \leftrightarrow x_N^\mu \tag{5.93}$$

and the jacobian is

$$\frac{\partial x_N^\mu}{\partial x_S^\nu} = -\frac{4l^2}{x_S^2} \left(\delta_\nu^\mu - 2\frac{x_S^\mu x_S^\nu}{x_S^2} \right) \tag{5.94}$$

Only functions which are invariant under the exchange of North and South pole stereographic coordinates are well defined on the sphere.

5.4 The two-sphere, S_2

Let us work out in detail the two dimensional case. Define dimensionless corrdinates asd

$$\begin{array}{ll} \xi_1 \equiv -\frac{x}{z+l} & \eta_1 = \frac{x}{l-z} \\ \xi_2 \equiv \frac{y}{z+l} & \eta_2 = \frac{y}{l-z} \end{array} \tag{5.95}$$

It then follows that

$$z = L\frac{1-\xi^2}{1+\xi^2} = l\frac{\eta^2 - 1}{\eta^2 - 1} \tag{5.96}$$

and the change N/S now reads

$$\begin{aligned} \eta^1 &= \tfrac{1}{\xi^2}\,\xi^1 \\ \eta^2 &= -\tfrac{1}{\xi^2}\,\xi^2 \end{aligned} \tag{5.97}$$

Then defining the complex variable

$$z \equiv \xi^1 + i\xi^2 \tag{5.98}$$

the change N/S reduces to

$$z \to w \equiv \frac{1}{z} \tag{5.99}$$

Consider now a field of vectors

$$l_n(z)\,\frac{\partial}{\partial z} = z^n\,\frac{\partial}{\partial z} = -\frac{1}{z^2}\,l_n(z)\,\frac{\partial}{\partial w} = -w^{2-n}\,\frac{\partial}{\partial w} \tag{5.100}$$

If we want the field to be non-singulkar for all values of z and w, then

$$n \geqslant 0 \quad \& \quad 2-n \geqslant 0 \tag{5.101}$$

so that the *only* vector fields globally defined on the two-sphete S_2 are

$$\left(a + bz + cz^2\right)\frac{\partial}{\partial z} = -\left(aw^2 + bw + c\right)\frac{\partial}{\partial w} \tag{5.102}$$

$$\tag{5.103}$$

The induced metric on the sphere reads

$$ds^2 = \frac{dx_S^2}{(1+\frac{x_S^2}{4l^2})^2} = \frac{dx_N^2}{(1+\frac{x_N^2}{4l^2})^2} \tag{5.104}$$

which is conformally flat. This is the main virtue of these coordinates, and the reason why cartographers are fond of them, We shall call a *frame* a basis on the tangent space to the sphere as a manifold. Let us define a frame through

$$\delta_{ab}e^a_\mu e^b_\nu = g_{\mu\nu} \tag{5.105}$$

The frames are given by

$$(e_S)^a_\mu = \delta^\mu_a \frac{1}{1+\frac{x_S^2}{4l^2}} \tag{5.106}$$

$$(e_N)^a_\mu = -\delta^\mu_a \frac{1}{1+\frac{x_N^2}{4l^2}} \tag{5.107}$$

It is easy to check that

$$L^a_b(x)\ \ (e_S)^b_\mu \equiv \frac{\delta^a_\mu - 2\frac{x^\mu_S x^S_a}{x^2_S}}{1+\frac{x^2_S}{4l^2}} = \frac{\partial x^\nu_N}{\partial x^\mu_S}\ \ (e_N)^a_\nu \tag{5.108}$$

where the position dependent rotation is given by

$$L^a_b \equiv \delta^a_b - 2\frac{x^a x_b}{x^2} \tag{5.109}$$

In fact this was the reason for the apparently arbitrary minus sign in front of the definition of e_N, which is unneccessary to reproduce the metric.

There are many reasons to be drawn from this example. First of all, it is *never* possible to cover a non trivial manifold with a single coordinate system. In this case we need at least two, namely North and South stereographic coordinates. Second, at each coordinate system, there is a frame in the tangent space, and if we refer all quantities to this frame formal operations are similar to the flat space ones.

6

Moving frames and curvature.

The analogous to the field strength tensor for gauge theories is then the *Riemann-Christoffel tensor*

$$R^{\mu}{}_{\nu\alpha\beta} \equiv \partial_{\alpha}\Gamma^{\mu}_{\nu\beta} - \partial_{\beta}\Gamma^{\mu}_{\nu\alpha} + \Gamma^{\mu}_{\sigma\alpha}\Gamma^{\sigma}_{\nu\beta} - \Gamma^{\mu}_{\sigma\beta}\Gamma^{\sigma}_{\nu\alpha} \tag{6.1}$$

The *Ricci tensor* is defined by contracting indices

$$R_{\mu\nu} \equiv R^{\lambda}{}_{\mu\lambda\nu} \tag{6.2}$$

Recall the algebraic Bianchi identity

$$R^{\mu}{}_{\alpha\beta\gamma} + R^{\mu}{}_{\gamma\alpha\beta} + R^{\mu}{}_{\beta\gamma\alpha} = 0 \tag{6.3}$$

Clever use of this identity allows to prove that

$$R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta} \tag{6.4}$$

Let us see it. We start with

$$\begin{aligned} R_{\alpha\lambda\mu\nu} + R_{\alpha\mu\nu\lambda} + R_{\alpha\nu\lambda\mu} = 0 \\ R_{\lambda\alpha\mu\nu} + R_{\lambda\nu\alpha\mu} + R_{\lambda\mu\nu\alpha} = 0 \end{aligned} \tag{6.5}$$

Substracting

$$2R_{\alpha\lambda\mu\nu} + R_{\alpha\mu\nu\lambda} + R_{\alpha\nu\lambda\mu} - R_{\lambda\nu\alpha\mu} - R_{\lambda\mu\nu\alpha} = 0 \tag{6.6}$$

The same equation with the indices interchanged

$$(\alpha\lambda) \rightarrow (\mu\nu) \tag{6.7}$$

$$2R_{\mu\nu\alpha\lambda} + R_{\mu\alpha\lambda\nu} + R_{\mu\lambda\nu\alpha} - R_{\nu\lambda\mu\alpha} - R_{\nu\alpha\lambda\mu} = 0 \tag{6.8}$$

conveys the fact that

$$R_{\mu\nu\alpha\lambda} = R_{\alpha\lambda\mu\nu} \tag{6.9}$$

We have then a symmetric tensor R_{IJ} where each index is in the antisymmetric $[\alpha\beta]$ (that is, $D \equiv \frac{n(n-1)}{2}$ values). This yields

$$\frac{D(D+1)}{2} - \binom{n}{4} = \frac{n^2(n^2-1)}{12} \tag{6.10}$$

(we withdraw $\binom{n}{4}$ because of the algebraic Bianchi identity) independent components. Id est, 20 in n=4 dimensions.

There are also some differential identities, the *Bianchi identities*

$$\nabla_\alpha R^\mu{}_{\nu\beta\gamma} + \nabla_\gamma R^\mu{}_{\nu\alpha\beta} + \nabla_\beta R^\mu{}_{\nu\gamma\alpha} = 0 \tag{6.11}$$

Contracting δ^β_μ

$$\nabla_\alpha R_{\nu\gamma} - \nabla_\gamma R_{\nu\alpha} + \nabla_\mu R^\mu{}_{\nu\gamma\alpha} = 0 \tag{6.12}$$

Contracting again $g^{\nu\alpha}$

$$\nabla^\alpha R_{\alpha\gamma} - \nabla_\gamma R + \nabla_\mu R^\mu_\gamma = 0 \tag{6.13}$$

We shall derive most of these equations in a short while. Many useful formulas of tensor calculus are to be found in Eisenhart's book, still indispensable.

Also very useful are the *Ricci identities* that state that

$$[\nabla_\alpha, \nabla_\beta]\,\omega_\gamma = \; R_{\alpha\beta\gamma\delta}\;\omega^\delta \tag{6.14}$$

This can actually be taken as the definition of the Riemann tensor, as is done in many books.

All this means that the geodesic equations can be written as

$$u^\mu \nabla_\mu u^\alpha = 0 \tag{6.15}$$

where the four-velocity of the massive particle is given by

$$u^\alpha \equiv \frac{dx^\alpha}{ds} \tag{6.16}$$

In general, the metric

$$da^2 = g_{\mu\nu}(x)\; dx^\mu\; dx^\nu \tag{6.17}$$

is not flat; to the extent that it differs from the flat metric, it indicates the presence of a gravitational field. At each point there are tensors (or spinors) that represent physical observables. For example, the energy momentum tensor

$$T_{\mu\nu}(x) \tag{6.18}$$

This tensor *live* in the tangent space; the set of all tangent spaces of the manifold is the tangent bundle. A *frame* is a basis of the tangent vector space at a given point of the space-time manifold. This four vectors are represented by

$$E_a^\mu \partial_\mu \tag{6.19}$$

where the index $a = 0, 1, 2, 3$ labels the four different vectors. The simplest possibility is to choose one of them timelike (this is the one labeled E_0) , and the other three spacelike. Furthermore, they can be normalized in such a way that

$$g_{\mu\nu} E_a^\mu E_b^\nu = \eta_{ab} \tag{6.20}$$

This is the reason why latin indices are dubbed *Lorentz* indices, whereas the ordinary spacetime indices are called Einstein indices. Such a frame is precisely a LIF (where FREFOS live) and the physical observables measured in the LIF are simply

$$T_{ab} \equiv T_{\mu\nu} E_a^\mu E_b^\nu \tag{6.21}$$

The determinant of E considered as a matrix cannot vanish. We can then *define* the *coframe* made out of the dual one-forms

$$e^a \left(E_b \right) = \delta_\beta^a \tag{6.22}$$

When indices are put in place, this is equivalent to computing the inverse matrix

$$\begin{aligned} e_\mu^a E_b^\mu &\equiv \delta_b^a \\ e_\mu^a E_a^\nu &= \delta_\mu^\nu \end{aligned} \tag{6.23}$$

From the normalization condition

$$g_{\mu\nu} E_a^\mu E_b^\nu = \eta_{ab}$$

and multiplying both members by the dual form e_σ^a

$$\Rightarrow e_\mu^a = g_{\mu\nu} \eta^{ab} E_b^\nu$$

This means that the dual form is simply the frame with the Einstein indices lowered with the spacetime metric, and the Lorentz indices raised with the Lorentz metric. Following most physicists we shall represent both the frame and the coframe with the same letter, although when neccessary we will indicate explicitly its nature, as in

$$\begin{aligned} \vec{e}_a &\equiv e_a^\mu \partial_\mu \\ \underline{e}_a &\equiv e_{a\mu} dx^\mu \end{aligned} \tag{6.24}$$

The parallel propagator is defined once frames at different points are selected by some mechanism

$$g^\alpha{}_{\beta'}(x, x') \equiv e_a^\alpha(x) e_{\alpha'}^a(x') \tag{6.25}$$

Then physical quantities at different points are related through

$$A^{\alpha}(x) \equiv g^{\alpha}{}_{\beta'}(x,x')a^{\alpha'}(x') \tag{6.26}$$

For n-dimensional spheres in stereographic coordinates

$$ds^2 = \Omega^2\ \delta_{\mu\nu}dx^{\mu}dx^{\nu} \tag{6.27}$$

where

$$\Omega \equiv \frac{1}{1+\frac{x^2}{4L^2}} \tag{6.28}$$

and the frame is defined by

$$e^a_{\mu} = \Omega\delta^a_{\mu} \tag{6.29}$$

in such a way that

$$e^{\mu}_a = \frac{1}{\Omega}\delta^{\mu}_a \tag{6.30}$$

The S_n Christoffels read

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{\Omega_{\beta}}{\Omega}\ \delta^{\alpha}_{\gamma} + \frac{\Omega_{\gamma}}{\Omega}\ \delta^{\alpha}_{\beta} - \frac{\Omega^{\alpha}}{\Omega}\ \delta_{\beta\gamma} \tag{6.31}$$

Under a local Lorentz transformation

$$E_{a'} = L_{a'}{}^{b}(x)E_b \tag{6.32}$$

$E^{\mu}_a(x)$ is a nonsingular square $n\times n$ matrix. The commutators are given by

$$[E_a, E_b] = C^c_{ab}E_c$$

It is a fact that

$$\begin{aligned} de^a &= \partial_{[\mu}e^a_{\rho]}dx^{\rho}\wedge dx^{\mu} = \frac{1}{2}\left(\partial_{\mu}e^a_{\nu} - \partial_{\nu}e^a_{\mu}\right)dx^{\mu}\wedge dx^{\nu} = \frac{1}{2}\left(\partial_{\mu}e^a_{\nu} - \partial_{\nu}e^a_{\mu}\right)e^{\mu}_c e^{\nu}_d e^c\wedge e^d = \\ &= \frac{1}{2}\left(e_c(e^a_{\nu})e^{\nu}_d - e_d(e^a_{\mu})e^{\mu}_c\right)e^c\wedge e^d = \frac{1}{2}\left(e_d(e^{\mu}_c)e^a_{\mu} - e^a_{\nu}e_c(e^{\nu}_d)\right)e^c\wedge e^d = \\ &= \frac{1}{2}\left[e_d, e_c\right]^{\mu}e^a_{\mu}e^c\wedge e^d = -\frac{1}{2}C^a_{cd}e^c\wedge e^d \end{aligned} \tag{6.33}$$

To be specific, the structure constants read

$$C^c_{ab} = e^c_{\mu}\left(e^{\lambda}_a\partial_{\lambda}e^{\mu}_b - e^{\lambda}_b\partial_{\lambda}e^{\mu}_a\right) = e^c_{\mu}\left(e^{\lambda}_a\nabla_{\lambda}e^{\mu}_b - e^{\lambda}_b\nabla_{\lambda}e^{\mu}_a\right) \tag{6.34}$$

(The Christoffels cancel when taking the antisymmetric part). In our S_n example,

$$C^c_{ab} = \frac{\Omega_b}{\Omega^2}\ \delta^c_a - \frac{\Omega_a}{\Omega^2}\ \delta^c_b \tag{6.35}$$

Under a local Lorentz transformation the vierbein transforms as

$$e^{a'} = L^a{}_b(x)e^b \tag{6.36}$$

This is not true of the derivatives of the vierbein, de^a, owing to the term in $dL^a{}_b$. We would like to introduce a gauge field (connection) in the LIF, the so called *spin connection*, such that the two-form

$$De^a \equiv de^a + \omega^a{}_b \wedge e^b \tag{6.37}$$

transforms as

$$(De^a)' = L^a\, bDe^b \tag{6.38}$$

For this to be true we need

$$d\left(L^a{}_b e^b\right) + \left(\omega'\right)^a{}_b \wedge \left(L^b{}_c e^c\right) = L^a{}_b \left(de^b + \omega^b{}_c \wedge e^c\right) \tag{6.39}$$

This is equivalent to

$$dL^a{}_b \wedge e^b + \left(\omega'\right)^a{}_b \wedge L^b{}_c e^c = L^a{}_b \omega^b{}_c \wedge e^c \tag{6.40}$$

which is kosher provided

$$dL^a{}_c + \left(\omega'\right)^a{}_b L^b{}_c = L^a{}_b \omega^b{}_c \tag{6.41}$$

Lorentz transformations are such that

$$L^{ac} L_{ad} = \delta^c_d = L^{ca} L_{da} \tag{6.42}$$

Finally we get the transformation law for the gauge field

$$\left(\omega'\right)^a{}_d = L^a{}_b\, \omega^b{}_c\, L_d{}^c - dL^a{}_c L_d{}^c \tag{6.43}$$

At the linear level, where

$$L_{ab} \equiv \eta_{ab} + \epsilon_{ab} \tag{6.44}$$

$$\delta\omega^a{}_{b\mu} = -\partial_\mu \omega^a{}_b + [\epsilon, \omega]^a{}_b \tag{6.45}$$

This should be valid for any field living in the LIF that transforms with a representation of the Lorentz group. But any field can be so represented. For example, a vector field, V^μ is projected on the LIF by a FREFO as $V^a \equiv e^a_\mu V^\mu$. We want that its Lorentz covariant derivative is also the projection of Einstein's covariant derivative, that is

$$\nabla_L \left(V^a\right) = e^a_\mu \left(\nabla_E V\right)^\mu \tag{6.46}$$

This physical requirement determines the relationship between Lorentz and Einstein connections to be

$$\omega^a{}_{b\sigma} = e^a_\lambda \Gamma^\lambda_{\mu\sigma} e^\mu_b - e^\rho_b \partial_\rho e^a_\sigma \tag{6.47}$$

It is a fact (confer [15]) that the *torsion* can be defined through the *connection* ω^a_b by

$$de^a + \omega^a_b \wedge e^b \equiv T^a \equiv \frac{1}{2} T^a_{bc} e^b \wedge e^c$$

Demanding that the tangent metric is covariantly constant we learn that

$$\nabla_a \eta_{bc} = 0 = -\omega^d_{ab}\eta_{dc} - \omega^d_{ac}\eta_{db} \equiv -\omega_{c|ab} - \omega_{b|ac} \tag{6.48}$$

When the torsion vanishes, and in tensor form

$$\partial_\rho e^a_\sigma - \partial_\sigma e^a_\rho + \omega^a{}_{\sigma\rho} - \omega^a{}_{\rho\sigma} = 0 \tag{6.49}$$

it follows that

$$\begin{aligned} \omega_{a|bc} - \omega_{a|cb} = (\partial_\rho e_{a\sigma} - \partial_\sigma e_{a\rho})\, e^\sigma_b e^\rho_c \equiv \vec{e}_b \partial_c \underline{e}_a - \vec{e}_c \partial_b \underline{e}_a = \underline{e}_a \partial_b \vec{e}_c - \underline{e}_a \partial_c \vec{e}_b = \\ = \underline{e}_a \cdot [\vec{e}_b, \vec{e}_c] = \underline{e}_a . C^d_{bc} \vec{e}_d \equiv C_{a|bc} \end{aligned} \tag{6.50}$$

where we have used the fact that

$$\vec{e}_b \partial_c \underline{e}_a = -\underline{e}_a \partial_c \vec{e}_b \tag{6.51}$$

This means that the torsion-free condition completely determines the antisymmetric part of the connection. One often is interested in the case when the connection lies in the Lie algebra of a simple group. For example, if $\omega_\mu \in \mathfrak{SO}(n))$

$$\omega_{\mu|ab} = -\omega_{\mu|ba} \tag{6.52}$$

For spheres we have

$$\omega_{a|bc} = \frac{1}{2}\left(\frac{\Omega_c}{\Omega^2}\ \delta_{ab} - \frac{\Omega_b}{\Omega^2}\ \delta_{ac}\right) \tag{6.53}$$

$$2\omega_{\mu|ab} = \frac{\Omega_b}{\Omega}\delta_{\mu a} - \frac{\Omega_a}{\Omega}\ \delta_{\mu b} = \left(\frac{\Omega_b}{\Omega}\ \delta_{a\mu} + \frac{\Omega_\mu}{\Omega}\ \delta_{ab} - \frac{\Omega_a}{\Omega}\ \delta_{b\mu}\right) - \frac{\Omega_\mu}{\Omega}\ \delta_{ab} \tag{6.54}$$

We see that this is equivalent to our physical postulate of FREFOs and FIDOS. The curvature of the connection is defined through

$$d\omega^a_b + \omega^a_c \wedge \omega^c_b \equiv R^a{}_b \equiv \frac{1}{2} R^a_{bcd} e^c \wedge e^d$$

It is asy to check that this a true Lorentz tensor; that is, under a local Lorentz transformation

$$R^a_b \rightarrow L^a{}_c R^c{}_d L_b{}^d \tag{6.55}$$

This leads immediately to Bianchi identities

$$\begin{aligned} & dT^a = d\omega^a_b \wedge e^b - \omega^a_b \wedge de^b = (R^a_b - \omega^a_c \wedge \omega^c_b) \wedge e^b - \omega^a_b \wedge (T^b - \omega^b_c \wedge e^c) = \\ & = R^a_b \wedge e^b - \omega^a_b \wedge T^b \\ & dR^a_b = d\omega^a_c \wedge \omega^c_b - \omega^a_c \wedge d\omega^c_b = \\ & (R^a_c - \omega^a_d \wedge \omega^d_c) \wedge \omega^c_b - \omega^a_c \wedge (R^c_b - \omega^c_d \wedge \omega^d_b) = R^a_c \wedge \omega^c_b - \omega^a_c \wedge R^c_b \end{aligned} \tag{6.56}$$

For a Levi-Civita connection the algebraic Bianchi identity in a natural basis reads

$$R^a_b \wedge e^b = 0 = \frac{1}{2} R^a_{b\mu\nu} e^b_\lambda dx^{\mu\nu\lambda} \tag{6.57}$$

In gory detail

$$R^\alpha{}_{[\lambda\mu\nu]} = 0 = R^\alpha{}_{\lambda\mu\nu} + R^\alpha{}_{\mu\nu\lambda} + R^\alpha{}_{\nu\lambda\mu} \tag{6.58}$$

Clever use of this identity allows to prove that

$$R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta} \tag{6.59}$$

Let us see it. We start with

$$\begin{aligned} R_{\alpha\lambda\mu\nu} + R_{\alpha\mu\nu\lambda} + R_{\alpha\nu\lambda\mu} = 0 \\ R_{\lambda\alpha\mu\nu} + R_{\lambda\nu\alpha\mu} + R_{\lambda\mu\nu\alpha} = 0 \end{aligned} \tag{6.60}$$

Substracting

$$2R_{\alpha\lambda\mu\nu} + R_{\alpha\mu\nu\lambda} + R_{\alpha\nu\lambda\mu} - R_{\lambda\nu\alpha\mu} - R_{\lambda\mu\nu\alpha} = 0 \tag{6.61}$$

The same equation with the indices interchanged

$$(\alpha\lambda) \rightarrow (\mu\nu) \tag{6.62}$$

$$2R_{\mu\nu\alpha\lambda} + R_{\mu\alpha\lambda\nu} + R_{\mu\lambda\nu\alpha} - R_{\nu\lambda\mu\alpha} - R_{\nu\alpha\lambda\mu} = 0 \tag{6.63}$$

conveys the fact that

$$R_{\mu\nu\alpha\lambda} = R_{\alpha\lambda\mu\nu} \tag{6.64}$$

We have then a symmetric tensor R_{IJ} where each index is in the antisymmetric $[\alpha\beta]$ (that is, $D \equiv \frac{n(n-1)}{2}$ values). This yields

$$\frac{D(D+1)}{2} - \binom{n}{4} = \frac{n^2(n^2-1)}{12} \tag{6.65}$$

(we withdraw $\binom{n}{4}$ because of the algebraic Bianchi identity) independent components. Id est, 20 in n=4 dimensions. The differential identity in a natural basis reads

$$\nabla_{[\alpha} R^\mu{}_{\overline{\beta}\gamma\delta]} \equiv \nabla_\alpha R^\mu{}_{\beta\gamma\delta} + \nabla_\gamma R^\mu{}_{\beta\delta\alpha} + \nabla_\delta R^\mu{}_{\beta\alpha\gamma} = 0 \tag{6.66}$$

where the overline on an index means that this particular index is absent from the antisymmetrization. Now

$$\begin{aligned} \nabla_{[\alpha} R^\mu{}_{\overline{\beta}\gamma\delta]} \equiv \partial_{[\alpha} R^\mu{}_{\overline{\beta}\gamma\delta]} + \Gamma^\mu_{[\alpha\overline{\sigma}} R^\sigma{}_{\overline{\beta}\gamma\delta]} - \Gamma^\sigma_{[\alpha\overline{\beta}} R^\mu{}_{\overline{\sigma}\gamma\delta]} - \Gamma^\sigma_{[\alpha\gamma} R^\mu{}_{\overline{\beta}\overline{s}\sigma\delta]} - \Gamma^\sigma_{\alpha\delta} R^\mu{}_{\overline{\beta}\gamma\overline{\sigma}]} = \\ = \partial_{[\alpha} R^\mu{}_{\overline{\beta}\gamma\delta]} + \Gamma^\mu_{[\alpha\overline{\sigma}} R^\sigma{}_{\overline{\beta}\gamma\delta]} - \Gamma^\sigma_{[\alpha\overline{\beta}} R^\mu{}_{\overline{\sigma}\gamma\delta]} \end{aligned} \tag{6.67}$$

Using the relationship between $\omega^a_{b\mu}$ and $\Gamma^\alpha_{\beta\mu}$ derived above we are done. On the other hand

$$\partial_\alpha R^\mu{}_{\beta\gamma\delta} = \partial_\alpha\left(e^\mu_a e^b_\beta R^a{}_{b\gamma\delta}\right) = \left(\partial_\alpha e^\mu_a\right) e^b_\beta R^a{}_{b\gamma\delta} + e^\mu_a\left(\partial_\alpha e^b_\beta\right) R^a{}_{b\gamma\delta} + e^\mu_a e^b_\beta \partial_\alpha R^a{}_{b\gamma\delta} \tag{6.68}$$

It is a fact of life that

$$\begin{aligned} &\nabla_{E_a}(E_b) \equiv \Gamma^c_{ab} E_c \\ &T^a_{bc} = \Gamma^a_{bc} - \Gamma^a_{cb} - C^a_{bc} \\ &R^a_{b,cd} = E_c \Gamma^a_{db} - E_d \Gamma^a_{cb} + \Gamma^e_{db}\Gamma^a_{ce} - \Gamma^e_{cb}\Gamma^a_{de} - C^e_{cd}\Gamma^a_{eb} \end{aligned} \tag{6.69}$$

It is nice exercise to check that the scalar curvature for a two-dimensional surface

$$\begin{aligned} R &= \frac{N}{D} \\ N &\equiv e\left[e_u g_v - 2 g_v f_u + g_u^2\right] + f\left[g_v e_u + 2 f_u\left(2 f_v - g_u\right) - e_v\left(2 f_v + g_u\right)\right] + \\ &+2 f^2\left[e_{vv} - 2 f_{uv} + g_{vv}\right] + g\left[e_v^2 + e_u\left(-2 f_v + g_u\right) - 2e\left(e_{vv} + 2 f_{uv} + g_{vv}\right)\right] \\ D &\equiv 2\left(f^2 - eg\right) \end{aligned} \tag{6.70}$$

7

The Gauss-Codazzi equations

Consider a codimension one hypersurface given by the embedding

$$\Sigma_{n-1} \hookrightarrow M_n \tag{7.1}$$

$$x^\alpha = \sigma^\alpha(y^i) \tag{7.2}$$

The indiced metric is given by

$$ds^2_{n-1} = h_{ij} dy^i dy^j = g_{\mu\nu}\partial_i\sigma^\mu\partial_j\sigma^\nu dy^i dy^j \tag{7.3}$$

There are then two metric connections: the n-dimensional one, ∇_g and the (n-1)-dimensional one associated to the induced metric, D_h. From the definition itself of the induced metric follows

$$0 = D_k h_{ij} = \partial_\rho g_{\alpha\beta} D_k\sigma^\rho\partial_i\sigma^\alpha\partial_j\sigma^b + g_{\alpha\beta}D_k(\partial_i\sigma^\alpha)\partial_j\sigma^\beta + g_{\alpha\beta}\partial_i\sigma^\alpha D_k(\partial_j\sigma^\beta) \tag{7.4}$$

Cyclic permutations

$$\partial_\rho g_{\alpha\beta}D_j\sigma^\rho\partial_k\sigma^\alpha\partial_i\sigma^\beta + g_{\alpha\beta}D_j(\partial_k\sigma^\alpha)\partial_i\sigma^\beta + g_{\alpha\beta}\partial_k\sigma^\alpha D_j(\partial_i\sigma^\beta) = 0$$
$$\partial_\rho g_{\alpha\beta}D_i\sigma^\rho\partial_j\sigma^\alpha\partial_k\sigma^\beta + g_{\alpha\beta}D_i(\partial_j\sigma^\alpha)\partial_i\sigma^\beta + g_{\alpha\beta}\partial_j\sigma^\alpha D_i(\partial_k\sigma^\beta) = 0$$

Adding 1+2-3 yields

$$\begin{aligned}
&0 = g_{\alpha\beta}D_jD_k\sigma^\alpha D_i\sigma^\beta + D_k\sigma^\rho D_i\sigma^\alpha D_j\sigma^\beta\frac{1}{2}\left(\partial_\rho g_{\alpha\beta} + \partial_\beta g_{\rho\alpha} - \partial_\alpha g_{\beta\rho}\right) = \\
&g_{\alpha\beta}D_jD_k\sigma^\alpha D_i\sigma^\beta + D_k\sigma^\rho D_i\sigma^\alpha D_j\sigma^\beta\{\alpha,\beta\rho\} = \\
&= g_{\alpha\beta}D_i\sigma^\beta\left(D_kD_j\sigma^\alpha + \{{}^{\alpha}_{\beta\rho}\}D_j\sigma^\beta D_k\sigma^\rho\right)
\end{aligned} \tag{7.5}$$

This means that

$$D_kD_j\sigma^\alpha = -\{{}^{\alpha}_{\beta\rho}\}D_j\sigma^\beta D_k\sigma^\rho + K_{jk}n^\alpha \tag{7.6}$$

where the normal component reads

$$K_{jk} \equiv n_\alpha\left(D_kD_j\sigma^\alpha + \{{}^{\alpha}_{\beta\rho}\}D_j\sigma^\beta D_k\sigma^\rho\right) \tag{7.7}$$

Taking the D_j

$$0 = D_j \left(g_{\alpha\beta} \partial_i \sigma^\alpha n^\beta \right) = D_j g_{\alpha\beta} \partial_i \sigma^\alpha n^\beta + g_{\alpha\beta} D_j D_i \sigma^\alpha n^\beta + g_{\alpha\beta} D_i \sigma^\alpha D_j n^\beta \tag{7.8}$$

On the other hand,

$$D_j g_{\alpha\beta} = D_j \sigma^\nu \partial_\nu g_{\alpha\beta} = D_j \sigma^\nu \left(\{\alpha\nu;\beta\} + \{\beta\nu;\alpha\} \right) \tag{7.9}$$

so that

$$K_{jk} = n_\alpha \{ {}^{\alpha}_{\beta\rho} \} D_j \sigma^\beta D_k \sigma^\rho - g_{\alpha\beta} D_j \sigma^\alpha D_k n^\beta - n^\beta D_k \sigma^\rho \left(\{\alpha\rho;\beta\} + \{\beta\rho;\alpha\} \right) D_j \sigma^\alpha = \\ -g_{\alpha\beta} D_j \sigma^\alpha D_k n^\beta - n^\beta D_k \sigma^\rho \{\beta\rho;\alpha\} D_j \sigma^\alpha = -\xi^\alpha_i \nabla_\rho n_\alpha \xi^\rho_j \tag{7.10}$$

This tensor is called the *extrinsic curvature*, and represents the derivative of the normal vector, projected on the surface.

Our purpose in life is now to relate the Riemann tensor on the hypersurface (computed with the induced metric) with the corresponding Riemann tensor of the spacetime manifold. Those are the famous Gauss-Codazzi equations, which we purport now to derive. They were one of the pillars of Gauss' *theorema egregium*, [15] which asserts that *If a curved surface is developed upon any other surface whatever the measure of curvature in each point remains unchanged.*

We start with

$$0 = D_j \left(g_{\alpha\beta} n^\alpha n^\beta \right) = D_j \sigma^\rho \left(\{\alpha\rho;\beta\} + \{\rho\beta;\alpha\} \right) n^\alpha n^\beta + g_{\alpha\beta} D_j n^\alpha n^\beta + g_{\alpha\beta} n^\alpha D_j n^\beta = \\ g_{\alpha\beta} n^\beta \left(D_j n^\alpha + \{ {}^{\alpha}_{\mu\nu} \} D_j \sigma^{(\mu} n^{\nu)} \right) = g_{\alpha\beta} n^\beta \nabla_\mu n^\alpha D_j \sigma^\mu = n_\alpha \nabla_\mu n^\alpha \xi^\mu_j \tag{7.11}$$

On the other hand, the explicit expression for the extrinsic curvature reads

$$K_{ij} = -\xi^\alpha_i \nabla_\rho n_\alpha \xi^\rho_j \tag{7.12}$$

First of all let us derive some properties of the extrinsic curvature. It is symmetric, $K_{ij} = K_{ji}$.

$$-K_{ij} = \nabla_\beta n_\alpha \xi^\alpha_i \xi^\beta_j = -n_\alpha \nabla_\beta \xi^\alpha_i \xi^\beta_j \tag{7.13}$$

But

$$\left[\xi^\beta_j, \xi^\alpha_i \right] = 0 \tag{7.14}$$

so that

$$-K_{ij} = -n_\alpha \xi^\alpha_i \nabla_\beta \xi^\alpha_j = \nabla_\beta n_\alpha \xi^\beta_i \xi^\alpha_l = K_{ji} \tag{7.15}$$

This symmetry implies a very useful formula for the extrinsic curvature, namely

$$-K_{ij} = \nabla_{(\beta} n_{\alpha)} \xi^\alpha_i \xi^\beta_j = \pounds(n) g_{\alpha\beta} \xi^\alpha_i \xi^\beta_j \tag{7.16}$$

By the way, in the physics jargon when $K_{ij} = 0$ it is said that it is a *moment of time symmetry*.

On the other hand, remembering that

$$\xi_i^\alpha \xi_\beta^i = g_\beta^\alpha - n^\alpha n_\beta \tag{7.17}$$

we deduce that

$$-K_{ij}\xi_\mu^i = -\left(g_\mu^\alpha - n^\alpha n_\mu\right)\nabla_\rho n_\alpha \xi_j^\rho = -\nabla_\rho n_\mu \xi_j^\rho \tag{7.18}$$

(because of [7.11]).

Let us analyze the definition of extrinsic curvature in even more detail.

$$\begin{aligned}&\left(D_k D_j D_i - D_j D_k D_i\right)\sigma^\alpha = \xi_m^\alpha h^{mh} R_{hijk} = D_k\left(-\{{}^{\alpha}_{\beta\rho}\}\xi_i^\beta \xi_j^\rho + K_{ij}n^\alpha\right) - \\ &-D_j\left(-\{{}^{\alpha}_{\beta\rho}\}\xi_i^\beta \xi_k^\rho + K_{ik}n^\alpha\right) = \partial_k\{{}^{\alpha}_{\beta\rho}\}\xi_i^\beta \xi_j^\rho - \{{}^{\alpha}_{\beta\rho}\}D_k\xi_i^\beta \xi_j^\rho - \{{}^{\alpha}_{\beta\rho}\}\xi_i^\beta D_k \xi_j^\rho + \\ &D_k K_{ij} n^\alpha + K_{ij} D_k n^\alpha + \partial_j\{{}^{\alpha}_{\beta\rho}\}\xi_i^\beta \xi_k^\rho - \{{}^{\alpha}_{\beta\rho}\}D_j\xi_i^\beta \xi_k^\rho + \{{}^{\alpha}_{\beta\rho}\xi_i^\beta D_j \xi_k^\rho - D_j K_{ik} - K_{ik} D_j n^\alpha\end{aligned}$$

and using again the defnition of the extrinsic curvature to eliminate the term with two derivatives,

$$\begin{aligned}&\xi_m^\alpha h^{mr} R_{rijk}[h] = -\partial_k\{{}^{\alpha}_{\beta\rho}\}\xi_i^\beta \xi_j^\rho - \{{}^{\alpha}_{\beta\rho}\}\xi_j^\rho\left(-\{{}^{\beta}_{\mu\nu}\}\xi_i^\mu \xi_k^\nu + K_{ik}n^\beta\right) + D_k K_{ij} n^\alpha + K_{ij} D_k n^\alpha + \\ &\partial_j\{{}^{\alpha}_{\beta\rho}\}\xi_i^\beta \xi_k^\rho + \{{}^{\alpha}_{\beta\rho}\}\xi_k^\rho\left(-\{{}^{\beta}_{\mu\nu}\}\xi_i^\mu \xi_j^\nu + K_{ij}n^\beta\right) - D_j K_{ik} n^\alpha - K_{ik} D_j n^\alpha = \\ &n^\alpha\left(D_k K_{ij} - D_j K_{ik}\right) + K_{ij}\left(D_k n^\alpha + \{{}^{\alpha}_{\beta\rho}\}n^\beta \xi_k^\rho\right) - K_{ik}\left(D_j n^\alpha + \{{}^{\alpha}_{\beta\rho} n^\beta \xi_j^\rho\right) - \\ &-\xi_i^\beta \xi_j^\rho \xi_k^\sigma\left(\partial_\sigma\{{}^{\alpha}_{\beta\rho}\} - \partial_\rho\{{}^{\alpha}_{\beta\sigma}\} - \{{}^{\alpha}_{\lambda\rho}\}\{{}^{\lambda}_{\beta\sigma} + \{{}^{\alpha}_{\lambda\sigma}\}\{{}^{\lambda}_{\beta\rho}\right)\end{aligned} \tag{7.19}$$

Using again the definition of the extrinsic curvature, as well as the one of the full Riemann tensor, we get

$$\xi_m^\alpha h^{mr}\left(R_{rijk}[h] + K_{ij}K_{rk} - K_{ik}K_{rj}\right) - n^\alpha\left(D_k K_{ij} - D_j K_{ik}\right) = -\xi_i^\beta \xi_j^\rho \xi_k^\sigma R^\alpha{}_{\beta\sigma\rho}[g]$$

This projects into the famous Gauss-Codazzi equations

$$R_{lijk}[h] + K_{il}K_{jk} - K_{ik}K_{lj} = \xi_l^\alpha \xi_i^\beta \xi_j^\rho \xi_k^\sigma R_{\alpha\beta\rho\sigma}[g] \tag{7.20}$$

as well as

$$D_j K_{ik} - D_k K_{ij} = -n^\alpha \xi_i^\beta \xi_j^\rho \xi_k^\sigma R_{\alpha\beta\sigma\rho}[g] \tag{7.21}$$

Please note that not all components of the full Riemann tensor can be recovered from the knowledge of the Riemann tensor computed on the hypersurface plus the extrinsic curvature. As a matter of fact,

$${}^{(n)}R = {}^{(n-1)}R^{ij}{}_{ij} + 2\,{}^{(n)}R^i{}_{nin} ={}^{(n-1)} R + K^2 - K_{ij}k^{ij} + 2\,{}^{(n)}R^i{}_{nin} \tag{7.22}$$

This means that an explicit computation of ${}^{(n)}R^{i}{}_{nin}$ is needed before the Einstein-Hilbert term could be written in the 1+(n-1) decomposition. To do that, consider Ricci's identity

$$\nabla_\gamma \nabla_\beta n_\alpha - \nabla_\beta \nabla_\gamma n_\alpha = R^\rho{}_{\alpha\beta\gamma} n_\rho \tag{7.23}$$

Now

$$n^\beta \left(\nabla_\gamma \nabla_\beta n^\gamma - \nabla_\beta \nabla_\gamma n^\gamma \right) = n^\beta g^{\alpha\gamma} R^\rho{}_{\alpha\beta\gamma} n^\rho \equiv R^{n\alpha}{}_{n\alpha} \tag{7.24}$$

Besides,

$$\begin{aligned} &\nabla_\gamma n^\beta \nabla_\beta n^\gamma = \nabla_\gamma n_\beta \left(n^\beta n^\mu + \xi_i^\beta \xi^{\mu i} \right) \left(n^\gamma n^\nu + \xi_j^\gamma \xi^{j\nu} \right) \nabla_\mu n_\nu = \\ &\nabla_\gamma n_\beta \xi_i^\beta \xi^{\mu i} \xi_j^\gamma \xi^{j\nu} \nabla_\mu n_\nu = -K_{ij} K^{ij} \end{aligned} \tag{7.25}$$

Summarizing,

$$\begin{aligned} R^{n\alpha}{}_{n\alpha} = n^\beta \nabla_\gamma \nabla_\beta n^\gamma - n^\beta \nabla_\beta \nabla_\gamma n^\gamma = \nabla_\gamma \left(n^\beta \nabla_\beta n^\gamma \right) - \nabla_\gamma n^\beta \nabla_\beta n^\gamma - \nabla_\beta \left(n^\beta \nabla_\gamma n^\gamma \right) + \\ + \nabla_\beta n^\beta \nabla_\gamma n^\gamma = \\ = \nabla_\gamma \left(n^\beta \nabla_\beta n^\gamma - n^\gamma \nabla_\beta n^\beta \right) + K_{ij} K^{ij} - K^2 \end{aligned} \tag{7.26}$$

Then

$${}^{(n)}R = {}^{(n-1)}R + K_{ij} K^{ij} - K^2 - \partial_\alpha V^\alpha \tag{7.27}$$

8

Distributions

Dirac introduced a function such that

$$\delta(x) = 0 \quad x \neq 0 \tag{8.1}$$

but

$$\int_{-\infty}^{\infty} dx\ \delta(x) = 1 \tag{8.2}$$

Consider the function $\phi(x, \epsilon)$ defined in such a way that

$$\begin{aligned} \phi(x, \epsilon) &= 0 \quad r \geqslant \epsilon \\ \phi(x, \epsilon) &= e^{-\frac{\epsilon^2}{\epsilon^2 - r^2}} \quad r \leqslant \epsilon \end{aligned} \tag{8.3}$$

It is clear that

$$f(0) = \frac{1}{e} \neq 0 \tag{8.4}$$

nevertheless

$$\int_{-\epsilon}^{\epsilon} \phi(x, \epsilon)\ d^3x = \int_{-\epsilon}^{\epsilon} e^{-\frac{\epsilon^2}{\epsilon^2 - r^2}}\ d^3x = 4\pi\epsilon^3 \int_0^1 e^{-\frac{1}{1-r^2}} dr = C\epsilon^3 \tag{8.5}$$

It is clear that Dirac's function cannot be a true function. Laurent Schwartz gave mathematical respectability to Dirac's ideas by introducing the concept of *distributions*. The main idea is to consider the dual of a convenient function space.

To begin, let us start with the *space of test functions* K, of real functions with continuous derivatives to all orders, and with compact support. It is not empty (actually, our recent friend, the function $\phi(x, \epsilon) \in K$).

It can be shown that given any continuous function f(x) with bounded support, there is always some $\phi(x) \in K$ arbitrarily close to it.

Define a *distribution* $d \in K'$ as a continuous linear functional on K

$$\forall \phi(x) \in K \quad \langle d, \phi(x)) \rangle \in \mathbb{R} \tag{8.6}$$

The two essential properties are

- $$\langle d, a_1\phi_1 + a_2\phi_2\rangle = \alpha_1\langle d, \phi_1\rangle + a_2\langle d, \phi_2\rangle \tag{8.7}$$

- If the sequence $\{\phi_n\}$ converges to 0 in K, then the sequence
$$\{\langle d, \phi_n\rangle\} \tag{8.8}$$
converges to zero.

It is plain that any locally summable function $f(x)$ is a particular case of a distribution, just by defining

$$\langle f, \phi\rangle \equiv \int_{-\infty}^{\infty} dx\ f(x)\ \phi(x) \tag{8.9}$$

those are called *regular distributions.*

But there are distributions (dubbed *singular*) which can not be written in such a way. The most important one is precisely the Dirac delta

$$\langle \delta(x), \phi(x)\rangle \equiv \phi(0) \tag{8.10}$$

- It is natural to define the behavior under a translation
$$\langle d(x-a), \phi(x)\rangle = \langle d(x), \phi(x+a)\rangle \tag{8.11}$$

- Under a reflexion
$$\langle d(-x), \phi(x)\rangle = \langle d(x), \phi(-x)\rangle \tag{8.12}$$

- Under a rescaling a regular distribution behaves as
$$\int dx\ f(x/\lambda)\phi(x) = \lambda \int\ dx\ f(x)\ \phi(\lambda x) \tag{8.13}$$
In general, we generalize this in n-dimensions to
$$\langle (\lambda d)\,(x), \phi(x)\rangle \equiv \lambda^n\langle d(x), \phi(\lambda x)\rangle \tag{8.14}$$

Let us now introduce the space S (Schwartz) of infinitely differentiable functions which, together with their derivatives, go to zero faster than any power of $\frac{1}{r}$ when $r \rightarrow \infty$. For example

$$e^{-r^2} \in \mathcal{S} \tag{8.15}$$

It is clear that *tempered distributions* are a subset of distributions

$$\mathcal{S}' \subset K' \tag{8.16}$$

(the bigger the starting space, the smaller its dual). The *derivative* of a distribution is defined as

$$\langle d', \phi(x)\rangle \equiv -\langle d, \phi'(x)\rangle \tag{8.17}$$

This result holds for regular distributions just by neglecting surface terms.

Let us work out some examples

- Consider the Heaviside function

$$\begin{aligned} &\theta(x) = 0 \quad x < 0 \\ &\theta(x) = 1 \quad x > 0 \end{aligned} \tag{8.18}$$

  Let us compute its derivative as a distribution.

$$\langle \theta'(x), \phi(x)\rangle \equiv -\langle \theta(x), \phi'(x) \equiv -\int_0^\infty \phi'(x)dx = \phi(0) \tag{8.19}$$

  This means that in this sense,

$$\theta'(x) = \delta(x) \tag{8.20}$$

- Let us find now the derivative of the distribution

$$x_+^\lambda \tag{8.21}$$

  defined for $-1 < \lambda < 0$ as

$$\begin{aligned} &x_+^\lambda = 0 \quad x \leqslant 0 \\ &x_+^\lambda = x^\lambda \quad x > 0 \end{aligned} \tag{8.22}$$

  This is locally summable, which is not the case with the ordinary derivative

$$\lambda x^{\lambda-1} \tag{8.23}$$

  We have to regularize the integral

$$\int_0^\infty \lambda x^{\lambda-1}\, dx \tag{8.24}$$

  According to the definition

$$\langle \left(x_+^\lambda\right)', \phi(x)\rangle = -\int_0^\infty x^\lambda \phi'(x)\, dx \equiv -\lim_{\epsilon\rightarrow 0}\int_\epsilon^\infty x^\lambda \phi'(x)\, dx \tag{8.25}$$

  Let us now integrate by parts with

$$\begin{aligned} &du = d\phi \quad \Longrightarrow \quad u = \phi + C \\ &v = x^\lambda \end{aligned} \tag{8.26}$$

This leads to

$$-\lim_{\epsilon\to 0}\left(\left.(\phi+C)\,\lambda x^{\lambda-1}\right|_{\epsilon}^{\infty}-\int_{\epsilon}^{\infty}\lambda x^{\lambda-1}\,(\phi+C)\;dx\right) \tag{8.27}$$

It is plain that in order for this to have a finite limit it is neccessary that

$$C=-\phi(0) \tag{8.28}$$

We are then led to the definition

$$\langle\left(x_{+}^{\lambda}\right)',\phi(x)\rangle\equiv\int_{0}^{\infty}\left(\phi(x)-\phi(0)\right)\lambda x^{\lambda-1}\;dx \tag{8.29}$$

- Let us compute the derivative of

$$\log\,(x+i0)\equiv\lim_{y\to 0}\log(x+iy) \tag{8.30}$$

Now

$$\log(x+i0)=\log\;|x|+i\pi\theta(-x) \tag{8.31}$$

We have seen that

$$\theta'(x)=\delta(x) \tag{8.32}$$

Now

$$\theta(x)+\theta(-x)=1\quad\Longrightarrow\quad\theta'(x)=-\delta(x) \tag{8.33}$$

as well as

$$x^2=|x|^2\quad\Longrightarrow\quad\frac{d}{dx}|x|=\frac{x}{|x|}\quad\Longrightarrow\quad\frac{d}{dx}\log|x|=\frac{1}{|x|}\frac{x}{|x|}=\frac{1}{x} \tag{8.34}$$

Then

$$\frac{d}{dx}\log(x+i0)=\frac{1}{x}-i\pi\delta(x) \tag{8.35}$$

- Let us explore $\frac{d}{dx}|x|$ in the sense of distributions

$$\begin{aligned}\langle|x|',f\rangle &\equiv-\textstyle\int|x|f'==\int_{-\infty}^{0}xf'-\int_{0}^{\infty}xf'=\\ &=\left.xf\right|_{-\infty}^{0}-\textstyle\int_{-\infty}^{0}f+\int_{0}^{\infty}f-\left.xf\right|_{0}^{\infty}=\int\sigma(x)f\end{aligned} \tag{8.36}$$

There is not delta-component of $|x|'$.

- Let is now compute the laplacian of the Newtonian potential.

$$\Delta\frac{1}{r} \tag{8.37}$$

Following the general definition

$$\langle\, \Delta\frac{1}{r}, \phi\, \rangle = \langle\, \frac{1}{r}, \Delta\phi\, \rangle = \int d^3x\ \frac{1}{r}\ \Delta\phi \equiv \lim_{\epsilon\to 0} \int_{r\geqslant\epsilon} d^3x\ \frac{1}{r}\ \Delta\phi =$$
$$\lim_{\epsilon\to 0} \int_{r\geqslant\epsilon} d^3x\ \left(\nabla_i \left(\frac{1}{r}\ \nabla^i\phi\right) - \left(\nabla_i\frac{1}{r}\right)\nabla^i\phi\right) =$$
$$= \lim_{\epsilon\to 0} \int_{r\geqslant\epsilon} d^3x\ \left(\nabla_i \left(\frac{1}{r}\ \nabla^i\phi\right) - \nabla_i\left(\nabla^i\frac{1}{r}\phi\right) + \left(\Delta\frac{1}{r}\right)\phi\right) \tag{8.38}$$

The volume term vanishes because $\frac{1}{r}$ is harmonic on $\mathbb{R}^3/0$. The surface term is

$$-\lim_{\epsilon\to 0} 4\pi\epsilon^2 \left(\frac{1}{\epsilon}\frac{d}{dn}\phi + \phi\frac{1}{\epsilon^2}\right) = -4\pi\phi(0) \tag{8.39}$$

Then

$$\Delta\frac{1}{r} = -4\pi\delta^3(x) \tag{8.40}$$

- It is illustrative to repeat this calculation for n=2.

$$\int_{r\geqslant\epsilon} \Delta\log\ r\ \phi(x)\ d^2x \equiv \int_{r\geqslant\epsilon} \log r\ \Delta\phi\ d^2x =$$
$$= \int_{r\geqslant\epsilon} \left\{\nabla\left(\log r\ \nabla\phi\right) - \Delta\log r\ \phi + \nabla\left(\nabla\log\ r\ \phi\right)\right\} =$$
$$= 0 + 0 + \int_0^{\epsilon} \frac{1}{r} 2\pi r\ dr\ \phi = 2\pi\ \phi(0) \tag{8.41}$$

- The derivative of a convergent sequence of differentiable functions also converges to the derivative of the limit, in the sense of distributions. For example, it is a fact that any series of the form

$$\sum_{-\infty}^{\infty} a_n\ e^{inx} \tag{8.42}$$

whose coefficients increase no faster that a power of n when $|n| \to \infty$ converges in the sense of distributions.

Proof. In fact such a series can always be obtained by sufficient number of term-by-term derivatives of another series of the type

$$\sum_{-\infty}^{\infty} \frac{a_n}{(in)^k}\ e^{inx} \tag{8.43}$$

QED

- A sequence of distributions

$$d_a, d_2 \dots d_n \tag{8.44}$$

is defined to converge to the distribution d if $\forall \phi(x) \in K$

$$\lim_{n\to\infty} \langle d_n, \phi \rangle = \langle d, \phi \rangle \tag{8.45}$$

It is a fact that every singular distribution is the limit of a sequence of regular functionals.

- Let us work out in detail an important example Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n} \sin\ nx \tag{8.46}$$

It is quite easy to check that it converges to the function

$$f(x) \equiv \frac{\pi - x}{2} \quad 0 < x < 2\pi \tag{8.47}$$

Consider the function over two periods

$$\begin{array}{ll} -\frac{\pi}{2} - \frac{x}{2} & -2\pi < x < 0 \\ \frac{\pi}{2} - \frac{x}{2} & 0 < x < 2\pi \end{array} \tag{8.48}$$

which has a 2π discontinuity at the origin. Its derivative is

$$\langle d', \phi \rangle \equiv -\int dx\ d(x)\ \phi'(x) = \int_{-\infty}^{0} dx\ \frac{\pi + x}{2}\ \phi'(x) - \int_{0}^{\infty} \frac{\pi - x}{2}\ \phi' =$$

$$= \frac{\pi}{2}\phi(0) - \frac{1}{2}\int_{-\infty}^{0} \phi\ dx + \frac{\pi}{2}\phi(0) - \frac{1}{2}\int_{0}^{\infty} \phi\ dx \tag{8.49}$$

Then

$$d' = \pi\ \delta(x) - \frac{1}{2} \tag{8.50}$$

Differentiating the whole series, we get a delta at each discontinuity

$$\sum \cos\ nx = -\frac{1}{2} + \pi \sum_{-\infty}^{\infty} \delta(x - 2\pi n) \tag{8.51}$$

Euler's formula now implies that

$$1 + e^{ix} + e^{2ix} + \dots + e^{-ix} + e^{-2ix} + \dots \equiv \sum_{-\infty}^{\infty} e^{inx} = 2\pi \sum_{-\infty}^{\infty} \delta(x - 2\pi n) \tag{8.52}$$

- A *delta convergent sequence* $\{f_i\}$, is one such that
  - 1.-For any $M > 0$ and for $|a| \leqslant M$ and $|b| \leqslant M$ the quantities

$$\left| \int_a^b f_j(\xi)\ d\xi \right| \leqslant C \tag{8.53}$$

    where C is independent of a, b, j, but it may depend on M.
  - 2.- For any fixed nonvanishing a and b

$$\lim_{j\to\infty} \int_a^b f_j(\xi)\ d\xi = 0 \quad a < b < 0 \quad \text{or} \quad 0 < a < b$$
$$\lim_{j\to\infty} \int_a^b f_j(\xi)\ d\xi = 1 \quad a < 0 < b \tag{8.54}$$

  There are many examples. One of them is

$$f_j = \frac{1}{\pi} \frac{\epsilon}{x^2 + \epsilon^2} \tag{8.55}$$

  as $\epsilon \to 0$.

8.1 Fourier transform

- The starting point in order to define the Fourier transform (FT) of distributions is Parseval's theorem. For regular distributions it asserts that

$$\int dx f^*(x) g(x)\ dx = \frac{1}{2\pi} \int dk\ \tilde{f}^*(k) \tilde{g}(k) \tag{8.56}$$

  where we have defined the FT as

$$\tilde{f}(k) \equiv \int_{-\infty}^{\infty} e^{ikx}\ f(x)\ dx \tag{8.57}$$

  This can be used to define a distribution in some space Z' (to be defined in a moment) for every distribution in K'. This is by definition the FT of the original distribution. Fourier transform establishes a one-to-one mapping

$$K \longleftrightarrow Z \tag{8.58}$$

  where Z is defined as follows. It consists of slowly increasing functions, that is, all entire functions $\psi(s)$ (where $s \equiv \sigma + i\tau$) such that

$$|s|^q |\psi(s)| \leqslant C e^{a|\tau|} \quad q = 0, 1, 2 \ldots \tag{8.59}$$

  where the constants a and C_q may depend on ψ.

- Let us compute the FT of Dirac's delta.

$$(\tilde{\delta}, \tilde{\phi}) \equiv 2\pi(\delta, \phi) = 2\pi\phi(0) = \int \tilde{\phi}(k) \equiv (1, \tilde{\phi}) \tag{8.60}$$

Ergo,

$$FT[\delta] = \tilde{\delta} = 1 \tag{8.61}$$

Also

$$(\tilde{1}, \tilde{\phi}) = 2\pi(1, \phi) = 2\pi \int \phi(x)\ dx = 2\pi\tilde{\phi}(0) = 2\pi(\delta, \phi) \tag{8.62}$$

$$FT[1] = \tilde{1} = 2\pi\delta \tag{8.63}$$

Similar computations lead to

$$\begin{aligned} FT[\delta^{(2m)}(x)] &= (-1)^m\ k^{2m} \\ FT[\delta^{(2m+1)}(x)] &= (-1)^{m+1}\ ik^{2m+1} \end{aligned} \tag{8.64}$$

- Consider now the function

$$f_\nu(x) \equiv \frac{1}{\pi}\ \frac{\sin\ \nu x}{x} \tag{8.65}$$

for $0 < \nu < \infty$). First of all,

$$\int_{-\infty}^{\infty} f_\nu(x)\ dx = 1 \tag{8.66}$$

It is easily done by using Cauchy's theorem to compute

$$\int \frac{e^{iz}}{z - i\epsilon} = 2\pi i e^{\epsilon} \tag{8.67}$$

Furthermore, for $0 < a < b$ the integrals

$$\int_a^b f_\nu(x)\ dx = \frac{1}{\pi} \int_{a\nu}^{b\nu} \frac{\sin\ y}{y} \tag{8.68}$$

go to zero as $\nu \to \infty$. Moreover, this same integral is bounded uniformly $\forall\nu$. Therefore we are dealing with a delta-convergent sequence.

$$\lim_{\nu\to\infty}\ f_\nu(x) = \delta(x) \tag{8.69}$$

Now observe that

$$\frac{\sin\ \nu x}{x} = \int_\nu^\nu \frac{e^{i\xi x}}{2\pi}\ d\xi \tag{8.70}$$

Then what we have just proved is that

$$\lim_{\nu\to\infty}\int_{-\nu}^{\nu} e^{i\xi x}\, d\xi = 2\pi\delta(x) \tag{8.71}$$

In fact it is a theorem that every integrable function $f(x)$ which does not grow at infinity faster that some power of $|x|$, has got a Fourier transform in the sense of distributions.

- Let us recall that the convolution of two ordinary functions (regular distributions) is defined as

$$\langle f * g, \phi(x)\rangle \equiv \int f(\xi)g(x-\xi)d\xi dx = \int d\xi d\eta f(\xi)g(\eta)\phi(\xi+\eta) \tag{8.72}$$

Let us examine the convolution of singular distributions. It is natural to generalize this last version of convolution to

$$\langle t_1 * t_2, \phi\rangle \equiv \langle t_1(x)t_2(y), \phi(x+y)\rangle \tag{8.73}$$

It is plain that

$$t_1 * t_2 = t_2 * t_1 \tag{8.74}$$

as well as

$$t_1 * (t_2 * t_3) = (t_1 * t_2) * t_3 \tag{8.75}$$

Let us now compute the convolution of Dirac's delta.

$$\langle \delta * t, \phi\rangle = \langle \delta(x)t(y), \phi(x+y)\rangle = \langle t(y), \phi(y)\rangle = \langle t, \phi\rangle \tag{8.76}$$

That is, Dirac's delta acts as a unit with respect to convolutions

$$\delta * t = t \tag{8.77}$$

Also,

$$\partial\,(t * s) = (\partial t) * s = t * (\partial s) \tag{8.78}$$

Indeed

$$\langle \partial\,(t * s)\,, \phi\rangle = -\langle t * s, \partial\phi\rangle \equiv -\langle t(x), (\langle s(y), \partial\phi(x+y)\rangle)\rangle = \langle \partial t * s, \phi\rangle \tag{8.79}$$

- Consider a linear differential equation with constant coefficients

$$P(\partial)\ y = J(x) \tag{8.80}$$

Define an *elementary solution* or *Green function* as

$$P(\partial)\ G = \delta \tag{8.81}$$

Then we can write solutions of our PDE as

$$y = J * G \tag{8.82}$$

because

$$P(\partial)\ y = J\ P(\partial)\ G = J * \delta = J \tag{8.83}$$

- Consider the ODE

$$\left(\frac{d^2}{dt^2} + \omega^2\right) \; x(t) = j(t) \tag{8.84}$$

Let us first show that the function

$$G(t) \equiv \frac{e^{i\omega|t|}}{2i\omega} \tag{8.85}$$

is an elementary solution. Indeed

$$\frac{d}{dt}G(t) = i\omega G(t).\sigma(t) \tag{8.86}$$

and

$$\frac{d^2}{dt^2}G(t) = \left(-\omega^2 + i\omega 2\delta(t)\right) G(t) \tag{8.87}$$

Incidentalt, the same thing happens with

$$G_R(t) = \theta(t)\frac{e^{i\omega t}}{2i\omega} \tag{8.88}$$

as well as

$$G_R(t) = -\theta(-t)\frac{e^{i\omega t}}{2i\omega} \tag{8.89}$$

- We have seen previously that

$$G(x) = -\frac{1}{(n-2)\Omega_n} \; \frac{1}{r^{n-2}} \quad (n > 2)$$
$$G(x) = -\frac{1}{2\pi} \; \log \; \frac{1}{r} \quad (n = 2) \tag{8.90}$$

is an elementary solution of the laplacian. This leads to Poisson's formula for the newtonian potential due to a density $\rho(x)$

$$\int V(x) = \int d\xi \rho(\xi) G(x-\xi) \tag{8.91}$$

That is,

$$V(x,y,z) = -\frac{1}{4\pi}\int \frac{\rho(\xi_1,\xi_2,\xi_3)}{\sqrt{(\xi_1-x)^2 + (\xi_2-y)^2 + (\xi_3-z)^2}} \; d\xi_1 d\xi_2 d\xi_3 \tag{8.92}$$

- We know that every periodic locally summable fuction $f(\theta)$ can be written in the form of a Fourier series

$$f(\theta) = \sum_{-\infty}^{\infty} c_n \; e^{in\theta} \tag{8.93}$$

Taking the Fourier transform term by term we easily get

$$\tilde{f}(k) = \sum_{-\infty}^{\infty} c_n \; \delta(k+n) \tag{8.94}$$

- Let us derive the marvelous Poisson summation formula. Starting from

$$\sum e^{inx} = 2\pi \sum \delta(x - 2\pi n) \tag{8.95}$$

we easily get

$$f(x) \equiv \sum e^{in\pi x/L} = 2\pi \sum \delta(\pi x/L - 2\pi n) = 2L \sum \delta(x - 2nL) \tag{8.96}$$

Its Fourier transform

$$\tilde{f}(k) \equiv \int dx e^{-2\pi ixk} \sum \delta(x - 2nL = 0 \sum_n e^{-2\pi ik(2nL)} = \frac{1}{2L} \sum \delta(k - \frac{n}{2L}) \tag{8.97}$$

Now, the transform of a gaussian

$$g(x) \equiv e^{-x^2} \tag{8.98}$$

is another gaussian

$$\tilde{f}(k) \equiv \sqrt{\pi} e^{-\pi^2 x^2} \tag{8.99}$$

Let us apply Parseval's theorem to this couple of functions

$$\int f(x) g(x) dx = \int \tilde{f}(k = \tilde{g}(-k) dk \tag{8.100}$$

We get Pôisson's formula

$$\sum e^{-4m^2 L^2} = \frac{1}{2L} \sum e^{-\frac{\pi^2 m^2}{4L^2}} \tag{8.101}$$

This has got plentiful physical applications.

8.2 Distributions on submanifolds.

- Consider

$$\delta(f(x)) \tag{8.102}$$

It is clear that in the simplest case

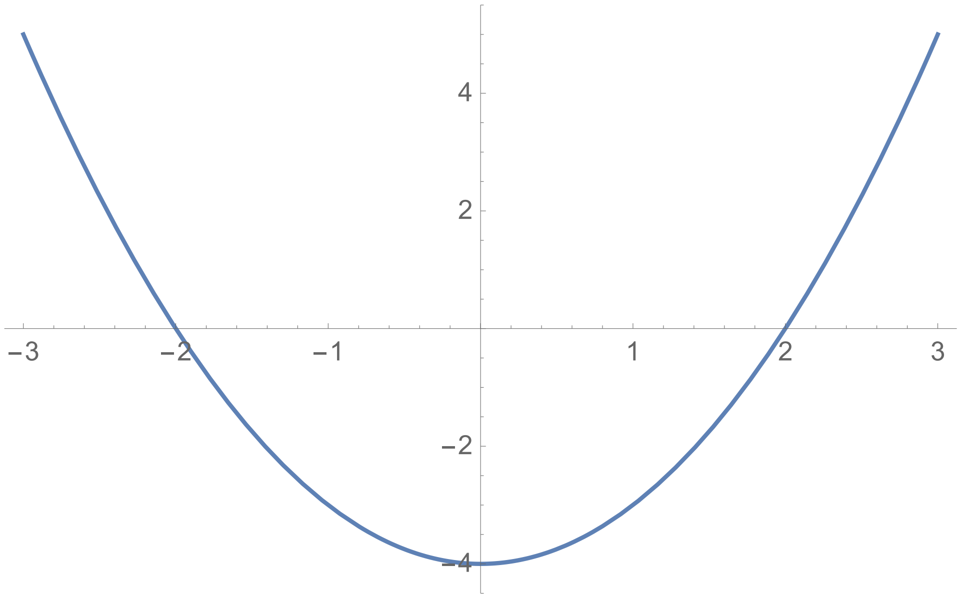
$$\int \delta(f(x)) g(x) dx = \int dt \delta(t) g(x(t)) \tag{8.103}$$

where

$$t \equiv f(x) \Rightarrow dt = f'(x) dx \tag{8.104}$$

In some case, some care must be taken. For example, consider

$$\delta(x^2 - m^2) \tag{8.105}$$



Descriptio 8.1: Exmaple.

Then

$$\int_{-\infty}^{\infty} dx = \int_{\infty}^{-\mu} \frac{dt}{-2\sqrt{t+\mu^2}} + \int_{-\mu}^{\infty} \frac{dt}{2\sqrt{t+\mu^2}} \tag{8.106}$$

Then

$$\int_{-\infty}^{\infty} dx\delta(x^2-\mu^2)\ g(x) = \frac{g(-\mu)}{2\mu} + \frac{g(\mu)}{2\mu} \tag{8.107}$$

- Every functional concentrated on a point is a linear combination of the delta function and its derivatives.
- Consider codimension one hypersurfaces given by

$$P(x_1 \ldots x_n) = 0 \tag{8.108}$$

We sould like to define such things as $\delta(P)$, etc. Let us assume that

$$\partial_\mu P|_P \neq 0 \tag{8.109}$$

Let us first define the *Leray form* ω as such that

$$dP \wedge \omega = d(vol) \tag{8.110}$$

Provided $\frac{\partial P}{\partial x^1} \neq 0$, there is always some coordinate system such that the equation of the surface reads

$$u^1 = P \quad u^2 = x^2 \quad \ldots \quad u^n = x^n \tag{8.111}$$

Then

$$d(vol) \equiv dx^1 \wedge dx^2 \wedge \ldots dx^n = \det\left(\frac{\partial x}{\partial u}\right)\; du^1 \wedge du^2 \ldots \wedge du^n =$$
$$\frac{1}{\frac{\partial P}{\partial x^1}}\; du^1 \wedge du^2 \ldots \wedge du^n \tag{8.112}$$

Ergo,

$$\omega = \frac{1}{\frac{\partial P}{\partial x^1}}\; dx^2 \ldots \wedge dx^n \tag{8.113}$$

In fact it can be shown that ω has an intrinsic meaning. It is only natural to define

$$\langle \delta(P), \phi \rangle \equiv \int_{P=0} \phi(x)\; \omega \tag{8.114}$$

As an example, let us work out

$$\delta(xy - c) \tag{8.115}$$

in two dimensions. Using the coordinates

$$u_1 = xy - c$$
$$u_2 = y \tag{8.116}$$

Then

$$\omega = \frac{dy}{y} \tag{8.117}$$

because

$$(xdy + ydx) \wedge \frac{dy}{y} = dx \wedge dy \tag{8.118}$$

It is a fact

$$\langle \delta\left(xy - c\right), \phi(x,y) \rangle = \int \phi\left(\frac{c}{y}, y\right)\; \frac{dy}{y} \tag{8.119}$$

Let us work out another example, namely $\delta(r - R)$ The Leray form coincides with the euclidean area element $R^{n-1}\; d\Omega$

$$\langle \delta(r - R), \phi \rangle = R^{n-1} \int_{r=R} \phi\; d\Omega \tag{8.120}$$

It is to be noted that this vanishes when $R = 0$, unless ϕ diverges in an appropiate way (in which case $(\delta^n(\vec{x}), \phi)$ would diverge). If it were instead $\delta(r^2 - c^2)$, we could define

$$u_1 = r^2 - R^2$$
$$u_2 = \theta_1$$
$$\ldots$$
$$u_n = \theta_{n-1} \tag{8.121}$$

This means that

$$\omega = \frac{1}{2r}\, d\Omega = \frac{1}{2R}\, R^{n-1} d\Omega \tag{8.122}$$

and

$$\langle \delta(r^2 - R^2), \phi \rangle = \frac{R^{n-2}}{2} \int_{r=R} \phi \, d\Omega \tag{8.123}$$

If we define a Heaviside function

$$\begin{aligned} \theta(P) = 1 &\Longleftrightarrow P(x) \geqslant 0 \\ \theta(P) = 0 &\Longleftrightarrow P(x) < 0 \end{aligned} \tag{8.124}$$

Then it can be shown that

$$\theta'(P) = \delta(P) \tag{8.125}$$

- Let us now introduce a family of forms that depend both on $\phi \in K$ and on P.

$$\begin{aligned} &\omega_0(\phi) \equiv \phi \, \omega \\ &d\omega_0 \equiv dP \wedge \omega_1(\phi) \\ &\quad \ldots \\ &d\omega_{k-1}(\phi) \equiv dP \wedge \omega_k(\phi) \\ &\quad \ldots \end{aligned} \tag{8.126}$$

Then we define the derivatives of the delta function as

$$\langle \delta^{(k)}(P), \phi \rangle \equiv (-1)^k \int_{P=0} \omega_k(\phi) \tag{8.127}$$

For example, let us compute $\delta^{(k)}(r - R)$. Using the same coordinates as before, we recover

$$\omega = r^{n-1}\, d\Omega \tag{8.128}$$

Then

$$\omega_0 = \phi \, r^{n-1}\, d\Omega \tag{8.129}$$

and

$$\omega_1(\phi) = \frac{\partial \left(\phi \, r^{n-1}\right)}{\partial r}\, d\Omega \tag{8.130}$$

In fact

$$\omega_k(\phi) = \frac{\partial^k \left(\phi \, r^{n-1}\right)}{\partial r^k}\, d\Omega \tag{8.131}$$

Then

$$\langle \delta^{(k)}(r - R), \phi \rangle = \frac{(-1)^k}{R^{n-1}} \int_{r=R} \frac{\partial^k \left(\phi \, r^{n-1}\right)}{\partial r^k}\, d\Omega \tag{8.132}$$

9

Finite groups.

A *group* G is a set with a product

$$G \times G \to G \tag{9.1}$$

such that

1.- $g_1, g_2 \in G \to g_1 g_2 \in G$
2.- The composition law is associative: $g_1 (g_2 g_3) = (g_1 g_2) g_3$.
3.- There is a unit $e \in G$, such that $eg = ge = g \ \forall g \in G$.
4.- Every element has got an inverse $g^{-1} g = g g^{-1} = e$

- A group is finite if the set has a finite number of elements. This is called the *order* of the group, $|G|$. *Cyclic* groups are particular instances such that

  $$\forall g \in G, \quad g^n = 1 \tag{9.2}$$

  for some integer n. For example, $\mathbb{Z}_3$ such that $a^3 = b^3 = e$, has got the multiplication table

	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

  (9.3)

  An *abelian* group obeys

  $$gh = hg \quad \forall g, h \in G \tag{9.4}$$

- A *representation* is a mapping

  $$g \in G \to D(g) \tag{9.5}$$

  where $D(g)$ is a linear operator acting in some linear space V and such that

– $D(e) = 1$

– $D(g_1)D(g_2) = D(g_1 g_2)$

The *dimension* of the representation is the dimension of the linear space V. For example a three dimensional representation of the cyclic group $\mathbb{Z}_3$ in $\mathbb{R}^3$ is

$$\begin{aligned} e \to D(e) &\equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ a \to D(a) &\equiv \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\ b \to D(b) &\equiv \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \end{aligned} \tag{9.6}$$

- This is in fact the *adjoint* representation. We associate the elements of the group with a basis in V_N, a vector space of dimension equal to the order of the group. For example,

$$\begin{aligned} e \leftrightarrow e(e) \equiv e_1 &\equiv \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ a \leftrightarrow e(a) \equiv e_2 &\equiv \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ b \leftrightarrow e(b) \equiv e_3 &\equiv \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{aligned} \tag{9.7}$$

Then we define

$$D^{ad}(g_1)e(g_2) \equiv e(g_1 g_2) \tag{9.8}$$

With the natural definition

$$D_{ij}(g) \equiv e_i^T D(g) e_j \tag{9.9}$$

It follows

$$\begin{aligned} & D_{ij}(gh) \equiv e_i^T D(gh) e_j = e_i^T D(g) D(h) e_j = e_i^T D(g) \sum_{k=1}^{3} e_k e_k^T D(h) e_j = \\ & = \sum_k D_{ik}(g) D_{kj}(h) \end{aligned} \tag{9.10}$$

If we change the basis of the linear space on which the representation acts
$$e_i \rightarrow \sum_j \tilde{e}_i \equiv S_i^j \; e_j \tag{9.11}$$
Then
$$\begin{aligned}
&\tilde{e}_i \rightarrow \tilde{D}_i^j \; \tilde{e}_j \\
&e_i \rightarrow D_i^j \; e_j \\
&\left(S^{-1}\right)_i^l \; \tilde{e}_l \rightarrow D_i^l \; \left(S^{-1}\right)\big)_j^k \; \tilde{e}_k \\
&\tilde{e} \rightarrow S D S^{-1} \; \tilde{e}
\end{aligned} \tag{9.12}$$
That is
$$\tilde{D} = S D S^{-1} \tag{9.13}$$
It is said that D and $\tilde{D}$ are equivalent representations.

- It is fact that all representations of finite groups are equivalent to unitary representations, that is, one such that
$$D D^{+} = D^{+} D = 1 \tag{9.14}$$
This is easy to show, by considering the positive semidefinite matrix
$$S \equiv \sum_{g \in G} D^{+}(g) D(g) = U^{-1} \Lambda U \tag{9.15}$$
where
$$\Lambda = \mathrm{diag}\,(\lambda_1 \ldots \lambda_n) \tag{9.16}$$
and all eigenvalues $\lambda_i \geqslant 0$. Actually all $\lambda_i > 0$ because if it were one zero eigenvalue, then there must be a vector such that
$$S v = 0 = v^{+} S v = \sum_{g \in G} ||D(g) v||^2 \tag{9.17}$$
which is impossible, because in particular $D(e) = 1$. This means that there is a matrix
$$S^{1/2} \equiv U^{-1} \Lambda^{\frac{1}{2}} U \tag{9.18}$$
and defining
$$\tilde{D}(g) \equiv S^{1/2} D(g) S^{-1/2} \tag{9.19}$$
we are done, because
$$\begin{aligned}
&\tilde{D}^{+}(g)\tilde{D}(g) = S^{-1/2}\left[D^{+}(g) S D(g)\right] S^{-1/2} = S^{-1/2}\left[D^{+}(g)\left(\sum_{h \in G} D^{+}(h) D(h)\right) D(g)\right] S^{-1/2} = \\
&= S^{-1/2}\left[\sum_{h \in G} D^{+}(hg) D(hg)\right] S^{-1/2} = S^{-1/2}\left[\sum_{k \in G} D^{+}(k) D(k)\right] S^{-1/2} = S^{-1/2} S S^{-1/2} = 1
\end{aligned} \tag{9.20}$$

- A representation is *reducible* if it has an invariant subspace. If $P^2 = P$ is the projector on this subspace, the condition is

$$\forall g \in G \quad PD(g)P = D(g)P \tag{9.21}$$

In the case of the regular representation of $\mathbb{Z}_3$, there is an invariant subspace with projector

$$P = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \tag{9.22}$$

Actually what happens is that

$$\forall g \in \mathbb{Z}_3 \quad D(g)P = P \tag{9.23}$$

The restriction of the representation to the subspace is itself a representation (in this case, the *trivial representation*). When this is not the case, the representation is *irreducible*. A representation is *completely reducible* if it can be written in block diagonal form as the direct sum of irreducible subrepresentations

$$D(g) = D_1(g) \oplus D_2(g) \oplus \ldots \oplus D_n(g) \tag{9.24}$$

Again, for finite groups, any representation is completely reducible. In our favorite $\mathbb{Z}_3$ example, defining

$$S = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \alpha^2 & \alpha \\ 1 & \alpha & \alpha^2 \end{pmatrix} \tag{9.25}$$

where

$$\alpha = e^{\frac{2\pi i}{3}} \tag{9.26}$$

Then

$$\begin{aligned} \tilde{D}(e) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \tilde{D}(a) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha^2 \end{pmatrix} \\ \tilde{D}(b) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha^2 & 0 \\ 0 & 0 & \alpha \end{pmatrix} \end{aligned} \tag{9.27}$$

In fact, every representation of a finite group is completely reducible. Let us work with the unitary form of the representation. It it is reducible, it means that there is a projector P such that

$$\forall g \quad PD(g)P = D(g)P \tag{9.28}$$

Taking the adjoint

$$\forall g \quad PD(g)^{+}P = PD(g)^{+} \tag{9.29}$$

But $D^{+}(g) = D^{-1}(g) = D(g^{-1})$ and g^{-1} runs over G as well as g does (because for every g, there is a unique g^{-1}. To summarize, we claim that

$$\forall g \quad PD(g)P = PD(g) \tag{9.30}$$

It follows that

$$\forall g \quad (1-P)\, D(g)(1-P) = D-PD-DP+PDP = D(g)(1-P) \tag{9.31}$$

and $1 - P$ also projects into an invariant subspace.

9.1 Normal subgroups

- Given a subgroup H, we define a *(left) coset*

$$gH \tag{9.32}$$

as the set of all elements

$$gh \quad \forall g \in G \quad \forall h \in H \tag{9.33}$$

It is plain that every element in G must be in one (and only one) coset, because we first pick $g_1 \notin H$ and construct the $|H|$ elements $g_1 H$ those are all different. Then we pick some $g_2 \notin H$, $g_2 \notin g_1 H$, and so on. This proves the *theorem of Lagrange.*

$$|gG| \times |H| = |G| \tag{9.34}$$

There is an equivalence relationship when $g_1, g_2 \in g_1 H$

$$g_1 \sim g_2 \Leftrightarrow g_1 = g_2 h, \quad h \in H \tag{9.35}$$

Then the quotient

$$G/\sim \tag{9.36}$$

is the set of those left cosets.

It is important to distinguish that from another equivalence relatioship

$$g_1 \sim g_2 \Leftrightarrow \exists h \in G, \quad h g_1 = g_2 h \tag{9.37}$$

The set of those (conjugacy) classes is

$$G/\sim \tag{9.38}$$

Those subgroups such that

$$\forall g \in G \quad gH = Hg \tag{9.39}$$

are dubbed *normal.* In this case, the coset space is also a group, because

$$(g_1 H)(g_2 H) = g_1 H g_1^{-1} g_1 g_2 H = H g_1 g_2 H = g_1 g_2 g_2^{-1} g_1^{-1} H g_1 g_2 H = g_1 g_2 H H = g_1 g_2 H \tag{9.40}$$

G/H is called the *factor group* of G by H.

- The *center* of a group, Z is the set of all elements that commute with all elements of the group,

$$z \in Z \leftrightarrow zg = gz \quad \forall g \in G \tag{9.41}$$

The center is an abelian invariant subgroup.

- Consider the permutation group S_3. Permutation groups are very important for a variety of reasons. One of them is *Cayley's theorem*: Eevery finite group $|G| = n$ is isomorphic so a subgroup of S_n. The elements of S_3 are

$$\begin{aligned} & e \\ & a_1 \equiv (123) \\ & a_2 \equiv (321) \\ & a_3 \equiv (12) \\ & a_4 \equiv (23) \\ & a_5 \equiv (31) \end{aligned} \tag{9.42}$$

The multiplicaction table is given by

e	e	a1=(123)	a2=(132)	a3=(12)	a4=(23)	a5=(13)
a1=(123)	a1	a2	e	a5	a3	a4
a2=(132)	a2	e	a1	a4	a5	a3
a3=(12)	a3	a4	a5	e	a1	a2
a4=(23)	a4	a5	a3	a2	e	a1
a5=(13)	a5	a3	a4	a1	a2	e

(9.43)

First of all, let us notice that the subset $\{e, a1, a2\}$ is a subgroup, the alternating group A_3, consisting on all even premutations, and which in this case happens to be isomorphic to $\mathbb{Z}_3$. Moreover

$$\begin{aligned} a_3\ A_3 &= \{a_3, a_4, a_5\} \\ a_4\ A_3 &= \{a_4, a_5, a_3\} \\ a_5\ A_3 &= \{a_5, a_3, a_4\} \end{aligned} \tag{9.44}$$

- Coming back to our S_3 example, $A_3 \sim \mathbb{Z}_3$ is a normal subgroup, because

$$\begin{aligned} A_3\ a_3 &= \{a_3, a_5, a_4\} \\ A_3\ a_4 &= \{a_4, a_3, a_5\} \\ A_3\ a_5 &= \{a_5, a_4, a_3\} \end{aligned} \tag{9.45}$$

Note however that there is another subgroup, $H \equiv \{e, a_4\}$ (remember that $a_4^2 = e$) which is not normal.

$$a_5 H = \{a_5, a_2\} \neq H a_5 = \{a_5, a_1\} \tag{9.46}$$

- The *conjugacy classes* are sets such that if they contain an element s, they also contain all its conjugates

$$S \equiv \{g^{-1} s g \quad \forall g \in G\} \tag{9.47}$$

It is a plain that for such a set

$$g^{-1} S g = S \tag{9.48}$$

e is always a conjugacy class. In S_3, taking into account that

$$\begin{aligned} &a_1^{-1} = a_2 \\ &a_3^2 = a_4^2 = a_5^2 = e \end{aligned} \tag{9.49}$$

the conjugacy classes are: frist the two three-cycles

$$\{a_1, a_2\} \tag{9.50}$$

and then the three two-cycles

$$\{a_3, a_4, a_5\} \tag{9.51}$$

We note here a general trend in the symmetric group: conjugate permutations have the same cycle structure; in particular the permutations in the same class are either all even or else all odd.

- For fixed $g \in G$, the mapping

$$h \in G \to ghg^{-1} \in G \tag{9.52}$$

is an *inner automorphism*. This is plain, because

$$\left(gg_1g^{-1}\right)\left(gg_2g^{-1}\right) = gg_1g_2g^{-1} \tag{9.53}$$

Besides, it is 1-1, because if

$$gg_1g^{-1} = gg_2g^{-1} \Rightarrow g_1 = g_2 \tag{9.54}$$

Outer automorphisms are all those automorphisms that are not inner.

9.2 Schur's lemma

- If there are two inequivalent irreducible representations D_1 and D_2 of a group G, such that there is a matrix A that obeys

$$D_1(g)A = AD_2(g) \quad \forall g \in G \tag{9.55}$$

then it follows that $A = 0$. In fact, assume there is a vector such that $Av = 0$. Then there is a projector P onto the subspace that annihilates A on the right. This subspace is invariant under D_2, because

$$AD_2P = D_1AP = 0 \forall g \in G \tag{9.56}$$

But D_2 is irreducible, so that $P = 1$ and $A = 0$. If A annihilates one state, it must annihilate them all.

If no vector annihilates A on either side, then it must be an invertible square matrix, Then

$$D_1 = AD_2A^{-1} \tag{9.57}$$

and the two representations are equivalent.

Another proof is as follows. Define

$$D(e_i) \equiv e_jD_{ji} \tag{9.58}$$

$$D^1_{ij}A_{ja} = A_{ib}D^2_{ba} \tag{9.59}$$

then

$$e_iD^1_{ij}A_{ja} = D^1\left(e_jA_{ja}\right) = e_iA_{ib}D^2_{ba} \tag{9.60}$$

Denoting

$$E_b \equiv e_iA_{ib} \tag{9.61}$$

this shows that

$$D^1(E_a) = E_bD^2_{ba} \tag{9.62}$$

which is not possible if D^1 is irreducible.

- On the other hand, it is a fact that if there is a finite dimensional irreducible representation D such that

$$D(g)A = AD(g) \quad \forall g \in G \tag{9.63}$$

then $A = 1$. This is obvious, because any finite dimensional matrix, A, has at least one eigenvalue Then

$$D(g)\,(A - \lambda 1) = (A - \lambda)\,D \tag{9.64}$$

and the matrix

$$A - \lambda 1 \tag{9.65}$$

has a null eigenvector. Then the former argument shows that

$$A - \lambda 1 = 0 \tag{9.66}$$

One consequence of Schur's lemma is that once the form of D is fixed, there is no further freedom to make nontrivial similarity transformations on the states.

9.3 Characters

Given an arbitrary matrix, let us say, X, consider the matrix

$$A \equiv \sum_{g\in G} D(g)\ X\ D(g^{-1}) \tag{9.67}$$

where $D(g)$ is an matrix irrep of G with dimension d_R. It is fact of life that

$$[D(h), A] = 0 \quad \forall h \in G \tag{9.68}$$

Indeed

$$D(h)A \equiv D(h)\sum_{g\in G} D(g)\ X\ D(g^{-1}) = \sum_{g\in G} D(hg)XD(g^{-1}) = \sum_{g\in G} D(hg)XD(g^{-1}h^{-1})D(h) = \\ = \sum_{g\in G} D(g)\ X\ D(g^{-1})D(h) \equiv AD(h) \tag{9.69}$$

Schur's now implies that

$$A = \lambda 1 \tag{9.70}$$

Let us now choose as starting point the particular matrix

$$X \equiv (E_{lm})_{ij} \equiv \delta_{il}\delta_{jm} \tag{9.71}$$

Then

$$\sum_{g\in G} D_{il}(g)D_{mj}(g^{-1}) = \lambda_{lm}\delta_{ij} \tag{9.72}$$

Taking the trace δ^{ij} we learn that

$$|G|\delta_{lm} = \lambda_{lm} d_R \tag{9.73}$$

This proves the *orthogonality relation*

$$\sum_{g\in G} D_{il}(g)D_{mj}(g^{-1}) = \frac{|G|}{d_R}\,\delta_{lm}\delta_{ij} \tag{9.74}$$

Let us now repeat the same procedure using two different representations, id est,

$$B \equiv \sum_{g\in G} D^2(g)XD^1(g^{-1}) \tag{9.75}$$

It is plain that

$$D^2(h)B = BD^1(h) \tag{9.76}$$

Schur's tell us that

$$B = 0 \tag{9.77}$$

and using

$$X \equiv E_{lm} \tag{9.78}$$

we learn that

$$\sum_{g\in G} D^2_{il}(g)\ D^1_{mj}(g^{-1}) = 0 \tag{9.79}$$

We can characterize both orthonomality relations in the following way. Consider the set of all irreps

$$D^{\mu}_{ij}(g) \tag{9.80}$$

This can be considered as a $|G|$-dimensional vector for every value of (μ, i, j). These vectors are orhogonal in the sense that

$$\sum_{g\in G} D^{\mu}_{il}(g)\ D^{\nu}_{mj}(g^{-1}) = \frac{|G|}{d_R}\,\delta^{\mu\nu}\delta_{lm}\delta_{ij} \tag{9.81}$$

For each irrep, μ there are d_R^2 mutually orthogonal vectors in $K_{|G|}$. This is possible only provided that

$$\sum_{\mu} d_{\mu}^2 \leqslant |G| \tag{9.82}$$

- The *character* of a given irrep is just the trace

$$\chi^\mu(g) \equiv Tr\ D^\mu(g) \equiv \sum_i D_{ij}(g) \tag{9.83}$$

The character is a *class function* because

$$tr\ D = tr\ hDh^{-1} \tag{9.84}$$

From our master relationship we learn that

$$\sum_{g\in G} \chi^\mu(g)\chi^\nu(g^{-1}) = |G|\ \delta^{\mu\nu} \tag{9.85}$$

Assume the classes of G are $K_1 \ldots K_C$; that is that there are C classes with number of elements

$$\sum_{i=1}^{C} d_{K_i} \ = \ |G| \tag{9.86}$$

Then restricting to unitaries

$$D(g)^+ D(g) = 1 \tag{9.87}$$

implies

$$\overline{\chi(g)} = \chi(g^{-1}) \tag{9.88}$$

$$\sum_i \chi_i^\mu\ \overline{\chi_i^\nu}\ d_{K_i} \ = \ d_G\ \delta^{\mu\nu} \tag{9.89}$$

This means that the number of irreps must be smaller or equal to the number of classes.

We can use the orthogonality relations to decompose the adjoint representation.

First of all, assume a reducible representation

$$D = D_1 \oplus D_2 \oplus \ldots \oplus D_k \tag{9.90}$$

so that

$$\chi = \sum \chi_j \tag{9.91}$$

The number of times the irrep (i) appears in this decomposition is equal to

$$\langle \chi | \chi_i \rangle \tag{9.92}$$

Remember that

$$D(s) e_t \equiv e_{st} \tag{9.93}$$

Now if $s \neq 1$ then $st \neq t$, so that the diagonal terms in the matrix $D(s)$ just vanish. Then

$$\chi(s \neq e) = 0 \tag{9.94}$$

and

$$\chi(e) = d_G \tag{9.95}$$

Then

$$\langle \chi^{ad} | \chi_i \rangle \equiv \frac{1}{d_G} \sum_{t \in G} \chi^{ad}(t^{-1}) \chi_i(t) = d_i \tag{9.96}$$

Ergo

$$\sum_i d_i^2 = d_G \tag{9.97}$$

Assume now a *central function* ($f(g) = g(hgh^{-1}) \forall h \in G$). Define a matrix in V aasociated to an irrep, D^R

$$D_f^R \equiv \sum_{t \in G} f(t) D^R(t) \tag{9.98}$$

It is plain that

$$[D_f^R, D(h)] = 0, \forall h \in G \tag{9.99}$$

Then by Schur's lemma

$$D_f^R = \lambda 1 \tag{9.100}$$

We can compute λ

$$d_R \lambda = \sum_{t \in G} f(t) \chi(t) \tag{9.101}$$

We now claim that the characters $\chi_1 \ldots \chi_h$ yield an orthonormal basis of H, the space of central functions on the group.

We need to prove that any element of H orthogonal to all the characters is zero.

Assume

$$\langle f | \chi^\mu \rangle = 0 \quad \forall \mu \tag{9.102}$$

This shows that

$$\lambda = 0 \tag{9.103}$$

for all irreps. Then

$$D_f^\mu = 0 \tag{9.104}$$

for all representations direct sum of irreps. Let us work this out for the regular representation.

$$0 = D_f^{ad} e_1 \equiv \textstyle\sum_{t \in G} f(t) D(t) e_1 = \sum_{t \in G} f(t) e_t \tag{9.105}$$

It follows that

$$f(t) = 0 \quad \forall t \in G \tag{9.106}$$

QED.

It follows that the number of irreps is equal to the number of classes.

In conclusion,

$$\sum_R d_R^2 = |G| \tag{9.107}$$

Let us check this in the abelian group $\mathbb{Z}_N \equiv z_0 \equiv e, z_1 \ldots z_{N-1}$

$$z_i z_j = z_{i+j (\text{mod } N)} \tag{9.108}$$

The irreps are given by

$$D_n(a_k) = e^{\frac{2\pi k}{N} i} \tag{9.109}$$

The orthogonality relationship means now that

$$\frac{1}{N} \sum_{k=0}^{N-1} e^{-\frac{2\pi n_1 k}{N} i} e^{\frac{2\pi n_2 k}{N} i} = \delta_{n_1 n_2} \tag{9.110}$$

It is actually vary easy to prove that all irreps of an abelian group are one-dimensional. Every element is a conjugacy class by itself. Then the number of irreps is equal to the order of the group. Each of them is got to be one-dimensional

- Let us repeat the former theorem in a different language. Given any class function, $F(g)$, we can expand it as

$$F(g) = \sum_{ajk} F_{ajk} D^a(g)_{jk} \tag{9.111}$$

We can actually write

$$\begin{aligned} F(g) &= \frac{1}{|G|} \sum_{h \in G} F(h^{-1} g h) = \frac{1}{|G|} \sum_{h \in G} \sum_{ajk} F_{ajk} D^a(h^{-1} g h)_{jk} = \\ &= \frac{1}{|G|} \sum_{h \in G} \sum_{ajk} F_{ajk} D^a(h^{-1})_{j j_1} \; D^a(g)_{j_1 j_2} \; D^a(h)_{j_2 k} = \sum \frac{1}{d_a} F_{ajk} \; D^a(g)_{j_1 j_2} \; \delta_{j_1 j_2} \delta_{jk} = \\ &= \sum_{ajk} \; \frac{1}{d_a} \; d_a F_{ajj} \; D^a_{kk}(g) = \sum_{aj} \frac{1}{d_a} f^a \chi_a(g) \end{aligned} \tag{9.112}$$

This means that the number of irreps is actually equal to the number of conjugacy classes. If we label conjugacy classes by α , $|\alpha|$ being the number of elements of the class α, then defining the square matrix

$$V_{\alpha a} \equiv \sqrt{\frac{|\alpha|}{|G|}} \chi_{D_a} \; (g_\alpha) \tag{9.113}$$

the orthogonality relation

$$\sum_{g\in G} \chi^*_{D_a}(g)\ \chi_{D_b}(g) = |G|\ \delta_{ab} \tag{9.114}$$

means that

$$VV^+ = 1 \quad \therefore \quad V^+V = 1 \tag{9.115}$$

To be specific

$$\sum_a \chi^*_{D_a}(g_\alpha)\chi_{D_a}(g_\beta) = \frac{|G|}{|\alpha|}\ \delta_{\alpha\beta} \tag{9.116}$$

Given any rep, it will containg all irreps D_a some number of times, m_a^D. This can be easily computed using

$$\sum_{g\in G} \chi_{D_a}(g)^*\chi_D(g) = |G| m_a^D \tag{9.117}$$

For example, the characters of the adjoint are

$$\begin{aligned} \chi(e) &= |G| \\ \chi(g \neq e) &= 0 \end{aligned} \tag{9.118}$$

Then

$$m_a^D = \chi_a(e) = |D_a| \tag{9.119}$$

Each irrep appears in the adjoint a number of times equal to its dimension.

Consider again the case of S_3. In this case $|G| = 6$. We know the one dimensional irrep

$$D(g) = 1 \tag{9.120}$$

It is such that

$$\chi_0(g) = 1 \tag{9.121}$$

Now

$$1 + \sum_{\mu\neq 0} n_\mu^2 = 6 \tag{9.122}$$

This means that

$$n_\mu = 1, 2 \tag{9.123}$$

Let us try and determine the characters using orthogonality

μ	{e}	$\{a_1, a_2\}$	$\{a_3, a_4, a_5\}$
0	1	1	1
1	1	1	-1
2	2	-1	0

(9.124)

Let us prove that given an arbitrary (in general reducible) representation, D, the operator

$$P_a \equiv \frac{d_a}{d_G} \sum_{g\in G} \bar{\chi}_{D_a}(g)\ D(g) \tag{9.125}$$

is a projector onto the subspace that transforms under the rep D_a.

In fact taking the trace of the first orthogonality relation, we learn that

$$\frac{d_a}{d_G} \sum_{g\in G} \chi^*_{D_a}(g)\ D^b_{lm}(g) = \delta^{ab} d_{lm} \tag{9.126}$$

Let us see how this works in the three dimensional rep of S_3

$$\begin{aligned}
e \to D(e) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
a_1 \to D(a_1) &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\
a_2 \to D(a_2) &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \\
a_3 \to D(a_3) &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
a_4 \to D(a_4) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
a_5 \to D(a_5) &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}
\end{aligned} \tag{9.127}$$

With this notation it is plain that

$$D(g)|j\rangle = \sum_k |k\rangle\langle k|D(g)|j\rangle \equiv \sum_k |k\rangle D_{kj}(g) \tag{9.128}$$

Let us now emply this three-dimensional rep to determine the projection operators

$$
\begin{aligned}
P_0 &= \frac{1}{6}\left(D_3(e) + \sum_{j=1}^{j=5} D_3(a_j)\right) = \frac{1}{3}\begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix}\\
P_1 &= \frac{1}{6}\left(D_3(e) + \sum_{j=1}^{j=2} D_3(a_j) - \sum_{j=3}^{j=5} D_3 a_j\right) = 0\\
P_2 &= \frac{1}{6}\left(2D_3(e) - \sum_{j=1}^{j=2} D_3(a_j)\right) = \frac{1}{3}\begin{pmatrix}2 & -1 & -1\\ -1 & 2 & -1\\ -1 & -1 & 2\end{pmatrix}
\end{aligned}
\tag{9.129}
$$

This makes vary explicit that

$$
D_3 = D_0 \oplus D_2 \tag{9.130}
$$

- Let us work out the regular representation of S_3

$$
D(e) = \begin{pmatrix}1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{pmatrix} \tag{9.131}
$$

$$
D(a_1) = \begin{pmatrix}0 & 0 & 1 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1 & 0 & 0\end{pmatrix} \tag{9.132}
$$

$$
D(a_2) = \begin{pmatrix}0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\end{pmatrix} \tag{9.133}
$$

$$
D(a_3) = \begin{pmatrix}0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\end{pmatrix} \tag{9.134}
$$

$$D(a_4) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{9.135}$$

$$D(a_5) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{9.136}$$

It is the case that

$$P_0 = \frac{1}{6} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \tag{9.137}$$

$$P_1 = \frac{1}{6} \begin{pmatrix} 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 & 1 & 1 \end{pmatrix} \tag{9.138}$$

$$P_2 = \frac{1}{3} \begin{pmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{pmatrix} \tag{9.139}$$

9.4 Partitions and representations of S_n

- Let us recall that a *cycle* is a cyclic permutation of a subset. An arbitray permutation has got k_j j-cycles, where

$$\sum_{j=1}^{j=n} j k_j = n \tag{9.140}$$

- Let us quickly revew a few general properties of cycles.

  Every cycle can be written as a product of transpositions, allowing for an index to appear several times:

$$(i_1 i_2 \dots i_n) = (i_1 i_2)(i_2 i_3) \dots (i_{n-1} i_n) \tag{9.141}$$

  The number of such transpositions is even for even permutations, and odd otherwise. The canonical way of writing a permutation is as a product of cycles without any common element. There are then n_1 one-cycles (usually not written down), n_2 two cycles, n_3 three-cycles, ad so on, in such a way that

$$n = k_1 + 2k_2 + 3k_3 + \dots + k_n n \tag{9.142}$$

  We say that the set of numbers $(n_1 \dots n n_n$ constitute a partition of the number n.)

  The cycle structure is invariant under conjugation. We claim that

$$\begin{pmatrix} 12 \dots n \\ s_1 s_2 \dots s_n \end{pmatrix} (a_1 \dots a_p) \begin{pmatrix} s_1 s_2 \dots s_n \\ 12 \dots n \end{pmatrix} = (b_1 \dots b_p) \tag{9.143}$$

  The reason is that for the numbers not involved in the cycle (let us say, 3) the cycle is irrelevant in the sense that

$$\begin{pmatrix} 3 \\ s_3 \end{pmatrix} \begin{pmatrix} s_3 \\ 3 \end{pmatrix} = (s_3) \tag{9.144}$$

  so that s_3 remains invariant.

  For example

$$(123)(12)(132) = (23) \tag{9.145}$$

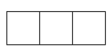
  Also

$$(12)(123)(12) = (132) \tag{9.146}$$

  How many elements are there in each conjugacy class? There are $n!$ permutations to begin with. But order is immaterial between cycles of the same length; so we must divide by $k_j|$. Also cyclic order does not matter within a cycle; this yields a factor j^{k_j}. Altogether we have

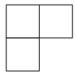
$$N_j = \frac{n!}{\prod_j j^{k_j} \; k_j!} \tag{9.147}$$

  It is useful to represent conjugacy classes by *Young frames*; columns of boxes of length j, top justified and arranged in decreasing j from left to right. For example, in S_3, the identity 1^3(with $\frac{3!}{3!} = 1$ element) is



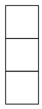
(9.148)

The class (2,1) (with $\frac{3!}{2} = 3$ elements)



(9.149)

and the class 3 (with $\frac{3!}{3} = 2$ elements)



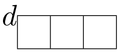
(9.150)

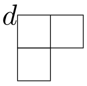
Altogether we recover the 6 elements of the group S_3.

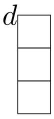
It is a fact that each tableau yields an dirrep of S_n with dimension

$$d_R = \frac{n!}{H} \tag{9.151}$$

where H is the hooks factor. To be specific,

d  $= \frac{3!}{3.2} = 1$ (9.152)

d  $= \frac{3!}{3} = 2$ (9.153)

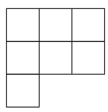
d  $= \frac{3!}{3.2} = 1$ (9.154)

- The inequivalent irreducible representations of S_n may be labelled by the partitions of the integer n. An unlabelled Young diagram or *Young frame* corresponds to a partition of the integer n, consisting of n boxes arranged in r rows

$$n = \sum_{i=1}^{r} \lambda_j \tag{9.155}$$

$$\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_r \tag{9.156}$$

The usual notation is

$\left\{3^2 1\right\} = \{331\} =$  (9.157)

- A *Young tableau*, or labelled Young diagram is an assignment of the numbers $1, 2 \dots n$ to the boxes of a Young frame. The tableau is *standard* if the numbers are increasing both along rows from left to right and along columns from top to bottom.

  The *Young operator* corrresponding to a given tableau is obtained by first symmetrizing rows (let us call p the horizontal permutations) and then antisymmetrizing columns (denote by q vertical permutations)

$$P = C \left(\prod_q \delta_\pi \pi \right) \left(\prod_p \pi \right) \tag{9.158}$$

  It is possible to check that this is a projector, and even to compute the constant C. We shall do it in some examples. We shall define a mapping from a given tableau to a state in the adjoint (that is, an element of S_n) by defining a em lexicografic ordering: from left to right and then top down, like reading a page in usual latin conventions. For example

$$\begin{array}{|c|c|c|} \hline 6 & 3 & 4 \\ \hline 5 & 1 & 2 \\ \hline 7 & 8 & \multicolumn{1}{c}{} \\ \cline{1-2} 9 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \quad \longrightarrow \quad (634512789) \tag{9.159}$$

- Let us now work in gory detail the case of S_3 First of all, consider the frame

$$\begin{array}{|c|c|c|} \hline & & \\ \hline \end{array} \tag{9.160}$$

  There is only one standard tableau,

$$\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} \quad \longrightarrow \quad (123) \tag{9.161}$$

  and six others (all possible permutations). The Young operator maps

$$(123) \quad \longrightarrow \quad Y_S \equiv C \Big(1 + (12) + (13) + (23) + (123) + (132) \Big) \tag{9.162}$$

  The projector is

$$P_S \equiv \frac{1}{6} Y_0 \tag{9.163}$$

  This is a one-dimensional subspace corresponding to the trivial representation

$$\pi \quad \longrightarrow \quad 1 \tag{9.164}$$

  Consider now the frame

$$\begin{array}{|c|} \hline \\ \hline \\ \hline \\ \hline \end{array} \tag{9.165}$$

Again, there is only one standard tableau

$$\boxed{1}\boxed{2}\boxed{3} \quad \longrightarrow \quad (123) \tag{9.166}$$

The Young operator maps

$$(123) \quad \longrightarrow \quad Y_A \equiv C\Big(1-(12)-(13)-(23)+(123)+(132)\Big) \tag{9.167}$$

Again, this is a one-dimensional subspace. It correspomds to the representation

$$P_A \equiv \frac{1}{6}Y_A \tag{9.168}$$

$$\pi \quad \longrightarrow \quad (-1)^\pi \tag{9.169}$$

Let us now turn to the hook.

$$\begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \tag{9.170}$$

There are two standard tableaux. Let us write them with their operators.

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad \longrightarrow \quad (123) \quad \longrightarrow \quad Y_1 \equiv C\Big(1-(13)\Big)\Big(1+(12)\Big) = C\Big\{1-(13)+(12)-(123)\Big\}$$

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad \longrightarrow \quad (132) \quad \longrightarrow \quad Y_2 \equiv C\Big(\left(1-(12)\right)\left(1+(13)\right)\Big) = C\Big(1+(13)-(12)-(132)\Big)$$

Let us compute

$$\begin{aligned} Y_1^2 = (1-(13)+(12)-(123))\left[1-(13)+(12)-(123)\right] = \left[1-(13)+(12)-(123)\right]+ \\ +\left[-(13)+1-(123)+(12)\right]+\left[(12)-(132)+1-(23))\right]+\left[-(13)+(32)-(13)+(132)\right]= \\ = 3Y_1 \end{aligned} \tag{9.171}$$

Also

$$Y_2^2 = 3Y_2 \tag{9.172}$$

This means that

$$\frac{1}{3}Y_1, \quad \frac{1}{3}Y_2 \tag{9.173}$$

are true projectors.

Besides

$$P_1.P_2 = \Big\{1-(13)+(12)-(123)\Big\}\Big\{1+(13)-(12)-(132)\Big\} = [1-(13)+(12)-(123)$$
$$+\,[+(13)-1+(132)-(23)]+[-(12)+(123)-1+(13)]+[-(132)+(23)-(13)+1]$$
$$P_1.P_S = P_1.P_A = 0$$
$$P_2.P_S = P_2.P_A = 0 \quad (9$$

Threre is closure, in the sense that

$$P_S + P_A + P_1 + P_2 = 1 \tag{9.175}$$

There are also four nonstandard ones

$$\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array} \longrightarrow (231) \longrightarrow P_3 \equiv [1-(12)]\,[1+(23)] = C\Big(1+(23)-(12)-(123)\Big)$$

$$\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \longrightarrow (213) \longrightarrow P_4 \equiv [1-(23)]\,[1+(12)] = C\Big(1+(12)-(23)-(132)\Big)$$

$$\begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \longrightarrow (312) \longrightarrow P_5 \equiv [1-(23)]\,[1+(13)] = C\Big(1+(13)-(23)-(123)\Big)$$

$$\begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 \\ \cline{1-1}\end{array} \longrightarrow (321) \longrightarrow P_6 \equiv [1-(13)]\,[1+(23)] = C\Big(1-(13)+(23)-(132)\Big)$$

Now by direct inspection we find that

$$\begin{aligned} P_1 + P_2 &= P_3 + P_4 \\ P_6 + P_5 &= P_1 + P_2 \end{aligned} \tag{9.176}$$

In the regular representation the Young operators read

$$Y_1 \equiv C \begin{pmatrix} 1 & 0 & -1 & 1 & 0 & -1 \\ -1 & 1 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 \\ 1 & -1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ -1 & 0 & 1 & -1 & 0 & 1 \end{pmatrix} \tag{9.177}$$

The structure of this matrix is

$$Y_1 = C \begin{pmatrix} A & A \\ B & B \end{pmatrix} \tag{9.178}$$

with

$$A \equiv \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \quad B \equiv \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix} \tag{9.179}$$

Eigenvectors read

$$\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \tag{9.180}$$

$$Y_2 \equiv C \begin{pmatrix} 1 & -1 & 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ -1 & 0 & 1 & 0 & 1 & -1 \\ -1 & 1 & 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 & -1 & 1 \end{pmatrix} \tag{9.181}$$

The structure is

$$Y_2 = C \begin{pmatrix} B & D \\ -B & -D \end{pmatrix} \tag{9.182}$$

with B as above and

$$D \equiv \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \tag{9.183}$$

Different tableaux corresponding to the same frame yield equivalent, although not identical representations.

- The dimension of a representation corresponding to a Young frame λ is computed by dividing $n!$ by the factorial of the hook length of each box in the first column of λ and multiply by the difference of each pair of such hook lengths. For example,

$$\dim \ydiagram{3,1} = \frac{4!}{4!1!}(4-1) = 3 \tag{9.184}$$

$$\dim \{p{+}2, 2\} \equiv \ydiagram{3,2} \cdots \ydiagram{3} = \frac{(p+4)!}{(p+3)!2!}(p{+}1) = \frac{(p+4)(p+1)}{2} \tag{9.185}$$

10

Lie groups.

Lie groups are particular instances of continuous groups, where each element $g \in G$ depends on a finite number of *real* continuous pareters

$$g(a) \equiv g(a_1 \ldots a_n) \tag{10.1}$$

in such a way that

$$g(a)g(b) = g(c) \tag{10.2}$$

where the functions

$$c_i = f_i\,(a_j, b_k) \tag{10.3}$$

are sufficiently regular. The canonical example is the thee-dimensional rotation group, $SO(3)$, where the parameters are the three Euler angles. In is often convenient to choose the parameters in such a way that

$$g(0) = e \tag{10.4}$$

Lie groups are n-dimensional manifolds (*symmetric spaces*). As is the case for all manifolds, Lie groups can be *compact* or *non compact*. (Remember that in $\mathbb{R}^n$ this means that a set is both closed and bounded). Compact groups (again SO(3) is the simplest nontrivial example) are much simpler, and in many ways analogous to finite groups.

The Lie magic is that many of the characteristics of a Lie group are determined by the properties of the neighborhood of the origin (which can be chose arbitrarily in the group manifold). The tangent space at the origin is dubbed the *Lie algebra*, $\underline{G}$. The relationship between a Lie group and its Lie algebra is the *exponential mapping*

$$g = e^{i \sum_{i=1}^{i=n} a_i T_i} \tag{10.5}$$

Indeed we can define the Lie algebra by analiticity

$$\lim_{\alpha \to 0} g(\alpha^i) = 1 + \alpha^i T_i + \ldots \tag{10.6}$$

On the other hand, it is plain that

$$g((t+s)\alpha^i) = g(t\alpha^i)g(s\alpha^i) \tag{10.7}$$

Taking the derivative with respect to $\left.\frac{d}{dt}\right|_{t=0}$ we get

$$\frac{d}{ds}g(s\alpha^i) = \alpha^i T_i\; g(s\alpha^i) \tag{10.8}$$

whose solution is the matrix exponential. Please note that

$$e^A e^B \neq e^{A+B} \tag{10.9}$$

(BCH)

Let us work out the composition law

$$\begin{gathered} e^{i\alpha^i T_i} e^{i\beta^j T_j} \equiv e^{i\gamma^i(\alpha,\beta)T_i} = \\ \left(1 + i\alpha^i T_i - \tfrac{\alpha^k\alpha^l}{2} T_k T_l + \ldots\right)\left(1 + i\beta^i T_i - \tfrac{\beta^k\beta^l}{2} T_k T_l + \ldots\right) = \\ = 1 + i\alpha^i T_i + i\beta^j T_j - \left(\tfrac{1}{2}\alpha^i\alpha^j + \tfrac{1}{2}\beta^i\beta^j + \alpha^i\beta^j\right) T_i T_j + \ldots \\ = 1 + i\left(\alpha^i + \beta^i\right) T_i + \tfrac{1}{2}\left(a^i + \beta^i\right)\left(\alpha^j + \beta^j\right)\; T_i T_j + \tfrac{1}{2}\left(\alpha^i\beta^j - \alpha^j\beta^i\right) T_i T_j + \ldots = \\ = 1 + i\left(\alpha^i + \beta^i\right) T_i + \tfrac{1}{2}\left(a^i + \beta^i\right)\left(\alpha^j + \beta^j\right)\; T_i T_j + \tfrac{1}{2}\alpha^i\beta^j\left[T_i T_j\right] + \ldots \end{gathered} \tag{10.10}$$

The elements T_i are a basis for the Lie algebra, which is nothing else than a vector space with an internal composition law, the commutator.

$$[T_i, T_j] = \sum_{k=1}^{k=n} C_{ij}^k T_k = \sum_{k=1}^{k=n} i\; f_{ij}^k T_k \tag{10.11}$$

In this way

$$\gamma^i(\alpha,\beta) = \alpha^i + \beta^i + f_{kl}^i \alpha^k \beta^l + \ldots \tag{10.12}$$

The constants C_{ij}^k (or f_{ij}^k) are denoted the *structure constants* of the algebra. A consequence of this is that the generators are traceless

$$Tr\; T_i = 0 \tag{10.13}$$

Jacobi's identity reads

$$[T_i,[T_j,T_k]]+[T_k,[T_i,T_j]]+[T_j,[T_k,T_i]] = 0 \quad \Longleftrightarrow \quad C_{il}^m\, C_{jk}^l + C_{kl}^m\, C_{ij}^l + C_{jl}^m\, C_{ki}^l = 0 \tag{10.14}$$

Let us now define the adjoint representation as

$$\left(T_k^{ad}\right)^i_{\;j} \equiv i f_{kj}^i \tag{10.15}$$

Let us check that this constitutes a representation. The first member is equal to

$$\begin{aligned} \left[T_k^{ad}, T_l^{ad}\right] = T_{kj}^i T_{lm}^j - T_{lj}^i T_{km}^j = f_{lj}^i f_{km}^j - f_{kj}^i f_{lm}^j = f_{lm}^j f_{jk}^i + f_{mk}^j f_{jl}^i = \\ = -f_{kl}^j f_{jm}^i = i f_{kl}^j \left(T_j^{ad}\right)^i_{\;m} \end{aligned} \tag{10.16}$$

Let us define a matrix in the algebra, $\mathfrak{G}$, the *Killing metric* as

$$g_{kl} \equiv tr\ T^{ad}_k T^{ad}_l \equiv -f^q_{kp} f^p_{lq} \tag{10.17}$$

It can be shown that for compact groups the Killing matrix is definite positive.

An intrinsic definition [13] is as follows. Consider an endomorphism of L:

$$\mathrm{Ad}(X) : Y \in L \rightarrow [X, Y] \in L \tag{10.18}$$

and the Killing form as

$$\kappa(X, Y) \equiv \mathrm{Tr}\ (Ad(X), Ad(Y)) = X^i Y^l C^k_{ij}\, C^j_{lk} \tag{10.19}$$

If we change basis in the Lie algebra

$$T_i \equiv M^a_i \tilde{T}_a \tag{10.20}$$

the new structure constants are

$$\tilde{C}^w_{uv} = \left(M^{-1}\right)^i_u \left(M^{-1}\right)^j_v C^k_{ij}\, M^w_k \tag{10.21}$$

and the new Killing form

$$\tilde{\kappa}_{ab} \equiv \left(M^{-1}\right)^i_a \left(M^{-1}\right)^j_b \kappa_{ij} \tag{10.22}$$

Then we can define

$$f_{ijk} \equiv f^l_{ij} g_{lk} = f^l_{ij} f^a_{lb} f^b_{ka} \tag{10.23}$$

We known of course, that

$$f^k_{(ij)} = 0 \tag{10.24}$$

Let us compute

$$f_{ijk} + f_{ikj} = f^l_{ij} f^a_{lb} f^b_{ka} + f^l_{ik} f^a_{lb} f^b_{ja} \tag{10.25}$$

But we can write

$$\left(f^a_{lb} f^l_{ij}\right) f^b_{ka} = -\left(f^a_{bl} f^l_{ij}\right) f^b_{ka} = \left(f^a_{il} f^l_{jb} + f^a_{jl} f^l_{bi}\right) f^b_{ka} \tag{10.26}$$

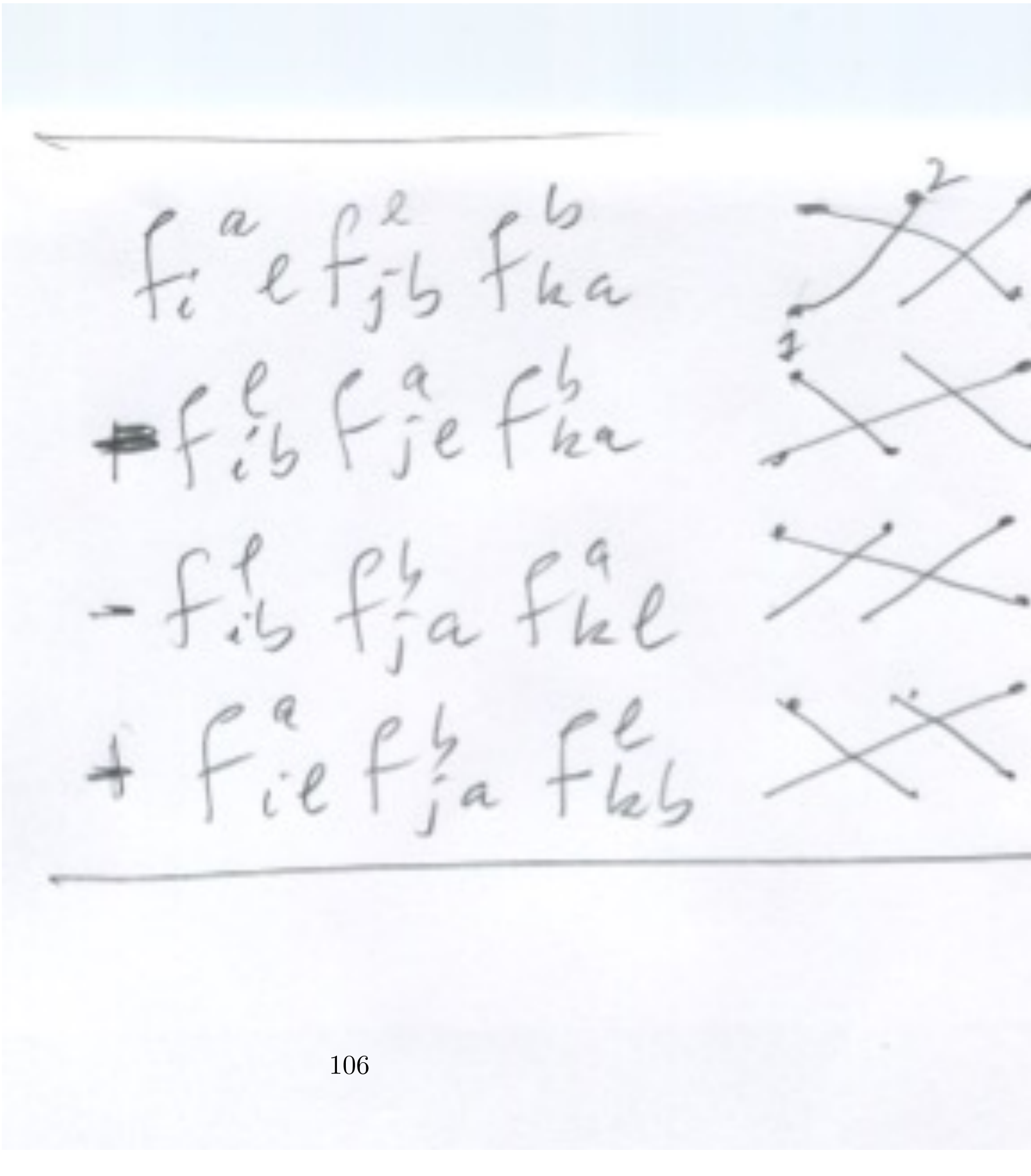
as well as

$$\left(f^l_{ik} f^a_{lb}\right) f^b_{ja} = \left(f^a_{bl} f^l_{ki}\right) f^b_{ja} = -\left(f^a_{ki} f^l_{ib} + f^a_{il} f^l_{bk}\right) f^b_{ja} \tag{10.27}$$

The structure constants thus are completely antisymmetric.

If is customary to define the quadratic Dynkin index as

$$tr\ T_a T_b \equiv I^{(2)}_R \delta_{ab} \tag{10.28}$$


$f_i{}^a{}_l f_j{}^l{}_b f_k{}^b{}_a$
$- f_i{}^l{}_b f_j{}^a{}_l f_k{}^b{}_a$
$- f_i{}^l{}_b f_j{}^b{}_a f_k{}^a{}_l$
$+ f_i{}^a{}_l f_j{}^b{}_a f_k{}^l{}_b$
2
1

We define an *invariant subalgebra*, $\mathfrak{A}$ as a set of generators that maps into itself under commutation with any element of the algebra $\mathfrak{G}$.That is, it is a subalgebra which is also an ideal in the algebraic sense. There are always two trivial invariant subalgebras, namely,e and $\mathfrak{G}$ itself.

A *simple* Lie algebra is such that it does not have any nontrivial invariant subalgebra. This the only type of Lie alegras we are going to study in this course.

In this case, the adjoint representation is irreducible. (Compare with finite groups). Assume there is an invariant subspace, generated by T_A. Call the other generators T_α. This means that

$$[\mathfrak{A}, \mathfrak{G}] \subset \mathfrak{A} \longrightarrow f_{iA\alpha} = 0 \tag{10.29}$$

Then by antisymmetry all structure constants with two indices A (in $\mathfrak{A}$) or with two indices α vanish; the only possibility is to have three A or three α; so that the algebra is not simple to begin with.

A *semisimple algebra* is such that there is no any abelian invariant subalgebra. They consist of direct products of simple algebras. The necessary and sufficient condition for an algebra to be semisimple is that the Killing form is non-degenerate, that is,

$$det\ g_{ij} \neq 0 \tag{10.30}$$

Let us prove the first part. Assume there is an invariant abelian subalgebra, $\mathfrak{B}$ generated by T_α (the full set of generators will be denoted by $i = (\alpha, A)$). This means that

$$[\mathfrak{B}, \mathfrak{G}] \subset \mathfrak{B} \longrightarrow f_{i\alpha A} = 0 \tag{10.31}$$

Then the row of the Killing metric corresponding to the subalgebra, that is,

$$g_{i\alpha} = f_{ijk} f_{\alpha kj} = f_{i\beta k} f_{\alpha k\beta} = 0 \tag{10.32}$$

This means that a whole row of the Killing matrix vanishes, and so does its determinant.

Semisimple Lie algebras are direct products of simple Lie algebras, such as

$$G = G_1 \times G_2 \times \ldots \tag{10.33}$$

where all G_i are simple.

10.1 Matrix groups

Most important are matrix groups.

- The group of n-dimensional nonsingular matrices in the field of complex (real) numbers is denoted as $GL_n(\mathbb{C})$ ($GL_n(\mathbb{R})$).

- The group of n-dimensional unimodular (unit determinant) matrices in the field of complex (real) numbers is denoted as $SL_n(\mathbb{C})$ ($SL_n(\mathbb{R})$).
- The group of n-dimensional unitary (complex matrices in the field of complex (real) numbers is denoted as $SU_n(\mathbb{C})$ (unitary group).

$$g^+ g = g g^+ = 1 \tag{10.34}$$

The unitary Lie algebra $\underline{SU(n)}$ is such that

$$e^{-iaT^+}\; e^{iaT} = 1 = 1 + ia\left(T - T^+\right) + O(a^2) \tag{10.35}$$

That is, elements of the Lie algebra are hermitian matrices. How many are those? The condition is

$$T_{ij} = T^*_{ji} \tag{10.36}$$

The n diagonal elements are real; and the $\frac{n^2-n}{2}$ complex elements below diagonal are deternined by those above; altogether we have (deleting the trace)

$$n + 2\frac{n^2 - n}{2} - 1 = n^2 - 1 \tag{10.37}$$

real parameters.

- The group of n-dimensional real orthogonal matrices is denoted as $SO(n)$ (orthogonal group).

$$g^T g = g g^T = 1 \tag{10.38}$$

The Lie algebra $\underline{SO(n)}$ is given by

$$e^{iaT^T}\; e^{iaT} = 1 + ia(T^T + T) + O(a^2) \tag{10.39}$$

antisymmetric matrices. The number of parameters is then

$$\frac{n(n-1)}{2} \tag{10.40}$$

- The group of matrices the leave invariant the diagonal quadratic form with p values of +1 and q values of -1

$$I_{p,q} \equiv \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ & & \ldots & & \\ 0 & 0 & \ldots & -1 & 0 \\ 0 & 0 & \ldots & 0 & -1 \end{pmatrix} \tag{10.41}$$

is denoted $SO(p, q)$ and they are non-compact as soon as either p or q are non-vanishing. The Lorentz group $SO(1, 3)$ belongs to this class.

- The group of matrices that leave invariant the quadratic form

$$J \equiv \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \tag{10.42}$$

where I is the $n \times n$ unit matrix is the symplectic group, $Sp(2n)$.

- The group of matruces that leave invariant the quadratic form

$$ds^2 = dz_1 d\bar{z}_1 + \ldots dz_n d\bar{z}_n \tag{10.43}$$

is the *unitary group* $U(n) \sim SU(n) \times U(1)$,

10.2 Representations of $SU(2)$ and $SO(3)$ through tensor methods.

First of all, it is a fact that

$$M \in SU(2) \Longrightarrow \overline{M} \in SU(2) \tag{10.44}$$

Consider u_a that transforms with $M \in SU(2)$ and u^a that does it with respect to $\overline{M}$. It so happens that

$$\delta^a_b \to \overline{M}^a_v \, M^w_b \, \delta^v_w = \left(MM^+\right)^a_b = \delta^a_b \tag{10.45}$$

also

$$\begin{gathered}\epsilon_{ab} \longrightarrow \epsilon_{ab} \, M^a_v \, M^b_w = M^1_v \, M^2_m - M^2_v \, M^1_w = \\ = \begin{pmatrix} M^1_1 M^2_1 - M^2_1 M^1_1 = 0 & M^1_1 M^2_1 - M^2_1 M^1_2 = \det \mathrm{M} \\ -\det \mathrm{M} & M^1_2 M^2_2 - M^2_2 M^1_2 = 0 \end{pmatrix} = \\ = \det \mathrm{M} \, \epsilon_{vw} = \epsilon_{vw}\end{gathered} \tag{10.46}$$

as well as

$$\epsilon^{ab} \longrightarrow \epsilon^{ab} \tag{10.47}$$

From an upper and a lower index we can always form a simpler representation with two indices less

$$T^{\alpha\beta}_\gamma = t^\alpha \delta^\beta_\gamma + h^\beta \delta^\alpha_\gamma \tag{10.48}$$

where

$$\begin{aligned} T^\alpha &\equiv T^{\alpha\gamma}_\gamma = 2t^\alpha_1 + t^\alpha_2 \\ T^{\cdot\beta} &\equiv T^{\gamma\beta}_\gamma = t^\alpha_1 + 2t^\alpha_2 \end{aligned} \tag{10.49}$$

Then

$$\begin{aligned} t^\alpha_1 &= \tfrac{1}{3}\left(2T^\alpha - T^{\cdot\beta}\right) \\ t^\alpha_2 &= \tfrac{1}{3}\left(2T^{\cdot\beta} - T^\alpha\right) \end{aligned} \tag{10.50}$$

We can then eliminate either contravariant or covariant indices; the only thing that matters is the difference, which we will write downstairs. Even then, from say

$$T_{\alpha\beta} \tag{10.51}$$

we can form

$$\epsilon^{\alpha\beta} T_{\alpha\beta} \tag{10.52}$$

unless it is totally symmetric. Those we cannot reduce further.

We know that the combinations with repetition of n objects taken m at a time is the number of ways of combining $n-1$ bars and m crosses; such that the number of crosses to the left of the first bar stands for the number of times times the first object appears; the number of crosses between the first and the second bar stands for the number of times the second object appears and so on. This is $\frac{(n+m-1)!}{m!(n-1)!} = \binom{n+m-1}{m}$.

In our case $n = 2$ and the irreps are generated by symmetric covariant tensors with m indices, of which there are $\binom{m+1}{m} = m + 1$.

In the case of $SO(3)$ all irreps are real, so we need to consider covariant indices only. In spite of that

$$\delta_{ij} \longrightarrow \delta_{ij}\, R^i_k\, R^j_l = \delta_{kl} \tag{10.53}$$

as well as

$$\epsilon_{ijk} \longrightarrow \epsilon_{ijk}\, R^i_l\, R^j_m\, R^k_m = \det \mathrm{R}\, \epsilon_{lmn} = \epsilon_{lmn} \tag{10.54}$$

So that irreps are generated by symmetric traceless covariant tensors with j indices. In our case $n = 3$, so this yields

$$\binom{j+2}{j} = \frac{(j+2)(j+1)}{2} \tag{10.55}$$

We have to withdraw all traces, of which there are

$$\binom{j}{j-2} = \frac{j(j-1)}{2} \tag{10.56}$$

The difference is just the dimension of the representation,

$$d = 2j + 1 \tag{10.57}$$

It is worth remarking that only when

$$m = 2j \tag{10.58}$$

(that is, m is an even integer) does the $SU(2)$ irrep be also an irrep of $SO(3)$.

10.3 Representations of $GL(n)$ through tensor methods.

Weyl's treatment of finite dimensional group representations rests on a simple fact. Consider any tensor that under $L \in GL(n)$ transforms as

$$T'_{\mu_1\mu_2\ldots\mu_n} = L^{\lambda_1}_{\mu_1}\ldots L^{\lambda_n}_{\mu_n}\, T_{\lambda_1\ldots\lambda_n} \tag{10.59}$$

Assume now that the tensor T is invariant under some permutation $\pi \in S_n$

$$T_{\lambda_{\pi(1)}\ldots\lambda_{\pi(n)}} = T_{\lambda_1\ldots\lambda_n} \tag{10.60}$$

Then

$$T'_{\mu_1\mu_2\ldots\mu_n} = L^{\lambda_1}_{\mu_1}\ldots L^{\lambda_n}_{\mu_n}\, T_{\lambda_1\ldots\lambda_n} = L^{\lambda_1}_{\mu_1}\ldots L^{\lambda_n}_{\mu_n}\, T_{\lambda_{\pi(1)}\ldots\lambda_{\pi(n)}} = L^{\lambda_1}_{\mu_{\pi(1)}}\ldots L^{\lambda_n}_{\mu_{\pi(n)}}\, T_{\lambda_1\ldots\lambda_n} \tag{10.61}$$

It follows that

$$T'_{\mu_{\pi(1)}\mu_{\pi(2)}\ldots\mu_{\pi(n)}} = T'_{\mu_1\mu_2\ldots\mu_n} \tag{10.62}$$

That is, the subspace of tensors invariant under any permutation symmetry is invariant under $GL(n)$ transformations. Let us perform now some elementary checks.

- $n = 2$. There are only two symmetry classes: antisymmetric

$$T_{\begin{array}{|c|}\hline \mu \\ \hline \nu \\ \hline \end{array}} \tag{10.63}$$

  and symmetric

$$T_{\begin{array}{|c|c|}\hline \mu & \nu \\ \hline \end{array}} \tag{10.64}$$

  The Young projectors are

$$\frac{1 \pm (\mu\nu)}{2} \tag{10.65}$$

- $n = 3$. There are now three classes.

$$\begin{array}{l} T_{\begin{array}{|c|c|c|}\hline \mu & \nu & \lambda \\ \hline \end{array}} \\ T_{\begin{array}{|c|}\hline \mu \\ \hline \nu \\ \hline \lambda \\ \hline \end{array}} \\ T_{\begin{array}{|c|c|}\hline \mu & \nu \\ \hline \lambda \\ \cline{1-1} \end{array}} \end{array} \tag{10.66}$$

  Let us work out this third case in detail. The Young operator is given by

$$Y \equiv PQ \equiv (1 - (\mu\lambda))\,(1 + (\mu\nu)) = 1 + (\mu\nu) - (\mu\lambda) - (\mu\lambda\nu) \tag{10.67}$$

$$(YT)_{\mu\nu\lambda} = T_{\mu\nu\lambda} + T_{\nu\mu\lambda} - T_{\lambda\nu\mu} - T_{\nu\lambda\mu} \tag{10.68}$$

It is clear that

$$Y^2 = 3Y \tag{10.69}$$

so that

$$P \equiv \frac{1}{3} Y \tag{10.70}$$

is a projector. Any tensor in this class is such that

$$2T_{\mu\nu\lambda} = T_{\nu\mu\lambda} - T_{\lambda\nu\mu} - T_{\nu\lambda\mu} \tag{10.71}$$

But then it is also a fact that

$$2T_{\mu'\nu'\lambda'} = L^{\mu}_{\mu'} L^{\nu}_{\nu'} L^{\lambda}_{\lambda'} \left(T_{\nu\mu\lambda} - T_{\lambda\nu\mu} - T_{\nu\lambda\mu}\right) = T_{\nu'\mu'\lambda'} - T_{\lambda'\nu'\mu'} - T_{\nu'\lambda'\mu'} \tag{10.72}$$

It is a fact that in general these tensors form a basis for an irreducible representation of $GL(n)$, without any further ado. Let us compute some dimensions of those representations. The dimension of the space

$$\begin{array}{|c|c|c|c|}\hline \alpha & \beta & \dots & \delta \\ \hline \end{array} \quad \text{(r slots)} \tag{10.73}$$

is the numer of combinations with repetition of n objects taken in packs of r. This can be computed as imagining (n-1) vertical lines and r crosses. The number is

$$CR^n_r = \frac{(n+r-1)!}{r!(n-1)!} = \binom{n+r-1}{r} \tag{10.74}$$

For example in the case $n = 3$, $r = 3$ this formula yields

$$D = 10 \tag{10.75}$$

To be specific, the components are

$$T_{111} \quad T_{112} \quad T_{113} \quad T_{122} \quad T_{123} \quad T_{133} \quad T_{222} \quad T_{223} \quad T_{233} \quad T_{333} \tag{10.76}$$

In order to count dimension for lower representations, it is useful to consider outer products. For example

$$\begin{array}{|c|}\hline \ \\ \hline \end{array} \otimes \begin{array}{|c|}\hline \ \\ \hline \end{array} = \begin{array}{|c|c|}\hline \ & \ \\ \hline \end{array} \oplus \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \end{array} \tag{10.77}$$

This just expresses the trivial identity

$$n^2 = \frac{n(n+1)}{2} + \frac{n(n-1)}{2} \tag{10.78}$$

One can also work out slightly more complicated examples; for example

$$\begin{array}{|c|c|}\hline \ & \ \\ \hline \end{array} \otimes \begin{array}{|c|}\hline \ \\ \hline \end{array} = \begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline \end{array} \oplus \begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \tag{10.79}$$

The dimension of the $\begin{smallmatrix}\square\square\\\square\end{smallmatrix}$ representetion easily follows:

$$\frac{n^2(n+1)}{2} = \binom{n+2}{3} + D_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}} \tag{10.80}$$

and we recover that

$$D_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}} = \frac{n(n^2-1)}{3} \tag{10.81}$$

10.4 Representations of $U(n)$

First of all, $U(n) \sim SU(n) \times U(1)$. It is plain that all irreps of GL(n) are also reps of any subgroup; but not necessarily irreps. As a trivial example, the tensor

$$T_{[ab]} \equiv T\ \epsilon_{ab} \tag{10.82}$$

transforms as a one-dimensional $[1^2] \equiv \begin{smallmatrix}\square\\\square\end{smallmatrix}$ irrep of $GL(2)$:

$$T_{[ij]} \to g^i_a g^j_b T_{[ij]} = T\ \epsilon_{ab}\ \det g \tag{10.83}$$

It is also the case that the diagram $[2^n]$ has only one standard tableau, for example $\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array}$. There is only one basis element. Actually the Young projector reads in this case

$$\begin{aligned}(P\ T)_{\alpha_1\alpha_2\alpha_3\alpha_4} &\sim T_{\alpha_1\alpha_2\alpha_3\alpha_4} + (T_{\alpha_2\alpha_1\alpha_3\alpha_4} + T_{\alpha_1\alpha_2\alpha_4\alpha_3}) - (T_{\alpha_3\alpha_2\alpha_1\alpha_4} + T_{\alpha_2\alpha_3\alpha_1\alpha_4} + T_{\alpha_3\alpha_2\alpha_4\alpha_1}) - \\ &- (T_{\alpha_1\alpha_4\alpha_3\alpha_2} + T_{\alpha_4\alpha_1\alpha_3\alpha_2} + T_{\alpha_1\alpha_4\alpha_2\alpha_3} - T_{\alpha_3\alpha_4\alpha_1\alpha_2} - T_{\alpha_4\alpha_3\alpha_1\alpha_2} - T_{\alpha_3\alpha_4\alpha_2\alpha_1})\end{aligned} \tag{10.84}$$

This result can be written as

$$(P\ T)_{\alpha_1\alpha_2\alpha_3\alpha_4} = T\ \epsilon_{\alpha_1\alpha_3}\ \epsilon_{\alpha_2\alpha_4} \tag{10.85}$$

Under the action of GL(n)

$$\begin{aligned}(P\ T)_{\alpha_1\alpha_2\alpha_3\alpha_4} \to \Big\{ & g^{\beta_1}_{\alpha_1} g^{\beta_2}_{\alpha_2} g^{\beta_3}_{\alpha_3} g^{\beta_4}_{\alpha_4} + \left(g^{\beta_1}_{\alpha_2} g^{\beta_2}_{\alpha_1} g^{\beta_3}_{\alpha_3} g^{\beta_4}_{\alpha_4} + g^{\beta_1}_{\alpha_1} g^{\beta_2}_{\alpha_2} g^{\beta_3}_{\alpha_4} g^{\beta_4}_{\alpha_3}\right) - \left(g^{\beta_1}_{\alpha_3} g^{\beta_2}_{\alpha_2} g^{\beta_3}_{\alpha_1} g^{\beta_4}_{\alpha_4} + g^{\beta_1}_{\alpha_2} g^{\beta_2}_{\alpha_3} g^{\beta_3}_{\alpha_1} g^{\beta_4}_{\alpha_4} + \right. \\ & \left. + g^{\beta_1}_{\alpha_3} g^{\beta_2}_{\alpha_2} g^{\beta_3}_{\alpha_4} g^{\beta_4}_{\alpha_1}\right) - \left(g^{\beta_1}_{\alpha_1} g^{\beta_2}_{\alpha_4} g^{\beta_3}_{\alpha_3} g^{\beta_4}_{\alpha_2} + g^{\beta_1}_{\alpha_4} g^{\beta_2}_{\alpha_1} g^{\beta_3}_{\alpha_3} g^{\beta_4}_{\alpha_2} + g^{\beta_1}_{\alpha_1} g^{\beta_2}_{\alpha_4} g^{\beta_3}_{\alpha_2} g^{\beta_4}_{\alpha_3} - g^{\beta_1}_{\alpha_3} g^{\beta_2}_{\alpha_4} g^{\beta_3}_{\alpha_1} g^{\beta_4}_{\alpha_2} + \right. \\ & \left. - g^{\beta_1}_{\alpha_4} g^{\beta_2}_{\alpha_3} g^{\beta_3}_{\alpha_1} g^{\beta_4}_{\alpha_2} - g^{\beta_1}_{\alpha_3} g^{\beta_2}_{\alpha_4} g^{\beta_3}_{\alpha_2} g^{\beta_4}_{\alpha_1} \right) \Big\}\ T_{\beta_1\beta_2\beta_3\beta_4}\end{aligned} \tag{10.86}$$

This condenses into

$$(P\ T)_{\alpha_1\alpha_2\alpha_3\alpha_4} \to (\det g)^2\ T\ \epsilon_{\alpha_1\alpha_3}\ \epsilon_{\alpha_2\alpha_4} \tag{10.87}$$

This result is actually quite generic. If we have a rep of GL(n), say $[\lambda_1 \ldots \lambda_n]$, and we add to it a column of n boxes to it, the only set of indices in standard order that can be inserted in the additional column of $[\lambda_1 + 1 \ldots \lambda_n + 1]$ is $(1, 2, \ldots, n)$.

Thus the number of standard tableaux of the representation $[\lambda_1 + 1 \ldots \lambda_n + 1]$ is the same as this number for $[\lambda_1 \ldots \lambda_n]$; the only change is a new factor of det g. This means that thse two patters are equivalent for unimodular groups.

Then for unimodular groups we need to consider only patters with fewer than n rows.

$$[\lambda_1 \ldots \lambda_n] = [\lambda_1 - \lambda_n \ldots \lambda_{n-1} - \lambda_n] \tag{10.88}$$

There is a second equivalence (related to Hodge duality), namely

$$\left[1^{n-1}\right] = [1] \tag{10.89}$$

This can be easily generalized to

$$\left[1^{n-p}\right] = [1^p] \tag{10.90}$$

The general theorem is that for unimodular transformations

$$[\lambda_1, \lambda_2, \ldots, \lambda_n] = [\lambda_1 - \lambda_n, \lambda_1 - \lambda_{n-1}, \ldots, \lambda_1 - \lambda_2] \tag{10.91}$$

Which is equivalent to

$$[\mu_1, \mu_2, \ldots, \mu_{n-1}] = [\lambda_1 \equiv \lambda, \lambda_2 \equiv \lambda - \mu_{n-1}, \lambda_3 \equiv \lambda - \mu_{n-2}, \ldots, \lambda_{n-1} \equiv \lambda - \mu_2, \lambda_n \equiv \lambda - \mu_1] \tag{10.92}$$

10.5 Representations of $O(n)$

This is the only case in which the Kronecker delta with two covariant or else two contravariant indices makes sense, because

$$g.g^T = 1 \quad \Longleftrightarrow \quad \delta^{ab}\ g^i_a g^j_b = \delta^{ij} \tag{10.93}$$

Contractions commute with group transformations

$$\delta^{ab}\ g^i_a g^j_b \ldots T_{ij\ldots} = \delta^{ij} \ldots T_{ij\ldots} \tag{10.94}$$

Traceless tensors are transformed into traceless tensors. There is a complete decomposition of an arbitary tensor into a traceless piece plus other terms containing Kronecker deltas. For example

$$T_{ij} = \left(T_{ij} - \frac{1}{n}\ T\ \delta_{ij}\right) + \frac{1}{n}\ T\ \delta_{ij} \equiv T^0_{ij} + \frac{1}{n}\ T\ \delta_{ij} \tag{10.95}$$

where

$$T \equiv \delta^{ij}\ T_{ij} \tag{10.96}$$

Another example

$$T_{ijk} \equiv T^0_{ijk} + \delta_{ij}\ A_k + \delta_{ik}\ B_j + \delta_{jk}\ C_i \tag{10.97}$$

Let us denote the three possible traces

$$\begin{aligned} T^{23}_i &\equiv \delta^{jk}\ T_{ijk} \\ T^{13}_j &\equiv \delta^{ik}\ T_{ijk} \\ T^{12}_k &\equiv \delta^{ij}\ T_{ijk} \end{aligned} \tag{10.98}$$

This means that

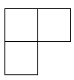
$$\begin{aligned} T^{23}_i &= A_i + B_i + n\ C_i \\ T^{13}_j &= A_j + nB_j + B_j \\ T^{12}_k &= n\ A_k + B_k + C_k \end{aligned} \tag{10.99}$$

then

$$\begin{aligned} A &= -\frac{1}{n^2+n-2}\left(T^{23} + T^{13} - (1+n)T^{13}\right) \\ B &= -\frac{1}{n^2+n-2}\left(T^{23} - (1+n)T^{13} + T^{13}\right) \\ C &= -\frac{1}{n^2+n-2}\left(-(1+n)T^{23} + T^{13} + T^{13}\right) \end{aligned} \tag{10.100}$$

It is plain that a permutation of the indices maps a traceless tensor into another traceless tensor. We can then apply Young operators to a traceless tensor to obtain traceless tensors of a given symmetry type.

In fact, there is a theorem that states that the traceless tensors corresponding to Young diagrams in which the sum of the lengths of the first two columns exceeds n must vanish.

Let us work out the hook  in n=2 dimensions (its traceless part should vanish in agreement with the preceding theorem). The action of the Young projector is proportional to

$$\left(P^{\text{Hook}}T\right)_{ijk} \equiv t_{ijk} + t_{jik} - t_{kji} - t_{jki} \tag{10.101}$$

Let us compute components in gory detail

$$
\begin{aligned}
T^H_{111} &= 0 \\
T^H_{112} &= t_{112} + t_{112} - t_{211} - t_{121} \\
T^H_{121} &= t_{121} + t_{211} - t_{121} - t_{211} = 0 \\
T^H_{122} &= t_{122} + t_{212} - t_{221} - t_{221} \\
T^H_{222} &= 0 \\
T^H_{211} &= t_{211} + t_{121} - t_{112} - t_{112} \\
T^H_{212} &= t_{212} + t_{122} - t_{212} - t_{122} = 0 \\
T^H_{221} &= t_{221} + t_{221} - t_{122} - t_{212}
\end{aligned} \tag{10.102}
$$

Of the four non-vanishing components only two are independent because

$$
\begin{aligned}
T^H_{211} &= -T^H_{112} \\
T^H_{221} &= -T^H_{122}
\end{aligned} \tag{10.103}
$$

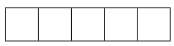
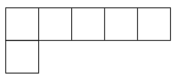
Imposing now tracelessness,

$$
\begin{aligned}
T^H_{112} + T^H_{222} &= T^H_{112} = 0 \\
T^H_{122} + T^H_{111} &= T^H_{122} = 0
\end{aligned} \tag{10.104}
$$

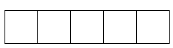
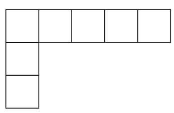
there is nothing left QED.

Let us define *associate diagrams*. Assume the length of the first column in T, say $a < n \leqslant 2$. Then the legth of the first colum of T' is $a' \equiv n - a$, and all othe columns of T and T' have the same length.

Fot example, for n=3

$T \equiv$  $\rightarrow T' \equiv$  (10.105)

In n=4

$T \equiv$  $\rightarrow T' \equiv$  (10.106)

In general the pattern T will contain a given number of indices , $r = 1, 2, \ldots$ and

$$
\mu_1 + \mu_2 + \ldots + \mu_\nu \equiv r \tag{10.107}
$$

indices (where as usual $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_\nu$). When n is an even number, then

$$
\nu \equiv \frac{n}{2} \tag{10.108}
$$

(in this case diagrams with $\frac{n}{2}$ rows are self-conjugate) and if n is an odd number, then

$$\nu \equiv \frac{n-1}{2} \tag{10.109}$$

In SO(n) the reps corresponding to *associate diagrams* T and T' are equivalent.For SO(3), $\nu = 1$, and irreps are desibed by the diagram

$$\square\square\ldots\square\square \tag{10.110}$$

(symmetric traceless tensors).

We shall denore by $\mathfrak{so}(\mathfrak{n})$ the Lie algebra of SO(n).

11

The rotation group $SO(3) \sim SU(2)/\mathbb{Z}_2$.

Assume we are interested in the matrix that relates two different orthonormal frames

$$\vec{e}_a = R_a{}^b \vec{e}_b \tag{11.1}$$

This a matrix $R \in SO(3)$. This means that

$$RR^T = R^T R = 1 \tag{11.2}$$

Put it into another form, this is the condition that

$$x^2 + y^2 + z^2 \tag{11.3}$$

remains invariant under such a linear transformation.

Any rotation is always a rotation around an axis, which is the locus of the fixed points of the rotation. Let us characterize the axis by a unit vector, $\hat{n}$.

Given any vector, $\vec{v} \in \mathbb{R}^3$, it is plain that the component of it in the direction of the axis, $\vec{v}_{||} \equiv (\vec{v}.\hat{n})\,\hat{n}$ will be unaffected, whereas the orthogonal component $\vec{v}_\perp \equiv \vec{v} - \vec{v}_{||}$ will become a combination of $\vec{v}_\perp$ and $\hat{n} \times \vec{v}$.

$$\begin{aligned} \vec{v}'_{||} &= \vec{v}_{||} \\ \vec{v}'_\perp &= \alpha\, \vec{v}_\perp + \beta\, \hat{n} \times \vec{v}_\perp \end{aligned} \tag{11.4}$$

The conservation of the norm implies that

$$\alpha^2 + \beta^2 = 1 \tag{11.5}$$

Altogether

$$\vec{v} \to \alpha\, \vec{v} + (1 - \alpha\,)\,(\vec{v}.\hat{n})\,\hat{n} + \beta\,\hat{n} \times \vec{v}_\perp \tag{11.6}$$

and the rotation matrix is

$$R_{\hat{n}} = \alpha\,\delta_{ij} + (1 - \alpha)\,n_i n_j + \beta\,\epsilon_{ikj} n^k \tag{11.7}$$

It is easy to check that this matrix is orthogonal,

$$\sum_j R_{ij} R_{jk} = \delta_{ik} \tag{11.8}$$

Choosing

$$\begin{aligned} \alpha &\equiv \cos\,\alpha \\ \beta &\equiv \sin\,\alpha \end{aligned} \tag{11.9}$$

$$R = \begin{pmatrix} n_1^2 + (n_2^2 + n_3^2)\cos\,\alpha & (1-\cos\,\alpha) n_1 n_2 - n_3 \sin\,\alpha & (1-\cos\,\alpha) n_1 n_3 + n_2 \sin\,\alpha \\ n_3 \sin\,\alpha + n_1 n_2\,(1-\cos\,\alpha) & n_2^2 + (n_1^2 + n_3^2)\cos\,\alpha & (1-\cos\,\alpha) n_2 n_3 - n_1 \sin\,\alpha \\ -n_2 \sin\,\alpha + (1-\cos\alpha) n_3 n_1 & (1-\cos\,\alpha) n_3 n_2 + n_1 \sin\,\alpha & n_3^2 + (n_2^2 + n_1^2)\cos\,\alpha \end{pmatrix} \tag{11.10}$$

All this yields, for $\hat{n} \equiv (0,0,1)$

$$R = \begin{pmatrix} \cos\,\alpha & \sin\,\alpha & 0 \\ -\sin\,\alpha & \cos\,\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{11.11}$$

which when $\alpha = \frac{\pi}{2}$ transforms the positive OX axis, $(1,0,0)$ into the *negative* OY axis, $(0,-1,0)$. The opposite sign corresponds to $\alpha \leftrightarrow -\alpha$.

For arbitrary $\hat{n}$ transforms the vector $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ into

$$R_n \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} (1-\cos\,\alpha) n_1 n_3 + n_2 \sin\,\alpha \\ (1-\cos\,\alpha) n_2 n_3 - n_1 \sin\,\alpha \\ n_3^2 + (n_2^2 + n_1^2)\cos\,\alpha \end{pmatrix} \tag{11.12}$$

This corresponds to the polar direction

$$\begin{aligned} \cos\,\Theta &= n_3^2 + (1 - n_3^2)\cos\,\alpha \\ \tan\,\Phi &= \tfrac{n_2 n_3 (1-\cos\,\alpha) - n_1 \sin\,\alpha}{n_1 n_3 (1-\cos\,\alpha) + n_2 \sin\,\alpha} \end{aligned} \tag{11.13}$$

This depends on three parameters, as it should: two from $\hat{n} \equiv \begin{pmatrix} \sin\,\theta\,\cos\,\phi \\ \sin\,\theta\,\sin\,\phi \\ \cos\,\theta \end{pmatrix}$ and another one from α.

$$\begin{aligned} \cos\,\Theta &= \cos^2\,\theta + (1 - \cos^2\,\theta)\cos\,\alpha \\ \tan\,\Phi &= \tfrac{\sin\,\phi\,\cos\,\theta(1-\cos\,\alpha) - \cos\,\phi \sin\,\alpha}{\cos\,\phi\,\cos\,\theta(1-\cos\,\alpha) + \sin\,\phi \sin\,\alpha} \end{aligned} \tag{11.14}$$

We can ask, for example, what is the rotation that transforms a given univ vector, say $\hat{n}_1$ into another one, say, $\hat{n}_2$. Let us denote

$$\hat{n_1}\hat{n_2} \equiv \cos\,\theta \tag{11.15}$$

It is plain that the axis of rotation will be

$$\hat{n} \equiv \frac{\hat{n}_1 \times \hat{n}_2}{\sin\, \theta} \tag{11.16}$$

We need

$$\hat{n}_2 = \cos\, \alpha\, \hat{n}_1 + \sin\, \alpha\, \frac{\hat{n}_2 - \cos\, \theta\, \hat{n}_1}{\sin \theta} \tag{11.17}$$

This clearly needs $\alpha = \theta$.

The groups $SO(3)$ and $SU(2)/\mathbb{Z}_2$ are intimately related. Indeed any unitary matrix can be parameterized as

$$u = \begin{pmatrix} \cos\, \alpha\, e^{i\beta} & \sin\, \alpha\, e^{i\gamma} \\ -\sin\, \alpha\, e^{-i\gamma} & \cos\alpha\, e^{-i\beta} \end{pmatrix} \tag{11.18}$$

It is clear that the range of the angles is

$$\begin{aligned} 0 \leqslant \beta \leqslant 2\pi \\ 0 \leqslant \alpha \leqslant \pi \\ 0 \leqslant \gamma \leqslant 2\pi \end{aligned} \tag{11.19}$$

Consider an arbitrary hermitian matrix

$$M \equiv \begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix} \tag{11.20}$$

Its determinant is

$$\det\, M = 1 - r^2 \tag{11.21}$$

It is plain that the transformation

$$M \rightarrow uMu^+ \tag{11.22}$$

leaves this determinant unchanged. Then there is a map

$$u \in SU(2) \rightarrow R \in SO(3) \tag{11.23}$$

It is plain that both $\pm u$ yield the same rotation; this is the reason for a factor $\mathbb{Z}_2$. To be specific, when $\beta = \gamma = 0$

$$uMu^+ = \begin{pmatrix} 1 + z\, \cos\, 2\alpha + x\, \sin\, 2\alpha & -i\, y + x\, \cos\, 2\alpha - z\, \sin\, 2\alpha \\ i\, y + x\, \cos\, 2\alpha - z\, \sin\, 2\alpha & 1 - z\, \cos\, 2\alpha - x\, \sin\, 2\alpha \end{pmatrix} \tag{11.24}$$

which means that

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\, 2\alpha & 0 & -\sin\, 2\alpha \\ 0 & 1 & 0 \\ -\sin\, 2\alpha & 0 & \cos\, 2\alpha \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{11.25}$$

It represents a rotation of angle 2α around the y axis, $R_2(-2\alpha)$. This rotation is negative, because when $2\alpha = \frac{\pi}{2}$ this yields

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \to \begin{pmatrix}0\\0\\1\end{pmatrix}$$
$$\begin{pmatrix}0\\0\\1\end{pmatrix} \to \begin{pmatrix}-1\\0\\0\end{pmatrix} \tag{11.26}$$

Also, when, $\alpha = 0$,

$$\begin{pmatrix}x'\\y'\\z'\end{pmatrix} = \begin{pmatrix}\cos 2\beta & \sin 2\beta & 0\\ -\sin 2\beta & \cos 2\beta & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} \tag{11.27}$$

namely, $R_3(-2\beta)$. It is curious that when

$$\alpha = \frac{\pi}{2}$$
$$\beta = 0 \tag{11.28}$$

we recover again a rotation $R_3(-2\gamma)$.

In the general case,

$$uMu^+ = \begin{pmatrix}1+z' & x'-iy'\\ x'+iy' & 1-z'\end{pmatrix} \tag{11.29}$$

$$1+z' \equiv 1+\ z\cos 2\alpha + \left(e^{i(\beta-\gamma)}(x-iy) + e^{i(\gamma-\beta)}(x+iy)\right)\sin 2\alpha$$
$$x'-iy' \equiv e^{2i\beta}(x-iy)\cos^2\alpha - e^{2i\gamma}(x+iy)\sin^2\alpha - e^{i(\beta+\gamma)}z\sin 2\alpha$$
$$x'+iy' \equiv e^{-2i\beta}(x+iy)\cos^2\alpha - e^{-2i\gamma}(x-iy)\sin^2\alpha - e^{-i(\beta+\gamma)}z\sin 2\alpha$$
$$1-z' \equiv 1 - z\cos 2\alpha - \left(e^{i(\beta-\gamma)}(x-iy) + e^{i(\gamma-\beta)}(x+iy)\right)\sin 2\alpha \tag{11.30}$$

That is

$$R = \begin{pmatrix}\cos^2\alpha\cos 2\beta - \sin^2\alpha\cos 2\gamma & -\left(\cos^2\alpha\sin 2\beta + \sin^2\alpha\sin 2\beta\right) & -\sin 2\alpha\cos(\beta+\gamma)\\ -\cos^2\alpha\sin\beta + \sin^2\alpha\sin 2\gamma & \cos^2\alpha\cos 2\beta + \sin^2\alpha\cos 2\gamma & \sin 2\alpha\sin(\beta+\gamma)\\ \sin 2\alpha\cos(\beta-\gamma) & \sin 2\alpha\sin(\beta-\gamma) & \cos 2\alpha\end{pmatrix} \tag{11.31}$$

This means that in order to go from the unit vector along the third axis, $\vec{e}_3$ to an arbitrary unit vector corresponding to the polar angles (θ, ϕ) all we have to do is identify

$$-\sin 2\alpha\cos(\beta+\gamma) \equiv \sin\theta\cos\phi$$
$$\sin 2\alpha\sin(\beta+\gamma) \equiv \sin\theta\sin\phi$$
$$\cos 2\alpha \equiv \cos\theta \tag{11.32}$$

which can be achieved by letting

$$\alpha = \frac{\theta}{2}$$
$$\beta + \gamma = \pi - \phi \tag{11.33}$$

in $SU(2)$ language

$$u = \begin{pmatrix} \cos \frac{\theta}{2} & -\sin \frac{\theta}{2} e^{-i\phi} \\ \sin \frac{\theta}{2} e^{i\phi} & \cos \frac{\theta}{2} \end{pmatrix} \tag{11.34}$$

Staring again at this formula, we learn that when precisely

$$\beta = 0$$
$$\gamma = \frac{\pi}{2} \tag{11.35}$$

we recover a rotation around the first axis, $R_1(2\alpha)$

$$x' = x$$
$$y' = y \cos 2\alpha + z \sin 2\alpha$$
$$z' = -y \sin 2\alpha + z \cos 2\alpha \tag{11.36}$$

Euler showed that every rotation $R \in SO(3)$ can be written in the form

$$R = R_3(\psi)\ R_1(\theta)\ R_3(\phi) \tag{11.37}$$

The range of the Euler angles is

$$0 \leqslant \phi \leqslant 2\pi$$
$$0 \leqslant \theta \leqslant \pi$$
$$0 \leqslant \psi \leqslant 2\pi \tag{11.38}$$

In our $SU(2)$ language this is

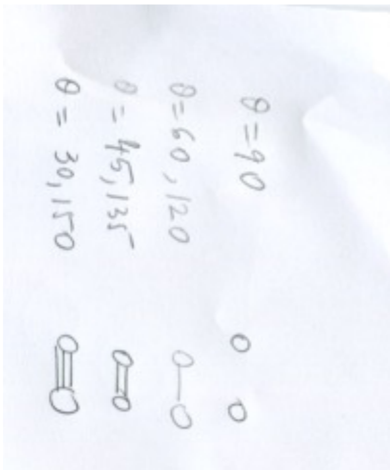
$$u = \begin{pmatrix} e^{i\frac{\psi}{2}} & 0 \\ 0 & e^{-i\frac{\psi}{2}} \end{pmatrix} \begin{pmatrix} \cos \frac{\theta}{2} & i \sin \frac{\theta}{2} \\ i \sin \frac{\theta}{2} & \cos \frac{\pi}{2} \end{pmatrix} \begin{pmatrix} e^{i\frac{\phi}{2}} & 0 \\ 0 & e^{-i\frac{\phi}{2}} \end{pmatrix} = \begin{pmatrix} e^{i\frac{\phi+\psi}{2}} \cos \frac{\theta}{2} & i\ e^{i\frac{\psi-\phi}{2}} \sin \frac{\theta}{2} \\ i\ e^{i\frac{\phi-\psi}{2}} \sin \frac{\theta}{2} & e^{-i\frac{\phi+\psi}{2}} \cos \frac{\theta}{2} \end{pmatrix}$$

It is plain that this covers the whole group manifold, provided

$$0 \leqslant \phi + \psi \leqslant 4\pi$$
$$0 \leqslant \phi - \psi \leqslant 4\pi$$
$$0 \leqslant \theta \leqslant \pi \tag{11.39}$$

Indeed

$$\psi + \phi = \beta$$
$$\psi - \phi = \gamma - \frac{\pi}{2}$$
$$\alpha = \frac{\theta}{2} \tag{11.40}$$



Descriptio 11.1: Euler angles

The relationship with Gel'fand's notation is

$$\begin{aligned}\psi &\rightarrow -\phi_1 \\ \phi &\rightarrow -\phi_2 \\ \theta &\rightarrow -\theta\end{aligned} \tag{11.41}$$

In $SO(3)$ language this is

$$R = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & \sin\phi\cos\psi + \cos\theta\sin\psi\cos\phi & \sin\theta\sin\psi \\ -\cos\phi\sin\psi - \cos\theta\cos\psi\sin\phi & -\sin\psi\sin\phi + \cos\theta\cos\psi\cos\phi & \sin\theta\cos\psi \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix}$$

Please note that this matriz transforms the unit vector along the third axis to the vector

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} \sin\theta\sin\psi \\ \sin\theta\cos\psi \\ \cos\theta \end{pmatrix} \tag{11.42}$$

corresponding to the direction $n = (\theta, \frac{\pi}{2} - \psi)$.

11.1 The Lie algebra $\mathfrak{SU}(2)$

Start with

$$T \in \mathfrak{SU}(2) \Longleftrightarrow T = T^{+} \quad \& \quad \operatorname{tr} T = 0 \tag{11.43}$$

The most general solotion is

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{11.44}$$

with

$$\begin{aligned} a &= \bar{a} \\ b &= \bar{c} \\ d &= -a \end{aligned} \tag{11.45}$$

That is

$$T = \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix} \equiv x\sigma_1 + y\sigma_2 + z\sigma_3 \equiv \vec{x}\vec{\sigma} \tag{11.46}$$

The Puli matrices generate the simplest Clifford algebra.

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij} \tag{11.47}$$

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \tag{11.48}$$

so that

$$\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k \tag{11.49}$$

11.2 Highest weight representations of $\mathfrak{SU}(2)$

Let us review the representations of the SU(2) algebra that you are already familiar with from quantum mechanics. We shall do it in a way which generalizes to arbitrary groups. The algebra reads

$$[J_i, J_j] = i\epsilon_{ijk} J_k \tag{11.50}$$

Define

$$J_\pm \equiv \frac{J_1 \pm J_2}{\sqrt{2}} \tag{11.51}$$

We are looking for finite dimensional unitary representations. Let us call j the highest value of J_3.

$$J_3|j,\alpha\rangle = j|j,\alpha\rangle \tag{11.52}$$

First of all, just because it is a highest weight state, we can easoily determine the value of the casimir

$$J^2 \equiv J_1^2 + J_2^2 + J_3^2 \tag{11.53}$$

$$J_- J_+|j,j\rangle = 0 = \left(J^2 - J_3^2 - J_3\right)|j,j\rangle \tag{11.54}$$

then

$$J^2|j,j\rangle = j(j+1)|j,j\rangle \tag{11.55}$$

Were there more than one highest weight state, we normalize as

$$\langle j\alpha|j\beta\rangle = \delta_{\alpha\beta} \tag{11.56}$$

If we define

$$J^\pm \equiv \frac{J_1 \pm iJ_2}{\sqrt{2}} \tag{11.57}$$

Then

$$\begin{aligned} [J_3, J^\pm] &= \pm J^\pm \\ [J^+, J^-] &= J_3 \end{aligned} \tag{11.58}$$

so that

$$J_3 J^\pm|m\rangle = J^\pm m|m\rangle \pm J^\pm|m\rangle = (m \pm 1)J^\pm|m\rangle \tag{11.59}$$

We have assumed from the beginning that there is no state with $J_3 = m+1$; then it must be the case that $\forall\alpha$

$$J^+|j,\alpha\rangle = 0 \tag{11.60}$$

as well as

$$J^-|j\alpha\rangle = N_j(\alpha)|j-1,\alpha\rangle \tag{11.61}$$

Let us compute

$$\overline{N_j(\beta)}N_j(\alpha)\langle j-1,\beta|j-1,\alpha\rangle = \langle j,\alpha|J^+J^-|j,\alpha\rangle = \langle j,\alpha|\left[J^+J^-\right]|j,\alpha\rangle = \\ = \langle j,\alpha|J_3 = |j,\alpha\rangle = j\delta_{\alpha\beta} \tag{11.62}$$

Then we learn that

$$N_j(\alpha) = N_j = \sqrt{j} \tag{11.63}$$

On the other hand

$$J^+|j-1,\alpha\rangle = \frac{1}{N_j}|J^+J^-|j,\alpha\rangle = \sqrt{j}|j,\alpha\rangle \tag{11.64}$$

In general

$$J^-|j-k,\alpha\rangle = N_{j-k}|j-k-1,\alpha\rangle \\ J^+|j-k-1,\alpha\rangle = \bar{N}_{j-k}|j-k,\alpha\rangle \tag{11.65}$$

Actually,

$$N_{j-k} = \langle j-k-1|J_-|j-k\rangle \\ \bar{N}_{j-k} = \langle j-k|J^+|j-k-1\rangle = N^*_{j-k} \tag{11.66}$$

We choose phases in such a way that

$$N_{j-k} = \bar{N}_{j-k} \tag{11.67}$$

$$|N_{j-k}|^2 = \langle j-k,\alpha|J^+J^-|j-k,\alpha\rangle = \langle j-k,\alpha|\left[J^+J^-\right]|j-k,\alpha\rangle + \langle j-k,\alpha|J^-J^+|j-k,\alpha\rangle = \\ = |N_{j-k-1}|^2 + j - k \tag{11.68}$$

Then we have a series of the type

$$a_k = a_{k-1} + j - k = a_{k-2} + j - k + j - k + j - (k-1) = \ldots = a_0 - kj - \frac{k(k+1)}{2} \tag{11.69}$$

that is

$$N^2_{j-k} = (k+1)j - \frac{k(k+1)}{2} = \frac{k+1}{2}(2j-k) \tag{11.70}$$

in other words,

$$N_m = \sqrt{\frac{(j+m)(j-m+1)}{2}} \tag{11.71}$$

We are looking for finite dimensional representations. This means that necessarily we must real some $m \equiv j - l$ such that

$$J^-|j-l,\alpha\rangle = 0 \tag{11.72}$$

This is only possible if there is a certain value of $k = l$ such that

$$0 = N_{j-l} = \sqrt{\frac{(2j-l)(l+1)}{2}} \tag{11.73}$$

which means

$$l = 2j \tag{11.74}$$

We learn that

$$j = \frac{l}{2} \tag{11.75}$$

where $l \in \mathbb{N}$, just because it counts the number of times we have applied the operator J_- . Besides, from now on we can drop the index α.

We can summarize, in the usual notation

$$\begin{aligned} \langle j, m'|J_3|j,m\rangle &= m\ \delta_{m',m} \\ \langle j, m'|J^+|j,m\rangle &= \sqrt{\tfrac{(j+m+1)(j-m}{2}}\delta_{m',m+1} \\ \langle j, m'|J^-|j,m\rangle &= \sqrt{\tfrac{(j+m)(j-m+1)}{2}}\delta_{m',m-1} \end{aligned} \tag{11.76}$$

11.3 Spherical Harmonics

Let us assume there is an action of G in M, that is

$$G \times M \to M \tag{11.77}$$

$$(g, x) \to g.x \tag{11.78}$$

Then there is a representation of the group in the space of functions on M, $\mathfrak{F}(\mathfrak{M})$

$$f \in \mathfrak{F}(\mathfrak{M}) \to (T_g f)(x) \equiv f\left(g^{-1}x\right) \in \mathfrak{F}(\mathfrak{M}) \tag{11.79}$$

It is indeed a representation, because

$$T_g (T_h f)(x) = T_g f\left(h^{-1}x\right) = f\left(h^{-1}g^{-1}x\right) = f\left((gh)^{-1}x\right) = (T_{gh} f)(x)) \tag{11.80}$$

Consider now the two-sphere, $M \equiv S_2$. Let us consider an infinitesimal (negative) rotation around the axis OZ. It must be so that

$$(T_g f)(\theta,\phi) \equiv f(\theta, \phi - \alpha) = f(\theta,\phi) - \alpha\frac{\partial f}{\partial \phi} + \ldots \tag{11.81}$$

Then

$$A_3 \equiv -\frac{\partial}{\partial \phi} \tag{11.82}$$

Now consider the (again, negative) rotation around the axis OX.

$$\begin{aligned} x' &= x \\ y' &= y\cos\,\alpha + z\sin\,\alpha \\ z' &= -y\sin\,\alpha + z\cos\,\alpha \end{aligned} \tag{11.83}$$

It follows that

$$\begin{aligned} \tfrac{dx}{d\alpha}\big|_{\alpha=0} &= 0 = \cos\,\theta\,\tfrac{d\theta}{d\alpha}\,\cos\,\phi - \sin\,\phi\,\tfrac{d\phi}{d\alpha}\,\sin\,\theta \\ \tfrac{dy}{d\alpha}\Big|_{\alpha=0} &= -z = -\cos\,\theta = \cos\,\theta\tfrac{d\theta}{d\alpha}\,\sin\,\phi + \sin\,\theta\,\cos\,\phi\,\tfrac{d\phi}{d\alpha} \\ \tfrac{dz}{d\alpha}\big|_{\alpha=0} &= y = \sin\,\theta\,\cos\,\phi = -\sin\,\theta\,\tfrac{d\theta}{d\alpha} \end{aligned} \tag{11.84}$$

which yields immediately

$$\begin{aligned} \tfrac{d\theta}{d\alpha} &= -\sin\,\phi \\ \tfrac{d\phi}{d\alpha} &= \tfrac{\cos\,\theta}{\sin\,\theta}\,\cos\,\phi \end{aligned} \tag{11.85}$$

so that

$$A_1 = \sin\,\phi\frac{\partial}{\partial\theta} + \cot\,\theta\,\cos\phi\,\frac{\partial}{\partial\phi} \tag{11.86}$$

In an analogous way we get

$$A_2 = -\cos\,\phi\,\frac{\partial}{\partial\theta} + \cot\,\theta\,\sin\,\phi\,\frac{\partial}{\partial\phi} \tag{11.87}$$

The hermiyian generators are

$$H_i \equiv iA_i \tag{11.88}$$

$$\begin{aligned} H_+ \equiv H_1 + iH_2 \equiv iA_1 - A_2 &= e^{i\phi}\left(\tfrac{\partial}{\partial\theta} + i\cot\,\theta\tfrac{\partial}{\partial\phi}\right) \\ H_- \equiv H_1 - iH_2 \equiv iA_1 + A_2 &= e^{-i\phi}\left(-\tfrac{\partial}{\partial\theta} + i\cot\,\theta\tfrac{\partial}{\partial\phi}\right) \\ H_3 = iA_3 &= -i\tfrac{\partial}{\partial\phi} \end{aligned} \tag{11.89}$$

Let us denot the $(2l+1)$ eigenfunctions corresponding to weight l by

$$Y_{lm}\,(\theta,\phi) \quad m = -l,\ldots,l \tag{11.90}$$

First of all, we want that

$$H_3 Y_{lm} = -i\frac{\partial}{\partial\phi}Y_{lm} = mY_{lm} \tag{11.91}$$

Then

$$Y_{lm}(\theta,\phi) = \frac{1}{\sqrt{2\pi}}e^{im\phi}\,F_l^m(\theta) \tag{11.92}$$

is normalized by

$$\int |Y_l^m(\theta,\phi)| \sin\,\theta\, d\theta\, d\phi = 1 \tag{11.93}$$

provided

$$\int_0^\pi \sin\,\theta\, d\theta\, |F_l^m(\theta)|^2 = 1 \tag{11.94}$$

Let us now impose that

$$H^2 Y_l^m = l(l+1) Y_l^m \tag{11.95}$$

$$-H^2 = \frac{1}{\sin\,\theta}\frac{\partial}{\partial\theta}\left(\sin\,\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\,\theta}\frac{\partial^2}{\partial\phi^2} \tag{11.96}$$

In terms of $\mu \equiv \cos\,\theta$, and defining $P_l^m(\mu) \equiv F_l^m(\cos\,\theta)$, the ODE reads

$$\frac{d}{d\mu}\left((1-\mu^2)\frac{dP_l^m(\mu)}{d\mu}\right) + \left(l(l+1) - \frac{m^2}{1-\mu^2}\right) P_l^m(\mu) = 0 \tag{11.97}$$

which defines the (normalized) *associated Legendre functions*

$$P_l^m(\mu) \equiv \sqrt{\frac{(l+m)!}{(l-m)!}}\sqrt{\frac{2l+1}{2}}\frac{1}{2^l l!}\left(1-\mu^2\right)^{-\frac{m}{2}}\frac{d^{l-m}}{d\mu^{l-m}}(\mu^2-1)^l \tag{11.98}$$

The functions $P_l(\mu) \equiv P_l^0(\mu)$ happen to be polynomials; the *Legendre polynomial* of order l.

$$P_l(\mu) \equiv \sqrt{\frac{2l+1}{2}}\frac{1}{2^l l!}\frac{d^l}{d\mu^l}\left(\mu^2-1\right)^l \tag{11.99}$$

11.4 Spinor representations

No all representations of $SU(2)$ are also representations of $SO(3)$, only those with $l \in \mathbb{N}$ qualify for that. The rest, that is, the ones such that $l \in \frac{2\mathbb{N}+1}{2}$ are the famous *spinor representations*, sometimes called somewhat confusingly, *bivalued representations* of $SO(3)$.

First of all, for the $s = \frac{1}{2}$ representation

$$v^\alpha \rightarrow \frac{1}{2}\left(\sigma^a\right)^\alpha_\beta v^\beta \tag{11.100}$$

In particular

$$H_3 \equiv -\frac{1}{2}\sigma_3\begin{pmatrix}1\\0\end{pmatrix} = -\frac{1}{2}\begin{pmatrix}1\\0\end{pmatrix} \tag{11.101}$$

$$H_3 \equiv -\frac{1}{2}\sigma_3\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}0\\1\end{pmatrix} \tag{11.102}$$

Let us denote

$$e_\alpha : \quad e_1 \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad e_2 \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{11.103}$$

It is easy to find the space of functions for such representations. It is the space of symmetric spinors with $n = 2s$ indices.

$$A = a^{(\alpha_1 \ldots \alpha_n)} e_{\alpha_1} \otimes \ldots \otimes e_{\alpha_n} \quad a_i = 1, 2. \tag{11.104}$$

The $SU(2)$ action is given by

$$a^{(\alpha'_1 \ldots \alpha'_{2s})} e_{\alpha'_1} \otimes \ldots \otimes e_{\alpha'_n} \equiv a^{(\alpha_1 \ldots \alpha_{2s})} \tau^{\alpha'_1}_{\alpha_1} \otimes \ldots \otimes \tau^{\alpha'_{2s}}_{\alpha_{2s}} e_{\alpha_1} \otimes \ldots \otimes e_{\alpha_n} \tag{11.105}$$

It is a fact that

$$H_3\, a^{(a_1 \ldots a_n)} e_{\alpha_1} \otimes \ldots \otimes e_{\alpha_n} \equiv a^{(a_1 \ldots a_n)} \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \otimes \ldots \otimes \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} a^{(a_1 \ldots a_n)} =$$
$$= \tfrac{p_2 - p_1}{2}\, a^{(a_1 \ldots a_n)} e_{\alpha_1} \otimes \ldots \otimes e_{\alpha_n} \tag{11.106}$$

where p_1 counts the number of times the value 1 appears amongst the set of indices, and p_2 likewise for the value 2.

We need

$$\begin{aligned} p_1 &= l - m \\ p_2 &= l + m \end{aligned} \tag{11.107}$$

in order that

$$H_3\, a^{(a_1 \ldots a_n)} e_{\alpha_1} \otimes \ldots \otimes e_{\alpha_n} = m\, a^{(a_1 \ldots a_n)} e_{\alpha_1} \otimes \ldots \otimes e_{\alpha_n} \tag{11.108}$$

- Let us compute $\vec{J}^2$ for $s = \frac{1}{2}$.

$$\vec{J}^2 \psi_a \equiv \frac{\vec{\sigma}^l_a}{2} \frac{\vec{\sigma}^j_l}{2} \psi_j = \frac{1}{4} 3 \psi_a = s(s+1)\psi_a \tag{11.109}$$

- Let us repeat now the computation for $s = 1$

$$\left(\vec{J}\psi\right)_{(ij)} \equiv \frac{\vec{\sigma}^a_{(i}}{2} \psi_{a|j)} + \frac{\vec{\sigma}^b_{(j}}{2} \psi_{i|b}$$
$$\left(\vec{J}^2\psi\right)_{(ij)} \equiv \frac{\vec{\sigma}^a_{(i}}{2} \left(\vec{J}\psi\right)_{a|j)} + \frac{\vec{\sigma}^b_{(j}}{2} \left(\vec{J}\psi\right)_{i|b} = \frac{\vec{\sigma}^a_i}{2} \left(\frac{\vec{\sigma}^u_a}{2}\psi_{uj} + \frac{\sigma^u_j}{2}\psi_{au}\right) +$$
$$+ \frac{\sigma^b_j}{2} \left(\frac{\vec{\sigma}^u_i}{2}\psi_{ub} + \frac{\sigma^u_b}{2}\psi_{iu}\right) = 2\psi_{ij} = s(s+1)\psi_{ij} \tag{11.110}$$

We have used repeteadly

$$\vec{\sigma}^j_i \vec{\sigma}^l_k = 2\delta^l_i \delta^j_k - \delta^j_i \delta^l_k \tag{11.111}$$

- Diagonalizing σ_3

$$(\sigma_3)_i^j\,\zeta_j^{\hat{z}\pm} \equiv \pm\zeta_i^{\bar{z}\pm} \tag{11.112}$$

An eigenstate pf J_3 with eigenvalue $j_{\hat{z}}$ will be given by

$$\psi^{sj_{\hat{z}}} = \left(\zeta^{\hat{z}+}\right)^{s+j_{\hat{z}}} \left(\zeta^{\hat{z}-}\right)^{s-j_{\hat{z}}} \tag{11.113}$$

We can simplify the notation as shown because all indices are totally symmetrized anyway.

Denote

$$\zeta_i \equiv \alpha_+\zeta_i^{\hat{z}+} + \alpha_-\zeta_i^{\hat{z}-} \tag{11.114}$$

Then

$$\zeta^{i_1}\ldots\zeta^{i_{2s}}\psi_{i_1\ldots i_{2s}} = \sum_{j_{\hat{z}}} \alpha_+^{s+j_{\hat{z}}}\alpha_l^{s-j_{\hat{z}}}\,\langle sj_{\hat{z}}|\psi\rangle \tag{11.115}$$

For example, when $s = 1/2$

$$\alpha_+\zeta^{+i}\psi_i + \alpha_-\zeta^{-i}\psi_i = \alpha_+^1\alpha_-^0\,\left\langle \frac{1}{2}\,\frac{1}{2}\right|\psi\left.\vphantom{\frac{1}{2}}\right\rangle + \alpha_+^0\alpha_-^1\,\left\langle \frac{1}{2}\,-\frac{1}{2}\right|\psi\left.\vphantom{\frac{1}{2}}\right\rangle \tag{11.116}$$

- For a general direction, $\hat{n} \equiv (\theta, \phi)$

$$\begin{pmatrix}\zeta^{\hat{n},+}\\ \zeta^{\hat{n},-}\end{pmatrix} = \begin{pmatrix}c & -s^*\\ s & c\end{pmatrix}\begin{pmatrix}\zeta^{\hat{z},+}\\ \zeta^{\hat{k},-}\end{pmatrix} \tag{11.117}$$

with

$$\begin{aligned} c &\equiv \cos\,\tfrac{\theta}{2}\\ s &\equiv e^{i\phi}\sin\,\tfrac{\theta}{2} \end{aligned} \tag{11.118}$$

Then

$$\begin{aligned}\psi^{s,j_{\hat{n}}} &= \left(\zeta^{\hat{n},+}\right)^{s+j_{\hat{n}}}\left(\zeta^{\hat{n},-}\right)^{s-j_{\hat{n}}} = \left(c\,\zeta^{\hat{z},+} - s\,\zeta^{\hat{z},-}\right)^{s+j_{\hat{n}}}\left(s^*\,\zeta^{\hat{z},+} + c\,\zeta^{\hat{z},-}\right)^{s-j_{\hat{n}}} =\\ &= \textstyle\sum_{pq}\binom{s+j_{\hat{n}}}{p}\binom{s-j_{\hat{n}}}{q}\left(c\zeta^{\hat{z},+}\right)^{s+j_{\hat{n}}-p}\left(-s\,\zeta^{\hat{z},-}\right)^p\left(s^*\,\zeta^{\hat{z},+}\right)^{s-j_{\hat{n}}-q}\left(c\,\zeta^{\hat{z},-}\right)^q \equiv\\ &\equiv \textstyle\sum_{j_{\hat{z}}} R^s_{j_{\hat{n}},\hat{z}}\,(\theta,\phi)\;\psi^{s,j_{\hat{z}}}\end{aligned} \tag{11.119}$$

where

$$R^s_{j_{\hat{n}},\hat{z}}\,(\theta,\phi) \equiv \sum_{m_\pm;\,m_++m_-=s+j_{\hat{z}}}\binom{s+j_{\hat{n}}}{m_+}\binom{s-j_{\hat{n}}}{m_-}\,c^{m_+}(-s)^{s+j_{\hat{n}}-m_+}\,c^{s-j_{\hat{n}}-m_-}\,(s^*)^{m_-} \tag{11.120}$$

- Let us revisit spherical harmonics from the spinor viewpoint. Let us define as usual

$$x_i^j \equiv \vec{x}\,\vec{\sigma}_i^j = \begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix} \tag{11.121}$$

AH4 also define

$$x_{ij} \equiv \epsilon_{ik} x_j^k = \begin{pmatrix} -(x-iy) & z \\ z & x+iy \end{pmatrix} \tag{11.122}$$

in order to build irreps

$$\zeta^{i_1}\zeta^{j_1}\ldots\zeta^{i_s}\zeta^{j_s} x_{i_1 j_1}\ldots x_{i_s j_s} \equiv (\zeta\zeta x)^s \tag{11.123}$$

Let us choose x as the unit vector with

$$\begin{aligned} x+iy &= \sin\,\theta e^{i\phi} \\ z &\equiv \cos\,\theta \\ \zeta_i &= (\alpha_+, \alpha_-) \\ \zeta^i &\equiv \epsilon^{ij}\zeta_j = (-\alpha_-, \alpha_+) \end{aligned} \tag{11.124}$$

Now it is a fact that

$$\begin{pmatrix} -\alpha_- & \alpha_+ \end{pmatrix} \begin{pmatrix} -s^* & c \\ c & s \end{pmatrix} \begin{pmatrix} -\alpha_- & \alpha_- \end{pmatrix} = -s^*\alpha_-^2 - 2c\alpha_+\alpha_- + s\alpha_+^2 \tag{11.125}$$

11.5 Product representations

It is possible to construct the tensor product of two irreps.

$$D(g) \equiv D_1 \otimes D_2 \tag{11.126}$$

The basis of the product space is just the tensor product of the two basis

$$e_1 \otimes e_2 \tag{11.127}$$

This is trivially a representation. Its action on the natural basis is given by

$$D_g\,(e_1 \otimes e_2) = e^{i\alpha A}\,(e_1 \otimes e_2) = e_1 \otimes e_2 + (\alpha A e_1) \otimes e_2 + e_1 \otimes (\alpha A e_2) + \ldots \tag{11.128}$$

This should be familiar from the addition of angular momentum in quantum mechanics. It is clear that the generators of

$$D \equiv (1+T) \otimes (1+T) \tag{11.129}$$

are

$$(1 \otimes T) \oplus (T \otimes 1) \tag{11.130}$$

In $\mathfrak{SU}(2)$, since we work in a basis where J_3 is diagonal, the values of J_3 just add.

$$J_3\left(|j_1 m_1\rangle \otimes |j_2 m_2\rangle\right) = (m_1 + m_2)_3 \;\; |j_1 m_1\rangle \otimes |j_2 m_2\rangle \tag{11.131}$$

Consider, for example, the product of the three-dimensional irrep with the two-dimensional one, $1 \otimes 1/2$. We shall analyze this tensor product by the familiar highest weight technique.

The highest weight state is unique

$$|3/2, 3/2\rangle \equiv |1/2, 1/2\rangle \otimes |1, 1\rangle \tag{11.132}$$

Now, remembering that

$$J^- |j, m\rangle = \sqrt{\frac{(j+m)(j-m+1)}{2}} |j, m-1\rangle \tag{11.133}$$

we get

$$J^- |3/2, 3/2\rangle = \sqrt{\frac{3}{2}} |3/2, 1/2\rangle = \sqrt{\frac{1}{2}} |1/2, -1/2\rangle \otimes |1, 1\rangle + |1/2, 1/2\rangle \otimes |1, 0\rangle \tag{11.134}$$

$$|3/2, 1/2\rangle = \sqrt{\frac{1}{3}} |1/2, -1/2\rangle \otimes |1, 1\rangle + \sqrt{\frac{2}{3}} |1/2, 1/2\rangle \otimes |1, 0\rangle \tag{11.135}$$

There is an state orthogonal

$$|\psi\rangle = \sqrt{\frac{2}{3}} |1/2, -1/2\rangle \otimes |1, 1\rangle - \sqrt{\frac{1}{3}} |1/2, 1/2\rangle \otimes |1, 0\rangle \tag{11.136}$$

This will be later taken as the highest weight of another chain.

$$J^- |3/2, 1/2\rangle = \sqrt{2} |3/2, -1/2\rangle = \sqrt{\tfrac{1}{3}} |1/2, -1/2\rangle \otimes |1, 0\rangle + \sqrt{\tfrac{2}{3}} \sqrt{\tfrac{1}{2}} |1/2, -1/2\rangle \otimes |1, 0\rangle + \sqrt{\tfrac{2}{3}} |1/2, 1/2\rangle \otimes |1, -1\rangle = \sqrt{\tfrac{4}{3}} |1/2, -1/2\rangle \otimes |1, 0\rangle + \sqrt{\tfrac{1}{3}} |1/2, 1/2\rangle \otimes |1, -1\rangle \tag{11.137}$$

then

$$|3/2, -1/2\rangle = \sqrt{\frac{2}{3}} |1/2, -1/2\rangle \otimes |1, 0\rangle + \sqrt{\frac{1}{3}} |1/2, 1/2\rangle \otimes |1, -1\rangle = \tag{11.138}$$

Here also there is another state orthogonal

$$|\chi\rangle \equiv \sqrt{\frac{2}{3}} |1/2, -1/2\rangle \otimes |1, 0\rangle - \sqrt{\frac{1}{3}} |1/2, 1/2\rangle \otimes |1, -1\rangle \tag{11.139}$$

Let us apply the operator J^- once more

$$J^-|3/2,-1/2\rangle = |3/2,-3/2\rangle = \sqrt{\frac{2}{3}}|1/2,-1/2\rangle\otimes|1,-1\rangle+\sqrt{\frac{1}{3}}|1/2,-1/2\rangle\otimes|1,-1\rangle = |1/2,-1/2\rangle\otimes|1,-1\rangle \tag{11.140}$$

Let us now check that the state $|\psi$ is a good candidate for a highest weight state. For this to be true it is necessary that

$$J^+|\psi\rangle \equiv 0 \tag{11.141}$$

then

$$\begin{aligned} J^-|\psi\rangle = \sqrt{\tfrac{2}{3}}|1/2,-1/2\rangle\otimes|1,0\rangle - \sqrt{\tfrac{1}{3}}\sqrt{\tfrac{1}{2}}|1/2,-1/2\rangle\otimes 1,0\rangle - \sqrt{\tfrac{1}{3}}|1/2,1/2\rangle\otimes|1,-1\rangle = \\ = \sqrt{\tfrac{2}{3}}|1/2,-1/2\rangle\otimes 1,0\rangle - \sqrt{\tfrac{1}{3}}|1/2,1/2\rangle\otimes|1,-1\rangle \end{aligned} \tag{11.142}$$

that is, we identify the two orthogonal states we have obtained as

$$\begin{aligned} \psi\rangle &\equiv |1/2,1/2\rangle \\ |\chi\rangle &\equiv |1/2,-1/2\rangle \end{aligned} \tag{11.143}$$

That is,

$$1\otimes 1/2 = 3/2\oplus 1/2 \tag{11.144}$$

11.6 Wigner-Eckart

A tensor operator $\mathcal{O}^s_l \quad l=-s\ldots+s)$ transforming under the spin-s representation of $\mathfrak{SU}(2)$ is a set of 2s+1 operators such that

$$[J_a,\mathcal{O}^s_l] = \mathcal{O}^s_m\,(J^s_a)_{ml} \tag{11.145}$$

In the standard basis

$$(J^s_3)_{ll'} = l\delta_{ll'} \tag{11.146}$$

$(-s\leqslant l,l'\leqslant s)$; so that

$$[J_3,\mathcal{O}^s_l] = l\mathcal{O}^s_l \tag{11.147}$$

A trivial example is a particle in an spherically symmetric potential. Then

$$J_a = L_a \equiv \epsilon_{abc}x_b p_c \tag{11.148}$$

and

$$[J_a,x_b] = -i\epsilon_{acb}x_c = x_c\left(J^{adj}_a\right)_{cb} \tag{11.149}$$

To go the canonical basis, first realize that

$$x_0 = x_3 \tag{11.150}$$

ans then

$$[J^{\pm}, x_0] \equiv x_{\pm 1} = \mp \frac{x_1 \pm i x_2}{\sqrt{2}} \tag{11.151}$$

Twnsor operators have got the interesting property that

$$J_a \mathcal{O}^s_l |jm\alpha\rangle = \mathcal{O}^s_{l'} |jm\alpha\rangle \left(J^s_a\right)_{l'l} + \mathcal{O}^s_l |jm'\alpha\rangle \left(J^j_a\right)_{m'm} \tag{11.152}$$

this is the transformation of the tensor product

$$s \otimes j \tag{11.153}$$

Note inprticular that

$$J_3 \mathcal{O}^s_l |jm\alpha\rangle = (l+m) \mathcal{O}^s_l |jm\alpha\rangle \tag{11.154}$$

The Wigner-Eckart theorem states that

$$\langle J, m', \beta | \mathcal{O}^s_l | j, m, \alpha\rangle = \delta_{m', l+m} \langle J, l+m | s, j, l, m\rangle \langle J, \beta | \mathcal{O}^s | j, \alpha\rangle \tag{11.155}$$

Let us work out an example in detail.Let us assume known the matrix element

$$\langle \frac{1}{2}, \frac{1}{2}, \alpha | x_3 | \frac{1}{2}, \frac{1}{2}, \beta\rangle \equiv A \tag{11.156}$$

and we would like to compute $\langle \frac{1}{2}, \frac{1}{2}, \alpha | x_1 | \frac{1}{2}, \frac{1}{2}, \beta\rangle$. First,

$$x_1 \equiv \frac{1}{\sqrt{2}} \left(-x_{+1} + x_{-1}\right) \tag{11.157}$$

Starting with the highest weight state

$$\left|\frac{3}{2}, \frac{3}{2}\right\rangle \equiv x_{+1} \left|\frac{1}{2}, \frac{1}{2}\right\rangle \tag{11.158}$$

we get

$$\left|\frac{3}{2}, \frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}} J^- \left|\frac{3}{2}, \frac{3}{2}\right\rangle = \sqrt{\frac{2}{3}} J^- x_{+1} \left|\frac{1}{2}, \frac{1}{2}\right\rangle \tag{11.159}$$

But using

$$J^- x_{+1} = x_0 + x_{+1} J^- \tag{11.160}$$

$$\left|\frac{3}{2}, \frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}} x_0 \left|\frac{1}{2}, \frac{1}{2}\right\rangle + \sqrt{\frac{1}{3}} x_{+1} \left|\frac{1}{2}, -\frac{1}{2}\right\rangle \tag{11.161}$$

Finally

$$0 = \langle \tfrac{1}{2}, \tfrac{1}{2} | \tfrac{3}{2}, \tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}} \langle \tfrac{1}{2}, \tfrac{1}{2} | x_0 | \tfrac{1}{2}, \tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{3}} \langle \tfrac{1}{2}, \tfrac{1}{2} | x_{+1} | \tfrac{1}{2}, -\tfrac{1}{2}\rangle \tag{11.162}$$

This implies that

$$\left\langle \frac{1}{2}, \frac{1}{2}\right| x_{+1} \left|\frac{1}{2}, -\frac{1}{2}\right\rangle = -\sqrt{2}\, A \tag{11.163}$$

and finally

$$\left\langle \frac{1}{2}, \frac{1}{2}\right| x_1 \left|\frac{1}{2}, -\frac{1}{2}\right\rangle = A \tag{11.164}$$

12

Roots and weights

A *Cartan subalgebra* is a maximal abelian subalgebra; that is, a set of commuting generators $H_i = H_i^+$ as large as possible

$$[H_i, H_j] = 0 \tag{12.1}$$

The dimension of the Cartan subalgebra is called the *rank* of the group. In the case of $SU(2)$, the rank is one and the only H is precisely J_3. The normalization in a given irrep is defined (Georgi) as

$$tr\ \ (H_i H_j) \equiv k_D \delta_{ij} \tag{12.2}$$

Humphreys defines a symmetric bilinear form as

$$\beta(X, Y) \equiv \text{tr}\ (D(X), D(Y)) \tag{12.3}$$

which then uses to define the dual basis of the Lie algebra $\mathfrak{L}$. .

In a more intrinsic way, $Ad_{\mathfrak{L}}\,\mathfrak{H}$ is simultaneously diagonalizable. That is, $\mathfrak{L}$ is the direct sum of the subspaces

$$\mathfrak{L}_\alpha \equiv \{\, X \in \mathfrak{L} \quad [H, X] = \alpha(H)\, X \quad \forall H \in \mathfrak{H}\} \tag{12.4}$$

where $\alpha \in H^*$. It is plain that $\mathfrak{L}_0$ is simply $C_{\mathfrak{L}}\,\mathfrak{H}$, the centralizer of $\mathfrak{H}$.

The set of nonzero roots $\alpha \in H^*$ is denoted by Φ. This yields the Cartan decomposition of the Lie algebra

$$\mathfrak{L} = C_{\mathfrak{L}}\,\mathfrak{H} \oplus \cup_{\alpha\in\Phi}\,\mathfrak{L}_\alpha \tag{12.5}$$

It can be proved that the restriction of κ to H is nondegenerate. This allows to identificate H with H^*:

$$\phi \in H* \rightarrow T_\phi \in H \tag{12.6}$$

such that

$$\phi(H) = \kappa(T_\phi, H) \tag{12.7}$$

The states of a given irrep will read

$$H_i|\mu\rangle = \mu_i|\mu\rangle \tag{12.8}$$

The eigenvalues are dubbed *weights*. They obey

$$\mu_i = \bar{\mu}_i \tag{12.9}$$

because they are eigenvalues of a hermitian operator. The vector $\mu_i \in \mathbb{R}_m$ is called a *weight vector*.

Let us remind ourselves of the *adjoint representation*. In order to define it, consider a linear space with an state associated to every generator

$$X_a \longleftrightarrow |X_a\rangle \tag{12.10}$$

with the scalar product defined as

$$\langle X_a|X_b\rangle \equiv \frac{1}{\lambda}\; tr\; X_a^+ X_b \tag{12.11}$$

in such a way that

$$\langle H_i|H_j\rangle = \delta_{ij} \tag{12.12}$$

It is plain that

$$\begin{aligned} X_a|X_b\rangle = \textstyle\sum_c |X_c\rangle\langle X_c|X_a|X_b\rangle = \sum_c X_c \left(D_a^{adj}\right)_{cb} \equiv -if_{acb}|X_c = \\ = if_{abc}|X_c\rangle = |if_{abc}X_c\rangle = |\,[X_a, X_b]\rangle \end{aligned} \tag{12.13}$$

It is plain that for the states corresponding to the Crtan generators the weight vanishes

$$H_i|H_j\rangle = |\,[H_i, H_j]\rangle = 0 \tag{12.14}$$

The other states have non zero weight vectors

$$H_i|E_\alpha\rangle = \alpha_i|E_\alpha\rangle \tag{12.15}$$

This equivalent to

$$[H_i, E_\alpha] = \alpha_i E_\alpha \tag{12.16}$$

This generators cannot be hermitian, because

$$\left[H_i, E_\alpha^+\right] = -\alpha_i E_\alpha^+ \tag{12.17}$$

which means that

$$E_{-\alpha} = E_\alpha^+ \tag{12.18}$$

This is the generalization of the well-known elements $J_\pm$ in the $SU(2)$ case. Is it possible to normalize in such a way that

$$\langle E_\alpha|E_\beta\rangle = \frac{1}{\lambda}\; tr\; E_\alpha^+ E_\beta = \delta_{\alpha\beta} \tag{12.19}$$

It is fact of life that the $E_{\pm\alpha}$ are lowering and raising operators for the weights. Starting from

$$H_i|\mu\rangle = \mu_i|\mu\rangle \tag{12.20}$$

we get

$$H_i E_{\pm\alpha}|\mu\rangle = \left([H_i, E_{\pm\alpha}] + E_{\pm\alpha}H_i\right)|\mu\rangle = \left(\pm\alpha_i + \mu_i\right)E_{\pm\alpha}|\mu\rangle \tag{12.21}$$

In particular, the state

$$E_\alpha|E_{-\alpha}\rangle \tag{12.22}$$

has zero weight, so that it must be a linear combination of Cartan generators.

$$E_\alpha|E_{-\alpha}\rangle = \beta_i|H_i\rangle = |\beta.H\rangle = |\left[E_\alpha, E_{-\alpha}\right]\rangle \tag{12.23}$$

The constants β_i are given by

$$\beta_i = \langle H_i|E_\alpha|E_{-\alpha}\rangle = \frac{1}{\lambda}\, tr\ \left(H_i\left[E_\alpha, E_{-\alpha}\right]\right) = \frac{1}{\lambda}\, tr\ \left(E_{-\alpha}\left[H_i, E_\alpha\right]\right) = \frac{1}{\lambda}\alpha_i\, tr\ E_{-\alpha}E_\alpha = \alpha_i \tag{12.24}$$

We conclude that

$$[E_\alpha, E_{-\alpha}] = \alpha.H \tag{12.25}$$

It so happens that for any non-zero pair of root vectors, $\pm\alpha$, there is an $SU(2)$ subalgebra, with generators

$$\begin{aligned} J_\pm &\equiv \tfrac{1}{|\alpha|}\, E_{\pm\alpha} \\ J_3 &\equiv \tfrac{\alpha.H}{|\alpha|^2} \end{aligned} \tag{12.26}$$

Indeed,

$$\begin{aligned} \left[\tfrac{1}{|\alpha|}E_\alpha, \tfrac{1}{|\alpha|}E_{-\alpha}\right] &= \tfrac{\alpha.H}{\alpha^2} \\ \left[\tfrac{\alpha.H}{|\alpha|^2}, E_{\pm\alpha}\right] &= E_{\pm\alpha} \end{aligned} \tag{12.27}$$

From that we can prove, for example, that **root vectors correspond to unique generators**.

Demonstratio. Let us assume that there are two, E_α and E'_α and we shall get a contradiction. Choose adequate linear combinations in such a way that

$$\langle E_\alpha|E'_\alpha\rangle \equiv \frac{1}{\lambda}\mathrm{tr}\left(E^+_\alpha E'_\alpha\right) = \frac{1}{\lambda}\mathrm{tr}\left(E_{-\alpha}E'_\alpha\right) = 0 \tag{12.28}$$

We now act with the J_-. This has zero weight vector, so that it is in the Cartan subalgebra. But

$$\langle H_i|J_-|E'_\alpha\rangle = \frac{1}{\lambda}\mathrm{tr}\left(H_i\left[J_-, E'_\alpha\right]\right) = -\frac{1}{\lambda}\mathrm{tr}\left(J_-\left[H_i, E'_\alpha\right]\right) = -\frac{\alpha_i}{\lambda}\mathrm{tr}\left(J_- E'_\alpha\right) = 0 \tag{12.29}$$

It follows that

$$J_-|E'_\alpha\rangle = 0 \tag{12.30}$$

But this is not possible, because

$$J_3|E'_\alpha\rangle = |E'_\alpha\rangle \tag{12.31}$$

and the lowest state in a spin 1 representation vcannot hace J_3 eigenvalue $+1$. □

More is true: **If α is a root, then no non-zero multiple of α (except $-\alpha$) is also a root**.

Demonstratio. It is not difficult to stablish a contradiction between the SU(2) associated to 2α and the SU(2) associated to α. □

Assume now we have a rep D with weights μ_i. Consider the action of the SU(2) associated to some root α

$$J_3|\mu\rangle \equiv \frac{\alpha.H}{|\alpha|^2}|\mu\rangle = \frac{\alpha.\mu}{\alpha^2}\,|\mu\rangle \tag{12.32}$$

But we know that the J_3 allowed values are either integers or half-integers. Ergo

$$\frac{2\alpha.\mu}{\alpha^2} \in \mathbb{Z} \tag{12.33}$$

Now the state $|\mu\rangle$ can always be written as a linear combination of states transforming according to definite reps of $SU(2)$. Assume the highest spin state appearing in this linear combination is j. There must necessarily exist an integer p such that

$$J_+^p|\mu\rangle \neq 0 \tag{12.34}$$

but

$$J_+^{p+1}|\mu\rangle = 0 \tag{12.35}$$

Then

$$\frac{\alpha.(\mu + p\alpha)}{\alpha^2} = \frac{\alpha.\mu}{\alpha^2} + p = j \tag{12.36}$$

Likewise, there must be another integer, q such that

$$J_-^q|\mu\rangle \neq 0 \tag{12.37}$$

but

$$J_-^{q+1}|\mu\rangle = 0 \tag{12.38}$$

Then

$$\frac{\alpha(\mu - q\alpha)}{\alpha^2} = \frac{\alpha.\mu}{\alpha^2} - q = -j \tag{12.39}$$

It follows that

$$\frac{a.\mu}{\alpha^2} = -\frac{p-q}{2} \tag{12.40}$$

One can also consider the α string through α itself. It is clear that in this case

$$q - p = 2 \tag{12.41}$$

But we know that $p = 0$, because 2α is not a root. Then q=2. This just reexpresses the fact that 0 and $-\alpha$ are also roots.

In [13] ia defined the *α-string through β* as the set of roots

$$\beta - q\alpha,\ \beta - (q-1)\alpha, \ldots, \beta, \ldots, \beta + p\alpha \tag{12.42}$$

and it is a fact that

$$\beta(H_\alpha) = -\left(p - q\right) \tag{12.43}$$

There is a formal identification of H^* with H:

$$\alpha \in \mathfrak{H}^* \Longleftrightarrow H_\alpha \in \mathfrak{H} \quad \text{such that} \quad \alpha(H) = \kappa\left(H_\alpha, H\right)\ \forall H \in \mathfrak{H} \tag{12.44}$$

Let us, against the famous Coleman's advice, belabor this point.

Given a basis $H_i \in \mathfrak{H}$ and the dual basis α^i in $\mathfrak{H}^*$

$$\alpha^i(H_j) \equiv \delta^i_j \tag{12.45}$$

Then any $H \in \mathfrak{H}, H = \sum h^i H_i$ and $\alpha(H) = h^i$ so that

$$h^i = \kappa_{kl}(H_{\alpha^i})^k h^l \tag{12.46}$$

and

$$\kappa_{kl}\left(H_{\alpha^i}\right)^k = \delta^i_l \quad \Rightarrow \quad \left(H_{\alpha^i}\right)^k = \kappa^{ki} \tag{12.47}$$

and the formal identification is explicitly given by

$$\beta = \beta_i \alpha^i \quad \Rightarrow \quad H_\beta = \sum \beta_i \kappa^{ki}\, H_k \equiv \sum \beta^k\, H_k \tag{12.48}$$

Also, a scalar product in the root space is defined through

$$(\alpha, \beta) \equiv \kappa\left(H_\alpha, H_\beta\right) \equiv \sum_i \alpha^i\, \beta_i \tag{12.49}$$

One immediate consequence is as follows. Defining the $\mathfrak{SU}(2)$ algebra with E_α

$$\frac{\alpha.\beta}{\alpha^2} = -\frac{1}{2}(p - q) \tag{12.50}$$

Defining the $\mathfrak{SU}(2)$ algebra with E_β yields

$$\frac{\beta.\alpha}{\beta^2} = -\frac{1}{2}(p' - q') \tag{12.51}$$

The angle between both roots then is determined by

$$\cos^2\theta \equiv \frac{(\alpha.\beta)^2}{\alpha^2\beta^2} = \frac{(p-q)(p'-q')}{4} \tag{12.52}$$

There are only four possibilities

$(p-q)(p'-q')$	θ
0	$90^0 = \frac{\pi}{2}$
1	$60^0 = \frac{\pi}{3};\quad \frac{2\pi}{3}$
2	$45^0 = \frac{\pi}{4};\quad \frac{3\pi}{4}$
3	$30^0 = \frac{\pi}{6};\quad \frac{5\pi}{6}$

(12.53)

12.1 $\mathfrak{SU}(3)$

Let us define the Gell-Mann matrices

$$\lambda_1 \equiv \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{12.54}$$

$$\lambda_2 \equiv \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{12.55}$$

$$\lambda_3 \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{12.56}$$

$$\lambda_4 \equiv \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{12.57}$$

$$\lambda_5 \equiv \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \tag{12.58}$$

$$\lambda_6 \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{12.59}$$

$$\lambda_7 \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \tag{12.60}$$

$$\lambda_8 \equiv \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{12.61}$$

We define the $\mathfrak{SU}(3)$ generators in such a way that

$$T_a \equiv \frac{1}{2}\,\lambda_a \tag{12.62}$$

and

$$\mathrm{tr}\, T_a T_b = \frac{1}{2}\delta_{ab} \tag{12.63}$$

It is clear that

$$\{T_1, T_2, T_3\} \tag{12.64}$$

generate an SU(2) subgroup.

Let us choose the Cartan subalgebra to be

$$\{H_1 \equiv T_3; \quad H_2 \equiv T_8\} \tag{12.65}$$

The weights in the fundamental representation are

$$\begin{aligned} e_1 &\rightarrow \left(\tfrac{1}{2}, \tfrac{\sqrt{3}}{6}\right) \\ e_2 &\rightarrow \left(-\tfrac{1}{2}, \tfrac{\sqrt{3}}{6}\right) \\ e_3 &\rightarrow \left(0, -\tfrac{\sqrt{3}}{3}\right) \end{aligned} \tag{12.66}$$

Weights for the vertices of an equilateral triangle of side 1 in the (H_1, H_2) plane

The roots are differences of weights. This often the best way to compute them.

$$\begin{aligned} e_1 - e_2 &= (1, 0) \\ e_1 - e_3 &= (\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}) \\ e_2 - e_3 &= (-\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}) \end{aligned} \tag{12.67}$$

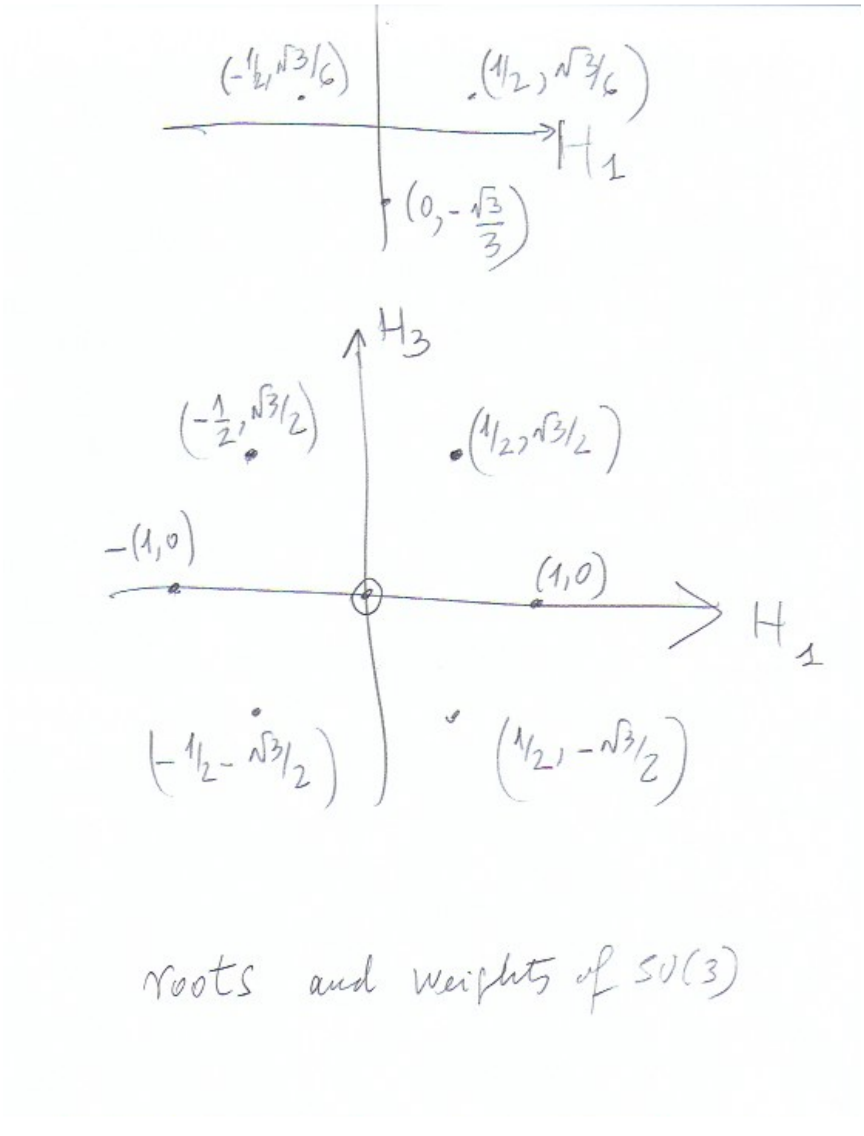
It is a fact that

$$\begin{aligned} E_{\pm 1,0} &\equiv \tfrac{T_1 \pm i T_2}{\sqrt{2}} \\ E_{\pm\frac{1}{2},\pm\frac{\sqrt{3}}{2}} &\equiv \tfrac{T_4 \pm i T_5}{\sqrt{2}} \\ E_{\mp\frac{1}{2},\pm\frac{\sqrt{3}}{2}} &\equiv \tfrac{T_6 \pm i T_7}{\sqrt{2}} \end{aligned} \tag{12.68}$$

Roots form a regular hexagon in the (H_1, H_2) plane.

In an arbitrary Lie algebra (and in some basis) we will say that a given weight μ is **positive** if its first non-zero component is positive, and **negative** if its first non-zero component is negative. This property defines an ordering, to wit

$$\mu > \nu \Longleftrightarrow \mu - \nu > 0 \tag{12.69}$$



Descriptio 12.1: Roots and weights of $SU(3)$.

The *highest weight* in a representation is then defined in an obvious way.

In the adjoint representation, positive roots will correspond to raising operators, and negative roots to lowering operators. The highest weight state must be annihilated by all positive roots.

In the particular case of $\mathfrak{SU}(3)$, positive roots are on the right half of the cartesian (H_1, H_2) plane, and negative roots are on the left hand side of it.

Again, in a general setting, we define **simple roots** as positive roots that cannot be written as sums of other positive roots. Let us call Δ the set of all simple roots. It is fact of life that from the geometry of the simple roots, it is possible to reconstruct the whole Lie algebra. Let us see how.

- If α and β are different simple roots, then $\alpha - \beta$ is not a root. Proof. This is so because otherwise either

$$\alpha = \beta + (\alpha - \beta) \tag{12.70}$$

  or else

$$\beta = \alpha + (\beta - \alpha) \tag{12.71}$$

  (depending on whether $\alpha - \beta > 0$ or $\beta - \alpha > 0$).

- This implies that

$$E_{-\alpha}|E_\beta\rangle = E_{-\beta}|E_\alpha\rangle = 0 \tag{12.72}$$

  Then using the master formula

$$\frac{\alpha.\beta}{\alpha^2} = -\frac{p-q}{2} \tag{12.73}$$

  we learn that $q = 0$. Also,

$$\frac{\beta.\alpha}{\beta^2} = -\frac{p'-q'}{2} \tag{12.74}$$

  implies that $q' = 0$. This means that we know the relative length of the roots, as well as the angle between them.

$$\frac{\beta^2}{\alpha^2} = \frac{p}{p'} \tag{12.75}$$

$$\cos\theta_{\alpha,\beta} = -\frac{\sqrt{pp'}}{2} \tag{12.76}$$

- A trivial consequence is that

$$\frac{\pi}{2} \leqslant \theta_{\alpha,\beta} \leqslant \pi \tag{12.77}$$

  (remember that simple roots are positive).

- Then all simple roots are linearly independent.

  Demonstratio. Assume
  $$\sum_{\alpha} C_\alpha \alpha = 0 \tag{12.78}$$
  which can be rewritten as
  $$\mu_+ = \mu_- \tag{12.79}$$
  with
  $$\mu_+ = \sum_{C_\alpha > 0} \alpha \tag{12.80}$$
  $$\mu_- = \sum_{C_\alpha < 0} \alpha \tag{12.81}$$
  But this cannot be, because
  $$(\mu_+ - \mu_-)^2 = \mu_+^2 + \mu_-^2 - 2\mu_+.\mu_- \geqslant \mu_+^2 + \mu_-^2 \geqslant 0 \tag{12.82}$$
  and $\mu_+\mu_- \leqslant 0$. □

- Any positive root can be written as a linear combination of simple roots with non-negative integer coefficients
  $$\phi = \sum_{\alpha} K_\alpha \alpha \tag{12.83}$$

- There are exactly l (rank) simple roots.

  Demonstratio. It this were not true there would be some vector ξ orthogonal to all simple roots (and therefore orthogonal to all roots),
  $$\forall \phi \in \Phi, \quad [\xi.H, E_\phi] = 0 \tag{12.84}$$
  This would mean that the algebra is not simple. □

- When we write
  $$\beta = \sum_{\alpha \in \Delta} K_\alpha \alpha \tag{12.85}$$
  Call the *height* of a root the number
  $$\mathrm{ht}\,\beta \equiv \sum K_\alpha \tag{12.86}$$
  If all $K_\alpha \geqslant 0$ we say that β is positive.

- Let us spell in detail how to build all roots out of the simple roots in the simple case of $\mathfrak{SU}(3)$; this then is easily generalized by induction. The simple roots are

$$\begin{aligned} \alpha_1 &= (\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}) \\ \alpha_2 &= (\tfrac{1}{2}, -\tfrac{\sqrt{3}}{2}) \end{aligned} \tag{12.87}$$

with

$$\begin{aligned} \alpha_1^2 &= \alpha_2^2 = 1 \\ \alpha_1.\alpha_2 &= -\tfrac{1}{2} \\ \tfrac{2\alpha_1.\alpha_2}{\alpha_1^2} &= \tfrac{2\alpha_2.\alpha_1}{\alpha_2^2} = -1 \end{aligned} \tag{12.88}$$

Thus p=1 for both α_1 acting on $|\alpha_2\rangle$ as well as for α_2 acting on $|\alpha_1\rangle$. Then

$$\alpha_1 + \alpha_2 \tag{12.89}$$

is a root, but neither $\alpha_1 + 2\alpha_2$ nor $\alpha_2 + 2\alpha_1$ are roots.

12.2 Dynkin diagrams

Remember that we found some time ago that

$$\begin{aligned} \tfrac{\alpha.\mu}{\alpha^2} + p &= j \\ \tfrac{\alpha.\mu}{\alpha^2} - q &= -j \end{aligned} \tag{12.90}$$

It could be the case that $|\mu\rangle$ has lower spin components; but j is the highest one. The value of j is determined by

$$p + q = 2j \tag{12.91}$$

In case $|\mu\rangle$ is a root $|\mu\rangle = |\beta\rangle$ in the adjoint representation, the situation is simpler, because we know that each root appears only once in the adjoint, and we conclude that

$$|\beta\rangle = |j; \quad \frac{\alpha.\beta}{\alpha^2}\rangle \tag{12.92}$$

which is completely determined up to a phase. Let us check this in the case of $\mathfrak{SU}(3)$. The root diagram is built out of

$$\{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, -\alpha_1, -\alpha_2, -\alpha_1 - \alpha_2\} \tag{12.93}$$

To this one has to add the two null roots to get the eight fimensions of the algebra. Besides, we know that

$$\begin{aligned} \alpha_1^2 = \alpha_2^2 = 1 \\ \alpha_1.\alpha_2 = -\tfrac{1}{2} \\ \tfrac{\alpha_1.\alpha_2}{\alpha_1^2} = \tfrac{\alpha_1.\alpha_2}{\alpha_2^2} = -\tfrac{1}{2} \end{aligned} \tag{12.94}$$

Also, we know how H_i commutes with everything, so that the only thing missing is the

$$[H_i, E_\alpha] \tag{12.95}$$

Consider the operator $E_{\alpha_1+\alpha_2}$. We know that $p = 1$ and $q = 0$, so that

$$p + q = 1 = 2j \tag{12.96}$$

We have

$$J(\alpha_1)^+|E_{\alpha_2}\rangle \equiv \tfrac{1}{|\alpha_1|} E_{\alpha_1}|E_{\alpha_2}\rangle = E_{\alpha_1}|E_{\alpha_2}\rangle = |\,[E_{a_1}, E_{\alpha_2}]\rangle \tag{12.97}$$

Under the $\mathfrak{SU}(2)_{\alpha_1}$

$$|E_{\alpha_2}\rangle = \left|\frac{1}{2}, -\frac{1}{2}\right\rangle \tag{12.98}$$

because

$$J_3^{\alpha_1}|E_{\alpha_2}\rangle = \frac{\alpha_1.\alpha_2}{\alpha_1^2}|E_{\alpha_2}\rangle = -\frac{1}{2}|E_{\alpha_2}\rangle \tag{12.99}$$

But we know that

$$J_+^{\alpha_1}\left|\frac{1}{2}, -\frac{1}{2}\right\rangle = \frac{1}{\sqrt{2}}\left|\frac{1}{2}, \frac{1}{2}\right\rangle \tag{12.100}$$

so that we learn that

$$\frac{1}{\sqrt{2}}\left|\frac{1}{2}, \frac{1}{2}\right\rangle = \frac{1}{\sqrt{2}}\eta\,|E_{\alpha_1+\alpha_2}\rangle \tag{12.101}$$

where η is a phase, which we can choose equal to 1, as our convention. It follows that

$$|E_{\alpha_1+\alpha_2}\rangle = \sqrt{2}\;|[E_{\alpha_1}, E_{\alpha_2}]\rangle \tag{12.102}$$

so that

$$E_{\alpha_1+\alpha_2} = \sqrt{2}\;[E_{\alpha_1}, E_{\alpha_2}] \tag{12.103}$$

The Jacobi identity applied to $[E_{-\alpha_2}, [E_{\alpha_1}, E_{\alpha_2}]] + \ldots$ now determines both

$$[E_{-\alpha_1}, E_{\alpha_1+\alpha_2}] = \frac{1}{\sqrt{2}}\,E_{\alpha_2} \tag{12.104}$$

(which is part of the $\mathfrak{SU}(2)^{\alpha_1}$ algebra, so that it was already known) as well as

$$[E_{-\alpha_2}, E_{\alpha_1+\alpha_2}] = -\frac{1}{\sqrt{2}} E_{\alpha_1} \tag{12.105}$$

The phase (-1) is fully determined now.

The **Dinkin diagram** associetes simple roots with open circles. Pairs of circles are connected by lines, depending on the angle between both roots:

- No line if the angle is $\frac{\pi}{2} = 90^0 \Longrightarrow |\alpha.\beta| = 0$
- One line if the angle is $\frac{2\pi}{3} = 120^0 \Longrightarrow |\alpha.\beta| = -\frac{1}{2}.|\alpha||\beta|$
- Two lines if the angle is $\frac{3\pi}{4} = 135^0 \Longrightarrow |\alpha.\beta| = -\frac{\sqrt{2}}{2}.|\alpha||\beta|$
- Three lines if the angle is $\frac{5\pi}{6} = 150^0 \Longrightarrow |\alpha.\beta| = -\frac{\sqrt{3}}{2}.|\alpha||\beta|$

In the figure we have indicated the real compact forms of the complex Lie algebra. There are also non-compact real forms of the same complex algebras, for example, a non-compact form of $SU(n)$ is $SL(n)$.

The dynkin diagrams evidences some isomorphisms between lower rank algebras.

$$\begin{aligned} SO(3) &\sim SU(2) \sim sp(2) \\ SO(4) &\sim SU(2) \times SU(2) \\ SO(6) &\sim SU(4) \\ SO(5) &\sim sp(4) \end{aligned} \tag{12.106}$$

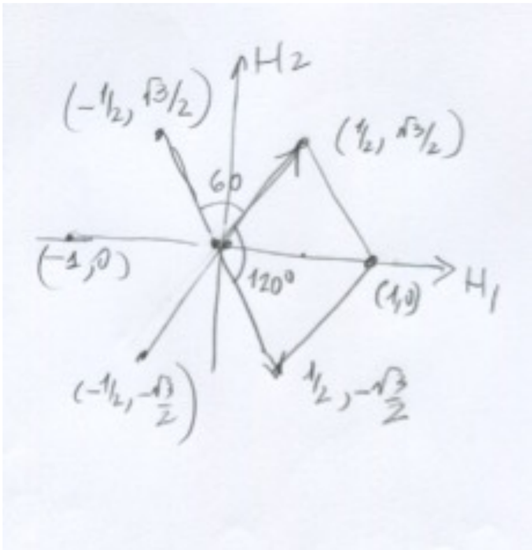
12.3 The exceptional algebra $\mathfrak{G}_2$

This algebra has got two simple roots

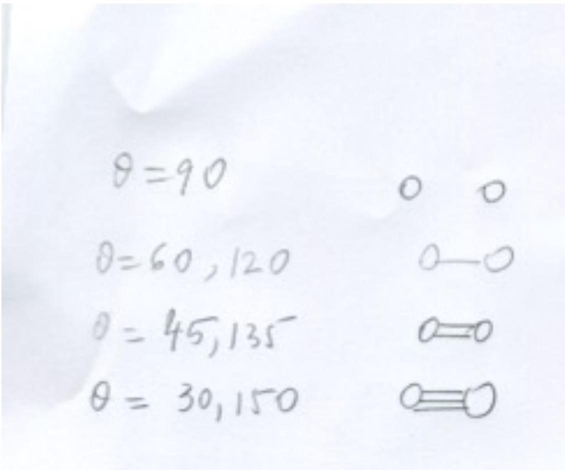
$$\begin{aligned} \alpha_1 &\equiv (0,1) \\ \alpha_2 &\equiv \left(\frac{\sqrt{3}}{2}, -\frac{3}{2}\right) \end{aligned} \tag{12.107}$$

It follows that

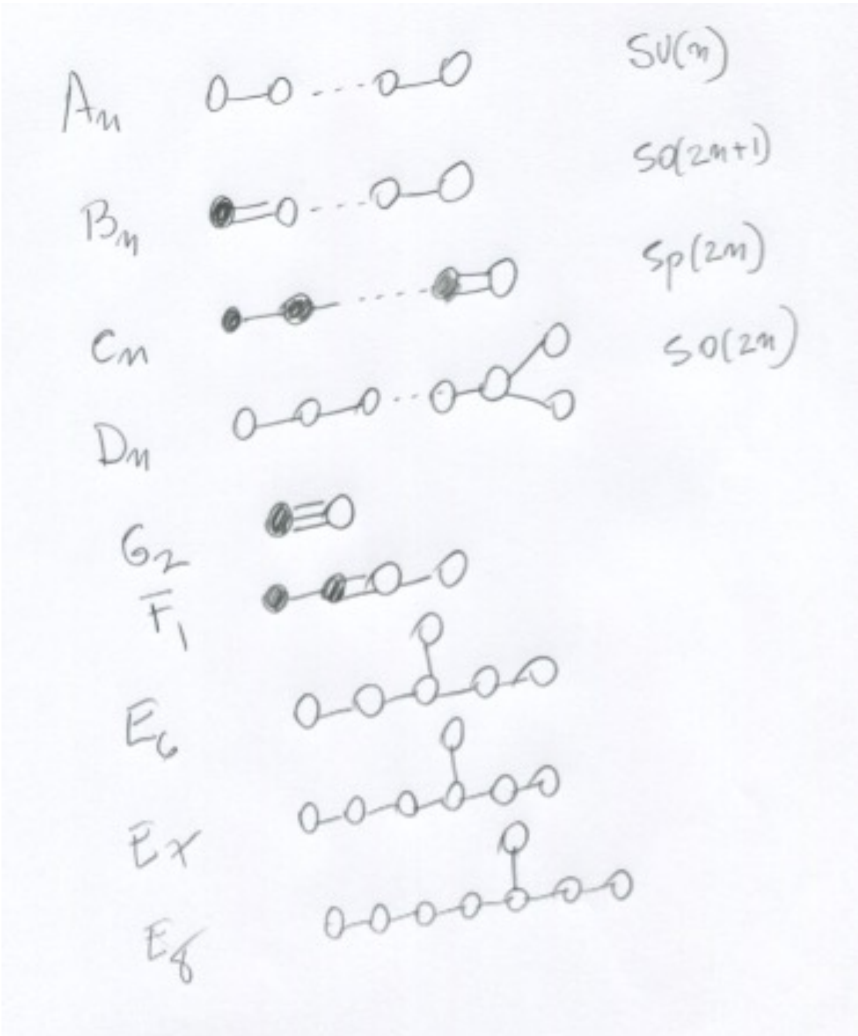
$$\begin{aligned} \alpha_1^2 &= 1 \\ \alpha_2^2 &= 3 \\ \alpha_1.\alpha_2 &= -\frac{3}{2} \\ \frac{2\alpha_1.\alpha_2}{\alpha_1^2} &= -3 \\ \frac{2\alpha_1.\alpha_2}{\alpha_2^2} &= -1 \\ \frac{\alpha_1.\alpha_2}{|\alpha_1|.|\alpha_2|} &\equiv \cos\,\theta_{12} = -\frac{\sqrt{3}}{2} \\ \theta_{12} &= 150^0 \end{aligned} \tag{12.108}$$



Descriptio 12.2: SU(3) simple roots.



Descriptio 12.3: Allowed angles between roots.



Descriptio 12.4: The Classification of simple groups.

The Dynkin diagram is simply two circles united by a triple line.

The α_1 string through α_2 has got $p = 3$. The α_2 string going through α_1 instead has $p = 1$. This means that

$$\begin{aligned} \phi_2 &\equiv \alpha_1 + \alpha_2 \\ \phi_3 &\equiv 2\alpha_1 + \alpha_2 \\ \phi_4 &\equiv 3\alpha_1 + \alpha_2 \end{aligned} \tag{12.109}$$

are all roots.

- We known that the ϕ_3 state is unique because $\alpha_1 + 2\alpha_2$ is not a root.
- In order to check whether there is another state at level 4, we have to check whether $2\alpha_1 + 2\alpha_2$ is a root (could it be reached by acting on ϕ_3 with a simple root (α_2)?)

$$\frac{2\alpha_2(2\alpha_1 + \alpha_2)}{\alpha_2^2} = -2 + 2 = 0 = -(p - q) \tag{12.110}$$

  But we already know that $q = 0$ because $2\alpha_1$ is not a root, so that $p = 0$ and $2\alpha_1 + 2\alpha_2$ is not a root. Another argument is that it is twice a root, namely $\alpha_1 + \alpha_2$, and no multiple of a root can ever be a root.
- We know that $4\alpha_1 + \alpha_2$ is not a root. The remaining possibility at level 5 is $3\alpha_1 + 2\alpha_2$.

$$\frac{2\alpha_2(3\alpha_1 + \alpha_2)}{\alpha_2^2} = -3 + 2 = -1 \tag{12.111}$$

  But we know that $q = 0$ which means that $p = 1$, so that $3\alpha_1 + 2\alpha_2$ is a root.
- Also, $3\alpha_1 + 3\alpha_2$ is not a root, so that at level 6 we only need to check $4\alpha_1 + 2\alpha_2$

$$\frac{2\alpha_1(3\alpha_1 + 2\alpha_2)}{\alpha_1^2} = 6 - 6 = 0 \tag{12.112}$$

  We know that $q = 0$, so that we are done.

We have uncovered the 12+2 roots of $\mathfrak{G}_2$.

In general, in order to keep track of the integers p_i and q_i cooresponding to the action of a simple root α_i on a state $|\phi\rangle$, Aassume that the positive root $\phi = \sum k_i \alpha_i$ (with $k_i > 0$); then

$$q_i - p_i = \frac{2\phi.\alpha_i}{\alpha_i^2} = \sum_j k_j \frac{2\alpha_j.\alpha_i}{\alpha_i^2} \equiv \sum_j k_j A_{ji} \tag{12.113}$$

where the Cartan matrix is defined as

$$\boxed{A_{ji} \equiv \frac{2\alpha_j.\alpha_i}{\alpha_i^2}} \tag{12.114}$$

Its diagnonal entries are all equal to 2. For $\mathfrak{SU}(3)$ the Cartan matrix is

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \tag{12.115}$$

And for $\mathfrak{G}_2$

$$A = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \tag{12.116}$$

Now when we go from ϕ to $\phi + \alpha_l$ by the action of the raising operator E_{α_l}, this changes k_l to $k_l + 1$ so that

$$q_i - p_i \quad \longrightarrow \quad q_i - p_i + A_{li} \tag{12.117}$$

It is now easy to work this out in gory detail in the $\mathfrak{SU}(3)$ case. The Cartan matrix gives the $q_i - p_i$, and we know the value of q_i, namely $q_i = 2$ for the root α_i itself (because it is the J_+ of an $\mathfrak{SU}(2)$), whereas $q_i = 0$ for any other root (because $\alpha_i - \alpha_j$ is not a root). Let us work out $A_2 = \mathfrak{SU}(3)$ again in detail. Cartan's matrix is

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \tag{12.118}$$

- Consider the α_2 string through α_1 (in this case q=0 because $\alpha_1 - \alpha_2$ is not a root).

  $$A_{12} = A_{21} = -1 = q - p \tag{12.119}$$

  Then $\alpha_1 + \alpha_2$ is a root; but neither $\alpha_1 + 2\alpha_2$ nor $\alpha_2 + 2\alpha_1$ are. We have then three roots (plus the negatives) plus two H; these exhaust the 8 dimensions of the algebra.

In the case of G_2, we start with

root	$q - p$
α_1	$[2, -1]$
α_2	$[-3, 2]$

(12.120)

This means, for the α_2-string through α_1 (q=0, because $\alpha_1 - \alpha_2$ is nort a root), that $p = 1$, so that $\alpha_1 + \alpha_2$ is indeed a root, but $\alpha_1 + 2\alpha_2$ is not a root. α_1 is then the highest weight of a doublet of $SU(2)_{\alpha_2}$. On the other hand α_2 is a triplet under $SU(2)_{\alpha_1}$: $\alpha_2 + 2\alpha_1$ and $\alpha_2 + 3\alpha_1$ are also roots (nut this is not the case with $\alpha_2 + 4\alpha_1$). Consider the α_2-string through $\beta \equiv a_2 + 3\alpha_1$. It so happens that

$$\frac{2\beta.\alpha_2}{\alpha_2^2} = \frac{2}{3}\left(3 + 3\left(\frac{-3}{2}\right)\right) = -1 = q - p \tag{12.121}$$

Giving the fact that we know that q=0 (because $3\alpha_1$ is not a root), this means that $\gamma \equiv 2\alpha_2 + 3\alpha_1$ is also a root. (β, γ) form a doublet under $SU(2)_{\alpha_2}$.

12.4 Fundamental weights

The highest weight of a rep is such that

$$\mu + \phi \tag{12.122}$$

is not a weight for any positive root ϕ. This is equivalkent to

$$E_{\alpha_i}|\mu\rangle = 0 \quad \Leftrightarrow \quad \frac{2\alpha_i.\mu}{\alpha_i^2} = l_i \geqslant 0 \tag{12.123}$$

The integers l_i are the *Dynkin coefficients*. It is useful to introduce the *fundamental weights* which are m vectors such that

$$\frac{2\alpha_i.\mu_j}{\alpha_i^2} = \delta_{ij} \tag{12.124}$$

The highest weight can the be written as

$$\mu = \sum l_i \mu_i \tag{12.125}$$

For example, for $A_2 \equiv \mathfrak{SU}(3)$, where the simple roots are

$$\begin{aligned} \alpha_1 &= (\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}) \\ \alpha_2 &= (\tfrac{1}{2}, -\tfrac{\sqrt{3}}{2}) \end{aligned} \tag{12.126}$$

they read

$$\begin{aligned} \mu_1 &= (\tfrac{1}{2}, \tfrac{\sqrt{3}}{6}) \\ \mu_2 &= (\tfrac{1}{2}, -\tfrac{\sqrt{3}}{6}) \end{aligned} \tag{12.127}$$

The defining representation generated by Gell-Mann's matrices has got μ_1 as its highest weight. Its Dynkin indices are then $(1, 0)$. Start with

$$\begin{aligned} H_1|\mu_1\rangle &= \tfrac{1}{2}|\mu_1\rangle \\ H_2|\mu_1\rangle &= \tfrac{\sqrt{3}}{6}|\mu_1\rangle \end{aligned} \tag{12.128}$$

It is clear that

$$E_{-\alpha_2}|\mu_1\rangle = 0 \tag{12.129}$$

because it is a highest weight state, and by definition, $\mu_1.\alpha_2 = 0$.

$$\frac{2\mu_1.\alpha_1}{\alpha_1^2} = 1 \tag{12.130}$$

This tells us that $\mu_1 - \alpha_1$ is a weight, but $\mu_1 - 2\alpha_2$ is not.

$$\begin{aligned} H_1 E_{-\alpha_1}|\mu_1\rangle &= E_{-\alpha_1}\tfrac{1}{2}|\mu_1\rangle - (\alpha_1)_1 E_{-\alpha_1}|\mu_1\rangle = 0 \\ H_2 E_{-\alpha_1}|\mu_1\rangle &= E_{-\alpha_1}\tfrac{\sqrt{3}}{6}|\mu_1\rangle - (\alpha_1)_2 E_{-\alpha_1}|\mu_1\rangle = \left(\tfrac{\sqrt{3}}{6} - \tfrac{\sqrt{3}}{2}\right)|\mu_1\rangle = \\ &= -\tfrac{1}{\sqrt{3}}|\mu_1\rangle \end{aligned} \tag{12.131}$$

This is then the weight

$$\mu_1 - \alpha_1 = (0, -\frac{1}{\sqrt{3}}) \tag{12.132}$$

Now

$$2\frac{(\mu_1 - \alpha_1).\alpha_2}{\alpha_2^2} = 1 \tag{12.133}$$

This tells us that $\mu_1 - \alpha_1 - \alpha_2$ must be a weight. We can represent this procedure as follos

$$\begin{array}{rl} \boxed{1 \quad 0} & \mu_1 \\ \boxed{-1 \quad 1} & \mu_1 - \alpha_1 \\ \boxed{0 \quad -1} & \mu_1 - \alpha_1 - \alpha_2 \end{array} \tag{12.134}$$

The rationale is as follows. The Cartan matrix is

$$\begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix} \tag{12.135}$$

We start with the highest weight which is the top of an α_1-doublet. We substract the first row of the Cartan matrix, and get to $\boxed{-1 \quad 1}$, which must be the top of an α_2 doublet. We then substract the second row of the Cartan matric and end up into $\boxed{0 \quad -1}$ and we are done

13

Representations.

Let us rewrite again the SU(3) Cartan matrix in another way (corresponding to level 0 and level one roots). The boxed numbers represent tha value of $q - p$

$$\begin{array}{c} k=1 \quad \boxed{2 \quad -1} \quad \boxed{-1 \quad 2} \\ k=0 \quad \boxed{0 \quad 0} \end{array} \tag{13.1}$$

Then we start, knowing that the q-values are

$$q = \boxed{2 \quad 0} \quad \boxed{0 \quad 2} \tag{13.2}$$

because $\alpha_1 - \alpha_2$ is not a root, and each root is in a $j = 1$ of its own $SU(2)$. From that, we can go up one step in level

$$\begin{array}{c} k=2 \quad \boxed{1 \quad 1} \quad \alpha_1 + \alpha_2 \\ k=1 \quad \boxed{2 \quad -1} \quad \boxed{-1 \quad 2} \quad \alpha_1 | \alpha_2 \\ k=0 \quad \boxed{0 \quad 0} \end{array} \tag{13.3}$$

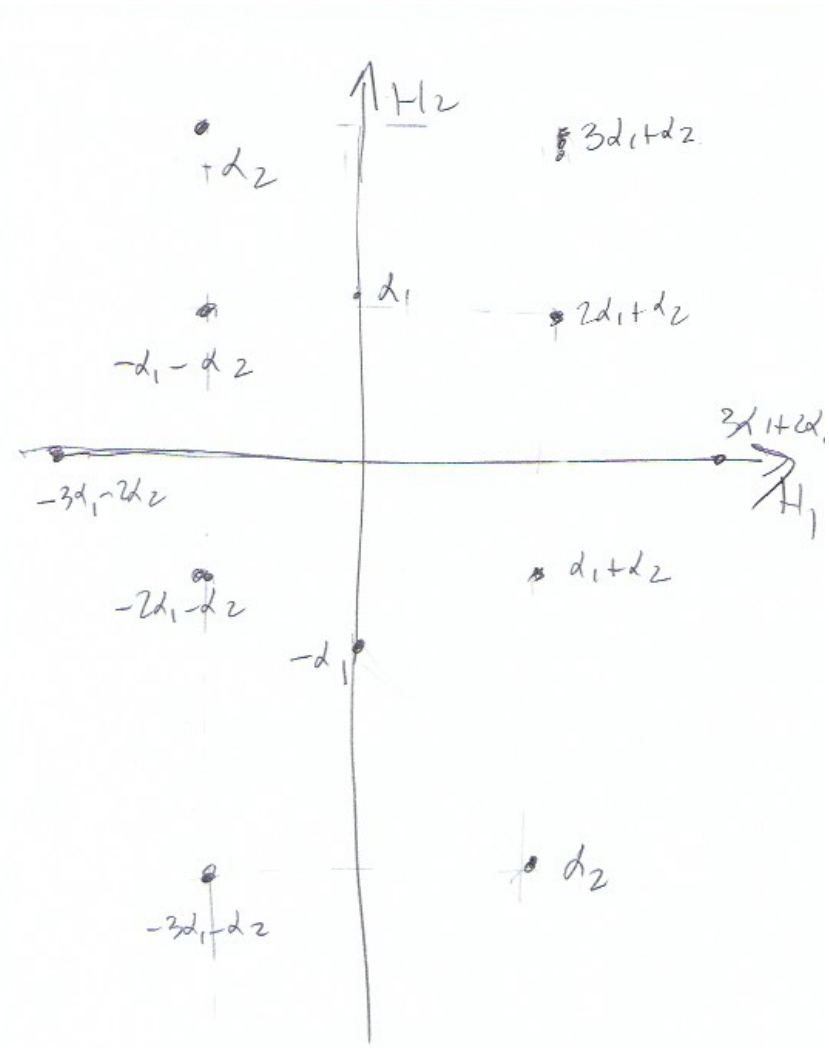
We know that $q = 1$ in both case, so that this is telling us that $p = 0$ and we are done with the positive roots.

To construct the μ_2 irrep (Dynkin indices $(0, 1)$) we proceed in a similar way, and get

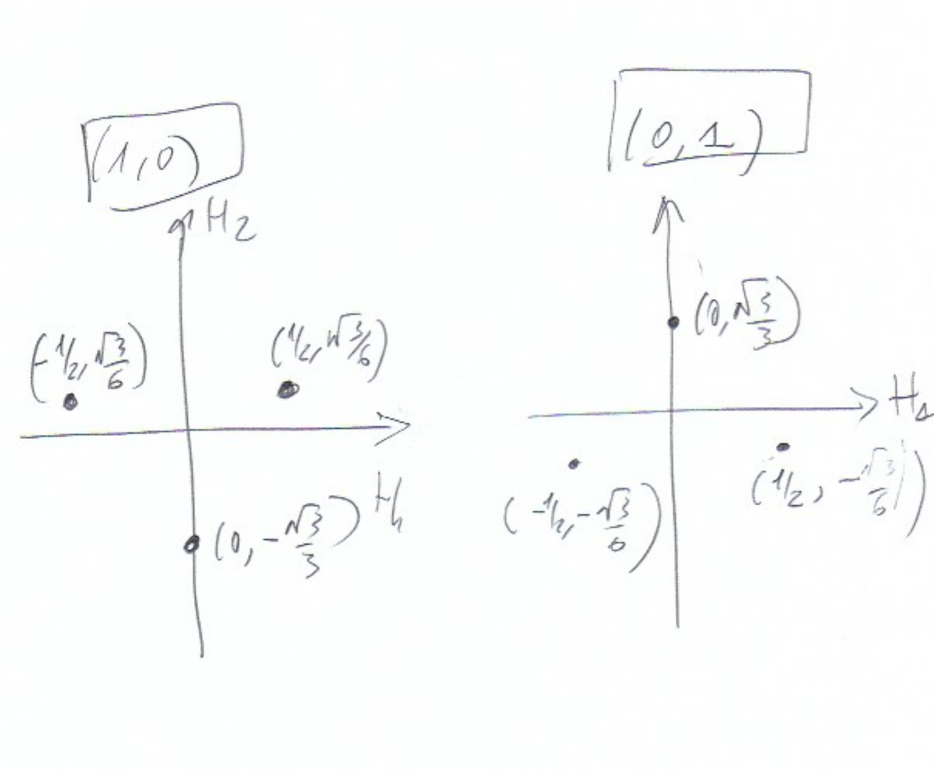
$$\begin{array}{c} \boxed{0 \quad 1} \quad \mu_2 \\ \boxed{1 \quad -1} \quad \mu_2 - \alpha_2 \\ \boxed{-1 \quad 0} \quad \mu_2 - \alpha_2 - \alpha_1 \end{array} \tag{13.4}$$

All states in a given irrep can easily be built out of the highest weight state as

$$E_{-\alpha^{a_n}} \ldots E_{-\alpha^{a_1}} |\mu\rangle \tag{13.5}$$



Descriptio 13.1: Roots of G_2.



Descriptio 13.2: Weights of the $\underline{3}$ and $\underline{\bar{3}}$ of $SU(3)$.

where

$$\alpha^{a_i} \in \Delta \tag{13.6}$$

A scalar product exists in this linear space which is such that given two subsets of Δ, $\vec{\alpha}, \vec{\beta} \subset \Delta$

$$\left\langle \mu \left| E_{\vec{\alpha}} E_{-\vec{\beta}} \right| \mu \right\rangle \sim \delta_{\vec{\alpha},\vec{\beta}} \tag{13.7}$$

The explicit computation of an orthonormal basis can become easily painful for large irreps.

13.1 The Weyl group

This is the set of all Weyl reflections. They stem for the fact that the SU(2) irreps are symmetrical under

$$J_3 \to -J_3 \tag{13.8}$$

Remember that

$$q - p = \frac{2\alpha.\mu}{\alpha^2} \tag{13.9}$$

so that

$$J_3 \left|\mu\right\rangle = \frac{\alpha.\mu}{\alpha^2} \left|\mu\right\rangle \quad \longrightarrow \quad \left|\mu - (q-p)\alpha\right\rangle = -\frac{\alpha.\mu}{\alpha^2} \left|\mu - (q-p)\alpha\right\rangle \equiv -\frac{\alpha.\mu}{\alpha^2} \left|\mu - \frac{2\alpha.\mu}{\alpha^2}\alpha\right\rangle$$

In slightly more formal terms we are multiplying the weigh by the idempotent

$$(I^\alpha)_i^j \equiv \delta_i^j - \frac{2\alpha_i \alpha^j}{\alpha^2} \tag{13.10}$$

It is easy to show that

$$I_\alpha^2 = 1 \tag{13.11}$$

$$I^\alpha.\alpha = -\alpha \tag{13.12}$$

In general, we can decompose any vector with respect to the direction of α

$$v = v_\perp + v_\parallel \tag{13.13}$$

then

$$I^\alpha.v = v_\perp - v_\parallel \tag{13.14}$$

In the particular case of the $\bar{3} \equiv (0,1)$ of SU(3), all weights are just the negative of the weights of the $3 \equiv (1,0)$. This means that the two irreps are related by **complex conjugation**.

$$[D_a, D_b] = i f_{abc} D_c \Longrightarrow [D_a, D_b]^* = -i f_{abc} D_c^* \Longrightarrow [-D_a^*, -D_b^*]^* = i f_{abc}(-D_c^*) \tag{13.15}$$

This irrep is usually dubbed $\bar{D}$. The irrep D is said to be **real** if it is equivalent to its complex conjugate. Otherwise, it is said to be **complex**.

Given the fact that $H_i^+ = H_i$ if μ is a weight in D, then $-\mu$ is a weight in $\bar{D}$. Then the lowest weight of $(1, o)$ is minus the highest weight of $(0, 1)$ and the other way around.

The highest weight of (n, m) is $n\mu_1 + m\mu_2$, and the lowest weight of (n, m) is $-n\mu_2 - m\mu_1$, so that that highest weight of (m, n) is $n\mu_2 + m\mu_1$. The irreps (n, m) ans (m, n) are complex conjugates.

Let us work out the $(2, 0)$ irrep of SU(3). Remember the $SU(3)$ Cartan matrix

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \tag{13.16}$$

Then the string of weights looks as follows

$$\begin{array}{ccc} & \boxed{2 \quad 0} \quad 2\mu_1 & \\ & \boxed{0 \quad 1} \quad 2\mu_1 - \alpha_1 & \\ \boxed{-1 \quad 2} \quad 2\mu_1 - 2\alpha_1 & & \boxed{1 \quad -1} \quad 2\mu_1 - \alpha_1 - \alpha_2 \end{array} \tag{13.17}$$

Let us now look at the Weyl reflections

$$J_3^{\alpha_1} |2\mu_1\rangle = \frac{2\alpha.\mu_1}{\alpha^2} |2\mu_1\rangle = |2\mu_1\rangle \tag{13.18}$$

Let us begin with the Weyl reflections of μ.

$$\begin{aligned} I^{\alpha_1} (\mu \equiv 2\mu_1) &= 2\mu_1 - 2\alpha_1 \\ I^{\alpha_2} (\mu \equiv 2\mu_1) &= \mu \\ I^{\alpha_2} (2\mu_1 - 2\alpha_1) &= 2\mu_1 - 2\alpha_1 - 2\alpha_2 \\ I^{\alpha_1} (2\mu_1 - 2\alpha_1 - 2\alpha_2) &= 2\mu_1 - 2\alpha_1 - 2\alpha_2 \end{aligned} \tag{13.19}$$

Let is now examine the Weyl reflections of $\mu - \alpha_1 \equiv 2\mu_1 - \alpha_1$. First of all, I^{α_1} leaves this weight invariant, because it is orthoginal to α_1. Otherwise

$$\begin{aligned} I^{\alpha_2} (2\mu_1 - \alpha_1) &= 2\mu_1 - \alpha_1 - \alpha_2 \\ I^{\alpha_1} (2\mu_1 - \alpha_1 - \alpha_2) &= 2\mu_1 - 2\alpha_1 - \alpha_2 \\ I^{\alpha_2} (2\mu_1 - 2\alpha_1 - \alpha_2) &= 2\mu_1 - 2\alpha_1 - \alpha_2 \end{aligned} \tag{13.20}$$

Altogether, this is a six-dimensional irrep

$$\boxed{(2, 0) = \underline{6}} \tag{13.21}$$

Consider now the irrep $(1,1)$. It so happens that

$$\mu_1 + \mu_1 = \alpha_1 + \alpha_2 \tag{13.22}$$

which is the highest weight of the adjoint of SU(3), already studied. We know thet the zero weight is doubly degenerate. Let us check now that the two ways of getting zero weight ara actually linearly independent.

$$\begin{aligned} |0_1\rangle &\equiv E_{-\alpha_1} E_{-\alpha_2} |\mu_1 + \mu_2\rangle \\ |0_2\rangle &\equiv E_{-\alpha_2} E_{-\alpha_1} |\mu_1 + \mu_2\rangle \end{aligned} \tag{13.23}$$

Our task is to show that

$$\langle 0_1 | 0_2 \rangle^2 \neq \langle 0_1 | 0_1 \rangle \langle 0_2 | 0_2 \rangle \tag{13.24}$$

Demonstratio. This is easy, because

$$\begin{aligned} \langle 0_1 | 0_1 \rangle &\equiv \langle \mu | E_{\alpha_2} E_{\alpha_1} E_{-\alpha_1} E_{-\alpha_2} | \mu \rangle = \langle \mu | E_{\alpha_2} \left(E_{-\alpha_1} E_{\alpha_1} + \alpha_1.H \right) E_{-\alpha_2} | \mu \rangle = \\ &= \langle \mu \left| E_{\alpha_2} \left(\alpha_1^i E_{-\alpha_2} H_i - \alpha_2^i E_{-\alpha_2} \right) \right| \mu \rangle = \mu.\alpha_1 - \alpha_1.\alpha_2 = \tfrac{1}{2} + \tfrac{1}{2} = 1 \end{aligned} \tag{13.25}$$

$$\begin{aligned} \langle 0_2 | 0_2 \rangle &\equiv \langle \mu | E_{\alpha_1} E_{\alpha_2} E_{-\alpha_2} E_{-\alpha_1} | \mu \rangle = \langle \mu | E_{\alpha_1} \left(E_{-\alpha_2} E_{\alpha_2} + \alpha_2.H \right) E_{-\alpha_1} | \mu \rangle = \\ &= \langle \mu \left| E_{\alpha_1} \left(\alpha_2^i E_{-\alpha_1} H_i - \alpha_1^i E_{-\alpha_1} \right) \right| \mu \rangle = \mu.\alpha_2 - \alpha_2.\alpha_1 = \tfrac{1}{2} + \tfrac{1}{2} = 1 \end{aligned} \tag{13.26}$$

$$\begin{aligned} \langle 0_1 | 0_2 \rangle &\equiv \langle \mu | E_{\alpha_2} E_{\alpha_1} E_{-\alpha_2} E_{-\alpha_1} | \mu \rangle = \langle \mu | E_{\alpha_2} E_{-\alpha_2} E_{\alpha_1} E_{-\alpha_1} | \mu \rangle = \\ &= \langle \mu | \alpha_2.H \alpha_1.H | \mu \rangle = (\alpha_2.\mu)(\alpha_1.\mu) = \tfrac{1}{2}.\tfrac{1}{2} = \tfrac{1}{4} \end{aligned} \tag{13.27}$$

□

The $(3,0)$, with highest weight $\mu = 3\mu_1$. It follows that the string of weights reads

$$\begin{array}{cccc} & \boxed{3 \quad 0} \;\; 3\mu_1 & & \\ & \boxed{1 \quad 1} \;\; 3\mu_1 - \alpha_1 & & \\ \boxed{-1 \quad 2} \;\; 3\mu_1 - \alpha_1 - \alpha_2 & & \boxed{2 \quad -1} \;\; 3\mu_1 - 2\alpha_1 & \\ \boxed{0 \quad 0} \;\; 3\mu_1 - 2\alpha_1 - \alpha_2 & & \boxed{-3 \quad 3} \;\; 3\mu_1 - 3\alpha_1 & \\ \boxed{1 \quad -2} \;\; 3\mu_1 - 2\alpha_1 - 2\alpha_2 & & \boxed{-2 \quad 1} \;\; 3\mu_1 - 3\alpha_1 - \alpha_2 & \\ & \boxed{-1 \quad 1} \;\; 3\mu_1 - 3\alpha_1 - 2\alpha_2 & & \\ & \boxed{0 \quad -3} \;\; 3\mu_1 - 2\alpha_1 - 3\alpha_2 & & \end{array} \tag{13.28}$$

All states are obviously unique except the $\boxed{0 \quad 0}$. But this is also unique because as you can undoubtly prove

$$E_{-\alpha_1} E_{-\alpha_2} E_{-\alpha_1} |3\mu_1\rangle \sim E_{-\alpha_2} E_{-\alpha_1} E_{-\alpha_1} |3\mu_1\rangle \tag{13.29}$$

So that $(3,0) = \underline{10}$. Its complex conjugate $\overline{10} = (0,3)$

14

The unitary groups $A_{N-1} = SU(N)$

Our normalization will be

$$\mathrm{tr}\,(T_a T_b) = \frac{1}{2}\delta_{ab} \tag{14.1}$$

The generators of the Cartan subalgebra in the fundamental irrep $\underline{N}$ are given by ($A = 1, \ldots, N-1; \quad i, j, k = 1 \ldots N$)

$$\left(H^A\right)_{ij} = \frac{1}{\sqrt{2A(A+1)}} \left(\sum_{k=1}^{A} \delta_{ik}\delta_{jk} - a\delta_{i,A+1}\delta_{j,A+1} \right) \tag{14.2}$$

The N weights (as many as the dimension of the fundamental) are ($N - 1$)-dimensional vectors, which are the eigenvalues of the H in the Cartan subalgebra

$$(\mu^a)_A \equiv \frac{1}{\sqrt{2A(A+1)}} \left(\sum_{b=1}^{b=A} \delta_{ab} - A\delta_{a,A+1} \right) \tag{14.3}$$

For example

$$\mu^1 = \left(\tfrac{1}{\sqrt{4}}, \quad \tfrac{1}{\sqrt{12}}, \quad \tfrac{1}{\sqrt{24}}, \quad \ldots, \quad \tfrac{1}{\sqrt{2N(N-1)}} \right) \tag{14.4}$$

$$\mu^2 = \left(-\tfrac{1}{\sqrt{4}}, \quad \tfrac{1}{\sqrt{12}}, \quad \tfrac{1}{\sqrt{24}} \quad \ldots, \quad \tfrac{1}{\sqrt{2N(N-1)}} \right) \tag{14.5}$$

$$\mu^3 = \left(0, \quad -\tfrac{2}{\sqrt{12}}, \quad \tfrac{1}{\sqrt{24}}, \quad \ldots, \quad \tfrac{1}{\sqrt{2N(N-1)}} \right) \tag{14.6}$$

$$\mu^4 = \left(0, \quad 0, \quad -2\tfrac{1}{\sqrt{24}}, \quad \ldots, \quad \tfrac{1}{\sqrt{2N(N-1)}} \right) \tag{14.7}$$

$$\ldots \tag{14.8}$$

$$\mu^N = \left(0, \quad 0, \quad 0, \quad \ldots, \quad 0, \quad \tfrac{1}{\sqrt{2N(N-1)}} \right) \tag{14.9}$$

We can compute the weight length

$$\mu^1.\mu^1 = \sum_{A=1}^{N-1} \frac{1}{2A(A+1)} = \frac{1}{2}\sum_{A=1}^{N-1}\left(\frac{1}{A} - \frac{1}{A+1}\right) \tag{14.10}$$

Let us dub

$$f(N) \equiv \sum_{A=1}^{N-1} \frac{1}{A} = 1 + \sum_{A=1}^{N-1} \frac{1}{A+1} - \frac{1}{N} \tag{14.11}$$

Then

$$\mu^1.\mu^1 = \frac{N-1}{2N} \tag{14.12}$$

and in fact this results holds for all other weights.

For $a < b$, for example,

$$\mu^1.\mu^2 = \left(\mu^1\right)^2 - \tfrac{1}{2} = -\tfrac{1}{2N} \tag{14.13}$$

Again, this results turns out to be generic. We can then write

$$\mu^a.\mu^b = -\frac{1}{2N} + \frac{1}{2}\delta_{ab} \tag{14.14}$$

We shall adopt here a backwards convention: a positive weight is one such that the last non-zero component is positive. Then

$$\mu^1 > \mu^2 > \ldots > \mu^N \tag{14.15}$$

The roots are differences of weights

$$\mu^a - \mu^b \quad (a \neq b) \tag{14.16}$$

Positive roots are

$$\mu^a - \mu^b \quad (a < b) \tag{14.17}$$

The simple roots are

$$\alpha^A \equiv \mu^A - \mu^{A+1} \tag{14.18}$$

It so happens that

$$\begin{aligned}\alpha^a.\alpha_b = &-\tfrac{1}{2N} + \tfrac{1}{2}\delta_{ab} - \left(-\tfrac{1}{2N} + \tfrac{1}{2}\delta_{a,b+1}\right) - \left(-\tfrac{1}{2N} + \tfrac{1}{2}\delta_{a+1,b}\right) + \left(-\tfrac{1}{2N} + \tfrac{1}{2}\delta_{a+1,b+1}\right) = \\ &= \tfrac{1}{2}\delta_{ab} - \tfrac{1}{2}\delta_{a,b+1} - \tfrac{1}{2}\delta_{a+1,b} + \tfrac{1}{2}\delta_{a+1,b+1} = \delta_{ab} - \tfrac{1}{2}\delta_{a,b+1} - \tfrac{1}{2}\delta_{a+1,b}\end{aligned} \tag{14.19}$$

This explains the shape of the Dynkin diagram, the simplest of them all. It is then plain that the fundamental weights are given by

$$M^A \equiv \sum_{a=1}^{a=A} \mu^a \tag{14.20}$$

Indeed

$$\begin{aligned} 2\alpha^B M^A &= 2\textstyle\sum_{a=1}^{a=A} \mu^a \left(\mu^B - \mu^{B+1}\right) = \\ &= \textstyle\sum_{a=1}^{a=A} \left(\delta_{aB} - \delta_{a,B+1}\right) = \delta_{AB} \end{aligned} \tag{14.21}$$

Oeing to the fact that the Cartan generators are traceless,

$$\sum_{a=1}^{a=N} \mu^a = 0 \tag{14.22}$$

Then

$$\mu^N = -\sum_{a=1}^{a=N-1} \mu^a = -\mu^{N-1} \tag{14.23}$$

Then

$$\overline{(1, 0 \dots 0)} = (0, \dots, 1) \tag{14.24}$$

and so on.

15

Orthogonal algebras

The Dynkin diagrams of $SO(2n)$ and $SO(2n+1)$ are different, and this reflects some important differences between the two sets of orthogonoal groups. Let us first examine the structure of both algebras.

15.1 $D_n = SO(2n)$

The Lie algebra consists on imaginary antisymmetric matrices of dimesion $2n$- There are $n(2n-1)$ of those. The Cartan generators in the fundamental representation can be chosen as

$$H^a_{jk} \equiv -i\left(\delta_{j,2m-1}\delta_{k,2m} - \delta_{k,2m-1}\delta_{j,2m}\right) \tag{15.1}$$

($a = 1 \ldots n$, the rank of D_n) For example, for D_2 in block form

$$\begin{aligned} H^1 &= \begin{pmatrix} \sigma_2 & 0 \\ 0 & 0 \end{pmatrix} \\ H^2 &= \begin{pmatrix} 0 & 0 \\ 0 & \sigma_2 \end{pmatrix} \end{aligned} \tag{15.2}$$

The corresponding eigenvectors are

$$\pm e^k \equiv \delta_{j,2k-1} \pm i\delta_{j,2k} \tag{15.3}$$

For example,

$$\pm e^1 \equiv \begin{pmatrix} 1 \\ \pm i \\ 0 \\ 0 \end{pmatrix} \tag{15.4}$$

$$\pm e^2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \pm i \end{pmatrix} \tag{15.5}$$

In general

$$H_a(\pm e^b) = \delta_a^b(\pm e^b) \tag{15.6}$$

This means that the weight vectors are given by

$$\left(\epsilon^a\right)_b \equiv \delta_b^a \tag{15.7}$$

Roots are given by

$$\pm\epsilon^a \pm \epsilon^b \quad (a \neq b) \tag{15.8}$$

There are $n(n-1)$ of those $(= n(2n-1) - n)$. The positive roots are given by

$$\epsilon^a \pm \epsilon^b \quad (a < b) \tag{15.9}$$

Finally, the simple roots are given by

$$\begin{array}{c} \epsilon^a - \epsilon^{a+1} \quad a = 1 \ldots n-1 \\ \epsilon^{n-1} + \epsilon^n \end{array} \tag{15.10}$$

It is plain that

$$\begin{array}{c} \left(\epsilon^a - \epsilon^{a+1}\right)^2 = 2 \\ \left(\epsilon^a - \epsilon^{a+1}\right)\left(\epsilon^{a+1} - \epsilon^{a+2}\right) = -1 \\ \cos\,\theta = -\frac{1}{4} \end{array} \tag{15.11}$$

On the other hand the two last simple roots are orthogonal

$$\left(\epsilon^{n-1} + \epsilon^n\right)\left(\epsilon^{n-1} - \epsilon^n\right) = 0 \tag{15.12}$$

15.2 $SO(2n+1) \equiv B_n$

This algebra has an extra one-dimensional subspace associated with a zero weight. The dimension of the algebra is $n(2n+1)$. The Cartan subalgebra is the same, with one extra row and column. For example

$$H = \begin{pmatrix} \sigma_2 & 0 \\ 0 & 0 \end{pmatrix} \tag{15.13}$$

There are extra roots connecting the extra dimensional subspace with the others:

$$\begin{array}{c} \pm\epsilon^a \pm \epsilon^b \\ \pm\epsilon^a \end{array} \tag{15.14}$$

Altogether, we have $2n$ extra roots, which is the difference between the dimensions of B_n and D_n. The positive roots are just

$$\begin{array}{c}\epsilon^a \pm \epsilon^b \quad (a < b)\\ \epsilon^a\end{array} \tag{15.15}$$

The simple roots

$$\begin{array}{c}\epsilon^a - \epsilon^{a+1} \quad (a = 1 \ldots n-1)\\ \epsilon^n\end{array} \tag{15.16}$$

What happens is that $\epsilon^{n-1} + \epsilon^n$ is not simple anymore, because it is $(\epsilon^{n-1} - \epsilon^n) + 2\epsilon^n$. This changes the angle between the two last roots

$$\begin{array}{c}(\epsilon^{n-1} - \epsilon^n).\epsilon^n = -1\\ \cos\theta = -\frac{1}{2}\end{array} \tag{15.17}$$

The fundamental weights are

$$M^a \equiv \sum_{i=1}^{i=a} \epsilon^i \quad a = 1 \ldots n-1 \tag{15.18}$$

$$M^n \equiv \frac{1}{2}\sum_{i=1}^{i=n} \epsilon^i \tag{15.19}$$

Indeed

$$\begin{array}{c}2M^a\alpha^b = \sum_{i=1}^{i=a} \epsilon^i \left(\epsilon^b - \epsilon^{b+1}\right) = \delta_{ab}\\ 2M^n\alpha^a = \frac{1}{2}\sum_{j=1}^{i=n} \epsilon^i \left(\epsilon^a - \epsilon^{a+1}\right) = 0\\ 2M^n\alpha^n = \sum_{j=1}^{i=n} \epsilon^i.\epsilon^n = 1\end{array} \tag{15.20}$$

Weyl reflexions of M^N on all roots ϵ^a yields the set of weights

$$I^{\epsilon^a} M^N = (1 - 2\epsilon^a \otimes \epsilon^a) . \frac{1}{2}\sum_c \epsilon^c = \frac{1}{2}\sum_c (\epsilon^c - 2\epsilon^a) \to \frac{1}{2}\left(\pm\epsilon^1 \pm \epsilon^2 \pm \ldots \pm \epsilon^n\right) \tag{15.21}$$

This is a 2^n dimensional representation, the spinor representation. We shall work in a rep space which the n-th tensor product of the two-dimensional space S of the spin 1/2 irrep.

$$S \otimes \ldots^{(n} \otimes S \tag{15.22}$$

Then the Cartan subalgebra is given by one $\frac{1}{2}\sigma_3$ in the j-th position

$$H_j \equiv 1 \otimes \ldots \otimes \frac{1}{2}\sigma_3 \otimes \ldots \otimes 1 \equiv \frac{1}{2}\sigma_3^j \tag{15.23}$$

Then it can be shown that

$$\begin{aligned} E_{\pm\epsilon^1} &= \tfrac{1}{2}\sigma_\pm^1 \\ E_{\pm\epsilon^2} &= \tfrac{1}{2}\sigma_3^1\sigma_\pm^2 \\ &\dots \\ E_{\pm\epsilon^j} &= \tfrac{1}{2}\sigma_3^1\dots\sigma_3^{j-1}\sigma_\pm^j \end{aligned} \tag{15.24}$$

To summarize,

$$\begin{aligned} M_{2j-1,2n+1} &= \tfrac{1}{2}\sigma_3^1\dots\sigma_3^{j-1}\sigma_1^j \\ M_{2j,2n+1} &= \tfrac{1}{2}\sigma_3^1\dots\sigma_3^{j-1}\sigma_2^j \end{aligned} \tag{15.25}$$

and then all other generators are determined by the algebra

$$M_{ab} \equiv -i\left[M_{a,2n-1}, M_{b,2n-1}\right] \tag{15.26}$$

In the case of $D_{n+1} \equiv SO(2n+2)$ the roots are

$$\begin{aligned} \alpha^j &= \epsilon^j - \epsilon^{j+1} \\ \alpha^{n+1} &\equiv \epsilon^n + \epsilon^{n+1} \end{aligned} \tag{15.27}$$

There are two special representations corresponding to the last two fundamental weights. Let us call them D^n and D^{n+1}

$$\begin{aligned} \mu^n &\equiv \tfrac{1}{2}\left(\epsilon^1 + \ldots + \epsilon^n - \epsilon^{n+1}\right) \\ \mu^{n+1} &\equiv \tfrac{1}{2}\left(\epsilon^1 + \ldots + \epsilon^n + \epsilon^{n+1}\right) \end{aligned} \tag{15.28}$$

Under the $SO(2n+1)$ subgroup generated by

$$M_{jk} \quad j,k \leqslant 2n+1 \tag{15.29}$$

both representations transform like the spinor representation. It can be shown that in D^n the extra generator in the Cartan subalgebra reads

$$H_{n+1} \equiv M_{2n+1,2n+2} = -\frac{1}{2}\sigma_3^1\dots\sigma_3^n \tag{15.30}$$

and in the other representation D^{n+1}

$$H_{n+1} \equiv M_{2n+1,2n+2} = \frac{1}{2}\sigma_3^1\dots\sigma_3^n \tag{15.31}$$

15.3 Clifford algebras

The simplest definition of a Clifford algebra is through the relations

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij} \quad i, j = 1, \ldots, N \tag{15.32}$$

Given a representation of the Clifford algebra, there is an associated representation of $SO(N)$ given by

$$M_{ij} \equiv -\frac{i}{4}\left[\gamma_i, \gamma_j\right] \tag{15.33}$$

The gamma matrices themselves transform with the fundamental $D^1 \equiv \underline{N}$ of $SO(N)$

$$\left[M_{jk}, \gamma_l\right] = i\left(\delta_{jl}\gamma_k - \delta_{kl}\gamma_j\right) \tag{15.34}$$

For $B_n \equiv SO(2n+1)$ there is an explicit representation of the Clifford algenbra that yields precisely the spinor representation of B_n.

$$\begin{aligned}
\gamma_1 &\equiv \sigma_2^1\sigma_3^2 \ldots \sigma_3^n \\
\gamma_2 &\equiv -\sigma_1^1\sigma_3^2 \ldots \sigma_3^n \\
\gamma_3 &\equiv \sigma_2^2\sigma_3^3 \ldots \sigma_3^n \\
\gamma_4 &\equiv -\sigma_1^2\sigma_3^3 \ldots \sigma_3^n \\
&\ldots \\
\gamma_{2n-1} &\equiv \sigma_2^n \\
\sigma_{2n} &\equiv -\sigma_1^n \\
\gamma_{2n+1} &\equiv \sigma_3^1\sigma_3^2 \ldots \sigma_3^n
\end{aligned} \tag{15.35}$$

It is fact of life that

$$\gamma_1\gamma_2 \ldots \gamma_{2n+1} = i^n \tag{15.36}$$

We do jot have enough elements to construct a representation of $SO(2n+2)$; but we can construct the $SO(2n)$ algebra just by leaving out γ_{2n+1}. This is a reducible representation; there is a nontrivial matrix that commutes with all the generator, namely γ_{2n+1} itself. There are two projectors. One onto D^{n-1}

$$\frac{1}{2}\left(1 - \gamma_{2n+1}\right) \tag{15.37}$$

and another onto D^n

$$\frac{1}{2}\left(1 + \gamma_{2n+1}\right) \tag{15.38}$$

There is a natural $SU(N)$ subgroup of $SO(2N)$. In fact from a Clifford algebra one can construcy the operators

$$\begin{aligned} a_j &\equiv \tfrac{1}{2}\left(\gamma_{2j-1} - i\gamma_{2j}\right) \\ a_j^+ &\equiv \tfrac{1}{2}\left(\gamma_{2j-1} + i\gamma_{2j}\right) \end{aligned} \tag{15.39}$$

They obey

$$\begin{aligned} \{a_j, a_k\} &= \left\{a_j^+, a_k^+\right\} = 0 \\ \left\{a_j, a_k^+\right\} &= \delta_{jk} \end{aligned} \tag{15.40}$$

Then out of the matrix elements in the $\underline{N}$ of $SU(N)$

$$T_a \equiv \sum_{ij} a_i^+ \left(T_a\right)_{ij} a_j \tag{15.41}$$

In order to show that this is in fact a subalgebra of $SO(2N)$, let us write

$$\begin{aligned} a_i^+ a_j = \tfrac{1}{2}\left\{a_i^+, a_j\right\} + \tfrac{1}{2}\left[a_i^+, a_j\right] = \tfrac{1}{2}\delta_{ij} + \tfrac{i}{2}M_{2i-1,2j-1} + \tfrac{1}{2}M_{2i-1,2j} - \\ -\tfrac{1}{2}M_{2i,2j-1} + \tfrac{i}{2}M_{2i,2j} \end{aligned} \tag{15.42}$$

The Fock states generate the representation D^N for N even, and D^{N-1} for Nodd. There is an $SO(2N)$ generator which commutes with the $SU(N)$ subgroup, namely

$$S \equiv \sum_{j=1}^{N} M_{2j-1,2j} = \sum_{i=1}^{N} a_i^+ a_i - \frac{N}{2} \tag{15.43}$$

This geberates a $U(1)$ algebra. Actually

$$S = \frac{1}{2}\sum_{i=1}^{N} \sigma_3^i \tag{15.44}$$

in such a way that

$$S\left|0\right\rangle = -\frac{N}{2}\left|0\right\rangle \tag{15.45}$$

16

Automorphisms

We shall dub *inner* such automorphisms that are equivalent to a conjugation

$$T_a \to R T_a R^{-1} \tag{16.1}$$

where

$$R \equiv e^{i\theta^a T_a} \tag{16.2}$$

All other automorphisms are called *outer*. Complex conjugation acts as

$$T_a \to -T_a^* \tag{16.3}$$

This means that an algebra can hace complex representations only it it enjoys nontrivial automorphisms. Sometimes this is trivial, like in the $SU(4)$ exchanging

$$E_{\alpha^1} \leftrightarrow E_{\alpha^3} \tag{16.4}$$

which exchanges the representation D^1 with the $D^3 \equiv \overline{D^1}$ which are non-equivalent. In fact all complex conjugations automorphisms can be obtained from reflexion symmetries of the Dynkin diagram. The opposite is not true: not all reflecion symmetries correspond to complex conjugation. The canonical example is $SO(8)$. Nontrivial automorphisms allow to classify all real forms of complex Lie algebras. Let us see how thos work for the complex algebra A_1. In order to do that it is better to forget about physicist's notation and wrote

$$g \equiv e^{\alpha . T} \tag{16.5}$$

This algebra is generated by

$$T_3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$T_1 \equiv \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$T_2 \equiv \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$
$$I_3 \equiv \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$
$$I_1 \equiv \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}$$
$$I_2 \equiv \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix} \tag{16.6}$$

Restriction to real matrices leaves the algebra of $\mathfrak{Sl}(2, R)$. The algebra is

$$\begin{aligned} [T_1 \equiv T_+, T_2 \equiv T_-] &= T_3 \\ [T_3, T_+] &= 2T_+ \\ [T_3, T_-] &= 2T_- \end{aligned} \tag{16.7}$$

This is exactly what we have been advocating for $SU(2)$. But were we to stick to real generators the algebra would really have been

$$J_i \equiv iH_i \tag{16.8}$$

$$[J_i, J_j] = i\epsilon_{ijk}J_k \rightarrow [H_i, H_j] = \epsilon_{ijk}H_k \tag{16.9}$$

Defining

$$H_\pm \equiv H_1 \pm H_2 \tag{16.10}$$

$$[H_3, H_\pm] = \mp H_\pm \quad]H_+, H_-] = -2H_3 \tag{16.11}$$

This algebra is almost the same as $\mathfrak{Sl}$(2,R). They differ only in

$$T_3 \rightarrow -H_3 \tag{16.12}$$

This can be interpresed as dua to the existence of an *involutive automosphism* in $\mathfrak{Sl}$(2,R)

$$\begin{aligned} \phi\,(T_\pm) &= -T_\pm \\ \phi(T_3) &= T_3 \end{aligned} \tag{16.13}$$

Now *Weyl's unitary trick* intruct to consider the algebra

$$T_{\pm} \to iH_{\pm} \tag{16.14}$$

and this is the *real compact form* of the complex Lie algebra. In this case, $\mathfrak{SU}(2) \subset \mathfrak{SL}(2, C)$. A complex Lie algebra includes menay real forms in general, although only one of them is compact.

Acknowledgments

We have been partially supported by the European Union FP7 ITN INVISIBLES (Marie Curie Actions, PITN- GA-2011- 289442)and (HPRN-CT-200-00148) as well as by FPA2012-31880 (Spain), FPA2014-54154-P, COST action MP1405 (Quantum Structure of Spacetime) and S2009ESP-1473 (CA Madrid). This work is supported by the Spanish Research Agency (Agencia Estatal de Investigaciń) through the grant IFT Centro de Excelencia Severo Ochoa SEV-2016-0597.

Conspectus librorum

[1] L. Schwartz, Methodes mathematiques pour les sciences physiques (Hermann)

[2] Dirk Struik, *Lectures in classical differential geometry* (Dover,1961)

[3] H. Flanders, Differential forms (Academic press,1963)

[4] Dubrovin, Novikok y Fomenko,
Geometria contemporánea (MIR, Moscu, 1979)

[5] M.J. Lighthill, An introduction to Fourier analysis and generalized functions (Cambridge,1958)

[6] IM Gelfand and GE Shilov, Generalized Functions, Vol 1 (Academic Press)

[7] P. Jacobson, *Lie algebras* (Dover)

[8] Hermann Weyl, The classical groups (Princeton)

[9] H. Borner Representations of groups (North Holland 1970).

[10] A. O. Barut and R. Raczka, "Theory Of Group Representations And Applications," Singapore, Singapore: World Scientific (1986) 717p

[11] M. Hamermesh, Group theory (Addison Wesley, 1964)

[12] JP. Serre, Representations lineaires des groupes finis (Hermann,Paris, 1998)

[13] James E. Humphreys, Introduction to Lie algebras and representation theory (Springer-Verlag,1980)

[14] H. Georgi, "Lie Algebras In Particle Physics. From Isospin To Unified Theories," Front. Phys. **54**, 1 (1982).

[15] S. Sternberg, Curvature in mathematics and physics (Dover,2012)
Group theory and physics (Cambridge, 1995)

[16] Pierre Ramond, Group Theory (Cambridge,2010)